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From coordinate subspaces over finite fields to ideal multipartite uniform clutters

Ahmad Abdi[1] **· Dabeen Lee**[2]

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Abstract

Take a prime power q, an integer $n \geq 2$, and a coordinate subspace $S \subseteq GF(q)^n$ over the Galois field $GF(q)$. One can associate with S an n-partite n-uniform clutter $\mathcal{C}$, where every part has size q and there is a bijection between the vectors in S and the members of $\mathcal{C}$. In this paper, we determine when the clutter $\mathcal{C}$ is *ideal*, a property developed in connection to Packing and Covering problems in the areas of Integer Programming and Combinatorial Optimization. Interestingly, the characterization differs depending on whether q is 2, 4, a higher power of 2, or otherwise. Each characterization uses crucially that idealness is a *minor-closed property*: first the list of excluded minors is identified, and only then is the global structure determined. A key insight is that idealness of $\mathcal{C}$ depends solely on the underlying matroid of S. Our theorems also extend from idealness to the stronger *max-flow min-cut* property. As a consequence, we prove the Replication and $\tau = 2$ Conjectures for this class of clutters.

Keywords Vector space over finite field · Multipartite uniform clutter · Ideal clutter · The max-flow min-cut property · Minor-closed property · Matroid

1 Introduction

Let V be a finite set of *elements*, and let $\mathcal{C}$ be a family of subsets of V called *members*. A *cover* is defined as a subset of V that intersects every member in $\mathcal{C}$. Given weights $w \in \mathbb{Z}_+^V$, a minimum weight cover can be computed by solving the integer program $\min\{w^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \in \mathbb{Z}_+^V\}$, where $M(\mathcal{C})$ is the incidence matrix of $\mathcal{C}$ whose columns are labeled by the elements and whose rows are the incidence vectors of the members. The linear programming relaxation of this integer program is the problem

✉ Dabeen Lee
dabeenl@kaist.ac.kr

Ahmad Abdi
a.abdi1@lse.ac.uk

1 Department of Mathematics, London School of Economics and Political Science, London WC2A 2AE, UK

2 Department of Industrial and Systems Engineering, KAIST, Daejeon 34126, Republic of Korea

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of minimizing $w^\top x$ over the *associated set covering polyhedron* given by $Q(\mathcal{C}) := \{x \in \mathbb{R}^V : M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}\}$. For the purpose of finding a minimum weight cover, we may assume without loss of generality that no member properly contains another, in which case we call $\mathcal{C}$ a *clutter* over ground set V [15]. A necessary and sufficient condition for the relaxation to return an integer solution for any $w \in \mathbb{Z}_+^V$, thereby giving a minimum weight cover, is that every extreme point of $Q(\mathcal{C})$ is an integral vector, in which case we say that $\mathcal{C}$ is *ideal* [12].

Every clutter whose members are pairwise disjoint is obviously ideal. Many non-trivial examples of ideal clutters can be found in Combinatorial Optimization – let us mention a few here: the clutter of st-paths of a graph [27], (inclusionwise) minimal st-cuts of a graph [14], minimal T-joins of a graph [17], minimal T-cuts of a graph [17], and odd circuits of a signed graph that has no odd-K_5 minor [18]. Each of these examples has as ground set the edge set of the associated graph. In general, it is co-NP-complete to decide whether a clutter is ideal [13], and understanding the various aspects of the theory of ideal clutters is one of the long-standing open research directions in the area: 11 of the 18 conjectures in the book *Combinatorial Optimization. Packing and Covering* [10] are directly about general or special instances of ideal clutters.

Very little is known about the structure of all ideal clutters (see [10][Sects. 1.1, 1.2, and 4]). As such, previous works focused on ideal clutters that arise from graphs and combinatorial optimization problems. In this paper, we introduce a novel approach to discover and understand ideal clutters, by studying the notion of *multipartite uniform clutters*. Our approach leads to a geometric framework to generate ideal clutters, thereby providing a new perspective for studying ideal clutters.

1.1 Multipartite uniform clutters and vector spaces

Multipartite uniform clutters A multipartite uniform clutter $\mathcal{C}$ is obtained as a family of hyperedges of an *n-partite* hypergraph whose vertices are partitioned into n nonempty disjoint subsets $V_1, \dots, V_n$ for some $n \geq 1$, and every hyperedge intersects each of the subsets exactly once. Then all members of $\mathcal{C}$ have an equal size n, and therefore, $\mathcal{C}$ is *n-uniform* (or simply *uniform*) and a *clutter*. In particular, in a multipartite uniform clutter, the size of a member is equal to the number of partitions. For example, Q_6, the clutter of triangles in K_4 given by $Q_6 := \{\{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}$, is a 3-partite 3-uniform clutter over ground set $\{1, \dots, 6\}$ partitioned into $\{1, 2\} \cup \{3, 4\} \cup \{5, 6\}$. The class of multipartite uniform clutters looks restricted, but in fact, it is general enough to understand the entire class of ideal clutters. More precisely, it was shown in [4] that if we had a characterization of when a multipartite uniform clutter is ideal, then this would in turn completely characterize ideal clutters. This is because any given clutter can be "locally embedded" in a multipartite uniform clutter [4][1]. This connection allows us to take a different angle on understanding idealness.

Vector spaces over $GF(q)$ Thanks to their special structure, one may take advantage of a geometric framework for constructing multipartite uniform clutters. To explain

[1] We discuss related ideas in Sect. 6.1.

it, take a prime power q and $GF(q)$, the *Galois field of order* q. For convention, we denote by 0 and 1 the additive and multiplicative identities of $GF(q)$, respectively. When q is a power of a prime number p, we call p the *characteristic* of $GF(q)$. $GF(q)^n$ for some $n \geq 1$ is the set of n-dimensional vectors whose coordinates are in $GF(q)$ and is called a *coordinate space*. We say that any vector subspace of the coordinate space over $GF(q)$ is a *coordinate subspace*. Throughout the paper, we refer to a coordinate subspace over $GF(q)$ as a *vector space over* $GF(q)$ or simply as a coordinate subspace. For any vector space $S \subseteq GF(q)^n$ over $GF(q)$, there exists a matrix A whose entries are in $GF(q)$ such that $S = \{x \in GF(q)^n : Ax = \mathbf{0}\}$ where $\mathbf{0}$ denotes the vector of all zeros of appropriate dimension and all equalities in the system $Ax = \mathbf{0}$ are over $GF(q)$. Given S, we construct a multipartite uniform clutter in the following way. Taking n disjoint copies $V_1, \dots, V_n$ of $GF(q)$, we can view $GF(q)^n$ as $V_1 \times \cdots \times V_n$ so that S is a subset of $V_1 \times \cdots \times V_n$. The *multipartite uniform clutter of* S is the clutter over ground set $V_1 \cup \cdots \cup V_n$ defined by

$$\mathrm{mult}(S) := \{\{x_1, \dots, x_n\} : (x_1, \dots, x_n) \in S, \ x_i \in V_i \ \text{ for } i \in [n]\}.$$

Here, the size of a member equals the number of partitions n, and $\mathrm{mult}(S)$ is an n-partite n-uniform clutter. For example, $R_{1,1} := \{(0, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ is a vector space over $GF(2)$, and $R_{1,1}$ is isomorphic to $\{(1, 3, 5), (1, 4, 6), (2, 3, 6), (2, 4, 5)\} \subseteq \{1, 2\} \times \{3, 4\} \times \{5, 6\}$.[2] So, $\mathrm{mult}(R_{1,1})$ is isomorphic[3] to Q_6. There is a one-to-one correspondence between the members of $\mathrm{mult}(S)$ and the vectors in S. Although we focus on vector spaces over a finite field, we remark that the definition of multipartite uniform clutters extends to any subset of the direct product of finite groups. We discuss this further in Sect. 2.1.

Binary spaces and clutter minors Abdi, Cornuéjols, Guričanová, and Lee [4] considered vector spaces over $GF(2)$, often referred to as *binary* spaces, and provided a characterization of when their multipartite uniform clutters are ideal. For example, $\mathrm{mult}(R_{1,1}) = Q_6$ is ideal [34]. The characterization is in terms of *clutter minors*, or simply *minors*. Given a clutter $\mathcal{C}$ over ground set V and disjoint subsets I, J of V, we define $\mathcal{C} \setminus I/J$ as the clutter over $V - (I \cup J)$ that consists of the minimal sets of $\{C - J : C \in \mathcal{C}, \ C \cap I = \emptyset\}$. Here, we say that $\mathcal{C} \setminus I/J$ is *the minor of clutter* $\mathcal{C}$ obtained after *deleting* I and *contracting* J. We call it a *proper* minor if $I \cup J \neq \emptyset$. It is well-known that if a clutter is ideal, then so is every minor [34].

Theorem 1.1 ([4]). *Let S be a binary space. Then* $\mathrm{mult}(S)$ *is ideal if, and only if,* $\mathrm{mult}(S)$ *has none of* $\mathbb{L}_7$, $\mathbb{O}_5$, $b(\mathbb{O}_5)$ *as a minor.*

Here, $\mathbb{L}_7$, $\mathbb{O}_5$, $b(\mathbb{O}_5)$ are some non-ideal clutters over at most 10 elements, which we define and explain in detail in Appendix A. The proof of Theorem 1.1 is based on the connection between binary spaces and *binary matroids*, by which we can apply Seymour's Theorem [32] on the *sums of circuits property*, introduced in [33].

[2] This holds because there is a natural bijection between $\{0, 1\}^3$ and $\{1, 2\} \times \{3, 4\} \times \{5, 6\}$.

[3] Given clutters $\mathcal{C}, \mathcal{C}'$, we say that $\mathcal{C}$ is *isomorphic* to $\mathcal{C}'$ and write $\mathcal{C} \cong \mathcal{C}'$ if $\mathcal{C}'$ can be obtained from $\mathcal{C}$ after relabeling the elements of $\mathcal{C}$.

1.2 Summary of our results

Results I Motivated by Theorem 1.1 for binary spaces, we consider the following question. Given a vector space S over an arbitrary finite field $GF(q)$, when is $\text{mult}(S)$ is ideal? In this paper, we completely answer this question. We divide our analysis into three cases. First, we consider prime powers that are odd, secondly the $q = 4$ case, and thirdly powers of 2 greater than 4. What follows is a summary of our main results for the three cases.

For our first result, we need two more definitions. The *dimension* of vector space S is defined as the maximum number of linearly independent vectors in S over $GF(q)$. Moreover, denote by Δ_3 the clutter over ground set $\{1, 2, 3\}$ whose members are $\{1, 2\}, \{2, 3\}, \{3, 1\}$. Notice that Δ_3 is the clutter of edges in a triangle and that Δ_3 is non-ideal because $(1/2, 1/2, 1/2)$ is a fractional extreme point of the associated set covering polyhedron $Q(\Delta_3)$.

Theorem 1.2 (proved in Sect. 5). *Take an odd prime power q, and let S be a vector space over $GF(q)$. Then the following statements are equivalent:*

i. $\text{mult}(S)$ *is ideal,*
ii. *S has the form $S = S_1 \times \cdots \times S_k$ where each S_i has dimension at most* 1*,*
iii. $\text{mult}(S)$ *contains no Δ_3 as a minor.*

The case of $GF(4)$ allows more general structures in the vector space. We say that row vectors $v^1, \ldots, v^r$ with $r \geq 2$ form a *sunflower* if, after permuting the coordinates, the vectors are of the form

$$\begin{array}{c} v^1 \\ v^2 \\ \vdots \\ v^r \end{array} \left[\begin{array}{c|c|c|c|c} u^0 & u^1 & \mathbf{0} & \cdots & \mathbf{0} \\ u^0 & \mathbf{0} & u^2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u^0 & \mathbf{0} & \mathbf{0} & \cdots & u^r \end{array}\right]$$

where $u^0, u^1 \ldots, u^r$ are some row vectors with nonzero entries and $\mathbf{0}$ denotes a row vector of all zeros of appropriate length.

Theorem 1.3 (proved in Sect. 7.1). *Let S be a vector space over $GF(4)$. Then the following statements are equivalent:*

i. $\text{mult}(S)$ *is ideal,*
ii. *S has the form $S = S_1 \times \cdots \times S_k$ where each S_i has dimension at most* 1 *or admits a sunflower basis,*
iii. $\text{mult}(S)$ *contains no Δ_3 as a minor.*

Lastly, for the case when q is a power of 2 greater than 4, we define another small non-ideal clutter. C_5^2 is the clutter over ground set $\{1, \ldots, 5\}$ whose members are $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}$. C_5^2 is the clutter of edges in a cycle of length 5, and notice that C_5^2 is non-ideal because $(1/2, 1/2, 1/2, 1/2, 1/2)$ is a fractional extreme point of the associated polyhedron $Q(C_5^2)$.

Theorem 1.4 (proved in Sect. 7.1). *Let q be a power of 2 such that $q > 4$, and let S be a vector space over $GF(q)$. Then the following statements are equivalent:*

i. mult(S) *is ideal,*
ii. S *has the form* $S = S_1 \times \cdots \times S_k$ *where each* S_i *has dimension at most* 1*,*
iii. mult(S) *contains no* C_5^2 *as a minor.*

Theorems 1.2 to 1.4 lead to the conclusion that when q is a prime power other than 2, the class of coordinate subspaces whose multipartite uniform clutter is ideal has restricted structures. Nevertheless, the main takeaway of this paper is that we propose a novel framework to study and generate idealness by multipartite uniform clutters and complete the analysis of the natural class of multipartite uniform clutters obtained from coordinate subspaces. Our analysis is based on an interesting interplay between the clutter and its underlying matroid.

Results II We take one step further to understand the *max-flow min-cut (MFMC) property* [34] for the multipartite uniform clutters from coordinate subspaces. While the idealness of a clutter corresponds to the integrality of the associated set covering polyhedron, the MFMC property is the analogue of *total dual integrality* [16, 19]. To formalize this, given a clutter $\mathcal{C}$ over ground set V with weights $w \in \mathbb{Z}_+^V$, we consider $\tau(\mathcal{C}, w) := \min\{w^\top x : M(\mathcal{C})x \geq \mathbf{1}, x \in \mathbb{Z}_+^V\}$ and $\nu(\mathcal{C}, w) := \max\left\{\mathbf{1}^\top y : M(\mathcal{C})^\top y \leq w, y \in \mathbb{Z}_+^{\mathcal{C}}\right\}$. Note that $\tau(\mathcal{C}, w)$ computes the minimum weight of a cover of $\mathcal{C}$, whereas $\nu(\mathcal{C}, w)$ computes the maximum size of a *packing* of members of $\mathcal{C}$ such that each element v appears in at most w_v members in the packing. Here, we say that $\mathcal{C}$ has the MFMC property if $\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w)$ holds for every $w \in \mathbb{Z}_+^V$. Hence, the MFMC property of $\mathcal{C}$ is equivalent to the total dual integrality of the linear system $M(\mathcal{C})x \geq \mathbf{1}, x \geq \mathbf{0}$, and therefore it follows that the MFMC property implies idealness. The following result provides a complete characterization of the MFMC property for the multipartite uniform clutters from vector spaces.

Theorem 1.5 (proved in Sect. 5). *Take any prime power* q*, and let* S *be a vector space over* $GF(q)$*. Then the following statements are equivalent:*

i. mult(S) *has the max-flow min-cut property,*
ii. S *has the form* $S = S_1 \times \cdots \times S_k$ *where each* S_i *has dimension at most* 1*,*
iii. mult(S) *has none of* Δ_3, Q_6 *as a minor.*

Here, Δ_3 does not have the MFMC property as it is non-ideal. While Q_6 is ideal, it does not have the MFMC property because $\tau(Q_6, \mathbf{1}) = 2 > 1 = \nu(Q_6, \mathbf{1})$.

As a corollary, idealness and the MFMC property coincide when q is an odd prime power or q is a power of 2 greater than 4. In contrast, there is an example of a vector space over $GF(4)$ whose multipartite uniform clutter is ideal but does not have the MFMC property. We demonstrate this example in Sect. 8. Theorem 1.5 also has a consequence on the *Replication Conjecture*, proposed by Conforti and Cornuéjols [9]. In particular, the Replication Conjecture is a set covering analogue of the Duplication Lemma for perfect graphs [25].

Corollary 1.6 (proved in Sect. 8). *The Replication Conjecture holds true for the class of multipartite uniform clutters from coordinate subspaces.*

Another corollary of Theorem 1.5 is on the $\tau = 2$ *Conjecture*, proposed by Cornuéjols, Guenin, and Margot [11]. They showed that if the $\tau = 2$ Conjecture holds, then so does the Replication Conjecture [11], providing a way of tackling the Replication Conjecture.

Corollary 1.7 (proved in Sect. 8). *The $\tau = 2$ Conjecture holds true for the class of multipartite uniform clutters from coordinate subspaces.*

We will formally state the Replication Conjecture and the $\tau = 2$ Conjecture along with the proofs of Corollaries 1.6 and 1.7 in Sect. 8.

1.3 Organizations of the paper

This paper provides a complete characterization of when the multipartite uniform clutter of a coordinate subspace is ideal and when it has the MFMC property. Recall that the proof for the binary space case (Theorem 1.1) is based on understanding connections between binary spaces and binary matroids. It turns out that extending this result to the case of vector spaces over $GF(q)$ for a general prime power q also requires characterizing relevant matroids that are representable over $GF(q)$. The proofs of our main results are divided into two steps. First, we characterize the underlying matroid of a vector space after certain minors are forbidden from its multipartite uniform clutter. Second, based on the theory of ideal clutters, we argue that the corresponding multipartite uniform clutter is ideal or have the max-flow min-cut property. Although we presented and categorized our results according to different cases of prime powers in Sect. 1.2, we organize and structure the paper based on the proof steps.

The first proof step that analyzes the underlying matroid is covered in Sects. 3 and 4. In Sect. 3, we provide structural characterizations for the underlying matroid of a vector space whose associated multipartite uniform clutters do not have Δ_3 as a minor. In Sect. 4, we study how such matroid structures shape the geometry of the vector space, providing structural characterizations of the vector space.

The second step for proving idealness is given in Sects. 5 to 7. In Sect. 5, we prove Theorem 1.5 which characterizes idealness for an odd prime power q. In fact, Theorem 1.2 for the MFMC property of the multipartite uniform clutter of a vector space over $GF(q)$ for any prime power q shares much of the proof with Theorem 1.5. Hence, we prove the two theorems in Sect. 5. For the idealness under the case of powers of 2, we need more techniques. In Sect. 6, we develop some tools for understanding vector spaces generated by a sunflower basis that appear for the case of powers of 2. We divide our analysis of the case of powers of 2 into the $q = 4$ case and the case of $q = 2^k$ for $k \geq 3$. The $q = 4$ case, Theorem 1.3, is covered in Sect. 7.1. The other case, Theorem 1.4, is presented in Sect. 7.2.

We conclude the paper by proving Corollaries 1.6 and 1.7 on the Replication Conjecture and the $\tau = 2$ Conjecture, respectively, for the class of multipartite clutters from coordinate subspaces in Sect. 8. Section 2 provides some basics of multipartite uniform clutters and matroid theory. More advanced concepts in matroid theory and the theory of ideal clutters are defined and explained whenever necessary.

2 Preliminaries

2.1 Basics of multipartite uniform clutters

In the introduction, we explained how to construct multipartite uniform clutters from vector spaces. In this section, we generalize this framework and discuss some basic properties of multipartite uniform clutters.

Multipartite uniform clutters from set systems Let $V_1, \ldots, V_n$ be n nonempty sets, and take a subset S of $V_1 \times \cdots \times V_n$. We would take $V_i = GF(q)$ for $i \in [n]$ for a vector space over $GF(q)$, but we may take arbitrary finite sets that do not necessarily have the same size. Then the multipartite uniform clutter of S, denoted $\text{mult}(S)$, is defined as the clutter over ground set $V_1 \cup \cdots \cup V_n$ whose members are $\{x_1, \ldots, x_n\}$ for $(x_1, \ldots, x_n) \in S$. Here, S need not be a vector space. When each V_i has size two, $\text{mult}(S)$ for $S \subseteq V_1 \times \cdots \times V_n$ coincides with the *cuboid of* S, denoted $\text{cuboid}(S)$ [4, 5]. In that case, $V_1 \times \cdots \times V_n$ is given by $\{0, 1\}^n$, so cuboids correspond to vertex subsets of the n-dimensional 0,1 hypercube, and this is how the name cuboid is coined. In particular, for a binary space S, we have that $\text{mult}(S) = \text{cuboid}(S)$. Hence, multipartite uniform clutters generalize cuboids.

Remark 2.1 Let $\mathcal{C}$ be a clutter, and let $V_1, \ldots, V_n$ be n non-empty sets. Then the following statements are equivalent:

(i) $\mathcal{C}$ is isomorphic to $\text{mult}(S)$ for some $S \subseteq V_1 \times \cdots \times V_n$,
(ii) the ground set of $\mathcal{C}$ can be partitioned into $V_1, \ldots, V_n$ so that for every $C \in \mathcal{C}$, $|C \cap V_i| = 1$ for all $i \in [n]$.

Remark 2.1 provides a different yet equivalent definition of multipartite uniform clutters. Now that we have seen Remark 2.1, we know that the incidence matrix of a multipartite uniform clutter can be partitioned. To be more precise, notice that if a multipartite uniform clutter's ground set is partitioned into n non-empty parts $V_1, \ldots, V_n$, then the columns of the member-element incidence matrix $M(\mathcal{C})$ of $\mathcal{C}$ can be partitioned into n groups, corresponding to $V_1, \ldots, V_n$, so that a row has precisely one nonzero entry in each group. For instance,

$$M(Q_6) = \begin{array}{c} \\ (0,0,0) \\ (0,1,1) \\ (1,0,1) \\ (1,1,0) \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 0 & 1 & 0 & 1 \end{array} \\ \left[\begin{array}{cc|cc|cc} 1 & & 1 & & 1 & \\ 1 & & & 1 & & 1 \\ & 1 & 1 & & & 1 \\ & 1 & & 1 & 1 & \end{array}\right] \end{array}.$$

As mentioned in Sect. 1, one can also view a multipartite uniform clutter with parts $V_1, \ldots, V_n$ as the clutter of hyperedges of an n-partite n-uniform hypergraph whose vertex set is partitioned into $V_1 \cup \cdots \cup V_n$.

Isomorphism We may define an isomorphism between two vector spaces by taking a bijection. Moreover, an isomorphism between two vector spaces leads to an isomorphism between their multipartite uniform clutters.

Remark 2.2 Take an integer $n \geq 1$ and a prime power q, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Let $f_i : GF(q) \to GF(q)$ be a bijection for $i \in [n]$, and $g : GF(q)^n \to GF(q)^n$ be the bijection defined as

$$g(x) := (f_1(x_1), \dots, f_n(x_n)), \quad x \in GF(q)^n.$$

Then S is isomorphic to $g(S)$, and moreover, $\text{mult}(S)$ is isomorphic to $\text{mult}\,(g(S))$.

Projection and restriction of set systems Take an integer $n \geq 1$. Let $V_1, \dots, V_n$ be n nonempty sets, and let S be a subset of $V_1 \times \cdots \times V_n$. Given $J \subseteq [n]$ and $x \in S$, x/J denote the subvector of x that consists of the coordinates not in J. Given $J \subseteq [n]$, we refer to the operation of taking $\{x/J : x \in S\}$ from S as *dropping the coordinates in J from S*. Here, any set obtained after dropping some set J of coordinates from S is referred as a *projection* of S. Next, we say that the points of a set $S \subseteq V_1 \times \cdots \times V_n$ *agree on* a coordinate $i \in [n]$ if there exists $v \in V_i$ such that $x_i = v$ for every $x \in S$. Let U_i be a nonempty subset of V_i for $i \in [n]$. Here, U_i need not be a proper subset of V_i. Throughout the paper, we consider the operation of taking $S \cap (U_1 \times \cdots \times U_n)$ and dropping the coordinates where the points of $S \cap (U_1 \times \cdots \times U_n)$ agree on. We call the operation *restricting S to $U_1 \times \cdots \times U_n$*. We will refer to a set obtained from S after restricting S to $U_1 \times \cdots \times U_n$ for some $U_1, \dots, U_n$ such that $U_i \subseteq V_i$ for $i \in [d]$ as a *restriction* of S.

Lemma 2.3 *Take an integer $n \geq 1$. Let $V_1, \dots, V_n$ be n nonempty sets, and let $S \subseteq V_1 \times \cdots \times V_n$. If S' be a set that is either a projection or a restriction of S, then* $\text{mult}(S')$ *is a minor of* $\text{mult}(S)$.

Proof Suppose first that S' is a projection, say for some $J \subseteq [n]$, S' is obtained from S after dropping the coordinates of J. Then $\text{mult}(S')$ is the minor of $\text{mult}(S)$ obtained after contracting the elements in V_j for $j \in J$.

Suppose next that S' is a restriction. Then S' is obtained after restricting S to $U_1 \times \cdots \times U_n$ for some $U_1, \dots, U_n$ such that $U_i \subseteq V_i$ for $i \in [n]$. Then $\text{mult}(S')$ is the minor of $\text{mult}(S)$ obtained after deleting the elements in $(V_i \setminus U_i)$ for $i \in [n]$ and contracting the elements in V_j for $j \in J$ where J is the set of coordinates where the points in $S \cap (U_1 \times \cdots \times U_n)$ agree on. □

2.2 Matroid theory for vector spaces

As mentioned in the introduction, understanding connections between vector spaces over $GF(q)$ and matroids representable over $GF(q)$ is the key to derive our main results. In this section, we provide some basic matroid theory concepts and tools.

Matroid basics A *matroid* is defined over some *ground set* E and some family $\mathcal{I}$ of subsets of E, called *independent sets*, that satisfy the following properties:

(1) $\emptyset \in \mathcal{I}$.
(2) Every subset of an independent set is an independent set, i.e., $B \in \mathcal{I}$ if $A \in \mathcal{I}$ and $B \subseteq A$.

(3) If $A, B \in \mathcal{I}$ and $|B| < |A|$, then there exists some $a \in A \setminus B$ such that $B \cup \{a\} \in \mathcal{I}$.

For example, given a matrix over a field F, one can construct a matroid over the set of columns of the matrix by taking any collection of linearly independent columns as an independent set.

A *basis* of a matroid is a maximal independent set. As one would expect, all bases in a matroid have the same number of elements, and this number is referred to as the (matroid) *rank*. A *dependent set* of a matroid is a subset of its ground set that is not an independent set, and a *circuit* is a (inclusion-wise) minimal dependent set.

Graphic matroids (also called cycle matroids) are another common class of matroids. Let G be a graph whose edge set is E. The graphic matroid of G, denoted $\text{Matroid}(G)$, is defined over ground set E, and its independent sets are (the edge sets of) the forests in G. Note that a circuit of $\text{Matroid}(G)$ is a cycle in G.

Matroids from vector spaces Take a prime power q, and consider the Galois field $GF(q)$ of order q, with additive and multiplicative identities denoted as 0 and 1, respectively. Take an integer $n \geq 1$, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Let A be a matrix over n columns with entries in $GF(q)$ such that $S = \{x \in GF(q)^n : Ax = \mathbf{0}\}$, where the equality in the linear system $Ax = \mathbf{0}$ holds over $GF(q)$. The *underlying matroid of* S, denoted $\text{Matroid}(S)$, is the matroid represented by A over $GF(q)$. Recall that the dimension of vector space S is defined as the maximum number of linearly independent vectors in S over $GF(q)$. Note that

$$\text{the dimension of } S = n - \text{rank}(A) = n - \text{rank}\ (\text{Matroid}(S))$$

where $\text{rank}(A)$ is the matrix rank of A over $GF(q)$ and $\text{rank}\ (\text{Matroid}(S))$ is the matroid rank of $\text{Matroid}(S)$ over $GF(q)$. Although the representation matrix A is not unique for vector space S, our terminology suggests that $\text{Matroid}(S)$ is. The remark below justifies this.

Remark 2.4 Take a prime power q, and let S be a vector space over $GF(q)$. Then the clutter of circuits of $\text{Matroid}(S)$ is the set of inclusion-wise minimal members of $\{\text{support}(x) : x \in S, x \neq \mathbf{0}\}$ where $\text{support}(x) = \{i \in [n] : x_i \neq 0\}$ denotes the *support* of a vector $x \in GF(q)^n$.

Given vectors $v^1, \ldots, v^r \in GF(q)^n$, let $\langle v^1, \ldots, v^r \rangle := \left\{\sum_{i \in [r]} \lambda_i v^i : \lambda_i \in GF(q) \text{ for } i \in [r]\right\}$, where addition is done over $GF(q)$. The set $\langle v^1, \ldots, v^r \rangle$, which we call the *span* of the vectors, is a vector space over $GF(q)$. A *basis* of a vector space S is an inclusion-wise minimal set of vectors whose span is S. In this section, we characterize in terms of the underlying matroid when a vector space is spanned by a set of vectors of disjoint supports, or a set of vectors that form a sunflower.

Matroid minors Matroid *deletions* and *contractions* in $\text{Matroid}(S)$ correspond to restrictions and projections in S. Let $\mathcal{M}$ be a matroid over ground set E. The matroid obtained after *deleting* a subset I of E is defined as the matroid over ground set $E \setminus I$ whose independent sets are the independent sets of $\mathcal{M}$ contained in $E \setminus I$, and we use notation $\mathcal{M} \setminus I$. The matroid obtained after *contracting* a subset J of E is defined as the matroid over ground set $E \setminus J$ and denoted as $\mathcal{M}/J$, and a set $U \subseteq E \setminus J$ is an independent set of $\mathcal{M}/J$ if $U \cup J'$ is an independent set of $\mathcal{M}$ for some subset J' of J. Here, we call a matroid obtained from $\mathcal{M}$ after a series of deletions and contractions a *matroid minor* of $\mathcal{M}$. For a matroid $\mathcal{M}$ and disjoint subsets I, J of the ground set of $\mathcal{M}$, we denote by $\mathcal{M} \setminus I/J$ the matroid minor of $\mathcal{M}$ obtained after deleting I and contracting J. Let $\mathcal{C}(\mathcal{M})$ denote the clutter of circuits of $\mathcal{M}$.

Lemma 2.5 *Take an integer $n \geq 1$ and a prime power q, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Then* $\text{Matroid}(S) \setminus I/J$ *for some disjoint $I, J \subseteq [n]$ is precisely* $\text{Matroid}(S')$ *where $S' \subseteq GF(q)^{n-|I|-|J|}$ is the vector space over $GF(q)$ obtained from $S \cap \{x \in GF(q)^n : x_i = 0 \ \forall i \in I\}$ after dropping coordinates in $I \cup J$.*

Proof It is clear that S' is a vector space over $GF(q)$, so $\text{Matroid}(S')$ is well-defined. To show that $\text{Matroid}(S)\setminus I/J = \text{Matroid}(S')$, we will argue that $\mathcal{C}\left(\text{Matroid}(S)\setminus I/J\right) = \mathcal{C}\left(\text{Matroid}(S')\right)$.

If $\mathcal{C}\left(\text{Matroid}(S) \setminus I/J\right) = \emptyset$, then every $C \in \mathcal{C}\left(\text{Matroid}(S)\right)$ intersects I, which means that $\text{support}(x)$ intersects I for every $x \in S - \{\mathbf{0}\}$. This implies that $S' = \{\mathbf{0}\}$, in which case $\mathcal{C}\left(\text{Matroid}(S')\right) = \emptyset$. Thus we may assume that $\mathcal{C}\left(\text{Matroid}(S)\setminus I/J\right) \neq \emptyset$.

Let $C_1 \in \mathcal{C}\left(\text{Matroid}(S) \setminus I/J\right)$. Then there exists $C \in \mathcal{C}\left(\text{Matroid}(S)\right)$ such that $C \cap I = \emptyset$ and $C_1 = C - J$. Then $C = \text{support}(x)$ for some $x \in S$ by the definition of $\text{Matroid}(S)$ (see also Remark 2.4). As $C \cap I = \emptyset$, it follows that $x_i = 0$ for $i \in I$, which implies that there exists $x' \in S' - \{\mathbf{0}\}$ such that $\text{support}(x') = \text{support}(x) - J = C - J$. So, there exists $C_2 \in \mathcal{C}\left(\text{Matroid}(S')\right)$ such that $C_2 \subseteq C_1$. Therefore, every member of $\mathcal{C}\left(\text{Matroid}(S)\setminus I/J\right)$ contains a member of $\mathcal{C}\left(\text{Matroid}(S')\right)$.

Let $C_2 \in \mathcal{C}\left(\text{Matroid}(S')\right)$. Then $C_2 = \text{support}(x')$ for some $x' \in S'$ by Remark 2.4. This implies that there is some $x \in S$ such that $x_i = 0$ for $i \in I$ and $\text{support}(x) - J = \text{support}(x')$. Since $\text{support}(x)$ contains a circuit of $\text{Matroid}(S)$ and $\text{support}(x) \cap I = \emptyset$, it follows that $C_2 = \text{support}(x')$ contains a circuit of $\text{Matroid}(S) \setminus I/J$. Therefore, we deduce that $\mathcal{C}\left(\text{Matroid}(S)\setminus I/J\right) = \mathcal{C}\left(\text{Matroid}(S')\right)$, as required. □

Matroid direct sum and graph block decomposition Consider matroids $\mathcal{M}_1, \ldots, \mathcal{M}_\ell$ over pairwise disjoint ground sets $E_1, \ldots, E_\ell$ and independent set families $\mathcal{I}_1, \ldots, \mathcal{I}_\ell$, respectively. The *direct sum* of $\mathcal{M}_1, \ldots, \mathcal{M}_\ell$, denoted $\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_\ell$, is the matroid over ground set $E_1 \cup \cdots \cup E_\ell$ whose independent set family is $\{I_1 \cup \cdots \cup I_\ell : I_i \in \mathcal{I}_i, i \in [\ell]\}$. We shall need the following basic remark about the direct sum of matroids. For the remark, we need to recall two notions. First, a *block* of a graph G is any maximal vertex-induced subgraph of G that is 2-vertex-connected. A bridge is a block that consists of a single edge, which is trivially 2-vertex-connected. Finally, we say that a vector space S is the *product* of vector spaces S_1 and S_2 if $S = \{(x, y) : x \in S_1, \ y \in S_2\} =: S_1 \times S_2$.

Lemma 2.6 *The following statements hold:*

1. *For a graph G, let $G_1, \ldots, G_k$ be the blocks of G. Then* $\text{Matroid}(G) = \text{Matroid}(G_1) \oplus \cdots \oplus \text{Matroid}(G_k)$.

2. *Take a prime power q and $GF(q)$-representable matroids $\mathcal{M}_1, \mathcal{M}_2$ over disjoint ground sets. If A_1 and A_2 are $GF(q)$-representations of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, then $\mathcal{M}_1 \oplus \mathcal{M}_2$ can be represented by $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$.*
3. *Take a prime power q and a vector space S over $GF(q)$. Then $S = S_1 \times S_2$ for some vector spaces S_1, S_2 over $GF(q)$ if and only if* $\text{Matroid}(S) = \text{Matroid}(S_1) \oplus \text{Matroid}(S_2)$.

Proof **(1), (2)**: See Chapters 4.1 and 4.2 of [28]. **(3)** follows immediately from (2). □

3 Matroid structures after forbidding non-ideal minors

In this section, we provide structural characterizations for the underlying matroid of a vector space over $GF(q)$. We start by proving a key tool, given in Lemma 3.1, that helps us to analyze the structure of the underlying matroid after excluding Δ_3 from the multipartite uniform clutter. Using this tool, in Sect. 3.1, we study the case where q is a power of 2 greater than 2. In Sect. 3.2, we consider the case when q is an odd prime power.

Let q be a power of a prime number p. Recall that we denote by 0 and 1 the additive and multiplicative identities of $GF(q)$. Then there must exist an integer ℓ such that $a + a + \cdots + a$ (ℓ times) equals 0 for all $a \in GF(q)$, and in fact, the smallest of such integers is p. Here, p is often referred to as the *characteristic* of $GF(q)$. Throughout this paper, we denote by $-v$ and v^{-1} the additive and multiplicative inverses of v for each $v \in GF(q) - \{0\}$.

Lemma 3.1 *Take an integer $n \geq 3$ and n non-empty sets $V_1, \ldots, V_n$, and let $S \subseteq V_1 \times \cdots \times V_n$. If* $\text{mult}(S)$ *contains no Δ_3 as a minor, then for any distinct $a, b, c \in S$ and distinct $i, j, k \in [n]$ such that*

$$a_i = b_i \neq c_i, \quad b_j = c_j \neq a_j, \quad c_k = a_k \neq b_k, \qquad (\star)$$

there exists $d \in S - \{a, b, c\}$ that satisfies the following:

(1) $d_\ell \in \{a_\ell, b_\ell, c_\ell\}$ for all $\ell \in [n]$, and
(2) at least two of $d_i = c_i$, $d_j = a_j$, and $d_k = b_k$ hold.

Proof Let V denote the ground set of $\text{mult}(S)$. We may assume that there exist three distinct points $a, b, c \in S$ satisfying ($\star$) for some distinct $i, j, k \in [n]$. Take subsets I, J of $[n]$ as follows:

$$I = V - \{a_\ell, b_\ell, c_\ell : \ell \in [n]\} \quad \text{and} \quad J = \{a_\ell, b_\ell, c_\ell : \ell \in [n] - \{i, j, k\}\}.$$

We will show that if $d \in S - \{a, b, c\}$ satisfying (1) and (2) does not exsit, then $\text{mult}(S) \setminus I / J$ contains Δ_3 as a minor.

Notice that $\text{mult}(S) \setminus I$ is $\text{mult}(R_0)$ where $R_0 = \{v \in S : v_\ell \in \{a_\ell, b_\ell, c_\ell\}$ for $\ell \in [n]\}$ and that each member of $\text{mult}(R_0)$ is $\{v_1, \ldots, v_n\}$ for some $v \in S$. Furthermore, each $v \in R_0$ satisfies $\{v_1, \ldots, v_n\} - J = \{v_i, v_j, v_k\}$, so $\{v_1, \ldots, v_n\} - J$

remains minimal after contracting J from $\mathrm{mult}(R_0)$. This in turn implies that $\mathrm{mult}(R_0)/J$ is equal to $\mathrm{mult}(R)$ where $R := \{(v_i, v_j, v_k) : v \in S,\ v_\ell \in \{a_\ell, b_\ell, c_\ell\} \text{ for } \ell \in [n]\}$. So, $\mathrm{mult}(S)\backslash I/J = \mathrm{mult}(R)$. By definition, R contains points (a_i, a_j, a_k), (b_i, b_j, b_k), and (c_i, c_j, c_k) that are obtained from a, b, c. Suppose that there is no $d \in S - \{a, b, c\}$ that satisfies (1) and (2). Let $d \in S$ with $d_\ell \in \{a_\ell, b_\ell, c_\ell\}$ for $\ell \in [n]$. Since d satisfies (1), d does not satisfy (2). Then (d_i, d_j, d_k) can be (c_i, b_j, c_k), (a_i, a_j, c_k), (a_i, b_j, b_k), or (a_i, b_j, c_k). To argue that $\mathrm{mult}(R)$ contains Δ_3 as a minor, let us look at the incidence matrix of $\mathrm{mult}(R)$:

$$\begin{array}{c} \\ a \\ b \\ c \\ \\ \end{array} \begin{pmatrix} a_i & \overbrace{\mathbf{c_i}} & \overbrace{\mathbf{a_j}} & b_j & c_k & \overbrace{\mathbf{b_k}} \\ 1 & \mathbf{0} & \mathbf{1} & 0 & 1 & \mathbf{0} \\ 1 & \mathbf{0} & \mathbf{0} & 1 & 0 & \mathbf{1} \\ 0 & \mathbf{1} & \mathbf{0} & 1 & 1 & \mathbf{0} \\ & & & \vdots & & \end{pmatrix}.$$

Observe that a row of $M(\mathrm{mult}(R))$ other than the ones for a, b, c, if any, has at least two ones in the columns for a_i, b_j, c_k. So, after contracting the columns for c_i, a_j, b_k and removing non-minimal rows, the resulting incidence matrix is precisely $M(\Delta_3)$. This implies that we obtain Δ_3 after contracting c_i, a_j, b_k from $\mathrm{mult}(R)$, a contradiction to the assumption that $\mathrm{mult}(S)$ has no Δ_3 minor. □

3.1 Excluding δ_3 for the case of characteristic 2

In this section, we prove Theorem 3.6 which provides an important tool for characterizing the idealness of $\mathrm{mult}(S)$ where S is a vector space over $GF(2^k)$ for $k \geq 2$. To be more specific, Theorem 3.6 characterizes the structure of the underlying matroid $\mathrm{Matroid}(S)$ when $\mathrm{mult}(S)$ has no Δ_3 as a minor.

Lemma 3.2 *Let q be a power of 2, and let $S \subseteq GF(q)^4$ be a vector space over $GF(q)$. If* $\mathrm{Matroid}(S)$ *is isomorphic*[4] *to $U_{2,4}$, then* $\mathrm{mult}(S)$ *has Δ_3 as a minor.*

Proof Suppose for a contradiction that $\mathrm{mult}(S)$ has no Δ_3 as a minor. Since the rank of $U_{2,4}$ is 2, the dimension of S is $4 - -2 = 2$. Let $v^1, v^2 \in GF(q)^4$ be two generators of S. By elementary row operations, we may assume that $(v_1^1, v_2^1) = (1, 0)$ and $(v_1^2, v_2^2) = (0, 1)$. Then

$$\begin{array}{c} v^1 \\ v^2 \end{array} \left[\begin{array}{cc|cc} 1 & 0 & x & y \\ 0 & 1 & z & w \end{array}\right]$$

where $x, y, z, w \in GF(q)$. Each circuit of $U_{2,4}$ has size 3, so $x, y, z, w \neq 0$. Then $a := (-x^{-1}z)v^1$, $b := v^2$, $c := a + b$ are vectors in S. Let us consider

$$\begin{array}{c} a \\ b \\ c \end{array} \left[\begin{array}{c|c|c|c} -x^{-1}z & 0 & -z & -x^{-1}yz \\ 0 & 1 & z & w \\ -x^{-1}z & 1 & 0 & -x^{-1}yz + w \end{array}\right]$$

[4] Matroids $\mathcal{M}, \mathcal{M}'$ are *isomorphic* if $\mathcal{M}'$ can be obtained from $\mathcal{M}$ after relabeling the elements of $\mathcal{M}$.

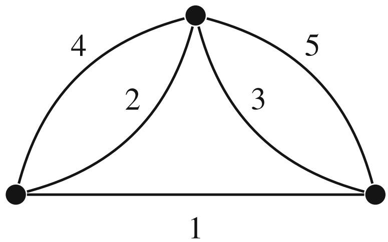

Fig. 1 K_4/e

and observe that $a_1 = c_1 \neq b_1$, $b_2 = c_2 \neq a_2$. We also have that $a_3 = b_3 \neq c_3$, because q being a power of 2 implies $z + z = 0$ and $z = -z$. By Lemma 3.1, there is a vector $d \in GF(q)^4$ that satisfies at least two of $d_1 = b_1 = 0$, $d_2 = a_2 = 0$, $d_3 = c_3 = 0$ and satisfies $d_4 \in \{-x^{-1}yz, w, -x^{-1}yz + w\}$. But then the support of d has size at most 2. Since every circuit of $U_{2,4}$ has size 3, $d = \mathbf{0}$, and therefore, $d_4 = -x^{-1}yz + w = 0$. This implies the support of c has size 2, a contradiction. □

Graph minors We say that a graph H is a *graph minor* of a graph G if H can be obtained from G after a series of edge deletions, edge contractions, and deletions of isolated vertices. If G is connected, then H is a graph minor of G if and only if for some disjoint subsets E_1, E_2 of $E(G)$, we can obtain H from G by deleting E_1 and contracting E_2. It is well-known that if H is a graph minor of G, then Matroid(H) is a matroid minor of Matroid(G) (see Chapter 3.2 in [28]).

K_4 is the complete graph on 4 vertices, and we denote by K_4/e what is obtained from K_4 after contracting an edge from it (see Fig. 1).

Lemma 3.3 *Let $q = 2^k$ for some $k \geq 2$, and let $S \subseteq GF(q)^5$ be a vector space over $GF(q)$. If* Matroid(S) *is isomorphic to* Matroid(K_4/e)*, then* mult(S) *has* Δ_3 *as a minor.*

Proof In Fig. 1, we can see that the fundamental cycles of K_4/e with respect to spanning tree $\{4, 5\}$ are $\{1, 4, 5\}, \{2, 4\}, \{3, 5\}$. Pick vectors $v^1, v^2, v^3 \in S$ whose supports are the three cycles. Notice that these vectors are linearly independent. Since the dimension of S is $5 - 2 = 3$, vectors v^1, v^2, v^3 generate S. After elementary row operations, S is generated by the 3 vectors v^1, v^2, v^3 of the following forms:

$$\begin{matrix} v^1 \\ v^2 \\ v^3 \end{matrix} \left[\begin{array}{ccc|cc} 1 & 0 & 0 & x & y \\ 0 & 1 & 0 & z & 0 \\ 0 & 0 & t & 0 & w \end{array}\right]$$

where $t, x, y, z, w \neq 0$. Since $q > 2$, we may assume that z and w are distinct nonzero elements in $GF(q)$. Now consider the restriction S' of S defined as follows:

$$S' := S \cap \left\{x \in GF(q)^5 : \ x_1 \in \{0, z, w\}, \ x_2 \in \{0, x\}, \ x_3 \in \{0, ty\}\right\}.$$

We will show that $\mathrm{mult}(S')$ has Δ_3 as a minor. Then as S' is a restriction of S, it follows from Lemma 2.3 that $\mathrm{mult}(S)$ also has Δ_3 as a minor. Notice that

$$S' = \left\{ \sum_{i=1}^{3} \lambda_i v^i \,:\, \lambda_1 \in \{0, z, w\},\ \lambda_2 \in \{0, x\},\ \lambda_3 \in \{0, y\} \right\}.$$

Consider three distinct points $a := zv^1$, $b := wv^1$, $c := xv^2 + yv^3$ in S':

$$\begin{matrix} a \\ b \\ c \end{matrix} \begin{bmatrix} z & 0 & 0 & zx & zy \\ w & 0 & 0 & wx & wy \\ 0 & x & ty & zx & wy \end{bmatrix}$$

As $z \neq w$, we have that $c_4 = a_4 \neq b_4$ and $b_5 = c_5 \neq a_5$. We also have $a_3 = b_3 \neq c_3$, because $ty \neq 0$. Suppose for a contradiction that $\mathrm{mult}(S')$ has no Δ_3 as a minor. By Lemma 3.1, there is $d \in S' - \{a, b, c\}$ that satisfies

(1) $d_1 \in \{0, z, w\}$, $d_2 \in \{0, x\}$, $d_3 \in \{0, ty\}$, $d_4 \in \{zx, wx\}$, $d_5 \in \{zy, wy\}$, and
(2) at least two of $d_3 = ty$, $d_4 = wx$, $d_5 = zy$ hold.

The points of $S' - \{a, b, c\}$ are the following:

$$S' - \{a, b, c\} = \left\{ \begin{matrix} (0,0,0,0,0), & (0,x,0,zx,0), & (0,0,ty,0,wy), \\ (z,x,0,0,zy), & (z,0,ty,zx,(z+w)y), & (w,x,0,(z+w)x,wy), \\ (w,0,ty,wx,0), & (z,x,ty,0,(z+w)y), & (w,x,ty,(z+w)x,0) \end{matrix} \right\}.$$

Since $z, w \neq 0$ and $z \neq w$, $(z+w)x \notin \{zx, wx\}$ and $(z+w)y \notin \{zy, wy\}$. Since $z, w, x, y \neq 0$, $0 \notin \{zx, wx\}$ and $0 \notin \{zy, wy\}$. This indicates that no point in $S' - \{a, b, c\}$ satisfies condition (1), a contradiction. Therefore, $\mathrm{mult}(S')$ has Δ_3 as a minor, and so does $\mathrm{mult}(S)$, as required. □

How does a graph with no K_4/e graph minor look like? We have the following result. Given an integer $t \geq 3$, denote by A_t the graph that consists of two vertices and t parallel edges connecting them. A *subdivision* of A_t is a graph obtained after adding vertices in between the edges of A_t.

Lemma 3.4 *Let $G = (V, E)$ be a connected graph. If G contains no K_4/e as a graph minor, then each block of G is a bridge, a cycle, or a subdivision of A_t for some $t \geq 3$.*

Proof See §B in the appendix. □

We call a graph a *series–parallel network* if each of its blocks is a series–parallel graph.

Theorem 3.5 ([8]). *Let $\mathcal{M}$ be a matroid. Then the following statements are equivalent:*

(i) $\mathcal{M}$ contains none of $U_{2,4}$ and Matroid(K_4) *as a matroid minor,*
(ii) $\mathcal{M}$ is the graphic matroid of a series–parallel network.

Theorem 3.6 *Let $q = 2^k$ for some $k \geq 2$, and let S be a vector space over $GF(q)$. If* $\mathrm{mult}(S)$ *has no Δ_3 as a minor, then for some $k \geq 1$,* $\mathrm{Matroid}(S) = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k$, *where each $\mathcal{M}_i$ is the graphic matroid of a bridge, a cycle, or a subdivision of A_t for some $t \geq 3$.*

Proof Assume that $\mathrm{mult}(S)$ has no Δ_3 as a minor. Suppose for a contradiction that $\mathrm{Matroid}(S)$ contains $U_{2,4}$ or $\mathrm{Matroid}(K_4/e)$ as a matroid minor. This in turn implies that there exists S' obtained from S after a series of restrictions and projections such that $\mathrm{Matroid}(S')$ is isomorphic to $U_{2,4}$ or $\mathrm{Matroid}(K_4/e)$ by Lemma 2.5. Here, $\mathrm{mult}(S')$ contains Δ_3 as a minor by Lemmas 3.2 and 3.3. As $\mathrm{mult}(S')$ is a minor of $\mathrm{mult}(S)$ due to Lemma 2.3, it follows that $\mathrm{mult}(S)$ also contains a Δ_3 as a minor, a contradiction. Hence, $\mathrm{Matroid}(S)$ contains none of $U_{2,4}$ and $\mathrm{Matroid}(K_4/e)$ as a matroid minor. As $\mathrm{Matroid}(K_4/e)$ is a matroid minor of $\mathrm{Matroid}(K_4)$, Theorem 3.5 implies that $\mathrm{Matroid}(S)$ is the graphic matroid of a series–parallel network not containing K_4/e as a graph minor. Then by Lemma 3.4, each block of the graph is a subdivision of A_t for some $t \geq 3$, a bridge, or a cycle. So, the assertion follows from Lemma 2.6, as required. □

3.2 Excluding Δ_3, Q_6 and odd prime powers

Theorem 3.6 characterized the case where q is a power of 2 greater than 2 and the multipartite uniform clutter contains no Δ_3 minor. In this section, we prove Theorem 3.9 which settles the case of odd prime powers. Theorem 3.9 also covers the case when q is a power of 2 and the multipartite uniform clutter contains none of Δ_3 and Q_6 as a minor, which will be the key to study the MFMC property later.

Lemma 3.7 *Take an integer $n \geq 1$ and a prime power q, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. If S does not admit a basis with vectors of pairwise disjoint supports, then* $\mathrm{mult}(S)$ *contains Δ_3 or Q_6 as a minor. Moreover, if q is an odd prime power, then* $\mathrm{mult}(S)$ *contains Δ_3 as a minor.*

Proof Assume that S does not admit a basis with vectors of pairwise disjoint supports. We will show that if $\mathrm{mult}(S)$ does not contain Δ_3 as a minor, then q is a power of 2 and $\mathrm{mult}(S)$ contains Q_6 as a minor.

Assume that $\mathrm{mult}(S)$ contains no Δ_3 as a minor. Let $v^1, \ldots, v^r \in GF(q)^n$ be a basis of S. After elementary arithmetic operations over $GF(q)$, we may assume that for each $i = 1, \ldots, r$,

$$v^i_i = 1 \quad \text{and} \quad v^i_j = 0 \quad \forall j \in [r] - \{i\}.$$

Since there is no basis of S with vectors of pairwise disjoint supports, we may assume that $v^1_{r+1}, v^2_{r+1} \neq 0$. This in turn implies that $n \geq 3$. Let x and y be the multiplicative inverses of v^1_{r+1} and v^2_{r+1} in $GF(q)$, respectively. Let $a := \mathbf{0} \in GF(q)^n$, $b := xv^1$, and $c := yv^2$. Notice that $a, b, c \in S$ and that a, b, c satisfy

$$(a_1, a_2, a_{r+1}) = (0, 0, 0), \quad (b_1, b_2, b_{r+1}) = (x, 0, 1), \quad (c_1, c_2, c_{r+1}) = (0, y, 1).$$

Now we consider $R = \{d \in S : d_j \in \{a_j, b_j, c_j\} \text{ for } j \in [n]\}$.

Claim 1 $R \subseteq \left\{\lambda_1 v^1 + \lambda_2 v^2 : \ \lambda_1 \in \{0, x\}, \ \lambda_2 \in \{0, y\}\right\}$.

Proof of Claim. Let $u \in R$. Then $u = \sum_{j=1}^{r} \lambda_j v^j$ for some $\lambda_1, \dots, \lambda_r \in GF(q)$. Since $a_j = b_j = c_j = 0$ for $j = 3, \dots, r$, it follows that $u_3 = \cdots = u_r = 0$, which implies that $\lambda_3 = \cdots = \lambda_r = 0$ and so $u = \lambda_1 v^1 + \lambda_2 v^2$. Notice that $\lambda_1 \in \{0, x\}$ and $\lambda_2 \in \{0, y\}$, because $a_1, b_1, c_1 \in \{0, x\}$ and $a_2, b_2, c_2 \in \{0, y\}$. □

Claim 2 *q is a power of 2 and* $R = \left\{\lambda_1 v^1 + \lambda_2 v^2 : \lambda_1 \in \{0, x\}, \lambda_2 \in \{0, y\}\right\}$.

Proof of Claim. By Lemma 3.1, R contains a vector $d \notin \{a, b, c\}$ such that (d_1, d_2, d_{r+1}) equals $(0, y, 0)$, $(x, 0, 0)$, $(x, y, 1)$, or $(x, y, 0)$. By Claim 1, $d \in \left\{\lambda_1 v^1 + \lambda_2 v^2 : \lambda_1 \in \{0, x\}, \lambda_2 \in \{0, y\}\right\}$. As $d \neq a, b, c$, it must be that $xv^1 + yv^2 = d$, so $xv^1 + yv^2 \in R$. In particular, $R = \left\{\lambda_1 v^1 + \lambda_2 v^2 : \lambda_1 \in \{0, x\}, \lambda_2 \in \{0, y\}\right\}$. Since $d = xv^1 + yv^2$, we obtain $(xv^1 + yv^2)_{r+1} = 1 + 1 = d_{r+1} \in \{0, 1\}$. Since $1 \neq 0$, we have $1 + 1 = 0$, so q is a power of 2, as required. □

By Claim 2, we deduce that R equals $\left\{\lambda_1 v^1 + \lambda_2 v^2 : \lambda_1 \in \{0, x\}, \lambda_2 \in \{0, y\}\right\}$ whose projection onto the space of coordinates $1, 2, r + 1$ is precisely $\{(0, 0, 0), (x, 0, 1), (0, y, 1), (x, y, 0)\}$, which is isomorphic to $R_{1,1} = \{(0, 0, 0), (1, 0, 1), (0, 1, 1), (1, 1, 0)\}$. Since $\text{mult}(R_{1,1}) = Q_6$, $\text{mult}(S)$ has Q_6 as a minor by Lemma 2.3. So, we have shown that if $\text{mult}(S)$ has no Δ_3 as a minor, then q is a power of 2 and $\text{mult}(S)$ contains Q_6 as a minor, as required. □

Lemma 3.7 tells us that forbidding Δ_3 for the case of odd prime powers and Δ_3, Q_6 for the case of powers of 2 implies that S is generated by vectors of pairwise disjoint supports. The next lemma characterizes the structure of the underlying matroid if S admits a basis with vectors of pairwise disjoint supports.

Lemma 3.8 *Take an integer $n \geq 1$ and a prime power q, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Then the following statements are equivalent:*

(i) *S has the form $S = \langle v^1, \dots, v^r \rangle$ where $v^1, \dots, v^r \in GF(q)^n$ have pairwise disjoint supports,*
(ii) $\text{Matroid}(S) = \text{Matroid}(G)$ *where every block of G is either a bridge or a cycle.*

Proof **(i)⇒(ii)**: Let $\mathcal{M}$ be a minor of $\text{Matroid}(S)$. Then it follows from Lemma 2.5 that $\mathcal{M}$ is isomorphic to $\text{Matroid}(S')$ where S' is obtained from $S \cap \{x \in GF(q)^n : x_i = 0 \ \forall i \in I\}$ after dropping coordinates in $I \cup J$ for some $I, J \subseteq [n]$ with $I \cap J = \emptyset$. Since S has a basis with vectors of pairwise disjoint supports, so does S', implying in turn that the circuits of $\text{Matroid}(S')$ are pairwise disjoint. Then the circuits of $\mathcal{M}$ are pairwise disjoint. Note that any of $U_{2,4}$, $\text{Matroid}(K_4/e)$, and $\text{Matroid}(A_3)$ have two circuits that intersect. Therefore, $\text{Matroid}(S)$ contains none of them as a minor. By Lemma 3.4 and Theorem 3.5, **(ii)** holds. **(ii)⇒(i)**: Note that the circuits of $\text{Matroid}(S)$ are pairwise disjoint, meaning that S is generated by vectors of pairwise disjoint supports. □

Theorem 3.9 *Let S be a vector space over $GF(q)$ for a prime power q. Suppose that one of the following holds.*

(a) *q is an odd prime power, and* $\text{mult}(S)$ *has no Δ_3 as a minor,*
(b) *q is a power of 2, and* $\text{mult}(S)$ *has none of Δ_3 and Q_6 as a minor.*

Then Matroid$(S) = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k$*, where each* $\mathcal{M}_i$ *is the graphic matroid of a bridge or a cycle.*

Proof Lemma 3.7 implies that if (a) or (b) holds, then S admits a basis with vectors of pairwise disjoint supports. Then it follows from Lemma 3.8 that Matroid(S) is Matroid(G) where every block of G is a bridge or a cycle. Then we deduce the assertion of this theorem from Lemma 2.6, as required. □

4 Vector space structure characterization

The characterizations of the underlying matroid provided in Theorems 3.6 and 3.9 are that the underlying matroid can be decomposed as the direct sum of the graphic matroids of some simple graphs. In Sect. 4.1, we consider how these matroid structures correspond to the geometry of vector spaces. In Sect. 4.2, we will see that these further lead to the decomposition of the associated multipartite uniform clutters.

4.1 Vector space decomposition

First, the following result considers the setting where the underlying matroid comes from a graph each of whose blocks is a bridge or a cycle.

Theorem 4.1 *Take an integer* $n \geq 1$ *and a prime power* q*, and let* $S \subseteq GF(q)^n$ *be a vector space over* $GF(q)$*. Then the following statements are equivalent:*

- *(i)* Matroid$(S) = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k$*, where each* $\mathcal{M}_i$ *is the graphic matroid of a bridge or a cycle,*
- *(ii)* S *has the form* $S = S_1 \times \cdots \times S_k$ *for some* k *where each* S_i *has dimension at most* 1.

Proof Note that Matroid$(\{0\})$ is the graphic matroid of a bridge and that $\{\mathbf{0}\} = \{0\} \times \cdots \times \{0\}$. Moreover, for a vector space T over $GF(q)$, T has dimension 1 if and only if Matroid(T) is the graphic matroid of a cycle. Then Lemma 2.6 implies that **(i)** holds if and only if **(ii)** holds, as required. □

In Theorem 3.6, we have another outcome, a subdivision of A_t for $t \geq 3$ when q is a power of 2 greater than 2. The following lemma provides a structural description of a vector space whose underlying matroid is the graphic matroid of a subdivision of A_t for some $t \geq 3$. We say that two elements e_1, e_2 of a matroid are *in series* if for every circuit C of the matroid, either $C \cap \{e_1, e_2\} = \{e_1, e_2\}$ or $C \cap \{e_1, e_2\} = \emptyset$ holds. In the context of graphic matroids, two edges e_1, e_2 are in series if for every cycle C, edge e_1 is on C if and only if e_2 is on C.

Lemma 4.2 *Take an integer* $n \geq 1$ *and a prime power* q*, and let* $T \subseteq GF(q)^n$ *be a vector space over* $GF(q)$*. Then* Matroid(T) *is the graphic matroid of a subdivision of* A_t *for some* $t \geq 3$ *if and only if* T *is generated by a sunflower basis.*

Proof (⇒): Assume that Matroid(T) = Matroid(G) where G is a subdivision of A_t for some $t \geq 3$. Notice that G consists of two vertices and t internally vertex-disjoint paths connecting them. Let $P_0, \dots, P_{t-1}$ denote the paths, and let $E(P_0), \dots, E(P_{t-1})$ denote their edge sets. Then it follows from Remark 2.4 that T contains a point whose support is $E(P_0) \cup E(P_i)$. Therefore, T contains $t-1$ points $v^1, \dots, v^{t-1}$ (in row vectors) of the following form:

$$\begin{array}{c} v^1 \\ v^2 \\ \vdots \\ v^{t-1} \end{array} \left[\begin{array}{c|c|c|c|c} u_1^0 & u^1 & \mathbf{0} & \cdots & \mathbf{0} \\ u_2^0 & \mathbf{0} & u^2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{t-1}^0 & \mathbf{0} & \mathbf{0} & \cdots & u^{t-1} \end{array} \right]$$

where $u_1^0, \dots, u_{t-1}^0 \in GF(q)^{|E(P_0)|}$ and $u^i \in GF(q)^{|E(P_i)|}$ for $i \in [n]$ are vectors of nonzero entries. As T is a vector space in $GF(q)^n$, Matroid(T) is over n elements, and therefore, G has n edges. Since G is a subdivision of A_t, a spanning tree of G has $n-(t-1)$ edges, which means that Matroid(T) = Matroid(G) has rank $n-(t-1)$. Then the dimension of T is n − Matroid$(T) = t-1$, so we have $T = \langle v^1, \dots, v^{t-1} \rangle$. Now, let us argue that we may assume that $u_1^0 = \cdots = u_{t-1}^0$ without loss of generality. As $P_1 \cup P_2$ is a cycle of G, Remark 2.4 implies that there is a point $v \in T$ whose support is $E(P_1) \cup E(P_2)$. Then v can be written as $v = \mu_1 v^1 + \mu_2 v^2$ for some $\mu_1, \mu_2 \in GF(q) - \{0\}$. As the support of v is $E(P_1) \cup E(P_2)$, we have that $\mu_1 u_1^0 + \mu_2 u_2^0 = 0$, which implies that $u_2^0 = \lambda_2 u_1^0$ for some nonzero λ_2. Similarly, we obtain $u_i^0 = \lambda_i u_1^0$ for some nonzero λ_i for $i \in [t-1]$, as required. Therefore, after scaling v^i's if necessary, we may assume that $u_1^0 = \cdots = u_{t-1}^0$, as required.

(⇐): Suppose $T = \langle v^1, \dots, v^{t-1} \rangle$ where $v^1, \dots, v^{t-1} \in GF(q)^n$ are vectors of the following form (in row vectors), after permuting the coordinates, for some $t \geq 3$:

$$\begin{array}{c} v^1 \\ v^2 \\ \vdots \\ v^{t-1} \end{array} \left[\begin{array}{c|c|c|c|c} u^0 & u^1 & \mathbf{0} & \cdots & \mathbf{0} \\ u^0 & \mathbf{0} & u^2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u^0 & \mathbf{0} & \mathbf{0} & \cdots & u^{t-1} \end{array} \right]$$

for some row vectors $u^0, u^1 \dots, u^{t-1}$ with no zero entries. Let E_i be the support of u^i for $i = 0, 1, \dots, t-1$. Let C be a circuit of Matroid(T). Then C = support(x) for some $x \in T$. Let $x = \sum_{i=1}^{t-1} \mu_i v^i$. Then x is of the form

$$x \left[\sum_{i=1}^{t-1} \mu_i u^0 \middle| \mu_1 u^1 \middle| \mu_2 u^2 \middle| \cdots \middle| \mu_{t-1} u^{t-1} \right].$$

If $C \cap E_0 \neq \emptyset$, then it means $\sum_{i=1}^{t-1} \mu_i \neq 0$, and therefore, $C \cap E_0 = E_0$. This implies that the elements in E_0 are in series. If $C \cap E_i \neq \emptyset$ for some $1 \leq i \leq t-1$, then $\mu_i \neq 0$. This indicates that $C \cap E_i = E_i$, implying in turn that the elements in E_i are in series.

Then consider the case where each u^i is 1-dimensional, under which we have $E_i = \{e_i\}$ is a singleton for $i = 0, \dots, t-1$. Observe that $|\text{support}(x)| \geq 2$ for any

$x \in T$. Then none of $\{e_0\}, \{e_1\}, \ldots, \{e_{t-1}\}$ is a circuit. However, we know that $\{e_0, e_i\}$ for $i = 1, \ldots, t-1$ are circuits of Matroid(T) because $v^1, \ldots, v^{t-1} \in T$, Moreover, $v^i + (q-1)v^j$ for $i \neq j$ has support $\{e_i, e_j\}$, and therefore, $\{e_i, e_j\}$ for distinct $i, j \in \{1, \ldots, t-1\}$ are all circuits. Then $\big\{\{e_i, e_j\} : i, j \in \{0, 1, \ldots, t-1\}, i \neq j\big\}$ is the family of circuits of Matroid(T) because any subset of the ground set of size at least 3 would contain $\{e_i, e_j\}$ for some $i \neq j$. Therefore, Matroid(T) is Matroid(A_t).

In general, as the elements of each E_i are in series, Matroid(T) is a series extension of Matroid(A_t), which is the graphic matroid of a subdivision of A_t, as required. □

Remark 4.2 implies the following characterization for the case of $q = 2^k$ for $k \geq 2$.

Theorem 4.3 *Take an integer $n \geq 1$ and a prime power q, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Then the following statements are equivalent:*

(i) Matroid$(S) = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k$, *where each $\mathcal{M}_i$ is the graphic matroid of a bridge, a cycle, or a subdivision of A_t for $t \geq 3$,*
(ii) S has the form $S = S_1 \times \cdots \times S_k$ where each S_i has dimension at most 1 *or admits a sunflower basis.*

Proof As argued in the proof of Theorem 4.1, a vector space T over $GF(q)$ has dimension at most 1 if and only if Matroid(T) is the graphic matroid of a bridge or a cycle. Moreover, by Remark 4.2, T admits a sunflower basis if and only if Matroid(T) is the graphic matroid of a subdivision of A_t for $t \geq 3$. Then the assertion follows from Lemma 2.6. □

4.2 Product decomposition of multipartite uniform clutters

In the previous subsection, we saw that the vector space can be decomposed as the product of smaller vector spaces. We will show that the associated multipartite uniform clutter can also be decomposed.

Products of set systems and clutters Take two integers $n_1, n_2 \geq 1$. Let $V_1, \ldots, V_{n_1}$ be n_1 nonempty sets, and let S_1 be a subset of $V_1 \times \cdots \times V_{n_1}$. Let $U_1, \ldots, U_{n_2}$ be n_2 nonempty sets, and let S_2 be a subset of $U_1 \times \cdots \times U_{n_2}$. Recall that the product of S_1 and S_2 is defined as $S_1 \times S_2 = \{(x_1, x_2) : \ x_1 \in S_1, \ x_2 \in S_2\}$. We also define products of clutters. Let $\mathcal{C}_1, \mathcal{C}_2$ be two clutters over disjoint ground sets E_1, E_2. The *product of $\mathcal{C}_1$ and $\mathcal{C}_2$*, denoted $\mathcal{C}_1 \times \mathcal{C}_2$, is defined as the clutter over ground set $E_1 \cup E_2$ whose members are $\mathcal{C}_1 \times \mathcal{C}_2 = \{C_1 \cup C_2 : \ C_1 \in \mathcal{C}_1, \ C_2 \in \mathcal{C}_2\}$. Having defined the product of two clutters, we define the product of two multipartite uniform clutters mult(S_1) and mult(S_2). In fact, we can show the following:

Lemma 4.4 *The following statements hold:*

1. mult$(S_1) \times$ mult$(S_2) =$ mult$(S_1 \times S_2)$.
2. *If* mult(S_1) *and* mult(S_2) *have the idealness (resp. MFMC) property, then so does* mult$(S_1 \times S_2)$.

Proof **(1)**: Let $C_1 \in$ mult(S_1) and $C_2 \in$ mult(S_2). Then $C_1 = \big\{x_1, \ldots, x_{n_1}\big\}$ for some $x = (x_1, \ldots, x_{n_1}) \in S_1$ and $C_2 = \big\{y_1, \ldots, y_{n_2}\big\}$ for some $y = (y_1, \ldots, y_{n_2}) \in$

S_2. Moreover, $(x, y) \in S_1 \times S_2$ and $C_1 \cup C_2 \in \mathrm{mult}(S_1 \times S_2)$. Conversely, any $C \in \mathrm{mult}(S_1 \times S_2)$ has the form $C = \{x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}\}$ for some $x = (x_1, \ldots, x_{n_1}) \in S_1$ and $y = (y_1, \ldots, y_{n_2}) \in S_2$. Then $C_1 = \{x_1, \ldots, x_{n_1}\} \in \mathrm{mult}(S_1)$ and $C_2 = \{y_1, \ldots, y_{n_2}\} \in \mathrm{mult}(S_2)$, which implies that $C = C_1 \cup C_2 \in \mathrm{mult}(S_1) \times \mathrm{mult}(S_2)$. Therefore, we obtain $\mathrm{mult}(S_1) \times \mathrm{mult}(S_2) = \mathrm{mult}(S_1 \times S_2)$. **(2)**: Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two clutters over disjoint ground sets. Then we deduce from [23, Proposition 8.3] that if $\mathcal{C}_1, \mathcal{C}_2$ have the idealness (resp. MFMC) property, then so does $\mathcal{C}_1 \times \mathcal{C}_2$. This implies that if $\mathrm{mult}(S_1)$ and $\mathrm{mult}(S_2)$ have the idealness (resp. MFMC) property, then so does $\mathrm{mult}(S_1) \times \mathrm{mult}(S_2)$ which equals $\mathrm{mult}(S_1 \times S_2)$ due to part (1), as required.

□

So, if a set can be represented as the product of some smaller sets, we can check if its multipartite uniform clutter is ideal by studying the smaller sets and their multipartite uniform clutters. In particular, we will use this lemma to show implication **(ii)**→**(i)** in Theorems 1.2 to 1.5.

5 The MFMC property and odd prime powers

In this section, we prove Theorem 1.5 that characterizes when the multipartite uniform clutter of a vector space has the MFMC property. Moreover, we prove Theorem 1.2 for the case when q is an odd prime power.

Lemma 5.1 *Take an integer $n \geq 1$ and a prime power q, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. If S has the form $S = S_1 \times \cdots \times S_k$ for some k where each S_i has dimension at most 1, then* $\mathrm{mult}(S)$ *has the MFMC property, and is therefore ideal.*

Proof We may assume that $S = \langle u^1 \rangle \times \cdots \times \langle u^r \rangle \times \{\mathbf{0}\}$ for some vectors $u^1, \ldots, u^r$ with no zero entries over $GF(q)$, by Theorem 4.1. Subsequently, $\mathrm{mult}(S) = \mathrm{mult}(\langle u^1 \rangle) \times \cdots \times \mathrm{mult}(\langle u^r \rangle) \times \mathrm{mult}(\{\mathbf{0}\})$, and to prove $\mathrm{mult}(S)$ has the MFMC property, it suffices to argue that $\mathrm{mult}(\langle u^i \rangle)$ for $i \in [r]$ and $\mathrm{mult}(\{\mathbf{0}\})$ have the MFMC property, by Lemma 4.4. First, notice that $\mathrm{mult}(\{\mathbf{0}\})$ has only one member, so it clearly has the MFMC property. In fact, we can argue that each $\mathrm{mult}(\langle u^i \rangle)$ has pairwise disjoint members as well. Notice that for any distinct $x, y \in GF(q)$, xu^i and yu^i do not have common coordinates, implying in turn that the members of $\mathrm{mult}\left(\langle u^i \rangle\right)$ corresponding to xu^i and yu^i are disjoint. That means that the members of $\mathrm{mult}\left(\langle u^i \rangle\right)$ are pairwise disjoint, implying in turn that it has the MFMC property, thereby proving that $\mathrm{mult}(S)$ has the MFMC property. □

Having proved Theorem 3.9, Theorem 4.1, and Lemma 5.1, we are now ready to show Theorem 1.5. The basic flow of our proof is as follows. Lemma 5.1 shows that if a vector space S is given by the product of some vector spaces of dimension at most 1, then $\mathrm{mult}(S)$ has the MFMC property. Conversely, it follows from Theorems 3.9 and 4.1 that if a vector space S cannot be written as such a product, then $\mathrm{mult}(S)$ has some minors certifying that the clutter does not have the MFMC property. More details are explained in the proof as follows.

Proof of Theorem 1.5 **(iii)⇒(ii)** follows from Theorems 3.9 and 4.1. **(ii)⇒(i)** follows from Lemma 5.1. **(i)⇒(iii)**: Assume that $\mathrm{mult}(S)$ has the MFMC property. Δ_3 is a non-ideal clutter, so it does not have the max-flow min-cut property. Recall that Q_6 is the clutter of triangles in K_4. Notice that the minimum number of edges required to intersect every triangle in K_4 is two and that the maximum number of disjoint triangles in K_4 is one. This implies that $\tau(Q_6, \mathbf{1}) = 2$ and $\nu(Q_6, \mathbf{1}) = 1$, so Q_6 does not have the max-flow min-cut property. Like idealness, the MFMC property is a minor-closed property [34]. Therefore, a clutter with the MFMC property contains none of Δ_3, Q_6 as a minor, implying in turn that $\mathrm{mult}(S)$ has none of Δ_3, Q_6 as a minor. □

The proof of Theorem 1.2 works similarly as that of Theorem 1.5. The additional component is that when q is an odd prime power and a vector space S over $GF(q)$ cannot be written as the product of some vector spaces of dimension at most 1, then $\mathrm{mult}(S)$ has a non-ideal minor due to Theorems 3.9 and 4.1.

Proof of Theorem 1.2 Take an integer $n \geq 1$ and an odd prime power q, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Since Δ_3 is non-ideal, direction **(i)⇒(iii)** is clear. Direction **(iii)⇒(ii)** follows from Theorems 3.9 and 4.1, and Lemma 5.1 shows direction **(ii)⇒(i)**. Therefore, **(i)–(iii)** are equivalent. □

6 Idealness and sunflower basis

In this section, we consider the case when $q = 2^k$ for $k \geq 2$. Excluding Δ_3 minor from $\mathrm{mult}(S)$, vector space S has the form $S = S_1 \times \cdots \times S_k$ where each S_i has dimension at most 1 or admits a sunflower basis by Theorem 4.3. If each S_i has dimension at most 1, then Lemma 5.1 implies that $\mathrm{mult}(S)$ is ideal. Hence, what remains is to study the case where some S_i is generated by a sunflower basis. In Sect. 6.1, we consider the notion of *localizations*, a tool for studying the idealness of $\mathrm{mult}(S)$. In Sect. 6.2, we use this tool to analyze the case where vector space S is generated by a sunflower basis.

6.1 Localization

We mentioned before that a clutter is ideal if and only if every minor of it is ideal. In this section, we will define and study *localizations* that appear as a minor of a multipartite uniform clutter.

Definition 6.1 Given a multipartite uniform clutter $\mathcal{C}$ whose ground set is partitioned into non-empty parts $V_1, \ldots, V_n$, a *localization of* $\mathcal{C}$ is any minor obtained from $\mathcal{C}$ after contracting precisely one element from each V_i.

Thus, a localization of $\mathcal{C}$ is obtained after contracting $v_1, \ldots, v_n$ for some $v = (v_1, \ldots, v_n) \in V_1 \times \cdots \times V_n$. As $\mathcal{C} = \mathrm{mult}(S)$ for some $S \subseteq V_1 \times \cdots \times V_n$ by Remark 2.1, the localization is equal to

$$\begin{aligned}\mathrm{local}(S, v) &:= \mathrm{mult}(S)/\{v_1, \ldots, v_n\}\\ &= \{\text{the minimal sets of } \{\{x_1, \ldots, x_n\} - \{v_1, \ldots, v_n\} : (x_1, \ldots, x_n) \in S\}\}.\end{aligned}$$

We call local(S, v) the *localization of* mult(S) *with respect to* v. So, every localization of $\mathcal{C}$ is equal to local(S, v) for some v and that local$(S, v) = \{\emptyset\}$ if $v \in S$. In [4], localizations of a cuboid are referred to as *induced clutters*.

It turns out that a multipartite uniform clutter is ideal if and only if all localizations are ideal; let us prove this in the remainder of this section. We say that a clutter is *minimally non-ideal* if it is non-ideal but every proper minor of it is ideal. We need the following lemma.

Lemma 6.2 *Let $\mathcal{C}$ be a minimally non-ideal clutter, and let V denote the ground set of $\mathcal{C}$. Then there is no subset U of V satisfying $|C \cap U| = 1$ for every member C of $\mathcal{C}$.*

Proof Since $\mathcal{C}$ is non-ideal, $P(\mathcal{C}) = \{\mathbf{1} \geq x \geq \mathbf{0} : M(\mathcal{C})x \geq \mathbf{1}\}$ has a fractional extreme point x^*. Let $v \in V$. Notice that $P(\mathcal{C}/v)$ and $P(\mathcal{C} \setminus v)$ are obtained from $P(\mathcal{C}) \cap \{x : x_v = 0\}$ and $P(\mathcal{C}) \cap \{x : x_v = 1\}$ after projecting out the variable x_v. As $\mathcal{C}/v$ and $\mathcal{C} \setminus v$ are ideal, $P(\mathcal{C}/v)$ and $P(\mathcal{C} \setminus v)$ are integral. Then both $P(\mathcal{C}) \cap \{x : x_v = 0\}$ and $P(\mathcal{C}) \cap \{x : x_v = 1\}$ are integral, implying in turn that x^* does not belong to any of these two. So, it follows that $0 < x_v^* < 1$ for each $v \in V$. Now, consider a nonsingular row submatrix A of $M(\mathcal{C})$ such that $Ax^* = \mathbf{1}$. Suppose that V has a subset U such that $|C \cap U| = 1$ for every member C of $\mathcal{C}$. Let χ_U denote the characteristic vector of U in $\{0, 1\}^V$. Since $|C \cap U| = 1$ for every member C of $\mathcal{C}$, we have that $M(\mathcal{C})\chi_U = \mathbf{1}$ and thus $A\chi_U = \mathbf{1}$. Since A is nonsingular, $Ax = \mathbf{1}$ has a unique solution, so it follows that $x^* = \chi_U$, a contradiction. Therefore, there is no such subset U of V, as required. □

Theorem 6.3 *A multipartite uniform clutter is ideal if and only if all of its localizations are ideal.*

Proof Let $\mathcal{C}$ be a multipartite uniform clutter whose ground set is partitioned into nonempty parts $V_1, \ldots, V_n$. ($\Rightarrow$): If $\mathcal{C}$ is ideal, every minor of $\mathcal{C}$ is ideal, and so are all of its localizations. ($\Leftarrow$): Assume that $\mathcal{C}$ is non-ideal. Then it has a minimally non-ideal minor $\mathcal{C}' := \mathcal{C} \setminus I/J$ obtained after deleting I and contracting J for some disjoint subsets $I, J \subseteq V_1 \cup \cdots \cup V_n$. Observe that $\mathcal{C} \setminus I$ is another multipartite uniform clutter whose ground set is partitioned into nonempty parts $U_1, \ldots, U_n$ where $U_i := V_i \setminus I$ for $i \in [n]$. In particular, every member C of $\mathcal{C} \setminus I$ satisfies $|C \cap U_i| = 1$ for $i \in [n]$. Suppose that $J \cap U_i = \emptyset$ for some $i \in [n]$. Then $|(C - J) \cap U_i| = |C \cap U_i| = 1$ for every member C of $\mathcal{C} \setminus I$. As $\mathcal{C}'$ is obtained after contracting J from $\mathcal{C} \setminus I$, we have $|C' \cap U_i| = 1$ for every member C' of $\mathcal{C}'$. This contradicts Lemma 6.2 due to our assumption that $\mathcal{C}'$ is minimally non-ideal. Therefore, for each $i \in [n]$, $J \cap U_i \neq \emptyset$, so we have that $J \cap V_i \neq \emptyset$. Let v_i denote some element in $J \cap V_i$ for $i \in [n]$. Since $\{v_1, \ldots, v_n\} \subseteq J$, $\mathcal{C}'$ is a minor of $\mathcal{C}/\{v_1, \ldots, v_n\}$, which is a localization. Therefore, one of $\mathcal{C}$'s localizations is non-ideal, as required. □

In contrast to idealness, even if all localizations have the MFMC property, a multipartite uniform clutter may not have the MFMC property. For example, all localizations of $Q_6 = \text{mult}(R_{1,1})$ are isomorphic to the clutter over ground set $\{1, 2, 3\}$ whose members are $\{1\}, \{2\}, \{3\}$. The clutter over 3 elements trivially has the MFMC property, but Q_6 does not [24, 34].

6.2 Fields of characteristic 2: a study of the localizations for A_t

Recall that a vector space S is generated by a sunflower basis if and only if Matroid(S) is the graphic matroid of a subdivision of A_t for some $t \geq 3$ by Remark 4.2. In this section, we consider the case when Matroid(S) = Matroid(A_t) for some $t \geq 3$, where A_t denotes the graph that consists of two vertices and t parallel edges connecting them. In particular, we prove three lemmas on properties of localizations of mult(S). Remark 6.4 identifies the structure of S for the case when Matroid(S) = Matroid(A_t) for $t \geq 3$. Lemma 6.5 characterizes the members of each localization of mult(S). Among those members, Lemma 6.6 specifies the members of size 1 or 2.

Lemma 6.4 *Take an integer $n \geq 3$ and a prime power q, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. Then* Matroid(S) = Matroid(A_n) *if and only if* $S \cong \{x \in GF(q)^n : x_1 + \cdots + x_n = 0\}$.

Proof Let $\{1, 2, 3 \ldots, n\}$ denote the edge set of A_n. Then $\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}$ are circuits of Matroid(A_n). ($\Leftarrow$): Let $\mathcal{S}$ be the clutter of the minimal supports of the points in $S - \{\mathbf{0}\}$. Then $\mathcal{S} = \{\{i, j\} : i \neq j\}$, so Matroid$(S)$ = Matroid(A_n) by Remark 2.4. ($\Rightarrow$): Since Matroid(S) = Matroid(A_n), S contains $n - 1$ points $u^1, \ldots, u^{n-1}$ whose supports are $\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}$, respectively. Notice that $u^1, \ldots, u^{n-1}$ are linearly independent over $GF(q)$, so the dimension of S is at least $n - 1$. On the other hand, the dimension is less than n, because $S \neq GF(q)^n$. Thus, $S = \langle u^1, \ldots, u^{n-1}\rangle$. After scaling the u^is, if necessary, we may assume that the first coordinate of each u^i is 1. Hence, $u^1, \ldots, u^{n-1}$ are of the form displayed below (left), where $\lambda_1, \ldots, \lambda_{n-1} \in GF(q) - \{0\}$. Notice that $\{x \in GF(q)^n : x_1 + \cdots + x_n = 0\} = \langle v^1, \ldots, v^{n-1}\rangle$ where $v^1, \ldots, v^{n-1}$ are displayed below (right):

$$\begin{matrix} u^1 \\ u^2 \\ \vdots \\ u^{n-1} \end{matrix} \begin{bmatrix} 1 & \lambda_1 & 0 & \cdots & 0 \\ 1 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \lambda_{n-1} \end{bmatrix} \qquad \begin{matrix} v^1 \\ v^2 \\ \vdots \\ v^{n-1} \end{matrix} \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix},$$

implying in turn that $\{x \in GF(q)^n : x_1 + \cdots + x_n = 0\} = \left\{(x_1, -\lambda_1^{-1}x_2, -\lambda_2^{-1}x_3, \ldots, -\lambda_{n-1}^{-1}x_n) : x \in S\right\}$. Therefore, $S \cong \{x \in GF(q)^n : x_1 + \cdots + x_n = 0\}$, as required. □

By Remark 6.4, we may focus on the set

$$S = \{x \in GF(q)^n : x_1 + \cdots + x_n = 0\}$$

to understand vector spaces whose underlying matroids are Matroid(A_n). Recall that a localization of mult(S) with respect to $\alpha \in GF(q)^n$, denoted local(S, α), is the minor of mult(S) after contracting the elements corresponding to α (see Sect. 2.1). mult(S) is defined over ground set $V_1 \cup \cdots \cup V_n$ where each V_i is a copy of $GF(q)$, and local(S, α)'s ground set is given by $U_1 \cup \cdots U_n$ where $U_i = V_i - \{\alpha_i\}$. The following lemma provides a characterization of the members of local(S, α) for any $\alpha \notin S$.

Lemma 6.5 *Take an integer $n \geq 3$. Let q be a power of 2, and let $\alpha \in GF(q)^n$ with $\sigma := \alpha_1 + \cdots + \alpha_n \neq 0$. Let $S = \{x \in GF(q)^n : x_1 + \cdots + x_n = 0\}$, and let $C \subseteq U_1 \cup \cdots \cup U_n$ where $U_i = GF(q) - \{\alpha_i\}$. Then the following statements are equivalent:*

(i) C is a member of $\mathrm{local}(S, \alpha)$.
(ii) C contains at most one element in U_i for each $i \in [n]$ and $\sum(v : v \in C) = \sigma + \sum(\alpha_i : C \cap U_i \neq \emptyset)$.

Proof **(i)⇒(ii)**: There exists $x = (x_1, \ldots, x_n) \in S$ such that $C = \{x_1, \ldots, x_n\} - \{\alpha_1, \ldots, \alpha_n\}$. Then $C \cap U_i = \{x_i\} - \{\alpha_i\}$, implying that $C \cap U_i$ has at most one element. Without loss of generality, we may assume that $x = (x_1, \ldots, x_k, \alpha_{k+1}, \ldots, \alpha_n)$ and $x_1 \neq \alpha_1, \ldots, x_k \neq \alpha_k$ for some $1 \leq k \leq n$. Then $C = \{x_1, \ldots, x_k\}$. Since $x \in S$, we have

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{k} x_i + \sum_{j=k+1}^{n} \alpha_j = 0.$$

As the characteristic of $GF(q)$ is 2, $\sum_{i=1}^{k} x_i = -\sum_{i=1}^{k} x_i$, implying in turn that $\sum_{i=1}^{k} x_i = \sum_{j=k+1}^{n} \alpha_j$. As $\sum_{i=1}^{n} \alpha_i = \sigma$, we also get $\sum_{j=k+1}^{n} \alpha_j = \sigma + \sum_{i=1}^{k} \alpha_i$, and therefore, we obtain $\sum_{i=1}^{k} x_i = \sigma + \sum_{i=1}^{k} \alpha_i$, as required.

(i)⇐(ii): Without loss of generality, we may assume that $C = \{x_1, \ldots, x_k\}$ where $x_i \in U_i$ for $i \in [k]$. Then $\{x_1, \ldots, x_k\} \cap \{\alpha_1, \ldots, \alpha_n\} = \emptyset$. Since $\sum_{i=1}^{k} x_i = \sigma + \sum_{i=1}^{k} \alpha_i$, we have $\sum_{i=1}^{k} x_i + \sum_{j=k+1}^{n} \alpha_j = \sigma + \sum_{i=1}^{n} \alpha_i = 0$, implying in turn that $(x_1, \ldots, x_k, \alpha_{k+1}, \ldots, \alpha_n) \in S$. As $C = \{x_1, \ldots, x_k, \alpha_{k+1}, \ldots, \alpha_n\} - \{\alpha_1, \ldots, \alpha_n\}$, it follows that C is a member of $\mathrm{local}(S, \alpha)$, as required. □

Using Lemma 6.5, we can show the following lemma providing a characterization of the members of size 1 and 2 in $\mathrm{local}(S, \alpha)$ for $\alpha \notin S$. Recall that $\mathrm{mult}(S)$ is given by $\{\{x_1, \ldots, x_n\} : (x_1, \ldots, x_n) \in S, x_i \in GF(q) \text{ for } i \in [n]\}$ whose ground set is $GF(q) \times \cdots \times GF(q)$. Here, any y with $y_i \equiv x_i \pmod q$ for $i \in [n]$ is equivalent to x. Similarly, if x_i is an element in $V_i = GF(q)$, then any y_i with $y_i \equiv x_i \pmod q$ refers to the same element x_i.

Lemma 6.6 *Take an integer $n \geq 3$. Let q be a power of 2, and let $\alpha \in GF(q)^n$ with $\sigma := \alpha_1 + \cdots + \alpha_n \neq 0$. Let $S = \{x \in GF(q)^n : x_1 + \cdots + x_n = 0\}$. Then the following statements hold:*

(1) the members of size 1 of $\mathrm{local}(S, \alpha)$ *are* $\{\alpha_1 + \sigma\}, \ldots, \{\alpha_n + \sigma\}$.
(2) the members of size 2 of $\mathrm{local}(S, \alpha)$ *form a graph that consists of $\frac{q}{2} - 1$ connected components $G_1, \ldots, G_{\frac{q}{2}-1}$ satisfying the following: for each $j = 1, \ldots, \frac{q}{2} - 1$,*

- *G_j's vertex set is $\left\{\beta_1^j, \beta_1^j + \sigma\right\} \cup \cdots \cup \left\{\beta_n^j, \beta_n^j + \sigma\right\}$ where $\left\{\beta_i^j, \beta_i^j + \sigma\right\} \subseteq U_i - \{\alpha_i + \sigma\} = GF(q) - \{\alpha_i, \alpha_i + \sigma\}$ for $i \in [n]$,*
- *G_j is a bipartite graph with bipartition $\left\{\beta_1^j, \ldots, \beta_n^j\right\} \cup \left\{\beta_1^j + \sigma, \ldots, \beta_n^j + \sigma\right\}$,*
- *$\beta_i^j = \beta_1^j + \alpha_1 + \alpha_i$ for $i \in [n]$, and*

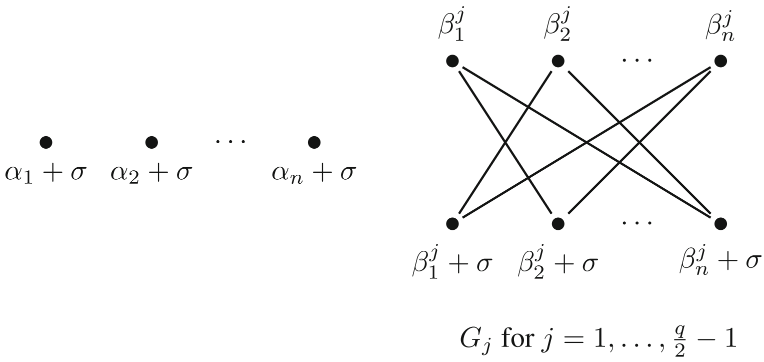

Fig. 2 Members of size 1 and 2 of local(S, α)

- *G_j's edge set is $\left\{\left\{\beta_i^j, \beta_k^j + \sigma\right\} : i \neq k\right\}$, i.e., G_j is obtained from a complete bipartite graph after removing the edges of a perfect matching (see Fig. 2 for an illustration).*

Proof See §C of the appendix. □

7 Characterizing idealness for powers of 2

In this section, based on our development from the previous sections, we consider a vector space S over $GF(2^k)$ for some $k \geq 2$ and show a characterization of when mult(S) is ideal. In Sect. 7.1, we prove Theorem 1.3 characterizing when the multipartite uniform clutter of a vector space over $GF(4)$ is ideal. In Sect. 7.2, we prove Theorem 1.4 which characterizes when the multipartite uniform clutter of a vector space S over $GF(2^k)$ with $k > 2$ is ideal.

7.1 The $q = 4$ case

The proof of Theorem 1.3 uses the following two lemmas. We first show Lemma 7.1 which implies that mult(T) is ideal if T is a vector space over $GF(4)$ such that Matroid$(T) \cong$ Matroid(A_n) for some $n \geq 3$. We then prove in Remark 7.2 that idealness is closed under series extensions.

Lemma 7.1 *Let $T = \{x \in GF(4)^n : x_1 + \cdots + x_n = 0\}$ for some $n \geq 3$. Then* mult(T) *is ideal.*

Proof By Theorem 6.3, it suffices to argue that all localizations of mult(T) are ideal. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \notin T$. We will show that the localization of mult(T) with respect to α, denoted local(T, α), is ideal. Let $\sigma = \alpha_1 + \cdots + \alpha_n \neq 0$. Note that local$(T, \alpha)$ has n members of cardinality 1, $\{\alpha_1 + \sigma\}, \ldots, \{\alpha_n + \sigma\}$ by Lemma 6.6 (1). By Lemma 6.6 (2), the members of cardinality 2 form a connected bipartite graph G where

- G is bipartite on $\{\beta_1, \ldots, \beta_n\} \cup \{\beta_1 + \sigma, \ldots, \beta_n + \sigma\}$ where $\{\beta_i, \beta_i + \sigma\} = GF(4) - \{\alpha_i, \alpha_i + \sigma\}$ for $i \in [n]$,
- $\beta_i = \beta_1 + \alpha_1 + \alpha_i$ for $i \in [n]$, and
- the edge set of G is $\{\{\beta_i, \beta_k + \sigma\} : i \neq k\}$.

We will show that there is no member of cardinality at least 3 in $\text{local}(T, \alpha)$. Suppose for a contradiction that $\text{local}(T, \alpha)$ has a member C whose cardinality is at least 3. As C does not contain any of the members of $\text{local}(T, \alpha)$ that have cardinality 1 or 2, $C \subseteq \{\beta_1, \ldots, \beta_n\}$ or $C \subseteq \{\beta_1 + \sigma, \ldots, \beta_n + \sigma\}$. Without loss of generality, we may assume that $C = \{\beta_1, \ldots, \beta_k\}$ for some $k \geq 3$. Then, by Lemma 6.5, we have $\sum_{i=1}^{k} \beta_i = \sigma + \sum_{i=1}^{k} \alpha_i$. Substituting $\beta_i = \beta_1 + \alpha_1 + \alpha_i$ for $i = 2, \ldots, k$, we obtain $\sum_{i=1}^{k} (\beta_1 + \alpha_1) = \sigma$. Since σ is nonzero and $\sum_{i=1}^{k} (\beta_1 + \alpha_1)$ is either $\beta_1 + \alpha_1$ or 0, we get $\sum_{i=1}^{k} (\beta_1 + \alpha_1) = \beta_1 + \alpha_1 = \sigma$. However, $\beta_1 + \alpha_1 = \sigma$ in turn implies that $\beta_i = \beta_1 + \alpha_1 + \alpha_i = \alpha_i + \sigma$, contradicting the assumption that $\beta_i \in GF(4) - \{\alpha_i, \alpha_i + \sigma\}$. Therefore, $\text{local}(T, \alpha)$ does not have a member of cardinality at least 3, as required.

Thus the members of $\text{local}(T, \alpha)$ have size either 1 or 2. Let $\mathcal{C}$ be what is obtained from $\text{local}(T, \alpha)$ after deleting every element that appears in a member of cardinality 1. As no minimally non-ideal clutter has a member of cardinality 1, $\text{local}(T, \alpha)$ is ideal if and only if $\mathcal{C}$ is ideal. Notice that $M(\mathcal{C})$, the incidence matrix of $\mathcal{C}$, is the edge - vertex incidence matrix of a bipartite graph. It follows from Kőnig's theorem for bipartite matching that $\mathcal{C}$ is ideal. Therefore, $\text{local}(T, \alpha)$ is ideal, and $\text{mult}(T)$ is ideal, as required. □

Lemma 7.2 *Suppose that S is a vector space over $GF(q)$ such that* $\text{Matroid}(S)$ *has elements in series. Let S' be a projection of S obtained after dropping one of the elements in series. Then* $\text{mult}(S)$ *is ideal if and only if* $\text{mult}(S')$ *is ideal.*

Proof Without loss of generality, assume that $\text{Matroid}(S)$ has n elements and that elements $n-1, n$ are in series. Let S' be defined as the projection of S obtained after dropping the n^{th} coordinate of the points in S. Then S' is a vector space in $GF(q)^{n-1}$, and by Lemma 2.5, $\text{Matroid}(T) = \text{Matroid}(S)/\{n\}$.

Let $x \in S$. Then $\text{support}(x)$ is the union of some circuits of $\text{Matroid}(S)$ by Remark 2.4. As $n-1, n$ are series elements, a circuit of $\text{Matroid}(S)$ contains $n-1$ if and only if it contains n, implying in turn that $n-1 \in \text{support}(x)$ if and only if $n \in \text{support}(x)$. Let $v^1, \ldots, v^r$ give rise to a basis of S. If $n \in \text{support}(x)$ for some $x \in S$, then $n \in \text{support}(v^\ell)$ for some $\ell \in [r]$, and thus, we may assume that $n \in \text{support}(v^1)$ and that $v_n^1 \neq 0$. After scaling the v^ℓ's, if necessary, we may assume that $v_n^\ell = 0$ for $\ell \in [r] - \{1\}$. Since $n-1 \in \text{support}(x)$ if and only if $n \in \text{support}(x)$ for $x \in S$, we have that $v_{n-1}^1 \neq 0$ and $v_{n-1}^\ell = 0$ for $\ell \in [r] - \{1\}$. Then for some $y, z \in GF(q) - \{0\}$,

$$\begin{matrix} v^1 \\ v^2 \\ \vdots \\ v^r \end{matrix} \begin{bmatrix} \cdots & y & z \\ \cdots & 0 & 0 \\ \vdots & 0 & 0 \\ \cdots & 0 & 0 \end{bmatrix}.$$

Then it follows that $S = \left\{(x_1, \ldots, x_{n-1}, zy^{-1}x_{n-1}) : (x_1, \ldots, x_{n-1}) \in S'\right\}$, and by Remark 2.2, $\text{mult}(S) \cong \text{mult}(T)$ where $T = \{(x_1, \ldots, x_{n-1}, x_{n-1}) :$

$(x_1, \dots, x_{n-1}) \in S'\big\}$. Let $V_1 \cup \dots \cup V_n$ be the ground set of mult(S) where each V_i is a copy of $GF(q)$. Then

$$\text{mult}(T) = \left\{C :\ C' \in \text{mult}(S'),\ C \cap (V_1 \cup \dots \cup V_{n-1}) = C',\ C \cap V_n = C' \cap V_{n-1}\right\}.$$

In words, mult(T) is obtained from mult(S') after duplicating the element in V_{n-1} of each member $C' \in$ mult(S'). Then the V_{n-1} part and the V_n part of the members of mult(T) are identical. Hence, mult(T) is ideal if and only if mult(S'). As mult(S) is isomorphic to mult(T), it follows that mult(S) is ideal if and only if mult(S') is ideal.

□

Now we are ready to prove Theorem 1.3. The proof first reduces to the case when the vector space T admits a sunflower basis. Then the idea is to show that Matroid(T) is a series extension of Matroid(T') where Matroid$(T') \cong$ Matroid(A_t) for some $t \geq 3$. We then use Lemmas 7.1 and 7.2 to show that mult(T) is ideal.

Proof of Theorem 1.3 Take an integer $n \geq 1$, and let $S \subseteq GF(4)^n$ be a vector space over $GF(4)$. First of all, **(i)⇒(iii)** is straightforward as Δ_3 is non-ideal. In what follows, we will show directions **(iii)⇒(ii)** and **(ii)⇒(i)**.

(iii)⇒(ii): By Theorem 3.6, Matroid$(S) = M_1 \oplus \dots \oplus M_k$ for some $k \geq 1$ where for each $i \in [k]$, M_i is the graphic matroid of a bridge, a cycle, or a subdivision A_t for some $t \geq 3$. Then it follows from Theorem 4.3 that S satisfies **(ii)**.

(ii)⇒(i): It suffices to show that mult(S_i) is ideal for every $i \in [k]$ due to Lemma 4.4. To this end, take an $i \in [k]$. If S_i has dimension at most 1, then $S_i = \{\mathbf{0}\}$ or $S_i = \langle v \rangle$ for some nonzero vector v, in which case it follows from Lemma 5.1 that S_i is ideal. Thus we may assume that $S_i = \langle v^1, \dots, v^r \rangle$ where $r \geq 2$ and $v^1, \dots, v^r$ give rise to a sunflower basis of S_i. Let $T' = \langle w^1, \dots, w^r \rangle$ where

$$\begin{matrix} w^1 \\ w^2 \\ \vdots \\ w^r \end{matrix} \left[\begin{array}{c|c|c|c|c} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{array}\right].$$

Then $T' = \left\{x \in GF(4)^{r+1} : x_1 + \dots + x_{r+1} = 0\right\}$, so by Lemma 7.1, mult$(T')$ is ideal. Suppose that v^i is of the form (u^0, u^i) for $i \in [r]$, and let d_ℓ denote the number of entries in u^ℓ for $\ell = 0, 1, \dots, r$. Then we define T as

$$T := \left\{ (\underbrace{x_1, \dots, x_1}_{d_0}, \underbrace{x_2, \dots, x_2}_{d_1}, \dots, \underbrace{x_{r+1}, \dots, x_{r+1}}_{d_r}) :\ (x_1, x_2, \dots, x_{r+1}) \in T' \right\}.$$

Then T is generated by $y^1, \ldots, y^r$ where

$$\begin{array}{c} y^1 \\ y^2 \\ \vdots \\ y^r \end{array} \left[\begin{array}{c|c|c|c|c} \overbrace{\mathbf{1}}^{d_0} & \overbrace{\mathbf{1}}^{d_1} & \overbrace{\mathbf{0}}^{d_2} & \cdots & \overbrace{\mathbf{0}}^{d_r} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} \end{array} \right].$$

Note that T' is a projection of T obtained after dropping the coordinates that correspond to some series elements of Matroid(T). As mult(T') is ideal, it follows from Remark 7.2 that mult(T) is ideal. Moreover, S_i can be obtained from T by taking coordinate-wise bijections. Hence, Remark 2.2 implies that mult$(S_i) \cong$ mult(T), thereby showing that mult(S_i) is ideal, as required. □

7.2 Powers of 2 greater than 4

We start by proving Lemmas 7.3 and 7.4 which imply that if mult(S) is ideal, then the underlying matroid Matroid(S) does not contain two distinct circuits that intersect. The proofs of the lemmas rely on the tools from Sect. 6.2. For the first lemma, recall that C_5^2 is the clutter of edges in a cycle of length 5, and that C_5^2 is non-ideal.

Lemma 7.3 *Let q be a power of 2 greater than 4, and let $S \subseteq GF(q)^3$ be a vector space over $GF(q)$ such that* Matroid(S) *is isomorphic to* Matroid(A_3). *Then* mult(S) *has C_5^2 as a minor.*

Proof By Remark 6.4, we may assume that $S = \{x \in GF(q)^3 : x_1 + x_2 + x_3 = 0\}$. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \notin S$. We will show that local(S, α) has C_5^2 as a minor. Let $\sigma = \alpha_1 + \alpha_2 + \alpha_3$, and we choose $a, b \in GF(q)$ such that $a \in GF(q) - \{\alpha_1, \alpha_1 + \sigma\}$ and $b \in GF(q) - \{\alpha_1, \alpha_1 + \sigma, a, a + \sigma\}$.

Claim 3 $a + b + \alpha_1 \in GF(q) - \{\alpha_1, \alpha_1 + \sigma, a, a + \sigma, b, b + \sigma\}$.

Proof of Claim. If $a + b + \alpha_1 = \alpha_1$ or $\alpha_1 + \sigma$, then $b = a$ or $b = a + \sigma$, contradicting the choice of b. If $a + b + \alpha_1 = a$ or $a + \sigma$, then $b = \alpha_1$ or $b = \alpha_1 + \sigma$, contradicting the choice of b. If $a + b + \alpha_1 = b$ or $b + \sigma$, then $a = \alpha_1$ or $a = \alpha_1 + \sigma$, a contradiction as $a \notin \{\alpha_1, \alpha_1 + \sigma\}$. Therefore, $a + b + \alpha_1 \notin \{\alpha_1, \alpha_1 + \sigma, a, a + \sigma, b, b + \sigma\}$, as required. □

By Lemma 6.6 (2), the members of cardinality 2 in local(S, α) form a graph with $\frac{q}{2} - 1$ connected components $G_1, \ldots, G_{\frac{q}{2}-1}$ where the vertex set of G_j is

$$\left\{\beta_1^j, \beta_1^j + \sigma\right\} \cup \left\{\beta_2^j, \beta_2^j + \sigma\right\} \cup \left\{\beta_3^j, \beta_3^j + \sigma\right\}$$

where $\beta_i^j, \beta_i^j + \sigma \in U_i - \{\alpha_i + \sigma\}$ and $U_i = GF(q) - \{\alpha_i\}$ for $i \in [3]$. Furthermore, $G_1, \ldots, G_{\frac{q}{2}-1}$ are 6-cycles by Lemma 6.6 (2) (see Fig. 3 for an illustration). As

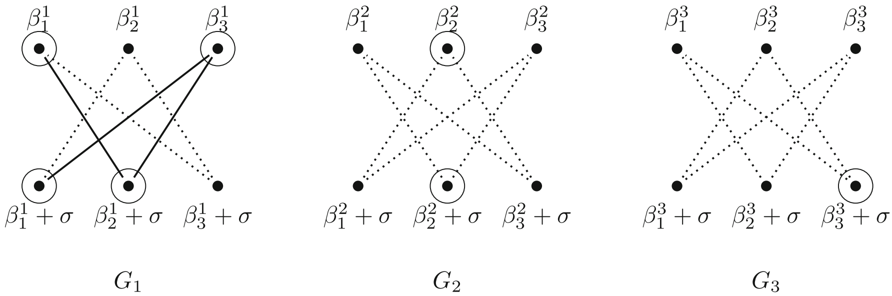

Fig. 3 The subgraph of $H_{n,\alpha}$ after deleting the vertices

$\frac{q}{2}-1 \geq 3$, without loss of generality, we may assume that $\beta_1^1 = a$, $\beta_1^2 = b$, and $\beta_1^3 = a+b+\alpha_1$, i.e., G_1, G_2, G_3 contain $a, b, a+b+\alpha_1 \in U_1 - \{\alpha_1+\sigma\}$, respectively.

Claim 4 *The following statements hold:*

(1) $\beta_1^1+\sigma = a+\sigma$, $\beta_2^1+\sigma = a+\alpha_1+\alpha_2+\sigma$, *and* $\beta_3^1 = a+\alpha_1+\alpha_3$,
(2) $\beta_2^2 = b+\alpha_1+\alpha_2$ *and* $\beta_2^2+\sigma = b+\alpha_1+\alpha_2+\sigma$, *and*
(3) $\beta_3^3+\sigma = a+b+\alpha_3+\sigma$.

Proof of Claim. The claim follows from Lemma 6.6 (2). □

Now keep elements $\beta_1^1, \beta_1^1+\sigma, \beta_2^1+\sigma, \beta_3^1$ in G_1, $\beta_2^2, \beta_2^2+\sigma$ in G_2, and $\beta_3^3+\sigma$ in G_3 and delete the other elements from local(S,α). (see Fig. 3 for an illustration; we keep only the circled elements). Let $\mathcal{C}$ denote the resulting minor of local(S,α).

As $\alpha_i+\sigma$ for $i \in [n]$ are deleted, we know from Lemma 6.6 (1) that $\mathcal{C}$ contains no member of size 1. By Lemma 6.6 (2), $\mathcal{C}$ has 3 members of size 2: $\left\{\beta_1^1, \beta_2^1+\sigma\right\}$, $\left\{\beta_3^1, \beta_1^1+\sigma\right\}$, $\left\{\beta_3^1, \beta_2^1+\sigma\right\}$, and these are the only ones. (see Fig. 3 for an illustration; the 3 thick edges represent the 3 members of size 2 in $\mathcal{C}$).

Claim 5 $\left\{\beta_1^1, \beta_2^2, \beta_3^3+\sigma\right\}$ *and* $\left\{\beta_1^1+\sigma, \beta_2^2+\sigma, \beta_3^3+\sigma\right\}$ *are the only members of size greater than 2 in* $\mathcal{C}$.

Proof of Claim. $\mathcal{C}$ contains at most one element in U_i for $i \in [3]$ by Lemma 6.5, so $\mathcal{C}$ has no member of size greater than 3. Moreover, a member of size 3 contains one element from each U_1, U_2, U_3. The subsets of size 3 that do not contain a member of size 2 but one element from each of U_1, U_2, U_3 are the following:

$$\left\{\beta_1^1, \beta_2^2, \beta_3^1\right\}, \left\{\beta_1^1, \beta_2^2+\sigma, \beta_3^1\right\}, \left\{\beta_1^1, \beta_2^2, \beta_3^3+\sigma\right\}, \left\{\beta_1^1, \beta_2^2+\sigma, \beta_3^3+\sigma\right\},$$

$$\left\{\beta_1^1+\sigma, \beta_2^1+\sigma, \beta_3^3+\sigma\right\}, \left\{\beta_1^1+\sigma, \beta_2^2, \beta_3^3+\sigma\right\}, \left\{\beta_1^1+\sigma, \beta_2^2+\sigma, \beta_3^3+\sigma\right\}.$$

By Lemma 6.5, a subset $\{x_1, x_2, x_3\}$ where $x_i \in U_i$ for $i = 1, 2, 3$ is a member if and only if $x_1+x_2+x_3 = \sigma+\alpha_1+\alpha_2+\alpha_3$. Notice that $\beta_1^1+\beta_2^2+\beta_3^1 = b+\alpha_2+\alpha_3$ cannot be $\sigma+\alpha_1+\alpha_2+\alpha_3$, because b is not $\alpha_1+\sigma$ by our choice of b. This implies that $\{\beta_1^1, \beta_2^2, \beta_3^1\}$ is not a member. Similarly, $\{\beta_1^1, \beta_2^2+\sigma, \beta_3^1\}$ is not a member, because $b \neq \alpha_1$. Notice also that $\{\beta_1^1+\sigma, \beta_2^1+\sigma, \beta_3^3+\sigma\}$ is not a member, because $\beta_1^1+\sigma+\beta_2^1+\sigma+\beta_3^3+\sigma = a+b+\alpha_1+\alpha_2+\alpha_3+\sigma$ cannot be $\sigma+\alpha_1+\alpha_2+\alpha_3$ by our assumption that $a \neq b$. Observe that $\beta_1^1+\beta_2^2+\beta_3^3+\sigma = \sigma+\alpha_1+\alpha_2+\alpha_3$, implying in turn that $\{\beta_1^1, \beta_2^2, \beta_3^3+\sigma\}$ and $\{\beta_1^1+\sigma, \beta_2^2+\sigma, \beta_3^3+\sigma\}$ are members, whereas $\{\beta_1^1, \beta_2^2+\sigma, \beta_3^3+\sigma\}$ and $\{\beta_1^1+\sigma, \beta_2^2, \beta_3^3+\sigma\}$ are not. Therefore, $\{\beta_1^1, \beta_2^2, \beta_3^3+\sigma\}$ and $\{\beta_1^1+\sigma, \beta_2^2+\sigma, \beta_3^3+\sigma\}$ are the only members of size at least 3 in $\mathcal{C}$, as required. □

Now that we have characterized all members of $\mathcal{C}$, we know that the incidence matrix of the corresponding minor $\mathcal{C}$ is the following 0,1 matrix:

$$\begin{array}{ccccccc} \beta_1^1 & \beta_2^1+\sigma & \beta_3^1 & \beta_1^1+\sigma & \beta_3^3+\sigma & \beta_2^2 & \beta_2^2+\sigma \end{array}$$
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Contracting the elements corresponding to $\beta_2^2, \beta_2^2+\sigma$ from $\mathcal{C}$, we obtain a C_5^2 minor. Since $\mathcal{C}$ is a minor of local(S, α), we deduce that local(S, α) also has C_5^2 as a minor, as required. □

Lemma 7.4 *Up to isomorphism,* Matroid(A_3) *is the unique minor-minimal matroid with distinct circuits that have a nonempty intersection. Consequently, if two distinct circuits of a matroid intersect, then the matroid has* Matroid(A_3) *as a minor.*

Proof Let M be a minor-minimal matroid over ground set E with distinct circuits that intersect.

Let C_1, C_2 be any pair of distinct circuits that intersect. Observe that $C_1 \cup C_2 = E$, for if not, $M \backslash \overline{C_1 \cup C_2}$ would a proper matroid minor with distinct circuits, namely C_1, C_2, that intersect, which cannot be the case. Observe further that $I := C_1 \cap C_2$, which by assumption is nonempty, has size one. For if not, for any $e \in I$, $M/(I - \{e\})$ would be a proper matroid minor with distinct circuits, namely $C_1 - (I - \{e\}), C_2 - (I - \{e\})$, that intersect, which cannot be the case.

In summary, every two circuits that intersect, have E as their union and an intersection of size one. Since M is a matroid, there is a circuit $C_3 \subseteq (C_1 \cup C_2) - \{e\}$. Clearly, C_3 intersects both C_1, C_2. Thus, $|C_1 \cap C_3| = |C_2 \cap C_3| = 1$ and $C_1 \cup C_3 = C_2 \cup C_3 = E$. It can be readily checked that $|C_1| = |C_2| = 2$, implying in turn that $M \cong$ Matroid(A_3), as required. □

Now we are ready to prove Theorem 1.4. The crux of the proof is outlined as follows. If mult(S) is ideal where S is a vector space over $GF(2^k)$ for some $k > 2$, then mult(S) has no C_5^2 as a minor. Then Matroid(S) has no two distinct circuits that intersect, by Lemmas 7.3 and 7.4. Then we use Theorem 4.1 to argue that S has a basis with vectors of pairwise disjoint supports.

Proof of Theorem 1.4 Take an integer $n \geq 1$. Let q be a power of 2 larger than 4, and let $S \subseteq GF(q)^n$ be a vector space over $GF(q)$. **(iii)⇒(ii)**: Since $\mathrm{mult}(S)$ contains no C_5^2 as a minor, $\mathrm{Matroid}(S)$ has no $\mathrm{Matroid}(A_3)$ as a matroid minor, by Lemma 7.3. Thus, every two distinct circuits of $\mathrm{Matroid}(S)$ must be disjoint, by Lemma 7.4. This implies that $\mathrm{Matroid}(S)$ is the graphic matroid of a graph whose blocks are bridges and cycles, so **(ii)** follows from Lemma 2.6 and Theorem 4.1. **(i)⇒(iii)** follows immediately from the fact that C_5^2 is non-ideal. **(ii)⇒(i)** follows immediately from Lemma 5.1. □

8 The replication and $\tau = 2$ conjectures

Let $\mathcal{C}$ be a clutter over ground set V. Given the weights of the elements $w \in \mathbb{Z}_+^V$, the minimum weight of a cover of $\mathcal{C}$ can be computed by the following integer linear program:

$$\tau(\mathcal{C}, w) = \min\left\{w^\top x : \; M(\mathcal{C})x \geq \mathbf{1}, \; x \in \mathbb{Z}_+^V\right\}.$$

A dual of this integer program is given by the following:

$$\nu(\mathcal{C}, w) = \max\left\{\mathbf{1}^\top y : \; M(\mathcal{C})^\top y \leq w, \; y \in \mathbb{Z}_+^{\mathcal{C}}\right\},$$

and this computes the maximum size of a *packing* of members of $\mathcal{C}$ such that each element v appears in at most w_v members in the packing. The linear programming relaxations of these two integer programs are the following primal-dual pair:

$$\begin{aligned} \tau^*(\mathcal{C}, w) = \text{minimize } & w^\top x \\ \text{subject to } & M(\mathcal{C})x \geq \mathbf{1}, \\ & x \geq \mathbf{0} \end{aligned} \qquad \begin{aligned} \nu^*(\mathcal{C}, w) = \text{maximize } & \mathbf{1}^\top y \\ \text{subject to } & M(\mathcal{C})^\top y \leq w\,. \\ & y \geq \mathbf{0} \end{aligned}$$

By linear programming duality, we have that

$$\tau(\mathcal{C}, w) \geq \tau^*(\mathcal{C}, w) = \nu^*(\mathcal{C}, w) \geq \nu(\mathcal{C}, w).$$

Although $\tau^*(\mathcal{C}, w) = \nu^*(\mathcal{C}, w)$ always holds, it is not always the case that $\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w)$. If $\tau(\mathcal{C}, w) = \nu(\mathcal{C}, w)$ holds for every $w \in \mathbb{Z}_+^V$, we say that $\mathcal{C}$ has the max-flow min-cut property. In fact, the max-flow min-cut property is equivalent to the *total dual integrality* for the integer program computing $\tau(\mathcal{C}, w)$. Namely, $\mathcal{C}$ has the max-flow min-cut property if and only if the linear system $M(\mathcal{C})x \geq \mathbf{1}, \; x \geq \mathbf{0}$ is *totally dual integral*. This implies that if $\mathcal{C}$ has the max-flow min-cut property, then $Q(\mathcal{C})$ is integral [16, 19] and thus $\mathcal{C}$ is ideal.

As the max-flow min-cut property is a special case of idealness, a natural question is as to when a clutter has the max-flow min-cut property. In this section, we characterize when the multipartite uniform clutter of a vector space over a finite field has the max-flow min-cut property.

The readers may have already noticed that Theorem 1.5 is similar to Theorem 1.2 and Theorem 1.4. As a direct corollary of these theorems, we obtain the following:

Theorem 8.1 *Take a prime power q other than* 2, 4*, and let S be a vector space over $GF(q)$. Then* $\text{mult}(S)$ *is ideal if and only if* $\text{mult}(S)$ *has the max-flow min-cut property.*

Unlike the case when $q \notin \{2, 4\}$, there is a vector space over $GF(4)$ whose multipartite uniform clutter is ideal but does not have the max-flow min-cut property. The element set of $GF(4)$ can be represented as $\{0, 1, a, b\}$ where a and b are the numbers satisfying the following addition and multiplication tables:

$$\begin{array}{c|cccc} + & 0 & 1 & a & b \\ \hline 0 & 0 & 1 & a & b \\ 1 & 1 & 0 & b & a \\ a & a & b & 0 & 1 \\ b & b & a & 1 & 0 \end{array} \qquad \begin{array}{c|cccc} \times & 0 & 1 & a & b \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & a & b \\ a & 0 & a & b & 1 \\ b & 0 & b & 1 & a \end{array}$$

Example Consider $S = \langle (1, 1, 0), (1, 0, 1) \rangle \subseteq GF(4)^3$. Then

$$S = \left\{ \begin{array}{l} (0,0,0),\ (1,1,0),\ (a,a,0),\ (b,b,0),\ (1,0,1),\ (0,1,1),\ (b,a,1),\ (a,b,1), \\ (a,0,a),\ (b,1,a),\ (0,a,a),\ (1,b,a),\ (b,0,b),\ (a,1,b),\ (1,a,b),\ (0,b,b) \end{array} \right\}.$$

One can check by using PORTA [29] that $\left\{x \in \mathbb{R}^{12}_{+} :\ M(\text{mult}(S))x \geq \mathbf{1}\right\}$ is an integral polyhedron, so $\text{mult}(S)$ is ideal. Notice further that $\text{mult}(S)$ does not have the max-flow min-cut property, since S contains

$$\{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\} \cong R_{1,1}$$

as a restriction and so $\text{mult}(S)$ has Q_6 as a minor by Lemma 2.3.

We say that clutter $\mathcal{C}$ *packs* if $\tau(\mathcal{C}, \mathbf{1}) = \nu(\mathcal{C}, \mathbf{1})$. We say that $\mathcal{C}$ has the *packing property* if every minor of $\mathcal{C}$ packs. It was observed in [11] that minimally non-ideal clutters do not pack due to Lehman's theorem [22] and that if a clutter has the packing property, then it is ideal. Moreover, notice that the packing property is a relaxed notion of the max-flow min-cut property. Here, the *Replication Conjecture* predicts that the packing property implies the max-flow min-cut property. We answer the conjecture in the affirmative for the class of multipartite uniform clutters from coordinate subspaces.

Proof of Corollary 1.6 Take a prime power q, and let S be a vector space over $GF(q)$. Suppose that $\text{mult}(S)$ has the packing property. Then every minor of $\text{mult}(S)$ packs and is ideal. Note that Δ_3 is non-ideal. Moreover, it is easy to check that $\tau(Q_6, \mathbf{1}) = 2$ and $\nu(Q_6, \mathbf{1}) = 1$, which means that Q_6 does not pack. Therefore, $\text{mult}(S)$ has none of Δ_3 and Q_6 as a minor. Then it follows from Theorem 1.5 that $\text{mult}(S)$ has the max-flow min-cut property. □

Next we consider the $\tau = 2$ *Conjecture* [11] which predicts that a stronger statement than the Replication Conjecture holds true. We call a clutter *minimally non-packing* if it does not have the packing property but every proper minor of it does. It is known that a minimally non-packing clutter is either ideal or minimally non-ideal [11]. Here, the $\tau = 2$ Conjecture is that if a clutter $\mathcal{C}$ is ideal and minimally non-packing, then its covering number, defined as $\tau(\mathcal{C}, \mathbf{1})$, is two. We show that if the multipartite uniform clutter of a coordinate subspace is ideal and minimally non-packing, then its covering number is two.

Proof of Corollary 1.7 Take a prime power q, and let S be a vector space over $GF(q)$. Suppose that $\mathrm{mult}(S)$ is ideal and minimally non-packing. As $\mathrm{mult}(S)$ does not pack, it does not have the max-flow min-cut property. Then by Theorem 1.5, $\mathrm{mult}(S)$ has Δ_3 or Q_6 as a minor. Note that as Δ_3 is non-ideal but $\mathrm{mult}(S)$ is ideal, $\mathrm{mult}(S)$ has no Δ_3 as a minor. Then it follows that $\mathrm{mult}(S)$ has Q_6 as a minor. Since Q_6 itself does not pack and every proper minor of $\mathrm{mult}(S)$ packs, $\mathrm{mult}(S)$ is isomorphic to Q_6. In fact, Q_6 is ideal and minimally non-packing, and it has covering number two, as required. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

A Multipartite uniform clutters from binary spaces

Recall that the associated matroid of a vector space S over a finite field is denoted as $\mathrm{Matroid}(S)$. For a binary space S, idealness of $\mathrm{mult}(S)$ can be characterized in terms of $\mathrm{Matroid}(S)$.

Theorem A.1 ([4]). *Take an integer $n \geq 1$, and let $S \subseteq GF(2)^n$ be a binary space. Then* $\mathrm{mult}(S)$ *is ideal if, and only if,* $\mathrm{Matroid}(S)$ *has the sums of circuits property.*

The *sums of circuits property* was introduced by Seymour [33]. Theorem A.1 is originally stated in terms of what is called the *cuboid* of S, defined in Sect. 2.1. To avoid confusion, let us stick to multipartite uniform clutters. In [32], Seymour proved that a binary matroid has the sums of circuits property if and only if it has none of F_7^*, R_{10}, $M(K_5)^*$ as a matroid minor, where F_7^* is the dual of the Fano matroid, R_{10} is the binary matroid whose graft representation is displayed in Fig. 4, and $M(K_5)^*$ is the cut matroid of K_5.

Fig. 4 R_{10}

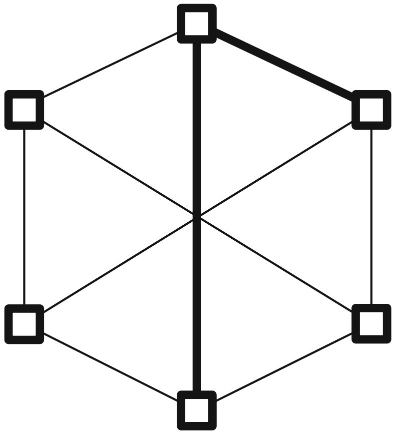
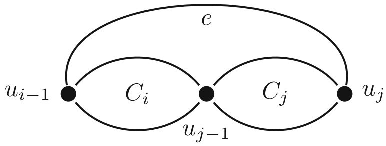


Fig. 5 The Fano plane



Theorem 1.1 provides a characterization of ideal multipartite uniform clutters from binary spaces, and it is in terms of excluded clutter minors. Recall that the following are two non-ideal clutters in the list of excluded clutter minors.

- $\mathbb{L}_7$ is the clutter over ground set $\{1, \ldots, 7\}$ whose members are $\{1, 2, 3\}$, $\{1, 4, 5\}$, $\{1, 6, 7\}$, $\{2, 4, 7\}$, $\{2, 5, 6\}$, $\{3, 4, 6\}$, $\{3, 5, 7\}$, and $\mathbb{L}_7$ is isomorphic to the clutter of lines of the *Fano plane* (Fig. 5).
- $\mathbb{O}_5$ is the clutter over ground set $E(K_5)$, the edge set of K_5, whose members are the odd cycles of K_5.

So, an ideal clutter contains none of $\mathbb{L}_7$, $\mathbb{O}_5$ as a minor.

A subset B of V is called a *cover* of a clutter $\mathcal{C}$ if B intersects every member of $\mathcal{C}$. The *blocker* of $\mathcal{C}$, denoted $b(\mathcal{C})$, is defined as the clutter over the same ground set V whose members are the minimal covers of $\mathcal{C}$. The following is a consequence of Lehman's *width-length inequality* [21]:

Theorem A.2 ([21]). *Let $\mathcal{C}$ be a clutter over ground set V. Then $\mathcal{C}$ is ideal if, and only if, $b(\mathcal{C})$ is ideal.*

Theorem A.2 implies that the blockers of $\mathbb{L}_7$ and $\mathbb{O}_5$ are non-ideal. It can be observed that the blocker of $\mathbb{L}_7$ is itself and that

- $b(\mathbb{O}_5)$ is the clutter over ground set $E(K_5)$ whose members are the cut complements of K_5.

As a consequence of Seymour's theorem [32] that a binary matroid has the sums of circuits property if and only if it has none of F_7^*, R_{10}, $M(K_5)^*$ as a matroid minor. The proof of Theorem 1.1 given in [4] is based on this result. We refer the reader to [4] for a formal proof.

B Proof of lemma 3.4

We will prove Lemma 3.4 that characterizes graphs with no K_4/e as a graph minor. Given a graph $G = (V, E)$ and its block decomposition, we may associate G with a bipartite graph $\mathcal{B}(G)$ where

- a part of the bipartition of $\mathcal{B}(G)$ consists of the cut-vertices of G,
- the other part consists of the blocks of G, and
- a cut-vertex u and a block B are adjacent in $\mathcal{B}(G)$ if u is a vertex in B.

It is well-known that $\mathcal{B}(G)$ is a tree all of whose leaves are blocks of G (see [7]). We call a vertex of G that is not a cut vertex an *internal vertex*.

Proof of Lemma 3.4 Assume that G contains no K_4/e as a graph minor. We will prove by induction on the number of edges that each block of G is a bridge, a cycle, or a subdivision of A_t for some $t \geq 3$. The base case is trivial. For the induction step, we may assume that G has at least 3 edges. If G has more than one block, a block of G has less edges than G does, so we may apply the induction hypothesis to each block of G. Thus we may assume that G is 2-vertex-connected, in which case, G has no loop.

Let e be an edge of G. By the induction hypothesis, each block of $G - \{e\}$ is a bridge, a cycle, or a subdivision of A_t for some $t \geq 3$. Moreover, since G has no loop, $G - \{e\}$ has no loop either. We first prove the following claim:

Claim 6 *Either $\mathcal{B}(G - \{e\})$ is a single vertex, i.e., $G - \{e\}$ is 2-vertex-connected, or $\mathcal{B}(G - \{e\})$ is a path whose two ends are blocks of G and e is incident to internal vertices of the two end blocks of the path.*

Proof of Claim. We may assume that $G - \{e\}$ has at least two blocks. Since G is 2-vertex-connected, e connects two distinct blocks B_1, B_2 of $G - \{e\}$. Recall that $\mathcal{B}(G - \{e\})$ is a tree, so there is a unique path between B_1 and B_2 in $\mathcal{B}(G - \{e\})$. Then, after putting e back, the blocks of $G - \{e\}$ on the path between B_1 and B_2 become a single block in G. In fact, since G is 2-vertex-connected, G has no other block. This implies that $G - \{e\}$ has no block other than the ones on C. So, $\mathcal{B}(G - \{e\})$ contains no vertex outside C, and therefore, $\mathcal{B}(G - \{e\})$ is a path where B_1, B_2 are its two ends. If e is not incident to an internal vertex of B_1, then e is incident to the cut-vertex of B_1, implying that B_1 is separated from B_2 in G, a contradiction. Thus e is incident to an internal vertex of B_1. Similarly, e is incident to an internal vertex of B_2, as required. □

Next, we claim the following:

Claim 7 *All but at most one block of $G - \{e\}$ are bridges.*

Proof of Claim. We may assume that $G - \{e\}$ has at least two blocks. Then, by Claim 1, $\mathcal{B}(G - \{e\})$ is a path $B_1, u_1, B_2, \ldots, u_{k-1}, B_k$ for some $k \geq 2$, where $B_1, \ldots, B_k$ are the blocks of $G - \{e\}$ and u_ℓ is the cut-vertex separating B_ℓ and $B_{\ell+1}$ for $\ell \in [k-1]$. Moreover, by Claim 1, $e = u_0 u_k$, where u_0 is an internal vertex of B_1 and u_k is an internal vertex of B_k.

Suppose for a contradiction that $G - \{e\}$ has two blocks that are not bridges. Then B_i, B_j for some distinct $i, j \in [k]$ are not bridges. In particular, B_i and B_j have cycles

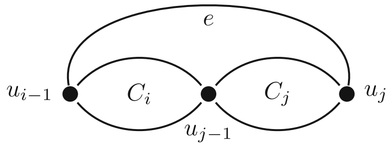


Fig. 6 $e = u_{i-1}u_j$

C_i and C_j, respectively. Here, both C_i and C_j have at least two edges as $G - \{e\}$ has no loop. After contracting the edges of B_ℓ for $\ell \in [k] - \{i, j\}$ from $G - \{e\}$, the vertices in $B_1, \ldots, B_{i-1}$ are identified with u_{i-1}, the vertices in $B_{i+1}, \ldots, B_{j-1}$ are identified with u_{j-1}, and the vertices in $B_{j+1}, \ldots, B_k$ are identified with u_j. Therefore, the resulting graph is $u_{i-1}, B_i, u_{j-1}, B_j, u_j$, where u_{i-1} and u_j are internal vertices of B_i and B_j, respectively, and u_{j-1} is the cut-vertex separating B_i, B_j. Notice that e connects u_{i-1} and u_j after the contraction, because u_0, u_k were identified with u_{i-1}, u_j, respectively (see Fig. 6 for an illustration). We then delete the edges outside of the cycles C_i, C_j. After adding e back, we obtain a subdivision of K_4/e, a contradiction as G has no K_4/e as a graph minor. Therefore, at most one block of $G - \{e\}$ is a bridge.

□

If every block of $G - \{e\}$ is a bridge, then it follows from Claim 1 that G is a cycle. Thus we may assume that a block B of $G - \{e\}$ is a cycle or a subdivision of A_t for some $t \geq 3$. Then, by Claim 2, the other blocks of $G - \{e\}$ are bridges.

Claim 8 *G is the union of B and a path P whose ends are two vertices in B and the other vertices are disjoint from $V(B)$.*

Proof of Claim. It follows from Claim 1 that e and the bridges of $G - \{e\}$ form a path P connecting two vertices of B. An interior vertex of P, if exists, is in a block of $G - \{e\}$ other than B, so it is not contained in $V(B)$, as required. □

As B is a cycle or a subdivision of A_t for some $t \geq 3$, B is a disjoint union of internally vertex-disjoint uv-paths for some distinct $u, v \in V(B)$. Let $P_1, \ldots, P_t$ be the uv-paths.

Claim 9 *If $t = 2$, G is a subdivision of A_3.*

Proof of Claim. If $t = 2$, B is a cycle and P connects two vertices on the cycle by Claim 3. So, G is the union of three internally vertex-disjoint paths connecting the two vertice, implying in turn that G is a subdivision of A_3. □

By Claim 4, we may assume that $t \geq 3$. We will show that P is also a path connecting u and v, thereby proving that G is a subdivision of A_{t+1}, obtained from uv-paths $P_1, \ldots, P_t, P$.

Claim 10 *P is an uv-path.*

Proof of Claim. Suppose for a contradiction that P is not a uv-path. Then one of P's two ends is not in $\{u, v\}$.

First, consider the case when one end of P is in $\{u, v\}$. Without loss of generality, we may assume that one end of P is u and the other end is $w \in V - \{u, v\}$. Without

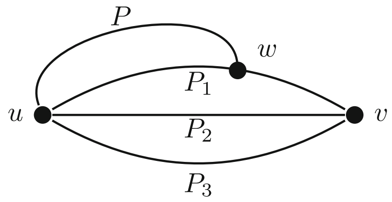
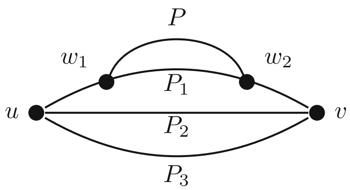


Fig. 7 $w \notin \{u, v\}$



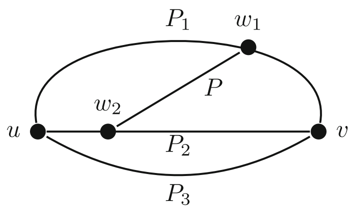


Fig. 8 $w_1, w_2 \notin \{u, v\}$

loss of generality, assume that w is on P_1. Then the subgraph of G obtained after deleting the edges $E - E(P) \cup E(P_1) \cup E(P_2) \cup E(P_3)$ (see Fig. 7 for an illustration) is a subdivision of K_4/e, contradicting the assumption that G has no K_4/e as a graph minor.

Now consider the case when both ends of P are not in $\{u, v\}$. Let the ends of P be $w_1, w_2 \in V - \{u, v\}$. There are two cases to consider: w_1, w_2 are on the same uv-path of B, or w_1, w_2 are on different uv-paths. If w_1, w_2 are on the same uv-path, we may assume that they are on P_1 without loss of generality. In this case, deleting the edges $E - E(P) \cup E(P_1) \cup E(P_2) \cup E(P_3)$ and contracting the edges of the uw_1-path on P_1 (see Fig. 8 for an illustration), we obtain a subdivision of K_4/e, a contradiction.

If w_1, w_2 are on different uv-paths, we may assume that w_1 is on P_1 and w_2 is on P_2 without loss of generality. Deleting the edges $E - E(P) \cup E(P_1) \cup E(P_2) \cup E(P_3)$ and contracting the edges of P (see Fig. 8 for an illustration), we obtain a subdivision of K_4/e, a contradiction as G has no K_4/e as a graph minor. □

By Claims 3 and 5, P is an uv-path that is internally vertex-disjoint from $P_1, \ldots, P_t$, implying in turn that G is a subdivision of A_{t+1}. This finishes the proof. □

C Proof of lemma 6.6

Proof of Lemma 6.6 **(1)** By Lemma 6.5, C is a member of size 1 if and only if $C = \{\sigma + \alpha_i\}$ for some $i \in [n]$. Therefore, $\{\alpha_1 + \sigma\}, \ldots, \{\alpha_n + \sigma\}$ are the members of size 1 in $\text{local}(S, \alpha)$, as required.

(2) First, we will argue that a member of cardinality 2 contains none of $\alpha_1 + \sigma, \ldots, \alpha_n + \sigma$. Let $\{u, v\}$ be a member of size 2 where $u \in U_i$ and $v \in U_j$ for some $i \neq j$. Then we get $u + v = \sigma + \alpha_i + \alpha_j$ by Lemma 6.5. If $u = \alpha_i + \sigma$, then $v = \alpha_j$, contradicting the assumption that $v \in U_j = GF(q) - \{\alpha_j\}$. Therefore, the members of cardinality 2 are contained in $U' := (U_1 - \{\alpha_1 + \sigma\}) \cup \cdots \cup (U_n - \{\alpha_n + \sigma\})$. Notice that we have preserved the symmetry between $U_1 - \{\alpha_1 + \sigma\}, \ldots, U_n - \{\alpha_n + \sigma\}$ and that $U_1 - \{\alpha_1 + \sigma\}$ is not different from the other $U_i - \{\alpha_i + \sigma\}$'s.

Observe that $U_1 - \{\alpha_1 + \sigma\} = GF(q) - \{\alpha_1, \alpha_1 + \sigma\}$ has $q - 2$ elements and that $U_1 - \{\alpha_1 + \sigma\}$ can be partitioned as $U_1 - \{\alpha_1 + \sigma\} = \left\{\beta_1^1, \beta_1^1 + \sigma\right\} \cup \cdots \cup \left\{\beta_1^{\frac{q}{2}-1}, \beta_1^{\frac{q}{2}-1} + \sigma\right\}$, with $\frac{q}{2} - 1$ sets of cardinality 2, where $\beta_1^1, \ldots, \beta_1^{\frac{q}{2}-1}$ are distinct elements. For $i = 2, \ldots, n$ and $j = 1, \ldots, \frac{q}{2} - 1$, we denote by $\beta_i^j \in U_i$ the element satisfying $\beta_i^j = \beta_1^j + \alpha_1 + \alpha_i$.

Claim 11 $U_i - \{\alpha_i + \sigma\} = \left\{\beta_i^1, \beta_i^1 + \sigma\right\} \cup \cdots \cup \left\{\beta_i^{\frac{q}{2}-1}, \beta_i^{\frac{q}{2}-1} + \sigma\right\}$ *for* $i = 1, \ldots, n$.

Proof of Claim. We may assume that $i \geq 2$. Let j, ℓ be distinct indices in $\left[\frac{q}{2} - 1\right]$. As $\beta_1^j \neq \beta_1^\ell$, we get $\beta_i^j \neq \beta_i^\ell$. Similarly, $\beta_1^j \neq \beta_1^\ell + \sigma$ implies $\beta_i^j \neq \beta_i^\ell + \sigma$. Therefore, $\beta_i^1, \beta_i^1 + \sigma, \ldots, \beta_i^{\frac{q}{2}-1}, \beta_i^{\frac{q}{2}-1} + \sigma$ are distinct elements, so $\left\{\beta_i^1, \beta_i^1 + \sigma\right\}, \cdots, \left\{\beta_i^{\frac{q}{2}-1}, \beta_i^{\frac{q}{2}-1} + \sigma\right\}$ partition $U_i - \{\alpha_i + \sigma\}$, as required. □

By Claim 1, each element in U' is β_i^j or $\beta_i^j + \sigma$ for some $i \in [n]$ and $j \in \left[\frac{q}{2} - 1\right]$. Now we are ready to characterize what the members of size 2 are.

Claim 12 *Let u, v be distinct elements in U'. Then $\{u, v\}$ is a member in* $\mathrm{local}(S, \alpha)$ *if and only if for some $j \in \left[\frac{q}{2} - 1\right]$ and distinct $i, k \in [n]$, we have $u = \beta_i^j$ and $v = \beta_k^j + \sigma$ or $u = \beta_i^j + \sigma$ and $v = \beta_k^j$.*

Proof of Claim. ($\Leftarrow$) Without loss of generality, we may assume that $j = 1, i = 1$, and $k = 2$. As $\beta_2^1 = \beta_1^1 + \alpha_1 + \alpha_2$, we have $\beta_1^1 + \beta_2^1 + \sigma = \alpha_1 + \alpha_2 + \sigma$. So, by Lemma 6.5, $\{u, v\}$ is a member.

($\Rightarrow$) Without loss of generality, we may assume that $u \in U_1$, $v \in U_2$. Then $u = \beta_1^j$ or $u = \beta_1^j + \sigma$ for some $j \in \left[\frac{q}{2} - 1\right]$. If $u = \beta_1^j$, then by Lemma 6.5, $v = \beta_1^j + \alpha_1 + \alpha_2 + \sigma = \beta_2^j + \sigma$. Similarly, if $u = \beta_1^j + \sigma$, we can argue that $v = \beta_2^j$, as required. □

For $j \in \left[\frac{q}{2} - 1\right]$, let G_j denote the graph induced by the elements in $\left\{\beta_1^j, \ldots, \beta_n^j\right\} \cup \left\{\beta_1^j + \sigma, \ldots, \beta_n^j + \sigma\right\}$. By Claim 2, the edge set of G_j is precisely $\left\{\left\{\beta_i^j, \beta_k^j + \sigma\right\} : i \neq k\right\}$. Moreover, Claim 2 also implies that there is no edge between G_j and G_ℓ if $j \neq \ell$, as required. □

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