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The Goldie Equation: III. Homomorphisms from functional equations

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Abstract. This is the second of three sequels to (Ostaszewski in Aequat Math 90:427–448, 2016)—the third of the resulting quartet—concerning the real-valued continuous solutions of the multivariate Goldie functional equation (GFE) below of Levi–Civita type. Following on from the preceding paper (Bingham and Ostaszewski in Homomorphisms from Functional Equations: II. The Goldie Equation, arXiv:1910.05816), in which these solutions are described explicitly, here we characterize (GFE) as expressing homomorphy (in all but some exceptional "improper" cases) between multivariate Popa groups, defined and characterized earlier in the sequence. The group operation involves a form of affine addition (with local scalar acceleration) similar to the circle operation of ring theory. We show the affine action in (GFE) may be replaced by a general continuous acceleration yielding a functional equation (GGE) which it emerges has the same solution structure as (GFE). The final member of the sequence (Bingham and Ostaszewski, The Gołąb–Schinzel and Goldie functional equations in Banach algebras, arXiv:2105.07794) considers the richer framework of a Banach algebra which allows vectorial acceleration, giving the closest possible similarity to the circle operation.

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1. Introduction

We begin by recalling the classic Gołąb–Schinzel equation

$$\eta(x+\eta(x)y)=\eta(x)\eta(y) \qquad (x,y\in X) \tag{GS}$$

for $X=\mathbb{R}$. Interpreted in the context of a real topological vector space X, the continuous solutions of (GS) are in one of the following two forms, both generalizing those for $X=\mathbb{R}$. One is

$$\eta(x)=1+\rho(x) \qquad (x\in X),$$

for some continuous linear functional $\rho\colon X\to\mathbb{R}$. The other is

$$\eta(x)=\max\{1+\rho(x),0\} \qquad (x\in X).$$

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(This is the Brillouët–Dhombres–Brzdęk theorem [17, Prop. 3], [18, Th. 4]; cf. [19].) Significantly for us, the latter form is non-negative. We assume henceforth that $\rho \neq 0$.

Next, recall from algebra the *circle operation* of ring theory (see e.g. Jacobson [Jac1951], II.3), which in a ring identifies the elements which have inverses, and so form a *group*:

$$a \circ b := a + b - ab.$$

Analogously, a *Popa group* on a real topological vector space X is obtained from a given $\rho \in X^*$, a continuous linear map $\rho\colon X \to \mathbb{R}$, by defining

$$u \circ_\rho v := u + v + \rho(u)v = u + \eta_\rho(u)v,$$

where

$$\eta_\rho(u) := 1 + \rho(u). \qquad (\eta_\rho)$$

This equips

$$\mathbb{G}^+(X) = \mathbb{G}_\rho(X) := \{x \in X : \eta_\rho(x) > 0\}$$

with a group structure. (Below the notation $\mathbb{G}^+(X)$ is useful whenever context dictates that η_ρ or some similar function, such as h below, is to be positive.) Its significance for (GS) was first recognized by Popa [36]. Its explanatory power as a fundamental tool in the study of regular variation is witnessed in several papers (both joint [6–11] and separate [32–35]) and especially in combination with shift-compactness – see [14] and the survey [15]; cf. [5].

We recall the associated *Goldie functional equation*, originally of (univariate) regular variation [7] for the pair (K, g):

$$K(u \circ_\rho v) = K(u) + g(u)K(v) \qquad (u, v \in \mathbb{G}_\rho(X)), \qquad (GFE)$$

where now with $g\colon X \to \mathbb{R}_+ := [0, \infty)$, $K\colon \mathbb{G}_\rho(X) \to Y$, for Y again a real topological vector space, and with both functions *continuous*.

Given the pair, K is called the *kernel,* g the (outer) *auxiliary* function of the kernel, following usage in regular variation theory, where this equation plays a key role, see e.g. [12]. It is helpful to view g as an *action* imparting local *acceleration* to the action of addition. The equation is a special case of the *Levi-Civita functional equation* [30], [38, Ch. 5] (for the wider literature, see [2, Ch. 14, 15]), usually studied on semi-groups (cf. [39]). We shall also study the *generalized Goldie equation* for the triple (K, h, g), now with inner and outer auxiliaries:

$$K(u + h(u)v) = K(u) + g(u)K(v) \qquad (u, v \in \mathbb{G}^+(X)), \qquad (GGE)$$

with $g, h\colon X \to [0, \infty)$, $K\colon X \to Y$ and all three *continuous* with

$$\mathbb{G}^+(X) = \mathbb{G}_h^+(X) := \{x \in X : h(x) > 0\},$$

using analogous notation. It emerges in Lemma 8.1 that h here may be replaced by g (i.e. briefly, that $\mathbb{G}_h^+(X) = \mathbb{G}_g^+(X)$).

When $\rho \in X^*$ is fixed, (GGE) reduces to (GFE) for $u, v \in \mathbb{G}_\rho(X)$ when

$$h(u) = \eta_\rho(u) = 1 + \rho(u).$$

For fixed $\rho \in X^*, K\colon \mathbb{G}_\rho(X) \to Y$ and a given continuous linear map $\sigma\colon Y \to \mathbb{R}$, take $g(u) = g^\sigma(u)$, for $u \in \mathbb{G}_\rho(X)$, where

$$g^\sigma(u) := \eta_\sigma(K(u)) = 1 + \sigma(K(u)). \qquad (g^\sigma)$$

Then (GFE) for (K, g) reduces to the Popa *homomorphism* [9] from $\mathbb{G}_\rho(X)$ to $\mathbb{G}_\sigma(Y)$:

$$K(u \circ_\rho v) = K(u) \circ_\sigma K(v) \quad (= K(u) + K(v) + \sigma(K(u))K(v)).$$

Main Theorems. These are as follows:
(i) The first is Theorem 7.2, asserting that a continuous solution (K, g) of the Goldie equation (GFE) defined on a Popa group $\mathbb{G}_\rho(X)$ of a (topological vector) space with values in a space Y is a homomorphy into a Popa group $\mathbb{G}_\sigma(Y)$ (with σ defined by g so that $g = g^\sigma$), unless the range $\mathcal{R}(K)$ collapses to the image of the null space of ρ, $K(\mathcal{N}(\rho))$.
(ii) The second is Theorem 8.1, asserting that a continuous solution (K, h, g) of the generalized equation (GGE) is a homomorphy between $\mathbb{G}_\rho$ and $\mathbb{G}_\sigma$ with ρ and σ defined respectively by h and g; thus (GGE) reduces to (GFE).

Key to the two results is Theorem 3.1, which asserts that in each case the kernel K maps any vector u in its domain radially to its multiple $\lambda(u)u$ and further identifies the linking function $\lambda(.)$.

Notational convention: Throughout below, X, Y are fixed real topological vector spaces; $\rho \in X^*$ may either be given in advance, or specifically constructed. For given $\rho \in X^*$, we say that the pair (K, g) *satisfies* (GFE) to mean that $g\colon X \to \mathbb{R}_+ := [0, \infty)$, $K\colon \mathbb{G}_\rho(X) \to Y$, with both functions *continuous,* and that (GFE) is satisfied by the pair (K, g). Always, '(GFE)' will imply the presence of a given $\rho \in X^*$ and the corresponding domain-restriction to $\mathbb{G}_\rho(X)$.

Likewise, we say that the triple (K, h, g) *satisfies* (GGE) to mean that $g, h\colon X \to \mathbb{R}_+$, $K\colon X \to Y$, with all three functions *continuous* and that the relation (GGE) is satisfied by the triple (K, h, g). In Sect. 8 it is shown that this implies that the inner auxiliary h is given by (η_ρ) above for some ρ.

For (fixed $\rho \in X^*$ and) a given pair (K, g) satisfying (GFE) the existence theorem, Theorem 7.2 below, asserts that, unless the range $\mathcal{R}(K)$ collapses to $K(\mathcal{N}(\rho))$, there *exists* a *unique* linear $\sigma = \sigma_g\colon Y \to \mathbb{R}$ with $g = g^\sigma$ as in (g^σ) above, which is *continuous* provided the range $K(\mathbb{G}_\rho(X))$ is a *closed complemented subspace* of Y (see §7). The argument is involved and begins by establishing necessary and sufficient conditions on any such $g\colon X \to \mathbb{R}_+$ that there exists a linear $\sigma\colon Y \to \mathbb{R}$ with $g = g^\sigma$ as above. In an intermediate step (see Proposition 7.1A and 7.1B) we deduce that, unless the range

$\mathcal{R}(K)$ collapses to $K(\mathcal{N}(\rho))$, such a σ always exists and is unique, so that we may refer to it as σ_g, and that this σ_g is continuous provided $K(\mathbb{G}_\rho(X))$ is a closed complemented subspace of Y. The result thus extends to continuous functions on arbitrary linear domains the one-dimensional homomorphy first recognized in [34, Th. 1].

It emerges that intersections of the null spaces of various additive maps on X are of central significance. So, given $g\colon X \to \mathbb{R}_+$, we put for $x \in \mathbb{G}_g^+(X)$

$$\gamma(x) := \log g(x),$$

and for additive $\alpha\colon \mathbb{G}_\rho(X) \to \mathbb{R}$ we write

$$\mathcal{N}(\alpha) := \{x \in X : \alpha(x) = 0\}, \qquad \mathcal{N}^*(\alpha) := \mathcal{N}(\alpha) \cap \mathcal{N}(\rho);$$

of particular interest here is

$$\mathcal{N}^*(\gamma) := \mathcal{N}(\gamma) \cap \mathcal{N}(\rho) = \{x : g(x) = 1\} \cap \mathcal{N}(\rho).$$

So we begin in Sect. 2 with a study of the auxiliary function g corresponding to a given pair (K, g) satisfying (GFE), leading us to Theorem 2.1, which we view as establishing an *index* (cf. [11]). Section 3, prompted by the radial properties of Popa homomorphisms established in [12], asserts in Theorem 3.1 analogous radiality properties of (GFE) kernels. The proof is delayed to Sect. 5, after establishing in Sect. 4 the density of sets canonically modelling the rationals m/n by appropriate 'steering' of m-fold g- or h-actions applied to u/n and computing related limits. This radiality implies (Corollary 6.2) that any kernel $K\colon \mathbb{G}_\rho(X) \to Y$ satisfies a dichotomy involving the null spaces $\mathcal{N}(\rho)$ and $\mathcal{N}(\alpha)$ for $\alpha\colon \mathbb{G}_\rho(X) \to \mathbb{R}$ an arbitrary additive map. The dichotomy concerning the two null spaces arises because two hyperplanes passing through the origin (representing the two null spaces) have intersection either with co-dimension 1, when they coincide, or 2 otherwise.

In Sect. 7 we induce a *Popa group structure* on the range space Y and prove our first main result, Theorem 7.2, on the existence of an appropriate functional $\sigma \in Y^*$ for transforming the right-hand side of (GFE) into the circle operation associated with $\mathbb{G}_\sigma(Y)$. Section 8 is devoted to establishing a reduction to (GFE) of the more general Goldie functional equation (GGE) above, wherein the accelerated summation $u + h(u)v$ on the left-hand side of (GGE) replaces the Popa operation $u \circ_\rho v$. The argument of Sect. 8 relies on radial behaviours and on reducing a known 'pexiderised variant' [1] of (GS) to (GS) itself, with a resulting *tetrachotomy* of possible Popa homomorphisms in any direction $u \in X$; the latter foursome can be interpreted as arising from the binary split into the vanishing or non-vanishing of Gateaux derivatives (along the radial direction u) of h and g, whose existence follows from h satisfying (GS) and g satisfying the pexiderized equation (PGS) (see Sect. 8). For the

[1] That is, additional function symbols replace instances in the functional equation (GS) of the function symbol of primary interest.

convenience of the reader we have chosen occasionally to omit simple routine checks, acknowledging as much; however, all such steps have been verified by us in the earlier arXiv publication.

2. Auxiliary functions: multiplicative property

We learn from Lemma 2.1 that the auxiliary function g of a kernel is ρ-multiplicative in the sense of (M) below, and so the corresponding $\gamma = \log g$ (see above) is ρ-additive. When context permits we omit the prefix.

Lemma 2.1 (cf. [9], [37, Prop. 5.8]). *If (K, g) satisfies (GFE) and K is non-zero, then g is ρ-multiplicative:*

$$g(u \circ_\rho v) = g(u)g(v) \qquad (u, v \in \mathbb{G}_\rho(X)), \tag{M}$$

and so $\gamma = \log g$ is ρ-additive:

$$\gamma(u \circ_\rho v) = \gamma(u) + \gamma(v) \qquad (u, v \in \mathbb{G}_\rho(X)). \tag{A}$$

Fix $u \neq 0$. For $t \in \mathbb{R}$, if $g(tu) \neq 1$ except at $t = 0$, then $g(tu)$ takes one of two forms:

$$g(tu) = \begin{cases} (1 + t\rho(u))^{\gamma(u)/\log(1+\rho(u))}, & \rho(u) > 0, \\ e^{\gamma(u)t}, & \rho(u) = 0. \end{cases}$$

Proof. To deduce (M) use the two ways to associate the three variables in $K(u \circ_\rho v \circ_\rho w)$; we omit the routine details. Put $g_u(t) := g(tu)$; then $g_u \colon \mathbb{G}_{\rho(u)}(\mathbb{R}) \to \mathbb{R}_+$ and satisfies (M) on the real line. This case is covered by established results, e.g. [9] or Th. BO below, yielding for some $\kappa(u)$

$$g_u(t) = (1 + \rho(u)t)^{\kappa(u)/\rho(u)},$$

whence the cited formulas. □

Remark. In §3 below we encounter the *link function* λ, and with it λ_u, in terms of which we will be able to write, in view of Theorem 3.1,

$$K_u(tu) := \lambda_u(t)K(u). \tag{K}$$

Then, for $K(u) \neq 0$,

$$\lambda_u(s \circ_\rho t) = \lambda_u(s) + g(su)\lambda_u(t)$$

whence, from the context of the reals in [9], $K_u(tu)$ is proportional to one of the two forms

$$\begin{cases} [(1 + t\rho(u))^{\gamma(u)} - 1], & \text{if } \rho(u) > 0, \\ [e^{\gamma(u)t} - 1], & \text{if } \rho(u) = 0. \end{cases}$$

This is implied by Lemma 6.1, itself a corollary of Theorem 2.1 below.

The radial case g_u of Lemma 2.1 shows that we need not restrict to scalars. We can further describe $g(x)$ explicitly by studying its associated *index* $\gamma(x)$, to borrow a term from extreme-value theory (EVT), for which see e.g. [11], [25, p. 295]. Below we distinguish between linearity (in the sense of vectors and scalars), which γ exhibits *only* on $\mathcal{N}(\rho)$, and its more general property of ρ-additivity. To see the difference note that, for distinct u and w with $\rho(u) = 1$ and $\rho(w) = 1$ (so that $w - u \in \mathcal{N}(\rho)$), taking $x = tw$ in the first display in Theorem 2.1 below gives

$$g(tw) = g(x) = e^{t\gamma(w-u)}(1+t)^{\gamma(u)/\log 2} = (1+t)^{\gamma(w)/\log 2} :$$
$$e^{t\gamma(w-u)} = (1+t)^{[\gamma(w)-\gamma(u)]/\log 2}.$$

We thus think of the following result as an Index theorem.

Theorem 2.1. *For (K, g) satisfying (GFE), the auxiliary function g is ρ -multiplicative and its index ρ-additive.*

So, as in Lemma 2.1, for any u with $\rho(u) = 1$,

$$g(x) = e^{\gamma(x-\rho(x)u)}(1+\rho(x))^{\gamma(u)/\log 2} \qquad (x \in \mathbb{G}_\rho(X)),$$

where, for $\gamma = \log g$,

$$\alpha(x) := \gamma(x - \rho(x)u)$$

is linear and $\alpha(u) = 0$.

Conversely, for any $\alpha : \mathbb{G}_\rho(X) \to \mathbb{R}$ additive and β a real parameter, the following function is multiplicative (satisfies (M)):

$$\bar{g}(x) = \bar{g}_{\alpha,\beta}(x) := e^{\alpha(x)}(1+\rho(x))^\beta.$$

Proof. By Lemma 2.1, γ satisfies (A). So $\gamma : \mathbb{G}_\rho(X) \to \mathbb{G}_0(\mathbb{R}) = (\mathbb{R}, +)$. Here $\gamma(\mathcal{N}(\rho)) \subseteq \mathbb{R} = \mathcal{N}(0)$. By a theorem of Chudziak [22, Th. 1] as amended in [12, Th. Ch., Th. 4A], for any u with $\rho(u) = 1$,

$$\gamma(x) = \gamma(x - \rho(x)u) + [\gamma(u)/\log 2]\log[(1+\rho(x))],$$
$$g(x) = e^{\gamma(x-\rho(x)u)}(1+\rho(x))^{\gamma(u)/\log 2}.$$

Then, taking $x = tu$,

$$g(tu) = (1+t)^{\gamma(u)/\log 2},$$

as by linearity $\rho(tu) = t$. For w with $\rho(w) > 0$, take $u = w/\rho(w)$. Then

$$g(tw) = g(t\rho(w)u) = (1+t\rho(w))^{\gamma(u)/\log 2}.$$

On the other hand, for w with $\rho(w) = 0$,

$$g(tw) = e^{\gamma(w)t}.$$

Given the form of $\bar{g}$, it is routine to check that (M) holds; we omit the details. □

Lemma 2.2. *For continuous (K,g) satisfying (GFE) and $\gamma=\log g, \mathcal{N}(\gamma)$ is a subgroup of $\mathbb{G}_\rho(X)$, and $\mathcal{N}^*(\gamma)$ is both a vector subspace and a $\mathbb{G}_\rho(X)$-subgroup of $\mathcal{N}(\rho)$.*

Proof. By Lemma 2.1, γ is additive on $\mathbb{G}_\rho(X)$ and so linear on $\mathcal{N}(\gamma)$, by continuity of γ. The remaining assertions are clear. □

3. Radiality

We write $\langle v\rangle$ for the linear span of a vector v, in X or Y. For $u\in\mathbb{G}_\rho(X)$, we write

$$\langle u\rangle_\rho := \langle u\rangle \cap \mathbb{G}_\rho(X) = \{tu : t\in\mathbb{R}, 1+\rho(tu)>0\}.$$

Our main result here and our later tool is Theorem 3.1 below. This asserts *radiality*, the property that the kernel function maps the points along $\langle u\rangle$ to points along $\langle K(u)\rangle$, and, furthermore, specifies precisely the *linkage* between the originating vector tu and its image $K(tu)=\lambda_u(t)K(u)$, as in (K). The dependence is uniform, through one and the same *link function* λ (below) but with its continuously varying parameters referring to what we term informally the *growth rates* (below) of the two auxiliaries along u at the origin. We use either the notation $\lambda(t;r,\theta)$ which identifies the parameters explicitly, or λ_u when the parameters are implied by the direction u.

To state it in a form adequate to cover both (GFE) and (GGE), we need several definitions and a lemma. Notice that (GGE) implies that

$$K(h(0)v)=K(0)+g(0)K(v),$$

so if $h(0)=0$ and $g(0)\neq 0$, the map K is trivial; similarly, if $g(0)=0$ and $h(0)\neq 0$, since Theorem 3.1 below asserts that $K(tv)=\lambda(t)K(v)$ for some monotone function λ (strictly monotone if $K(v)\neq 0$). Thus without loss of generality (briefly, w.l.o.g.) we standardize the auxiliary functions in (GGE), by taking $h(0)=g(0)=1$. This then coincides with the corresponding conditions for (GFE) in Theorem 2.1. It now easily follows that $K(0)=0$, since

$$K(0)=K(0+h(0)0)=K(0)+g(0)K(0).$$

Definition. 1. Following Th. 2.1 above, taking as parameters $r\geqslant 0, \theta\in\mathbb{R}$, the standard *multiplicative radial function* $g_{r,\theta}(t)$ is defined for $t>-1/r$ (the latter interpreted for $r=0$ as $-\infty$) by

$$g_{r,\theta}(t) := \begin{cases} (1+tr)^{\theta/r}, & \text{if } r>0,\\ e^{\theta t}, & \text{if } r=0.\end{cases}$$

Thus $g_{r,\theta}$ is Gateaux differentiable at $t=0$. This definition blends two possible instances of the function $\bar{g}$ of Theorem 2.1 consistently with the L'Hospital convention (as $g_{0,\theta}=\lim_{r\searrow 0} g_{r,\theta}$), constantly applied below implicitly.

2. Taking again as parameters $r \geqslant 0$ and $\theta \in \mathbb{R}$, we define below the function $\lambda(t; r, \theta)$ for $t > -1/r$ (where for $r = 0$, we interpret $-1/r$ as $-\infty$), which we call the *Popa link* function on account of its role in Theorem 3.1 below:

$$\lambda(t) = \lambda(t; r, \theta) := \begin{cases} [(1+rt)^{\theta/r} - 1]/[(1+r)^{\theta/r} - 1], & \text{if } r > 0, \theta \neq 0, \\ \ln(1+rt)/\ln(1+r), & \text{if } r > 0, \theta = 0, \\ (e^{t\theta} - 1)/(e^{\theta} - 1), & \text{if } r = 0, \theta \neq 0, \\ t, & \text{if } r = \theta \in \{0, \pm\infty\}. \end{cases} \tag{†}$$

This function first arises in the context of (GFE) as a map $\mathbb{G}_r(\mathbb{R}) \to \mathbb{R}$ with parameters r, θ and where λ satisfies an analogous equation (GFE_λ) below. But as it plays an equivalent role in (GGE), it is convenient to derive its properties in the more general setting. Its key elementary properties are summarized in the following lemma; see also Lemma 8.4.

Lemma 3.1. *For $r \in [0, \infty), \theta \in \mathbb{R}$, the Popa link function λ is separately continuous in t and in its parameters, with $\lambda(0) = 0$ and $\lambda(1) = 1$, and satisfies the equation*

$$\lambda(s \circ_r t) = \lambda(s) + g_{r,\theta}(s)\lambda(t) \qquad (s, t \in \mathbb{G}_r(\mathbb{R})), \tag{GFE_λ}$$

equivalently

$$\lambda(s \circ_r t) = \lambda(s) \circ_\sigma \lambda(t), \ \textit{for } \sigma = g_{r,\theta}(1) - 1.$$

Except for $r = \theta = 0$, the equation $\lambda(t) = t$ has a unique solution $t = 1$.

Proof. Routine: we omit the details. □

The main result of this section is Theorem 3.1 below, asserting the radiality property of the kernel function in (GFE) and (GGE) that, for some scalar function $\lambda = \lambda_u$ which we determine,

$$K(su) = \lambda(s)K(u) \text{ for } s \geqslant 0.$$

It is convenient to use notation bringing together some arguments common to g and h. Put

$$\begin{aligned} \delta_n(u) &:= \delta_n^g = g(u/n) - 1, \text{ or } \delta_n^h = h(u/n) - 1, \\ \gamma_g(u) &:= \lim\nolimits_n n\delta_n^g(u), \qquad \text{and } \gamma_h(u) = \lim\nolimits_n n\delta_n^g(u), \end{aligned}$$

whenever these limits exists, possibly $\pm\infty$.

In the context $h(u) = 1 + \rho(u)$ of (GFE), we obtain $n\delta_n^h = \rho(u)$, so that $\gamma_h(u) = \rho(u)$. Recall from §2 that $\gamma = \log g$; here $g(su) = e^{\gamma(su)}$ gives $\gamma_g(u) = \gamma(u)$, motivating the common notation.

Theorem 3.1. *If* (K,h,g) *satisfies* (GGE) *with* K,g,h *continuous and*

$$n\delta_n^g(u) \to \gamma_g(u) \in \mathbb{R} \text{ and } n\delta_n^h \to \gamma_h(u) \in \mathbb{R},$$

then for u *with* $K(u) \neq 0$

$$K\left(\frac{e^{\gamma_h(u)t}-1}{\gamma_h(u)}u\right) = \frac{e^{\gamma_g(u)t}-1}{e^{\gamma_g(u)}-1}K\left(\frac{e^{t\gamma_h(u)}-1}{\gamma_h(u)}u\right).$$

In particular, if $\gamma_g(u) = \gamma_h(u)$, *then*

$$K(tu) = tK(u) \qquad (t \in \mathbb{R}).$$

This applies also if one or other limit is infinity, both then being equal. Furthermore, the usual L'Hospital convention applies when either of the limits $\gamma_g(u), \gamma_h(u)$ *is zero. Thus*

$$K(tu) = \lambda_u(t)K(u) \qquad \text{for } \lambda_u(t) := \lambda(t; \gamma_h(u), \gamma_g(u)). \qquad (Rad)$$

In particular, K *is Gateaux differentiable in direction* u*:*

$$K'(tu) = \lambda_u'(t)K(u),$$

and the function λ_u *satisfies a corresponding scalar* (GGE)*:*

$$\lambda_u(s + h(su)t) = \lambda_u(s) + g_{\gamma_g(u),\gamma_h(u)}(su)\lambda_u(t).$$

The proof, based on Lemma 4.1 below, must be delayed until after a series of limit calculations ending in Proposition 4.3. These rest only on the assumption that g and h are *continuous*. A stronger assumption (invoking *Gateaux differentiability* of g and h at 0) allows a much shorter proof, directly from Lemma 4.1; see §9.4 (Appendix) in the arXiv version.

We will refer informally to the limits $\gamma_g(u)$ and $\gamma_h(u)$ above as the *growth rates* of g and h.

4. Density preliminaries for Theorem 3.1

The general approach of using a dense set to identify an unknown function is familiar (see [1, Ch. 2, Ch. 6]), although here it is more involved, in view of only a latent group structure (cf. [37]). Our first step is Lemma 4.1 for which we need a definition. This is phrased with a view to generalizations beyond real-valued functions, allowing for (GFE) to be interpreted also in a Banach algebra setting, cf. [13].

Definition. The polynomials and rational polynomials (in the indeterminate x) $\wp_n$ and $[\wp_m/\wp_n]$ for $m,n \in \mathbb{N}$ are defined by:

$$\wp_m(x) := 1 + x + \ldots + x^{m-1}, \qquad [\wp_m/\wp_n](x) := \wp_m(x)/\wp_n(x).$$

So $\wp_m(1) = m$, and so $[\wp_m/\wp_n](1) = m/n$, and also $\wp_m(t) = (t^m - 1)(t-1)^{-1}$ when $t - 1$ is invertible.

Lemma 4.1. *If (K, h, g) satisfies (GGE), then for u with $K(u) \neq 0$:*

$$K(\wp_m(h(u/n))u/n) = \wp_m(g(u/n))K(u/n), \tag{$*$}$$

$$K(\wp_m(h(u/n))u/n) = [\wp_m/\wp_n](g(u/n))K(\wp_n(h(u/n))u/n). \tag{$**$}$$

Proof. For $x, y \in X$, and z an indeterminate ranging over either $\mathbb{R}$ (as here) or a unital commutative Banach algebra $\mathbb{A}$ (as in [13]), put

$$x \circ_z y := x + yz.$$

Starting from u and $v := K(u)$, we define a pair of sequences of 'powers', by iterating the operation $\circ_z$ for respectively $z = h(u)$ and $z = g(u)$. These iterates are defined inductively:

$$u_h^{n+1} = u \circ_{h(u)} u_h^n, \qquad v_g^{n+1} = v \circ_{g(u)} v_g^n, \qquad \text{with } u_h^1 = u, \qquad v_g^1 = v.$$

Then, for $n \geqslant 1$,

$$K(u_h^{n+1}) = K(u) + g(u)K(u_h^n) = K(u)_g^{n+1} = v_g^{n+1}. \tag{$***$}$$

Motivated by the case

$$K(u_h^2) = K([1 + h(u)]u) = K(u) + g(u)K(u) = [1 + g(u)]K(u),$$

the recurrence $(***)$ justifies associating with the iterates above sequences of 'coefficients' $(g_n(.))$, $(h_n(.))$, by writing

$$v_g^n = g_n(u)K(u), \qquad u_h^n = h_n(u)u : \qquad K(h_n(u)u) = g_n(u)K(u).$$

Solving appropriate recurrences arising from $(***)$ for the iterations $u_h^{n+1} = u \circ_h u_h^n$ and $v_g^{n+1} = v \circ_g v_g^n$ gives

$$u_h^n = h_n(u)u, \text{ and } v_g^n = g_n(u)K(u),$$

where

$$h_n(u) := \wp_n(h(u)) = \begin{cases} \frac{h(u)^n - 1}{h(u) - 1}, & h(u) \neq 1, \\ n, & h(u) = 1, \end{cases}$$

$$g_n(u) := \wp_n(g(u)) = \begin{cases} \frac{g(u)^n - 1}{g(u) - 1}, & g(u) \neq 1, \\ n, & g(u) = 1. \end{cases}$$

Note that $g_m(u/n) \neq 0$, since $g(u/n)^m = 1$ implies $g_m(u/n) = m/n$. Replacing n by m and u by u/n yields $(*)$ above. Hence

$$K(h_m(u/n)u/n) = g_m(u/n)K(u/n) = g_m(u/n)g_n(u/n)^{-1}K(h_n(u/n)u/n),$$

giving $(**)$ above. □

By Lemma 4.1 above for any $m, n \in \mathbb{N}$

$$K(h_m(u/n)u/n) = g_m(u/n)K(u/n).$$

Taking $m = kn$ and, separately, $m = \bar{k}n$ gives

$$K(uh_{kn}(u/n)/n) = g_{kn}(u/n)K(u/n) \text{ and } K(uh_{\bar{k}n}(u/n)/n) = g_{\bar{k}n}(u/n)K(u/n).$$

Eliminating $K(u/n)$ gives, as $g_{\bar{k}n}(u/n) \neq 0$ (see above),

$$K(uh_{kn}(u/n)/n) = g_{kn}(u/n)g_{\bar{k}n}(u/n)^{-1}K(uh_{\bar{k}n}(u/n)/n).$$

We will deduce the radiality property from this equation by studying the sets

$$Q^g{:=}\{g_{kn}(u/n) : n \in \mathbb{N}, k > 0\}, \quad Q^h{:=}\{h_{kn}(u/n) : n \in \mathbb{N}, k > 0\},$$

which it emerges are *dense* in $[0, \infty)$. The terms in both sets take the form

$$q_n(k){:=}\frac{(1+\delta_n)^{kn} - 1}{n\delta_n} > 0,$$

where respectively, as above,

$$\delta_n = \delta_n^g = g(u/n) - 1, \text{ or } \delta_n^h = h(u/n) - 1,$$

both of which tend to 0 (by the continuity of g and h at 0). We put

$$Q{:=}\{q_n(k) : n \in \mathbb{N}, k > 0\}.$$

Writing $m = kn$ and noting that for $\delta_n \neq 0$

$$\frac{(1+\delta_n)^m - 1}{n\delta_n} = \frac{m}{n} + \frac{\delta_n}{n}c_2^m + \ldots + \frac{\delta_n^{m-1}}{n},$$

we may again use the *L'Hospital convention* to interpret $q_n(k)$ as $m/n = k$ whenever $\delta_n = 0$. In Proposition 4.1 below, we show that Q is dense in $[0, \infty)$ when the sequence $n\delta_n$ is *convergent*. See the Remark immediately below for the significance of this assumption in terms of differentiability. The proof of density is achieved by identifying

$$\ell(k){:=} \lim_{n\to\infty} q_n(k),$$

which emerges as a simple increasing continuous injection for $k > 0$. This gives an immediate way of *steering* $q_n(k)$ into approaching any point s of $[0, \infty)$ by use of the inverse function $k(s)$ of $\ell(k)$. We call $k(s)$ a *steering function.*

If $n\delta_n$ is divergent to $\pm\infty$, the proof for Proposition 4.2 offers a more complicated steering function for approaching through Q all the points s of $[0, \infty)$.

By taking limits relative to an appropriate subset $\mathbb{N}'_u \subseteq \mathbb{N}$, we may arrange that each of the sequences $\{n\delta_n^g\}_{n\in\mathbb{N}'_u}$ and $\{n\delta_n^h\}_{n\in\mathbb{N}'_u}$ is either convergent or divergent. There is thus no loss of generality in assuming below that $\mathbb{N}' = \mathbb{N}$. We apply these results in Proposition 4.3 to prove the promised radiality property of continuous solutions (K, g, h) of the (GGE), and as corollaries compute the forms that the radial link function may take.

Remark. For g, h the auxiliaries of the Goldie equation with $g(0) = h(0) = 1$, the case(s) under Proposition 4.1 below corresponds to a differentiability assumption. Thus, for example,

$$g(u/n) = 1 + \delta_n,$$
$$g'(0) = \lim_{n\to\infty} \frac{g(u/n) - 1}{1/n} = \lim_n n\delta_n = \gamma_g(u),$$
$$g(tu) = 1 + t\gamma_g(u) + o(t).$$

Proposition 4.1. *Assume $\delta_n \to 0$ with $\gamma := \lim_n n\delta_n \in \mathbb{R}$. Then*

$$\ell(k) := \lim_n q_n(k) = \frac{e^{k\gamma} - 1}{\gamma}, \text{ with steering provided by}$$
$$k(s) := \frac{\log(1 + s\gamma)}{\gamma}.$$

The case $\gamma = 0$ falls under the L'Hospital convention as $\ell(k) = k$ with inverse $k(s) = s$. Given these assumptions, Q is dense in $\mathbb{R}_+$.

Remarks. The case $\gamma = 0$ requires separate proof. Since $\ell(k)$ is a (non-constant) continuous function of k, its range is an interval J, and so the set Q is dense in J.

Proof of Proposition 4.1. Put $\gamma_n := n\delta_n$.

Case 1. $\gamma_n \to \gamma \neq 0$. Here

$$q_n(k) := \frac{(1 + \frac{\gamma_n}{n})^{kn} - 1}{\gamma_n} \to \ell(k) := \frac{e^{k\gamma} - 1}{\gamma} = k + \frac{1}{2}\gamma k^2 + \dots.$$

So $\ell(0+) = 0$ and $\ell'(k) = e^{k\gamma} > 0$. Hence $\{\ell(k) : k \geqslant 0\} = [0, \infty)$, so that $\{q_n(k) : n, k > 0\}$ forms a dense set.

Case 2. $\gamma_n = n\delta_n \to 0$. Expanding $\log(1 + t)$ around $t = 0$ gives

$$\log(1 + t) = t\frac{1}{1 + d(t)},$$

for some $d(t)$ between 0 and t. Take $t = \delta_n$, so that $d(\delta_n) \to 0$, and put

$$k_n := \frac{k}{1 + d(\delta_n)} \to k, \text{ and } t_n := kn\log(1 + \delta_n).$$

Then

$$t_n = n\delta_n k_n.$$

Likewise expanding $\exp(t)$ around $t = 0$:

$$\exp(t) = 1 + te^{p(t)},$$

for some $p(t)$ between 0 and t, gives

$$q_n(k) = \frac{(1+\delta_n)^{kn}-1}{n\delta_n} = \frac{\exp(kn\log(1+\delta_n))-1}{n\delta_n}$$
$$= \frac{t_n e^{p(t_n)}}{n\delta_n} = k_n e^{p(t_n)} \to ke^0 = k,$$

since $n\delta_n k_n \to 0$. That is,

$$\ell(k) = k,$$

as asserted, and again Q is dense in $[0,\infty)$. □

Proposition 4.2. *Assume* $\delta_n \to 0$ *with* $\lim_n n\delta_n = \pm\infty$*. Then* Q *is dense in* $[0,\infty)$ *and the steering functions*

$$k_n(s) := \frac{\log(1+sn\delta_n)}{n\log(1+\delta_n)}$$

secure a sequence in Q *approaching* s*.*

Proof. In view of the denominator oddness in δ_n, it suffices to consider the case when $n\delta_n \to +\infty$. Furthermore, it suffices to consider the density of

$$\bar{q}_n(k) = \frac{(1+\delta_n)^{kn}}{n\delta_n},$$

since under the current assumptions

$$\bar{q}_n(k) - q_n(k) = \frac{1}{n\delta_n} \to 0.$$

We put

$$r_k(n) = \log \bar{q}_n(k) = kn\log(1+\delta_n) - \log(n\delta_n),$$

which reduces the density consideration of Q in $[0,\infty)$ to that of $\bar{Q}:=\{r_k(n) : n \in \mathbb{N}, k > 0\}$ in $\mathbb{R}$. For any $r \in \mathbb{R}$, consider any n with $r + \log n\delta_n > 0$. Solving $r = r_k(n)$ for k gives a steering function

$$k = \kappa_n(r) := \frac{(r + \log n\delta_n)}{n\log(1+\delta_n)} > 0.$$

That is, any $r \in \mathbb{R}$ is *achieved* as being exactly $r_k(n) \in \bar{Q}$ for any n by the choice $k = \kappa_n(r) > 0$, since

$$r_k(n) = \log \bar{q}_n(\kappa_n(r)) = \frac{(r+\log n\delta_n)}{n\log(1+\delta_n)} n\log(1+\delta_n) - \log(n\delta_n) = r.$$

Furthermore, as $\kappa_n(r)$ is increasing in r, setting $r' := r + 1/n$, with n large enough so that $r' + \log n\delta_n > 0$, gives $\kappa_n(r') > 0$ and

$$r_n(\kappa_n(r') - r_n(\kappa_n(r)) = (\kappa_n(r') - \kappa_n(r))n\log(1+\delta_n)$$
$$= \frac{(r'-r)}{n\log(1+\delta_n)} n\log(1+\delta_n) = \frac{1}{n} \to 0.$$

It now follows that any $r \in \mathbb{R}$ is achieved as a limit of points $r_k(n)$ by taking $k = \kappa_n(r+1/n)$ with n large enough. Hence $\bar{Q}$ is dense in $\mathbb{R}$, and so Q is dense in $\mathbb{R}_+$. In particular, for $s > 0$ and

$$r = \log\left(s + \frac{1}{n\delta_n}\right),$$

we have

$$\log\left(s + \frac{1}{n\delta_n}\right) = \log \bar{q}_n(\kappa_n(r)) : \qquad s = \bar{q}_n(\kappa_n(r)) - \frac{1}{n\delta_n} = q_n(\kappa_n(r)).$$

Putting

$$k_n(s){:=}\kappa_n\left(\log\left(s + \frac{1}{n\delta_n}\right)\right) = \frac{\log(s + \frac{1}{n\delta_n}) + \log n\delta_n}{n\log(1+\delta_n)} = \frac{\log(sn\delta_n + 1)}{n\log(1+\delta_n)}$$

provides steering of the sequence $q_n(k_n(s))$ in Q to the limit s. $\square$

Remark. Above, $k_n(r) \to 0$ as $n \to \infty$. To see this, note that $n\log(1+\delta_n)$ is asymptotic to $n\delta_n$ to first order (cf. Case 2 in Proposition 4.1).

Proposition 4.3 is the next step in identifying the radiality property explicitly, by reference to a function which we temporarily denote by $\bar{\lambda}_u(s)$ and which we subsequently show in Corollary. 4.1 below to be the link function $\lambda_u(s)$ of Sect. 3.

Proposition 4.3. *For continuous* (K, h, g) *solving* (*GGE*), *such that* $n\delta_n^g$ *and* $n\delta_n^h$ *are each either convergent or divergent sequences, if* $K(u) \neq 0$, *then there exists* $\bar{\lambda}_u(s) \geqslant 0$ *defined for* $s > 0$ *with*

$$K(su) = \bar{\lambda}_u(s)K(u).$$

Proof. Fix u with $K(u) \neq 0$. By Lemma 4.1,

$$K(h_m(u/n)u/n) = g_m(u/n)K(u/n).$$

Recall that

$$\delta_n = g(u/n) - 1 = \delta_n^g \text{ resp. } h(u/n) - 1 = \delta_n^h$$

gives rise to

$$q_n^g(k) = g_{kn}(u/n)/n, \text{ resp. } q_n^h(k) = h_{kn}(u/n)/n.$$

Taking $m = kn$ and separately $m = \bar{k}n$ gives

$$K(uh_{kn}(u/n)/n) = g_{kn}(u/n)K(u/n),$$
$$K(uh_{\bar{k}n}(u/n)/n) = g_{\bar{k}n}(u/n)K(u/n),$$

and so

$$K(uh_{kn}(u/n)/n) = g_{kn}(u/n)K(u/n) = g_{kn}(u/n)g_{\bar{k}n}(u/n)^{-1}K(uh_{\bar{k}n}(u/n)/n).$$

Applying one or other of Propositions 4.1 and 4.2 and taking the relevant steering functions, write $k = k(s)$ (resp. $k_n(s)$) with $h_{kn}(u/n)/n \to s > 0$, and likewise write $\bar{k}$ (resp. $\bar{k}_n(1)$) with $h_{\bar{k}n}(u/n)/n \to 1$. By continuity of K,

$$K(su) = \lim_n K(uh_{kn}(u/n)/n) \text{ and } K(u) = \lim_n K(uh_{\bar{k}n}(u/n)/n).$$

So

$$K(su) = \bar{\lambda}_u(s)K(u),$$

where the limit

$$\bar{\lambda}_u(s) := \lim_n g_{kn}(u/n) g_{\bar{k}n}(u/n)^{-1} = \lim_n q_n^g(k) q_n^g(\bar{k})^{-1}$$

exists, as $K(u) \neq 0$. Indeed, $q_n^g(k)q_n^g(\bar{k})^{-1}$ remain bounded over n, otherwise $K(su)$ is undefined. So $\lim_{n\in\mathbb{M}} q_n^g(k)q_n^g(\bar{k})^{-1}$ is identical for each infinite $\mathbb{M} \subseteq \mathbb{N}$.

Note that

$$q_n^g(k)q_n^g(\bar{k})^{-1} = \begin{cases} \frac{g(u/n)^{kn}-1}{g(u/n)^{\bar{k}n}-1}, & g(u/n) \neq 1, \\ \frac{k}{\bar{k}}, & g(u/n) = 1. \end{cases}$$

□

Conclusions from density considerations

Our next result recovers from Proposition 4.1 (i.e. the two cases $\lim n\delta_n$ zero or not) all the defining clauses of the link function of Lemma 3.1, giving explicit form to the radiality result in Proposition 4.3.

Corollary 4.1. *For continuous (K, h, g) solving (GGE) and u with $K(u) \neq 0$, if $n\delta_n^h \to \gamma_h = \gamma_h(u) \in \mathbb{R}$ and $n\delta_n^g \to \gamma_g = \gamma_g(u) \in \mathbb{R}$, then*

$$\bar{\lambda}_u(s) = \begin{cases} [(1+s\gamma_h)^{\gamma_g/\gamma_h} - 1]/[(1+\gamma_h)^{\gamma_g/\gamma_h} - 1], & \text{if } \gamma_g \neq 0 \text{ and } \gamma_h \neq 0, \\ (e^{s\gamma_g} - 1)/(e^{\gamma_g} - 1), & \text{if } \gamma_g \neq 0 \text{ and } \gamma_h = 0, \\ \log(1+s\gamma_h)/\log(1+\gamma_h), & \text{if } \gamma_g = 0 \text{ and } \gamma_h \neq 0, \\ s, & \text{if } \gamma_h = 0 = \gamma_g, \end{cases}$$

so that generally if $\gamma_h = \gamma_g$, then

$$\bar{\lambda}_u(s) = s.$$

In all these cases $\bar{\lambda}_u(s) = \lambda(s; \gamma_h(u), \gamma_g(u))$ and is differentiable.

Proof. We argue by cases. Henceforth we omit the overbar to lighten the notation, as no misunderstanding can arise. First suppose $\gamma_g \neq 0$ and $\gamma_h \neq 0$. As above $(1+\delta_n^h)^n \to e^{\gamma_h}$ and $(1+\delta_n^g)^n \to e^{\gamma_g}$ and when as in Proposition 4.3 $k, \bar{k}$ are constants, we use logarithmic steering:

$$k^h = k^h(s) = \frac{1}{\gamma_h}\log(1+s\gamma_h) \text{ and } \bar{k}^h = k^h(1) = \frac{1}{\gamma_h}\log(1+\gamma_h).$$

So, as $\gamma_g \neq 0$,

$$\lambda_u(s) = \lim_n \frac{q_n^g(k^h)}{q_n^g(\bar{k}^h)} = \frac{e^{k^h(s)\gamma_g} - 1}{e^{\bar{k}^h(1)\gamma_g} - 1} = \frac{(1+s\gamma_h)^{\gamma_g/\gamma_h} - 1}{(1+\gamma_h)^{\gamma_g/\gamma_h} - 1},$$

as required. Likewise, if $\gamma_g \neq 0$ and $\gamma_h = 0$, then

$$\lambda_u(s) = \frac{e^{k^h(s)\gamma_g} - 1}{e^{\bar{k}^h(1)\gamma_g} - 1} = \frac{e^{s\gamma_g} - 1}{e^{\gamma_g} - 1},$$

giving the second case.

As in Proposition 4.1 Case 2,

$$\begin{aligned} q_n(k) &= \frac{(1+\delta_n)^{kn} - 1}{n\delta_n} = \frac{\exp(kn\log(1+\delta_n)) - 1}{n\delta_n} \\ &= \frac{t_n e^{p(t_n)}}{n\delta_n} = k_n e^{p(t_n)} \to ke^0 = k. \end{aligned}$$

So suppose now that $\gamma_h \neq 0$ but $\gamma_g = 0$. To compute λ we need a combination of k^h with δ_n^g. Here with $t_n^g := k^h n \log(1+\delta_n^g)$

$$\begin{aligned} q_n^g(k^g) &= \frac{(1+\delta_n^g)^{k^g n} - 1}{n\delta_n^g} = \frac{\exp(k^h n\log(1+\delta_n^g)) - 1}{n\delta_n^g} \\ &= \frac{t_n^g e^{p(t_n^g)}}{n\delta_n^g} = k^g e^{p(t_n^g)} \to k^g e^0 = k^g. \end{aligned}$$

Similarly, with $t_n^g = n\delta_n^g k_n^h$

$$\exp(k^h n\log(1+\delta_n^g)) - 1 = n\delta_n^g k_n^h e^{p(t_n^g)},$$

$$k_n^h := k^h/(1+d(\delta_n)) = \frac{\log(1+s\gamma)/\gamma}{1+d(\delta_n)} \to k^h = \log(1+s\gamma_h)/\gamma_h.$$

From here,

$$\lambda_u(s) := \lim_n \frac{(1+\delta_n^g)^{kn} - 1}{(1+\delta_n^g)^{\bar{k}n} - 1} = \frac{n\delta_n^g k_n^h e^{p(t_n^g)}}{n\delta_n^g \bar{k}_n^h e^{p(\bar{t}_n^g)}} = \frac{k^h(s)e^{p(t_n^g)}}{k^h(1)e^{p(\bar{t}_n^g)}} \to \frac{\log(1+s\gamma_h)}{\log(1+\gamma_h)},$$

as $t_n^g \to 0$, since $n\delta_n^g \to \gamma_g = 0$ here.

Finally, suppose $\gamma_h = \gamma_g = 0$. Then $k^h(s) = s$ and we have

$$\lambda_u(s) = s,$$

as required in the last case. □

We now consider the cases in which one of the sequences is divergent. For these, it is helpful in the context of (GGE), and hence also of (GFE), to note that differentiability at 0 of K in direction u for $h(u) \neq 0$ implies differentiability of K elsewhere along u, as the following shows.

$$\begin{aligned} K(u+h(u)tu)-K(u) &= g(u)K(tu), \\ \frac{K(u+h(u)tu)-K(u)}{th(u)} &= \frac{g(u)}{h(u)}\frac{K(tu)-K(0)}{t}. \end{aligned}$$

Given this observation, the connection between g and h in (GGE) is such that either one of them is differentiable at the origin iff the other is. Indeed, from

$$K([t+h(tu)]u) = K(tu+h(tu)u) = K(tu)+g(tu)K(u),$$

it follows, after subtracting $K(u)$ from each side (and since $K(0)=0$), that

$$\begin{aligned} &\frac{K((t+h(tu))u)-K(h(0)u)}{(t+h(tu)-1)}\cdot\frac{t+(h(tu)-1)}{t} \\ &\quad = \frac{K(tu)-K(0)}{t}+\frac{(g(tu)-1)}{t}K(u), \end{aligned}$$

for all small enough $t>0$. Given radiality and so differentiability of K_u established in Cor. 4.1, the preceding equation also yields, for non-zero $K(u)$,

$$\lambda_u'(1)\cdot\left(1+\lim_{t\to 0}\frac{(h(tu)-1)}{t}\right) = \lambda_u'(0)+\lim_{t\to 0}\frac{(g(tu)-1)}{t},$$

implying, since $\lambda_u'>0$, that $n\delta_n^h\to\infty$ iff $n\delta_n^g\to\infty$, as is borne out below.

Corollary 4.2. *For continuous (K,h,g) solving (GGE) and u with $K(u)\neq 0$,*

$$n\delta_n^g\to\infty \text{ iff } n\delta_n^h\to\infty,$$

and then for $s>0$

$$\lambda_u(s)=s.$$

Thus λ_u is differentiable in all cases and so K_u is differentiable.

Proof. First we suppose that $n\delta_n^h\to\infty$. Here, according to Proposition 4.2, steering through Q towards $s>0$ is provided by

$$k_n^h(s) := \frac{\log(1+sn\delta_n^h)}{n\log(1+\delta_n^h)}.$$

So

$$\begin{aligned} \log\bar{q}_n(k_n^h(s)) &= \frac{\log\left(s+\frac{1}{n\delta_n^h}\right)+\log n\delta_n^h}{n\log(1+\delta_n^h)}n\log(1+\delta_n^h)-\log(n\delta_n^h) \\ &= \log\left(s+\frac{1}{n\delta_n^h}\right): \\ \bar{q}_n(k_n^h(s)) &= s+\frac{1}{n\delta_n^h}. \end{aligned}$$

We now consider that

$$\frac{(1+\delta_n^g)^{kn}-1}{n\delta_n^g}=q_n^g(k_n^h)=\bar{q}_n^g(k_n^h)-\frac{1}{n\delta_n^g}=\left(s+\frac{1}{n\delta_n^h}\right)-\frac{1}{n\delta_n^g}.$$

Hence, with k for $k^h(s)$ and $\bar{k}$ for $k^h(1)$,

$$\lambda(s)=\lim_n\frac{q_n^g(k)}{q_n^g(\bar{k})}=\lim_n\frac{(1+\delta_n^g)^{kn}-1}{(1+\delta_n^g)^{\bar{k}n}-1}=\lim_n\frac{n\delta_n^g\left(s+\frac{1}{n\delta_n^h}\right)-1}{n\delta_n^g\left(1+\frac{1}{n\delta_n^h}\right)-1}$$

$$=\lim_n\frac{\left(s+\frac{1}{n\delta_n^h}\right)-1/(n\delta_n^g)}{\left(1+\frac{1}{n\delta_n^h}\right)-1/(n\delta_n^g)}.$$

So, provided $\gamma_g\neq 1$, we conclude that since $n\delta_n^h\to+\infty$,

$$\lambda_u(s)=\frac{\gamma_g s-1}{\gamma_g-1}\text{ or }s,\ \text{according as }n\delta_n^g\to\gamma_g\in\mathbb{R}\text{ or }n\delta_n^g\to\infty.$$

Taking $s\neq 1$, Proposition 4.3 implies that the case $n\delta_n^g\to\gamma_g=1$ cannot arise as $K(u)\neq 0$ yields the finiteness of $\lambda_u(s)>0$. We 'park this case' temporarily, but will eventually show that also $n\delta_n^g\to\gamma_g\in\mathbb{R}$ cannot occur.

Now suppose that $n\delta_n^h\to\gamma_h\in\mathbb{R}$ but $n\delta_n^g\to+\infty$. Taking

$$k^h(s)=\frac{\log(1+s\gamma_h)}{\gamma_h}$$

gives

$$\lambda_u(s)=\lim_n\frac{q_n^g(k^h)}{q_n^g(\bar{k}^h)}=\lim_n\frac{\bar{q}_n^g(k^h)}{\bar{q}_n^g(\bar{k}^h)}=\lim_n\frac{(1+\delta_n^g)^{kn}}{(1+\delta_n^g)^{\bar{k}n}}.$$

So again, since $\lambda_u(s)>0$ as $K(u)\neq 0$,

$$\log\lambda_u(s)=\left(\frac{\log(1+s\gamma^b)/(1+\gamma^b)}{\gamma^b}\right)n\log(1+\delta_n^g).$$

Here the RHS is divergent, since as above

$$n\log(1+\delta_n^g)=\frac{n\delta_n^g}{1+d(\delta_n^g)}\to\infty.$$

Again, by Proposition 4.3, this case cannot arise.

But now that all forms of $\lambda_u(s)$ are known, we see that λ_u is differentiable and, by the radiality property, so is K_u.

It now follows that in the 'parked case' with $n\delta_n^h\to+\infty$ and $n\delta_n^g\to\gamma_g\in\mathbb{R}$, in fact g is differentiable at 0. So by our initial observations in the introductory paragraph, h is differentiable at 0, contrary to $n\delta_n^h\to+\infty$. That is, the first case stated in the parked case does not in fact arise. □

Remark. The case $n\delta_n^h \to \gamma_h$ but $n\delta_n^g \to +\infty$ above, ruled out by Proposition 4.3, is also ruled out by the fact that h and K being differentiable implies g is differentiable.

Our next result addresses the partially pexiderized (GS) functional equation below, whose solution we will need later.

$$g(su + h(su)tu) = g(su)g(tu).$$

For $g > 0$, taking $\kappa(tu) = \log g(tu)$ gives rise to a special case of (GGE) identified below (with κ replacing K).

Proposition 4.4. *For continuous (K, h, g) solving (GGE), the kernel K is differentiable along u for $h(u) > 0$, $K(u) \neq 0$ and is strictly increasing along u. Hence if $K(u) = 0$, and $h(su) > 0$ with $s > 0$, then*

$$K(su) = 0, \text{ i.e. } K(su) = \lambda_u(s)K(u) \text{ with } \lambda_u(s) \equiv 0.$$

Furthermore, if either auxiliary generates a Popa binary operation along u, then so does the other.

In particular, continuous solutions of the equation

$$\kappa(a + h(a)b) = \kappa(a) + \kappa(b) \text{ for } a, b \in \langle u\rangle$$

have $h(tu) \equiv (1 + t\rho)u$ for some $\rho \in \mathbb{R}$.

Proof. The first statement follows from Corollaries 4.1 and 4.2 and Proposition 4.3. Suppose that $K(u) = 0$. We claim that $K(su) = 0$ for all $s > 0$. Suppose not and that $K(ru) \neq 0$ for some $r > 0$. Then with $s = r^{-1}$

$$0 = K(u) = K(sru) = \lambda_{ru}(r^{-1})K(ru) \neq 0.$$

Since by Corollaries 4.1 and 4.2, $\lambda_{ru}(r^{-1}) > 0$, this is a contradiction. So $K(ru) = 0$ for all $r > 0$.

Putting $U = K(u), V = K(v), H = hK^{-1}$ and $G = gK^{-1}$,

$$K(u + h(u)v) = K(u) + g(u)K(v),$$
$$U + H(U)V = K^{-1}[U + G(U)V].$$

Suppose $h(u) = 1 + \rho u$. Then, as $u + h(u)v = u + v + \rho uv$ is commutative in u, v,

$$K(u \circ_\rho v) = K(u) + g(u)K(v) = K(v) + g(v)K(u),$$
$$(g(u) - 1)K(v) = K(u)(g(v) - 1),$$
$$K(v)/(g(v) - 1) = K(u)/(g(u) - 1) = c, \text{ say},$$
$$g(u) = 1 + cK(u),$$
$$G(x) = g(K^{-1}(x)) = 1 + cx.$$

Conversely, suppose $G(U) = 1+cU$. Then as above $U+G(U)V = U+V+cUV$, so

$$\begin{aligned} U + H(U)V &= K^{-1}[U + G(U)V] = V + H(V)U, \\ (H(U) - 1)V &= (H(V) - 1)U, \\ (H(U) - 1)/U &= (H(V) - 1)/V = \rho, \text{ say}, \\ H(U) &= 1 + \rho U. \end{aligned}$$

In either case commutativity implies that the auxiliary on the other side of the equation generates a binary group operation.

The final assertion arises by a specialization of (GGE) to $g(x) \equiv 1$, permitted by Proposition 4.3. □

Corollary 4.3. (i) *For* $s, t > 0$,

$$\lambda_{su}(t) = \lambda_u(st)/\lambda_u(s) \text{ and } \lambda'_{su}(t) = \lambda'_u(st)s/\lambda_u(s).$$

(ii) *For* u *with* $h(u) > 0$ *and* $h(-u) > 0$,

$$K(-u) = -g(-u)\lambda_u(1/h(-u))K(u).$$

Proof. (i) As for the first assertion, $\lambda_u(st) = \lambda_{su}(t)\lambda_u(s)$, this holds, since

$$K(tsu) = \lambda_u(st)K(u) = \lambda_{su}(t)K(su) = \lambda_{su}(t)\lambda_u(s)K(u), \text{ for } s, t \geqslant 0.$$

The second assertion follows from Proposition 4.4 by differentiation with respect to t.

(ii) This is immediate from

$$0 = K(-u + h(-u)u/h(-u)) = K(-u) + g(-u)K(u/h(-u)).$$

□

5. Proof of Theorem 3.1

Fix u and t. Recall from Proposition 4.3 that

$$K(uh_{kn}(u/n)/n) = g_{kn}(u/n)g_{\bar{k}n}(u/n)^{-1}K(uh_{\bar{k}n}(u/n)/n).$$

Again as in Proposition 4.3 with $k = t$, $\bar{k} = 1$ and integers $m(n)$ with $m(n)/n \to t$ writing $\gamma(u)$ for $\gamma_g(u)$, as in Proposition 4.1 and Proposition 4.2,

$$\frac{g(u/n)^{tn} - 1}{g(u/n) - 1} \to \frac{e^{t\gamma(u)} - 1}{\gamma(u)} \text{and} \frac{g(u/n)^{m(n)} - 1}{g(u/n)^n - 1} \to \frac{e^{t\gamma(u)} - 1}{e^{\gamma(u)} - 1},$$

appropriately interpreted for $\gamma = 0$. Similarly for h, and writing $\rho(u)$ for $\gamma_h(u)$,

$$\frac{h(u/n)^{tn} - 1}{h(u/n) - 1} \to \frac{e^{t\rho(u)} - 1}{\rho(u)}.$$

By continuity of K, these limit relations lead to the first claims of Theorem 3.1, that

$$K\left(\frac{e^{t\rho(u)}-1}{\rho(u)}u\right)=\frac{e^{t\gamma(u)}-1}{e^{\gamma(u)}-1}K\left(\frac{e^{t\rho(u)}-1}{\rho(u)}\right).$$

To proceed further, reparametrize the coefficient of u by taking $s{:=}(e^{t\rho(u)}-1)/\rho(u)$ and let $s=\mu=\mu(u){:=}(e^{\rho(u)}-1)/\rho(u)$ correspond to $t=1$. Solving for t in terms of s gives $e^{t\rho(u)}=(1+s\rho(u))$ and so

$$K(su)=\frac{(1+\rho(u)s)^{\gamma(u)/\rho(u)}-1}{e^{\gamma(u)}-1}K(\mu u):\quad K(u)=\frac{(1+\rho(u))^{\gamma(u)/\rho(u)}-1}{e^{\gamma(u)}-1}K(\mu u),$$

the latter by specializing the former to $s=1$. After cross-substitution,

$$K(su)=\frac{(1+s\rho(u))^{\gamma(u)/\rho(u)}-1}{(1+\rho(u))^{\gamma(u)/\rho(u)}-1}K(u)=\lambda_u(s,\rho(u),\gamma(u))K(u).$$

Here if $\gamma(u)=\rho(u)$, then $K(su)=sK(u)$ and then $\{u:K(su)=sK(u)\}=\{u:\gamma(u)=\rho(u)\}$ is a linear subspace. $\square$

6. Shuffling and switching

The defining formula (†) (Sect. 3, Definition 2) of the link function involves all the standard homomorphisms between different *scalar* Popa groups, i.e. Popa groups on $\mathbb{R}$ with (η_ρ) in Sect. 1 specialized to $\rho(t)=rt$ for $r,t\in\mathbb{R}$, so between $\mathbb{G}_r$ and $\mathbb{G}_\theta$ for $r,\theta\in[0,\infty]$. These are summarized in the table below for $\mathbb{G}_r=\mathbb{G}_r(\mathbb{R})$ and $\mathbb{G}_\theta=\mathbb{G}_\theta(\mathbb{R})$, reproduced for convenience from [12] Theorem BO (cf. [9,10,34]).

Popa parameter	$\theta=0$	$\theta\in(0,\infty)$	$\theta=\infty$
$r=0$	κt	$\eta_\theta^{-1}(e^{\theta\kappa t})$	$e^{\kappa t}$
$r\in(0,\infty)$	$\log\eta_r(t)^{\kappa/r}$	$\eta_\theta^{-1}(\eta_r(t)^{\theta\kappa/r})$	$\eta_r(t)^{\kappa/r}$
$r=\infty$	$\log t^\kappa$	$\eta_\theta^{-1}(t^{\theta\kappa})$	t^κ

Theorem 3.1 may now be interpreted as a *shuffling*, via the link function, of these Popa homomorphisms. Explicitly, Theorem 3.1 may be read as saying (see Corollary 6.1 below) that for a given pair (K,g) satisfying (GFE), the kernel K induces a map between the canonical scalar Popa homomorphisms.

Corollary 6.1. *For (K,g) satisfying $(GFE),\gamma=\log g$, and fixed $u\in X$, with $\rho(u)>0$ and $\gamma(u)=1$, put*

$$a(u){:=}e^{\rho(u)}-1.$$

Then:

(i) *The map* $a : \mathbb{G}_\rho(X) \to \mathbb{G}_\iota(\mathbb{R})$ *is additive, for* $\iota(t){:=}t$ $(t \in \mathbb{R})$; *equivalently, with*

$$b(u){:=}\log(1 + a(u)),$$

b is linear on $\mathbb{G}_\rho(X)$:

$$a(u + v) = a(u) \circ_\iota a(v), \qquad b(u + v) = b(u) + b(v).$$

(ii) *Put* $c(u){:=}\gamma(u)/\rho(u)$; *then, up to the constant factor* $\eta^{-1}_{a(u)}(\eta_{a(u)}(1)^{c(u)})$ *below and with* $\sim$ *denoting isomorphism,*

$$K_u : \langle u \rangle_\rho \sim \mathbb{G}_{a(u)}(\mathbb{R}) \to \langle K(u) \rangle_{a(u)}$$

induces a map $\mathbb{G}_{a(u)}(\mathbb{R}) \to \mathbb{G}_{a(u)}(\mathbb{R})$:

$$\eta^{-1}_{a(u)}(\eta_{a(u)}(1)^{c(u)}) \cdot K(su) = \eta^{-1}_{a(u)}(\eta_{a(u)}(s)^{c(u)})K(u).$$

So again

$$K(\langle u \rangle_\rho) \subseteq \langle K(u) \rangle_Y.$$

(iii) *Taking, for any* $w \in \mathbb{G}_\rho(X)$ *and* $b \in \mathbb{R}$,

$$b_K(w){:=}e^{c(w)\log[1+\rho(w)]} - 1, \qquad \psi_b(t){=}\eta_b^{-1}(e^{t\log[1+b]}),$$

the kernel function K induces a map between homomorphisms $\mathbb{G}_{\rho(w)}(\mathbb{R}) \to \mathbb{G}_{b_K(w)}(\mathbb{R})$:

$$K(\psi_{\rho(w)}(t)w) = \begin{cases} \psi_{b_K(w)}(t)K(w), & c(w) \neq 0, \\ tK(w), & c(w) = 0. \end{cases}$$

Proof. (i) Since ρ is additive, by definition of a :

$$(1 + a(u))(1 + a(v)) = e^{\rho(u)+\rho(v)} = e^{\rho(u+v)} = 1 + a(u + v) :$$

$$a(u + v) = a(u) + a(v) + a(u)a(v) = a(u) \circ_1 a(v) :$$

$$b(u) + b(v) = \log(1 + a(u))(1 + a(v)) = \log(1 + a(u + v)) = b(u + v).$$

(ii) For $h_u(t) = 1 + t\rho(u)$ with $h'_u(t) \equiv \rho(u)$, take $w = w(u){:=}(e^{\rho(u)} - 1)u/\rho(u) \in \langle u \rangle_\rho$ (as $\rho(w) > 0$). By homogeneity of directional derivatives,

$$\rho(w(u)) = e^{\rho(u)} - 1, \qquad \gamma(w(u)) = (e^{\rho(u)} - 1)\gamma(u)/\rho(u) = \rho(w(u))\gamma(u)/\rho(u) :$$

$$c(u){:=}\gamma(u)/\rho(u) = \gamma(w(u))/\rho(w(u)) = c(w(u)).$$

The operation $\circ_\rho$ on $\langle w \rangle_\rho$ is the same as $\circ_{a(u)}$ on $\mathbb{R}$, since $\rho(w(u)) = e^{\rho(u)} - 1 = a(u)$: indeed,

$$sw \circ_\rho tw = sw + tw + stw\rho(w) = [s + t + sta(u)]w = [s \circ_{a(u)} t]w.$$

As $\alpha{:=}a(u) \neq 0$, we may write $\eta_\alpha(s){:=}1 + a(u)s$, and put

$$t = \frac{\log \eta_\alpha(s)}{\log \eta_\alpha(1)} = \frac{\log[1 + s(e^{\rho(u)} - 1)]}{\rho(u)} : \qquad s = \frac{e^{\rho(u)t} - 1}{e^{\rho(u)} - 1}.$$

As $a(u){:=}e^{\rho(u)}-1$ and $c(u){:=}\gamma(u)/\rho(u)$, as in Theorem 3.1, with w for $w(u)$:

$$K(sw(u))=\frac{e^{\gamma(u)t}-1}{e^{\gamma(u)}-1}K(w)=\frac{\eta_\alpha(s)^{c(u)}-1}{\eta_\alpha(1)^{c(u)}-1}K(w)$$
$$=\frac{[\eta_\alpha(s)^{c(u)}-1]/\alpha}{[\eta_\alpha(1)^{c(u)}-1]/\alpha}K(w)=\frac{\eta_\alpha^{-1}(\eta_\alpha(s)^{c(u)})}{\eta_\alpha^{-1}(\eta_\alpha(1)^{c(u)})}K(w(u)).$$

(iii) With $w(u)$ as in (ii) above,

$$\rho(w(u))=e^{\rho(u)}-1(=a(u)):\qquad \rho(u)=\log[1+\rho(w(u))]:$$
$$\log[1+b_K(w)]=\rho(u):\qquad b_K(w){:=}e^{\log[1+\rho(w(u))]}-1.$$

Substitution into the formula (*Rad*) of Theorem 3.1 yields the assertion. □

As a further corollary of Theorem 3.1, we now have the following radial version of a familiar result (see e.g. [4, Proof of Lemma 3.2.1], [7, Th. 1(ii)], [16, (2.2)], [3]), here written as result on *switching* between tu and u. (We will encounter a skeletal version within the proof of Corollary 6.2 below.)

Lemma 6.1. *For* (K,g) *satisfying* (GFE) *with* $K(u)\neq 0$ *and with* $g\neq 1$ *on* $\langle u\rangle_\rho$ *except at* 0 :

$$(g(tu)-1)K(u)=(g(u)-1)K(tu)\qquad (tu\in\mathbb{G}_\rho(X)),$$

that is,

$$(g(x)-1)K(u)=(g(u)-1)K(x)\qquad (x\in\langle u\rangle_\rho).$$

Proof. Here $u\neq 0$ (since $K(0)=0$, by (GFE)). As $\langle u\rangle_\rho$ is abelian ([12, §3 Lemma]),

$$K(su\circ_\rho tu)=K(su)+g(su)K(tu)=K(tu)+g(tu)K(su).$$

As $K(u)\neq 0$ and $g(su)\neq 1$ for $s\neq 0$, Theorem 3.1 yields

$$K(tu)=\lambda_u(t)K(u),\tag{R}$$

whence

$$K(tu)[g(su)-1]=[g(tu)-1]K(su):\qquad \lambda_u(t)/[g(tu)-1]=\lambda_u(s)/[g(su)-1].$$

So this is constant, say $k(u)$. Hence

$$[g(tu)-1]K(u)=k(u)\lambda_u(t)K(u)=k(u)K(tu),$$

again using (R). Take $t=1$; then

$$[g(u)-1]K(u)=k(u)K(u):\qquad [g(u)-1]=k(u).$$

□

Lemma 6.2 secures the non-triviality of the radial function $g_u(t){:=}g(tu)$.

Lemma 6.2. *For* g *continuous satisfying* (M), *if* $g(u)\neq 1$ *and* $\rho(u)=1$, *then* $g(tu)\neq 1$ *for* $t\neq 0$.

Proof. From Lemma 2.1, $\gamma := \log g$ satisfies (GFE) in the simpler additive form (A). So Theorem 3.1 here yields

$$\gamma\left(\frac{e^t-1}{e-1}u\right)=\frac{e^{\gamma(u)t}-1}{e^{\gamma(u)}-1}\gamma(u),$$

as $\gamma(u)\neq 0$. So for $t\neq 0$, $\gamma(tu)\neq 0$, and so $g(tu)\neq 1$. □

Theorem 7.1 and Corollary 6.2 below will immediately imply our first main result, Theorem 7.2 below, on the existence of σ_g. As noted in the Introduction, the dichotomy below concerning two null spaces arises because two hyperplanes passing through the origin (representing the pair of null spaces of interest) have intersection with co-dimension 1, when coincident, but co-dimension 2 otherwise.

Corollary 6.2. *Suppose* (K,g) *satisfies* (GFE), *so that for some* $\alpha:\mathbb{G}_\rho(X)\to\mathbb{R}$ *additive,* $\beta\in\mathbb{R}$, *g is characterized in Theorem 2.1 as having the form*

$$g(x)=\bar{g}_{\alpha,\beta}(x):=e^{\alpha(x)}(1+\rho(x))^{\beta}.$$

Then the restriction of the kernel $K|\mathcal{N}(\rho)$ *is linear on* $\mathcal{N}^*(\alpha)=\mathcal{N}(\alpha)\cap\{x:\bar{g}(x)=1\}$. *Furthermore, either*

$$\mathcal{N}(\rho)=\mathcal{N}(\alpha) \qquad (N_A^=)$$

holds, or else $K|\mathcal{N}^*(\alpha)=0$, *and then*

$$K(\mathcal{N}(\rho))=\langle K(v)\rangle \text{ for some } v\in\mathcal{N}(\rho)\backslash\mathcal{N}(\alpha). \qquad (N_B^=)$$

Proof. Since $N_A^=$ holds and

$$\bar{g}(x)=e^{\alpha(x)}(1+\rho(x))^{\beta},$$

$g|\mathcal{N}^*(\alpha)\equiv 1$ (as $\rho(x)=0$ here), and so additivity of K on $\mathcal{N}^*(\alpha)$ and hence (by continuity) its linearity is immediate. If $\mathcal{N}(\rho)=\mathcal{N}(\alpha)$, then $K|\mathcal{N}(\rho)$ is linear. Otherwise $\mathcal{N}^*(\alpha)$ is of co-dimension 1 in the subspace $\mathcal{N}(\rho)$ (see e.g. [24, 3.5.1]). In particular, we may choose and fix $v_2\in\mathcal{N}(\rho)\backslash\mathcal{N}(\alpha)$. Now take $v_1\in\mathcal{N}^*(\alpha)$ arbitrarily. Then as $v_1,v_2\in\mathcal{N}(\rho)$ by commutativity and (GFE) :

$$K(v_2)+e^{\alpha(v_2)}K(v_1)=K(v_1+v_2)=K(v_1)+e^{\alpha(v_1)}K(v_2),$$
$$K(v_1)[e^{\alpha(v_2)}-1]=K(v_2)[e^{\alpha(v_1)}-1]=0 \qquad (\text{as } e^{\alpha(v_1)}=1):$$
$$K(v_1)=0 \qquad (\text{as } e^{\alpha(v_2)}\neq 1).$$

That is, $K|\mathcal{N}^*(\alpha)=0$, and, by Theorem 3.1, $K(\mathcal{N}(\rho))=\langle K(v_2)\rangle$. □

7. Inducing a Popa structure in Y from (GFE)

We now turn our attention to inducing a Popa-group structure on Y from a pair (K, g) satisfying (GFE). Recall that $\langle\Sigma\rangle$ denotes the *linear span* of Σ.

Our first main result, Theorem 7.2 below, is motivated by attempting an operation on the image $K(X)$ utilizing a solution (K, g) of the (GFE) via:

$$y \circ y' = y + g(x)y' \text{ for some } x \text{ with } y = K(x),$$

which faces an obstruction, unless $K(x_1) = K(x_2)$ implies $g(x_1) = g(x_2)$, i.e. $g(x_1 - x_2) = 1$. This is resolved in the following

Theorem 7.1. *For (K, g) satisfying (GFE) with $g \neq 1$, there exists $\sigma\colon Y \to \mathbb{R}$ such that $g = g^\sigma$ iff one of the following two conditions holds:*

$$\mathcal{N}(\rho) \subseteq \mathcal{N}(\gamma), \tag{N_A}$$

for $\gamma = \log g$ together with the range condition

$$\mathcal{R}(K) \neq K(\mathcal{N}(\rho)),$$

or

$$K(\mathcal{N}(\rho)) \subseteq \langle K(u)\rangle \text{ for some } u \text{ with } g(u) \neq 1. \tag{N_B}$$

Then σ is uniquely determined on $K(X)$.

Proof. We first establish necessity. We suppose the pair (K, g) satisfies (GFE) with $g = g^\sigma$ for some continuous linear $\sigma\colon Y \to \mathbb{R}$. The result follows from the Abelian Dichotomy of [12, §6] that either

(i) $K(\mathcal{N}(\rho)) \subseteq \mathcal{N}(\sigma)$. or
(ii) $K(\mathcal{N}(\rho)) \subseteq \langle K(u)\rangle_\sigma$ for some $u \in X$ with $\sigma(K(u)) \neq 0$.

In case (i),

$$\sigma(K(u)) = 0 \text{ for } u \in \mathcal{N}(\rho),$$

so that, for such u, $g(u) = 1$, i.e. $\gamma(u) = 0$. Thus $\mathcal{N}(\rho) \subseteq \mathcal{N}(\gamma)$ and so (N_A) holds. Furthermore, if the range condition were to fail, then $\mathcal{R}(K) = K(\mathcal{N}(\rho)) \subseteq \mathcal{N}(\sigma)$, implying that $\sigma(K(x)) = 0$ for all x, i.e. that $g = 1$ and

$$K(x \circ_\rho y) = K(x) + K(y).$$

Otherwise (ii) holds, i.e. $K(\mathcal{N}(\rho)) \subseteq \langle K(u)\rangle_\sigma$ for some $u \in X$ with $\sigma(K(u)) \neq 0$, so in particular with $g^\sigma(u) \neq 1$, and a fortiori (N_B) holds.

Note that (N_B) needs no subscript on $\langle K(u)\rangle$ as $K|\mathcal{N}(\rho))$ is linear, so $K(\mathcal{N}(\rho))$ is a subspace of Y. This completes the proof of necessity. □

The converse direction requires the construction of σ from g, so is quite involved. Theorem 7.2 below asserts uniqueness and sufficiency, with the latter following from Proposition 7.1A and 7.1B, our next results. This will involve *complemented* subspaces. We note that in the context of Y a Banach space, *algebraically* complementary spaces are *topologically* complementary [24, Th. 13.1]. See also [31]. We recall our blanket assumption that $\rho \neq 0$.

Proposition 7.1A. *If (K, g) satisfies both (GFE) with $g \neq 1$ and also (N_A), that is,*

$$\mathcal{N}(\rho) \subseteq \mathcal{N}(\gamma),$$

for $\gamma = \log g$ (on $\mathbb{G}^*(X)$), then a necessary and sufficient condition that $g = g^\sigma$ for some linear $\sigma\colon Y \to \mathbb{R}$ is the range condition

$$\mathcal{R}(K) \neq K(\mathcal{N}(\rho)).$$

In this case σ is continuous provided $K(\mathcal{N}(\gamma))$ is closed and complemented.

Proof. Here $K|\mathcal{N}(\rho)$ is linear, and so $K(\mathcal{N}(\rho))$ is a vector subspace of Y.

Suppose first that $\mathcal{R}(K) \neq K(\mathcal{N}(\rho))$. Then there is $u \in X$ with $K(u) \notin K(\mathcal{N}(\rho))$. So $K(u) \neq 0$ and $\rho(u) \neq 0$, so that $g(u) \neq 1$. Without loss of generality we may assume that $\rho(u) = 1$. Indeed, by Theorem 3.1, $K(u/\rho(u)) = \lambda_u(\rho(u)^{-1})K(u)$, and so $K(u/\rho(u)) \notin K(\mathcal{N}(\rho))$ by linearity of $K|\mathcal{N}(\rho)$.

Step 1. We first prove the result under the assumption that $Y = \langle K(X)\rangle$, the span here being assumed a closed subspace.

We begin by defining a continuous linear map σ by setting:

$$\sigma(y) := \begin{cases} 0, & y \in K(\mathcal{N}(\rho)), \\ t(g(u) - 1), & y = tK(u). \end{cases} \qquad (\sigma_A)$$

The two clauses are thus mutually exclusive and so

$$\sigma K(x) = g(x) - 1 \qquad (Eq)$$

certainly holds for the one vector $x = u$.

We first decompose K into summands and likewise g into factors, by projecting along $\langle u\rangle$. On these we act with σ, as σ has non-zero effect only on the K-image $\langle u\rangle$. Thereafter we reassemble the components.

As $\rho(x - \rho(x)u) = 0$ and $g(x - \rho(x)u) = 1$,

$$K(x) = K([x - \rho(x)u] \circ_\rho \rho(x)u) = K(x - \rho(x)u) + K(\rho(x)u).$$

Since $x - \rho(x)u \in \mathcal{N}(\rho)$ and as $\sigma = 0$ on $K(\mathcal{N}(\rho))$ and σ is linear, applying σ gives

$$\sigma K(x) = \sigma K(\rho(x)u). \qquad (A1)$$

As $K(u) \neq 0$ and $g(u) \neq 1$ we may put (by Theorem 3.1)

$$K(\rho(x)u) = \lambda_w(\rho(x))K(u). \qquad (A2)$$

Here λ_w is defined by the formula of Theorem 3.1. By $(A1)$ and applying σ to $(A2)$,

$$\sigma K(x)=\sigma K(\rho(x)u)=\lambda_w(\rho(x))\sigma K(u)=\lambda_w(\rho(x))[g(u)-1]. \tag{$A3$}$$

This completes the action on the K side.

We decompose g similarly by (M), as $\mathcal{N}(\rho)=\mathcal{N}(\gamma)$:

$$g(x)=g(x-\rho(x)u)\cdot g(\rho(x)u): \qquad g(x)=g(\rho(x)u). \tag{$A4$}$$

Here again $\rho(x-\rho(x)u)=0$, so $(x-\rho(x)u)\circ_\rho \rho(x)u=x$.

We now act on the g side.

By Lemma 6.2 on non-triviality, $g_u(tu)\neq 0$ for $t\neq 0$, so we may apply Lemma 6.1 (on switching). So

$$\begin{aligned}(g(\rho(x)u)-1)K(u)&=K(\rho(x)u)[g(u)-1]\\&=[g(u)-1]\lambda_w(\rho(x))K(u) \qquad \text{(from } (A2)),\\(g(\rho(x)u)-1)[g(u)-1]&=\lambda_w(\rho(x))[g(u)-1]^2 \qquad \text{(apply } \sigma \text{ and } (\sigma_A)):\end{aligned}$$

$$(g(\rho(x)u)-1)=\lambda_w(\rho(x))[g(u)-1] \qquad \text{(cancelling)}, \tag{$A5$}$$

as $g(u)-1\neq 0$.

We now reassemble the components. Combining, $(A5)$ with $(A4)$ and $(A3)$ gives

$$(g(x)-1)=\lambda_w(\rho(x))[g(u)-1]=\sigma(K(\rho(x)u))=\sigma K(x).$$

So (Eq) holds for *all* vectors $x\in X$. This completes the reassembly.

Step 2. If $Y\neq\langle K(X)\rangle$, choose in Y a subspace Z complementary to $\langle K(X)\rangle$ and define σ as above on $\langle K(X)\rangle$; then extend by taking $\sigma=0$ on Z.

We turn to the converse and suppose now that $g=g^\sigma$ for some linear $\sigma\colon Y\to\mathbb{R}$. We show that (σ_A) holds for some $u\in X$, from which the range condition will follow. By (GFE), $\sigma(y)=g(x)-1$ whenever $y=K(x)$. Further, as (N_A) holds, $g(x)=1$ for $x\in\mathcal{N}(\rho)$ and so, since

$$K(x)+K(x)=K(x)+g(x)K(x)=K(x\circ_\rho x)=K(x)+K(x)+\sigma(K(x))K(x),$$

we conclude that

$$\sigma(K(x))=0,$$

whether or not $K(x)=0$. That is, $\sigma(y)=0$ for $y\in K(\mathcal{N}(\rho))$, as in the first clause of (σ_A).

Since $g\neq 1$ there is $u\in X$ with $g(u)\neq 1$. By (N_A), $\rho(u)\neq 0$. The kernel $\mathcal{N}(\rho)$ is of co-dimension 1 in the space X, so that X is the span of u and $\mathcal{N}(\rho)$. The assumption $g=g^\sigma$ with σ linear now gives for $t\neq 0$

$$\sigma(K(u))=g(u))-1\neq 0: \qquad \sigma(tK(u)))=t(g(u)-1),$$

as in the second clause of (σ_A). As the two clauses are exclusive, $K(u) \notin K(\mathcal{N}(\rho))$. $\square$

Since, as above, the kernel $\mathcal{N}(\rho)$ is of co-dimension 1 in the space X, it seems natural to view the condition

$$\mathcal{R}(K) = K(\mathcal{N}(\rho))$$

as a type of degeneracy which we will refer to here (only) as yielding *improper* solutions of (GFE).

An improper example where $\mathcal{R}(K) = K(\mathcal{N}(\rho))$ is given by taking $X = Y = \mathbb{R}^2, \rho(x) = x_1$ and $g(x) = 1 + x_1$ and $K(x) = (0, x_2)$ for $x = (x_1, x_2)$. Then (GFE) is satisfied, since

$$K(x \circ_\rho y) = (0, x_2) + (1 + x_1)(0, y_2).$$

Here $\mathcal{N}(\rho) = \{x : x_1 = 0\} = \mathcal{N}(\gamma)$, so that $\mathcal{R}(K) = K(\mathcal{N}(\rho)) = \mathcal{N}(\rho)$, and the contradiction that $\sigma(0, x_2) = x_1$ follows from the condition

$$x_2 + (1 + x_1)y_2 = x_2 + (1 + \sigma(0, x_2))y_2.$$

In the case (N_A) Proposition 7.1A above shows that all but the improper solutions of (GFE) are homomorphies. In the alternative case (N_B) all solutions of (GFE) are homomorphies, as we now show.

Proposition 7.1B. *If (K, g) satisfies (GFE) and (N_B), that is,*

$$K(\mathcal{N}(\rho)) \subseteq \langle K(w)\rangle_Y,$$

for some $w \in \mathcal{N}(\rho)$, then $g = g^\sigma$ for some linear $\sigma\colon Y \to \mathbb{R}$ which is continuous, provided $K(\mathcal{N}^*(\gamma))$ is closed complemented.

Proof. Here $V_0{:=}\mathcal{N}^*(\gamma) = \mathcal{N}(\gamma) \cap \mathcal{N}(\rho)$ is a subgroup of $\mathbb{G}_\rho(X)$, as

$$K(x + y) = K(x \circ_\rho y) = K(x) + g(x)K(y) = K(x) + K(y),$$

and so $K|V_0$ is a homomorphism from $\mathbb{G}_\rho(X)$ to Y. Since V_0 is a subspace of $\mathcal{N}(\rho)$, we copy the argument of Proposition 7.1A working with the linear map $K|V_0$ with $V_0 \subseteq \mathcal{N}(\gamma)$ as a replacement for $K|\mathcal{N}(\rho)$; so if $g = g^\sigma$ is to hold, then $K|V_0 : V_0 \to \mathcal{N}(\sigma)$.

In $\mathcal{N}(\rho)$ choose a subspace V_1 complementary to V_0, and let $\pi_i : X \to V_i$ denote projection onto V_i. Notice that for any $v \in \mathcal{N}(\rho)$, as $\pi_0(v) \in \mathcal{N}(\rho)$ and $\pi_0(v) \in \mathcal{N}(\gamma)$

$$\begin{aligned} K(v) &= K(\pi_0(v) \circ_\rho \pi_1(v)) = K(\pi_0(v)) + g(\pi_0(v))K(\pi_1(v)) \\ &= K(\pi_0(v)) + K(\pi_1(v)). \end{aligned}$$

Fix a non-zero $v_1 \in V_1 \subseteq \mathcal{N}(\rho)$; then $V_1 = \langle v_1\rangle$, since by Lemma 2.1 $\gamma = \log g$ is linear on $\mathcal{N}(\rho)$ and so $\mathcal{N}^*(\gamma)$ either equals $\mathcal{N}(\rho)$ or has co-dimension 1 in $\mathcal{N}(\rho)$. We can see this directly as follows. Since $v_1 \notin V_0$, $\gamma(v_1) \neq 0$, so replacing v_1 by $v_1/\gamma(v_1)$, w.l.o.g. $\gamma(v_1) = 1$. For any $z \in \mathcal{N}(\rho)$, $z - \gamma(z)v_1 \in \mathcal{N}(\gamma)$, as

$$\gamma(z - \gamma(z)v_1) = \gamma(z) - \gamma(z) = 0.$$

Likewise, for such z, as $\rho(v_1) = 0$,

$$\rho(z - \gamma(z)v_1) = \rho(z) - \gamma(z)\rho(v_1) = 0 - 0 = 0,$$

i.e. $v_0{:=}z - \gamma(z)v_1 \in V_0$ and so

$$z = v_0 + \gamma(z)v_1, \text{ i.e. } \mathcal{N}(\rho) = V_0 + \langle v_1 \rangle.$$

If ρ is not identically zero, again fix $u \in X$ with $\rho(u) = 1$. Then $x \mapsto \pi_u(x) = x - \rho(x)u$ is again (linear) projection onto $\mathcal{N}(\rho)$. If $\rho \equiv 0$, set u below to 0. Whether or not $\rho \equiv 0$, as $\rho(x - \rho(x)u) = 0$ take $z{:=}x - \rho(x)u \in \mathcal{N}(\rho)$; then for some $v_0 \in V_0$ and some scalar α

$$x = v_0 + \alpha v_1 + \rho(x)u = v \circ_\rho \rho(x)u.$$

So w.l.o.g. provided $K(v_1) \neq 0 \neq K(u)$

$$Y = \langle K(V_0), K(v_1), K(u) \rangle.$$

We first show that these "generators" are distinct. Recall that $g(v_1) \neq 1$ as $v_1 \notin V_0$ and that for some w with $K(w) \neq 0$

$$K(\mathcal{N}(\rho)) = K(V_0 + \langle v_1 \rangle) \subseteq \langle K(w) \rangle_Y.$$

Suppose first that $K(u) = K(v)$ for some $v \in V_0 + \langle v_1 \rangle$. Then, as $K(u) = K(v) \in \langle K(w) \rangle_Y$ and $K(w) \neq 0$,

$$K(u) = \lambda_w(u)K(w) = \lambda_w(v)K(w).$$

So, since λ_w is montonic,

$$u = v \in \mathcal{N}(\rho),$$

contradicting that $\rho(u) = 1$.

Next suppose that $K(v_1) = K(v_0)$, for some $v_0 \in V_0$. Then, since $-v_0 + v_1 = -v_0 \circ_\rho v_1$ and $g(v_0) = 1$,

$$\begin{aligned} 0 &= -K(v_0) + g(v_0)K(v_1) = K(-v_0 + v_1) \\ &= K(v_1) - g(v_1)K(v_0) = K(v_1) - g(v_1)K(v_1) \\ &= (1 - g(v_1))K(v_1). \end{aligned}$$

So $K(v_1) = 0$, a contradiction.

So the following defines a continuous linear map $\sigma\colon Y \to \mathbb{R}$:

$$\sigma(y) = \begin{cases} 0, & y \in K(V_0), \\ t(g(v_1) - 1), & y = tK(v_1), \\ t(g(u) - 1), & y = tK(u). \end{cases} \qquad (\sigma_B)$$

So (Eq) holds for the two vectors $x = v_1$ and $x = u$.

As with $(A1)$ in Proposition 7.1A, via Lemma 6.1 (on switching),

$$\sigma K(\rho(x)u) = g(\rho(x)u) - 1, \qquad (B1)$$

$$\sigma K(\alpha v_1) = g(\alpha v_1) - 1. \tag{B2}$$

Since v_i are in $\mathcal{N}(\rho)$,

$$\begin{aligned} K(x) &= K(v_0 + \alpha v_1 + \rho(x)u) \\ &= K(v_0) + K(\alpha v_1) + g(\alpha v_1)K(\rho(x)u). \end{aligned}$$

Using (σ_B) and applying σ gives

$$\begin{aligned} \sigma K(x) &= 0 + [g(\alpha v_1) - 1] + g(\alpha v_1)[g(\rho(x)u) - 1] \qquad (\text{by } (B2) \text{ and } (B1)) \\ &= [g(\alpha v_1) - 1] - g(\alpha v_1) + g(\alpha v_1)g(\rho(x)u) \\ &= g(v_0)g(\alpha v_1)g(\rho(x)u) - 1 \\ &= g(x) - 1. \end{aligned}$$

If $Y \neq \langle K(V_0), K(v_1), K(u)\rangle$, this span being assumed a closed subspace, choose in Y a subspace Z complementary to $\langle K(V_0), K(v_1), K(u)\rangle$, and define σ as above on $\langle K(V_0), K(v_1), K(u)\rangle$; then extend by taking $\sigma = 0$ on Z. □

In Lemma 7.1 below we refer to the defining equation (g^σ) in §1.

Lemma 7.1. *If (K, g) satisfies (GFE) non-trivially, then g is uniquely determined by K. In particular, if $g^\sigma = g = g^\tau$, then $\sigma = \tau$ on $K(X)$. Furthermore,*

$$\sigma(K(u \circ_\rho v)) = \sigma(K(u)) \circ_\iota \sigma(K(v)) \qquad (\iota(t) \equiv t).$$

Proof. For given K, suppose both (K, g) and (K, h) satisfy (GFE). As K is non-trivial, we fix $v \in X$ with $K(v) \neq 0$. Then for arbitrary $u \in X$

$$K(u) + h(u)K(v) = K(u \circ_\rho v) = K(u) + g(u)K(v),$$

so $g(u) = h(u)$. So if $g^\tau = g = g^\sigma$, then

$$\sigma(K(u)) = g^\sigma(u) - 1 = g^\tau(u) - 1 = \tau(K(u)),$$

as claimed. Checking the final assertion is routine and omitted here. □

We may now pass to the key existence theorem, our first Main Theorem.

Theorem 7.2. *If (K, g) satisfies (GFE), then, unless $\mathcal{R}(K) = K(\mathcal{N}(\rho))$, there is a unique linear map $\sigma\colon Y \to \mathbb{R}$ such that*

$$\sigma(K(x)) + 1 = g(x) \qquad (x \in X).$$

The map σ is continuous, provided K has closed complemented range.

Proof. By Corollary 6.2, one of $(N_A^{=})$ or $(N_B^{=})$ holds, and so either Proposition 7.1A or 7.1B implies the existence of σ, and its continuity conditional on K having closed range. Its uniqueness is assured by Lemma 7.1. □

8. The generalized Goldie equation

This section is devoted to demonstrating in Theorem 8.1 below that (GGE) is reducible to (GFE). Our main tool is Theorem 3.1, and we will also use Theorem 2.1 (the Index Theorem). For (GGE) to conform with our study of (GFE), we assume here that, just like η_ρ, the non-negative inner auxiliary $h\colon X \to [0,\infty)$ preserves positivity on $\mathbb{G}_h^+(X){:=}\{x \in X : h(x) > 0\}$:

$$h(u + h(u)v) > 0 \text{ for } u, v \in \mathbb{G}_h^+(X).$$

This assumption prompts the question of for which continuous functions $h\colon X \to [0,\infty)$ does the binary operation $x \circ_h y{:=}x + h(x)y$ *preserve positivity*, i.e.

$$h(x), h(y) > 0 \Longrightarrow h(x + h(x)y) > 0.$$

It emerges that strengthening $\Longrightarrow$ above to $\Longleftrightarrow$ yields Chudziak's theorem, that h satisfies (GS) (i.e. $h = \eta_\rho$ for some ρ). For details see [21] (cf. [23]). Our hypothesis is thus weaker; however, this preservation combined with (GGE) yields some similar connections with (GS) below.

We will need to know the connection between the null spaces of the inner and outer auxiliaries. Recall that $K(0) = 0$.

Lemma 8.1. *If (K, g, h) satisfies (GGE) and $K(w) \neq 0$ for some w, then*

$$g(x) = 0 \Longleftrightarrow h(x) = 0 \qquad (x \in X),$$

so that

$$\mathbb{G}_h^+(X){:=}\{x \in X : h(x) > 0\} = \mathbb{G}^+(X) = \mathbb{G}_g^+(X){:=}\{x \in X : g(x) > 0\}.$$

Proof. If $h(a) = 0$, then $g(a) = 0$, since $K(w) \neq 0$ and

$$K(a) = K(a + h(a)w) = K(a) + g(a)K(w).$$

If one had $g(a) = 0$ but $h(a) \neq 0$, then, for any x, taking $b{:=}h(a)^{-1}(x - a)$ gives

$$K(a) = K(a) + g(a)K(b) = K(a + h(a)b) = K(x).$$

Hence K is constant. But $K(0) = 0$, so $K(w) = 0$, a contradiction. □

Our first result identifies a known partially 'pexiderized' variant of the Gołąb–Schinzel equation, (PGS) below, studied in [20, 26]; see [27] for a fully pexiderized equation (cf. [28]). The gist of the matter is in Proposition 4.4 and the solution is given by

$$h_u(t){:=}\begin{cases} 1 + rt, \text{ for } r \neq 0, \text{ and then: } & g_u = g_{r,\theta} = (1 + rt)^{\theta/r}; \\ \quad 1, \quad \text{ for } r = 0, \text{ and then: } & \qquad g_u = g_{0,\theta} = e^{\theta t}. \end{cases}$$

Here $g_u(t) = g(tu)$ and $h_u(t) = h(tu)$. The next proposition links the behaviour of the two auxiliary functions g, h via the linking function λ_w. Motivated by the notation $\langle u\rangle_\rho$ in Popa groups, but now in the context of a Javor group, we write for $u \in X$

$$\langle u\rangle_h = \mathbb{G}_h^+(X) \cap \mathrm{Lin}\{u\}.$$

Proposition 8.1. *For (K, h, g) satisfying (GGE) and for each $w \in X$ with $K(w) \neq 0$, and $u \in \mathbb{G}^+(X)$,*

$$\lambda_w\left(\frac{h(a+h(a)b)}{h(a)h(b)}\right) = \frac{g(a+h(a)b)}{g(a)g(b)} \qquad (a, b \in \mathbb{G}_h^+(X)).$$

In particular, if the auxiliary h_u satisfies the Gołąb–Schinzel equation, then g_u satisfies a partially pexiderized Gołąb–Schinzel equation:

$$g(a+h(a)b) = g(a)g(b) \qquad (a, b \in \langle u\rangle_h). \tag{PGS}$$

So $g = g_{\gamma(u),\rho(u)}$ (for some appropriate parameters), and conversely if g has this form, then g satisfies (PGS) for $h_u = 1 + \rho(u)$.

Proof. We approach the action of K on

$$a + h(a)b + h(a+h(a)b)h(a)h(b)w$$

in two ways. For the approach to be valid we need $h(a+h(a)b) > 0$, which comes from the assumed preservation of positivity (cf. Lemma 8.1). We consider the two sides of the equality

$$\begin{aligned} &K(a + h(a)b + h(a+h(a)b)h(a)h(b)w) \\ &\quad = K(a+h(a)b) + g(a+h(a)b)K(h(a)h(b)w). \end{aligned}$$

Here, with LHS for left-hand side etc.,

$$\begin{aligned} LHS &= K(a + h(a)[b + h(a+h(a)b)h(b)w]) \\ &= K(a) + g(a)K(b + h(b)h(a+h(a)b)w) \\ &= K(a) + g(a)[K(b) + g(b)K(h(a+h(a)b)w) \\ &= K(a) + g(a)K(b) + g(a)g(b)K(h(a+h(a)b)w); \\ RHS &= K(a) + g(a)K(b) + g(a+h(a)b)K(h(a)h(b)w). \end{aligned}$$

Cancelling common terms on the two sides gives, in view of $g(a)g(b) \neq 0$, that

$$g(a)g(b)K(h(a+h(a)b)w) = g(a+h(a)b)K(h(a)h(b)w):$$

$$K\left(\frac{h(a+h(a)b)}{h(a)h(b)}w\right) = \frac{g(a+h(a)b)}{g(a)g(b)}K(w),$$

on replacing w appropriately (since $h(a)h(b) \neq 0$). Now for $K(w) \neq 0$, apply Theorem 3.1. So if h satisfies (GS), then g satisfies (PGS). □

Corollary 8.1. *If (K, h, g) satisfies (GGE), then either K is linear and $g|\langle w\rangle_h = h|\langle w\rangle_h$ for each w, or h satisfies (GS) and g satisfies (PGS).*

Before deducing this result we need to characterize the 'contour behaviour' of the link functions $\lambda_u(t)$ in response to parameter changes.

We recall that when the kernel function is non-zero at u (i.e. $K(u) \neq 0$) the link function $\lambda_u(t)$ is either the identity function $\mathrm{id}(t) = t$, or for some $r > 0$ $\lambda_u(t) = \varphi(rt)/\varphi(r)$, where $\varphi(x)$ takes one of three *contour types*, the exponential, logarithmic, or power-c, all with domain parameter r :

$$\begin{array}{ll} \varphi(x){:=}e^x - 1: & \lambda_u(t) = (e^{rt} - 1)/(e^r - 1), \\ \varphi(x){:=}1 + \log x: & \lambda_u(t) = \log(1 + rt)/\log(1 + r), \\ \varphi(x){:=}(1 + x)^c - 1: & \lambda_u(t) = [(1 + rt)^c - 1]/[(1 + r)^c - 1]. \end{array}$$

These are strictly monotone in t and either convex or concave shaped. In the power-c type convexity arises for $c > 1$ and (like the exponential) is separated from its concave inverse function by the linear variant $\lambda_u(t) = t$ arising from $c = 1$.

Indeed, it is enough to note the relevant second derivative, $\varphi''(rt)$, which according to type is

$$r^2 e^{rt}, \qquad -r^2(1 + rt)^{-2}, \qquad r^2 c(c - 1)(1 + rt)^{c-2}.$$

Lemma 8.2. *For any point (x, y) in the positive quadrant other than $(1, 1)$, there is at most one curve λ_u in any of the three contour types with $\lambda_u(x) = y$.*

Proof. For $s > 0$ and u with $K(u) \neq 0$ as above, recall Corollary 4.3(i):

$$\lambda_{su}(t) = \lambda_u(st)/\lambda_u(s).$$

From here it follows that scaling the vector u by $s > 0$ does not alter the *contour type* of the link function λ_{su} but merely scales its domain parameter from r to sr. Indeed, given the tabulation above, this follows from:

$$\lambda_u(st)/\lambda_u(s) = \frac{\varphi(rst)}{\varphi(r)} \Big/ \frac{\varphi(rs)}{\varphi(r)} = \frac{\varphi(rst)}{\varphi(rs)}.$$

The contours all have $(1, 1)$ as fixed point, but on each side of $t = 1$ the curves of any one type are strictly *monotone* in the domain parameter r, as some routine calculus readily shows (Lemma 8.4 below). For example, the exponential type curves decrease with r to the left of $t = 1$ and increase with r to the right of $t = 1$.

Hence for any point (x, y) in the positive quadrant other than $(1, 1)$ there is at most one curve λ_u in each type with $\lambda_u(x) = y$. □

Proof of Corollary 8.1. We apply Lemma 8.2 and consider w with $K(w) \neq 0$. By Proposition 8.1, for all $s > 0$ and $a, b \in \mathbb{G}_h^+(X) = \{x : h(x) > 0\}$, since $K(sw) = \lambda_w(s)K(w) \neq 0$,

$$\lambda_{sw}\left(\frac{h(a + h(a)b)}{h(a)h(b)}\right) = \frac{g(a + h(a)b)}{g(a)g(b)} = \lambda_w\left(\frac{h(a + h(a)b)}{h(a)h(b)}\right):$$

$$\lambda_{sw}\left(\frac{h(a+h(a)b)}{h(a)h(b)}\right)=\lambda_w\left(\frac{h(a+h(a)b)}{h(a)h(b)}\right). \qquad (*)$$

In the case when $\lambda_w(t)\equiv t$ this last equation holds, no matter the value of s, since also $\lambda_{sw}(t)\equiv t$. But that is a very special case, which leads to a linear K. To begin with, K is homogeneous, i.e. for all $t>0$ and all w in $\mathbb{G}_+^h(X)$

$$K(tw)=tK(w).$$

But in this case $g=h$ on $\langle w\rangle_h$. Indeed, for $s,t>0$,

$$sK(w)+g_w(s)tK(w)=K(sw+h_w(s)tw)=(s+h_w(s)t)K(w).$$

This implies that K is additive and so linear (by homogeneity):

$$K(a+b)=K\left(a+h(a)\frac{b}{h(a)}\right)=K(a)+\frac{g(a)}{h(a)}K(b)=K(a)+K(b).$$

Here h may be arbitrary with $g=h$ on each ray $\langle w\rangle_h$.

So suppose now that for some w we have $\lambda_w\neq \mathrm{id}$. Then for $s>0$ all the curves λ_{sw} are of one contour type differing only in their domain parameter. So $(*)$ contradicts Lemma 8.2 in that there may be at most one contour in any contour type passing through a point unless that point is $(1,1)$. Hence

$$\frac{h(a+h(a)b)}{h(a)h(b)}=1=\frac{g(a+h(a)b)}{g(a)g(b)}.$$

It now follows that h satisfies the (GS) equation and g satisfies the pexiderized variant (PGS). □

We need a further (folk-lore) result.[2]

Lemma 8.3. *For ρ homogeneous on $\mathbb{G}_\rho(X)$, if $\eta_\rho(u)=1+\rho(u)$ satisfies (GS), then ρ is linear on $\mathbb{G}_\rho(X)$.*

Proof. Fix $u,v\in\mathbb{G}_h^+(X)$ and consider α,β with $t{=:}\eta_\rho(\alpha u)=1+\rho(\alpha u)>0$ and $\eta_\rho(\beta u)>0$. Then

$$\begin{aligned}\rho(\alpha u+\beta v)&=\eta_\rho(\alpha u+\eta_\rho(\alpha u)\beta v/t)-1=(1+\rho(\alpha u))(1+\rho(\beta v/t))-1\\&=\rho(\alpha u)+(1+\rho(\alpha u))\rho(\beta v/t)=\alpha\rho(u)+t\rho(\beta v/t)\\&=\alpha\rho(u)+\beta\rho(v).\end{aligned}$$

□

Our second Main Theorem is formally a corollary of earlier results.

[2] Recalled by Prof Chudziak at the 20th ICFE.

Theorem 8.1. *Suppose X is a normed vector space and (K, h, g) satisfies (GGE) with K non-trivial (i.e. there is $w \in X$ with $K(w) \neq 0$). Then either K is linear or else, for some continuous linear ρ,*

$$h(su) = 1 + s\rho(u) \qquad (s > 0, u \notin \mathcal{N}(K)).$$

In particular, (K, g) satisfies (GFE) and so, for any u with $\rho(u) = 1$,

$$g(x) = e^{\alpha(x)}(1 + \rho(x))^{\beta} \qquad (x \in \mathbb{G}_\rho(X)),$$

where, for $\gamma = \log g$,

$$\alpha(x) := \gamma(x - \rho(x)u) \quad (x \in \mathbb{G}_\rho(X))$$

is linear and $\alpha(u) = 0$ and $\beta = \gamma(u)/\log 2$. Thus there are four cases:

$$\begin{array}{ll} h_u(s) = g_u(s) = 1, & \rho(u) = \gamma(u) = 0, \\ h(su) = 1, \qquad g(su) = e^{s\gamma(u)}, & \rho(u) = 0 \text{ and } \gamma(u) \neq 0, \\ h_u(s) = 1 + s\rho(u), \qquad g_u(s) = 1, & \rho(u) \neq 0 \text{ and } \gamma(u) = 0, \\ h_u(s) = 1 + s\rho(u), \qquad g_u(s) = (1 + s\rho(u))^{(\gamma/\rho)}, & \gamma(u) \neq 0 \neq \rho(u). \end{array}$$

The form of K may be read off from [12, Th 4A, 4B].

Proof. By Corollary 5.1, $h_u(s) = 1 + s\rho(u) = h(su) = h_{su}(1) = 1 + \rho(su)$, so that $\rho(u)$ is homogeneous. By Lemma 8.3, ρ is linear and by assumption continuous. Hence the equation (GGE) has the form (GFE), i.e.

$$K(u \circ_\rho v) = K(u) + g(u)K(v).$$

The remaining assertions follow from Theorem 2.1 (the Index Theorem). □

The four cases above can also be reached from Theorem 3.1 by four direct but laborious computations using Proposition 4.4. Theorem 7.2 puts these last conclusions into perspective, since (GFE) above reduces to a homomorphism between $\mathbb{G}_\rho(X)$ and $\mathbb{G}_\sigma(Y)$ for some σ, unless the range condition is violated. Recall from [12, Th.2] that, being abelian, $K(\mathcal{N}(\rho))$ is either included in $\mathcal{N}(\sigma)$ (the N_A-case of Theorem 7.1 with $\mathcal{N}(\rho) \subseteq \mathcal{N}(\gamma)$) or lies along a radius in Y (the N_B-case). In the first case, by [12, Th 4A], K exhibits linear and power types of behaviour (the latter only if $\rho(u) = 1$ for some u, the behaviour becoming logarithmic in the limit when $\sigma(K(u)) = 0$). Otherwise, by [12, Th 4B], K exhibits exponential and power type behaviour (the latter, again if $\rho(u) = 1$ for some u, becoming logarithmic when $\sigma(K(u)) = 0$).

We close by verifying how the $\lambda_u(t; r)$ contour rises or falls as the domain parameter r rises, according to contour type, and according to which side of $t = 1$; it is here at $t = 1$ where behaviour reverses (cf. $\lambda_u(t) \simeq e^{r(t-1)}$, using a large r approximation). The calculations check for monotonicity with a simple scheme based on the form $\lambda(t) = \varphi(rt)/\varphi(r)$.

Lemma 8.4. *(a) Consider $\lambda_u(t; r) = \varphi(rt)/\varphi(r)$ with $\varphi(x) = e^x - 1$. Then:*

i) *for* $0 < t < 1$ *the* λ *contours fall as* r *rises: if* $0 < r < R$, *then*

$$\lambda(t; R) < \lambda(t; r);$$

ii) *by reciprocation, at any* $s > 1$, *if* $0 < r < R$, *then*

$$\lambda(s; r) < \lambda(t; R);$$

iii) *by inversion, the results in (i) and (ii) are reversed for* $\varphi(x) = \log(1+x)$.

(b) Consider $\lambda_u(t; r) = \varphi(rt)/\varphi(r)$ *with* $\varphi(x) = (1 + x)^c - 1$ *and* $c < 1$. *Then:*

i) *for* $0 < t < 1$ *the* λ *contours fall as* r *rises: if* $0 < r < R$, *then*

$$\lambda(t; R) < \lambda(t; r);$$

ii) *by reciprocation, at any* $s > 1$, *if* $0 < r < R$, *then*

$$\lambda(s; r) < \lambda(t; R);$$

iii) *by inversion, the results in (i) and (ii) are reversed for* $c > 1$.

Proof. We begin by computing that

$$\frac{\partial}{\partial r}\left(\frac{\varphi(rt)}{\varphi(r)}\right) = \frac{t\varphi'(rt)\varphi(r) - \varphi(rt)\varphi'(r)}{\varphi(r)^2} = \frac{\varphi(rt)\varphi(r)}{\varphi(r)^2}\left(t\frac{\varphi'(rt)}{\varphi(rt)} - \frac{\varphi'(r)}{\varphi(r)}\right),$$

$$\frac{\partial}{\partial r}\left(\frac{\varphi'(rt)}{\varphi(rt)}\right) = \frac{t\varphi''(rt)\varphi(rt) - t\varphi'(rt)\varphi'(rt)}{\varphi(rt)^2}.$$

(a) (i) As $\varphi' = \varphi'' = e^x$, we have for $0 < t < 1$

$$te^{rt}(e^{rt} - 1) - te^{rt}e^{rt} = -te^{rt} < 0 : \qquad \frac{\partial}{\partial r}\left(t\frac{\varphi'(rt)}{\varphi(rt)}\right) < 0.$$

Since $rt < r$ for $0 < t < 1$, we have

$$t\frac{\varphi'(rt)}{\varphi(rt)} < \frac{\varphi'(r)}{\varphi(r)}, \text{ so } \frac{\partial}{\partial r}\left(\frac{\varphi(rt)}{\varphi(r)}\right) < 0 : \qquad \frac{\varphi(rt)}{\varphi(r)} \text{ is decreasing in } r.$$

Hence, if $0 < r < R$ and $0 < t < 1$, then

$$\varphi(rt)/\varphi(r) > \varphi(Rt)/\varphi(R).$$

(ii) With $0 < t < 1$, note that $\varphi(r)/\varphi(rt)$ is increasing in $r > 0$. Writing $\rho = rt$ and $s = 1/t > 1$, we see that $\varphi(\rho s)/\varphi(\rho)$ is increasing in ρ (t being fixed.)

(iii) The final assertion follows because $y = \varphi(x) = e^x - 1$ has inverse $x = \log(1 + y)$.

(b) Here $\varphi' = c(1 + x)^{c-1}$ and $\varphi'' = c(c - 1)(1 + x)^{c-2}$, so for $c, t < 1$

$$t\varphi''(rt)\varphi(rt) - \varphi'(rt)\varphi'(rt) = tc(c-1)(1+rt)^{c-2}[(1+rt)^c - 1] - c^2(1+rt)^{2c-2} < 0,$$

since $(1 + rt) > 1$. This leads to (i) and (ii), exactly as in (a).

(iii) Since $y = \varphi_c(x) = (1+x)^c - 1$ has as inverse $x = \varphi_{1/c}(y) = (1+y)^{1/c} - 1$, the assertion (iii) now follows from (i) and (ii) with reversal. □

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