Persuading large investors ☆

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ARTICLE INFO

ABSTRACT

A regulator who designs a public stress test to avert default of a distressed bank via private investment must account for large investors' private information on the bank's state. We provide conditions for crowding-in (crowding-out) so that the regulator offers an endogenous more (less) informative signal to better-informed investors. We show that crowding-in occurs as long as investors remain responsive to public news and they are sufficiently well informed: the regulator's test perfectly reveals the state as investors' become privately perfectly informed. Investors' value from more precise private signals may come from their effect on the public test's precision.

1. Introduction

> *We didn't intend to preemptively nationalize major banks, and we didn't intend to let them fail...the centerpiece of our approach was a "stress test"...if an unhealthy firm couldn't raise enough money from private investors, government would forcibly inject the missing capital.* Geithner (2014, p. 11).

Failure of a systemic bank can send tremors across the economy leading to bankruptcies, job losses, and even a recession. During the financial crisis of 2007-08 it was believed that default of certain "financial bombs" could even lead to "a reprise of the Depression," substantiating their "too big to fail" moniker (see Geithner (2014, pp. 4–5)). However, even when the regulator stands ready to recapitalize any bank,[1] she is strongly averse to doing so using public funds.[2] A way out of this conundrum is for private, institutional investors to come to the rescue. Direct communication of the regulator with private investors, as well as targeted monetary incentives

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(e.g., tax cuts) to foster private investment, might appear to violate the public need for transparency and frugality. The channel that then naturally arises (see above quote) is the public disclosure of stress test results by regulators.[3,4]

However, regulators and private investors do not necessarily have the same objectives. The regulator, in deciding whether to recapitalize a distressed bank, considers all stakeholders, including the bank's existing creditors and employees, as well as any spillovers to the economy. Private investors maximize their own shareholders' value. This can create a conflict of interest, which leads to cases where a regulator may prefer that, for example, default of an insolvent bank is averted because that will safeguard jobs, while private investors disagree.

A key characteristic of private investors who may come to the rescue of a large financial institution is that they are themselves *large*, including other banks,[5] pension funds,[6] and (foreign) sovereign wealth funds.[7] By virtue of their size they can each have a pivotal effect on the success or failure of any recapitalization effort. Moreover, private investors have information on market activities that naturally place them at an advantage to evaluate the health of peer banks, especially when they share common exposures or financial linkages (e.g., via the interbank lending market, affinity of business models, overlapping clients, etc.). Finally, the recapitalization effort can, in principle, be driven by more than one large investor; this may be due to not enough "deep pockets" of any single private investor, risk-sharing motives, and a pursuit of competition policy by the regulator. Therefore, there exists a coordination motive (i.e., an externality) in investors' recapitalization decisions.

Against this background, we consider the optimal stress test design by a regulator and analyze its dependence on private investors' own information. We consider a model (see Section 2) with the following ingredients: a single bank in distress whose financial health can be in one of two possible states: *good* (e.g., with liquidity issues) and *bad* (e.g., with solvency issues); a regulator who designs a stress test to minimize the state-contingent probability of default via private investment; and two investors with imperfect private information on the bank's state—who observe the public stress test—and can take two possible actions *invest* (recapitalize) or *not*[8]; joint investment is necessary to avert default and doing so results in a positive externality, regardless of the state. Averting default in any state is beneficial given the systemic nature of the bank.

The centerpiece of our analysis is the interplay between private and public information in the presence of a coordination motive: how the accuracy of investors' (exogenous) private signals (their *expertise*)[9] drives the informational content of the (endogenous) stress test. In particular, we ask: does the regulator disclose more informative tests to investors with higher expertise—so that privately better-informed investors "crowd-in" the regulators' public test—or instead offers less informative tests—leading to "crowding-out" of the regulator's test?

We summarize our results as follows. First consider the case where the stress test is absent. Then, investors coordinate their decisions on the basis of the precision of their private signals (i.e., their expertise); optimal investment rules take the form of threshold strategies, i.e., investing if private signals exceed some level. Thus, the regulator must account for this expertise when deciding what to disclose about the bank's state. Full disclosure (i.e., performing a test that perfectly reveals the bank's state) is always an option and makes investors' expertise irrelevant: they would disregard their own research and jointly act on the basis of the public test's outcome. However, full disclosure is never optimal if the regulator prefers to avert default regardless of the state. In this case, private information of especially large influential investors is a key feature and affects directly the regulators' optimal design (see Section 3).

Two salient types of tests deserve special consideration: Critical-Fault (CF) tests reveal that the bank is in the bad state, discouraging private investment, while Coordinated-Investment (CI) tests provide strong evidence of a good state, prompting investors to disregard their signals and jointly invest. The distinguishing feature of these tests is that they dispel strategic uncertainty (while fundamental uncertainty may remain), a finding that we contrast with the relevant literature (e.g., Inostroza and Pavan (2023)). Which type of test the regulator ultimately runs depends on the ex-ante beliefs of market participants (investors): Critical-Fault tests are optimal when investors are (ex-ante) pessimistic, while Coordinated-Investment tests are selected only if investors are moderately (ex-ante) optimistic.

Now, to elucidate if there is crowding-in or out, which is our main focus, it is instructive to decompose the regulator's problem in two steps. First, the regulator solves for the optimal threshold given investors' private signals. Second, the regulator solves for the test outcome that would lead investors in equilibrium to select that optimal threshold. While the first step depends on the distributional properties of investors' signals, the second step is made easier if investors become more responsive to public news. We then show in Section 4 that investors' expertise crowds in optimal CF-tests if better-informed investors invest less aggressively than the regulator's revised optimal threshold.

Our results become more transparent when considering the limit case as private signals become perfectly informative of the state. In Section 4.3 we show that, under general conditions, the regulator provides a perfectly informative test as investors' signals

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perfectly reveal the bank's returns–we refer to this case as *asymptotic crowding-in*. This follows as long as investors remain responsive to public news, which in turn requires a bound on how locally informative private signals become. For instance, we show that asymptotic crowding-in always obtains if investors' signals are discrete, and generically whenever investors signals always reveal one state with positive probability (a case we dub "Conclusive News Signals"). In sum, asymptotic crowding-in implies that private information reinforces public information: regardless of their prior belief, the regulator provides more information to investors if they are *sufficiently* better-informed.

Our interest in the question of crowding-in or -out is twofold. First, the effect of changes in expertise on investors' welfare is indirectly informed by the impact on the (endogenous) stress test. Indeed, we show in Section 5.2 that better informed investors can benefit more from the test if the regulator preemptively provides a more informative test. Second, private improvements in expertise may increase or decrease the information provided to other potential investors and stakeholders by the public disclosure of the stress test. The regulator in designing her test does not maximize the information provided; if she did, then full disclosure would be optimal—an environment that is both theoretically uninteresting but also practically implausible[10]; her purpose is to prevent financial havoc, which makes averting default necessary. However, many investors value more rather than less information to reach their optimal decision.[11] Therefore, our results on crowding-in show that, under general conditions, there are positive informational spillovers from improvements in private information of large private investors on other market participants, see Section 5.2.

As we conclude in Section 6, we offer takeaways that can guide empirical analysis studying how the regulator's and investors' information is related to stress test announcements. Our main result is that a regulator should provide more (public) informative signals to sufficiently better (privately) informed investors (i.e., with sufficiently higher expertise), as long as, they are responsive to public news. Empirically, we can proxy the level of information in the public signal with the (stock) market reaction to its announcement. In turn, we can measure the level of investors' expertise by the number of analysts they employee (e.g., see Cowen et al. (2006) for data on analysts and their employers); as expected larger investors employ more analysts and hence have higher expertise, on average. Finally, an investor's reactiveness to public information can be assessed by their trading decisions around the announcement.[12]

The Appendix contains all the proofs. In the Online Appendix we: i) study the equilibrium payoffs of investors at the optimal test (Section OA.1), ii) analyze the case of "small" correlated mistakes (Section OA.2, iii) study the case of negative externalities (Section OA.3), iv) perform a comparative statics analysis of the CF and CI tests with respect to their coordination motive (Section OA.5) and with respect to investors' expertise (Sections OA.6 and OA.7).

Finally, while motivated by the optimal design of stress tests in a financial setting, our analysis with positive investment externalities sheds light on the optimal disclosure policy when persuading large privately informed players to act in other scenarios as well, e.g., currency crises where the regulator wants to maintain a peg (Corsetti et al. (2004)). Moreover, the case of negative externalities (Section OA.3) can be applied in settings where the regulator pursues joint investment (not driven by the goal to avert default) in a market with competitive forces. For example, our "investors" could be two pharmaceutical companies considering investing in developing the same vaccine, where the state reflects the (market) need for such innovation (+1 "high" or −1 "low"), while the relevant policy maker has precautionary motives and considers the wider public health. Another application is the design of voting recommendations by proxy advisors (e.g., ISS); in such corporate voting setting there are multiple institutional shareholders/voters (e.g., mutual funds) with substantial holdings and private information (Malenko et al. (2021)) via their potential links with the firm (Cvijanović et al. (2016)).

1.1. Related literature

Our paper is related to, and borrows from, different literatures. First is the theoretical literature on Information Design and Bayesian Persuasion (Kamenica and Gentzkow (2011)).[13] The extant papers study how one or multiple senders can further their goals by selectively disclosing information to other players (receivers) (see Bergemann and Morris (2019) and Kamenica (2019) for excellent overviews). We contribute to the strand of this literature that studies persuasion of multiple, privately informed, receivers. There have been many papers studying each of these cases separately.[14] Combining in this paper both features (multiple receivers that observe exogenous private signals of a payoff-relevant state) allows us to address an important issue in information design: the interaction of (endogenous) public and private information.

A few papers consider, like ours, public persuasion of privately informed receivers. Bergemann and Morris (2016b) and Bergemann and Morris (2016a) provide a characterization of the outcome distributions induced in coordination games by arbitrary information structures. Theirs is a feasibility study; we instead look at the optimal choice of disclosure when a regulator maximizes social welfare. Goldstein and Huang (2016) consider a setting with atomistic investors in which the sender can announce whether or not the fundamentals exceed a given threshold (see also Laclau and Renou (2017)). We allow for more general public disclosure policies and focus on the effect of large investors on stress testing.

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[10] Two empirical papers that study regulatory discretion in the design of the stress test are: Bird et al. (2015), who investigate the possibility that the Federal Reserve might bias the disclosed results of the stress tests in order to promote desirable market outcomes); and Agarwal et al. (2014), who investigate whether regulatory effectiveness depends on rules and/or regulators' incentives.

[11] The famous exception to this rule is the Hirshleifer effect, see, e.g., Goldstein and Leitner (2018) in the context of stress tests.

[12] The trades need not be in the considered bank, as long, as they can be attributed to the announcement. For example, financial institutions ofter trade in US Treasuries if they fear a systemic bank may fail, in a "flight to quality" trade (e.g., see Dicks and Fulghieri (2019)).

[13] See also Brocas and Carrillo (2007), and Rayo and Segal (2010).

[14] For instance, the case with multiple, albeit initially uninformed, receivers is the focus of Alonso and Câmara (2016b), Bardhi and Guo (2018), Mathevet et al. (2020), Bergemann and Morris (2016b), Che and Hörner (2018), Arieli and Babichenko (2019), Li et al. (2019), and Taneva (2019) among others. In turn, persuasion of a single, privately-informed receiver is explored in Kolotilin et al. (2017), Kolotilin (2018), Alonso and Câmara (2016a), and Guo and Shmaya (2019), among others.

The closest paper to ours is Inostroza and Pavan (2023), who analyze the robust design of stress tests in a global game of regime change. While Inostroza and Pavan (2023) also consider financial regulation as their main application (see also Inostroza (2019)), there are two main differences with our paper. First, by considering a robust approach when coordinating atomistic investors they find that the regulator's optimal stress test has the perfect coordination property: all investors select the same action for any realization of the test,[15]—i.e., optimal testing dispels any strategic uncertainty about investors' actions. We, instead, consider persuasion of large investors and optimal tests will seldom eliminate all strategic uncertainty—in fact, we find that investors' reliance on their expertise will typically lead them to an uncoordinated response. Second, we solve for the optimal stress test as a function of investors' private signals which allows us to assess how the information conveyed by the public test depends on these signals.[16,17]

We also contribute to the theoretical literature on Stress Testing and Financial Disclosures.[18] For an excellent overview of stress test design and regulatory disclosure in the financial system see Goldstein and Sapra (2014). In Gick et al. (2014) a regulator discloses information over the health of the banking sector to induce an optimal share of investors to invest. Bouvard et al. (2015) study a credit rollover setting where a policy maker must choose between transparency (full disclosure) and opacity (no disclosure) but cannot commit to a disclosure policy. Castro et al. (2014) demonstrate that stress tests will be more informative when the regulator has a strong (fiscal) position. Orlov et al. (2018) focus on a regulator who designs a stress tests to reveal information over the value of a risky asset held by multiple banks. Goldstein and Leitner (2018) characterize the optimal stress test when the regulator tries to promote risk-sharing opportunities amongst banks. Williams (2017) analyzes the impact of stress tests on banks' portfolio choices. Quigley and Walther (2020) study parametrized stress test design when banks can preemptively engage in costly, certifiable disclosure of the state, and show that reducing the noise in the stress test crowds-out voluntary disclosure by banks. Shapiro and Zeng (2020) focus on capital requirements and banks' endogenous choice of risk as key elements of stress testing. Ding et al. (2022) study the incentives of a regulator to coarsen the information they provide in a stress test, in order to encourage information production in the financial markets.

Our paper differs from all the above in several important aspects: (i) there is no constrain on the design of the optimal test; (ii) the analysis is applicable to positive and negative externalities amongst large—i.e., non-atomistic—and privately informed players; (iii) we provide conditions for public information to either crowd-in or crowd-out private information.

The third literature strand we relate to is the theoretical literature on Endogenous Public Signals in Global Games.[19] Applications include: a central bank strategically shaping its public announcements (as in Morris and Shin (2002), Cornand and Heinemann (2008), Angeletos and Pavan (2009), James and Lawler (2011), and Chahrour (2014)); policy makers either signaling their private information through their policy choices (Angeletos et al. (2006)); or directly shaping the informativeness of public announcements (Edmond (2013), Cornand and Heinemann (2008)); or the nonstrategic disclosure of summary statistics of agents' actions, for instance as conveyed by market prices (Vives (2017), Bayona (2018)). Our paper differs both in methodology (information design, which imposes no parametric restrictions on stress test designs) and also focus (the impact of private signals on the informational content of public tests).

2. Model

We consider a stylized model of financial regulation with: a single distressed bank with a risky asset and outstanding liabilities; a regulator (she) who can disclose information about the asset's return through a stress test, in order to minimize the state-contingent probability of default via private investment; two large private investors that must decide whether to invest in the bank; investment is subject to a positive externality and sensitive to investors' private information as well as the public information revealed by the stress test. We abstract from many institutional details as we focus on the interplay between public and private information.

Investors preferences and private information: There are two time periods $t = 0, 1$ and no time-discounting between them for either investors or regulator. A single representative bank has an illiquid risky asset in-place, which yields random (net) return $\omega \in \{-1, 1\}$ at $t = 1$. The bank also has outstanding debt liabilities (e.g., short-term debt) maturing at $t = 0$ of two units of capital; it has no liquidity to service this liability on its own and is in need of outside investment (i.e., it is in financial distress). We refer to ω as the *state* and its possible values $-1, 1$ as "bad" (e.g., the bank is insolvent) and "good" (e.g., illiquid but solvent). If the liability is not paid by the end of $t = 0$, then the bank defaults at $t = 1$, after ω is realized; if it is paid in full then the bank is "saved" and at $t = 1$ on top of ω claimholders also receive a known (net) return $\gamma \in (0, 1)$ that reflects the (expected) going concern value of the bank.

Two expected-utility maximizing, risk-neutral investors $i = 1, 2$ simultaneously consider to either invest or not a unit of capital, $a_i \in \{1, 0\}$, in the bank at the start of $t = 0$. If investor i chooses $a_i = 1$, then the firm can service half of its debt and the investor has

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[15] Our analysis also allows for a robust approach, i.e., to look at the worst equilibrium for the regulator. However, even then we do not have perfect coordination. For such result the assumption of atomistic investors in Inostroza and Pavan (2023) is key.

[16] See also Basak and Zhou (2020) for an analysis of the optimal dynamic disclosure policy in global games of regime change. Alonso and Câmara (2016a) consider persuasion of non-interacting receivers with heterogeneous priors where the coordination motive studied here is absent.

[17] In the case of adversarial selection and with discriminatory disclosure Li et al. (2023) and Morris et al. (2023) find that optimal information policy in coordination games should eliminate strategic uncertainty and perfectly coordinate investors' actions; Dai et al. (2022) find that with selection power the optimal policy still eliminates strategic uncertainty but may benefit from miscoordinating the market. None of the above papers, however, feature *bank specific* private information for investors.

[18] There is some natural overlap with the first literature category and, to an extent, the separation between the two is artificial.

[19] Two empirical papers that study public information as a coordination mechanism in a financial context are Chen et al. (2010) for a study in mutual fund complementarities) and Hertzberg et al. (2011) for the amplified role of public information in a related context).

Table 1
Investor Payoffs.

		a_2	
		1	0
a_1	1	$\omega+\gamma, \omega+\gamma$	$\omega,0$
	0	$0,\omega$	$0,0$

a claim on any return at $t=1$; the payoff of non-investment is normalized to zero. Given two investors and two units of outstanding debt, default can be averted only if they both invest (e.g., because none of them has enough "deep pockets"); this is a stylized version in our two-player setup of the fact that a sufficient amount of capital needs to be raised when a bank is in distress and investors have position limits (e.g., see Inostroza and Pavan (2023), p. 29) for similar assumptions). Hence, return γ captures the coordination motive in averting default by co-investment, and we refer to it as (positive) *externality*.[20] Summarizing, investor i's ex-post (i.e., at the end of $t=1$) utility is $u_i^{ex-post} = a_i(\omega + \gamma a_j)$, as seen in Table 1.[21]

The stage game at $t=0$ is equivalent to the baseline model in global games (Carlsson and Van Damme (1993); Morris and Shin (2001)) and other coordination games (Baliga and Morris (2002); Mathevet et al. (2020)). Investor behavior is easily predicted if they know the state: investing is a dominant action if $\omega=1$ and is dominated if $\omega=-1$.[22] All players process information according to Bayes' rule and hold a common prior belief $\mu_0 = \Pr[\omega=1]$. In the absence of additional private or public information, investors would perfectly coordinate their decisions if their prior belief is sufficiently precise: investing is dominant whenever $\mu_0 > 1/2$ while non-investing is dominant if $\mu_0 < (1-\gamma)/2$. Hence, the coordination problem arises only under incomplete information.[23]

Each investor $i=1,2$ observes a private signal $x_i \in X \subseteq [0,1]$ (any compact subset of $\mathbb{R}$ would suffice). We concentrate on the symmetric case, so that signal distributions do not depend on the investor's identity. Hence, define:

$$F(x) \equiv \Pr[x_i \leq x | \omega = 1], \quad G(x) \equiv \Pr[x_i \leq x | \omega = -1], \qquad i = 1, 2,$$

$$F(y|x) \equiv \Pr[x_i \leq y | \omega = 1, x_j = x], \quad G(y|x) \equiv \Pr[x_i \leq y | \omega = -1, x_j = x], \quad i \neq j.$$

Moreover, f, g denote the corresponding densities, and $\overline{F}$, $\overline{G}$ the complementary distributions. We introduce some additional notation to describe the distributions F and G. Let $\lambda(x) \equiv f(x)/g(x)$ denote the likelihood ratio of a good state, with $\overline{\Lambda}(x) \equiv \overline{F}(x)/\overline{G}(x)$, while $h_F(x) \equiv f(x)/\overline{F}(x)$ and $h_G(x) \equiv g(x)/\overline{G}(x)$ represent the hazard rates of F and G. We will assume that f and g are continuous in X, while $F(y|x)$ and $G(y|x)$ are continuously differentiable in X^2.[24] We further assume the following:

A1. $\lambda(x)$ is strictly increasing for $x \in \{t \in [0,1] : f(t) > 0 \text{ and } g(t) > 0\}$ (MLRP).
A2. $\overline{F}(k|x)$ and $\overline{G}(k|x)$ are non-decreasing in x for $x \in [0,1]$.

The assumption of a monotone likelihood ratio (Assumption A1) implies that higher realizations of x_i are indicative of a high state, and it is a standard assumption in signaling models. Assumption A2 implies that (for our case of a positive externality) investors' signals exhibit positive dependence for each state: conditional on ω, a higher realization of x_i makes it more likely that the other investor's signal exceeds a fixed threshold. Note that it is satisfied in the important case that signals are conditionally independent, so that for all $(x,y) \in X^2$, $\overline{F}(y|x) = \overline{F}(y)$ and $\overline{G}(y|x) = \overline{G}(y)$.

Stress testing (strategic experimentation): The regulator's preferences over investors' choices are given by:

$$u_R^{ex-post} = a_i a_j [\eta \mathbb{I}(\omega = -1) + \mathbb{I}(\omega = 1)], \eta \in [-1, 1],$$

where $\mathbb{I}(\cdot)$ is the indicator function. Hence, the regulator wants to minimize the state-contingent probability of default or, equivalently, to maximize the probability of joint investment—e.g., as in Inostroza and Pavan (2023)—with η representing her payoff from doing so when the state is bad.[25] For instance, if $\eta < 0$ the regulator is fully aligned with players in that she would like them to invest iff $\omega = 1$, while if $\eta > 0$ the regulator is pro-investment regardless of the state.

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[20] We extend the formal analysis for $\gamma \in (-1, 0)$, see Section OA.3 in the Online Appendix. In the context of the example in the Introduction, just before Section 1.1, γ is then the loss in individual profits from competition.

[21] The assumption we make here is that a solvent bank which defaults because of liquidity issues is more valuable than an insolvent bank under default—for instance, because a solvent bank has assets that, once liquidity is restored, can be sold to compensate creditors/investors even within default.

[22] Moreover, the unique investment equilibrium is also Pareto efficient (from the investors' perspective).

[23] Other closely related papers consider instead a continuum of agents (e.g., the regime-change global game of Inostroza and Pavan (2023) based on Rochet and Vives (2004); the linear-quadratic global game of Bouvard et al. (2015) and Quigley and Walther (2020) based on Morris and Shin (2000)). Our finite player setup aims to emphasize the pivotal role of large investors in recapitalizing systemic institutions.

[24] We depart from these smoothness conditions on distributions in Section 4.3 when we study the case of discrete signals. We will also appeal to stricter smoothness conditions as required for some of our results.

[25] The regulator's preferences capture in a stark way two general properties: they are i) increasing in the investors' actions; and ii) weakly increasing in the state (see also Bouvard et al. (2015), Mathevet et al. (2020), and Quigley and Walther (2020)).

In particular, the regulator and investors agree that a bank with $\omega = 1$ (e.g., illiquid but solvent) is a worthwhile investment; in this case default can be averted by the regulator announcing $\omega = 1$ which leads to both investing.[26] The disagreement between regulator and investors may be for banks with $\omega = -1$ (e.g., with a large proportion of non-performing loans): investors would not invest if they knew the state, because that would hurt their own shareholders; while the regulator might still want to avert default of an $\omega = -1$ bank because she takes into account the welfare of the bank's other claimholders as well as the health of the overall financial system (see, e.g., Kupiec and Ramirez (2013) for the large costs of bank failures to the economy as a whole). Parameter η then encapsulates the *conflict of interest* between investors and regulator.

The regulator can selectively disclose information about asset returns in the form of a stress test π, in order to influence investment decisions. The test consists of a finite realization space $S(\pi)$ and a family of likelihood functions over $S(\pi)$, $\{\pi(\cdot|\omega)\}_{\omega \in \{-1,1\}}$, with $\pi(\cdot|\omega) \in \Delta(S(\pi))$. Given the common prior, we can without loss take $S(\pi) \subset \Delta(\{-1,1\})$, so that we can represent $\pi = \{q_i, \Pr[q_i]\}_{i \in I_\pi}$ as a distribution over posterior beliefs $q_i \in S(\pi)$ induced by observing the test's outcome, where I_π indexes all possible outcomes. The test's realization is conditionally (on ω) independent of investors' signals, so that the test only allows investors to better coordinate their investments by dispelling fundamental uncertainty.

We make two important assumptions regarding the test design. First, as in Kamenica and Gentzkow (2011), the regulator can commit to *any test* that is correlated with the state. Second, we abstract from the costs of designing, implementing and disclosing the test, as the regulator perceives all tests to be costless.[27]

Timing: The regulator publicly selects $\pi = \{q_i, \Pr[q_i]\}_{i \in I_\pi}$. Investor i observes the public realization of π and privately observes x_i, updates his beliefs according to Bayes' rule, and both investors simultaneously make investment decisions. We look for Perfect Bayesian Equilibria (PBE). If there are multiple PBEs we concentrate on the equilibrium that maximizes the regulator's ex-ante expected utility.[28] This will be relevant only if $\eta > 0$, in which case we select the equilibrium that maximizes the probability of joint investment.

3. Optimal tests

The regulator can ensure that both investors disregard their signals and jointly invest if $\omega = 1$ by providing a perfectly informative test, albeit foregoing at the same time any investment when $\omega = -1$. This is optimal for the regulator whenever $\eta \leq 0$ as she then wants to match private investment to the state—see Lemma 2 below. However, when $\eta > 0$ the regulator prefers to avert default regardless of the state. To understand how she can boost joint investment through selective disclosure, we first analyze equilibrium investment behavior when the test outcome leads investors to an interim common posterior $\mu = \Pr[\omega = 1]$.

3.1. Investors' equilibrium behavior

Suppose that investor j follows a threshold strategy and invests if his private signal exceeds a threshold, $x_j \geq k_j$. For what values of (k_i, k_j) is this an equilibrium of the investment subgame? Investor $i's$ expected interim value from investing after observing $x_i = x$ and given $j's$ threshold rule is

$$v_i(k_j, x; \mu) \equiv \mathbb{E}[\omega|x_i = x] + \gamma \mathbb{E}[a_j|x_i = x]$$
$$= \left(1 + \gamma \overline{F}(k_j|x)\right) \Pr[\omega = 1|x_i = x] - \left(1 - \gamma \overline{G}(k_j|x)\right) \Pr[\omega = -1|x_i = x].$$

Assumptions A1 and A2 guarantee that $v_i(k_j, x; \mu)$ is strictly single-crossing in x, so that investor's i best response must also be a threshold strategy. Then, any interior equilibrium (k_i, k_j) requires $v_i(k_j, k_i; \mu) = 0$ and $v_j(k_i, k_j; \mu) = 0$, that is,

$$\frac{\Pr[\omega = -1|x_i = k_i]}{\Pr[\omega = 1|x_i = k_i]} = \frac{1 + \gamma \overline{F}(k_j|k_i)}{1 - \gamma \overline{G}(k_j|k_i)}, \text{ and } \frac{\Pr[\omega = -1|x_j = k_j]}{\Pr[\omega = 1|x_j = k_j]} = \frac{1 + \gamma \overline{F}(k_i|k_j)}{1 - \gamma \overline{G}(k_i|k_j)}.$$

In words, at the equilibrium thresholds, the odds of the bad state equals the ratio of the gains from investing if returns turn out to be high to the investment losses if returns are actually low, given the revised likelihood that the other investor also invests. Since the odds of the bad state following $x_i = x$ are $(1 - \mu)/\mu\lambda(x)$, we can succinctly express these conditions as:

$$\lambda(k_i)\frac{1 + \gamma \overline{F}(k_j|k_i)}{1 - \gamma \overline{G}(k_j|k_i)} = \frac{1 - \mu}{\mu} = \lambda(k_j)\frac{1 + \gamma \overline{F}(k_i|k_j)}{1 - \gamma \overline{G}(k_i|k_j)}. \tag{1}$$

For instance, when considering symmetric threshold strategies $k_i = k_j = k$ and defining

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[26] Voluntary bail-ins (which are isomorphic to investing here) can substitute bailouts from the regulator, as long as, the probability of the latter is not certain, see Bernard et al. (2022)); the market-perceived probability of bailouts has indeed fallen following the Dobb-Frank Act, see Acharya et al. (2016).

[27] In any case, costly experimentation would not be a serious limitation if all experiments impose the same cost as this would not affect the regulator's choice if he decides to experiment.

[28] Our focus on the regulator-preferred PBE is not essential for our analysis. In Section 3.1 we show how our framework also allows for an adversarial approach (see Inostroza and Pavan (2023)) in which investors coordinate on the worst equilibrium for the regulator.
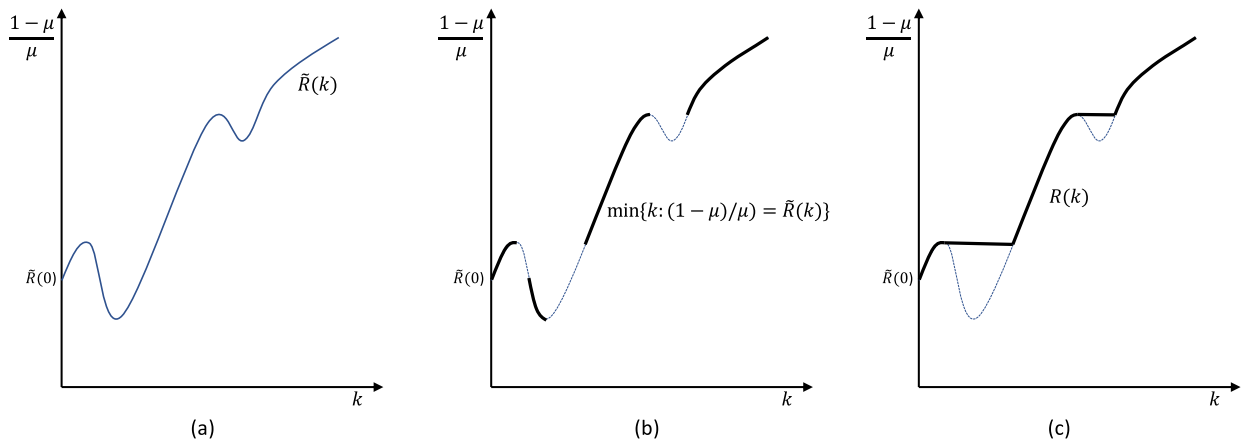
Fig. 1. (a) Function $\widetilde{R}(k)$ for $\gamma > 0$, (b) Equilibrium threshold that maximizes joint investment rates $k(\mu) = \min\left\{k : (1-\mu)/\mu = \widetilde{R}(k)\right\}$, (c) Function $R(k)$.

$$
\widetilde{R}(k) \equiv \overbrace{\lambda(k)}^{\text{Likelihood}} \underbrace{\frac{1 + \gamma \overline{F}(k|k)}{1 - \gamma \overline{G}(k|k)}}_{\text{Coordination}},
\tag{2}
$$

then (1) collapses to the condition

$$
(1 - \mu)/\mu = \widetilde{R}(k).
\tag{3}
$$

Any interior symmetric BNE must satisfy (3). The function $\widetilde{R}(\cdot)$ is the product of two terms: the first is the likelihood ratio of a favorable state at the investment threshold—which, by assumption A1, is always increasing—while the second captures the effect of coordination amongst investors. The following lemma characterizes equilibria $k(\mu)$ of the investment subgame for any interim $\mu = \Pr[\omega = 1]$.

Lemma 1. *Suppose that Assumptions 1 and 2 hold. Then,*

(i) If $\lambda'(x)$ is bounded away from zero for $x \in \{t \in [0,1] : f(t) > 0 \text{ and } g(t) > 0\}$, then there exists $\overline{\gamma} > 0$, so that whenever $\gamma < \overline{\gamma}$ there is a unique equilibrium of the investment subgame, which is symmetric, while for $\gamma > \overline{\gamma}$ there are multiple equilibria, all of them symmetric. In the case of multiple equilibria, we select the regulator-preferred equilibrium, which for $\eta > 0$ induces the highest joint investment, i.e., the lowest investment threshold.[29]

(ii) Define the non-decreasing function

$$
R(k) \equiv \max_{0 \le k' \le k} \widetilde{R}(k'),
\tag{4}
$$

with $R^{-1}(y) = \min\left\{k : y = \widetilde{R}(k)\right\}$. Then, for $\mu \in \left(0, \frac{1}{1+R(0)}\right)$, the equilibrium investment threshold is

$$
k(\mu) = \begin{cases} R^{-1}\left((1-\mu)/\mu\right), & \text{if } \frac{1}{1+R(1)} \le \mu \le \frac{1}{1+R(0)}, \\ 1, & \text{if } \mu \le \frac{1}{1+R(1)}. \end{cases}
\tag{5}
$$

If $\mu > 1/\left(1 + \min_{k \in (0,1)} \widetilde{R}(k)\right)$, then $k(\mu) = 0$.

(iii) $k(\mu)$ is non-increasing, and, whenever differentiable, satisfies

$$
k'(\mu) = -\frac{1}{\mu^2 \, R'(k)|_{k=k(\mu)}}.
\tag{6}
$$

With positive externalities, investors invest more often (i.e., set a lower threshold) if they expect others to increase investment rates. Strategic complementarity implies that all equilibria are symmetric (see Vives (1999, sec. 2.2.3 and fn. 23)), but opens the door for multiplicity of equilibria. This is only possible, however, under a strong coordination motive ($\gamma > \overline{\gamma}$). Fig. 1-a shows that multiplicity is linked to non-monotonicity of the function $\widetilde{R}(k)$. As long as $\eta > 0$, the regulator's preferred equilibrium corresponds to the one with the lowest threshold as it maximizes the likelihood of joint investment. As shown in Fig. 1-b, this corresponds

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[29] As we show in Lemma 2, if $\eta \le 0$ a fully informative test is optimal for the regulator.

to selecting the investment threshold $k(\mu) = \min \left\{ k : (1 - \mu)/\mu = \widetilde{R}(k) \right\}$. Investors are willing to invest regardless of their private signals if $k(\mu) = 0$—i.e., if $\mu = 1/(1 + \widetilde{R}(0))$—and the regulator will never select a stress test with realizations $\mu > 1/(1 + \widetilde{R}(0))$ if $\eta > 0$. Therefore, when restricting to undominated stress tests for $\eta > 0$, equilibrium investment can be expressed as $k(\mu) = R^{-1}((1 - \mu)/\mu)$ with R defined by (4); this function is depicted in Fig. 1-c.

In summary, $R(k)$ embodies the equilibrium sensitivity of private to public information, as *investor's responsiveness to public news*—as given by (6)—is inversely related to the slope of $R(k)$. Moreover, when the evidence produced by the test is sufficiently compelling, investors disregard their private signals when making investment decisions: if $\mu = 1/(1 + R(0))$ investors always invest, while if $\mu \leq 1/(1 + R(1))$ they never invest, irrespective of the realization x_i.

Our focus on the regulator's preferred equilibrium is not essential for the analysis: for any non-increasing equilibrium selection $k_s(\mu)$, so that $(1 - \mu)/\mu = \widetilde{R}(k_s(\mu))$, we can replace $R(k)$ with $R_s(k)$ satisfying $R_s^{-1}((1 - \mu)/\mu) = k_s(\mu)$ and our characterization of the optimal tests and comparative statics in Section 4.1 would still hold. For instance, we could study an adversarial approach (see Inostroza and Pavan (2023)) in which investors coordinate on the equilibrium that minimizes the regulator's payoff by instead considering the non-decreasing function $\underline{R}(k) \equiv \min_{k \leq k' \leq 1} \widetilde{R}(k')$.

3.2. Characterization of optimal tests

Lemma 1 captures the persuasive properties of the stress test: the regulator can induce an investment threshold k by providing evidence that lead to an interim belief $\mu = 1/(1 + R(k))$. Of course, the regulator's ability to influence investors is limited as the average interim belief must be equal to the prior; that is, the test must be Bayesian plausible.

To study the interplay between private and public information, we concentrate on the conditionally independent case and leave a treatment of the case of correlated signals for Section OA.2 in the Online Appendix.[30] The regulator's interim expected utility when investors follow a k-rule is then $\widetilde{u}_R(k; \mu) \equiv \overline{F}^2(k)\mu + \eta \overline{G}^2(k)(1 - \mu)$ or for the equilibrium $k(\mu)$ in Lemma 1

$$u_R(\mu) \equiv \widetilde{u}_R(k(\mu); \mu) = \overline{F}^2(k(\mu))\mu + \eta \overline{G}^2(k(\mu))(1 - \mu). \tag{7}$$

Adapting Kamenica and Gentzkow (2011) to our setting, the following corollary (of their results), characterizes the regulator's optimal stress test by computing the concave closure of u_R.

Corollary 1. *The regulator's test can be described with the help of a partition of $[0, 1] = D \cup N$ into two disjoint sets of intervals defined by signal realizations $\left\{ \mu_i \right\}_{i=1,\ldots,I}$ as follows:*
(i) Interval $[\mu_i, \mu_{i+1}] \subseteq N$ ("non-disclosure") if for prior $\mu_0 \in [\mu_i, \mu_{i+1}]$ there exists $\gamma(\mu_0)$ such that:

$$u_R(\mu) \leq u_R(\mu_0) + \gamma(\mu_0)(\mu - \mu_0), \ \mu \in [0, 1]. \tag{8}$$

For all priors μ_0 in this interval a completely uninformative test is optimal.
(ii) Interval $(\mu_i, \mu_{i+1}) \subseteq D$ ("disclosure") if

$$\mu_{i+1} \in \arg \max_{\mu_i < \mu \leq 1} \frac{u_R(\mu) - u_R(\mu_i)}{\mu - \mu_i}, \tag{9}$$

$$\mu_i \in \arg \min_{0 \leq \mu < \mu_{i+1}} \frac{u_R(\mu_{i+1}) - u_R(\mu)}{\mu_{i+1} - \mu}. \tag{10}$$

For all priors μ_0 in this interval a test with two realizations that induces public posteriors μ_i and μ_{i+1} is optimal.

Corollary 1 can be described with the help of Fig. 2, which depicts an example of a u_R and its associated concave closure U_R for $\eta > 0$, where the partition in Corollary 1 comprises the non-disclosure region $N = \{\{0\}, [\mu_1, \mu_2], [\mu_3, \mu_4], [\mu_5, 1]\}$ and the disclosure region $D = \{(0, \mu_1), (\mu_2, \mu_3), (\mu_4, \mu_5)\}$.

There are situations where the regulator prefers to dispel any uncertainty and disclose the state. For instance, if $\eta \leq 0$, then she would like to avert default only under a good state, so that it is clear that the optimal test fully discloses ω. However, a fully informative test ceases to be optimal if she prefers investors to invest regardless of the state (that is, when $\eta > 0$).

Lemma 2. *(i) For $\eta \in [-1, 0]$ a fully informative test is optimal. (ii) For $\eta \in (0, 1]$ a fully informative test is never optimal, as long as $g(0) > 0$ and $g'(0)$ is bounded.*

3.3. "Critical-Fault" and "Coordinated-Investment" tests

While the optimal stress test may vary subtly with the prior belief of investors—as it is the case, for instance, in Fig. 2—there are two regions in which the optimal stress test leads investors to perfectly coordinate their actions. For sufficiently low priors, the

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[30] We show that allowing for some correlation between investors' signals does not change our main results as the regulator's optimal test is invariant to the presence of "small" correlated mistakes.
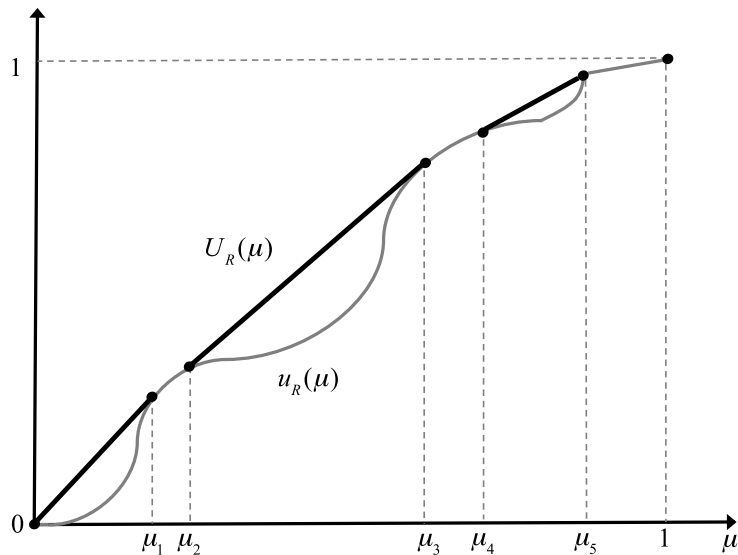
Fig. 2. Regulator's indirect utility u_R and associated concave closure U_R, defined via the cutoffs $\{\mu_1, \ldots, \mu_5\}$.

"Critical-Fault" test (CF-test) may conclusively establish that returns are low and discourage investment, regardless of the investors' private signals,[31] while, for sufficiently high priors, the "Coordinated-Investment" test (CI-test) leads investors to disregard their private signal and simultaneously invest after a favorable test outcome.[32]

Our interest in these salient tests is twofold. First, it allows us to contrast our findings to the literature that singles out tests with the perfect-coordination property (see Inostroza and Pavan (2023)). Second, they form the basis for our analysis of the interplay between private-public information in Section 4. We now discuss these tests in more detail and provide conditions for their existence. In what follows we assume that R is differentiable almost everywhere so that u_R is also.

Critical-Fault test   The CF-test is defined by a realization $\underline{\mu}$ such that for prior $\mu_0 \in (0, \underline{\mu})$ the test either discloses that the state is $\omega = -1$, or improves investors' interim belief to $\underline{\mu}$. This affords a simple practical implementation: the regulator specifies a range of scenarios to simulate and the test reveals whether the bank would pass all (in which case investors' posterior is $\underline{\mu}$) or would fail some scenario, thus conclusively revealing that $\omega = -1$.

Optimality of the CF-test requires that (9) holds with $\mu_i = 0$ and $\mu_{i+1} = \underline{\mu}$. Alternatively, let $A_R(k)$ be the regulator's average utility when the test's outcome induces threshold k,

$$A_R(k) \equiv \frac{\widetilde{u}_R(k; \mu(k))}{\mu(k)} = \overline{F}^2(k) + \eta R(k)\overline{G}^2(k). \tag{11}$$

Then (9) can be rewritten as:

$$\underline{\mu} \in \arg\max_{0 < \mu \leq 1} \frac{u_R(\mu)}{\mu} = A_R(k(\mu)). \tag{12}$$

That is, the CF-test is found in two steps: The regulator derives the threshold k^* that maximizes the regulator's average utility, $A_R(k^*) \equiv \max_{0 \leq k \leq 1} A_R(k)$, and then computes the test outcome $\underline{\mu}$ that induces investors to select k^* in equilibrium, i.e., $\underline{\mu} = 1/(1 + R(k^*))$.

Lemma 1 shows that for a sufficiently low prior—i.e., when $\mu_0 < 1/(1 + R(1))$—investors will never invest in the absence of a test. Thus, $u_R(\mu_0) = 0$ and the regulator can always increase average joint investment through the CF-test. However, $1/(1 + R(1)) > 0$ requires that private signals are never perfectly informative of $\omega = 1$ (i.e., $g(1) > 0$). Therefore, a sufficient condition for the existence of the CF-test is that investors' signals do not reveal that returns are high.

Lemma 3. *If $\eta > 0$, then the CF-test exists if $\lambda(1)$ is finite.*

Coordinated-Investment test   The CI-test is defined by realizations $\mu_{CI} = 1/(1 + R(0))$ and $\overline{\mu}$ such that for prior $\mu_0 \in (\overline{\mu}, \mu_{CI})$ it leads to the bad news $\overline{\mu}$ or to coordinated investment after the favorable news μ_{CI}—see Lemma 1. One possibility is for the test to dispel any fundamental uncertainty and reveal that $\omega = 1$. This is the case if $R(0) = 0$, which requires $\lambda(0) = 0$ so that investors' private signals

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[31] The regulator could induce investors to perfectly coordinate in not investing even if they remain uncertain about the state by disclosing μ' such that $0 < \mu' < 1/(1 + R(1))$ (see Lemma 1). From Corollary 1, this is never optimal for the regulator as she can increase the likelihood of joint investment by disclosing $\mu = 0$ instead.

[32] To ease the exposition, we simply talk of the "CF-test" or "CI-test" when referring to both the actual signal realizations of those tests and the regions in the prior belief space where the regulator's optimal test comprises those realizations.

can detect the occurrence of low returns. If $R(0) > 0$, however, the CI-test eliminates any strategic uncertainty while fundamental uncertainty remains: favorable evidence after outcome $\mu_{CI} < 1$ is so compelling that investors invest regardless of their signals.

Optimality of the CI-test is given by (10) with $\mu_{i+1} = \mu_{CI} \equiv 1/(1 + R(0))$. In the example of Fig. 2 both CF- and CI-tests exist with $\underline{\mu} = \mu_1$, $\overline{\mu} = \mu_4$, and $\mu_{CI} = \mu_5$. Trivially, whenever a fully informative stress test is optimal, it would fall under both categories. The following lemma provides sufficient conditions for the existence of the CI-test.

Lemma 4. *Suppose that $\eta \in (0, 1]$. (i) If f and g are twice-continuously differentiable in $[0, 1]$, then the optimal test never reveals $\omega = 1$. (ii) If $R(0) > 0$, then, the CI-test exists if*

$$R'(0) < 2f(0)(1 + R(0))\left(\frac{1}{\eta} + \frac{1 + \gamma}{1 - \gamma}\right). \tag{13}$$

The role of stress tests that perfectly coordinate investors has been highlighted by Inostroza and Pavan (2023). They show in a model with a continuum of investors (i.e., "small" investors) that for the worst-case scenario, in which investors select the most adversarial profile of rationalizable strategies—in our case, the profile that minimizes joint investment—it is without loss to consider tests that perfectly coordinate investors' actions for every test outcome. This hinges on the regulator's ability to predict, given investors' behavior, whether default is averted from knowledge of the state and the test outcome only; this is possible with a continuum of investors by appealing to the law of large numbers—see Inostroza and Pavan (2023). Then, the regulator can append the previous test with information of the default outcome given investors' adversarial behavior; this extra information dispels any outcome uncertainty regarding default and induces all investors to coordinate their actions. Our assumption of "large" investors, however, implies that for a given realization of the test $\mu \in (0, 1)$, the state-contingent default outcome depends on investors' signals, so that the regulator cannot predict whether default will be averted only on the basis of (ω, μ). That is, investors are "large" in that their aggregate behavior cannot be predicted solely on the basis of the banks' underlying fundamentals or, more generally, on the basis of the information available to the regulator.

4. Investors expertise and optimal tests

Our main goal is to clarify whether better-informed large investors elicit more disclosure from the regulator in the form of a more informative stress test. To characterize investors' expertise, we consider a family of signals $X(\alpha)$ indexed by $\alpha \in [0, 1]$ –with distributions $F(x; \alpha)$ and $G(x; \alpha)$– that are Blackwell-ordered: signal $X(\alpha')$ is Blackwell-more informative than signal $X(\alpha)$ whenever $\alpha' > \alpha$.

We ask: do private signals reinforce the (endogenous) public information? That is, does the regulator react by providing more (or less) information when investors expertise is higher? We restrict attention to the case $\eta > 0$ and we take a dual approach to answering this question.[33]

First, we consider the effect of increasing investors' expertise by restricting attention to the set of prior beliefs for which the optimal test is the CF-test $\{0, \underline{\mu}(\alpha)\}$.[34] We say that investors expertise *locally crowds-in* the regulator's test if the stress test becomes more Blackwell-informative after a marginal increase in α, while *local crowding-out* follows if instead the test is less Blackwell-informative. Our analysis in Section 4.1 shows that the interplay between private and public information in the CF-test hinges on investors' responsiveness to public information and we illustrate with examples the effect of expertise in Section 4.2.

Second, we consider the asymptotic effect of expertise on the regulator's optimal test for any prior belief $\mu_0 \in (0, 1)$. We say that investors' expertise *asymptotically crowds-in* public disclosure if the regulator's test becomes fully informative of the state, while it *asymptotically crowds-out* public disclosure if it becomes completely uninformative, as private signals perfectly reveal the bank's state. In Section 4.3 we show that asymptotic crowding-in obtains under general conditions: regardless of investors' prior belief, the regulator provides a perfectly informative public test as investors become very-well informed as long as they remain responsive to public information.

Our interest in the interplay between private and (endogenous) public information is twofold. First, the effect of changes in expertise on investors' welfare is indirectly informed by the impact on the (endogenous) stress test. Indeed, we show in Section 5.2 that better informed investors can benefit more from stress testing if the regulator pre-emptively provides a more informative test. Second, private improvements in expertise generate informational spillovers on other stakeholders channeled through the regulator's test: improvements in the information of large investors may increase or decrease the public information available to these stakeholders, depending on whether the regulator becomes more or less transparent of the state.

4.1. Local crowding-in and crowding-out

Using Corollary 1, consider a prior $\mu_0 \in (\mu_i(\alpha), \mu_{i+1}(\alpha))$ in a disclosure interval. Increasing expertise leads to a more or less informative signal only if the two realizations $\mu_i(\alpha)$ and $\mu_{i+1}(\alpha)$ move in opposite directions; stress tests are not Blackwell-comparable if both realizations increase or decrease, rendering the question of the test informativeness moot. To sidestep this issue, we focus on

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[33] Recall that if $\eta \leq 0$, then expertise is irrelevant as the optimal test always reveals the state—see Lemma 2.

[34] We relegate these questions for the CI-test $\{\overline{\mu}(\alpha), \mu_{CI}(\alpha)\}$ to the Appendix.

the region of prior belief where the CF-test $\left\{0, \underline{\mu}(\alpha)\right\}$ is optimal and note that these tests are Blackwell-ranked as a function of α: local crowding-in holds if $d\underline{\mu}(\alpha)/d\alpha \geq 0$, while local crowding-out follows if $d\underline{\mu}(\alpha)/d\alpha < 0$.

We always have crowding-in if investors did not adjust their investment thresholds when privately observing more informative signals. Indeed, average expected utility $\tilde{u}_R(k; \mu)/\mu = \overline{F}^2(k, \alpha) + \eta \overline{G}^2(k, \alpha)(1 - \mu)/\mu$ is supermodular in (μ, α) implying that the regulator would increase μ following an increase in α if investment thresholds remained unchanged.[35] However, investors equilibrium behavior does vary with the informativeness of their signals. Thus, to derive conditions for crowding-in/-out we take a more indirect approach: we first find the regulator-optimal investors' threshold as a function of investors' private information and then derive the public signal that induces that threshold in equilibrium.

Recall that the CF-test satisfies $\underline{\mu}(\alpha) = \frac{1}{1+R(k^*(\alpha),\alpha)}$ with the threshold $k^*(\alpha)$ solving

$$-2f(k,\alpha)\overline{F}(k,\alpha) - 2\eta g(k,\alpha)\overline{G}(k,\alpha)R(k,\alpha) + \eta \overline{G}^2(k,\alpha)\frac{\partial R(k,\alpha)}{\partial k}\bigg|_{k=k^*(\alpha)} = 0, \qquad (14)$$

which is the optimality condition $dA_R(k^*(\alpha))/dk = 0$—see (12). Thus, $k^*(\alpha)$ balances the gain from increasing average joint investment by lowering investors' threshold vs the cost of under-weighting joint investment when the state is $\omega = -1$. However, the optimal test must also account for the fact that investors' response to public news $k(\mu, \alpha)$, as given by (5), varies with their expertise. The next Proposition shows that local crowding-in obtains whenever better-informed investors' invest less aggressively than the regulator's revised optimal threshold.

Proposition 1. *Let $k(\mu, \alpha)$ be the equilibrium threshold derived in Lemma 1 as a function of $\alpha \in [0,1]$. A marginal increase in investors' expertise crowds-in the regulator's test iff*

$$\frac{dk^*(\alpha)}{d\alpha} \leq \frac{\partial k(\mu, \alpha)}{\partial \alpha}\bigg|_{\mu=\underline{\mu}(\alpha)}. \qquad (15)$$

While Proposition 1 requires first the computation of the optimal threshold $k^*(\alpha)$, (15) allows us to derive some intuition on the role of investor responsiveness to public news in crowding-in/-out. As we illustrate in Section 4.2, increasing investors' expertise may increase or decrease the regulator's optimal threshold $k^*(\alpha)$: the revised threshold satisfying (14) depends on the distributional properties of the signal $X(\alpha)$. However, crowding-in may not follow in either case, even if the regulator prefers a lower threshold—the reason in this case been that investors may lower their investment thresholds and invest more aggressively in response to Blackwell-more informative signals, so that the regulator may prefer a less informative test even though she wants to increase investment rates. Thus, with more informative signals, crowding-in requires that investors do not significantly lower their investment thresholds if $dk^*(\alpha)/d\alpha < 0$, while it requires investors to aggressively raise investment thresholds when $dk^*(\alpha)/d\alpha > 0$.

4.2. Examples: Conclusive News Signals

To gain some intuition on Proposition 1, we introduce two classes of signals—Conclusive-Good-News (CGN) and Conclusive-Bad-News (CBN)—for which investors privately learn the state with some probability and that will shed some light on condition (15).

1. **Conclusive-Good-News Signal, $X_+(\alpha)$.** Fix a decreasing (in k) function $m(k, \alpha) \geq 0$ defined in $[0, 1/2]$ with $\int_0^{1/2} m(t, \alpha)dt = 1$. Then, $X_+(\alpha)$ is defined in terms of the densities

$$f(k; \alpha) = \begin{cases} 2(1-\alpha), & k < 1/2, \\ 2\alpha, & k \geq 1/2. \end{cases}$$

$$g(k) = \begin{cases} m(k, \alpha), & k < 1/2, \\ 0, & k \geq 1/2. \end{cases}$$

   In other words, each investor learns that $\omega = +1$ upon observing $k \geq 1/2$, and their posterior that $\omega = +1$ increases monotonically for $k < 1/2$—Fig. 3a shows the likelihood ratio $\lambda(k; \alpha) = 2(1-\alpha)/m(k, \alpha)$ for two different choices of m—$m(k, \alpha) = \sqrt{1/2 - k}$ and $m(k, \alpha) = (1/2 - k)^2$. Finally, the parameter α is the probability that each investor learns the state conditional on $\omega = +1$.

2. **Conclusive-Bad-News Signal, $X_-(\alpha)$.** Fix an increasing (in k) function $h(k, \alpha) \geq 0$ defined in $[1/2, 1]$ with $\int_{1/2}^1 h(t, \alpha)dt = 1$. Then, $X_-(\alpha)$ is characterized by densities f and g as follows: G is uniform in $[0, 1/2]$ and $[1/2, 1]$ while F has density h in $[1/2, 1]$. That is,

$$f(k) = \begin{cases} h(k, \alpha), & k \geq 1/2, \\ 0, & k < 1/2. \end{cases}$$

$$g(k; \alpha) = \begin{cases} 2(1-\alpha), & k \geq 1/2, \\ 2\alpha, & k < 1/2. \end{cases}$$

---

[35] This is true as long as $\overline{G}(k, \alpha)$ decreases with α. As we show when discussing Assumption A3, there is always a parametrization of a signal that satisfies this condition.

(a) Likelihood Ratios

(b) $\underline{\mu}(\alpha)$

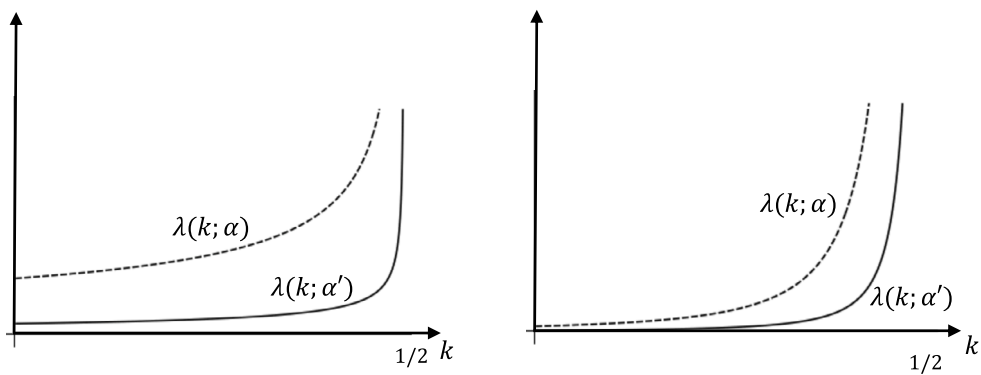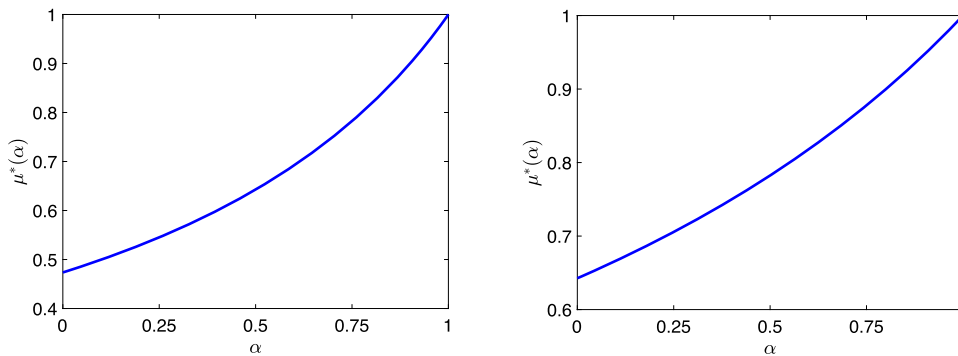**Fig. 3.** Conclusive Good News signals with $m(k, \alpha) = \sqrt{k - 1/2}$ and $m(k, \alpha) = (k - 1/2)^2$.

In other words, each investor learns that $\omega = -1$ upon observing $k \leq 1/2$, while their posterior belief regarding $\omega = +1$ increases monotonically for $k > 1/2$—Fig. 4a shows the likelihood ratio $\lambda(k; \alpha) = h(k, \alpha)/2(1 - \alpha)$ for two different choices of h—$h(k, \alpha) = \sqrt{k - 1/2}$ and $h(k, \alpha) = (k - 1/2)^2$. Finally, the parameter α is the probability that each investor learns the state conditional on $\omega = -1$.

These parametrizations of Conclusive Signals are without loss of generality when considering Blackwell-ordered signals for which investors, with positive probability, learn one of the states—with that probability strictly increasing in the parameter α.[36] While both classes of signals appear symmetric in their informativeness (as captured by the functions h and m), the fact that the state being revealed either makes investing dominant or not investing dominant has quite different implications in terms of local crowding-in. To illustrate this, Figs. 3b and 4b show the realization $\underline{\mu}(\alpha)$ of the optimal CF-test for two functions $m(k, \alpha)$—$m(k, \alpha) = \sqrt{k - 1/2}$ and $m(k, \alpha) = (k - 1/2)^2$—and $h(k, \alpha)$—$h(k, \alpha) = \sqrt{1/2 - k}$ and $h(k, \alpha) = (1/2 - k)^2$—for $\eta = 1$. We obtain local crowding-in for all the examples of Conclusive Good News signals in Fig. 3b. In contrast, expertise can systematically crowd-out public disclosure for Conclusive Bad News signals—see $h(k, \alpha) = \sqrt{1/2 - k}$ in 4b—or even have a non-monotonic effect on the CF-test—see $h(k; \alpha) = (1/2 - k)^2$ in 4b.

The Online Appendix provides a more detailed treatment of local crowding-in and -out in terms of the distributions F and G evaluated at $\underline{\mu}(\alpha)$. We show that both phenomena can occur for both types of classes of signals depending on properties of F and G. However, the asymptotic properties of both classes are better behaved, as we show in the next Section.

4.3. Asymptotic crowding-in and crowding-out

Consider now the effect of expertise on public disclosure when investors are well- (but not perfectly-) informed. Our main result is that under mild conditions, private signals reinforce the public test as the regulator fully discloses the state when investors become

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[36] To see this, suppose that $\tilde{X}$ reveals $\omega = -1$ with probability α and never reveals $\omega = 1$. Letting $F_{\tilde{X}}(\tilde{x}) \equiv \Pr[\tilde{X} \leq \tilde{x} | \omega = 1]$ and $G_{\tilde{X}}(\tilde{x}) \equiv \Pr[\tilde{X} \leq \tilde{x} | \omega = -1]$, with K_F the support of $F_{\tilde{X}}(\tilde{x})$, define $X'(\alpha) \equiv G_{\tilde{X}}(\tilde{X}) - \alpha$ if the realization $\tilde{x} \in K_F$. Translating K_F to the interval $[1/2, 1]$ and assigning probability α uniformly in $[0, 1/2)$ if $\omega = -1$, one obtains a CBN signal $X(\alpha)$. Using instead $X'(\alpha) \equiv F_{\tilde{X}}(\tilde{X}) - \alpha$ if $\tilde{X}$ reveals $\omega = 1$ with probability α one obtains a CGN signal.
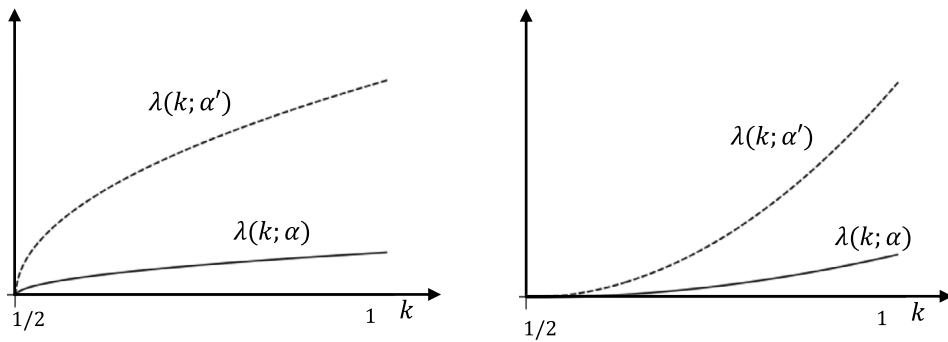
(a) Likelihood Ratios
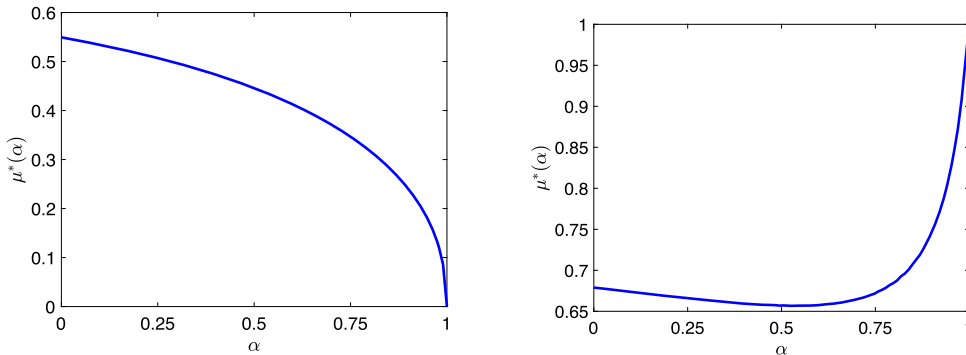
(b) $\underline{\mu}(\alpha)$

Fig. 4. Conclusive Bad News signals with $h(k,\alpha) = \sqrt{1/2-k}$ and $h(k,\alpha) = (1/2-k)^2$.

perfectly informed. This is always true if private signals are discrete, while it is true with continuous signals as long as investors remain responsive to public news.

For our analysis in this Section, we consider signals with the following characteristics.

A3. Signal $X(\alpha')$ is Blackwell-more informative than signal $X(\alpha)$ with $\overline{G}(k;\alpha') \le \overline{G}(k;\alpha)$ whenever $\alpha' > \alpha$, and $X(\alpha)$ fully reveals the state as α tends to 1. In particular, there exists $k_{crit} \in (0,1)$ so that (i) $\lim_{\alpha \to 1} \overline{F}(k_{crit};\alpha) = 1$ with $\lim_{\alpha \to 1} \overline{F}(k;\alpha) < 1, k > k_{crit}$, and (ii) $\lim_{\alpha \to 1} \overline{G}(k_{crit};\alpha) = 0$, with $\lim_{\alpha \to 1} \overline{G}(k;\alpha) > 0, k < k_{crit}$.[37]

Assumption A3 ensures both that expertise increases with α and that investors learn the state as α tends to 1, but also guarantees that their investment thresholds converge to a common value k_{crit} as α tends to 1 for any (fixed) inconclusive outcome of the stress-test. Note that the stress-test is irrelevant if $\alpha = 1$ as investors perfectly coordinate their decisions on the basis of their private signals —so any test is optimal for the regulator. Our interest is in how much information she discloses if investors are well informed but not perfectly informed (so $1 - \epsilon < \alpha < 1$ with $\epsilon > 0$ an arbitrarily small constant). Finally, both CBN and CGN signals introduced in Section 4.2 satisfy this Assumption (with $k_{crit} = 1/2$).

4.3.1. Continuous signals

Assumption 3 guarantees that $X(\alpha)$ becomes very discriminating of the state at k_{crit} and that the equilibrium investment thresholds $k(\mu,\alpha)$ converge to k_{crit} regardless of the interim belief $\mu \in (0,1)$. However, it also implies that the regulator's optimal test will induce interior thresholds that also converge to k_{crit}; for instance, the CF-test threshold $k^*(\alpha)$ converges to k_{crit} as $\alpha \to 1$. To compare the rates at which $k(\mu,\alpha)$ and $k^*(\alpha)$ converge—which ultimately dictates the regulator's asymptotic optimal test—we define

$$\tau(k,\alpha) \equiv -\frac{\partial \ln \widetilde{R}(k,\alpha)/\partial k}{\partial \ln \overline{G}(k,\alpha)/\partial k} = \frac{(\partial \widetilde{R}(k;\alpha)/\partial k)/\widetilde{R}(k;\alpha)}{g(k;\alpha)/\overline{G}(k;\alpha)}. \tag{16}$$

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[37] The condition that $\overline{G}(k;\alpha') \le \overline{G}(k;\alpha)$ whenever $\alpha' > \alpha$ is without loss of generality as we can always find a reparametrization of $X(\alpha)$ that satisfies it—for instance, the integral probability transform $\widetilde{X} = G(X;\alpha)$ would suffice as then signal $\widetilde{X}|\omega = -1$ is uniformly distributed for all α— and we impose it as it makes our results more transparent.

Recall that the (possibly non-monotonic) function $\widetilde{R}(k,\alpha)$, defined in (2), characterizes all equilibrium thresholds and, in particular, we used it to construct R which corresponds to the regulator's preferred equilibrium—i.e., the equilibrium with the lowest threshold in case of multiplicity. We revert now to this function as Proposition 2 below will not require the assumption of the regulator's *selection power* over investment equilibria.

We can interpret $\tau(k,\alpha)$ as a cost-benefit ratio when considering the CF-test. The regulator's marginal expected payoff from a more informative CF-test, $d(u_R(k(\underline{\mu}),\underline{\mu})\Pr[\underline{\mu}])/d\underline{\mu}$, can be decomposed as

$$\left.\frac{\partial\left(u_R(k,\underline{\mu})\Pr[\underline{\mu}]\right)}{\partial\underline{\mu}}\right|_{k=k(\underline{\mu})}$$

$$+ \left.\Pr[\underline{\mu}]\frac{dk}{d\underline{\mu}}\left(\mathbb{E}\left[\left.\frac{\partial u_R(k,\underline{\mu})}{\partial k}\right|\omega=1\right]\Pr[\omega=1]+\mathbb{E}\left[\left.\frac{\partial u_R(k,\underline{\mu})}{\partial k}\right|\omega=-1\right]\Pr[\omega=-1]\right)\right|_{k=k(\underline{\mu})}.$$

(17)

The first term is always negative and represents the reduction in joint investment from a lower probability of outcome $\overline{\mu}$ when investors do not revise their investment thresholds; the second and third terms are positive and represent the state-contingent increase in joint investment as investors respond to a higher $\underline{\mu}$ by lowering their investment thresholds. Then, $\tau(k,\alpha)$ is simply (twice) the ratio of the cost—first term, which equals $\eta\overline{G}^2(k)$—to the benefit in the bad state—third term, which equals $2\eta g\widetilde{R}(k)\overline{G}/\widetilde{R}'(k)$. The importance of this quantity is that asymptotic crowding-in holds if this ratio remains bounded.

Proposition 2. *Suppose that for some $\alpha' < 1$, f and g are continuously differentiable and $\tau(k,\alpha)$ is uniformly bounded, in $(k,\alpha)\in[0,1]\times[\alpha',1)$. For each $\alpha\geq\alpha'$, consider a selection $k_s(\mu,\alpha)$ of investment equilibria with associated $R_s(k,\alpha)$, so that $R_s(k,\alpha)=\widetilde{R}(k,\alpha)$ for each $k\in\{k:k=k_s(\mu,\alpha),\mu\in(0,1)\}$. Then, for every $\mu_0\in(0,1)$ the regulator's test perfectly reveals the state as $\alpha\to1$.*

We can interpret the boundedness condition on $\tau(k,\alpha)$ as imposing a lower bound on investors' responsiveness to public news as given by (6). To see this, consider an interim belief μ so that the equilibrium selection $R_s(k,\alpha)$ is strictly increasing at threshold k that satisfies $(1-\mu)/\mu=R_s(k,\alpha)(=\widetilde{R}(k,\alpha))$. Then, $\tau(k,\alpha)\leq M$ implies that

$$\left|\frac{\partial k(\mu,\alpha)}{\partial\mu}\right|=\frac{1}{\mu^2\left.\widetilde{R}'(k,\alpha)\right|_{k=k(\mu,\alpha)}}=\frac{\overline{G}(k;\alpha)/g(k;\alpha)}{\mu(1-\mu)\tau(k,\alpha)}\geq\frac{\overline{G}(k;\alpha)/g(k;\alpha)}{\mu(1-\mu)M}.$$

(18)

Therefore, we can rephrase Proposition 2 as showing that asymptotic crowding-in follows if investors remain responsive to the stress test—as required by (18). In fact, it follows from Proposition 2 that for any prior belief μ_0 and expertise $\alpha\in(0,1)$ there exists a minimum expertise $\underline{\alpha}(\alpha,\mu_0)$ such that the regulator provides more information to investors if their expertise increases to any $\alpha'\geq\underline{\alpha}(\alpha,\mu_0)$. In other words, if investors remain responsive to public news, then for any prior belief the regulator provides more information to *sufficiently* better-informed investors.

The proof of the proposition shows that the CF-test has $\underline{\mu}(\alpha)\to1$, so that for any prior $\mu_0\in(0,1)$, (i) there is an $\alpha(\mu_0)<1$ such that μ_0 falls in the region in which the CF-test is optimal for all $\alpha>\alpha(\mu_0)$, and (ii) the CF-test becomes fully revealing as $\alpha\to1$. To see this, consider the regulator's marginal payoff (17). The key observation is that the ratio of the benefit from increased investment in state $\omega=1$ to the benefit in state $\omega=-1$ is proportional to $\overline{F}(k^*,\alpha)/\overline{G}(k^*,\alpha)$, which diverges without bound as investors' signals perfectly reveal the state. Since the ratio of the costs to the benefits in state $\omega=-1$ remains bounded, then (17) must necessarily become positive as $\alpha\to1$. In other words, as private signals become more discriminating, the relative benefit of increasing investment rates looms larger than the relative costs of lowering the probability of a favorable test outcome. This implies that the CF-test must necessarily have $\underline{\mu}(\alpha)\to1$; if instead we conjecture $\underline{\mu}(\alpha)\to\underline{\mu}\neq1$ then the regulator eventually benefits from increasing investment rates for α close to 1 and it is not optimal for her to settle on $\underline{\mu}<1$.

We now showcase Proposition 2 by studying the optimal test for the Conclusive Signals introduced in Section 4.2.

Conclusive-Good-News signal, $X_+(\alpha)$ For CGN signals, signal realizations are inconclusive whenever $k\in[0,1/2)$. However, they perfectly reveal $\omega=1$ as $k\to1/2$ if $m(1/2,\alpha)=0$. In fact, in this case not only is the likelihood ratio $\lambda(k,\alpha)=2(1-\alpha)/m(k,\alpha)$ unbounded at $k=1/2$, but also $\lambda'(k,\alpha)/\lambda(k,\alpha)=-m'(k,\alpha)/m(k,\alpha)$ whenever $m'(1/2,\alpha)\neq0$. Therefore, $\partial\ln\widetilde{R}(k,\alpha)/\partial k$ can become unbounded (reflecting the fact that, for $\alpha<1$, CGN signals can be very discriminating at $k<1/2$). Nevertheless, the hazard rate of the bad state, $g(k,\alpha)/\overline{G}(k,\alpha)=m(k,\alpha)/\int_k^{1/2}m(s,\alpha)ds$ diverges without bound as $k\to1/2$. Combining these two effects and applying Proposition 2 we find that asymptotic crowding-in holds under very general conditions for CGN signals, and in particular whenever $\partial m(1/2,\alpha)/\partial k$ is finite.

Corollary 2. *Suppose that investors' signals provide conclusive good news $X_+(\alpha)$ and there is a constant M so that $|\partial m(k,\alpha)/\partial k|\leq M/(1/2-k)$, $k\in[0,1/2]$. Then, for any $\mu_0\in(0,1)$ the regulator's test becomes perfectly informative as $\alpha\to1$.*

Consistent with this corollary, Fig. 3b shows that $\underline{\mu}(\alpha)\to1$ for all the examples considered.

Conclusive-Bad-News signal   Similarly to CGN signals, CBN signal realizations are inconclusive whenever $k \in (1/2, 1]$ but can become very discriminating as $k \to 1/2$ if $h(1/2, \alpha) = 0$—in fact, in this case the likelihood ratio $\lambda(k, \alpha) = h(k, \alpha)/2(1 - \alpha)$ is always zero at $k = 1/2$, so that, if $h'(k, \alpha) \neq 0$, $\lambda'(k, \alpha)/\lambda(k, \alpha) = h'(k, \alpha)/h(k, \alpha)$ becomes unbounded as $k \to 1/2$. Thus, both CBN and CGN signals can become very discriminating of the state as $k \to 1/2$.

If $h(1/2, \alpha)$ is bounded away from zero then investors remain responsive to the state in the sense that $\partial \ln \widetilde{R}(k, \alpha)/\partial k$ remains bounded and asymptotic crowding-in follows. However, $h(1/2, \alpha) = 0$ can lead to situations in which the limit test is not fully revealing. Indeed, the hazard rate of the bad state, $g(k, \alpha)/\overline{G}(k, \alpha) = 1/(1 - k)$ remains bounded in this case, implying that $\tau(k, \alpha)$ becomes unbounded. In fact, expertise can asymptotically crowd out public disclosure for CBN signals.

Corollary 3. *Suppose that $X_-(\alpha)$ satisfies Assumption A3, then*

(i) *If $\max_{1/2 \leq k \leq 1} \partial h(k, \alpha)/\partial k \leq M$ and $h(1/2, \alpha) \geq \varepsilon > 0$ for $\alpha \in [0, 1]$ and $M \in \mathbb{R}_+$, then the regulator's test fully reveals the state as $\alpha \to 1$.*

(ii) *Suppose that $h(1/2, \alpha) = 0$ and that $\lim_{\alpha \to 1} \partial_+ h(1/2, \alpha)/\partial k = \overline{h}'$. Define*

$$\underline{\mu}_h = \frac{1}{1 + \frac{\eta(1+\gamma)^2 \overline{h}'}{8}}. \tag{19}$$

Then, the regulator's optimal test converges to $\left\{ 0, \underline{\mu}_h \right\}$ if $\mu_0 \in \left(0, \underline{\mu}_h \right)$ and is completely uninformative if $\mu_0 \geq \underline{\mu}_h$. In particular, we have asymptotic crowding-in of the public test iff $\partial_+ h(1/2, \alpha)/\partial k$ tends to 0, while we have asymptotic crowding-out of the public test iff $\partial_+ h(1/2, \alpha)/\partial k$ becomes unbounded, as $\alpha \to 1$.

This effect is illustrated in Fig. 4b that shows the CF-test converging to $\left\{ 0, \underline{\mu}_h \right\}$ for the particular examples shown in that figure. Consistent with our main insight, the regulator is willing to provide a perfectly informative signal as $\alpha \to 1$ as long as investors remain responsive to the public signal.

4.3.2. Discrete signals

Suppose that investors' signals are discrete—i.e., the support of $X(\alpha)$ is finite. Then, the regulator always resorts to perfectly disclosing the state as $\alpha \to 1$.

Proposition 3. *Let $support(X(\alpha)) = \left\{ x_1, \ldots, x_n \right\}$ and $f_i(\alpha) = \Pr\left[x_i | \omega = +1; \alpha \right]$ and $g_i(\alpha) = \Pr\left[x_i | \omega = -1; \alpha \right]$. Suppose that for $i \in \{1, \ldots, n\}$ and $\alpha < 1$: (i) either $f_i(\alpha) > 0$ or $g_i(\alpha) > 0$, and (ii) either $\lim_{\alpha \to 1} f_i(\alpha)/g_i(\alpha) = 0$ or $\lim_{\alpha \to 1} g_i(\alpha)/f_i(\alpha) = 0$. Then, for every prior $\mu_0 \in (0, 1)$, the regulator's optimal test converges to a fully informative public signal as $\alpha \to 1$.*

The conditions in the proposition simply require that for some $i^* \in \{2, \ldots, n\}$, any signal realization $x_{i'}, i' \geq i^*$, becomes strong evidence of $\omega = +1$, while $x_{i'}, i' < i^*$, becomes strong evidence of $\omega = -1$. For α close to 1, if the regulator does not reveal any information, then investors will set an investment threshold $k(\mu_0; \alpha) = x_{i^*}$. As we show in the proof, investors would, however, switch to a lower threshold x_{i^*-1} if the regulator's test reveals sufficiently good news about the state. It is this responsiveness to the regulator's test in the case of discrete signals that leads to the asymptotic crowding-in of the public test.

5. Value of stress testing

We now study the social and private value (for investors) of stress-testing and how it varies with their expertise.

5.1. Private value of stress testing

In the absence of a stress test, well-informed investors can coordinate their investment decisions through the precision of their private signals. However, investment externalities and dispersed information still lead to investment frictions when these signals are not perfectly informative of the underlying state.[38] Given these frictions, do investors benefit from the regulator's test? To answer this question, let

$$u_i(\mu) \equiv \mu \left(\overline{F}(k(\mu)) + \gamma \overline{F}^2(k(\mu)) \right) + (1 - \mu) \left(-\overline{G}(k(\mu)) + \gamma \overline{G}^2(k(\mu)) \right),$$

be investor-$i's$ interim equilibrium expected utility after realization μ of the public test. If $\{ \{ \mu_i \}_{i \in I}, D, N \}$ is the regulator's test in Corollary 1, then the private value of the test when $\mu_0 \in \left(\mu_l, \mu_{l+1} \right) \subset D$ is

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[38] Recall that if the state is commonly known, then the unique investment equilibrium maximizes investors joint surplus. Thus, investment inefficiencies are a result of imperfect information.

$$V_i^{test} \equiv \Pr\left[s = \mu_{l+1}\right]\left(u_i(\mu_{l+1}) - u_i(\mu_0)\right) + \Pr\left[s = \mu_l\right]\left(u_i(\mu_l) - u_i(\mu_0)\right). \tag{20}$$

While the regulator always provides a weakly informative test, her focus on inducing co-investment may exacerbate investment distortions and, actually, make investors worse-off. In Section OA.1 in the Online Appendix we provide a full analysis of the impact on investors' welfare of stress testing by studying conditions for $V_i^{test} > 0$. We now briefly summarize some of the main findings focusing on CF-tests and CI-tests. Investors benefit from *any* binary test that reveals that the state is $\omega = -1$. This is intuitive: if investors know that $\omega = -1$, then they will refrain from investing which is the investor-optimal response, while if the test reveals $\mu > \mu_0$ it boosts joint investment and reduces the positive externality. Thus, investors *always* benefit from a CF-test. The same is not true, however, for a CI-test $\{\overline{\mu}, \mu_{CI}\}$ as revealing $\mu = \overline{\mu}$ may exacerbate the underinvestment problem relative to the prior $\mu = \mu_0$.

5.2. Social and private value of expertise

How do improvements in investors' expertise affect both the private value and the social value of the stress test (as captured by the regulator's payoff)? We show that expertise has, in general, an ambiguous effect both on social welfare and on investors' payoffs once one accounts for the regulator's optimal response to changes in expertise.

Suppose that investors' signals are indexed by α and Assumption A3 holds. Expertise has no impact on equilibrium payoffs when $\eta \leq 0$ as the regulator always provides a perfectly informative test. Thus, we study the case $\eta > 0$ and concentrate on CF-tests. To clarify the effect of expertise, let $A_I(k)$ represent investors' interim equilibrium average expected utility when the test leads to interim belief $\mu(k) = 1/(1 + R(k))$ after which investors select a threshold k,

$$A_I(k) \equiv \frac{u_i(\mu(k))}{\mu(k)} = \overline{F}(k) - R(k)\overline{G}(k) + \gamma\left(\overline{F}^2(k) + R(k)\overline{G}^2(k)\right). \tag{21}$$

Using (11) and (21), the regulator and investors' expected payoff from a CF-test are $U_R^{CF}(\alpha) \equiv \mu_0 A_R(k(\underline{\mu}(\alpha); \alpha); \alpha)$ and $U_I^{CF}(\alpha) \equiv \mu_0 A_I(k(\underline{\mu}(\alpha); \alpha); \alpha)$ and we can use (12) and the properties of $k(\mu)$ in Lemma 1 to evaluate the impact of expertise on welfare.

Consider first the impact of α on U_R^{CF}. Appealing to the envelope theorem,[39] we have

$$\frac{1}{\mu_0}\frac{dU_R^{CF}}{d\alpha} = \left.\frac{\partial A_R(k; \alpha)}{\partial \alpha}\right|_{k=k(\underline{\mu}(\alpha); \alpha)}. \tag{22}$$

This is intuitive: as the regulator can optimally adjust the test in response to investors' improved information, expertise has only a direct effect on the regulator's payoff. This effect is positive if, and only if, increasing expertise leads investors to raise average joint investment while holding constant the investment threshold $k(\underline{\mu}(\alpha); \alpha)$.

While the sign of (22) is in general ambiguous, it is clear that increases in expertise must be detrimental to the regulator for some parameter values. Indeed, as $\underline{\mu}(\alpha)$ maximizes the regulator's average utility, U_R^{CF} can only decrease when agent's perfectly learn the state—which corresponds to the limit as $\alpha \to 1$. For example, when private signals can detect low returns, increases in expertise are always detrimental to the regulator.

Remark 1. Consider the Conclusive Bad News signals in Section 4.2. Then $dU_R^{CF}/d\alpha < 0$ so that an increase in investors' expertise makes the regulator worse off.

Consider now the impact of expertise on investors' equilibrium payoffs. Do better-informed investors benefit more from public disclosure of the stress tests? To answer this question, let $\widetilde{A}_I(\mu; \alpha) \equiv A_I(k(\mu, \alpha); \alpha)$ and consider the marginal impact of expertise on the value of the test V_i^{test} defined in (20),

$$\frac{dV_i^{test}(\alpha)}{d\alpha} = \underbrace{\left(\frac{\partial \widetilde{A}_I(\underline{\mu}(\alpha); \alpha)}{\partial \alpha} - \frac{\partial \widetilde{A}_I(\mu_0; \alpha)}{\partial \alpha}\right)}_{\text{direct effect}} + \underbrace{\frac{\partial \widetilde{A}_I(\underline{\mu}(\alpha); \alpha)}{\partial \mu}\frac{\partial \underline{\mu}(\alpha)}{\partial \alpha}}_{\text{indirect effect}}. \tag{23}$$

The effect of expertise on V_i^{test} is decomposed into two effects. The first term in (23) is the direct effect of increased expertise on V_i^{test} holding constant the regulator's test. Note that the sign of this effect is in principle ambiguous as it depends on the comparison between the effect of expertise on investors average utility at the realization $\underline{\mu}(\alpha)$ and at the prior μ_0. For instance, if investors underinvest (relative to full information) under μ_0 and improvements in private information ameliorate this underinvestment, then the direct effect will be positive.

The second term in (23) describes the indirect effect as improvements in expertise will lead the regulator to pre-emptively change the public test. This indirect effect depends on whether expertise crowds-in or -out the stress test (i.e., whether $\partial \underline{\mu}(\alpha)/\partial \alpha \gtrless 0\}$, and on the effect of public information on investors' average utility (i.e., on the sign of $\partial \widetilde{A}_I(\mu, \alpha)/\partial \mu$).

While one would expect the test to be less valuable to better-informed investors–as both sources of information on returns may act as substitutes–the sign of the direct effect in (23) depends in general on the functional form of signals $X(\alpha)$. Nevertheless, better-

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[39] The optimality condition (12) guarantees that $(\partial A_R(k, \alpha)/\partial k)|_{k(\underline{\mu}(\alpha); \alpha)} = 0$.

informed investors may benefit more from the test if it compels the regulator to be more transparent. To see this, we study situations in which the direct effect in (23) is small.

Proposition 4. *Let $\eta = 1$ and suppose that a CF-test $\{0, \underline{\mu}(\alpha)\}$ exists. Then, there is $\epsilon > 0$ so that whenever $\underline{\mu}(\alpha) - \epsilon < \mu_0 < \underline{\mu}(\alpha)$, we have $dV_i^{test}/d\alpha > 0$ if and only if $d\underline{\mu}(\alpha)/d\alpha > 0$.*

When the prior belief is close to the realization $\underline{\mu}(\alpha)$, a CF-test is unlikely to change investors views on fundamentals—indeed, the probability that the test reveals that the state is $\omega = -1$ is then $\left(\underline{\mu}(\alpha) - \mu_0\right)/\underline{\mu}(\alpha)$—and investors derive little value from the test. Thus, improvement in private information render the direct effect in (23) as second order, and the total effect is driven by the indirect effect. Proposition 4 shows that the indirect effect is positive if $\underline{\mu}(\alpha)$ increases: better-informed investors gain more from the test if and only if the regulator pre-emptively discloses more information. Thus, Proposition 4 links our results in Section 4 on the interplay between private and (endogenous) public information to the welfare effect of the test.

The key observation in Proposition 4 is that, irrespective of the sign of the investment externality, investors average utility is increasing at the test outcome $\underline{\mu}(\alpha)$. To see this, write (21) as $A_I(k) = \mathcal{E}(k) + \gamma A_{R,1}(k)$ where $\mathcal{E}(k) \equiv \mathbb{E}[\omega 1_{\{x_i \geq k\}}]/\mu(k)$ captures how adapted are investments to the state and $A_{R,1}(k)$ is the regulator's average utility when $\eta = 1$. The fact that the CF-test satisfies $\partial A_R((k(k, \alpha); \alpha))/\partial \mu = 0$—see (12)—means that the effect of a more informative CF-test on investor's average utility is given by $\partial \mathcal{E}((k(\underline{\mu}(\alpha), \alpha); \alpha))/\partial \mu$, i.e., whether a more informative test leads to a more adapted decision to the state. The proof then shows that $\partial \mathcal{E}((k(\underline{\mu}(\alpha), \alpha); \alpha))/\partial \mu > 0$. In summary, if $d\underline{\mu}(\alpha)/d\alpha > 0$, then investors benefit from more precise private signals from the increased adaptation to the state that follows a more informative stress test.

6. Concluding remarks

In this paper we studied the stress test design by a regulator in the presence of large private investors. The regulator would like to elicit private investment in order to avert default of a distressed bank. The large private investors, besides the public stress test, rely on the quality of their private research (expertise) to decide on whether to fund banks seeking capital, where averting default creates a coordination motive.

We characterized the optimal public test and provided conditions for this test to, with positive probability, perfectly coordinate investment choices; nevertheless, perfect coordination is not a generic feature. Our main results concern the interplay between private and (endogenous) public information. We showed that, as long as investors remain responsive to public news, we have asymptotic crowding-in. Thus, the regulator provides more information to sufficiently better-informed investors. For instance, asymptotic crowding-in always obtains if investors' signals are discrete or generically if private signals reveal one state with positive probability—i.e., the case of Conclusive News Signals.

Our results have implications on empirical exercises aiming to relate public information released from stress tests to investors' private information. Practically, our asymptotic result of Proposition 2 holds very close to the limit or for the most informed of investors; moreover, our graphs in Figs. 3 and 4 support that crowding-in should also hold further away from the limit. Hence, our main empirical prediction is that in settings where large investors are sufficiently more informed (as proxied by the number of analysts they employee or follow) but still reactive to public news (as measured by their trading activity around announcements) then the regulator will provide an even more informative test (as assessed by the market reaction to the release of public information), so we get crowding-in.[40]

CRediT authorship contribution statement

Ricardo Alonso: Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Konstantinos E. Zachariadis:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A

Proof of Lemma 1. Let

$$\widetilde{R}(x; k) \equiv \lambda(x)\widetilde{C}(x; k), \text{ with } \widetilde{C}(x; k) \equiv \left(1 + \gamma \overline{F}(k|x)\right)/\left(1 - \gamma \overline{G}(k|x)\right). \tag{A.1}$$

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[40] On the other hand, in models with investors' information acquisition (see, e.g., Diamond (1985) and Goldstein and Yang (2017)) or banks' information release (see, e.g., Quigley and Walther (2020)), there is crowding-out of private by public information.

Suppose that Assumptions A1 and A2 hold. Assumption A2 guarantees that $\widetilde{C}(x;k)$ is non-decreasing in x as the product of two positive and non-decreasing functions, therefore $v_i(k,x;\mu)$ is again strictly single-crossing in x as the product of a positive function and an strictly increasing function. Therefore, the best response is a threshold strategy with unique threshold satisfying $v_i(k,x;\mu) = 0$. Moreover, all equilibria are symmetric as this is a symmetric game with strategic complementarities (see Vives (1999, sec. 2.2.3 and fn. 23)).

We now show that the equilibrium is unique for low values of γ. If $\gamma = 0$ then $\widetilde{R}(k) = \lambda(k)$ is strictly increasing from assumption A1. The assumption that $\lambda'(k)$ is bounded away from zero implies that $\widetilde{R}'(k) > 0$ for γ in a neighborhood of zero. Therefore, there exists $\overline{\gamma} > 0$ for which $\widetilde{R}(k)$ is still increasing for all k and hence a unique symmetric equilibrium exists for $\gamma < \overline{\gamma}$.

Now, for $\gamma > \overline{\gamma}$, $\widetilde{R}(k)$ is in general non-monotone, so that there are multiple solutions to (3). If we define $R(k) \equiv \max_{0 \leq k' \leq k} \widetilde{R}(k')$ as in (4), then (i) $R(k)$ is non-decreasing, and (ii) for a given $y = (1-\mu)/\mu$, $R^{-1}(y) = \min\left\{ k : y = \widetilde{R}(k) \right\}$ provides the symmetric equilibrium corresponding to the highest probability of joint investment, which is the regulator's preferred equilibrium. Therefore, under this equilibrium selection criteria, the equilibrium threshold satisfies $(1-\mu)/\mu = R(k)$ whenever $1/(1 + R(1)) < \mu < 1/(1 + R(0))$. Moreover, if $\min_{k \in [0,1]} \widetilde{R}(k) = \widetilde{R}(0)$, then whenever $(1-\mu)/\mu \leq R(0)$ (i.e., $\mu \geq 1/(1 + R(0))$) an investor would invest for any signal and hence we set her threshold to the lower bound zero. If $(1-\mu)/\mu \geq R(1)$–that is, $\mu \leq 1/(1 + R(1))$–then an investor would never invest and we set his threshold to one.

ii-Note that $R^{-1}(y)$ is non-decreasing, so that $k(\mu) = R^{-1}((1-\mu)/\mu)$ must be non-increasing. Differentiating (5) gives (6). $\quad\square$

Lemma A.1 (Properties of $R(k)$ and $k(\mu)$). (i) If f and g are twice continuously differentiable in $[0,1]$ then $k''(1)$ is bounded. (ii) We have $\partial R(k;\gamma)/\partial\gamma \geq 0$ for all (k,γ).

Proof. (i) To compute $k''(1)$, we first differentiate each term in

$$k'(\mu) = \frac{1}{\mu} \frac{g(k(\mu))(1 - \gamma\overline{G}(k(\mu)))}{(1-\mu)g'(k(\mu))(1 - \gamma\overline{G}(k(\mu))) - \mu f'(k(\mu))(1 + \gamma\overline{F}(k(\mu))) + \gamma\left[(1-\mu)g^2(k(\mu)) + \mu f^2(k(\mu))\right]}, \tag{A.2}$$

and then compute the limit as $\mu \to 1$. We have in turn:

$$\lim_{\mu \to 1}\left[\overbrace{\frac{1}{\mu}}^{I_1(\mu)}\right]' = \lim_{\mu \to 1} -\frac{1}{\mu^2} = \overbrace{-1}^{I_1'(1)},$$

$$\lim_{\mu \to 1}\left[\overbrace{g(k(\mu))\left(1 - \gamma\overline{G}(k(\mu))\right)}^{I_2(\mu)}\right]' = \lim_{\mu \to 1} g'(k(\mu))\left(1 - \gamma\overline{G}(k(\mu))\right) + \gamma g^2(k(\mu)) = \overbrace{g'(0)(1-\gamma) + \gamma g^2(0)}^{I_2'(1)},$$

$$\lim_{\mu \to 1}\left[\overbrace{(1-\mu)g'(k(\mu))(1 - \gamma\overline{G}(k(\mu))) - \mu f'(k(\mu))(1 + \gamma\overline{F}(k(\mu))) + \gamma\left[(1-\mu)g^2(k(\mu)) + \mu f^2(k(\mu))\right]}^{I_3(\mu)}\right]' =$$

$$\lim_{\mu \to 1} -g'(k(\mu))\left(1 - \gamma\overline{G}(k(\mu))\right) + (1-\mu)g''(k(\mu))\left(1 - \gamma\overline{G}(k(\mu))\right) + (1-\mu)\gamma g'(k(\mu))g(k(\mu))$$

$$-f'(k(\mu))\left(1 + \gamma\overline{F}(k(\mu))\right) - \mu f''(k(\mu))\left(1 + \gamma\overline{F}(k(\mu))\right) + \mu f'(k(\mu))\gamma f(k(\mu))$$

$$-\gamma g^2(k(\mu)) + \gamma(1-\mu)2g(k(\mu))g'(k(\mu)) + \gamma f^2(k(\mu)) + 2\gamma\mu f(k(\mu))f'(k(\mu)) =$$

$$-g'(0)(1-\gamma) + 0g''(0)(1-\gamma) + 0\gamma g'(0)g(0) - f'(0)(1+\gamma) - f''(0)(1+\gamma) + f'(0)\gamma 0 - \gamma g^2(0)$$

$$+\gamma 0 2g(0)g'(0) + \gamma 0 + 2\gamma 0 f'(0) =$$

$$\overbrace{-g'(0)(1-\gamma) - f'(0)(1+\gamma) - f''(0)(1+\gamma) - \gamma g^2(0)}^{I_3'(1)},$$

where the last equality follows from assuming that $f''(0), g''(0) < \infty$. Given that, as defined above, $k'' = I_1'I_2/I_3 + I_1(I_2'I_3 - I_2I_3')/I_3^2$, and all the terms are finite from the above calculations we have $k''(1) < \infty$.

(ii) Recall from (2) that $R(k;\gamma) = \max_{0 \leq k' \leq k} \widetilde{R}(k';\gamma)$. Let:

$$k^*(k,\gamma) \equiv \arg\max_{0 \leq k' \leq k} \widetilde{R}(k';\gamma).$$

Now, pick $\gamma_1 > \gamma_2$. We want to show that for all k:

$$R(k;\gamma_1) = \widetilde{R}(k^*(k,\gamma_1);\gamma_1) \geq \widetilde{R}(k^*(k,\gamma_2);\gamma_2) = R(k;\gamma_2). \tag{A.3}$$

We claim that:

$$\widetilde{R}(k^*(k,\gamma_1);\gamma_1) \geq \widetilde{R}(k^*(k,\gamma_2);\gamma_1). \tag{A.4}$$

This follows from optimality of $k^*(k,\gamma_1)$ for all $k' < k$ and the definition of $k^*(k,\gamma_1)$, which guarantees that it is also less that k.

From the fact that $\partial \widetilde{R}(k;\gamma)/\partial\gamma > 0$ for all (k,γ) (see (2)) we also have that:

$$\widetilde{R}(k^*(k,\gamma_2);\gamma_1) \geq \widetilde{R}(k^*(k,\gamma_2);\gamma_2). \tag{A.5}$$

Then from (A.4) and (A.5) we have $\widetilde{R}(k^*(k,\gamma_1);\gamma_1) \geq \widetilde{R}(k^*(k,\gamma_2);\gamma_2)$ which establishes (A.3). $\square$

Proof of Corollary 1. Since the state is binary and u_R is upper semicontinuous, Kamenica and Gentzkow (2011, p. 2596) guarantee the existence of an optimal test that for every prior μ_0 induces at most two different posterior beliefs. Let $U_R(\mu_0)$ be the regulator's expected payoff for prior μ_0 under an optimal test. As shown in Kamenica and Gentzkow (2011, p. 2596) $U_R(\mu_0)$ is given by the value of the concave closure of $u_R(\mu)$ at $\mu = \mu_0$.

First, for prior μ_0 to belong in a non-disclosure interval $[\mu_i, \mu_{i+1}]$ it must be that $U_R(\mu_0) = u_R(\mu_0)$. Geometrically, this means there is a supporting hyperplane through the point $(\mu_0, u_R(\mu_0))$ that majorizes u_R. Formally, for every $\mu_0 \in [\mu_i, \mu_{i+1}]$, there is $\gamma(\mu_0)$ such that (8) holds for any $\mu \in [0,1]$. Moreover, if u_R is differentiable for $\mu = \mu_0$, then the supporting line is the tangent at μ_0 and so $\gamma(\mu_0) = u_R'(\mu_0)$.

Second, for prior μ_0 to belong to the disclosure interval (μ_i, μ_{i+1}) it must be that $U_R(\mu_0) > u_R(\mu_0)$. Let the optimal test induce two different posteriors μ_i and μ_{i+1} so that $0 \leq \mu_i < \mu_0 < \mu_{i+1} \leq 1$. Bayesian feasibility requires that $\Pr[\mu_i] + \Pr[\mu_{i+1}] = 1$ and

$$\Pr[\mu_i]\,\mu_i + \Pr[\mu_{i+1}]\,\mu_{i+1} = \mu_0 \Rightarrow \Pr[\mu_{i+1}] = \frac{\mu_0 - \mu_i}{\mu_{i+1} - \mu_i}.$$

The value to the regulator of this test is then:

$$U_R(\mu_0) = \overbrace{\frac{\mu_0 - \mu_i}{\mu_{i+1} - \mu_i}}^{\Pr[\mu_{i+1}]} u_R(\mu_{i+1}) + \overbrace{\frac{\mu_{i+1} - \mu_0}{\mu_{i+1} - \mu_i}}^{\Pr[\mu_i]} u_R(\mu_i)$$

$$= u_R(\mu_i) + (\mu_0 - \mu_i)\frac{u_R(\mu_{i+1}) - u_R(\mu_i)}{\mu_{i+1} - \mu_i} \tag{A.6}$$

$$= u_R(\mu_{i+1}) - (\mu_{i+1} - \mu_0)\frac{u_R(\mu_{i+1}) - u_R(\mu_i)}{\mu_{i+1} - \mu_i}. \tag{A.7}$$

The second and third lines above follow from re-arranging terms in the first line, their use will be clear below. Geometrically, in the disclosure interval case the concave closure of u_R coincides with the line that connects the points $(\mu_i, u_R(\mu_i))$ and $(\mu_{i+1}, u_R(\mu_{i+1}))$. To make sure all the values of u_R for the signals in the interval are below this line we must pick the infimum and supremum of the interval as follows. Fix μ_i then from (A.6) we have that in order to maximize the value to the regulator we need to pick μ_{i+1} so that the slope of the line from $(\mu_i, u_R(\mu_i))$ to $(\mu_{i+1}, u_R(\mu_{i+1}))$ is maximized. In turn, fix μ_{i+1} then from (A.7) we have that in order to maximize the value to the regulator we need to pick μ_i so that the slope of the line from $(\mu_{i+1}, u_R(\mu_{i+1}))$ to $(\mu_i, u_R(\mu_i))$ is minimized. These statements correspond to (9)-(10). Note also that for $\mu_i > 0$ and $\mu_{i+1} < 1$ there are $\gamma(\mu_i)$ and $\gamma(\mu_{i+1})$, respectively, for which (8) is satisfied, while for any point $\mu_0 \in (\mu_i, \mu_{i+1})$ there is no such $\gamma(\mu_0)$. In words, the infimum and supremum of a disclosure interval are the last and first point, respectively of a non-disclosure interval. $\square$

Proof of Lemma 2. *(i)* For $\eta < 0$ the regulator wants to induce joint investment only if $\omega = 1$. Assume the optimal test leads to an interior interim belief $\mu \in (0,1)$. Then investors might jointly invest when $\omega = -1$ or refrain from investing when $\omega = 1$–both are possible if their signals do not perfectly reveal the state. Both these possibilities are eliminated by fully revealing the state, in which case partial disclosure cannot be optimal. [41]

(ii) For $\eta > 0$, we look at the value $R(0) = \widetilde{R}(0) = \lambda(0)(1+\gamma)/(1-\gamma)$ to show that for all priors in some neighborhood of 1 the optimal test leads to no disclosure. Then consider two cases:

Case 1. Let $f(0) > 0$ so that $R(0) > 0$. Then for all priors $\mu_0 \in (1/(1+R(0)), 1)$ either (i) $k(\mu_0) = 0$,[42] or (ii) a test that discloses $1/(1+R(0)), 1$ leads investors to always invest regardless of the test outcome; and so the regulator can achieve utility $\mu_0 + \eta(1-\mu_0) > \mu_0$, since $\eta > 0$. Note, that μ_0 is the utility of the regulator at prior μ_0 under a fully informative test; given the inequality we just established such a test cannot be optimal.

Case 2. Let $f(0) = 0$ and assume that $g(0) > 0$ so that $R(0) = 0$. Then, for μ_0 close to 1

$$u_R(\mu_0) \approx u_R(1) + (\mu_0 - 1)u_R'(1). \tag{A.8}$$

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[41] If investors' signals perfectly reveal the state, then investment outcomes are independent on the stress test, so any test (and in particular a fully informative one) is optimal.

[42] This would always be the case for $\mu_0 \in (1/(1+R(0)), 1)$ if $\min_{k \in [0,1]} \widetilde{R}(k) = \widetilde{R}(0)$–see Lemma 1.

Now, $u_R(1) = 1$ and from (7) the regulator's equilibrium marginal indirect utility is

$$u'_R(\mu) = \overline{F}^2(k) - \eta \overline{G}^2(k) - 2k'(\mu)\left(\mu f(k)\overline{F}(k) + \eta(1-\mu)g(k)\overline{G}(k)\right)\Big|_{k=k(\mu)}, \tag{A.9}$$

so that

$$u'_R(1) = 1 - \eta - 2k'(1)f(0), \tag{A.10}$$

where we used $k(1) = 0$ (see (5)) and $g(0) < \infty$. Now, from (6), $k'(1) < \infty$ iff $R'(0) > 0$. For $R'(0) > 0$ we need $\widetilde{R}'(0) > 0$ (see (4)). Bearing in mind that $f(0) = 0$, $g(0) > 0$, and $g(0) < \infty$ we have that $\widetilde{R}'(0) = \lambda'(0)(1 + \gamma)/(1 - \gamma)$; given our assumptions, $\lambda'(0) > 0$ obtains as long as we have $f'(0) > 0$ and $g'(0) < \infty$.

The above implies that in (A.10) we have $u'_R(1) = 1 - \eta$, where we used $k'(1) < \infty$ and $f(0) = 0$; this in turn leads from (A.8) to $u_R(\mu_0) \approx 1 + (\mu_0 - 1)(1 - \eta) = \mu_0 + \eta(1 - \mu_0) > \mu_0$, for $\eta > 0$. Hence, as in case 1, a fully informative test is never optimal. $\quad\square$

Proof of Lemma 3. For $0 \le \mu \le 1/(1 + R(1))$ investors never invest—see (5)—and so $u_R(\mu) = 0$. Thus, disclosure must be necessarily optimal for any η—i.e., $D_0 \equiv (0, 1/(1 + R(1)) \subseteq D$ for the set D in Corollary 1. For D_0 to be non-empty we need $1/(1 + R(1)) > 0$ or $R(1) = \lambda(1) < \infty$. $\quad\square$

Proof of Lemma 4. *(i)* From Lemma 2 we know that the regulator would never disclose both $\omega = -1$ and $\omega = 1$ if $\eta > 0$. Moreover, for $R(0) > 0$, the regulator can induce both investors to invest by disclosing $\mu_{CI} < 1$ whenever $\mu_0 < \mu_{CI}$, so she never reveals $\omega = 1$. Note that if $R(0) > 0$, then $\tilde{R}(k) > 0$ for $k \in [0, 1]$,[43] and the regulator can induce both investors to invest by disclosing $1/(1 + \min_{k\in[0,1]} \tilde{R}(k)) < 1$ or by non-disclosure if $\mu_0 > \mu_{CI}$, so that, again she never reveals $\omega = 1$. Hence, our focus here will be on the case $R(0) = 0$, which for bounded densities in $[0, 1]$ requires $f(0) = 0$ and $g(0) > 0$. We will show that there is a neighborhood of 1 which belongs to the non-disclosure set in Corollary 1. This requires that (8) must hold for μ_0 close to 1. We will show that it suffices that $u_R(\mu)$ is locally concave at $\mu = 1$ for (8) to hold, i.e., there exists $\mu_{ND} < 1$ so that for $\mu \in (\mu_{ND}, 1)$ the indirect utility $u_R(\mu)$ is concave.

From (7) we have

$$\begin{aligned}
u''_R(\mu) = {} & 2f^2(k(\mu))\left(k'(\mu)\right)^2\mu - 2\overline{F}(k(\mu))f'(k(\mu))\left(k'(\mu)\right)^2\mu - 2\overline{F}(k(\mu))f(k(\mu))k''(\mu)\mu \\
& - 2\overline{F}(k(\mu))f(k(\mu))k'(\mu) - 2\overline{F}(k(\mu))f(k(\mu))k'(\mu) + 2\eta g^2(k(\mu))\left(k'(\mu)\right)^2(1 - \mu) \\
& - 2\eta\overline{G}(k(\mu))g'(k(\mu))\left(k'(\mu)\right)^2(1 - \mu) - 2\eta\overline{G}(k(\mu))g(k(\mu))k''(\mu)(1 - \mu) \\
& + 2\eta\overline{G}(k(\mu))g(k(\mu))k'(\mu) + 2\eta\overline{G}(k(\mu))g(k(\mu))k'(\mu).
\end{aligned}$$

Recall that we are considering the case $R(0) = \widetilde{R}(0) = f(0) = 0$ and that as $\mu \to 1$ we have $k(1) = 0$, $\overline{F}(k(1)) = \overline{G}(k(1)) = 1$. Therefore, if $k'(1), k''(1), f'(0)$ and $g'(0)$ are bounded, then

$$\lim_{\mu \to 1} u''_R(\mu) = -2f'(0)\left(k'(1)\right)^2 + 2\eta g(0)k'(1).$$

The assumptions we made to write the preceding equation already guarantee that $k'(1)$ is bounded. In Lemma A.1 above we also show that $k''(1) < \infty$. Moreover, since $f(0) = 0$ then $\lambda'(0) > 0$–see Assumption 1– implies that $f'(0) > 0$. Therefore, for $\eta > 0$ it follows that $\lim_{\mu\to1} u''_R(\mu) < 0$; hence, there exists $\mu_{ND} < 1$ so that $u_R(\mu)$ is concave in $(\mu_{ND}, 1)$.

To complete the proof, we will find a neighborhood of 1 in which (8) holds with $\gamma_0 = u'_R(\mu_0)$, so that non-disclosure is optimal for all priors in this neighborhood. Using expression (A.9) in the proof of Lemma 2 we have:

$$\lim_{\mu \to 1} u'_R(\mu) = 1 - \eta < 1.$$

Using this and the fact that $\overline{F}^2, \overline{G}^2 < 1$ for $\mu < 1$ we have

$$u_R(1) + u'_R(1)(\mu - 1) = 1 + (1 - \eta)(\mu - 1) = \mu + \eta(1 - \mu) > u_R(\mu).$$

Now, let $\Delta = \max_{\mu \le \mu_{ND}} \mu + \eta(1 - \mu) - u_R(\mu) > 0$. Then, if u_R is continuously differentiable in $\mu < 1$, then through a continuity argument we can find μ^*_{ND} so that for $\mu_0 > \mu^*_{ND}$

$$\max_{\mu \le \mu^*_{ND}} \left| u_R(1) + u'_R(1)(\mu - 1) - \left(u_R(\mu_0) + u'_R(\mu_0)(\mu - \mu_0)\right)\right| < \Delta.$$

So that for any $\mu_0 > \mu^*_{ND}$, the triangle inequality implies.

$$\max_{\mu \le \mu^*_{ND}} \left| u_R(\mu_0) + u'_R(\mu_0)(\mu - \mu_0) - u_R(\mu)\right| > 0.$$

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[43] This follows from (2) as the second term is always strictly positive for $k \in [0, 1]$.

In summary, for any $\mu_0 \in (\min\{\mu_{ND}, \mu_{ND}^*\}, 1)$, we have that (8) holds and non-disclosure is optimal.

(ii) Now, suppose that $R(0) > 0$ so that $\mu_{CI} < 1$. From the proof of Lemma 1.ii we have that, whenever $k(\mu)$ is differentiable, $k'(\mu) = -1/\mu^2 R'(k(\mu))$. Given the above and using expression (A.9) in the proof of Lemma 2 we have:

$$\lim_{\mu \to \mu_{CI}} u_R'(\mu) = 1 - \eta + \frac{2f(0)}{\mu_{CI}^2 R'(0)} \left(1 + \eta \frac{1+\gamma}{1-\gamma}\right),$$

where we have also used $R(0) = f(0)(1+\gamma)/(g(0)(1-\gamma))$ and $\mu_{CI} = 1/(1 + R(0))$. Note that the regulator's expected utility from a (non-necessarily optimal) test that discloses $\{0, \mu_{CI}\}$ is $\Pr[s = \mu_{CI}]u_R(\mu_{CI}) = (\mu_0/\mu_{CI})(\mu_{CI} + \eta(1 - \mu_{CI}))$ while non-disclosure would generate $u_R(\mu_0) \approx u_R(\mu_{CI}) + (\mu_0 - \mu_{CI})u_R'(\mu_{CI})$. A sufficient condition for disclosure to be optimal is that $\Pr[s = \mu_{CI}]u_R(\mu_{CI}) > u_R(\mu_0)$, or in other words $u_R'(\mu_{CI}) > u_R(\mu_{CI})/\mu_{CI}$. Using the expression of $u_R'(\mu_{CI})$, disclosure is optimal if

$$1 - \eta + \frac{2f(0)}{\mu_{CI}^2 R'(0)} \left(1 + \eta \frac{1+\gamma}{1-\gamma}\right) > 1 + \eta R(0),$$

which implies (13). $\quad\square$

Proof of Proposition 1. The optimal CF-test $\{0, \underline{\mu}(\alpha)\}$ satisfies $\underline{\mu}(\alpha) = 1/(1 + R(k^*(\alpha), \alpha)$ with $k^*(\alpha)$ solving the first order condition $dA_R(k^*(\alpha); \alpha)/dk = 0$. Totally differentiating,

$$\frac{d\underline{\mu}(\alpha)}{d\alpha} = -\frac{\partial R(k, \alpha)/\partial\alpha}{(1 + R(k, \alpha))^2} - \frac{\partial R(k, \alpha)/\partial k}{(1 + R(k, \alpha))^2} \frac{dk^*(\alpha)}{d\alpha}\bigg|_{k=k^*(\alpha)}$$

$$= \frac{\partial R(k, \alpha)/\partial k}{(1 + R(k, \alpha))^2} \left(-\frac{\partial R(k, \alpha)/\partial\alpha}{\partial R(k, \alpha)/\partial k} - \frac{dk^*(\alpha)}{d\alpha}\right)\bigg|_{k=k^*(\alpha)} \tag{A.11}$$

Let $k(\mu, \alpha)$ be the equilibrium threshold defined in (5). Implicitly differentiating,

$$\frac{\partial k(\mu, \alpha)}{\partial\alpha} = -\frac{\partial R(k, \alpha)/\partial\alpha}{\partial R(k, \alpha)/\partial k}$$

so that (A.11) translates to

$$\frac{d\underline{\mu}(\alpha)}{d\alpha} = \frac{\partial R(k, \alpha)/\partial k}{(1 + R(k, \alpha))^2} \left(\frac{\partial k(\mu, \alpha)}{\partial\alpha} - \frac{dk^*(\alpha)}{d\alpha}\right)\bigg|_{k=k^*(\alpha)}.$$

By definition, $R(k, \alpha)$ is non-decreasing in k. Thus, $d\underline{\mu}(\alpha)/d\alpha \geq 0$ iff

$$\frac{\partial k(\mu, \alpha)}{\partial\alpha}\bigg|_{k=k^*(\alpha)} \geq \frac{dk^*(\alpha)}{d\alpha}. \quad\square$$

Proof of Proposition 2. For each α, consider an equilibrium selection $k_s(\mu, \alpha)$ with associated $R_s(k, \alpha)$ defined in $K_s(\alpha) \equiv \{k : k = k_s(\mu, \alpha), \mu \in (0, 1)\}$, so that for each $k \in K_s(\alpha)$ $R_s(k, \alpha) = \widetilde{R}(k, \alpha)$—with $\widetilde{R}$ defined in (2). As we will analyze the optimal CF-test, it suffices to only consider non-increasing (in μ) equilibrium selections; the reason being that the optimal CF-test would never select a realization $\underline{\mu}(\alpha)$ such that $\partial k_s(\underline{\mu}(\alpha), \alpha)/\partial\mu > 0$ as then the regulator can increase the probability of joint investment by switching to a lower realization $\underline{\mu}'(\alpha) < \underline{\mu}(\alpha)$—this realization induces a lower threshold and occurs with higher probability. Thus, to solve for the regulator's problem we can replace any equilibrium selection with a selection represented by a non-decreasing (in k) $R_s(k, \alpha)$.

To simplify notation, let $t = (1 - \mu)/\mu$ be the odds of state $\omega = -1$ so that the equilibrium threshold (5) can be written as $k_s(t, \alpha) = R_s^{-1}(t, \alpha)$. Then the regulator's average payoff (11) can be expressed as a function of t

$$A_R(t; \alpha) \equiv A_R(\mu(t); \alpha) = \overline{F}^2(k_s(t; \alpha); \alpha) + \eta\overline{G}^2(k_s(t; \alpha); \alpha)t. \tag{A.12}$$

We proceed by contradiction: suppose that the optimal CF-test satisfies $\liminf_{\alpha \to 1} \underline{\mu}(\alpha) = \underline{\mu}'' < 1$. We will show that for any $\underline{\mu}' \in (\underline{\mu}'', 1)$, with $t' = (1 - \underline{\mu}')/\underline{\mu}'$, there exists $\tilde{\alpha} < 1$ such that, whenever it exists, $dA_R(t'; \alpha)/dt < 0$ for $\alpha > \tilde{\alpha}$. Thus, $t' > 0$ cannot maximize $A_R(t; \alpha)$ for $\alpha > \tilde{\alpha}$, thus reaching the contradiction $\liminf_{\alpha \to 1} \underline{\mu}(\alpha) > \underline{\mu}''$. Therefore, we must have $\liminf_{\alpha \to 1} \underline{\mu}(\alpha) = 1$ and the optimal asymptotic CF-test perfectly reveals the state.

Differentiating $A_R(t; \alpha)$, where we drop the dependence on α to ease notation,

$$\frac{\partial A_R}{\partial t} = \left(-2f(k)\overline{F}(k) - 2\eta g(k)\overline{G}(k)t\right)\frac{\partial k_s}{\partial t} + \eta\overline{G}^2(k)\bigg|_{k=R_s^{-1}(t)}$$

$$= \eta t g(k)\overline{G}(k)\frac{\partial k_s}{\partial t}\left(-\frac{2}{\eta}\frac{\lambda(k)}{t}\frac{\overline{F}(k)}{\overline{G}(k)} - 2 + \frac{\overline{G}(k)}{t g(k)\frac{\partial k_s}{\partial t}}\right)\bigg|_{k=R_s^{-1}(t)}$$

Using, (i) as $k_s(\mu,\alpha)$ satisfies $t = R_s(k,\alpha)(=\widetilde{R}(k,\alpha))$, we have $\frac{\lambda(k)}{t} = \frac{1+\gamma\overline{F}(k)}{1-\gamma\overline{G}(k)}$, and (ii) $\frac{\partial k_s}{\partial t} = \frac{1}{\partial R_s/\partial k}\Big|_{k=k_s(t)} = \frac{1}{\partial\widetilde{R}/\partial k}\Big|_{k=k_s(t)}$ for any t such that $k_s(t,\alpha)$ is in the interior of $K_s(\alpha)$; (iii) using (16) we can establish

$$\frac{\overline{G}(k)}{tg(k)\frac{\partial k_s}{\partial t}} = \frac{\overline{G}(k)\partial\widetilde{R}(k)/\partial k}{g(k)\widetilde{R}(k)}\Bigg|_{k=k_s(t,\alpha)} = \tau.$$

Therefore,

$$\frac{dA_R}{dt}\bigg|_{t=t'} = \eta tg(k)\overline{G}(k)\frac{\partial k_s}{\partial t}\left(-\frac{2}{\eta}\frac{1+\gamma\overline{F}(k)}{1-\gamma\overline{G}(k)}\frac{\overline{F}(k)}{\overline{G}(k)} - 2 + \tau\right)\Bigg|_{k=k_s(t',\alpha)}. \tag{A.13}$$

We make the following three observations regarding the asymptotic behavior of the terms in (A.13):

(i) For $t' > 0$, $\overline{F}(k_s(t',\alpha),\alpha) \to 1$ and $\overline{G}(k_s(t',\alpha),\alpha) \to 0$ as $\alpha \to 1$–see Assumption A3. This captures the idea that investors will always invest if $\omega = 1$ and never invest if $\omega = -1$ as their private signals become perfectly informative of the state. Therefore,

$$\lim_{\alpha\to1}\frac{1+\gamma\overline{F}(k_s(t',\alpha),\alpha)}{1-\gamma\overline{G}(k_s(t',\alpha),\alpha)} = 1 + \gamma > 0.$$

(ii) Assumption A3 implies that $\overline{F}(k_s(t',\alpha),\alpha)/\overline{G}(k_s(t',\alpha),\alpha)$ diverges as $\alpha \to 1$.

(iii) By the boundedness assumption on τ, we have, $\tau(k_s(t,\alpha),\alpha) \leq M$ for $t > 0, \alpha \in [0,1]$.

Using these three observations, we have that the term in parentheses in (A.13) becomes strictly negative for any $t' > 0$ and α close to 1. Then, for α close to 1, the optimal CF-test must reveal more information that $\underline{\mu}'$. □

Proof of Corollary 2. We show that for each $\underline{\mu} \in (0,1)$

$$\lim_{\alpha\to1}\overline{G}(k,\alpha)\tau(k,\alpha)\Big|_{k=k(\underline{\mu},\alpha)} = 0.$$

This guarantees in the proof of Proposition 2 that there is an α' such that $dA_R(\underline{t};\alpha)/dt < 0$ for $\alpha > \alpha'$ and $\underline{t} = \frac{1-\underline{\mu}}{\underline{\mu}} > 0$, thus guaranteeing asymptotic crowding-in.

Note that any equilibrium threshold k satisfies $k \leq 1/2$ and $\lambda(k,\alpha) = 2(1-\alpha)/m(k,\alpha)$ for $k \leq 1/2$. Furthermore, $\overline{G}(k,\alpha) \equiv \int_k^{1/2} m(s,\alpha)ds \leq m(k,\alpha)(1/2 - k)$ as $m(k,\alpha)$ is decreasing. Then

$$\overline{G}(k,\alpha)\tau(k,\alpha) = \frac{\overline{G}^2(k,\alpha)}{g(k,\alpha)}\left(\frac{\partial\lambda(k,\alpha)/\partial k}{\lambda(k,\alpha)} + \frac{\partial\left(1+\gamma\overline{F}(k,\alpha)\right)/\partial k}{1+\gamma\overline{F}(k,\alpha)} - \frac{\partial\left(1-\gamma\overline{G}(k,\alpha)\right)/\partial k}{1-\gamma\overline{G}(k,\alpha)}\right).$$

$$= \frac{\overline{G}^2(k,\alpha)}{g(k,\alpha)}\left(-\frac{\partial m(k,\alpha)/\partial k}{m(k,\alpha)} - \frac{\gamma 2(1-\alpha)}{1+\gamma\overline{F}(k,\alpha)} - \frac{m(k,\alpha)}{1-\gamma\overline{G}(k,\alpha)}\right)$$

$$\leq (1/2-k)^2\left(|\partial m(k,\alpha)/\partial k| - \frac{\gamma 2(1-\alpha)m(k,\alpha)}{1+\gamma\overline{F}(k,\alpha)} - \frac{m^2(k,\alpha)}{1-\gamma\overline{G}(k,\alpha)}\right)$$

$$\leq (1/2-k)M$$

where in the last inequality we used $m(k,\alpha) \geq m(1/2,\alpha) \geq 0$ and the assumption $(1/2-k)|\partial m(k,\alpha)/\partial k| \leq M$. The proof is complete by noting that $\lim_{\alpha\to1}k(\underline{\mu},\alpha) = 1/2$. □

Proof of Corollary 3. (i) We apply Proposition 2 by showing that $\tau(k,\alpha)$ is bounded. Indeed,

$$\tau(k,\alpha) = \frac{\overline{G}(k,\alpha)}{g(k,\alpha)}\left(\frac{\partial\lambda(k,\alpha)/\partial k}{\lambda(k,\alpha)} + \frac{\partial\left(1+\gamma\overline{F}(k,\alpha)\right)/\partial k}{1+\gamma\overline{F}(k,\alpha)} - \frac{\partial\left(1-\gamma\overline{G}(k,\alpha)\right)/\partial k}{1-\gamma\overline{G}(k,\alpha)}\right)$$

$$= \frac{2(1-\alpha)(1-k)}{2(1-\alpha)}\left(\frac{\partial h(k,\alpha)/\partial k}{h(k,\alpha)} - \frac{\gamma h(k,\alpha)}{1+\gamma\overline{F}(k,\alpha)} - \frac{\gamma 2(1-\alpha)}{1-\gamma\overline{G}(k,\alpha)}\right)$$

$$\leq (1-k)\frac{\partial h(k,\alpha)/\partial k}{h(k,\alpha)} \leq \frac{1-k}{\varepsilon}\max_{k\in[0,1/2]}\partial h(k,\alpha)/\partial k,$$

where we used the assumption $h(1/2,\alpha) \geq \varepsilon$, $\max\partial h(k,\alpha)/\partial k \leq M$, and $h(k,\alpha)$ being increasing (in k).

(ii) To simplify notation, we let $\widetilde{\alpha} = 2(1 - \alpha)$, so that $g(k, \alpha) = \widetilde{\alpha}$ and $\overline{G}(k, \alpha) = (1 - k)\widetilde{\alpha}$, and we denote $\overline{H}(k, \alpha) \equiv \overline{F}(k, \alpha)$. Using the representation (A.12) of A_R as a function of the odds of a low state $t = (1 - \mu)/\mu$, the optimality condition for the optimal CF-test $\underline{\mu} = 1/(1 + \underline{t})$ is

$$\left| \left(-2f(k, \alpha)\overline{F}(k, \alpha) - 2\eta g(k, \alpha)\overline{G}(k, \alpha)\underline{t} \right) \frac{\partial k}{\partial t} + \eta \overline{G}^2(k, \alpha) \right|_{k=k(\underline{t}, \alpha)} = 0. \tag{A.14}$$

We will use the following

$$\frac{\partial k}{\partial t} = \left| \frac{1}{\partial R(k, \alpha)/\partial k} \right|_{k=k(t, \alpha)},$$

$$\frac{\partial R(k, \alpha)}{\partial k} = R(k, \alpha) \left(\frac{\partial h(k, \alpha)/\partial k}{h(k, \alpha)} - \frac{\gamma h(k, \alpha)}{1 + \gamma \overline{H}(k, \alpha)} - \gamma \frac{g(k, \alpha)}{1 - \gamma \overline{G}(k, \alpha)} \right)$$

$$= t \left(\frac{\partial h(k, \alpha)/\partial k}{h(k, \alpha)} - \gamma \frac{h(k, \alpha)}{1 + \gamma \overline{H}(k, \alpha)} - \gamma \frac{\widetilde{\alpha}}{1 - \gamma \widetilde{\alpha}(1 - k)} \right).$$

Replacing these values in (A.14) and solving for $\underline{t}$ we have

$$\underline{t}(\alpha) = \frac{2h(k, \alpha)\overline{H}(k, \alpha)}{\eta \widetilde{\alpha}^2 (1 - k)^2 \left(\frac{\partial h(k,\alpha)/\partial k}{h(k,\alpha)} - \gamma \frac{h(k,\alpha)}{1+\gamma \overline{H}(k,\alpha)} - \gamma \frac{\widetilde{\alpha}}{1-\gamma\widetilde{\alpha}(1-k)} \right) - 2\eta \widetilde{\alpha}^2 (1-k)}$$

$$= \frac{h^2(k, \alpha)}{\widetilde{\alpha}^2} \frac{2\overline{H}(k, \alpha)}{\eta(1-k)^2 \frac{\partial h(k,\alpha)}{\partial k} - \eta h(k, \alpha) \left(\gamma \frac{h(k,\alpha)}{1+\gamma \overline{H}(k,\alpha)} + \gamma \frac{\widetilde{\alpha}}{1-\gamma\widetilde{\alpha}(1-k)} - 2\eta(1-k) \right)}. \tag{A.15}$$

Let $t_h \equiv \lim_{\alpha \to 1} \underline{t}(\alpha)$, and from the statement of the proposition $\overline{h}' = \lim_{\alpha \to 1} \partial_+ h(1/2, \alpha)/\partial k$. Note that as $\alpha \to 1$, $k(t, \alpha) \to 1/2$ so

$$\lim_{\alpha \to 1} \left| \frac{2\overline{H}(k, \alpha)}{\eta(1-k)^2 \frac{\partial h(k,\alpha)}{\partial k} - \eta h(k, \alpha) \left(\gamma \frac{h(k,\alpha)}{1+\gamma \overline{H}(k,\alpha)} + \gamma \frac{\widetilde{\alpha}}{1-\gamma\widetilde{\alpha}(1-k)} - 2\eta(1-k) \right)} \right|_{k(\underline{t}(\alpha), \alpha)} = \frac{2}{\eta(1-1/2)\overline{h}'}$$

Taking limits on both sides of (A.15) we have,

$$t_h = \frac{2}{\eta(1-1/2)^2 \overline{h}'} \lim_{\alpha \to 1} \left| \left(\frac{h(k, \alpha)}{\widetilde{\alpha}} \right)^2 \right|_{k(\underline{t}(\alpha), \alpha)}$$

Since $t = R(k(t, \alpha), \alpha)$, we have

$$t_h^2 = \lim_{\alpha \to 1} \left| \left(\frac{h(k, \alpha)}{\widetilde{\alpha}} \frac{1 + \gamma \overline{H}(k, \alpha)}{1 - \gamma \widetilde{\alpha}(1 - k)} \right)^2 \right|_{k=k(\underline{t}(\alpha), \alpha)} = (1 + \gamma)^2 \lim_{\alpha \to 1} \left| \left(\frac{h(k, \alpha)}{\widetilde{\alpha}} \right)^2 \right|_{k(\underline{t}(\alpha), \alpha)}$$

Therefore, we have

$$t_h = \frac{\eta(1 + \gamma)^2 \overline{h}'}{8} \to \underline{\mu}_h = \frac{1}{1 + t_h} = \frac{1}{1 + \frac{\eta(1+\gamma)^2 \overline{h}'}{8}} \qquad \square$$

Proof of Proposition 3. With $k(\mu; \alpha) \in \{x_1, \ldots, x_n\}$ the investment threshold in a symmetric equilibrium, define $\underline{\mu}(i; \alpha)$ as the minimum prior belief so that investors are willing to invest if they observe $x = x_i$,

$$\underline{\mu}(i; \alpha) \equiv \min \{ \mu : k(\mu; \alpha) \leq x_i \}.$$

The equilibrium conditions (1) adapted to this discrete case imply that

$$\underline{\mu}(i; \alpha) = \frac{1}{1 + \frac{f_i(\alpha)}{g_i(\alpha)} \frac{1 + \gamma \overline{F}_i(\alpha)}{1 - \gamma \overline{G}_i(\alpha)}}, \tag{A.16}$$

where $\overline{F}_i(\alpha) = \sum_{j=i}^n f_j(\alpha)$ and $\overline{G}_i(\alpha) = \sum_{j=i}^n g_j(\alpha)$.

Fix a prior $\mu_0 \in (0, 1)$. Under the conditions of the proposition, there exists $i^* \in \{2, \ldots, n\}$ so that $\lim_{\alpha \to 1} f_i(\alpha)/g_i(\alpha) = 0$ for $i < i^*$ and $\lim_{\alpha \to 1} g_i(\alpha)/f_i(\alpha) = 0$ for $i \geq i^*$, which using (A.16) implies

$$\lim_{\alpha \to 1} \underline{\mu}(i; \alpha) = 1 \text{ for } i < i^* \text{ and } \lim_{\alpha \to 1} \underline{\mu}(i; \alpha) = 0 \text{ for } i \geq i^*, \tag{A.17}$$

and we can find $\tilde{\alpha} < 1$ so that $\max_{i \geq i^*} \underline{\mu}(i; \alpha) \leq \mu_0 < \min_{i < i^*} \underline{\mu}(i; \alpha)$ for $\alpha \geq \tilde{\alpha}$.

We first show that for $\alpha \geq \tilde{\alpha}$, the optimal test leads to some disclosure—i.e., for the optimal test in Proposition 1 adapted to the discrete case, $\mu_0 \in D$. We then show that the optimal test converges to a fully revealing test as $\alpha \to 1$.

Let $\underline{i}(\alpha)$ and $\bar{i}(\alpha)$ be defined by $\underline{\mu}(\underline{i}(\alpha); \alpha) = \min_{i < i^*} \underline{\mu}(i; \alpha)$ and $\underline{\mu}(\bar{i}(\alpha); \alpha) = \max_{i \geq i^*} \underline{\mu}(i; \alpha)$—note that if $\widetilde{R}(i; \alpha) \equiv \frac{f_i(\alpha)}{g_i(\alpha)} \frac{1 + \gamma \overline{F}_i(\alpha)}{1 - \gamma \overline{G}_i(\alpha)}$ increases in i then we would immediately have $\bar{i}(\alpha) = i^*$ and $\underline{i}(\alpha) = i^* - 1$. Then, for $\alpha \geq \tilde{\alpha}$ we have $k(\mu; \alpha) = x_{\bar{i}(\alpha)}$ and the regulator's payoff if she provides no information to investors is

$$\mu \left(\overline{F}_{\bar{i}(\alpha)}(\alpha) \right)^2 + (1 - \mu) \eta \left(\overline{G}_{\bar{i}(\alpha)}(\alpha) \right)^2.$$

We now show that the test $\pi_{\left\{ \underline{\mu}(\bar{i}(\alpha); \alpha), \underline{\mu}(\underline{i}(\alpha); \alpha) \right\}}$, supported on $\underline{\mu}(\bar{i}(\alpha); \alpha)$ and $\underline{\mu}(\underline{i}(\alpha); \alpha)$ strictly outperforms the test $\pi_\varnothing$ where the regulator provides no public information. Note that any other potential test that may dominate $\pi_{\left\{ \underline{\mu}(\bar{i}(\alpha); \alpha), \underline{\mu}(\underline{i}(\alpha); \alpha) \right\}}$ can only be supported on $\left\{ \mu \leq \underline{\mu}(\bar{i}(\alpha); \alpha) \right\} \cup \left\{ \mu \geq \underline{\mu}(\underline{i}(\alpha); \alpha) \right\}$. Then, from (A.17), we have that any of these tests, including $\pi_{\left\{ \underline{\mu}(\bar{i}(\alpha); \alpha), \underline{\mu}(\underline{i}(\alpha); \alpha) \right\}}$, converges to a fully informative test as $\alpha \to 1$, concluding our proof.

Let Δ denote the difference between the regulator's payoff under $\pi_{\left\{ \underline{\mu}(\bar{i}(\alpha); \alpha), \underline{\mu}(\underline{i}(\alpha); \alpha) \right\}}$ and $\pi_\varnothing$:

$$\Delta \equiv \Pr\left[\underline{\mu}(\bar{i}(\alpha); \alpha) \right] \left(\underline{\mu}(\bar{i}(\alpha); \alpha) \left(\overline{F}_{\bar{i}(\alpha)}(\alpha) \right)^2 + \left(1 - \underline{\mu}(\bar{i}(\alpha); \alpha) \right) \eta \left(\overline{G}_{\bar{i}(\alpha)}(\alpha) \right)^2 \right) +$$

$$\Pr\left[\underline{\mu}(\underline{i}(\alpha); \alpha) \right] \left(\underline{\mu}(\underline{i}(\alpha); \alpha) \left(\overline{F}_{\underline{i}(\alpha)}(\alpha) \right)^2 + \left(1 - \underline{\mu}(\underline{i}(\alpha); \alpha) \right) \eta \left(\overline{G}_{\underline{i}(\alpha)}(\alpha) \right)^2 \right)$$

$$- \left(\mu \left(\overline{F}_{\bar{i}(\alpha)}(\alpha) \right)^2 + (1 - \mu) \eta \left(\overline{G}_{\bar{i}(\alpha)}(\alpha) \right)^2 \right)$$

$$= \Pr\left[\underline{\mu}(\underline{i}(\alpha); \alpha) \right] \underline{\mu}(\underline{i}(\alpha); \alpha) \left(\left(\overline{F}_{\underline{i}(\alpha)}(\alpha) \right)^2 - \left(\overline{F}_{\bar{i}(\alpha)}(\alpha) \right)^2 \right) +$$

$$+ \Pr\left[\underline{\mu}(\underline{i}(\alpha); \alpha) \right] \left(1 - \underline{\mu}(\underline{i}(\alpha); \alpha) \right) \eta \left(\left(\overline{G}_{\underline{i}(\alpha)}(\alpha) \right)^2 - \left(\overline{G}_{\bar{i}(\alpha)}(\alpha) \right)^2 \right).$$

By assumption, either $f_i(\alpha) > 0$ for $\alpha < 1$ implying that

$$\left(\left(\overline{F}_{\underline{i}(\alpha)}(\alpha) \right)^2 - \left(\overline{F}_{\bar{i}(\alpha)}(\alpha) \right)^2 \right) = \left(\sum_{j=\underline{i}(\alpha)}^{\bar{i}(\alpha)} f_j(\alpha) \right) \left(\overline{F}_{\bar{i}(\alpha)}(\alpha) + \overline{F}_{\underline{i}(\alpha)}(\alpha) \right) > 0,$$

or $g_i(\alpha) > 0$ for $\alpha < 1$, implying that

$$\left(\left(\overline{G}_{\underline{i}(\alpha)}(\alpha) \right)^2 - \left(\overline{G}_{\bar{i}(\alpha)}(\alpha) \right)^2 \right) = \left(\sum_{j=\underline{i}(\alpha)}^{\bar{i}(\alpha)} g_j(\alpha) \right) \left(\overline{G}_{\bar{i}(\alpha)}(\alpha) + \overline{G}_{\underline{i}(\alpha)}(\alpha) \right) > 0.$$

Therefore, $\Delta > 0$ for $1 > \alpha \geq \tilde{\alpha}$. $\square$

Proof of Remark 1. Using the change of variable $x = (k - \alpha)/(1 - \alpha)$ in (11) we can express the regulator's expected average utility as

$$A_R(x; \alpha) = \overline{F}^2(x; \alpha) + \eta \overline{G}^2(x; \alpha) R(x; \alpha) = \overline{H}^2(x) + \eta ((1 - \alpha)(1 - x))^2 R(x; \alpha)$$

$$= \overline{H}^2(x) + \eta \frac{(1 - \alpha)}{1 - \gamma (1 - \alpha)(1 - x)} h(x) \left(1 + \gamma \overline{H}(x) \right) (1 - x)^2.$$

Differentiating with respect to α, we obtain

$$\frac{\partial A_R(x; \alpha)}{\partial \alpha} = -\eta \frac{1}{(1 - \gamma (1 - \alpha)(1 - x))^2} h(x) \left(1 + \gamma \overline{H}(x) \right) (1 - x)^2 \leq 0. \quad \square$$

Proof of Proposition 4. For $\mu_0 \in (\underline{\mu}(\alpha) - \epsilon, \underline{\mu}(\alpha))$ for some $\epsilon > 0$, the sign of (23) is

$$sign \left[\frac{dV_i^{test}(\alpha)}{d\alpha} \right] = sign \left[\frac{\partial \widetilde{A}_I(\underline{\mu}(\alpha), \alpha)}{\partial \mu} \frac{\partial \underline{\mu}(\alpha)}{\partial \alpha} \right]$$

We show that, evaluated at $\underline{\mu}(\alpha)$, average investor's utility is increasing, i.e., $\partial \widetilde{A}_I(\underline{\mu}(\alpha)), \alpha)/\partial \mu > 0$, so that

$$sign \left[\frac{dV_i^{test}(\alpha)}{d\alpha} \right] = sign \left[\frac{\partial \underline{\mu}(\alpha)}{\partial \alpha} \right].$$

Using $A_I(k; \alpha) = \mathcal{E}(k; \alpha) + \gamma A_{R,1}(k; \alpha)$ where $\mathcal{E}(k; \alpha) \equiv \mathbb{E}_{X(\alpha)}[\omega 1_{\{x_i \geq k\}}]/\mu(k; \alpha)$ captures how adapted are investments to fundamentals and $A_{R,1}(k; \alpha)$ is the regulator's average utility when $\eta = 1$, we can write

$$\frac{\partial \widetilde{A}_I(\mu, \alpha)}{\partial \mu} = \frac{\partial \mathcal{E}(k(\mu, \alpha); \alpha)}{\partial \mu} + \gamma \frac{\partial A_R(k(\mu, \alpha); \alpha)}{\partial \mu}.$$

Optimality conditions for CF-tests (12) imply that $\partial A_R(k(\underline{\mu}(\alpha), \alpha); \alpha)/\partial \mu = 0$ so that

$$\frac{\partial \widetilde{A}_I(\underline{\mu}(\alpha), \alpha)}{\partial \mu} = \frac{\partial \mathcal{E}(k(\underline{\mu}(\alpha), \alpha); \alpha)}{\partial \mu}.$$

We next show that $\partial \mathcal{E}(k(\mu, \alpha); \alpha)/\partial \mu > 0$ when evaluated at $\mu = \underline{\mu}(\alpha)$. Differentiating $\mathcal{E}(k; \alpha) = \overline{F}(k; \alpha) - R(k; \alpha)\overline{G}(k; \alpha)$,

$$\frac{\partial \mathcal{E}(k(\underline{\mu}(\alpha), \alpha); \alpha)}{\partial \mu} = -f(k; \alpha) + R(k; \alpha)g(k; \alpha) - \frac{\partial R(k; \alpha)}{\partial k}\overline{G}(k; \alpha)\bigg|_{k=k(\underline{\mu}(\alpha), \alpha)} \frac{\partial k(\underline{\mu}(\alpha), \alpha)}{\partial \mu} \qquad (A.18)$$

The necessary FOC for a critical-fault tests—see (12)— requires that at $k = k(\underline{\mu}(\alpha), \alpha)$

$$\frac{\partial R(k; \alpha)}{\partial k}\overline{G}(k; \alpha) = 2f(k; \alpha)\frac{\overline{F}(k; \alpha)}{\overline{G}(k; \alpha)} + 2g(k; \alpha)R(k; \alpha)g(k; \alpha),$$

and replacing $(\partial R(k; \alpha)/\partial k)\overline{G}(k; \alpha)$ in (A.18), and recalling from Lemma 1-iii that $\partial k(\mu)/\partial \mu < 0$ we have

$$\frac{\partial \mathcal{E}(k(\underline{\mu}(\alpha), \alpha); \alpha)}{\partial \mu} = -f(k; \alpha)\left(1 + 2\frac{\overline{F}(k; \alpha)}{\overline{G}(k; \alpha)}\right) - R(k; \alpha)g(k; \alpha)\bigg|_{k=k(\underline{\mu}(\alpha), \alpha)} \frac{\partial k(\underline{\mu}(\alpha), \alpha)}{\partial \mu} > 0.$$

For all $\gamma \in (0, 1)$, and irrespective of α. $\quad \square$

Appendix B. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jet.2024.105933.

Data availability

No data was used for the research described in the article.

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