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Quickest Change-point Detection Problems for Multidimensional Wiener Processes

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Abstract
We study the quickest change-point (disorder) detection problems for an observable multidimensional Wiener process with the constantly correlated components changing their drift rates at certain unobservable random (change-point) times. These problems seek to determine the times of alarms which should be as close as possible to the unknown change-point times at which some of the components have changed their drift rates. The optimal stopping times of alarm are shown to be the first times at which the appropriate posterior probability processes exit certain regions restricted by the stopping boundaries. We characterise the value functions and optimal boundaries as unique solutions to the associated free-boundary problems for partial differential equations. It is observed that the optimal stopping boundaries can also be uniquely specified by means of the equivalent nonlinear Fredholm integral equations in the class of continuous functions of bounded variation. We also provide estimates for the value functions and boundaries which are solutions to the appropriately constructed ordinary differential free-boundary problems.

Keywords Quickest change-point (disorder) detection problem · Multidimensional Wiener process · Optimal stopping problem · Stochastic boundary · Partial differential free-boundary problem · A change-of-variable formula with local times on surfaces · Nonlinear Fredholm integral equation

Mathematics Subject Classification (2010) Primary 60G40, 60J60, 34K10 · Secondary 62M20, 62C10, 62L15

1 Introduction

The quickest change-point (disorder) detection problems for observable multidimensional Wiener process seek to determine the times of alarm at which some of the components of the process change their local drift rates as soon as possible and with minimal error probabilities. More precisely, we consider the classical Bayesian formulation of this problem,

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Published online: 06 December 2024 Springer which consists of the minimisation of linear combinations of the probabilities of false alarms and the expected linear penalty costs in the detection of the change-points correctly. It is customary assumed that the change-point times are independent exponentially distributed random variables.

The quickest change-point detection problem for observable one-dimensional Wiener processes is well-understood in its classical formulation (see, e.g. Shiryaev (1961, 1963, 1965) for a solution to the original problem and Shiryaev (2019) for the introducti on to the area). The standard Poisson disorder problem, in which the intensity of the observable process changes from one value to another, was solved in full generality in Peskir and Shiryaev (2002) (see also (Peskir and Shiryaev 2006; Chapter VI, Sections 23 and 24)). Dayanik and Sezer (2006) solved the quickest disorder detection problem for observable compound Poisson processes, in which the changing characteristics were the intensity and distribution of jumps. Other formulations based on the exponential delay penalty setting were studied in Beibel (2000) for an observable Wiener process and in Bayraktar and Dayanik (2006) for an observable Poisson process. The standard Poisson disorder problem with various types of probabilities of false alarms and delay penalty functions was studied in Bayraktar et al. (2005, 2006). These problem settings are suitable when modelling the situations in which other measures of the errors due to false alarms are preferable or the costs of delay in disorder detection are not necessarily linear (e.g. continuous compounding by the interest rates in financial applications). Further extensions of the quickest change-point detection problem were studied for observable Wiener processes in Gapeev and Peskir (2006) in the finite horizon setting and, for certain time-homogeneous diffusion processes in Gapeev and Shiryaev (2013), on infinite time intervals. The aim of all the problems for one-dimensional observable processes mentioned above was to determine stopping times of alarm as close as possible to the times of change of the local characteristics of the observations.

Multidimensional versions of the quickest change-point (disorder) detection problems naturally arise when one models real-world systems described by several stochastic processes, which may have dependent components. Bayraktar et al. (2007) solved the problem for two observable independent Poisson processes, in which stopping times were sought as close as possible to the earliest of the two appropriate disorder times. Dayanik et al. (2008) solved the problem for observable multidimensional Wiener and Poisson processes with independent components, which change their local characteristics simultaneously. In this paper, we study the multidimensional Wiener quickest change-point detection problem. Our setting is closer to the one of Bayraktar et al. (2007), since the change-point times of the components are different, but is more general in the sense that we observe multiple correlated components. The multidimensional structure of the problem is clearly exhibited when making the reduction to the appropriate optimal stopping problem. Possible applications of the solutions to these quickest detection problems include: assembly line breakdown in plant production of an item when we aim to detect the earliest of all change-point times (see Bayraktar et al. 2007); abnormal returns in one of many stocks when we aim to detect just one of the change-point times; total system breakdown when we aim to detect the latest of all change-point times. More recently, some results for multidimensional sequential testing and detection problems were obtained in Ekström and Wang (2022), where the independent driving Brownian motions were considered. Ernst and Peskir (2022) and Ernst et al. (2024) studied the quickest real-time detection of a Brownian coordinate drift for a model with the observable two-dimensional and multidimensional standard Brownian motion with independent components and changing drift rates. It was shown that the optimal stopping boundaries for the appropriate posterior probability processes are determined as unique solutions to the nonlinear Fredholm integral equations in the class of continuous functions of bounded variation.

We begin by reducing the original change-point detection problem for a multidimensional Wiener process with constantly correlated components to an optimal stopping problem for a multidimensional Markov diffusion process. The components of the diffusion form a family of the posterior probability processes which correspond to every subset of change-point times and play the role of sufficient statistics for the original optimal stopping problem. For the reduction, we use the ideas from Gapeev and Jeanblanc (2010), where the filtering equations for the posterior probabilities are derived for two observable constantly correlated Wiener processes. It is shown that the optimal stopping times of alarms are the first times at which one of the posterior probability processes exits from the regions restricted by stochastic boundary surfaces determined by the current values of the other sufficient statistics. We formulate the equivalent free-boundary problem and prove a verification theorem which identifies its unique solution with the value function of the optimal stopping problem. We note that the optimal stopping boundaries can also be uniquely specified by means of the equivalent nonlinear Fredholm integral equations in the class of continuous functions of bounded variation. From the methodological point of view, the main complication in our setting arises from the higher dimensions of the sufficient statistics processes, which are needed to formulate the optimal stopping problem for a Markov process, due to the presence of the multiple disorder times. Moreover, the correlation structure of the observable processes needs to be taken into account when deriving the filtering equations. The proof of the verification theorem uses the change-of-variable formula with local time on surfaces from Peskir (2007). We also provide lower estimates for the value functions, which inherently construct the upper estimates for the stochastic boundary surfaces, in the case in which we aim to detect the earliest of the change-point times of the components. These estimates are given as solutions to the appropriate one-dimensional free-boundary problems for ordinary differential equations.

In Section 2, we introduce the setting of the model for the quickest change-point detection problem for observable multidimensional Wiener processes with constantly correlated components. We derive stochastic differential equations for a family of posterior probability processes corresponding to the appropriate subsets of the disorder times, by means of generalised Bayes' formula from (Liptser and Shiryaev 2001; Chapter VII, Theorem 7.23). In Section 3, we construct the associated optimal stopping problem for the posterior probability processes and formulate the equivalent multidimensional free-boundary problem. The verification theorem is proved providing an appropriate characterisation of the optimal stopping boundary surface as the unique solution to the free-boundary problem. Moreover, we observe that the optimal stopping boundary surface is uniquely characterised by means of the appropriate nonlinear Fredholm integral equations in the class of continuous functions of bounded variation. Finally, in Section 4, we provide estimates for the original solution to the problem of detection of the earliest of all change-point times considered in the model. The main results of the paper are stated and proved in Section 3.

2 The Problem Formulation

Let us consider a probability space $(\Omega, \mathcal{G}, P_{\vec{\pi}})$ with constantly correlated standard Wiener processes (Brownian motions) $B^i = (B^i_t)_{t \geq 0}$, for $i = 1, \ldots, n$, for some $n \in \mathbb{N}$, with the quadratic covariation $\langle B^i, B^j \rangle_t = \rho_{i,j} t$, for some constants $\rho_{i,j} \in (-1, 1)$ given and fixed, for $i, j = 1, \ldots, n$, and nonnegative random variables θ_j, for $j = 1, \ldots, n$, such that $P_{\vec{\pi}}(\theta_j = 0) = \pi_j$ and $P_{\vec{\pi}}(\theta_i > t \,|\, \theta_i > 0) = e^{-\lambda_i t}$ with $\lambda_i > 0$, for $t \geq 0$. Suppose that the variables θ_i and θ_j are independent of each other, for $i, j = 1, \ldots, n$, $i \neq j$, and of the processes B^k, for $k = 1, \ldots, n$. Hereafter, $\vec{\pi}$ denotes an n-dimensional vector

$\vec{\pi} = (\pi_1, \ldots, \pi_n) \in [0, 1]^n$, for $n \in \mathbb{N}$. Assume that we observe the continuous processes $X^i = (X_t^i)_{t \geq 0}$, for $i = 1, \ldots, n$, of the form

$$X_t^i = \mu_i \, (t - \theta_i)^+ + \nu_i \, B_t^i, \tag{2.1}$$

where the constants $\mu_i, \nu_i > 0$, for $i = 1, \ldots, n$, are given and fixed. Our aim is to determine an optimal stopping time of alarm τ_* with respect to the observable filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by all X^i, for $i = 1, \ldots, n$, that is, $\mathcal{F}_t = \vee_{i=1}^n \sigma(X_s^i \, | \, 0 \leq s \leq t)$, for $t \geq 0$, which is as close as possible to the random variable $f_k(\theta_1, \ldots, \theta_n)$, for any given (continuous) function $f_k : [0, \infty)^n \mapsto [0, \infty)$, for some $k = 1, \ldots, m$ and some $m \in \mathbb{N}$. Specifically, the quickest change-point (disorder) detection problem for a multidimensional Wiener process is to compute the Bayesian risk function

$$V_*(\vec{\pi}) = \inf_\tau \left(\sum_{k=1}^m \Big(b_k \, P_{\vec{\pi}}\big(\tau < f_k(\theta_1, \ldots, \theta_n)\big) + c_k \, E_{\vec{\pi}}\big[\big(\tau - f_k(\theta_1, \ldots, \theta_n)\big)^+\big] \Big) \right) \tag{2.2}$$

and find the optimal stopping time τ_* at which the infimum is attained in Eq. 2.2, where $b_k, c_k > 0$ are given constants, for $k = 1, \ldots, m$. Here, $P_{\vec{\pi}}(\tau < f_k(\theta_1, \ldots, \theta_n))$ represents the probability of false alarm and $E_{\vec{\pi}}[(\tau - f_k(\theta_1, \ldots, \theta_n))^+]$ represents the average delay of detecting the function $f_k(\theta_1, \ldots, \theta_n)$ correctly, for $k = 1, \ldots, m$.

By using standard arguments (see (Shiryaev 1978; Pages 195-197)), we get that

$$\begin{aligned} &P_{\vec{\pi}}\big(\tau < f_k(\theta_1, \ldots, \theta_n)\big) = E_{\vec{\pi}}\big[I\big(\tau < f_k(\theta_1, \ldots, \theta_n)\big)\big] \\ &= E_{\vec{\pi}}\big[E_{\vec{\pi}}\big[I\big(\tau < f_k(\theta_1, \ldots, \theta_n)\big) \,\big|\, \mathcal{F}_\tau\big]\big] = E_{\vec{\pi}}\big[P_{\vec{\pi}}\big(\tau < f_k(\theta_1, \ldots, \theta_n) \,\big|\, \mathcal{F}_\tau\big)\big] \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &E_{\vec{\pi}}\big[\big(\tau - f_k(\theta_1, \ldots, \theta_n)\big)^+\big] \\ &= E_{\vec{\pi}} \int_0^\tau I\big(f_k(\theta_1, \ldots, \theta_n) \leq t\big) \, dt = E_{\vec{\pi}} \int_0^\infty I\big(f_k(\theta_1, \ldots, \theta_n) \leq t \leq \tau\big) \, dt \\ &= E_{\vec{\pi}} \int_0^\infty E_{\vec{\pi}}\big[I\big(f_k(\theta_1, \ldots, \theta_n) \leq t \leq \tau\big) \,\big|\, \mathcal{F}_t\big] \, dt \\ &= E_{\vec{\pi}} \int_0^\tau P_{\vec{\pi}}\big(f_k(\theta_1, \ldots, \theta_n) \leq t \,\big|\, \mathcal{F}_t\big) \, dt \end{aligned} \tag{2.4}$$

holds, for any stopping time τ with respect to the observable filtration $(\mathcal{F}_t)_{t \geq 0}$, for $k = 1, \ldots, m$, where $I(\cdot)$ denotes the indicator function.

2.1 Sufficient Statistics and Filtering Equations

Let us now reduce the original problem of Eq. 2.2 to an optimal stopping problem for a multidimensional (strong) Markov process. We define the posterior probability processes $(\Pi_t^{*,k})_{t \geq 0}$ by $\Pi_t^{*,k} = P_{\vec{\pi}}(f_k(\theta_1, \ldots, \theta_n) \leq t \,|\, \mathcal{F}_t)$, for $t \geq 0$ and $k = 1, \ldots, m$, and observe that it follows from Eqs. 2.3-2.4 that the Bayesian risk function in Eq. 2.2 can be represented as

$$V_*(\vec{\pi}) = \inf_\tau E_{\vec{\pi}} \left[\sum_{k=1}^m \left(b_k \, (1 - \Pi_\tau^{*,k}) + c_k \int_0^\tau \Pi_t^{*,k} \, dt \right) \right]. \tag{2.5}$$

For each (ordered) sequence $J = \{j_1, \ldots, j_n\}$, $j_1 \leq \ldots \leq j_n$, for $n \in \mathbb{N}$, such that $J \subseteq N$, where we set $N = \{1, \ldots, n\}$, we define the posterior probability process $(\Pi_t^J)_{t \geq 0}$ as $\Pi_t^J := P_{\vec{\pi}}(\bigcap_{i \in J} \{\theta_i \leq t\} \,|\, \mathcal{F}_t)$, for $t \geq 0$. In order to simplify the notation, we will order the processes Π^J by choosing an arbitrary integer-valued bijection $O : \{1, \ldots, 2^n\} \mapsto 2^N$ from the set of integers $\{1, \ldots, 2^n\}$ to the power set (i.e. the set of all subsets) of N and denoting by $\vec{\Pi} = (\Pi^1, \ldots, \Pi^{2^n})$ the 2^n-dimensional process with components given by $\Pi_t^j = \Pi_t^{O(j)}$, for $t \geq 0$ and $j = 1, \ldots, 2^n$. Let us now assume that the functions $f_k(\theta_1, \ldots, \theta_n)$ are such that every process $\Pi^{*,k} = (\Pi_t^{*,k})_{t \geq 0}$ is of the form

$$\Pi_t^{*,k} \equiv P_{\vec{\pi}}\big(f_k(\theta_1, \ldots, \theta_n) \leq t \,\big|\, \mathcal{F}_t\big) = \sum_{j=1}^{2^n} a_{k,j}\, \Pi_t^j, \tag{2.6}$$

for $t \geq 0$ and some constants $a_{k,j}$, as well as for every $k = 1, \ldots, m$ and $j = 1, \ldots, 2^n$ (examples of such functions $f_k(\theta_1, \ldots, \theta_n)$ will be provided in Section 4 below). In what follows, we prove that the 2^n-dimensional process $\vec{\Pi}$ has the strong Markov property.

Let us now introduce the probability measure $P^J(\cdot) := P_{\vec{\pi}}(\cdot \mid \bigcap_{i \in J}\{\theta_i = 0\} \bigcap \bigcap_{j \in N \setminus J}\{\theta_j = \infty\})$ and the (weighted) density process $Z^J = (Z_t^J)_{t \geq 0}$ by

$$Z_t^J := \exp\left(\sum_{i \in J} \lambda_i\, t\right) \frac{d(P^J \,|\, \mathcal{F}_t)}{d(P^{\varnothing} \,|\, \mathcal{F}_t)}, \tag{2.7}$$

for $t \geq 0$ and $J \subseteq N$, where $P^J \,|\, \mathcal{F}_t$ denotes the restriction of the measure P^J to $\mathcal{F}_t$, for $t \geq 0$. Let the correlation matrix $\Sigma = (\sigma_{i,j})_{i,j \in N}$ of the n-dimensional process $X = (X^1, \ldots, X^n)$ be given by

$$\sigma_{i,j} = \frac{\langle X^i, X^j \rangle_1}{\nu_i \nu_j}, \tag{2.8}$$

for $i, j \in N$, and denote the entries of the inverse correlation matrix by $\Sigma^{-1} = (\eta_{i,j})_{i,j \in N}$, which exists, because Σ is a symmetric and positive definite matrix. We can express the density process from Eq. 2.7 in terms of processes adapted to the observable filtration $(\mathcal{F}_t)_{t \geq 0}$, and these processes will be linear combinations of the observable processes X^i, for $i \in N$, as the following lemma shows. The arguments are essentially based on the application of Girsanov's theorem for a multidimensional Wiener process.

Lemma 2.1 *In the model for a quickest change-point detection in the drift rates of multidimensional Wiener processes stated above, we have*

$$Z_t^J = \exp\left(\sum_{i \in J}(\lambda_i\, t + Y_t^i) - \frac{1}{2}\left(\sum_{i,j \in J} \frac{\mu_i \mu_j}{\nu_i \nu_j}\, \eta_{i,j}\right) t\right), \tag{2.9}$$

for $t \geq 0$ and $J \subseteq N$, where we have set

$$Y_t^i := \frac{\mu_i}{\nu_i} \sum_{j=1}^{n} \frac{\eta_{i,j}}{\nu_j}\, X_t^j, \tag{2.10}$$

for $t \geq 0$ and every $i \in N$.

Proof See Appendix below. □

Let us now define the process $(\Phi_t^{\alpha,L})_{t \geq 0}$ recursively by

$$\Phi_t^{\alpha,L} := \lambda_{\alpha_k} \int_0^t \Phi_u^{[\alpha_1,\ldots,\alpha_{k-1}],L}\, \frac{Z_t^{K \cup L}}{Z_u^{K \cup L}} du, \quad \Phi_t^{\varnothing,L} := \pi^L Z_t^L, \quad \Phi^{\varnothing,\varnothing} \equiv 1, \tag{2.11}$$

for $t \geq 0$ and $K, L \subseteq N$ such that $K \neq \varnothing$, $K \cap L = \varnothing$, and any permutation $\alpha := [\alpha_1, \ldots, \alpha_k] \in \mathrm{Perm}(K)$, where $\mathrm{Perm}(K)$ denotes the set of all permutations of K, and

$\pi^L := \prod_{l\in L}\pi_l$. The process $\Phi^{\alpha,L}$ can be regarded as a (weighted) likelihood ratio process corresponding to the event $\bigcap_{l\in L}\{\theta_l = 0\}\bigcap\{0 < \theta_{\alpha_1} \le \cdots \le \theta_{\alpha_k} \le t\}\bigcap\bigcap_{i\in N\setminus(K\cup L)}\{t < \theta_i\}$, since it can be written in the form

$$\Phi_t^{\alpha,L} = \pi^L \exp\left(\sum_{i\in N}\lambda_i\, t\right)\int_{A_t}\frac{d(P^{u,L}\,|\,\mathcal{F}_t)}{d(P^{\varnothing}\,|\,\mathcal{F}_t)}\prod_{i=1}^{k+r}\lambda_{\alpha_i}e^{-u_i\lambda_{\alpha_i}}\,d^{k+r}\,\vec{u}, \tag{2.12}$$

for $t \ge 0$, where $r \in \mathbb{N}$ is the number of elements of the set $N \setminus (K \cup L)$ and

$$\{\alpha_{k+1},\ldots,\alpha_{k+r}\} = N \setminus (K \cup L), \tag{2.13}$$

$$A_t = \{x \in \mathbb{R}^{k+r} \mid 0 < x_1 \le \cdots \le x_k \le t \text{ and } t < x_{k+i} \text{ for } i = 1,\ldots,r\}, \tag{2.14}$$

$$P^{u,L}(\cdot) = P_{\vec{\pi}}(\cdot \mid \textstyle\bigcap_{i\in L}\{\theta_i = 0\}\bigcap\bigcap_{j=1,\ldots,k+r}\{\theta_{\alpha_j} = u_j\}), \tag{2.15}$$

for $t \ge 0$ and $\vec{u} = (u_1,\ldots,u_{k+r}) \in \mathbb{R}^{k+r}$. Therefore, the processes $\Psi^{J,L} = (\Psi_t^{J,L})_{t\ge0}$ and $\Psi^J = (\Psi_t^J)_{t\ge0}$ defined by

$$\Psi_t^{J,L} := \sum_{J\subseteq K\subseteq N\setminus L}\;\sum_{\alpha\in\mathrm{Perm}(K)}\Phi_t^{\alpha,L} \quad\text{and}\quad \Psi_t^J := \sum_{L_1\subseteq N\setminus J,\, L_2\subseteq J}\Psi_t^{J\setminus L_2, L_1\cup L_2}, \tag{2.16}$$

for $t \ge 0$ and $J, L \subseteq N$ such that $J \cap L = \varnothing$, can be regarded as a (weighted) likelihood ratio processes corresponding to the events $\{(\theta_l = 0)_{l\in L}\}\bigcap\{(0 < \theta_i \le t)_{i\in J}\}\bigcap\{(0 < \theta_i)_{i\in N\setminus(J\cup L)}\}$ and $\{(\theta_i \le t)_{i\in J}\}$, respectively. Hence, by using the generalised Bayes' formula from (Liptser and Shiryaev 2001; Chapter VII, Theorem 7.23), we obtain that the posterior probability process $\Pi^J = (\Pi_t^J)_{t\ge0}$ takes the form

$$\Pi_t^J = \frac{\Psi_t^J}{\Psi_t^{\varnothing}}, \tag{2.17}$$

for $t \ge 0$ and $J \subseteq N$.

It follows from the expression in Eq. 2.9 that Z^J satisfies the following stochastic differential equation

$$dZ_t^J = Z_t^J\left(\sum_{i\in J}\lambda_i\,dt + \sum_{i\in J}dY_t^i\right), \tag{2.18}$$

for $t \ge 0$ and $J \subseteq N$. By applying Itô's formula from (Liptser and Shiryaev 2001; Chapter IV, Theorem 4.4) to the expressions in Eqs. 2.18 and 2.11, we get

$$d\Phi_t^{\alpha,L} = \left(\lambda_{\alpha_k}\Phi_t^{[\alpha_1,\ldots,\alpha_{k-1}],L} + \sum_{i\in K\cup L}\lambda_i\,\Phi_t^{\alpha,L}\right)dt + \sum_{i\in K\cup L}\Phi_t^{\alpha,L}\,dY_t^i, \tag{2.19}$$

$$d\Phi_t^{\varnothing,L} = \sum_{i\in L}\lambda_i\,\Phi_t^{\varnothing,L}\,dt + \sum_{i\in L}\Phi_t^{\varnothing,L}\,dY_t^i, \tag{2.20}$$

for $t \ge 0$ and $K, L \subseteq N$ such that $K \neq \varnothing$, $K \cap L = \varnothing$, and any $\alpha := [\alpha_1,\ldots,\alpha_k] \in \mathrm{Perm}(K)$. Therefore, by using the expression in Eq. 2.16, we further obtain

$$d\Psi_t^{J,L} = \left(\sum_{i\in J}\lambda_i\,\Psi_t^{J\setminus\{i\},L} + \sum_{i\notin J}\lambda_i\,\Psi_t^{J,L}\right)dt + \sum_{i\in J\cup L}\Psi_t^{J,L}\,dY_t^i + \sum_{i\notin J\cup L}\Psi_t^{J\cup\{i\},L}\,dY_t^i \tag{2.21}$$

for $t \ge 0$, and, by summing up the related equations, we get

$$d\Psi_t^J = \left(\sum_{i\in J}\lambda_i\,\Psi_t^{J\setminus\{i\}} + \sum_{i\notin J}\lambda_i\,\Psi_t^J\right)dt + \sum_{i\in J}\Psi_t^J\,dY_t^i + \sum_{i\notin J}\Psi_t^{J\cup\{i\}}\,dY_t^i, \tag{2.22}$$

for $t \geq 0$ and $J, L \subseteq N$ such that $J \cap L = \varnothing$. Hence, by applying Itô's formula to the expression in Eq. 2.17, we conclude that

$$d\Pi_t^J = \sum_{i \in J} \lambda_i \left(\Pi_t^{J \setminus \{i\}} - \Pi_t^J \right) dt + \sum_{i \in N} \left(\Pi_t^{J \cup \{i\}} - \Pi_t^J \, \Pi_t^{\{i\}} \right) \left(dY_t^i - \sum_{j=1}^n \Pi_t^{\{j\}} \, d\langle Y^i, Y^j \rangle_t \right), \tag{2.23}$$

for $t \geq 0$ and $J \subseteq N$.

Furthermore, we get from Eq. 2.10 that

$$\langle Y^i, Y^j \rangle_t = \frac{\mu_i \mu_j}{\nu_i \nu_j} \sum_{k,l=1}^n \eta_{i,k} \, \eta_{j,l} \, \sigma_{k,l} \, t = \frac{\mu_i \mu_j}{\nu_i \nu_j} \, \eta_{i,j} \, t \tag{2.24}$$

holds, for all $t \geq 0$, and therefore, we can write the equation in Eq. 2.23 as

$$d\Pi_t^J = \sum_{i \in J} \lambda_i \left(\Pi_t^{J \setminus \{i\}} - \Pi_t^J \right) dt + \sum_{i \in N} \left(\Pi_t^{J \cup \{i\}} - \Pi_t^J \, \Pi_t^{\{i\}} \right) \frac{\mu_i}{\nu_i} \sum_{j=1}^n \frac{\eta_{i,j}}{\nu_j} \left(dX_t^j - \mu_j \, \Pi_t^{\{j\}} \, dt \right) \tag{2.25}$$

for $t \geq 0$. Defining the innovation processes $\overline{B}^i = (\overline{B}_t^i)_{t \geq 0}$, for $i = 1, \dots, n$, by

$$\overline{B}_t^i := \frac{X_t^i}{\nu_i} - \frac{\mu_i}{\nu_i} \int_0^t \Pi_s^{\{i\}} \, ds, \tag{2.26}$$

for $t \geq 0$, and using the P. Lévy's characterisation theorem (Liptser and Shiryaev 2001; Chapter IV, Theorem 4.1), we see that $\overline{B}^i$ is a standard Brownian motion with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ under the probability measure $P_{\vec{\pi}}$. Moreover, we have $\langle \overline{B}^i, \overline{B}^j \rangle_t = \sigma_{i,j} t$, for all $t \geq 0$ and every $i, j \in N$, and we can rewrite Eq. 2.25 as

$$d\Pi_t^J = \sum_{i \in J} \lambda_i \left(\Pi_t^{J \setminus \{i\}} - \Pi_t^J \right) dt + \sum_{i \in N} \left(\Pi_t^{J \cup \{i\}} - \Pi_t^J \, \Pi_t^{\{i\}} \right) \frac{\mu_i}{\nu_i} \sum_{j=1}^n \eta_{i,j} \, d\overline{B}_t^j \tag{2.27}$$

for $t \geq 0$. Alternatively, by defining the processes $\widehat{B}^i = (\widehat{B}_t^i)_{t \geq 0}$, for $i = 1, \dots, n$, as

$$\widehat{B}_t^i := \frac{Y_t^i - \sum_{j=1}^n \int_0^t \Pi_s^{\{j\}} \, d\langle Y^i, Y^j \rangle_s}{\sqrt{\langle Y^i, Y^i \rangle_t / t}} = \left(Y_t^i - \frac{\mu_i}{\nu_i} \sum_{j=1}^n \int_0^t \Pi_s^{\{j\}} \, \frac{\mu_j}{\nu_j} \, \eta_{i,j} \, ds \right) \frac{\nu_i}{\mu_i \sqrt{\eta_{i,i}}}, \tag{2.28}$$

for $t \geq 0$, and, by using the P. Lévy's characterisation theorem, we see that $\widehat{B}^i$ is a Brownian motion with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ under the probability measure $P_{\vec{\pi}}$. Moreover, taking into account the expression in Eq. 2.24, we have

$$\langle \widehat{B}^i, \widehat{B}^j \rangle_t = \frac{\eta_{i,j}}{\sqrt{\eta_{i,i} \eta_{j,j}}} \, t, \tag{2.29}$$

for all $t \geq 0$ and $i, j \in N$, and thus, we can rewrite Eq. 2.23 as

$$d\Pi_t^J = \sum_{i \in J} \lambda_i \left(\Pi_t^{J \setminus \{i\}} - \Pi_t^J \right) dt + \sum_{i \in N} \left(\Pi_t^{J \cup \{i\}} - \Pi_t^J \, \Pi_t^{\{i\}} \right) \frac{\mu_i}{\nu_i} \sqrt{\eta_{i,i}} \, d\widehat{B}_t^i, \tag{2.30}$$

for $t \geq 0$. Therefore, by using either the expression in Eq. 2.27 or the one in Eq. 2.30, we obtain that the process $\vec{\Pi}$ satisfies the conditions of (Øksendal 1998; Chapter V, Theorem 5.2.1]) about the existence and uniqueness of strong solutions of stochastic differential equations, and thus, by virtue of (Øksendal 1998; Chapter VII, Theorem 7.2.4), it has the strong Markov property with respect to its natural filtration, which coincides with $(\mathcal{F}_t)_{t\geq 0}$. Moreover, since we have the representations

$$\Pi_t^J \equiv P_{\vec{\pi}}(\bigcap_{i\in J}\{\theta_i \leq t\} \,|\, \mathcal{F}_t) = \sum_{J\subseteq K\subseteq N} P_{\vec{\pi}}(\bigcap_{i\in K}\{\theta_i \leq t\} \bigcap \bigcap_{i\in N\setminus K}\{t < \theta_i\} \,|\, \mathcal{F}_t), \tag{2.31}$$

$$P_{\vec{\pi}}(\bigcap_{i\in K}\{\theta_i \leq t\} \bigcap \bigcap_{i\in N\setminus K}\{t < \theta_i\} \,|\, \mathcal{F}_t) = \Pi_t^K - \sum_{i\in N\setminus K} \Pi_t^{K\cup\{i\}} + \sum_{i\neq j\in N\setminus K} \Pi_t^{K\cup\{i,j\}} + \dots \tag{2.32}$$
$$\dots + (-1)^{n-k-1} \sum_{i\in N\setminus K} \Pi_t^{N\setminus\{i\}} + (-1)^{n-k}\Pi_t^N,$$

for $t \geq 0$ and $J, K \subseteq N$, where k is the number of elements of K and

$$\sum_{K\subseteq N} P_{\vec{\pi}}(\{(\theta_i \leq t)_{i\in K}\} \bigcap \{(t < \theta_i)_{i\in N\setminus K}\} \,|\, \mathcal{F}_t) = 1, \tag{2.33}$$

holds as well, it follows that the state space of the process $\vec{\Pi}$ is given by

$$\mathcal{D} := \Bigg\{ \vec{\pi} \in [0,1]^{2^n} \,\Bigg|\, \text{for some } \vec{\pi}\,' \in [0,1]^{2^n} \text{ with } \sum_{j=1}^{2^n} \pi'_j = 1 \tag{2.34}$$
$$\text{we have that } \pi_i = \sum_{O(i)\subseteq O(j)\subseteq N} \pi'_j \quad \text{for } i = 1, \dots, 2^n \Bigg\}.$$

Finally, by using the expressions in Eqs. 2.5-2.6 and the strong Markov property of the process $\vec{\Pi}$, we can reduce the problem of Eq. 2.2 to the optimal stopping problem

$$V_*(\vec{\pi}) = \inf_\tau E_{\vec{\pi}}\Bigg[\sum_{j=1}^{m} b_j \Bigg(1 - \sum_{i=1}^{2^n} a_{i,j}\, \Pi_\tau^i \Bigg) + c_j \int_0^\tau \sum_{i=1}^{2^n} a_{i,j}\, \Pi_t^i \, dt \Bigg], \tag{2.35}$$

where the infimum is taken over all stopping times τ with respect to $(\mathcal{F}_t)_{t\geq 0}$ such that the integrals above have finite expectation, so that $E_{\vec{\pi}}\,\tau < \infty$ (see, e.g. (Shiryaev 1978; Chapter IV, Section 4) and (Peskir and Shiryaev 2006; Chapter VI, Section 22). Here, the process $\vec{\Pi}$ starts at some $\vec{\pi} \in \mathcal{D}$ under the probability measure $P_{\vec{\pi}}$. Note that, from the linearity of the representations in Eqs. 2.31-2.32, it follows that the value function $V_*(\vec{\pi})$ is concave.

3 Main Results

The main results of the paper are presented in this section. We obtain certain properties of the optimal stopping time and the optimal boundaries in the problem of Eq. 2.35 and provide the characterisation of the value function V_* and optimal stopping boundary surface as the unique solution to a multidimensional free-boundary problem. We also formulate an equivalent nonlinear Fredholm integral equation for the optimal stopping boundary surface.

Let us first introduce some further notations. For any $j = 1, \dots, 2^n$, we denote by J the subset of N corresponding to the index j, that is $J := O(j) \subseteq N$. For any subset $K \subseteq N$, we denote the number of its elements by $|K|$, and we put $\lambda(K) := \sum_{k\in K} \lambda_k$.

3.1 The Structure of the Optimal Stopping Time

Define the linear function $F^j(\vec{\pi})$ by

$$F^j(\vec{\pi}) = \sum_{i=1}^{2^n} f_{j,i}\,\pi_i, \tag{3.1}$$

where the constants $f_{j,i}$ are given by

$$f_{j,j} = -\frac{1}{\lambda(J)}, \quad \text{if } J \neq \varnothing, \tag{3.2}$$

$$f_{j,i} = -\frac{\prod_{k\in(J\setminus O(i))}\lambda_k}{\lambda(O(i))} \sum_{\alpha\in\mathrm{Perm}(J\setminus O(i))} \prod_{q=1}^{|J\setminus O(i)|} \frac{1}{\lambda(O(i)) + \sum_{r=1}^{q}\lambda_{\alpha_r}}, \quad \text{if } \varnothing \neq O(i) \subset J, \tag{3.3}$$

$$f_{j,i} = 0, \quad \text{otherwise}, \tag{3.4}$$

for any $\vec{\pi} \in \mathcal{D}$ and $j = 1, \ldots, 2^n$. By applying Itô's formula to the expression $F^j(\vec{\Pi}_t)$ as well as the optional sampling theorem (see, e.g. (Liptser and Shiryaev 2001; Chapter III, Theorem 3.6) or (Karatzas and Shreve 2012; Chapter I, Theorem 3.22), by using the expression in Eq. 2.30, we can see that the expression

$$E_{\vec{\pi}}\big[F^j(\vec{\Pi}_\tau)\big] = F^j(\vec{\pi}) + E_{\vec{\pi}}\bigg[\int_0^\tau \Pi_t^j\,dt - \tau\bigg], \tag{3.5}$$

holds, for any stopping time τ such that $E_{\vec{\pi}}\,\tau < \infty$, and any $\vec{\pi} \in \mathcal{D}$ and $j = 1, \ldots, 2^n$. Therefore, the value function of the optimal stopping problem in Eq. 2.35 can be rewritten as

$$\overline{V}_*(\vec{\pi}) := V_*(\vec{\pi}) + \sum_{k=1}^{m}\Bigg(\sum_{i=1}^{2^n} c_k\,a_{i,k}\,F^i(\vec{\pi}) - b_k\Bigg) = \inf_\tau E_{\vec{\pi}}\big[G(\vec{\Pi}_\tau) + c\,\tau\big], \tag{3.6}$$

where we have defined

$$G(\vec{\pi}) := \sum_{k=1}^{m}\sum_{i=1}^{2^n}\big(c_k\,a_{i,k}\,F^i(\vec{\pi}) - b_k\,a_{i,k}\,\pi_i\big) \quad \text{and} \quad c := \sum_{k=1}^{m}\sum_{i=1}^{2^n} c_k\,a_{i,k}, \tag{3.7}$$

for $\vec{\pi} \in \mathcal{D}$. Note that we can conclude from Eq. 2.6 that the constants $a_{j,i}$ satisfy

$$0 \le \sum_{j=1}^{2^n} a_{j,i}\,\pi_j \le 1, \tag{3.8}$$

for $\vec{\pi} \in \mathcal{D}$ and $i = 1, \ldots, m$, and we obtain that $c \ge 0$, so that the optimal stopping problem in Eq. 3.6 is well-posed. Moreover, by using the expression in Eq. 3.1, we can rewrite $G(\vec{\pi})$ as

$$G(\vec{\pi}) = \sum_{i=1}^{2^n} g_i\,\pi_i \quad \text{with} \quad g_i = \sum_{k=1}^{m}\Bigg(\sum_{j=1}^{2^n} c_k\,a_{j,k}\,f_{j,i} - b_k\,a_{i,k}\Bigg), \tag{3.9}$$

and from the concavity of the function $V_*(\vec{\pi})$ and the linearity of the function $F^j(\vec{\pi})$, for $j = 1, \ldots, 2^n$, we also get that the value function $\overline{V}_*(\vec{\pi})$ is concave.

From the general optimal stopping theory for Markov processes (see, e.g. (Peskir and Shiryaev 2006; Chapter I, Section 2.2) and the form of the value function in Eq. 3.6, we know that the optimal stopping time in Eq. 2.35 is given by

$$\tau_* = \inf \left\{ t \geq 0 \,\middle|\, \overline{V}_*(\vec{\Pi}_t) = G(\vec{\Pi}_t) \right\}, \tag{3.10}$$

whenever it exists.

Let us now choose an integer l such that $l = 1, \ldots, 2^n$ and denote by $\vec{\Pi}^{-l}$ the process $\vec{\Pi}$ without its l-th component, and by $\vec{\pi}_{-l}$ the vector $\vec{\pi} \in \mathcal{D}$ without its l-th component π_l. Assume that $g_l < 0$ holds (the case $g_l > 0$ can be considered similarly) and $G(\vec{\pi})$ achieves its minimum at $\pi_l = 1$, for all $\vec{\pi} \in \mathcal{D}$. We see from Eq. 3.9 that the linear function $G(\vec{\pi})$ is decreasing in π_l, and by the concavity of the function $\overline{V}_*(\vec{\pi})$ and the fact that $\overline{V}_*(\vec{\pi}) = G(\vec{\pi})$, for all $\vec{\pi} \in \mathcal{D}$, we get that the optimal stopping time from Eq. 3.10 is of the form

$$\tau_* = \inf \left\{ t \geq 0 \,\middle|\, \Pi_t^l \geq b_*(\vec{\Pi}_t^{-l}) \right\}, \tag{3.11}$$

whenever it exists, for some function $0 \leq b_*(\vec{\pi}_{-l}) \leq 1$ and all $\vec{\pi} \in \mathcal{D}$.

Summarising the facts proved above, we are now in a position to state the following result.

Lemma 3.1 *Let the posterior probability processes $\Pi^{*,k}$ be such that the expression in* Eq. 2.6 *holds, for $k = 1, \ldots, m$. Assume there exists some $l = 1, \ldots, 2^n$ such that $g_l < 0$, and the function $G(\vec{\pi})$ achieves its minimum at $\pi_l = 1$, for all $\vec{\pi} \in \mathcal{D}$. Then, the optimal stopping time τ_* in the problems* Eqs. 2.35 and 3.6 *is of the form* Eq. 3.11, *whenever it exists, for some function $0 \leq b_*(\vec{\pi}_{-l}) \leq 1$ and all $\vec{\pi} \in \mathcal{D}$.*

In what follows, we work under the assumptions of Lemma 3.1.

3.2 The Location and Structure of the Optimal Stopping Boundary

Let us define the linear function $H^j(\vec{\pi})$ as

$$H^j(\vec{\pi}) = \sum_{i \in J} \lambda_i \left(\pi_{O^{-1}(J \setminus \{i\})} - \pi_j \right) = \sum_{i=1}^{2^n} h_{j,i}\, \pi_i, \tag{3.12}$$

where the constants $h_{j,i}$ are given by

$$h_{j,j} = -\lambda(J), \quad \text{for } J \neq \varnothing, \tag{3.13}$$

$$h_{j,i} = \lambda_k, \quad \text{if } O(i) = J \setminus \{k\} \text{ with } k \in J, \tag{3.14}$$

$$h_{j,i} = 0, \quad \text{otherwise.} \tag{3.15}$$

for any $\vec{\pi} \in \mathcal{D}$ and $j = 1, \ldots, 2^n$. By using the expression in Eq. 2.30 and the optional sampling theorem, we get

$$E_{\vec{\pi}} \int_0^\tau H^j(\vec{\Pi}_t)\, dt + \pi_j = E_{\vec{\pi}}\, \Pi_\tau^j, \tag{3.16}$$

for any stopping time τ such that $E_{\vec{\pi}}\, \tau < \infty$, and any $\vec{\pi} \in \mathcal{D}$ and $j = 1, \ldots, 2^n$. Therefore, the optimal stopping problem of Eq. 2.35, and thus Eq. 3.6, is equivalent to

$$\widetilde{V}_*(\vec{\pi}) := V_*(\vec{\pi}) + \sum_{k=1}^m \left(\sum_{i=1}^{2^n} b_k\, a_{i,k} - b_k \right) = \inf_\tau E_{\vec{\pi}} \int_0^\tau H(\vec{\Pi}_t)\, dt, \tag{3.17}$$

where we denote

$$H(\vec{\pi}) = \sum_{k=1}^{m} \left(\sum_{i=1}^{2^n} c_k \, a_{i,k} \, \pi_i - b_k \, a_{i,k} \, H^i(\vec{\pi}) \right), \tag{3.18}$$

for $\vec{\pi} \in \mathcal{D}$. By using the expression in Eq. 3.12, we can rewrite the function $H(\vec{\pi})$ in the form

$$H(\vec{\pi}) = \sum_{i=1}^{2^n} h_i \, \pi_i \quad \text{with} \quad h_i = \sum_{k=1}^{m} \left(c_k \, a_{i,k} - \sum_{j=1}^{2^n} b_k \, a_{j,k} \, h_{j,i} \right). \tag{3.19}$$

It is seen from Eq. 3.17 that, whenever $H(\vec{\Pi}_t) < 0$, for $t \geq 0$, it is not optimal to stop the observations, or, equivalently

$$H(\vec{\pi}) \geq 0 \quad \text{for} \quad \vec{\pi} \in S, \tag{3.20}$$

where the stopping region S is defined as (compare with Eq. 3.11)

$$S := \left\{ \vec{\pi} \in \mathcal{D} \,\middle|\, \pi_l \geq b_*(\vec{\pi}_{-l}) \right\}. \tag{3.21}$$

By virtue of the expression in Eq. 3.19, this fact means that the set

$$\left\{ \vec{\pi} \in \mathcal{D} \,\middle|\, \sum_{i=1}^{2^n} h_i \, \pi_i < 0 \right\} \tag{3.22}$$

belongs to the continuation region C defined by

$$C := \left\{ \vec{\pi} \in \mathcal{D} \,\middle|\, \pi_l < b_*(\vec{\pi}_{-l}) \right\}. \tag{3.23}$$

If we assume that $h_l > 0$, the expression above leads to the inequality

$$b_*(\vec{\pi}_{-l}) \geq \overline{b}_*(\vec{\pi}_{-l}) \equiv \frac{h_l \pi_l - \sum_{i=1}^{2^n} h_i \pi_i}{h_l}, \tag{3.24}$$

so that $\overline{b}_*(\vec{\pi}_{-l}) \leq b_*(\vec{\pi}_{-l})$ holds, for all $\vec{\pi}_{-l} \in [0, 1]^{2^n - 1}$ such that $\vec{\pi} \in \mathcal{D}$. Therefore we call *admissible* the parameters of the model that satisfy Eq. 3.24, because otherwise, the optimal stopping time is not of the form Eq. 3.11, whenever it exists.

Let us take $\vec{\pi}, \vec{\pi}' \in \mathcal{D}$ such that $\pi_l < b_*(\vec{\pi}_{-l})$, $\pi'_k \leq \pi_k$, and $\vec{\pi}'_k = \vec{\pi}_k$, for some $k = 1, \ldots, 2^n$ such that $k \neq l$, and assume that $h_k > 0$ holds. Then, using the fact that $\vec{\Pi}$ is a (time-homogeneous continuous) strong Markov process and taking into account the comparison results for solutions of multidimensional stochastic differential equations in Veretennikov (1980), we get

$$\begin{aligned} \overline{V}_*(\vec{\pi}') - G(\vec{\pi}') \equiv \widetilde{V}_*(\vec{\pi}') \leq E_{\vec{\pi}'} \int_0^{\tau_*(\vec{\pi})} H(\vec{\Pi}_t)\, dt \leq E_{\vec{\pi}} \int_0^{\tau_*(\vec{\pi})} H(\vec{\Pi}_t)\, dt \\ = \widetilde{V}_*(\vec{\pi}) \equiv \overline{V}_*(\vec{\pi}) - G(\vec{\pi}) < 0, \end{aligned} \tag{3.25}$$

which leads to the fact that $\pi_l \equiv \pi'_l < b_*(\vec{\pi}'_{-l})$. Since we can choose π_l arbitrarily close to $b_*(\vec{\pi}_{-l})$, it follows that $b_*(\vec{\pi}_{-l}) \leq b_*(\vec{\pi}'_{-l})$, and therefore, the boundary $b_*(\vec{\pi}_{-l})$ is decreasing in π_k, for $k = 1, \ldots, 2^n$ such that $k \neq l$. The case in which $h_k < 0$ is considered similarly and leads to the fact that $b_*(\vec{\pi}_{-l})$ is increasing in π_k, for $k = 1, \ldots, 2^n$.

Let us summarise the results proved above in the following assertion.

Lemma 3.2 *Suppose that the assumptions of* Lemma 3.1 *hold. Then, under the assumption that* $h_l > 0$ *holds, for some* $l = 1, \ldots, 2^n$*, the inequality in* Eq. 3.24 *holds and the parameters* *of the model are admissible. Moreover, if the inequality $h_k > 0$ ($h_k < 0$) holds, for some $k = 1, \dots, 2^n$ such that $k \neq l$, then the boundary $b_*(\vec{\pi}_{-l})$ is decreasing (increasing) in π_k, for $\vec{\pi} \in \mathcal{D}$.*

3.3 The Free-boundary Problem

By means of standard arguments (see, e.g. (Karatzas and Shreve, 2012; Chapter V, Section 5.1) and (Øksendal 1998; Theorem 7.5.4), it can be seen from the expression in Eq. 2.30 that the infinitesimal operator $\mathbb{L}$ of the process $\vec{\Pi}$ is given by the expression

$$\mathbb{L} = \sum_{j=1}^{2^n} \sum_{i \in J} \lambda_i \, (\pi_{O^{-1}(J \setminus \{i\})} - \pi_j) \, \partial_{\pi_j} \\ + \frac{1}{2} \sum_{j=1}^{2^n} \sum_{i=1}^{2^n} \sum_{k,l \in N} \frac{\mu_k \mu_l}{\nu_k \nu_l} \, \eta_{k,l} \, (\pi_{O^{-1}(J \cup \{k\})} - \pi_j \pi_{O^{-1}(\{k\})}) \, (\pi_{O^{-1}(O(i) \cup \{l\})} - \pi_i \pi_{O^{-1}(\{l\})}) \, \partial^2_{\pi_j \pi_i}, \tag{3.26}$$

for all $\vec{\pi} \in \mathcal{D}$. In order to find analytic expressions for the unknown value function $\overline{V}_*(\vec{\pi})$ from Eq. 3.6 and the unknown boundary $b_*(\vec{\pi}_{-l})$ from Eq. 3.11, we will use results from the general theory of optimal stopping problems for (time-homogeneous continuous strong) Markov processes (see, e.g. (Shiryaev 1978; Chapter III, Section 8) and (Peskir and Shiryaev 2006; Chapter IV, Section 8). Specifically, we formulate the associated free-boundary problem

$$(\mathbb{L}V)(\vec{\pi}) = -c \quad \text{for} \quad \pi_l < b(\vec{\pi}_{-l}), \tag{3.27}$$

$$V(\pi_1, \dots, \pi_{l-1}, b(\vec{\pi}_{-l})-, \pi_{l+1}, \dots, \pi_{2^n}) = G(\pi_1, \dots, \pi_{l-1}, b(\vec{\pi}_{-l}), \pi_{l+1}, \dots, \pi_{2^n}), \tag{3.28}$$

$$V(\vec{\pi}) = G(\vec{\pi}) \quad \text{for} \quad \pi_l > b(\vec{\pi}_{-l}), \tag{3.29}$$

$$V(\vec{\pi}) < G(\vec{\pi}) \quad \text{for} \quad \pi_l < b(\vec{\pi}_{-l}), \tag{3.30}$$

$$(\mathbb{L}V)(\vec{\pi}) > -c \quad \text{for} \quad \pi_l > b(\vec{\pi}_{-l}), \tag{3.31}$$

for some $0 \leq b(\vec{\pi}_{-l}) \leq 1$, where the *instantaneous stopping* condition of Eq. 3.28 is satisfied at $b(\vec{\pi}_{-l})$, for all $\vec{\pi}_{-l} \in [0, 1]^{2^n - 1}$ such that $\vec{\pi} \in \mathcal{D}$. Since the problem formulated in Eqs. 3.27-3.31 may admit multiple solutions, we need to use some additional conditions which would specify the appropriate solution, and thus, provide the value function and the optimal stopping boundary for the initial problem of Eqs. 3.6 and 2.35. Therefore, we will assume that

$$\partial_{\pi_k} V(\pi_1, \dots, \pi_{l-1}, \pi_l, \pi_{l+1}, \dots, \pi_{2^n})\Big|_{\pi_l = b(\vec{\pi}_{-l})-} = g_k \quad (\textit{smooth fit}) \tag{3.32}$$

holds, for all $k = 1, \dots, 2^n$ and $\vec{\pi} \in \mathcal{D}$. Note that the *smooth-fit* conditions of Eq. 3.32 are naturally used for the value function at the optimal stopping boundary, whenever the general payoff function $G(\vec{\pi})$ is continuously differentiable in π_l at the boundary $b(\vec{\pi}_{-l})$, for $l = 1, \dots, 2^n$ fixed (see (Peskir and Shiryaev 2006; Chapter IV, Section 9) for an extensive overview).

We further search for analytic solutions of the elliptic-type free-boundary problem in Eqs. 3.27-3.30 satisfying the conditions of Eqs. 3.31-3.32 and such that the resulting boundary is continuous and of bounded variation. Since such free boundary problems cannot normally be solved explicitly, the existence and uniqueness of classical as well as viscosity solutions of the variational inequalities arising in the context of optimal stopping problems have been extensively studied in the literature (see, e.g. Friedman (1976), Bensoussan et al. (1982), Krylov (1980), or Øksendal (1998)). Although the necessary conditions for existence and uniqueness of such solutions in (Friedman 1976; Chapter XVI, Theorem 11.1), (Krylov 1980; Chapter V, Section 3, Theorem 14) with (Krylov 1980; Chapter VI, Section 4, Theorem 12), and (Øksendal 1998; Chapter X, Theorem 10.4.1) can be verified by virtue of the properties of the coefficients of the process $\vec{\Pi}$, the application of these classical results would still have a rather inexplicit character.

We therefore continue with the following verification assertion related to the free-boundary problem formulated above.

Theorem 3.1 *Suppose that the assumptions of* Lemmata 3.1 and 3.2 *hold. Assume that* $V(\vec{\pi}; b_*(\vec{\pi}_{-l}))$ *together with* $0 \le b_*(\vec{\pi}_{-l}) \le 1$ *form a solution of the free-boundary problem in* Eqs. 3.27-3.31, *and the boundary* $b_*(\vec{\pi}_{-l})$ *is continuous and of bounded variation. Define the stopping time* τ_* *as the first exit time of the process* Π^l *from the interval* $[0, b_*(\vec{\Pi}^{-l}))$ *as in* Eq. 3.11, *and assume that* $E_{\vec{\pi}}\, \tau_* < \infty$ *holds, for* $\vec{\pi} \in \mathcal{D}$. *Then, the value function* $\overline{V}_*(\vec{\pi})$ *from* Eq. 3.6 *takes the form*

$$\overline{V}_*(\vec{\pi}) = \begin{cases} V(\vec{\pi}; b_*(\vec{\pi}_{-l})), & \text{if } \pi_l < b_*(\vec{\pi}_{-l}) \\ G(\vec{\pi}), & \text{if } \pi_l \ge b_*(\vec{\pi}_{-l}) \end{cases} \tag{3.33}$$

with

$$V(\vec{\pi}; b_*(\vec{\pi}_{-l})) = E_{\vec{\pi}}\big[G(\vec{\Pi}_{\tau_*}) + c\,\tau_*\big], \tag{3.34}$$

and the boundary $b_*(\vec{\pi}_{-l})$ *is uniquely determined by the smooth-fit condition of* Eq. 3.32.

Proof In order to verify the assertions stated above, let us denote by $V(\vec{\pi})$ the right-hand side of the expression in Eq. 3.33. Then, using the fact that the function $V(\vec{\pi})$ satisfies the conditions of Eqs. 3.29-3.30 by construction, we can apply the local time-space formula from (Peskir 2007; Theorem 3.1) (see also (Peskir and Shiryaev 2006; Chapter II, Section 3.5) for a summary of the related results and further references) to obtain

$$V(\vec{\Pi}_t) + c\,t = V(\vec{\pi}) + M_t + L_t + \int_0^t \big((\mathbb{L}V)(\vec{\Pi}_s) + c\big)\, I\big(\Pi_s^l \ge b_*(\vec{\Pi}_s^{-l})\big)\, ds, \tag{3.35}$$

for all $t \ge 0$, where the process $M = (M_t)_{t \ge 0}$ defined by

$$M_t = \sum_{i=1}^{2^n} \sum_{k \in N} \int_0^t V_{\pi_i}(\vec{\Pi}_s)\, \frac{\mu_k}{\nu_k}\, \sqrt{\eta_{k,k}}\, \Big(\Pi_s^{O^{-1}(O(i)\cup\{k\})} - \Pi_s^i \Pi_s^{O^{-1}(\{k\})}\Big)\, I\big(\Pi_s^l \ne b_*(\vec{\Pi}_s^{-l})\big)\, d\widehat{B}_s^k, \tag{3.36}$$

for $t \ge 0$, is a continuous local martingale under the probability measure $P_{\vec{\pi}}$ with respect to the filtration $(\mathcal{F}_t)_{t \ge 0}$. Here, the process $L = (L_t)_{t \ge 0}$ is given by

$$L_t = \frac{1}{2} \int_0^t \Delta_{\pi_l} V(\vec{\Pi}_s)\, I\big(\Pi_s^l = b_*(\vec{\Pi}_s^{-l})\big)\, d\ell_s^l, \tag{3.37}$$

where the function $\Delta_{\pi_l} V(\vec{\pi})$ is given by

$$\Delta_{\pi_l} V(\vec{\pi}) = V_{\pi_l}(\pi_1, \ldots, \pi_{l-1}, \pi_l+, \pi_{l+1}, \ldots, \pi_{2^n}) - V_{\pi_l}(\pi_1, \ldots, \pi_{l-1}, \pi_l-, \pi_{l+1}, \ldots, \pi_{2^n}), \tag{3.38}$$

and the process $\ell^l = (\ell_t^l)_{t \ge 0}$ defined by

$$\ell_t^l = P_{\vec{\pi}} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I\Big(b_*(\vec{\Pi}_s^{-l}) - \varepsilon < \Pi_s^l < b_*(\vec{\Pi}_s^{-l}) + \varepsilon\Big)\, d\langle \Pi^l - b_*(\vec{\Pi}^{-l})\rangle_s, \tag{3.39}$$

for $t \ge 0$, is the local time of Π^l at the surface $b_*(\vec{\Pi}^{-l})$, at which the partial derivative $V_{\pi_l}(\vec{\pi})$ may not exist. It follows from the fact that the gain function $G(\vec{\pi})$ in Eq. 3.6 is decreasing in π_l with the minimum at $\pi_l = 1$ and the conditions Eqs. 3.29-3.30 that the inequality $\Delta_{\pi_l} V(\vec{\pi}) \leq 0$ should hold for all $\vec{\pi} \in \mathcal{D}$, so that the continuous process L defined in Eq. 3.37 is non-increasing. We may therefore conclude that $L_t = 0$ can hold, for all $t \geq 0$, if and only if the smooth-fit conditions of Eq. 3.32 is satisfied.

By using the assumption that the inequality in Eq. 3.31 holds with the boundary $b_*(\vec{\pi}_{-l})$, we conclude from the condition in Eq. 3.29 that $(\mathbb{L}V)(\vec{\pi}) + c \geq 0$ holds, for any $\vec{\pi} \in \mathcal{D}$ such that $\pi_l \neq b_*(\vec{\pi}_{-l})$. Moreover, it follows from the conditions of Eqs. 3.28-3.30 that the inequality $V(\vec{\pi}) \leq G(\vec{\pi})$ holds, for all $\vec{\pi} \in \mathcal{D}$. Thus, the expression in (3.35) yields that the inequalities

$$G(\vec{\Pi}_\tau) + c\,\tau - L_\tau \geq V(\vec{\Pi}_\tau) + c\,\tau - L_\tau \geq V(\vec{\pi}) + M_\tau, \tag{3.40}$$

hold, for any stopping time τ such that $E_{\vec{\pi}}\,\tau < \infty$ and $E_{\vec{\pi}}\,L_\tau > -\infty$, and all $\vec{\pi} \in \mathcal{D}$. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence of stopping times for the process M such that $\tau_n = \inf\{t \geq 0 \,|\, |M_t| \geq n\}$. Taking the expectations with respect to the probability measure $P_{\vec{\pi}}$ in Eq. 3.40, by means of the optional sampling theorem, we get the inequalities

$$\begin{aligned} E_{\vec{\pi}}\big[G(\vec{\Pi}_{\tau\wedge\tau_n}) + c\,(\tau \wedge \tau_n) - L_{\tau\wedge\tau_n}\big] &\geq E_{\vec{\pi}}\big[V(\vec{\Pi}_{\tau\wedge\tau_n}) + c\,(\tau \wedge \tau_n) - L_{\tau\wedge\tau_n}\big] \\ &\geq V(\vec{\pi}) + E_{\vec{\pi}}\,M_{\tau\wedge\tau_n} = V(\vec{\pi}). \end{aligned} \tag{3.41}$$

Hence, letting n go to infinity and using Fatou's lemma, we obtain

$$E_{\vec{\pi}}\big[G(\vec{\Pi}_\tau) + c\,\tau - L_\tau\big] \geq E_{\vec{\pi}}\big[V(\vec{\Pi}_\tau) + c\,\tau - L_\tau\big] \geq V(\vec{\pi}), \tag{3.42}$$

for any stopping time τ such that $E_{\vec{\pi}}\,\tau < \infty$ and $E_{\vec{\pi}}\,L_\tau > -\infty$, and all $\vec{\pi} \in \mathcal{D}$, where $L_\tau = 0$ holds, whenever the condition of Eq. 3.32 is satisfied. By virtue of the structure of the stopping time in Eq. 3.11 and the condition Eq. 3.29, it is readily seen that the equalities in Eq. 3.40 hold with τ_* instead of τ when $\pi_l \geq b_*(\vec{\pi}_{-l})$.

Let us now show that the equalities are attained in Eq. 3.42, for $\pi_l < b_*(\vec{\pi}_{-l})$, when τ_* replaces τ, and the smooth-fit conditions of Eq. 3.32 hold. By virtue of the fact that the function $V(\vec{\pi})$ and the continuous boundary of bounded variation $b_*(\vec{\pi}_{-l})$ solve the partial differential equation in Eq. 3.27 and satisfy the conditions in Eqs. 3.28 and 3.32, it follows from the expression in Eq. 3.35 and the structure of the stopping time in (3.11) that

$$G(\vec{\Pi}_{\tau_*\wedge\tau_n}) + c\,(\tau_* \wedge \tau_n) = V(\vec{\Pi}_{\tau_*\wedge\tau_n}) + c\,(\tau_* \wedge \tau_n) = V(\vec{\pi}) + M_{\tau_*\wedge\tau_n}, \tag{3.43}$$

holds for $\pi_l < b_*(\vec{\pi}_{-l})$. Hence, taking the expectations and letting n go to infinity in Eq. 3.43, using the facts that $G(\vec{\pi})$ is bounded and $E_{\vec{\pi}}\,\tau_* < \infty$, we apply the Lebesgue dominated convergence theorem to obtain the equality

$$E_{\vec{\pi}}\big[G(\vec{\Pi}_{\tau_*}) + c\,\tau_*\big] = V(\vec{\pi}), \tag{3.44}$$

for all $\vec{\pi} \in \mathcal{D}$. We may therefore conclude that the function $V(\vec{\pi})$ coincides with the value function $\overline{V}_*(\vec{\pi})$ of the optimal stopping problem in Eq. 3.6 whenever the smooth-fit condition of Eq. 3.32 holds.

In order to prove the uniqueness of the value function $\overline{V}_*(\vec{\pi})$ and the boundary $b_*(\vec{\pi}_{-l})$ as solutions to the free-boundary problem in Eqs. 3.27-3.31 with the smooth-fit condition of Eq. 3.32, let us assume that there exists another continuous boundary of bounded variation $b'(\vec{\pi}_{-l})$ such that $0 \leq b'(\vec{\pi}_{-l}) \leq 1$ holds. Then, define the function $V'(\vec{\pi})$ as in Eq. 3.33 with $V'(\vec{\pi}; b'(\vec{\pi}_{-l}))$ satisfying Eqa. 3.27-3.31 and the stopping time τ' as in Eq. 3.11 with $b'(\vec{\pi}_{-l})$ instead of $b_*(\vec{\pi}_{-l})$, such that $E_{\vec{\pi}}\,\tau' < \infty$ holds. In this case, following the arguments from the previous part of the proof and using the fact that the function $V'(\vec{\pi})$ solves the partial differential equation in Eq. 3.27 and satisfies the conditions of Eqs. 3.28 and 3.32 with

$b'(\vec{\pi}_{-l})$ instead of $b_*(\vec{\pi}_{-l})$ by construction, we apply the change-of-variable formula from Peskir (2007) to get

$$V'(\vec{\Pi}_t) + c\,t = V'(\vec{\pi}) + M'_t + \int_0^t \big((\mathbb{L}V')(\vec{\Pi}_s) + c\big)\, I\big(\Pi^l_s \geq b'(\vec{\Pi}^{-l}_s)\big)\, ds, \tag{3.45}$$

for all $t \geq 0$, where the process $M' = (M'_t)_{t\geq 0}$ defined as in Eq. 3.36 with $V'_{\pi_i}(\vec{\pi})$ instead of $V_{\pi_i}(\vec{\pi})$ is a continuous local martingale with respect to the probability measure $P_{\vec{\pi}}$. Thus, taking into account the structure of the stopping time τ', we obtain from Eq. 3.45 that

$$G(\vec{\Pi}_{\tau'\wedge\tau'_n}) + c\,(\tau' \wedge \tau'_n) \geq V'(\vec{\Pi}_{\tau'\wedge\tau'_n}) + c\,(\tau' \wedge \tau'_n) = V'(\vec{\pi}) + M'_{\tau'\wedge\tau'_n}, \tag{3.46}$$

holds, for $\pi_l < b'(\vec{\pi}_{-l})$ and any localising sequence $(\tau'_n)_{n\in\mathbb{N}}$ of M'. Hence, taking expectations and letting n go to infinity in Eq. 3.46, using the fact that $G(\vec{\pi})$ is bounded and $E_{\vec{\pi}}\,\tau' < \infty$, by means of the Lebesgue dominated convergence theorem, we have that the equality

$$E_{\vec{\pi}}\big[G(\vec{\Pi}_{\tau'}) + c\,\tau'\big] = V'(\vec{\pi}), \tag{3.47}$$

is satisfied. Therefore, recalling the fact that τ_* is an optimal stopping time in Eq. 3.6 and comparing the expressions in Eqs. 3.44 and 3.47, we see that the inequality $V'(\vec{\pi}) \geq V(\vec{\pi})$ should hold, for all $\vec{\pi} \in \mathcal{D}$.

Finally, we show that $b'(\vec{\pi}_{-l})$ should coincide with $b_*(\vec{\pi}_{-l})$, for all $\vec{\pi}_{-l} \in [0,1]^{2^n-1}$ such that $\vec{\pi} \in \mathcal{D}$. By using the fact that $V'(\vec{\pi})$ and $V(\vec{\pi})$ satisfy Eqs. 3.28-3.30, and $V'(\vec{\pi}) \geq V(\vec{\pi})$ holds, for all $\vec{\pi} \in \mathcal{D}$, we get that $b'(\vec{\pi}_{-l}) \leq b_*(\vec{\pi}_{-l})$. Then, by inserting $\tau_* \wedge \tau'_n$ into Eq. 3.45 in place of t and applying arguments similar to the ones used above, we obtain

$$E_{\vec{\pi}}\big[V'(\vec{\Pi}_{\tau_*}) + c\,\tau_*\big] = V'(\vec{\pi}) + E_{\vec{\pi}} \int_0^{\tau_*} \big((\mathbb{L}V')(\vec{\Pi}_s) + c\big)\, I\big(\Pi^l_s \geq b'(\vec{\Pi}^{-l}_s)\big)\, ds, \tag{3.48}$$

for all $\vec{\pi} \in \mathcal{D}$. Thus, since we have $V'(\vec{\pi}) = V(\vec{\pi}) = G(\vec{\pi})$, for $\pi_l = b_*(\vec{\pi}_{-l})$, and $V'(\vec{\pi}) \geq V(\vec{\pi})$, we see from the expressions in Eqs. 3.44 and 3.48 that the inequality

$$E_{\vec{\pi}} \int_0^{\tau_*} \big((\mathbb{L}V')(\vec{\Pi}_s) + c\big)\, I\big(\Pi^l_s \geq b'(\vec{\Pi}^{-l}_s)\big)\, ds \leq 0 \tag{3.49}$$

should hold. Due to the assumption of continuity of $b'(\vec{\pi}_{-l})$, we may therefore conclude that $b_*(\vec{\pi}_{-l}) = b'(\vec{\pi}_{-l})$, so that $V'(\vec{\pi})$ coincides with $V(\vec{\pi})$, for all $\vec{\pi} \in \mathcal{D}$. □

Corollary 3.1 *It is shown by means of the same arguments as in* Ernst and Peskir (2022) and Ernst et al. (2024) *that the expression in* Eq. 3.17 *takes the form*

$$\widetilde{V}_*(\vec{\pi}; b_*(\vec{\pi}_{-l})) = E_{\vec{\pi}} \int_0^{\tau_*} H(\vec{\Pi}_t)\, dt, \tag{3.50}$$

with the optimal stopping time of alarm τ_* from Eq. 3.11*, and thus, the equality*

$$\widetilde{V}_*(\vec{\pi}; b_*(\vec{\pi}_{-l})) = \int_0^{\infty} E_{\vec{\pi}}\big[H(\vec{\Pi}_t)\, I\big(\Pi^l_t < b_*(\vec{\Pi}^{-l}_t)\big)\big]\, dt \tag{3.51}$$

holds, while the optimal stopping boundary $b_*(\vec{\pi}_{-l})$ provides a unique solution of the nonlinear Fredholm integral equation

$$\int_0^{\infty} E_{(\vec{\pi}_{-l}, b(\vec{\pi}_{-l}))}\big[H(\vec{\Pi}_t)\, I\big(\Pi^l_t < b(\vec{\Pi}^{-l}_t)\big)\big]\, dt \tag{3.52}$$

for all $\vec{\pi}_{-l} \in [0,1]^{2^n-1}$ such that $\vec{\pi} \in \mathcal{D}$, in the class of continuous functions of bounded variation.

4 Examples and Estimates for the Value Function

In the previous sections we characterised the Bayesian risk function of Eq. 2.2 as the solution to the optimal stopping problem in Eq. 2.35 and, under certain assumptions, to the free-boundary problem in Eqs. 3.27-3.32. However, explicit solutions to such a complicated multidimensional free-boundary problem are generally not available. Therefore, in what follows, we first study specific examples that satisfy the assumptions in Lemma 3.1 and Proposition 3.2, and then provide estimates for the value function and optimal boundaries in Eq. 2.35, which are easier to compute. We assume for the notational convenience that the bijection O satisfies $O(1) = \varnothing$, so that we have $\Pi^1 = \Pi^{\varnothing} \equiv 1$.

4.1 The Cases of Earliest and Latest Change-points

Let us now present an example, in which we can can indeed find $l = 1, \ldots, 2^n$ such that $g_l < 0$ and $h_l > 0$ holds, and $G(\vec{\pi})$ achieves its minimum at $\pi_l = 1$, for all $\vec{\pi} \in \mathcal{D}$. Let $m = 2$ and the functions $f_1(\theta_1, \ldots, \theta_n)$ and $f_2(\theta_1, \ldots, \theta_n)$ in Eq. 2.2 be given by $f_1(\theta_1, \ldots, \theta_n) = \bigwedge_{i \in N} \theta_i$ and $f_2(\theta_1, \ldots, \theta_n) = \bigvee_{i \in K} \theta_i$, for some $\varnothing \neq K \subseteq N$. This means that the posterior probability processes $\Pi^{*,1}$ and $\Pi^{*,2}$ from Eq. 2.5 are of the form Eq. 2.6 with

$$a_{1,1} = 0, \quad a_{j,1} = (-1)^{|O(j)|-1} \quad \text{for } j = 2, 3, \ldots, 2^n, \tag{4.1}$$
$$a_{k,2} = 1, \quad a_{j,2} = 0 \quad \text{for } j = 1, \ldots, k-1, k+1, \ldots, 2^n, \tag{4.2}$$

where we have taken $k \in \mathbb{N}$ such that $2 \leq k \leq 2^n$ to be such that $O(k) = K$. Notice that, by virtue of the expressions in Eqs. 3.2-3.4, we have

$$\sum_{K \subseteq O(j) \subseteq N} (-1)^{|O(j) \setminus K|} f_{j,k} = \frac{1}{\lambda(N)}, \tag{4.3}$$

and, by using the expressions in Eqs. 3.9, 3.19 and 4.1-4.2, we get

$$g_j = -a_{j,1} \left(b_1 + \frac{c_1}{\lambda(N)} \right) - b_2\, a_{j,2} + c_2\, f_{k,j} \quad \text{if } O(j) \subseteq K, \tag{4.4}$$
$$g_j = -a_{j,1} \left(b_1 + \frac{c_1}{\lambda(N)} \right) \qquad \text{otherwise}, \tag{4.5}$$

and

$$h_k = a_{k,1}\, (b_1\, \lambda(N) + c_1) + b_2\, \lambda(K) + c_2, \tag{4.6}$$
$$h_j = a_{j,1}\, (b_1\, \lambda(N) + c_1) - b_2\, \lambda_i \quad \text{if } \varnothing \neq O(j) = K \setminus \{i\} \text{ with } i \in K, \tag{4.7}$$
$$h_1 = -b_1\, \lambda(N) - b_2\, \lambda_i \qquad \text{if } K \equiv \{i\}, \tag{4.8}$$
$$h_j = a_{j,1}\, (b_1\, \lambda(N) + c_1) \qquad \text{if } \varnothing \neq O(j) \neq K \setminus \{i\} \text{ with } i \in K, \tag{4.9}$$
$$h_1 = -b_1\, \lambda(N) \qquad \text{if } K \not\equiv \{i\}. \tag{4.10}$$

In the case in which $|K|$ is an odd number, we can choose $l \equiv k$, and from Eqs. 4.4-4.10 and the fact that $a_{l,1} \equiv a_{k,1} = 1$, it follows that $g_l < 0$ and $h_l > 0$ holds. In the case in which $|K|$ is an even number and $K \neq N$, we can choose l such that $O(l) = K \cup \{\overline{k}\}$ with $\overline{k} \in N \setminus K$, and from Eqs. 4.4-4.10 and the fact that $a_{l,1} = 1$, it follows that $g_l < 0$ and $h_l > 0$

holds. In the case in which $K \equiv N$ and $|K|$ is an even number, we additionally assume that

$$b_1 - b_2 + \frac{c_1 - c_2}{\lambda(N)} < 0. \tag{4.11}$$

Therefore, we can choose $l \equiv k$ again and, from Eqs. 4.4-4.10 with Eq. 4.11 and the fact that $a_{l,1} \equiv a_{k,1} = -1$, it follows that the inequalities $g_l < 0$ and $h_l > 0$ hold.

By using the definition of $\mathcal{D}$ in Eq. 2.34, we obtain that

$$\pi_j = 1 \quad \text{if } O(j) \subseteq O(l), \tag{4.12}$$

$$\pi_j = \pi_i \quad \text{if } O(j) = O(i) \cup \{r\} \text{ with } r \in O(l), \tag{4.13}$$

holds, for all $\vec{\pi} \in \mathcal{D}$. Therefore, by using the fact that $a_{j,1} = -a_{i,1}$, for $O(i) = O(j) \setminus \{r\}$ with $r \in O(j)$, we get that $\sum_{j=1}^{2^n} a_{j,1}\pi_j = 1$. If we choose j such that $O(j) \subseteq K$, it follows that $f_{k,j}$ is negative and $K \subseteq O(l)$ implies that $\pi_j = 1$. Hence, we conclude from Eqs. 3.8, 3.9 and 4.4-4.5 that $G(\vec{\pi})$ attains its minimum at $\pi_l = 1$, for all $\vec{\pi} \in \mathcal{D}$.

Let us finally note that, in the case when $m = 1$ and the function $f_1(\theta_1, \ldots, \theta_n)$ is defined as above, we can choose $l = 2, 3, \ldots, 2^n$, such that $|O(l)| = 1$, and we will have that $g_l < 0$ and $h_l > 0$ holds, and $G(\vec{\pi})$ attains its minimum at $\pi_l = 1$, for all $\vec{\pi} \in \mathcal{D}$.

4.2 Estimates for the Solution in the Earliest Change-point Case

In order to find estimates for the value function $V_*(\vec{\pi})$ from Eq. 2.35 and the boundary $b_*(\vec{\pi}_{-l})$ from Eq. 3.11, we will use the solution to the ordinary free-boundary problem from (Shiryaev 1978; pages 203-204) (see also (Peskir and Shiryaev 2006; Chapter VI, Section 22.1)). We assume that $m = 1$, the function $f_1(\theta_1, \ldots, \theta_n)$ is given as in Section 4.1 and $b_1 = 1$ in Eq. 2.2. Therefore, the problem in Eq. 2.2 is reduced to finding a stopping time of alarm τ_* with respect to the observable filtration $(\mathcal{F}_t)_{t \geq 0}$, which is as close as possible to the earliest of all considered change-point times.

Let $\varkappa_i = \mu_i \sqrt{\eta_{i,i}}/\nu_i$, for $i \in N$, and define the ordinary differential operator $\mathbb{L}_*$ by

$$\mathbb{L}_* := \lambda(N)\,(1 - \pi_*)\,\frac{d}{d\pi_*} + \frac{\pi_*^2(1 - \pi_*)^2}{2} \sum_{i,j \in N} |\varkappa_i \varkappa_j|\,\frac{d^2}{d\pi_*^2}, \tag{4.14}$$

for $\pi_* \in (0, 1)$. Let us formulate the ordinary free-boundary problem

$$(\mathbb{L}_* V_1)(\pi_*) = -c_1\,\pi_* \quad \text{for} \quad \pi_* \in [0, h), \tag{4.15}$$

$$V_1(h-) = 1 - h \quad (\textit{continuous fit}), \tag{4.16}$$

$$V_1'(h-) = -1 \quad (\textit{smooth fit}), \tag{4.17}$$

$$V_1(\pi_*) < 1 - \pi_* \quad \text{for} \quad \pi_* \in [0, h), \tag{4.18}$$

$$V_1(\pi_*) = 1 - \pi_* \quad \text{for} \quad \pi_* \in (h, 1], \tag{4.19}$$

for some $0 \leq h \leq 1$. It is shown in (Shiryaev 1978; pages 203-204) that there exist a unique concave solution $V_1(\pi_*)$ to the problem in Eqs. 4.15-4.19 with the property that the $V_1'(0+) = 0$ holds, which could be equivalently simplified to $|V_1'(0+)| < \infty$. In particular, the solution is given by

$$V_1(\pi_*) = \begin{cases} 1 - h - \int_{\pi_*}^{h} \psi(x)dx & \text{if } \pi_* \in [0, h), \\ 1 - \pi_* & \text{if } \pi_* \in [h, 1], \end{cases} \tag{4.20}$$

and the constant h is the unique root of the equation

$$\psi(h) = -1, \tag{4.21}$$

which satisfies $h \geq \lambda(N)/(\lambda(N) + c_1)$, where we set

$$\psi(\pi_*) := -\frac{c_1}{\gamma}\, e^{-\lambda(N)\delta(\pi_*)/\gamma} \int_0^{\pi_*} \frac{e^{\delta(x)}}{x(1-x)^2} dx, \tag{4.22}$$

$$\delta(\pi_*) := \log \frac{\pi_*}{1-\pi_*} - \frac{1}{\pi_*}, \quad \gamma := \frac{\sum_{i,j \in N} |\varkappa_i \varkappa_j|}{2}, \tag{4.23}$$

for $\pi_* \in (0, 1)$. By using the fact that $V_1(\pi_*)$ satisfies Eq. 4.19, we obtain

$$(\mathbb{L}_* V_1)(\pi_*) \geq -c_1\, \pi_*, \tag{4.24}$$

for $\pi_* \in (\lambda(N)/(\lambda(N)+c_1), 1]$, and hence, for all $\pi_* \in [0, h) \cup (h, 1]$, since $V_1(\pi_*)$ satisfies Eq. 4.15 and $h \geq \lambda(N)/(\lambda(N) + c_1)$.

Letting $\Pi^* \equiv \Pi^{*,1}$, we obtain from Eqs. 2.6 and 4.1 that

$$\Pi_t^* \equiv P_{\vec{\pi}}(\theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n \leq t \,|\, \mathcal{F}_t) = \sum_{i \in N} \Pi_t^{\{i\}} - \sum_{i,j \in N, i \neq j} \Pi_t^{\{i,j\}}$$
$$+ \sum_{i,j,k \in N, i \neq j, i \neq k, j \neq k} \Pi_t^{\{i,j,k\}} - \ldots + (-1)^{n-2} \sum_{i \in N} \Pi_t^{N \setminus \{i\}} + (-1)^{n-1} \Pi_t^N, \tag{4.25}$$

while applying Itô's formula, by using the expressions in Eqs. 2.30 and 4.1-4.2, we can see that the process Π^* satisfies the stochastic differential equation

$$d\Pi_t^* = \sum_{i \in N} \lambda_i\, (1 - \Pi_t^*)\, dt + \sum_{i \in N} \varkappa_i\, \Pi_t^{\{i\}} (1 - \Pi_t^*)\, d\widehat{B}_t^i, \tag{4.26}$$

for all $t \geq 0$. Therefore, using the fact that the function $V_1(\pi_*)$ satisfies the smooth-fit condition Eqs. 4.17 and 4.19, we can apply the local time-space formula from Peskir (2007) to obtain

$$V_1(\Pi_t^*) = V_1(\Pi_0^*) + \int_0^t V_1'(\Pi_s^*)\; \lambda(N)\, (1 - \Pi_s^*)\, ds + \sum_{i \in N} \int_0^t V_1'(\Pi_s^*)\, \varkappa_i\; \Pi_s^{\{i\}} (1 - \Pi_s^*)\, d\widehat{B}_s^i \tag{4.27}$$
$$+ \frac{1}{2} \int_0^t V_1''(\Pi_s^*) \sum_{i,j \in N} \left(\frac{\varkappa_i \varkappa_j \eta_{i,j}}{\sqrt{\eta_{i,i} \eta_{j,j}}}\, \Pi_s^{\{i\}}\, \Pi_s^{\{j\}} \right) (1 - \Pi_s^*)^2\, I(\Pi_s^* \neq h)\, ds,$$

for all $t \geq 0$. From the expressions in Eqs. 4.18-4.19, by means of the optional sampling theorem, we get that the expression

$$E_{\vec{\pi}}\left[1 - \Pi_\tau^* + c_1 \int_0^\tau \Pi_t^*\, dt\right] \geq E_{\vec{\pi}}\left[V_1(\Pi_\tau^*) + c_1 \int_0^\tau \Pi_t^*\, dt\right] \tag{4.28}$$
$$= V_1(\Pi_0^*) + E_{\vec{\pi}} \int_0^\tau \left(V_1'(\Pi_t^*)\, \lambda(N)\, (1 - \Pi_t^*) + c_1\, \Pi_t^* \right) dt$$
$$+ \frac{1}{2}\, E_{\vec{\pi}} \int_0^\tau V_1''(\Pi_t^*) \sum_{i,j \in N} \left(\frac{\varkappa_i \varkappa_j \eta_{i,j}}{\sqrt{\eta_{i,i} \eta_{j,j}}}\, \Pi_t^{\{i\}}\, \Pi_t^{\{j\}} \right) (1 - \Pi_t^*)^2\, I(\Pi_t^* \neq h)\, dt$$

is satisfied, for any stopping time τ such that $E_{\vec{\pi}}\, \tau < \infty$, and all $\vec{\pi} \in \mathcal{D}$. Since the function $V_1(\pi_*)$ is twice continuously differentiable and concave, we have that the inequality

$V_1''(\pi_*) \le 0$ holds, for $\pi_* \in [0, h) \cup (h, 1]$. From Eq. 4.28 and the fact that the inequalities $-1 < \eta_{i,j}/\sqrt{\eta_{i,i}\eta_{j,j}} < 1$ hold, we therefore obtain

$$E_{\vec{\pi}}\bigg[1 - \Pi_\tau^* + c_1 \int_0^\tau \Pi_t^*\, dt\bigg] \ge V_1(\Pi_0^*) + E_{\vec{\pi}} \int_0^\tau \Big(V_1'(\Pi_t^*)\, \lambda(N)\, (1 - \Pi_t^*) + c_1\, \Pi_t^*\Big)\, dt \tag{4.29}$$

$$+ \frac{1}{2} E_{\vec{\pi}} \int_0^\tau V_1''(\Pi_t^*) \sum_{i,j \in N} \Big(|\varkappa_i \varkappa_j|\, \Pi_t^{\{i\}}\, \Pi_t^{\{j\}}\Big)\, (1 - \Pi_t^*)^2\, I(\Pi_t^* \ne h)\, dt.$$

By using the fact that

$$\Pi_t^{\{i\}} \equiv P_{\vec{p}}(\theta_i \le t \,|\, \mathcal{F}_t) \le P_{\vec{p}}(\theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n \le t \,|\, \mathcal{F}_t) \equiv \Pi_t^* \tag{4.30}$$

holds, for all $t \ge 0$ and any $i \in N$, while the expression in Eq. 4.24 is satisfied, we obtain

$$\begin{aligned} &E_{\vec{\pi}}\bigg[1 - \Pi_\tau^* + c_1 \int_0^\tau \Pi_t^*\, dt\bigg] \qquad (4.31)\\ &\ge V_1(\Pi_0^*) + E_{\vec{\pi}} \int_0^\tau \Big((\mathbb{L}_* V_1)(\Pi_t^*) + c_1 \Pi_t^*\Big)\, I(\Pi_t^* \ne h)\, dt \ge V_1(\Pi_0^*), \end{aligned}$$

for any stopping time τ such that $E_{\vec{\pi}}\, \tau < \infty$, and all $\vec{\pi} \in \mathcal{D}$. Since $\Pi_0^* = \sum_{j=1}^{2^n} a_{j,1}\pi_j$, under the measure $P_{\vec{\pi}}$, by using the expression in Eq. 2.35, we have

$$V_*(\vec{\pi}) \equiv \inf_\tau E_{\vec{\pi}}\bigg[1 - \Pi_\tau^* + c_1 \int_0^\tau \Pi_t^*\, dt\bigg] \ge V_1\bigg(\sum_{j=1}^{2^n} a_{j,1}\, \pi_j\bigg), \tag{4.32}$$

for $\vec{\pi} \in \mathcal{D}$.

By using the results from Section 4.1 in the case $m = 1$, we can choose $l = 1, \ldots, 2^n$, where $O(l) = \{r\}$, for some $r \in N$, and apply Lemma 3.1 to obtain that the optimal stopping time τ_* is of the form Eq. 3.11. Therefore, by using the fact that Π^* is of the form Eq. 4.25, we have that $a_{l,1} = 1$, and hence, the optimal stopping time τ_* is of the form

$$\tau_* = \inf\big\{t \ge 0 \,\big|\, \Pi_t^* \ge g_1^*(\vec{\Pi}_t)\big\}, \tag{4.33}$$

with $g_1^*(\vec{\pi})$ given by

$$g_1^*(\vec{\pi}) = b_*(\vec{\pi}_{-l}) + \sum_{j=1}^{2^n} a_{j,1}\, \pi_j - \pi_l, \tag{4.34}$$

for $\vec{\pi} \in \mathcal{D}$. Moreover, from the expressions in Eq. 3.24 and Eqs. 4.6-4.10, we obtain that

$$b_*(\vec{\pi}_{-l}) \ge \overline{b}_*(\vec{\pi}_{-l}) = \pi_l - \sum_{j=1}^{2^n} a_{j,1}\, \pi_j + \frac{\lambda(N)}{\lambda(N) + c_1}, \tag{4.35}$$

and it follows that $0 < \lambda(N)/(\lambda(N) + c_1) \le g_1^*(\vec{\pi})$ for $\vec{\pi} \in \mathcal{D}$.

We can also deduce from Theorem 3.1 that the function $\overline{V}_*(\vec{\pi})$ defined in Eq. 3.6 satisfies the conditions of Eqs. 3.29-3.30, and therefore, by using the expression in Eq. 4.34, we have that $V_*(\vec{\pi}) < 1 - \sum_{j=1}^{2^n} a_{j,1}\pi_j$ holds, for all $\vec{\pi} \in \mathcal{D}$ such that $0 \le \sum_{j=1}^{2^n} a_{j,1}\pi_j < g_1^*(\vec{\pi})$. Since $V_1(\pi_*)$ satisfies the conditions of Eqs. 4.18-4.19, it follows from Eq. 4.32 that $g_1^*(\vec{\pi}) \le$

h, and we also get from Eq. 4.34 that

$$b_*(\vec{\pi}_{-l}) \leq h + \pi_l - \sum_{j=1}^{2^n} a_{j,1} \pi_j, \tag{4.36}$$

for $\vec{\pi} \in \mathcal{D}$.

Summarising the facts proved above, we are now ready to state the main result of this section.

Corollary 4.1 *Suppose that the assumptions of* Theorem 3.1 *hold. Assume that the function* $V_1(\pi_*)$ *is concave and, together with the constant* $h \in [0, 1]$*, solves the ordinary free-boundary problem in* Eqs. 4.15-4.19*. Then, we have that the lower bound in* Eq. 4.32 *holds for the value function* $V_*(\vec{\pi})$ *from* Eq. 2.35*, while the upper bound in* Eq. 4.36 *holds for the boundary* $b_*(\vec{\pi}_{-l})$ *from* Eq. 3.11*. Moreover, the optimal stopping time in* Eq. 2.35 *can be written in the form of* Eq. 4.33*, where the optimal boundary* $g_1^*(\vec{\pi})$ *is such that* $0 < \lambda(N)/(\lambda(N) + c_1) \leq g_1^*(\vec{\pi}) \leq h \leq 1$*, for* $\vec{\pi} \in \mathcal{D}$*.*

Appendix

A.1 Proof of Lemma 2.1

Define the n-dimensional row vector $\mu^J = (\mu_1^J, \dots, \mu_n^J)$ and the row process $\overline{X} = (\overline{X}^1, \dots, \overline{X}^n)$ as

$$\mu_i^J = \frac{\mu_i}{\nu_i} \quad \text{for} \quad i \in J, \quad \mu_i^J = 0 \quad \text{for} \quad i \in N \setminus J, \quad \overline{X}_t^i = \frac{X_t^i}{\nu_i} \quad \text{for} \quad i \in N, \tag{A.1}$$

for $t \geq 0$. From the definition of X in Eq. 2.1, under the measure $P^{\varnothing}$, we have

$$\frac{X_t^i}{\nu_i} = B_t^i \quad \text{for} \quad i \in N, \tag{A.2}$$

and under the measure P^J we have

$$\frac{X_t^i}{\nu_i} = \frac{\mu_i}{\nu_i} t + B_t^i \quad \text{for} \quad i \in J, \quad \frac{X_t^i}{\nu_i} = B_t^i \quad \text{for} \quad i \in N \setminus J, \tag{A.3}$$

for $t \geq 0$. Therefore, by the Girsanov theorem for n-dimensional Brownian motion (see, e.g. (Liptser and Shiryaev 2001; Chapter VI, Theorem 6.4)), we conclude that the weighted density process Z^J satisfies

$$Z_t^J = \exp\left(\sum_{i \in J} \lambda_i t\right) \frac{d(P^J | \mathcal{F}_t)}{d(P^{\varnothing} | \mathcal{F}_t)} = \exp\left(\sum_{i \in J} \lambda_i t + \mu^J \Sigma^{-1} (\overline{X}_t)^T - \frac{1}{2} \mu^J \Sigma^{-1} (\mu^J)^T t\right) \tag{A.4}$$

$$= \exp\left(\sum_{i \in J} \lambda_i t + \sum_{i \in J} \frac{\mu_i}{\nu_i} \sum_{j=1}^n \frac{\eta_{i,j}}{\nu_j} X_t^j - \frac{1}{2} \sum_{i,j \in J} \frac{\mu_i \mu_j}{\nu_i \nu_j} \eta_{l,j} t\right)$$

$$= \exp\left(\sum_{i \in J} \left(\lambda_i t + Y_t^i\right) - \frac{1}{2} \sum_{i,j \in J} \frac{\mu_i \mu_j}{\nu_i \nu_j} \eta_{l,j} t\right),$$

for $t \geq 0$, where the processes Y^i are defined as in Eq. 2.10, for $i \in N$, and $(\cdot)^T$ denotes the vector transpose. □

A.2 Sufficient Statistics in the Case of Exponential Delay Penalty Costs

We describe here the sufficient statistics and their corresponding stochastic differential (filtering) equations in the case of exponential delay penalty costs. We are interested in detecting the so-called k^{th}-to-default event, which is a generalisation of the earliest and the latest of all disorder times. Specifically, keeping the notation from Section 2, let $m = 1$ and let the Bayesian risk function from Eq. 2.2 be of the form

$$V_*(\vec{p}) = \inf_\tau \Big(b_1 \, P_{\vec{p}}\big(\tau < f_1(\theta_1, \dots, \theta_n)\big) + c_1 \, E_{\vec{p}}\big[e^{\beta(\tau - f_1(\theta_1,\dots,\theta_n))^+} - 1\big]\Big), \tag{A.5}$$

where $\beta > 0$ and the function $f_1(\theta_1, \dots, \theta_n)$ is equal to the k-th element θ_{i_k} in the ordering $\theta_{i_1} \leq \theta_{i_2} \leq \dots \leq \theta_{i_n}$ of the elements of $(\theta_1, \dots, \theta_n)$, that is, it is given by

$$f_1(\theta_1, \dots, \theta_n) = \bigwedge_{J \subseteq N, |J|=k} \bigvee_{j \in J} \theta_j, \tag{A.6}$$

for some $k \in N$. The term $E_{\vec{p}}[e^{\beta(\tau - f_1(\theta_1,\dots,\theta_n))^+} - 1]$ represents the average *exponential* delay of detecting the function $f_1(\theta_1, \dots, \theta_n)$. We also note that

$$E_{\vec{\pi}}\big[e^{\beta(\tau - f_1(\theta_1,\dots,\theta_n))^+} - 1\big] = E_{\vec{\pi}} \int_0^\infty I(f_1(\theta_1, \dots, \theta_n) \leq t \leq \tau)\, \beta \, e^{\beta(t - f_1(\theta_1,\dots,\theta_n))} \, dt \tag{A.7}$$

$$= E_{\vec{\pi}} \int_0^\infty E_{\vec{\pi}}\big[I\big(f_1(\theta_1, \dots, \theta_n) \leq t \leq \tau\big)\, \beta \, e^{\beta(t - f_1(\theta_1,\dots,\theta_n))} \,\big|\, \mathcal{F}_t\big] \, dt$$

$$= E_{\vec{\pi}} \int_0^\tau \beta E_{\vec{\pi}}\big[I\big(f_1(\theta_1, \dots, \theta_n) \leq t\big)\, e^{\beta(t - f_1(\theta_1,\dots,\theta_n))} \,\big|\, \mathcal{F}_t\big] \, dt.$$

In order to reduce the problem in Eq. A.5 to an optimal stopping problem for a multidimensional Markov process, we define the process $\widetilde{\Pi}^{*,1} = (\widetilde{\Pi}^{*,1}_t)_{t \geq 0}$ by

$$\widetilde{\Pi}^{*,1}_t := E_{\vec{\pi}}[I(f_1(\theta_1, \dots, \theta_n) \leq t)\, e^{\beta(t - f_1(\theta_1,\dots,\theta_n))} \,|\, \mathcal{F}_t], \tag{A.8}$$

for $t \geq 0$. Hence, from Eqs. 2.3 and A.7, it follows that the Bayesian risk function in Eq. A.5 can be written as

$$V_*(\vec{\pi}) = \inf_\tau E_{\vec{\pi}}\bigg[b_1 \, (1 - \Pi^{*,1}_\tau) + c_1 \int_0^\tau \beta \, \widetilde{\Pi}^{*,1}_t \, dt\bigg]. \tag{A.9}$$

Let us also define the posterior probability process $\widetilde{\Pi}^J = (\widetilde{\Pi}^J_t)_{t \geq 0}$ by

$$\widetilde{\Pi}^J_t := E_{\vec{\pi}}\bigg[I\bigg(\bigcap_{i \in J} \{\theta_i \leq t\}\bigg) \, e^{\beta(t - f_1(\theta_1,\dots,\theta_n))^+} \,\bigg|\, \mathcal{F}_t\bigg], \tag{A.10}$$

for $t \geq 0$ and $J \subseteq N$, and denote by $\widetilde{\Pi} = (\widetilde{\Pi}^1, \dots, \widetilde{\Pi}^{2^n})$ the 2^n-dimensional process with components given by $\widetilde{\Pi}^j = \widetilde{\Pi}^{O(j)}$, for $j = 1, \dots, 2^n$. Note that, by the inclusion-exclusion principle, we have that

$$I\big(f_1(\theta_1, \dots, \theta_n) \leq t\big) = \sum_{i=k}^n (-1)^{i-k} \frac{(i-1)!}{(k-1)!(i-k)!} \sum_{J \subseteq N, |J|=i} I\big(\textstyle\bigcap_{j \in J} \{\theta_j \leq t\}\big), \tag{A.11}$$

and therefore, the representation in Eq. 2.6 is satisfied and the process $\widetilde{\Pi}^{*,1} = (\widetilde{\Pi}^{*,1}_t)_{t\geq 0}$ is of the form

$$\widetilde{\Pi}^{*,1}_t \equiv E_{\vec{\pi}}\big[I\big(f_1(\theta_1,\ldots,\theta_n) \leq t\big)\, e^{\beta(t-f_1(\theta_1,\ldots,\theta_n))}\,\big|\,\mathcal{F}_t\big] = \sum_{j=1}^{2^n} a_{j,1}\,\widetilde{\Pi}^j_t, \tag{A.12}$$

where we have

$$a_{j,1} = (-1)^{i-k}\,\frac{(i-1)!}{(k-1)!(i-k)!} \quad \text{for} \quad k = 1,\ldots,|O(j)| = i, \quad a_{j,1} = 0 \quad \text{otherwise}, \tag{A.13}$$

for $j = 1,\ldots,2^n$. Moreover, by using the fact that the equalities

$$\begin{aligned} &I\big(\textstyle\bigcap_{i\in J}\{\theta_i \leq t\} \bigcap \{f_1(\theta_1,\ldots,\theta_n) \leq t\}\big) \\ &= \sum_{i=k}^{n}(-1)^{i-k}\,\frac{(i-1)!}{(k-1)!(i-k)!} \sum_{L\subseteq N,|L|=i} I\big(\textstyle\bigcap_{j\in L\cup J}\{\theta_j \leq t\}\big), \end{aligned} \tag{A.14}$$

$$\begin{aligned} &I\big(\textstyle\bigcap_{i\in J}\{\theta_i \leq t\}\big)\, e^{\beta(t-f_1(\theta_1,\ldots,\theta_n))^+} \\ &= I\big(\textstyle\bigcap_{i\in J}\{\theta_i \leq t\} \bigcap \{f_1(\theta_1,\ldots,\theta_n) \leq t\}\big) e^{\beta(t-f_1(\theta_1,\ldots,\theta_n))} \\ &\quad + \big(1 - I(f_1(\theta_1,\ldots,\theta_n) \leq t)\big)\, I\big(\textstyle\bigcap_{i\in J}\{\theta_i \leq t\}\big), \end{aligned} \tag{A.15}$$

hold, we get that

$$\widetilde{\Pi}^J_t = \Pi^J_t + \sum_{i=k}^{n}(-1)^{i-k}\,\frac{(i-1)!}{(k-1)!(i-k)!} \sum_{L\subseteq N,|L|=i} \big(\widetilde{\Pi}^{J\cup L}_t - \Pi^{J\cup L}_t\big), \tag{A.16}$$

for $t \geq 0$ and $J \subseteq N$. It follows that, for any $J \subseteq N$ such that $|J| < k$, the process $\widetilde{\Pi}^J$ can be written as a linear combination of the processes Π^J, $\Pi^{J\cup L}$ and $\widetilde{\Pi}^{J\cup L}$, where $L \subseteq N$ and $|J \cup L| \geq k$. Therefore, we only need to obtain the stochastic differential equations satisfied by the processes $\widetilde{\Pi}^J$, for all $J \subseteq N$ such that $|J| \geq k$.

For any $R, L \subseteq N$ such that $R \neq \varnothing$, $R \cap L = \varnothing$ and any permutation $\alpha := [\alpha_1,\ldots,\alpha_r] \in \mathrm{Perm}(R)$, we define the process $(\widetilde{\Phi}^{\alpha,L}_t)_{t\geq 0}$ recursively by

$$\widetilde{\Phi}^{\alpha,L}_t := \lambda_{\alpha_r} \int_0^t \widetilde{\Phi}^{[\alpha_1,\ldots,\alpha_{r-1}],L}_u\, \frac{Z^{R\cup L}_t e^{\beta t}}{Z^{R\cup L}_u e^{\beta u}}\, du \quad \text{for} \quad |R \cup L| \geq k, \tag{A.17}$$

$$\widetilde{\Phi}^{\alpha,L}_t := \Phi^{\alpha,L}_t \quad \text{for} \quad |R \cup L| < k, \quad \widetilde{\Phi}^{\varnothing,L}_t := \pi^L\, e^{\beta t} Z^L_t \quad \text{for} \quad |L| \geq k, \tag{A.18}$$

where Z^L and $\Phi^{\alpha,L}$ are given by Eqs. 2.7 and 2.11. By analogy with the arguments in Section 2 above, from the generalised Bayes formula in (Liptser and Shiryaev 2001; Chapter VII, Theorem 7.23), we obtain that the posterior probability process $(\widetilde{\Pi}^J_t)_{t\geq 0}$ takes the form

$$\widetilde{\Pi}^J_t = \frac{\widetilde{\Psi}^J_t}{\Psi^{\varnothing}_t}, \tag{A.19}$$

for $t \geq 0$, where we set

$$\widetilde{\Psi}^J_t := \sum_{\substack{L_1\subseteq N\setminus J \\ L_2\subseteq J}} \sum_{\substack{R\supseteq J\setminus L_2 \\ R\subseteq N\setminus(L_1\cup L_2)}} \sum_{\alpha\in\mathrm{Perm}(R)} \widetilde{\Phi}^{\alpha,L_1\cup L_2}_t, \tag{A.20}$$

for $J \subseteq N$ and $\Psi^{\varnothing}$ as in Eq. 2.16. By using Itô's formula, from Eqs. 2.18 and A.17, we get

$$d\widetilde{\Phi}_t^{\alpha,L} = \left(\lambda_{\alpha_r}\widetilde{\Phi}_t^{[\alpha_1,\ldots,\alpha_{r-1}],L} + \left(\beta + \sum_{i\in R\cup L}\lambda_i\right)\widetilde{\Phi}_t^{\alpha,L}\right)dt + \sum_{i\in R\cup L}\widetilde{\Phi}_t^{\alpha,L}\,dY_t^i, \tag{A.21}$$

for all $t \geq 0$ and $R, L \subseteq N$ such that $R \neq \varnothing$, $R \cap L = \varnothing$ and $|R \cup L| \geq k$, and any $\alpha := [\alpha_1, \ldots, \alpha_r] \in \mathrm{Perm}(R)$. We also obtain from Eq. A.18 that

$$d\widetilde{\Phi}_t^{\varnothing,L} = \left(\beta + \sum_{i\in L}\lambda_i\right)\widetilde{\Phi}_t^{\varnothing,L}\,dt + \widetilde{\Phi}_t^{\varnothing,L}\sum_{i\in L}dY_t^i \tag{A.22}$$

holds, for $t \geq 0$ and $L \subseteq N$ such that $|L| \geq k$. Therefore, by using the expression in Eq. A.20 and summing up the related expressions, we further obtain

$$d\widetilde{\Psi}_t^J = \left(\sum_{i\in J}\lambda_i\widetilde{\Psi}_t^{J\setminus\{i\}} + \left(\beta + \sum_{i\notin J}\lambda_i\right)\widetilde{\Psi}_t^J\right)dt + \sum_{i\in J}\widetilde{\Psi}_t^J\,dY_t^i + \sum_{i\notin J}\widetilde{\Psi}_t^{J\cup\{i\}}\,dY_t^i, \tag{A.23}$$

for $t \geq 0$. Hence, by applying Itô's formula to the expression in Eq. A.19 and arguments similar to the ones used in Section 2, we conclude that

$$d\widetilde{\Pi}_t^J = \left(\sum_{i\in J}\lambda_i\widetilde{\Pi}_t^{J\setminus\{i\}} + \left(\beta - \sum_{i\in J}\lambda_i\right)\widetilde{\Pi}_t^J\right)dt + \sum_{i\in N}\left(\widetilde{\Pi}_t^{J\cup\{i\}} - \widetilde{\Pi}_t^J\,\Pi_t^{\{i\}}\right)\frac{\mu_i}{\nu_i}\sqrt{\eta_{i,i}}\,d\widehat{B}_t^i, \tag{A.24}$$

for $t \geq 0$ and $J \subseteq N$ such that $|J| \geq k$. It follows that $(\vec{\Pi}, \widetilde{\Pi})$ is a (time-homogeneous strong) Markov process, even after removing all the components $\widetilde{\Pi}^J$, where $J \subseteq N$ and $|J| < k$.

Finally, by using the expressions in Eqs. A.9, 2.6 and A.12, we can reduce the problem of Eq. A.5 to the optimal stopping problem

$$V_*(\vec{\pi}) = \inf_\tau E_{\vec{\pi}}\left[b_1\left(1 - \sum_{i=1}^{2^n}a_{i,1}\,\Pi_\tau^i\right) + c_1\int_0^\tau\sum_{i=1}^{2^n}a_{i,1}\,\widetilde{\Pi}_t^i\,dt\right]. \tag{A.25}$$

Here, the processes $\vec{\Pi}$ and $\widetilde{\Pi}$ start at the same $\vec{\pi} \in \mathcal{D}$ under the probability measure $P_{\vec{\pi}}$.

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