The square of a Hamilton cycle in randomly perturbed graphs

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Abstract: We investigate the appearance of the square of a Hamilton cycle in the model of randomly perturbed graphs, which is, for a given $\alpha \in (0,1)$, the union of any n-vertex graph with minimum degree αn and the binomial random graph $G(n,p)$. This is known when $\alpha > 1/2$, and we determine the exact perturbed threshold probability in all the remaining cases, i.e., for each $\alpha \leq 1/2$. Our result has implications on the perturbed threshold for 2-universality, where we also fully address all open cases.

1 Introduction

Our goal is to completely settle the question when the square of a Hamilton cycle appears in randomly perturbed graphs. Given a graph H and a non-negative integer $m \in \mathbb{N}$, the m-th power H^m of H is the graph on vertex set $V(H)$ in which two vertices are adjacent if and only if their distance in H is at most m. As randomly perturbed graphs interpolate between random graph theory and extremal graph theory, before stating our results, we recall what is already known in these two fields on the containment problem for C_n^m, the m-th power of a cycle on n vertices.

We start with the binomial random graph $G(n,p)$. Since the expected number of copies of C_n^m in $G(n,p)$ is $\frac{1}{2}(n-1)!p^{nm}$, the threshold for the appearance of a copy of C_n^m is at least $n^{-1/m}$. For $m = 1$, a famous result by Pósa [**16**] shows that the threshold for the containment of a Hamilton cycle is $n^{-1}\log n$. For $m \geq 3$ a more general result of Riordan [**17**], that is proved using the second moment method, gives that $n^{-1/m}$ is indeed the threshold. The case of the square is more subtle: applications of the second moment method for $p = \Theta(n^{-1/2})$ were not successful and variants of the absorption technique only gave the threshold within a polylog-term. It was only recently proved by Kahn, Narayanan, and Park [**12**] that also in this case the lower bound from above is the truth.

Let us now turn to minimum degree conditions in dense graph. Given $0 \leq \alpha \leq 1$, let G_α be any n-vertex graph with minimum degree $\delta(G_\alpha) \geq \alpha n$. For fixed $m \in \mathbb{N}$, we are interested in conditions on α that guarantee the containment of C_n^m in any such graph G_α. The case $m = 1$ is Dirac's theorem [**8**]: $\alpha \geq 1/2$ suffices and is best possible. For larger values of m, it was conjectured by Pósa that $\alpha \geq 2/3$ suffices when $m = 2$, and this conjecture was further generalised by Seymour to all m with $\alpha \geq \frac{m}{m+1}$. The conjecture is tight and was solved for all m by Komlós, Sarközy, and Szemerédi [**13**] for large enough n (depending on m).

One question that recently obtained quite some attention is how many random edges need to be added to any graph G_α with $\alpha \geq \frac{m}{m+1}$ to guarantee the containment of an even larger power of a Hamilton cycle. We formalise this question and state related results for the model of randomly perturbed graphs, which was introduced by Bohman, Frieze, and Martin [**5**] and allows to investigate how containment properties change if random edges are added. A *randomly perturbed graph* $G_\alpha \cup G(n,p)$ is the graph obtained by adding to a deterministic graph G_α on n vertices with minimum degree at least αn a random graph graph $G(n,p)$ on the same vertex set.

Definition 1 (perturbed threshold) Let $m > 0$ be an integer and let $\alpha \in (0,1)$. The *perturbed threshold* for the containment of the m-th power of a Hamilton cycle is $\hat{p} = \hat{p}(n,\alpha,m)$ if there exist constants C and c (depending on m and α) such that the following holds. For any $p \geq C\hat{p}$ and for any sequence of graphs G_n with $\delta(G_n) \geq \alpha n$ we have $\lim_{n\to\infty} \mathbb{P}\big(C_n^m \subseteq G_n \cup G(n,p)\big) = 1$, and for any $p \leq c\hat{p}$ there exists a sequence of graphs G_n with $\delta(G_n) \geq \alpha n$ such that $\lim_{n\to\infty} \mathbb{P}\big(C_n^m \subseteq G_n \cup G(n,p)\big) = 0$.

Bohman, Frieze, and Martin studied when Hamilton cycles appear in randomly perturbed graphs. They showed in [**5**] that for any $\alpha \in (0,1/2)$, there is a constant C such that a.a.s. for any n-vertex graph G_α, the perturbed graph $G_\alpha \cup G(n,p)$ is Hamiltonian, provided $p \geq C/n$. Moreover, this condition on p is optimal as the graph $K_{\alpha n,(1-\alpha)n}$ has minimum degree αn and misses a linear number of edges to be Hamiltonian. Therefore, using the notation of Definition 1, they showed that $\hat{p}(n,\alpha,1) = n^{-1}$ for any $\alpha \in (0,1/2)$. For higher powers of Hamilton cycles, one of the first results was obtained in [**6**], which showed that for any $\alpha \in (0,1)$ there exists $\eta > 0$ such that $\hat{p}(n,\alpha,m) \leq n^{-1/m-\eta}$, and asked for the optimal η.

In the range $\alpha \in (1/2,2/3)$, Bennett, Dudek, and Frieze [**4**] showed that $\hat{p}(n,\alpha,2) \leq n^{-2/3}(\log n)^{1/3}$. This was improved and generalised by Dudek, Reiher, Ruciński, and Schacht [**9**]. They showed that for $\alpha \in (\frac{m}{m+1},\frac{m+1}{m+2})$, not only G_α contains the m-th power of a Hamilton cycle, but adding a linear number of random edges suffices to get the $(m+1)$-st power, that is, $\hat{p}(n,\alpha,m+1) = n^{-1}$. Nenadov and Trujić [**14**] then proved that in fact, with α in the same range, this suffices for the $(2m+1)$-st power and thus $\hat{p}(n,\alpha,2m+1) = n^{-1}$. They also conjectured that $\hat{p}(n,\frac{m}{m+1},2m+1) = n^{-1}\log n$ for $\alpha = \frac{m}{m+1}$. When $\alpha > 1/2$, even higher powers have been studied by Antoniuk, Dudek, Reiher, Ruciński, and Schacht [**3**], who proved that in many cases the threshold is guided by the largest clique required from $G(n,p)$.

Observe that the exact results obtained so far all deal with the range $\alpha \in (1/2,1)$ and already [**3**] asked for similar exact results for the case $\alpha \in (0,1/2]$ and, in particular, for $m = 2$. We give the first optimal results on the perturbed threshold of the square of a Hamilton cycles for $\alpha \in (0,1/2]$, answering the questions from [**3**] in a strong from.

Theorem 2 *We have*

$$\hat{p}(n,\alpha,2) = \begin{cases} n^{-1} & \text{for } \alpha \in (\frac{1}{3},\frac{2}{3}), \\ n^{-1}\log n & \text{for } \alpha = \frac{1}{3}. \end{cases}$$

Note that our result allows $\alpha \in (1/2,1/3)$, but this was already covered in [**9**]. Theorem 2 has immediate consequences for the *2-universality* of randomly perturbed graphs, that is, the containment of all graphs of maximum degree two. Indeed, it is easy to see that the square of the Hamilton cycle on n vertices contains each n-vertex graph with maximum degree two as a subgraph. This significantly strengthens one of our results from [**7**] on the containment of a triangle factor under the same conditions, and is optimal (see the discussion after Theorems 1.3 and 2.2 in [**7**]). The threshold for 2-universality in randomly perturbed graphs was studied in [**15**], which showed that for $\alpha \in (0, 1/3)$ the perturbed threshold is $n^{-2/3}$. In $G(n,p)$ alone, Ferber, Kronenberg, and Luh [**10**] showed that the threshold is $n^{-2/3}(\log n)^{1/3}$. Moreover, Aigner and Brandt [**1**] showed that for $\alpha \geq 2/3$, the graph G_α is already 2-universal. Thus, our Theorem 2, together with these results, fully solves the question for 2-universality in randomly perturbed graphs.

When α gets smaller than 1/3, the thresholds for the square of a Hamilton cycle and that for 2-universality behave differently, as in the former case we need to increase the probability to ensure that we can find many copies of the square of a short path (see also the beginning of Section 2). However, we are still able to determine precisely the perturbed threshold for the square of a Hamilton cycle for all remaining α.

Theorem 3 *For any integer $k \geq 2$ we have*

$$\hat{p}(n, \alpha, 2) = \begin{cases} n^{-(k-1)/(2k-3)} & \text{for } \alpha \in (\frac{1}{k+1}, \frac{1}{k}), \\ n^{-(k-1)/(2k-3)}(\log n)^{1/(2k-3)} & \text{for } \alpha = \frac{1}{k+1}. \end{cases}$$

Observe that Theorem 2 is a direct consequence of Theorem 3 and [**9**]. In the next section we provide an overview of the proof of this result and also explain what is the intuition behind the probabilities appearing there. Our theorem shows that the perturbed threshold $\hat{p}(n, \alpha, 2)$, regarded as a function of α, exhibits countable many jumps at $\alpha = 2/3$ and $\alpha = 1/k$ for each integer $k \geq 2$. Moreover for α tending to zero (i.e. for k tending to infinity), $\hat{p}(n, \alpha, 2)$ tends to $n^{-1/2}$, which is precisely the threshold for the square of a Hamilton cycle in $G(n,p)$ alone as discussed above.

It would be interesting to investigate larger powers of Hamilton cycles for $\alpha \leq 1/2$. A natural candidate to start with is the third power of a Hamilton cycle, for $\alpha \geq 1/4$ and $p \geq Cn^{-1/2}$. However, this seems to be a more challenging problem, as it requires working with the square of a Hamilton cycle in $G(n,p)$ at the threshold of appearance.

2 Proof overview

In this section we will sketch the proof of Theorem 3. We start with some notation, discuss the idea of our embedding strategy and explain how this leads to the threshold probabilities given in Theorem 3. We then turn to the arguments for the lower bound on $\hat{p}(n, \alpha, 2)$, and afterwards split the upper bound into two theorems depending on the structure of the dense graph G_α.

Let F be the square of a path P_k^2 with vertices $v_1, v_2, \ldots, v_k$ and edges $v_i v_j$, $1 \leq |i-j| \leq 2$. We call (v_1, v_2) the *start-tuple* of F and (v_{k-1}, v_k) the *end-tuple* of F. We also refer to v_i as the i-th vertex of F. Given $k \geq 2$, $\alpha, p \in [0,1]$, and any n-vertex graph G with minimum degree αn, we want to find the square of a Hamilton cycle C_n^2 in the graph $G \cup G(n,p)$. We now describe a *decomposition* of the edges of the square of a long path or a cycle into deterministic edges (to be embedded to G) and random edges (to be embedded to $G(n,p)$) that we will use in our proof(s). We would like vertex disjoint copies $F_1, \ldots, F_t$ of the square of a path on k vertices P_k^2 in the random graph $G(n,p)$ such that the following holds. For each $i = 1, \ldots, t-1$, if we denote by (x_i, y_i) and (u_i, w_i) the start-tuple and end-tuple of F_i, then $w_i x_{i+1}$ is also an edge in $G(n,p)$.

Moreover, there exist $t-1$ additional vertices $v_1, \ldots, v_{t-1}$ such that, for $i = 1, \ldots, t-1$, all four edges $v_iu_i, v_iw_i, v_ix_{i+1}, v_iy_{i+1}$ are edges in G. This gives the square of a path on $t(k+1)-1$ vertices with edges of $G \cup G(n,p)$. Note that by requiring the edge w_tx_1 from $G(n,p)$ and adding another vertex v_t joined to u_t, w_t, x_1, y_1 in G, we get the square of a cycle on $t(k+1)$ vertices. In order to find C_n^2 and for some technical reasons, our proof(s) will allow some of $F_1, \ldots, F_t$ to be the squares of paths of different lengths.

This decomposition already justifies the probabilities that appear in Theorem 3. Indeed, $n^{-(k-1)/(2k-3)}$ is the threshold in $G(n,p)$ for a linear number of copies of P_k^2 (by a standard application of Janson's inequality), while $n^{-(k-1)/(2k-3)}(\log n)^{1/(2k-3)}$ is the threshold in $G(n,p)$ for the existence of a P_k^2-factor (this follows from a general result of Johannson, Kahn, and Vu [**11**]).

2.1 Lower bounds

For any $\alpha \in (0, 1/2)$, let H_α be the complete bipartite n-vertex graph with vertex classes A and B of size αn and $(1-\alpha)n$, respectively. We start with a sketch for the lower bound on $\hat{p}(n, \alpha, 2)$ for $\alpha \in (\frac{1}{k+1}, \frac{1}{k})$. We want to argue that for some constant $c \in (0,1)$ depending on α and $p \leq cn^{-(k-1)/(2k-3)}$ a.a.s. $H_\alpha \cup G(n,p)$ does not contain C_n^2. In B there are a.a.s. at most $2cn$ copies of P_k^2 (by an upper tail bound on the distribution of small subgraphs [**18**]) and at most $o(n)$ copies of P_{k+1}^2 (by the first moment method). On the other hand, in any embedding of C_n^2 into $H_\alpha \cup G(n,p)$, an α-fraction of the vertices is mapped into A and, because of the bound on the number of P_{k+1}^2 in B, two such vertices can only rarely be of distance more than $k+1$. From this it follows that there are at least $\frac{1-\alpha k}{2}n$ copies of P_k^2 in B, which is a contradiction if $c < \frac{1-\alpha k}{4}$.

We argue similarly for the lower bound of $\hat{p}(n, \frac{1}{k+1}, 2)$. We show that with $p \leq \frac{1}{2}n^{-(k-1)/(2k-3)}(\log n)^{1/(2k-3)}$ a.a.s. $H_{1/(k+1)} \cup G(n,p)$ does not contain C_n^2. Indeed, in this regime with $c = \frac{1}{2k}$, a.a.s. (by the first moment method) at least n^{1-2c} vertices from B are not contained in any copy of P_k^2 within B and B contains at most n^{1-c} copies of P_{k+1}^2. Therefore, in any embedding of C_n^2, the distance between two vertices mapped into A can only n^{1-c} often be larger thank $k+1$, but exactly one in $k+1$ vertices is mapped into A. This implies that all but n^{1-c} vertices from B are contained in a copy of P_k^2 within B, which gives a contradiction.

2.2 Stability

In turns out that the additional $(\log n)^{1/(2k-3)}$-term in $\hat{p}(n, \frac{1}{k+1}, 2)$ is only necessary if the deterministic graph G is really *close* to $H_{1/(k+1)}$. The next definition formalises what we mean by *close*.

Definition 4 For $\alpha, \beta > 0$ we say that an n-vertex graph G is (α, β)-*stable* if there exists a partition of $V(G)$ into two sets A and B of size $|A| = (\alpha \pm \beta)n$ and $|B| = (1-\alpha \pm \beta)n$ such that the minimum degree of the bipartite subgraph $G[A,B]$ of G induced by A and B is at least $\frac{1}{4}\alpha n$, all but at most βn vertices from A have degree at least $|B| - \beta n$ into B, all but at most βn vertices from B have degree at least $|A| - \beta n$ into A, and $G[B]$ contains at most βn^2 edges.

We can prove the following stability version for the upper bound on $\hat{p}(n, \frac{1}{k+1}, 2)$ in Theorem 3.

Theorem 5 *For every $k \geq 2$ and every $0 < \beta < \frac{1}{6k}$, there exists $\gamma > 0$ and $C > 0$ such that the following holds. Let G be any n-vertex graph with minimum degree at least $(\frac{1}{k+1} - \gamma)n$ that is not $(\frac{1}{k+1}, \beta)$-stable. Then a.a.s. $G \cup G(n,p)$ contains the square of a Hamilton cycle, provided that $p \geq Cn^{-(k-1)/(2k-3)}$.*

Only when the graph G is stable we need the $(\log n)^{1/(2k-3)}$-term. This case is treated by the following theorem.

Theorem 6 *For every $k \geq 2$ there exists $\beta > 0$ and $C > 0$ such that the following holds. Let G be any n-vertex graph with minimum degree at least $\frac{1}{k+1}n$ that is $(\frac{1}{k+1}, \beta)$-stable. Then a.a.s. $G \cup G(n,p)$ contains the square of a Hamilton cycle, provided that $p \geq Cn^{-(k-1)/(2k-3)}(\log n)^{1/(2k-3)}$.*

We sketch the ideas for the proof of these two theorems in the following two subsections. Together with the lower bounds, Theorem 5 and 6 imply Theorem 3.

2.3 Extremal case

We now sketch the proof of Theorem 6. Suppose that G is an n-vertex $(\frac{1}{k+1}, \beta)$-stable graph, and let $p \geq C(\log n)^{1/(2k-3)}n^{-(k-1)/(2k-3)}$. From the stability we get a partition $A \cup B$ of $V(G)$ as in Definition 4. We would like to embed copies F_i of P_k^2 into B and vertices v_i into A, as described in the decomposition above. However this is only possible if $|B| = k|A|$ and, therefore, we first embed squares of short paths of different lengths to ensure this is the case. Moreover, we cover similarly all vertices in A and B that do not have high degree to the other part. Then we cover the remaining vertices in B with copies of P_k^2, which is possible by [**11**] with our p. We let $\mathcal{F}$ be the set of the copies of squares of paths that we obtain during these steps and for each $F \in \mathcal{F}$, denote its start-tuple by (x_F, y_F) and its end-tuple by (u_F, w_F).

To turn this into an embedding of the square of a Hamilton cycle, we now reveal additional edges of $G(n,p)$ and encode this in an auxiliary directed graph $\mathcal{T}$ on vertex set $\mathcal{F}$ as follows. There is a directed edge (F, F') if and only if the edge $w_F x_{F'}$ appears in $G(n,p)$. It is easy to see that all directed edges in $\mathcal{T}$ are revealed with probability p independently of all the others and, therefore, with [**2**], we can find a directed Hamilton cycle $\overrightarrow{C}$ in $\mathcal{T}$. We finally match to each edge (F, F') of $\overrightarrow{C}$ a vertex $v \in A$ not yet covered by any $F \in \mathcal{F}$ such that $u_F, w_F, x_{F'}, y_{F'}$ are all neighbours of v in the graph G. Owing to the minimum degree conditions, this easily follows from Hall's matching theorem. Thus we get the square of a Hamilton cycle, as desired.

2.4 Non-extremal case.

We now turn to the proof of Theorem 5. Assume that G is not $(\frac{1}{1+k}, \beta)$-stable and let $p \geq Cn^{-(k-1)/(2k-3)}$. After applying the regularity lemma to G, with the help of a variant of [**7**, Lemma 4.4] is not hard to show that the reduced graph R can be vertex-partitioned into copies of stars $K_{1,k}$ and matching edges $K_{1,1}$, such that there are not too many stars. We would like to cover each such star and matching edge with the square of a Hamilton path, and then connect these paths to get the square of a Hamilton cycle. However since we do not have an additional log-term in p, we need the centre cluster of each star to be larger than the other clusters. Moreover, to ensure that we can connect the Hamilton paths, we need to setup some connections between the stars and matching edges in advance.

Therefore, we first remove some vertices from the leaf cluster of each star to make it unbalanced and ensure that all pairs are super-regular. We then label the stars and matching edges arbitrarily as $S_1, \dots, S_s$ and for $i = 1, \dots, s$ find the square of a short path, that we denote by Q_i, with start- and end-tuple in leaf clusters of S_i and S_{i+1} (where indices are modulo s). We let V_0 be the sets of vertices not any more contained in any of the stars or matching edges. We cover V_0 by appending its vertices to the paths Q_i. Here we use that any vertex $v \in V_0$ has degree at least $(\frac{1}{k+1} - \alpha)n$ in G and, as we do not have too many stars, v has also many neighbours in some clusters which are not centres of stars. This is crucial to ensure that in each star the centre cluster from each star remains large enough in comparison to the leaf clusters.

Then, for any star S_i, we connect the end-tuple of Q_{i-1} with the start-tuple of Q_i while covering all vertices in all clusters of S_i. We emphasise again that, since our p does not have log-terms, this is only possible since each centre cluster is larger than the leaf-clusters and so we do not need to cover all vertices in the leaf clusters with copies of P_k^2. For any matching edge S_i, we split its clusters and obtain two stars $K_{1,k}$ that also allow us to connect Q_{i-1} to Q_i and covering all vertices of S_i, as before. This gives the square of a Hamilton cycle in $G \cup G(n,p)$ we wanted.

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