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The law of one price in quadratic hedging and mean–variance portfolio selection

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Abstract
The *law of one price (LOP)* broadly asserts that identical financial flows should command the same price. We show that when properly formulated, the LOP is the minimal condition for a well-defined mean–variance portfolio allocation framework without degeneracy. Crucially, the paper identifies a new mechanism through which the LOP can fail in a continuous-time L^2-setting without frictions, namely "trading from just before a predictable stopping time", which surprisingly identifies LOP violations even for continuous price processes. Closing this loophole allows us to give a version of the "fundamental theorem of asset pricing" appropriate in the quadratic context, establishing the equivalence of the economic concept of the LOP with the probabilistic property of the existence of a local $\mathscr{E}$-martingale state price density. The latter provides unique prices for all square-integrable contingent claims in an extended market and subsequently plays an important role in mean–variance portfolio selection and quadratic hedging. Mathematically, we formulate a novel variant of the uniform boundedness principle for conditionally linear functionals on the L^0-module of conditionally square-integrable random variables. We then study the representation of time-consistent families of such functionals in terms of stochastic exponentials of a fixed local martingale.

Keywords Law of one price · $\mathscr{E}$-density · Efficient frontier · Mean–variance portfolio selection · Quadratic hedging

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1 Introduction

In this paper, we take a fresh look at the mathematical foundations of quadratic hedging. For a fixed time horizon $T > 0$, a locally square-integrable semimartingale price process S, and a square-integrable contingent claim H, the goal of quadratic hedging is to

$$\text{minimise } E\left[\left(v + \int_{(0,T]} \vartheta_u dS_u - H\right)^2\right] \tag{1.1}$$

over initial wealth $v \in \mathbb{R}$ and all reasonable trading strategies ϑ; see Pham [27] and Schweizer [32, 33] for literature overviews. The task (1.1), first formulated and analysed in the seminal paper of Schweizer [30], was abstracted from an early work on mean–variance hedging by Duffie and Richardson [17]; it is thus intimately related to efficient portfolio allocation in the sense of Markowitz [24].

Our point of departure is the mean–variance hedging framework of Černý and Kallsen [4], or rather its extension by Czichowsky and Schweizer [11] to settings that admit some arbitrage opportunities, properly formulated as "L^2 free lunches" (Schachermayer [29, Definition 1.3]). The aim is to obtain minimal conditions on S and ϑ under which the solution of (1.1) exists and retains the form reported in [4, 11]. This leads us to study an appropriate version of the *law of one price (LOP)* and its implication for the existence of suitably modified state price densities in an L^2-setting.

Our analysis adds an important qualifier to the classical textbook folklore asserting that the existence of a state price density is sufficient for the LOP in continuous-time models; cf. Cochrane [9, Sect. 4.3]. We show that it is not enough to consider signed conditional state price densities as in the seminal paper of Hansen and Richard [19], but one must also require certain limiting properties at predictable stopping times. Our main result (Theorem 3.2) provides a version of the "fundamental theorem of asset pricing" appropriate in the quadratic context, establishing the equivalence of the economic concept of the LOP with the probabilistic property of the existence of a local $\mathscr{E}$-martingale state price density (stemming from the notion of $\mathscr{E}$-martingales introduced by Choulli et al. [8], which we slightly extend here). The latter gives unique prices for all square-integrable contingent claims in an extended market and can be chosen such that the passage to the extended market does not alter the efficient frontier (Remark 3.17). Mathematically, we formulate a novel variant of the uniform boundedness principle for conditionally linear functionals on the L^0-module of conditionally square-integrable random variables (Proposition 4.1). We then study the representation of time-consistent families of such functionals in terms of stochastic exponentials of a fixed local martingale (Proposition 4.2).

There are very few theoretical studies of the law of one price in frictionless markets and none specifically in the context of quadratic hedging. On closer reading, the existing studies are also limited in scope. Courtault et al. [10] examine finite-discrete-time models, observing in Sect. 3 that

> *... [the discrete-time] results have no natural counterparts for continuous-time models. "Natural" here means "for the standard concept of admissibility". The latter requires that the value process is bounded from below.*

Bättig and Jarrow [2] study wealth transfers between two fixed dates in a complete market setting. Their assumption of a signed pricing measure in [2, Theorem 1] implicitly invokes the law of one price on L^∞ with proximity given by the weak-star topology. One should observe that the space $L^\infty(P)$ depends on P only through the null sets, while $L^2(P)$ lacks such an invariance property. This paper provides, for the first time in the literature, a complete analysis of the LOP in a continuous-time L^2-setting without frictions. The repercussion of our findings for arbitrage theory on L^∞ are left for future work. For continuous price processes, we show that the LOP is equivalent to the existence of an equivalent local martingale measure with square-integrable density (Proposition 3.4) and hence to the condition of no "L^2 free lunch" as shown in Stricker [34, Theorems 2 and 3]. In Example 3.5, we illustrate that the LOP, which we consider here only in the L^2-sense, can fail for continuous price processes that satisfy the no-arbitrage (NA) condition (Schachermayer [29, Definition 1.1]) both in the L^2- and L^∞-sense.

The paper is organised as follows. In Sect. 2, after establishing notation (Sect. 2.1), market dynamics and admissible strategies (Sect. 2.2), we introduce alternative descriptions of the law of one price by means of (i) the price process S; (ii) pricing functionals; and (iii) state price densities (Sects. 2.3–2.5). We conclude Sect. 2 with a review of $\mathscr{E}$-densities and $\mathscr{E}$-martingales (Sect. 2.6). Section 3 contains the main results of the paper. In Sect. 3.1, we pull the various notions of the LOP together to demonstrate their equivalence (Theorem 3.2). Section 3.2 illustrates several phenomena that arise when the law of one price fails. In Sect. 3.3, we offer some intuition for the LOP and interpret Theorem 3.2 as a market extension theorem. We next examine the consequences of the LOP for quadratic hedging (Sect. 3.4), extend its applicability to the conditional framework of Hansen and Richard [19] (Sect. 3.5), and study the resulting mean–variance portfolio allocation in the presence of a contingent claim (Sect. 3.6). Section 4 contains the proof of the main theorem presented via several partial statements of independent interest.

2 Problem formulation

2.1 Preliminaries

We work on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq T}, P)$ with $\mathscr{F}_T = \mathscr{F}$, satisfying the usual conditions of right-continuity and completeness. For $p \in [0, \infty]$, we frequently write $L^p(P)$ or just L^p as a shorthand for $L^p(\mathscr{F}_T, P)$. The L^2-closure of an arbitrary $\mathcal{A} \subseteq L^2$ is denoted by $\operatorname{cl}\mathcal{A}$.

Throughout the paper, we consider generalised *conditional expectations* as defined for example in Jacod and Shiryaev [21, I.1.1]. That is, for a random variable X and $\mathscr{G} \subseteq \mathscr{F}$, the conditional expectation $E[X|\mathscr{G}]$ is finite if and only if there exists a $\mathscr{G}$-measurable random variable $K > 0$ such that KX is integrable (He et al. [20, Theorem 1.16]), in which case one has $E[X|\mathscr{G}] = \frac{E[KX|\mathscr{G}]}{K}$, where the right-hand side now features the standard conditional expectation. In particular, X may have finite conditional expectation without being integrable. We call X *conditionally square-integrable* and write $X \in L^2(P|\mathscr{G})$ if $E[X^2|\mathscr{G}] < \infty$ P-a.s.

The set of all $[0,T]$-valued *stopping times* is denoted by $\mathcal{T}$. An increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ in $\mathcal{T}$ is called a *localising sequence* if it converges *stationarily* to T, i.e., if $P[\tau_n < T] \to 0$; see Dellacherie and Meyer [15, VII.99]. For a class of stochastic processes $\mathscr{C}$, we say with [15, Definition VI.27] that X belongs to $\mathscr{C}$ *locally*, writing $X \in \mathscr{C}_{\text{loc}}$, if there is a localising sequence $(\tau_n)_{n\in\mathbb{N}}$ such that $X^{\tau_n}\mathbb{1}_{\{\tau_n>0\}} \in \mathscr{C}$ for each $n \in \mathbb{N}$. This notion of localisation is slightly broader than that in Jacod and Shiryaev [21, I.1.34].

For a semimartingale X, we denote by $L(X)$ the space of all X-integrable predictable processes $\vartheta = (\vartheta_t)_{0\le t\le T}$ and write

$$\vartheta \cdot X_t := \int_{(0,t]} \vartheta_u dX_u$$

for their stochastic integral up to time $t \in [0,T]$. The symbol $\mathscr{E}(N)$ denotes the *stochastic exponential* of a semimartingale N, i.e., the unique strong solution of the stochastic differential equation $\mathscr{E}(N) = 1 + \mathscr{E}(N)_- \cdot N$; see Doléans-Dade [16].

Using the convention that $\inf\emptyset = \infty$, we say that a semimartingale X *does not reach zero continuously and is absorbed in zero* if for $\sigma^X := \inf\{t > 0 : X_t = 0\} \wedge T$, one has

$$X_- \neq 0 \text{ on } [\![0,\sigma^X]\!] \quad \text{and} \quad X = 0 \text{ on } [\![\sigma^X, T]\!] \text{ on the event } \{\sigma^X < T\}. \tag{2.1}$$

For such X, the *stochastic logarithm* $\mathcal{L}(X)$ is well defined by $\mathcal{L}(X) = \frac{\mathbb{1}_{[\![0,\sigma^X]\!]}}{X_-} \cdot X$ and has the property $X = X_0\mathscr{E}(\mathcal{L}(X))$; see [8, Proposition 2.2].

For a special semimartingale X, we write

$$X = X_0 + M^X + B^X$$

for the canonical decomposition of X into its local martingale component M^X and its finite-variation predictable component B^X, both null at zero. We denote by $\mathcal{H}^2$ the class of all special semimartingales X with $E[[M^X, M^X]_T] + E[(\int_0^T |dB_u^X|)^2] < \infty$; see Protter [28, Chap. IV].

2.2 Asset prices and trading strategies

Consider a financial market consisting of one riskless asset with constant value 1 and d risky assets described by an $\mathbb{R}^d$-valued locally square-integrable semimartingale S. Denote the running supremum of $|S|$ by S^*.

Definition 2.1 A trading strategy ϑ is called *simple* if it is of the form

$$\vartheta = \sum_{i=1}^{m-1} \xi_i \mathbb{1}_{]\!]\sigma_i,\sigma_{i+1}]\!]}$$

with stopping times $0 \le \sigma_1 \le \cdots \le \sigma_m \le \tau_n$ for some $n \in \mathbb{N}$, some bounded $\mathbb{R}^d$-valued $\mathscr{F}_{\sigma_i}$-measurable random variables ξ_i for $i = 1,\dots,m-1$ and some localising sequence $(\tau_n)_{n\in\mathbb{N}}$ such that $S^*_{\tau_n} \in L^2$. The set of all simple trading strategies is denoted by Θ.

We further introduce strategies that trade from a stopping time τ or from just before a predictable stopping time τ by setting, respectively,

$$\Theta_\tau := \{\vartheta \in \Theta : \vartheta \mathbb{1}_{[\![0,\tau]\!]} = 0\},$$
$$\Theta_{\tau-} := \{\vartheta \in \Theta : \vartheta \mathbb{1}_{[\![0,\tau[\![} = 0\}.$$

As is well known, the set of terminal wealths generated by simple strategies need not be L^2-closed; see Monat and Stricker [26], Delbaen et al. [12] and Choulli et al. [8]. The next definition taken from [11] and building upon [4] offers a way out.

Definition 2.2 A trading strategy $\vartheta \in L(S)$ is called *admissible* if there exists a sequence $(\vartheta^n)_{n\in\mathbb{N}}$ of simple trading strategies $\vartheta^n = (\vartheta^n_t)_{0\le t\le T}$, called *approximating sequence for* ϑ, such that

(1) $\vartheta^n \cdot S_T \to \vartheta \cdot S_T$ in L^2;

(2) $\vartheta^n \cdot S_\tau \to \vartheta \cdot S_\tau$ in L^0 for all $\tau \in \mathcal{T}$.

The set of all admissible trading strategies is denoted by $\overline{\Theta}$, and we further let

$$\overline{\Theta}_\tau := \{\vartheta \in \overline{\Theta} : \vartheta \mathbb{1}_{[\![0,\tau]\!]} = 0\} \qquad \text{for } \tau \in \mathcal{T},$$
$$\overline{\Theta}_{\tau-} := \{\vartheta \in \overline{\Theta} : \vartheta \mathbb{1}_{[\![0,\tau[\![} = 0\} \qquad \text{for predictable } \tau \in \mathcal{T}.$$

2.3 The law of one price for the price process S

In layperson's terms, the law of one price states that portfolios generating the same terminal payoff by trading in the market S should have the same value at all earlier times, thus eliminating the most conspicuous arbitrage opportunities. Then by the linearity of portfolio formation, all portfolios generating a zero payoff at maturity T should have price zero at all earlier times. Since we wish to consider pricing rules that are continuous in the L^2-space of payoffs, the law of one price must be true also approximately in L^2. This yields the requirement (1) below for trading between an arbitrary stopping time τ and the terminal date T.

For fixed τ, this line of reasoning is very natural and appears, for example, in Hansen and Richard [19] in the context of static trading with infinitely many assets. Observe that in a finite-discrete-time setup, the continuity requirement is unnecessary since the set of terminal wealth distributions attainable by trading is closed in L^0, hence also in L^2.

The crucial step in this paper is a deeper analysis of predictable times. For a predictable time, say σ, it is possible to start trading immediately before σ, at time $\sigma-$ so to say. Requirement (2) below formulates the law of one price for such trades. Here it becomes important that the approximation that was previously static takes on a temporal dimension over a sequence of announcing times.

Definition 2.3 We say that *the price process* $S = (S_t)_{0\le t\le T}$ *satisfies the law of one price* if the following conditions hold:

(1) For all stopping times $\tau \in \mathcal{T}$, all $\mathscr{F}_\tau$-measurable endowments x_τ and all sequences $(\vartheta^n)_{n\in\mathbb{N}}$ of simple trading strategies such that $x_\tau + \vartheta^n \mathbb{1}_{]\!]\tau,T]\!]} \cdot S_T \to 0$ in L^2, we have $x_\tau = 0$.

(2) Let $\sigma \in \mathcal{T}$ be a predictable stopping time and $(\sigma_n)_{n\in\mathbb{N}}$ any announcing sequence of stopping times for σ. Then for all sequences of $\mathscr{F}_{\sigma_n}$-measurable endowments $(x^n_{\sigma_n})_{n\in\mathbb{N}}$ and $(\vartheta^n)_{n\in\mathbb{N}}$ of simple trading strategies such that

$$x^n_{\sigma_n} + \vartheta^n \mathbb{1}_{]\!]\sigma_n,T]\!]} \cdot S_T \to 0 \quad \text{in } L^2 \qquad \text{and} \qquad x^n_{\sigma_n} \to x_{\sigma-} \quad \text{in } L^0$$

for some random variable $x_{\sigma-}$, we have $x_{\sigma-} = 0$.

Remark 2.4 The law of one price for S has the following easy consequences:

– Since simple strategies approximate admissible strategies in L^2 at maturity and in L^0 at intermediate times, the LOP for S extends to admissible strategies by a diagonal argument.

– The wealth process of an admissible strategy is uniquely determined by its terminal value, i.e., if one has $\vartheta \cdot S_T = \tilde{\vartheta} \cdot S_T$ for $\vartheta, \tilde{\vartheta} \in \overline{\Theta}$, then $\vartheta \cdot S$ and $\tilde{\vartheta} \cdot S$ are indistinguishable.

– In the setting of (2) in Definition 2.3, the L^0-limit of the sequence $(\vartheta^n \cdot S_{\sigma_n})$, should it exist for a given sequence of strategies $\vartheta^n \in \overline{\Theta}$, does not depend on the announcing sequence $(\sigma_n)_{n\in\mathbb{N}}$ of stopping times.

2.4 Price systems

Definition 2.5 A family $(p_\tau)_{\tau\in\mathcal{T}}$ of operators $p_\tau : L^2(\mathscr{F}_T, P) \to L^0(\mathscr{F}_\tau, P)$ is a *price system satisfying the law of one price* if the following properties hold for every $\tau \in \mathcal{T}$:

(1) *Correct pricing of the riskless asset*. One has $p_\tau(1) = 1$.

(2) *Time-consistency*. For all $\sigma \leq \tau$ in $\mathcal{T}$, one has $p_\sigma(p_\tau(H)) = p_\sigma(H)$ for each $H \in L^2(\mathscr{F}_T, P)$ such that $p_\tau(H) \in L^2(\mathscr{F}_\tau, P)$.

(3a) *Conditional linearity*. One has $p_\tau(a_1 H_1 + a_2 H_2) = a_1 p_\tau(H_1) + a_2 p_\tau(H_2)$ for all $H_1, H_2 \in L^2(\mathscr{F}_T, P)$ and $a_1, a_2 \in L^\infty(\mathscr{F}_\tau, P)$.

(3b) *Conditional continuity*. Let $(H_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(\mathscr{F}_T, P)$ that converges to some H in $L^2(\mathscr{F}_T, P)$. Then $p_\tau(H_n)$ converges to $p_\tau(H)$ in $L^0(\mathscr{F}_\tau, P)$.

(4) *Left limits at predictable stopping times*. For predictable $\tau \in \mathcal{T}$, any announcing sequence $(\tau_n)_{n\in\mathbb{N}}$ for τ and any $H \in L^2(\mathscr{F}_T, P)$, we have that $p_{\tau_n}(H)$ converges to some random variable in $L^0(\mathscr{F}_T, P)$.

Furthermore, a family $(p_\tau)_{\tau\in\mathcal{T}}$ of operators $p_\tau : L^2(\mathscr{F}_T, P) \to L^0(\mathscr{F}_\tau, P)$ is said to be *compatible* with S if $p_\tau(S_{\tau_n}) = S_{\tau_n\wedge\tau}$ for all $n \in \mathbb{N}$, all $\tau \in \mathcal{T}$ and any localising sequence $(\tau_n)_{n\in\mathbb{N}}$ such that $S^*_{\tau_n} \in L^2(\mathscr{F}_T, P)$ for all $n \in \mathbb{N}$.

Definition 2.5 does not require that $p_\tau(H)$ is positive for every positive payoff $H \in L^2(\mathscr{F}_T, P)$. Instead, it imposes the law of one price along the following lines. Observe that $p_\tau(H)$ is the terminal wealth obtained by investing the time-τ price of H into the risk-free asset. To uphold the LOP, the prices of H and $p_\tau(H)$ must therefore be the same at time τ. This indeed follows from conditions (1) and (3a) of Definition 2.5. Time-consistency (2) further asserts that the two portfolios $p_\tau(H)$ and H have the same value at all *earlier* stopping times $\sigma \leq \tau$. This natural LOP condition does not follow from (1) and (3a). Conditions (1), (2), (3a) and (3b) jointly yield that the limiting variable in condition (4) does not depend on the announcing sequence of stopping times. Denoting this limit by $p_{\tau-}(H)$, one has $p_{\sigma-} = p_{\sigma-} \circ p_{\tau-}$ for all predictable $\sigma, \tau \in \mathcal{T}$ with $\sigma \le \tau$.

Conditions (3a), (3b) and (4) reflect, in this order, increasing generality of modelling frameworks. Combined with (1) and (2), the conditional linearity (3a) gives the LOP on finite probability spaces (see e.g. Koch-Medina and Munari [22, Proposition 15.2.2]) and also in finite-discrete-time models with finitely many assets (see e.g. Courtault et al. [10]). Finite discrete time with infinitely many assets needs also continuity (3b) (see e.g. Hansen and Richard [19]); continuity at predictable times (4) has not been studied previously.

2.5 State price densities

Fix $\tau \in \mathcal{T}$. An operator $p_\tau : L^2(\mathscr{F}_T, P) \to L^0(\mathscr{F}_\tau, P)$ that is conditionally linear and conditionally continuous in the sense of (3a) and (3b) in Definition 2.5 can naturally be represented as a conditional expectation involving a conditionally square-integrable random variable. Namely, by a conditional version of the Riesz representation theorem for Hilbert spaces in Hansen and Richard [19, Theorem 2.1], there is an $\mathscr{F}_T$-measurable random variable ${}^\tau Z_T$ such that $E[{}^\tau Z_T^2|\mathscr{F}_\tau] < \infty$ and $p_\tau(H) = E[{}^\tau Z_T H|\mathscr{F}_\tau]$ for all contingent claims $H \in L^2(\mathscr{F}_T, P)$. This ${}^\tau Z_T$ is commonly known as a *state price density*.

Henceforth, "state price density" refers to any $\mathscr{F}_T$-measurable random variable which is $\mathscr{F}_\tau$-conditionally square-integrable for some $\tau \in \mathcal{T}$. Let us now rephrase the notion of a "price system satisfying the law of one price" in terms of the corresponding family of state price densities $({}^\tau Z_T)_{\tau\in\mathcal{T}}$. Since the pricing functionals we consider are not necessarily positive, the next definition allows the state price densities to take negative values.

Definition 2.6 We say that a family $({}^\tau Z_T)_{\tau\in\mathcal{T}}$ is a family of *state price densities satisfying the law of one price* if for each $\tau \in \mathcal{T}$, the random variable ${}^\tau Z_T$ is $\mathscr{F}_T$-measurable and the following conditions hold:

(1) *Correct pricing of the risk-free asset*. One has ${}^\tau Z_\tau := E[{}^\tau Z_T|\mathscr{F}_\tau] = 1$.

(2) *Time-consistency*. For all $\sigma \le \tau$ in $\mathcal{T}$, one has ${}^\sigma Z_T = {}^\sigma Z_\tau {}^\tau Z_T$ with

$$ {}^\sigma Z_\tau := E[{}^\sigma Z_T|\mathscr{F}_\tau]. $$

(3) *Conditional square-integrability*. One has $E[{}^\tau Z_T^2|\mathscr{F}_\tau] < \infty$.

(4) *Bounded conditional second moments before predictable stopping times*. For any predictable τ and any announcing sequence $(\tau_n)_{n\in\mathbb{N}}$ for τ, the $\mathscr{F}_{\tau-}$-measurable random variable $C := \sup_{n\in\mathbb{N}} E[{}^{\tau_n} Z_T^2|\mathscr{F}_{\tau_n}]$ is finite.

Furthermore, we say that a family of state price densities $({}^\tau Z_T)_{\tau\in\mathcal{T}}$ is *compatible* with S if $E[S_{\tau_n} {}^\tau Z_T|\mathscr{F}_\tau] = S_{\tau_n\wedge\tau}$ for all $n \in \mathbb{N}$, all $\tau \in \mathcal{T}$ and any localising sequence $(\tau_n)_{n\in\mathbb{N}}$ such that $S^*_{\tau_n} \in L^2(\mathscr{F}_T, P)$ for all $n \in \mathbb{N}$.

Remark 2.7 The process ${}^\tau Z = ({}^\tau Z_t)_{0\le t\le T}$ that arises in (1) and (2) of Definition 2.6 by setting

$$ {}^\tau Z_t = E[{}^\tau Z_T|\mathscr{F}_t] $$

need not be a martingale. However, for $K := E[|{}^\tau Z_T||\mathscr{F}_\tau] \vee 1$, the adapted process $\frac{{}^\tau Z}{K}\mathbb{1}_{]\!]\tau,T]\!]}$ coincides on $]\!]\tau, T]\!]$ with the uniformly integrable martingale closed by $\frac{{}^\tau Z_T}{K}$. This also shows that ${}^\tau Z$ is a semimartingale.

2.6 $\mathscr{E}$-densities and $\mathscr{E}$-martingales

We next recall and slightly adapt the concept of $\mathscr{E}$-martingales introduced by Choulli et al. [8]. For any stopping time $\tau \in \mathcal{T}$, we denote the process Y stopped at τ by Y^τ and, for a semimartingale N, we set ${}^\tau\mathscr{E}(N) = \mathscr{E}(N - N^\tau)$. The stochastic exponential ${}^\tau\mathscr{E}(N)$ therefore denotes a multiplicative restarting of $\mathscr{E}(N)$ in the sense that $\mathscr{E}(N) = \mathscr{E}(N)^\tau\, {}^\tau\mathscr{E}(N)$. We now study the family of processes $({}^\tau\mathscr{E}(N))_{\tau\in\mathcal{T}}$.

Definition 2.8 We say that the family $({}^\tau\mathscr{E}(N))_{\tau\in\mathcal{T}}$ is an *$\mathscr{E}$-density* if for all $\tau \in \mathcal{T}$, one has $E[{}^\tau\mathscr{E}(N)_T|\mathscr{F}_\tau] = 1$. An $\mathscr{E}$-density $({}^\tau\mathscr{E}(N))_{\tau\in\mathcal{T}}$ is called *square-integrable* if for all $\tau \in \mathcal{T}$, one has $E[{}^\tau\mathscr{E}(N)_T^2|\mathscr{F}_\tau] < \infty$.

Definition 2.9 An adapted RCLL process Y is an *$\mathscr{E}(N)$-martingale* if for all $\tau \in \mathcal{T}$, one has $Y_\tau = E[{}^\tau\mathscr{E}(N)_T Y_T|\mathscr{F}_\tau]$. We say Y is an *$\mathscr{E}(N)$-local martingale* if there is a localising sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times such that Y^{τ_n} is an $\mathscr{E}(N)$-martingale for each $n \in \mathbb{N}$.

Remark 2.10 We have slightly generalised the original definition of $\mathscr{E}$-martingales (see [8, Definition 3.11]) by imposing milder integrability conditions. To see this, observe that by [8, Sect. 1], for each semimartingale N, the sequence of stopping times given by $T_0 = 0$ and $T_{m+1} = \inf\{t > T_m : {}^{T_m}\mathscr{E}(N)_t = 0\} \wedge T$ for $m \in \mathbb{N}_0$ increases stationarily to T. The following are then equivalent:

(i) Y is an $\mathscr{E}(N)$-martingale in the sense of Definition 2.9.

(ii) There is a sequence of strictly positive, $\mathscr{F}_{T_m}$-measurable random variables K_m with

$$E[|Y_{T_m} K_m\, {}^{T_m}\mathscr{E}(N)_{T_{m+1}}|] < \infty$$

and such that $(Y - Y^{T_m})K_m\, {}^{T_m}\mathscr{E}(N)$ is a P-martingale for all $m \in \mathbb{N}_0$.

In [8], the random variables $(K_m)_{m\in\mathbb{N}_0}$ are not needed since the combination of the assumptions that Y is square-integrable and $({}^\tau\mathscr{E}(N))_{\tau\in\mathcal{T}}$ satisfies a reverse Hölder inequality permits taking $K_m \equiv 1$ for all $m \in \mathbb{N}_0$.

The next two results, which mirror [8, Proposition 3.15, Corollaries 3.16 and 3.17], are reminiscent of the Girsanov theorem for absolutely continuous measure changes; the original proofs in [8] still work for our generalisations.

Proposition 2.11 *For a semimartingale Y and an $\mathscr{E}$-density $({}^\tau\mathscr{E}(N))_{\tau\in\mathcal{T}}$, the following are equivalent*:

(i) *Y is an $\mathscr{E}(N)$-local martingale.*

(ii) *$Y + [Y, N]$ is a P-local martingale.*

Furthermore, if either of the conditions holds and Y is special (*with local martingale part M^Y*), *then $Y = Y_0 + M^Y - \langle M^Y, N\rangle$.*

Proposition 2.12 *Assume $({}^{\tau}\mathscr{E}(N))_{\tau\in\mathcal{T}}$ is an $\mathscr{E}$-density and Y is an $\mathscr{E}(N)$-local martingale. Consider the sequence $(T_m)_{m\in\mathbb{N}_0}$ of stopping times from Remark 2.10. If there is a sequence of positive $\mathscr{F}_{T_m}$-measurable random variables $(K_m)_{m\in\mathbb{N}_0}$ such that*

$$E\big[K_m Y_T^*\big({}^{T_m}\mathscr{E}(N)\big)_T^*\big] < \infty \qquad \text{for all } m \in \mathbb{N}_0,$$

then Y is an $\mathscr{E}(N)$-martingale.

To assist the reader, we conclude this section by linking $\mathscr{E}$-martingales and $\mathscr{E}$-densities to equivalent measures and their densities.

Remark 2.13 For a positive $\mathscr{E}$-density $({}^{\tau}\mathscr{E}(N))_{\tau\in\mathcal{T}}$, the following are equivalent:

(i) Y is a Q-martingale for the equivalent measure Q given by $\frac{dQ}{dP} = \mathscr{E}(N)_T$.

(ii) Y is an $\mathscr{E}(N)$-martingale and Y_0 is integrable.

Thus for an $\mathscr{E}$-density $({}^{\tau}\mathscr{E}(N))_{\tau\in\mathcal{T}}$ with $\mathscr{E}(N) > 0$, an $\mathscr{E}(N)$-martingale coincides with the notion of "generalised Q-martingale" in Dellacherie and Meyer [15, Remark V.2(d)].

3 Main results and counterexamples

3.1 Equivalent characterisations of the law of one price

We first recall an important concept from the quadratic hedging literature.

Definition 3.1 The process $L = (L_t)_{0\le t\le T}$ given by

$$L_t := \operatorname*{ess\,inf}_{\vartheta\in\Theta_t} E[(1-\vartheta\cdot S_T)^2|\mathscr{F}_t] \tag{3.1}$$

is called the *opportunity process*.

Theorem 3.2 *For a locally square-integrable semimartingale S, the following are equivalent:*

(i) *The law of one price holds for the price process S (Definition 2.3).*

(ii) *The semimartingale S admits a compatible price system satisfying the LOP (Definition 2.5).*

(iii) *The semimartingale S admits a compatible family of state price densities satisfying the LOP (Definition 2.6).*

(iv) *There exists a semimartingale N such that S is an $\mathscr{E}(N)$-local martingale and $({}^{\tau}\mathscr{E}(N))_{\tau\in\mathcal{T}}$ is a square-integrable $\mathscr{E}$-density (Definitions 2.8 and 2.9).*

(v) *The opportunity process L and its left limit L_- are strictly positive (Definition 3.1).*

Furthermore, if any of the above conditions holds, then:

(vi) *The subspace $\{\vartheta\cdot S_T : \vartheta\in\overline{\Theta}\}$ is closed in L^2. Hence a unique (up to a null strategy) solution of (1.1) exists in $\overline{\Theta}$.*

(vii) *For every $\tau\in\mathcal{T}$, the set $\{\vartheta\cdot S_T : \vartheta\in\overline{\Theta}_\tau\}$ is closed in L^2.*

(viii) *For every predictable stopping time* $\sigma \in \mathcal{T}$ *with announcing sequence* $(\sigma_n)_{n\in\mathbb{N}}$, *one has* $\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}_{\sigma-}\} = \bigcap_{n\in\mathbb{N}}\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}_{\sigma_n}\}$.

In practice, the criterion (iv) is easily verified if there is an equivalent martingale measure with a square-integrable density (Remark 2.13). Failing that, the easiest criterion to check is the strict positivity of the opportunity process and its left limit in (v). Theorem 3.9 below simplifies the verification procedure further in concrete models by dispensing with the need to find the actual opportunity process in favour of the easier task of identifying just a candidate opportunity process.

Apart from the novel definition of the LOP and its link to $\mathscr{E}$-martingales, the key improvement in Theorem 3.2 compared to Czichowsky and Schweizer [11, Theorem 6.2] is that *a priori* one does not need the solutions of

$$\min_{W\in\mathrm{cl}\{\vartheta\cdot S_T:\vartheta\in\Theta_\tau\}} E[(1-W)^2|\mathscr{F}_\tau] \tag{3.2}$$

with $\tau \in \mathcal{T}$ to be realised by trading strategies in $\overline{\Theta}_\tau$. Instead, with a further argument, one obtains that the set $\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}_\tau\}$ is closed *a posteriori* purely on the strength of any of the items (i)–(v). Observe that the conditions (i)–(v) of Theorem 3.2 are not necessary for the L^2-closedness of $\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}_\tau\}$. This can be seen in finite discrete time from the results of Melnikov and Nechaev [25]. In continuous time, a further counterexample appears in Delbaen et al. [12, Example 6.4]. In this example, $\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}\} = L^2$ and $\mathscr{F}_0$ is trivial. Therefore $1 \in \{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}\}$ and hence $L_0 = 0$. Thus if (i)–(v) of Theorem 3.2 fail, no firm conclusions on the closedness of $\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}_\tau\}$ can be drawn.

Remark 3.3 Because the law of one price for wealth transfers between 0 and T does not imply the LOP on subintervals (e.g. $L_0 > 0$ does not automatically yield $L_t > 0$ for $t > 0$), the proof of Theorem 3.2 is not based, unlike other variants of "fundamental theorems of asset pricing", on an application of a separating hyperplane theorem. Rather, the proof constructs via Lemma 4.6 a specific family of state price densities $({}^\tau\hat{Z}_T)_{\tau\in\mathcal{T}}$ compatible with S whose elements ${}^\tau\hat{Z}_T$ are characterised by having the smallest conditional second moment. Because one has $E[{}^\tau Z_T|\mathscr{F}_\tau] = 1$ for all $\tau \in \mathcal{T}$ and all families $({}^\tau Z_T)_{\tau\in\mathcal{T}}$ satisfying the LOP by Definition 2.6(1), the elements of the minimal family $({}^\tau\hat{Z}_T)_{\tau\in\mathcal{T}}$ then also have the smallest conditional variance $\mathrm{Var}[{}^\tau\hat{Z}_T|\mathscr{F}_\tau]$ among all compatible families of state price densities satisfying the LOP. The family $({}^\tau\hat{Z}_T)_{\tau\in\mathcal{T}}$ is in this sense *variance-optimal*; cf. Schweizer [31, Sect. 1]. In the same vein, for N such that ${}^\tau\hat{Z}_T = {}^\tau\mathscr{E}(N)_T$, $\tau \in \mathcal{T}$, one may speak of the *variance-optimal* $\mathscr{E}$*-density*.

It is shown in 2) of Theorem 3.6 and (4.15) below that under the LOP for S, the variance-optimal $\mathscr{E}$-density has the form

$${}^\tau\hat{Z} = \frac{{}^\tau\mathscr{E}(-a\cdot S)L}{L^\tau} = {}^\tau\mathscr{E}\big(-a\cdot S + \mathcal{L}(L) - [a\cdot S, \mathcal{L}(L)]\big)$$

for some $a \in L(S)$. For continuous S, this yields ${}^0\hat{Z} > 0$ and hence that S admits an equivalent local martingale measure with square-integrable density. Thus for continuous S, the LOP for S is equivalent to the condition of *no "L^2 free lunch"* by Stricker [34, Theorems 2 and 3].

Proposition 3.4 *For a continuous price process S, the following are equivalent*:

(i) *The process S satisfies the law of one price.*

(ii) *The process S admits an equivalent local martingale measure with square-integrable density.*

(iii) *The variance-optimal* (*signed*) *local martingale measure for S exists and is positive.*

(iv) *There is no "L^2 free lunch", i.e.*,

$$\mathrm{cl}(\{\vartheta \cdot S_T : \vartheta \in \Theta\} - L^2_+) \cap L^2_+ = \{0\}.$$

(v) *There is no arbitrage in the L^2-closure of simple strategies, i.e.*,

$$\mathrm{cl}\{\vartheta \cdot S_T : \vartheta \in \Theta\} \cap L^2_+ = \{0\}.$$

This recovers and extends the celebrated result of Delbaen and Schachermayer [14], i.e., the equivalence of (ii) and (iii). Observe that in [14], (ii) is assumed, while here (ii) and (iii) follow from the generally weaker LOP assumption (i). In contrast to Proposition 3.4, Theorem 3.2 applies also in situations where the LOP holds, but there is no equivalent martingale measure with a square-integrable density.

3.2 Counterexample

Example 3.5 below illustrates various phenomena that arise when $L_- > 0$ does not hold, hence the law of one price fails. It is striking that the example operates with a *continuous* price process.

(A) For $\sigma = \inf\{t > 0 : L_{t-} = 0\}$, the event $F := \{L_{\sigma-} = 0\} \in \mathscr{F}_{\sigma-}$ occurs with positive probability, while $L_t > 0$ for all $t \in [0, T]$.

(B) There is a family $({}^\tau \hat{Z}_T)_{\tau \in \mathcal{T}}$ of state price densities compatible with S that satisfies properties (1)–(3) of Definition 2.6. Furthermore, this family of random variables can be chosen such that $E[({}^\tau \hat{Z}_T)^2 | \mathscr{F}_\tau] = \frac{1}{L_\tau}$ for all $\tau \in \mathcal{T}$. However, $({}^\tau \hat{Z}_T)_{\tau \in \mathcal{T}}$ does not have bounded conditional second moments before predictable stopping times, that is, property (4) of Definition 2.6 fails.

(C) The price process satisfies the no-arbitrage (NA) condition on the whole time interval $[0, T]$ for L^∞-admissible as well as for L^2-admissible strategies, that is,

$$\{\vartheta \cdot S_T : \vartheta \in \Theta^\infty\} \cap L^0_+ = \{0\} \qquad \text{and} \qquad \{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}\} \cap L^2_+ = \{0\},$$

where $\Theta^\infty = \{\vartheta \in L(S) : \inf_{t \in [0,T]} \vartheta \cdot S_t \in L^\infty\}$.

(D) The subspace $\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}\}$ fails to be closed in L^2, even though $L_t > 0$ for all $t \in [0, T]$.

(E) There is an announcing sequence $(\sigma_n)_{n \in \mathbb{N}}$ for the predictable stopping time σ in (A) such that

$$\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}_{\sigma-}\} \neq \bigcap_{n \in \mathbb{N}} \mathrm{cl}\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}_{\sigma_n}\}.$$

(F) The price process admits an absolutely continuous local martingale measure Q with square-integrable density, but no equivalent martingale measure. In particular, $Q[F] = 0$ for the non-null event $F \in \mathscr{F}_{\sigma-}$ defined in (A). Indeed, S starts at 1 and terminates at 0 on F.

(G) In contrast to (C), there is a "free lunch with vanishing risk", i.e., a sequence of zero-cost trading strategies with wealth bounded below by $-\frac{1}{n}$ such that the wealth of each strategy is 1 on F. Likewise, there is an "L^2 free lunch", i.e., a sequence of simple zero-cost strategies which after disposal of an L^2-integrable nonnegative amount converges in L^2 to a nonzero element of L^2_+, i.e.,

$$\mathrm{cl}(\{\vartheta \cdot S_T : \vartheta \in \Theta\} - L^2_+) \cap L^2_+ \neq \{0\}.$$

Example 3.5 Let W be a Brownian motion in its natural filtration. For $T := 1$ and $t \in [0, T]$, we set $X_t = (T - t)\mathscr{E}(W)_t$. Let τ be an independent stopping time such that $P[\tau = T] = p \in (0, 1)$ and τ is uniformly distributed on $[0, T)$ with probability $1 - p$. Define the stock price by $S = X^\tau$. Then

$$\frac{\mathrm{d}S_t}{S_t} = \mu_t \mathrm{d}t + \mathrm{d}W_t \qquad \text{for } t \in [0, T),$$

where $\mu_t = -\frac{1}{T-t}\mathbb{1}_{[\![0,\tau[\![}$ and we used that $T - t = \mathrm{e}^{\log(T-t)} = \mathrm{e}^{-\int_0^t \frac{1}{T-s}\mathrm{d}s}$ for $t \in [0, T)$.

Let us highlight the key points in the construction of the example. The continuous process X starts at 1, is positive on $[0, T)$, and equals 0 at T. It is constructed to admit an equivalent local martingale measure with square-integrable density on each closed subinterval of $[0, T)$. Hence trading in X fails the (NA) condition on $[0, T]$, but satisfies it on $[0, s]$ for every $s < T$. The stopping time τ is chosen to satisfy $P[t < \tau < T|\mathscr{F}_t] > 0$ for every $t \in [0, T)$, which yields that the stopped process $S = X^\tau$ satisfies (NA) on the whole time interval $[0, T]$. Intuitively, in order to realise an arbitrage opportunity, the trading must start at some stopping time ϱ that satisfies $P[\varrho < \tau] > 0$. However, because τ is totally inaccessible on $[0, T)$, such a strategy is active also on the smaller non-null event $\{\varrho < \tau < T\}$, where the trading gains take both signs as X is independent of τ. A rigorous proof is supplied in item (C).

The situation changes as soon as one considers the concept of a "free lunch with vanishing risk" (FLVR). While an (NA) violation is realised by a single strategy, FLVR allows one to choose an entire sequence of strategies that get increasingly closer to an arbitrage opportunity. In this example, it is significant that an arbitrarily small strictly positive initial capital can be turned into 1 at maturity on the event $\{\tau = T\}$ by an admissible strategy whose wealth never drops below zero. The FLVR strategy borrows $\frac{1}{n}$ at the risk-free rate at time zero and places this amount into a closed-end fund whose policy is to have proportion $-\frac{1}{T-t}\mathbb{1}_{[\![0,\tau[\![}$ invested in the risky asset S. The increasingly larger short position in the risky asset exploits the fact that S is drifting strongly towards zero on the predictable set $\{\tau = T\}$ as $t \to T$, while the conditional probability $P[\tau = T|\tau > t]$ increases to 1. The fund is liquidated if it ever reaches the value of 1, which is guaranteed to happen on the event $\{\tau = T\}$. We now dispose of the fund value built up on the complement $\{\tau < T\}$. After repaying the risk-free borrowing, this yields the FLVR sequence of terminal wealths

$(\mathbb{1}_{\{\tau=T\}} - \frac{1}{n})_{n\in\mathbb{N}}$. The strategy earns $1-\frac{1}{n}$ with probability $P[\tau=T]\equiv 1-p>0$ and loses no more than $\frac{1}{n}$ with probability of no more than $p>0$. The details are found in item (G).

(A): Let $\varphi^n = -\mathscr{E}(-(\frac{\mu}{S}\mathbb{1}_{[\![0,T-\frac{1}{n}]\!]})\cdot S)\frac{\mu}{S}\mathbb{1}_{[\![0,T-\frac{1}{n}]\!]}$ so that

$$1+\varphi^n\cdot S_T = \mathscr{E}\Big(-\Big(\frac{\mu}{S}\mathbb{1}_{[\![0,T-\frac{1}{n}]\!]}\Big)\cdot S\Big)_T = \mathscr{E}\Big(-\frac{\mu}{S}\cdot S\Big)_{T-\frac{1}{n}}.$$

Observe that $\varphi^n\cdot S$ is an $\mathcal{H}^2$-semimartingale (Protter [28, Chap. IV]) and therefore can be approximated in $\mathcal{H}^2$ by stochastic integrals of simple strategies so that each $\varphi^n\in\overline{\Theta}$ by [28, Theorem IV.2]. The opportunity process $L=(L_t)_{0\le t\le T}$ is given by

$$\begin{aligned}
L_t &= \lim_{n\to\infty} E\Big[{}^t\mathscr{E}\Big(-\frac{\mu}{S}\cdot S\Big)^2_{T-\frac{1}{n}}\Big|\mathscr{F}_t\Big]\\
&= \lim_{n\to\infty} E[\mathrm{e}^{-\int_{\tau\wedge t}^{\tau\wedge(T-1/n)}\mu_s^2\mathrm{d}s}|\mathscr{F}_t]\\
&= E[\mathrm{e}^{-\int_{\tau\wedge t}^{\tau}\mu_s^2\mathrm{d}s}\mathbb{1}_{\{\tau<T\}}|\mathscr{F}_t]\\
&= \mathbb{1}_{\{\tau\le t\}} + \mathbb{1}_{\{\tau>t\}}(1-p)\mathrm{e}^{(T-t)^{-1}}\int_{[t,T)}\mathrm{e}^{-(T-u)^{-1}}\mathrm{d}u\\
&\le \mathbb{1}_{\{\tau\le t\}} + \mathbb{1}_{\{\tau>t\}}(1-p)(T-t),
\end{aligned}$$

yielding $L_t>0$ for all $t\in[0,T]$ and $L_{T-}=\lim_{t\uparrow T}L_t=0$ on $\{\tau=T\}$ with

$$P[\tau=T]=p>0.$$

(B): Note that $\frac{\mu}{S}\mathbb{1}_{[\![0,T-\frac{1}{n}]\!]}$ is the standard adjustment process on $[0,T-\frac{1}{n}]$ for any $n\in\mathbb{N}$, but $\frac{\mu}{S}$ itself is not in $L(S)$. Nonetheless, since $\mathscr{E}(-\frac{\mu}{S}\cdot S)_{T-\frac{1}{n}}\to 0$ on $\{\tau=T\}$, one can define $\varphi := -\frac{\mu}{S}\mathscr{E}(-\frac{\mu}{S}\cdot S)\mathbb{1}_{[\![0,T[\![} = \lim_{n\to\infty}\varphi^n$, which gives an integrand in $L(S)$ such that $1+\varphi\cdot S_T=\lim_{n\to\infty}\mathscr{E}(-\frac{\mu}{S}\cdot S)_{T-\frac{1}{n}}$. See also Liptser and Shiryaev [23, Sect. 6.1.4] for details about stochastic exponentials of stochastic integrals of Brownian motion hitting zero. Because of the independence of W and τ, we have

$$E[(1+\varphi\cdot S_T)^2] = E\Big[\exp\Big(-\int_0^\tau \mu_s^2\mathrm{d}s\Big)\Big] = E\Big[\exp\Big(-\int_0^\tau\mu_s^2\mathrm{d}s\Big)\mathbb{1}_{\{\tau<T\}}\Big] = L_0.$$

This yields $1+\varphi^n\cdot S_T=\mathscr{E}(-\frac{\mu}{S}\cdot S)_{T-\frac{1}{n}}\to 1+\varphi\cdot S_T$ in L^2 and therefore $\varphi\in\overline{\Theta}$ by approximating φ^n with simple strategies and extracting a diagonal sequence. Property (B) follows directly from Lemma 4.6 below and the fact that

$$P[L_{\sigma-}=0]=P[\tau=T]=p>0$$

by (A), because $L>0$. This also yields that Q defined via

$$\frac{dQ}{dP}=\frac{1+\varphi\cdot S_T}{E[1+\varphi\cdot S_T]}={}^0\hat{Z}_T\ge 0$$

is an absolutely continuous local martingale measure (ACLMM) for S with square-integrable density process ${}^0\hat{Z} = ({}^0\hat{Z}_t)_{0\le t\le T}$ and

$$\left\{\frac{dQ}{dP} = 0\right\} = \{L_{\sigma-} = 0\} = \{\tau = T\}.$$

(C): We begin by showing the L^∞-(NA) property, that is,

$$\{\vartheta \cdot S_T : \vartheta \in \Theta^\infty\} \cap L^0_+ = \{0\}.$$

For a proof by contradiction, suppose that this property fails, that is, there is $\psi \in \Theta^\infty$ such that $\psi \cdot S_T \in L^0_+ \setminus \{0\}$. Denote by ${}^0\hat{Z}$ the square-integrable density process of the ACLMM Q from (B). Since $\psi \cdot S$ has a uniform lower bound, the local martingale ${}^0\hat{Z}(\psi \cdot S)$ is a supermartingale. Non-negativity of ${}^0\hat{Z}_T$ and $\psi \cdot S_T$ together with the supermartingale property yield ${}^0\hat{Z}(\psi \cdot S) = 0$ on $[0, T]$. This in turn gives $\psi \cdot S_t = 0$ for all $t \in [0, T)$ since ${}^0\hat{Z} > 0$ on $[0, T)$. By the continuity of S, one has $\psi \cdot S_T = 0$, which contradicts $P[\psi \cdot S_T > 0] > 0$.

The proof of the L^2-(NA) property, that is, $\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}\} \cap L^2_+ = \{0\}$, proceeds similarly. Indeed, suppose again, for a proof by contradiction, that the property fails and there is $\psi \in \overline{\Theta}$ such that $\psi \cdot S_T = f \in L^2_+ \setminus \{0\}$. Then ${}^0\hat{Z}(\psi \cdot S)$ is a martingale by Lemma 4.6 below and hence $E[{}^0\hat{Z}_T(\psi \cdot S_T)] = 0$. As before, the latter contradicts the assumption that $P[\psi \cdot S_T > 0] > 0$.

(D) and (E): Recall that $\varphi = -\frac{\mu}{S}\mathscr{E}(-\frac{\mu}{S} \cdot S)\mathbb{1}_{[\![0,T[\![}$ is in $\overline{\Theta}$ and that

$$1 + \varphi \cdot S = \mathscr{E}\left(-\frac{\mu}{S} \cdot S\right)^\tau \mathbb{1}_{\{\tau<T\}}.$$

Likewise, $\vartheta^n := -\frac{\mu}{S}\mathbb{1}_{\{T-\frac{1}{n}<\tau\}}{}^{T-\frac{1}{n}}\mathscr{E}(-\frac{\mu}{S} \cdot S)\mathbb{1}_{[\![0,T[\![}$ is in $\overline{\Theta}$ and one has

$$\mathbb{1}_{\{T-\frac{1}{n}<\tau\}} + \vartheta^n \cdot S = {}^{T-\frac{1}{n}}\mathscr{E}\left(-\frac{\mu}{S} \cdot S\right)^\tau \mathbb{1}_{\{T-\frac{1}{n}<\tau<T\}}.$$

Observe that ϑ^n starts trading at

$$\sigma_n := \left(T - \frac{1}{n}\right)\mathbb{1}_{\{\tau>T-\frac{1}{n}\}} + T\mathbb{1}_{\{\tau\le(T-\frac{1}{n})\}},$$

which is an announcing sequence for the predictable stopping time $\sigma := T$. Therefore the above yields

$$E[(\mathbb{1}_{\{T-\frac{1}{n}<\tau\}} + \vartheta^n \cdot S_T)^2] = E\left[\exp\left(-\int_{T-\frac{1}{n}}^{\tau} \mu_s^2 \mathrm{d}s\right)\mathbb{1}_{\{T-\frac{1}{n}<\tau<T\}}\right] \longrightarrow 0$$

so that $-\vartheta^n \cdot S_T \to \mathbb{1}_{\{\tau=T\}}$ in L^2. Thus $\mathbb{1}_{\{\tau=T\}} \in \mathrm{cl}\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}\}$, but there is no $\psi \in \overline{\Theta}$ such that $\psi \cdot S_T = \mathbb{1}_{\{\tau=T\}}$ by (C) because such a $\psi \in \overline{\Theta}$ would violate the L^2-(NA) property. This gives (D).

Moreover, because S is continuous and $\sigma = T$, we have $\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}_{\sigma-}\} = \{0\}$ while $\mathbb{1}_{\{\tau=T\}} \in \bigcap_{n\in\mathbb{N}} \mathrm{cl}\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}_{\sigma_n}\}$. This gives (E).

(F): The ACLMM for S has been constructed in part (B), where it is also shown that $Q[F]=0$. On the other hand, by the fundamental theorem of asset pricing as for example in Delbaen and Schachermayer [13, Theorem 1.1], there cannot be an ELMM for S.

(G): Observe that the non-negative local martingale $\mathscr{E}(-\mu \cdot W) = 1/\mathscr{E}(\frac{\mu}{S} \cdot S)$ on $[0,T)$ has a finite left limit at T and that this limit is 0 on $\{\tau = T\}$. For the stopping times $(\sigma_n)_{n\in\mathbb{N}}$ given by

$$\sigma_n := \inf\left\{t > 0 : \mathscr{E}\left(\frac{\mu}{S} \cdot S\right)_t > n\right\} \wedge T,$$

we thus have that $\sigma_n < T$ on the event $\{\tau = T\}$ and that $\frac{1}{n}(\mathscr{E}(\frac{\mu}{S} \cdot S)^{\sigma_n} - 1)$ is the wealth of an L^∞-admissible trading strategy

$$\psi^n := \frac{1}{n}\frac{\mu}{S}\mathscr{E}\left(\frac{\mu}{S} \cdot S\right)\mathbb{1}_{[\![0,\sigma_n]\!]} \in \Theta^\infty.$$

Since

$$\begin{aligned}\psi^n \cdot S_T &= \frac{1}{n}\left(\mathscr{E}\left(\frac{\mu}{S} \cdot S\right)_T^{\sigma_n} - 1\right)\\ &\geq \frac{1}{n}\left(\mathscr{E}\left(\frac{\mu}{S} \cdot S\right)_T^{\sigma_n}\mathbb{1}_{\{\tau=T\}} - 1\right) = \mathbb{1}_{\{\tau=T\}} - \frac{1}{n} \longrightarrow \mathbb{1}_{\{\tau=T\}} \qquad \text{in } L^\infty\end{aligned}$$

and $P[\tau = T] > 0$, this sequence yields a "free lunch with vanishing risk".

In the specific setting of this example, one could modify the sequence $(\psi^n)_{n\in\mathbb{N}}$ by starting to trade at $T - \frac{1}{n}$ solely on the event $\{\tau > T - \frac{1}{n}\}$, which yields a sequence of terminal wealths where the gain is still $(1-\frac{1}{n})\mathbb{1}_{\{\tau=T\}}$ and the worst loss is still $\frac{1}{n}$, but the probability of loss decreases from at most $P[\tau < T]$ to at most $\frac{P[\tau<T]}{n}$.

It remains to argue that the strategies ψ^n are not only in Θ^∞ but also in $\overline{\Theta}$. To this end, we recall that the wealth process $\psi^n \cdot S$ is valued in $[-1/n, 1]$ for each $n \in \mathbb{N}$ and hence a square-integrable semimartingale. Because S is locally square-integrable and $\psi^n \cdot S$ is square-integrable, the stochastic integral $\psi^n \cdot S$ can be approximated in $\mathcal{H}^2(P)$ by stochastic integrals $\psi^{n,m} \cdot S$ with bounded simple integrands $(\psi^{n,m})_{m\in\mathbb{N}}$. This gives $\psi^n \in \overline{\Theta}$.

3.3 The law of one price and wealth transfers

Let us offer some intuition for the connection between the values of the opportunity process L on the one hand and the law of one price for S on the other hand, as announced in Theorem 3.2. Recall that $\mathcal{G} = \{\vartheta \cdot S_T : \vartheta \in \Theta\}$ consists of the terminal wealth distributions attainable by simple trading with zero initial wealth. The statement

$$\text{"the affine subspaces } v + \mathcal{G} \text{ are disjoint for different values of } v \in \mathbb{R}\text{"} \tag{3.3}$$

can be seen as the LOP for simple wealth transfers between time 0 and time T. Indeed, the terminal wealths in $v + \mathcal{G}$ are obtainable at the initial price v. If the *same* wealth is obtainable at two distinct initial prices, the law of one price no longer applies. Observe that (3.3) can be restated more compactly as $1 \notin \mathcal{G}$. Since $\mathcal{G}$ is not necessarily closed, one must strengthen this requirement to

$$1 \notin \operatorname{cl}\mathcal{G}; \tag{3.4}$$

this condition fails if there exists a fixed random variable in L^2 that can be approximated arbitrarily well by elements of $v + \mathcal{G}$ for two different values of $v \in \mathbb{R}$.

Let us generalise these observations to trading between an arbitrary stopping time $\tau \in \mathcal{T}$ and T. To this end, let $\mathcal{G}_\tau$ consist of terminal wealths of simple zero-cost strategies that do not trade on the interval $[\![0, \tau]\!]$, i.e., $\mathcal{G}_\tau := \{\vartheta \cdot S_T : \vartheta \in \Theta_\tau\}$. The appropriate condition now reads

$$\mathbb{1}_A \notin \operatorname{cl}\mathcal{G}_\tau \qquad \text{for all } A \in \mathscr{F}_\tau \text{ with } P[A] > 0. \tag{3.5}$$

In analogy with (3.4), the requirement (3.5) may be interpreted as the *law of one price for wealth transfers between times* τ *and* T: if (3.5) fails for some non-null event $A \in \mathscr{F}_\tau$, then there are strategies with different (constant on A) initial wealth at time τ that approximate the same terminal wealth.

Given this interpretation, it is not difficult to see (at least on an intuitive level) that (3.5) corresponds to the condition (1) of Definition 2.3 and that it yields $L > 0$ via (3.1). Theorem 3.2 further shows that Definition 2.3(2) corresponds to the requirement that for every predictable $\tau \in \mathcal{T}$ and some (equivalently every) announcing sequence $(\tau_n)_{n\in\mathbb{N}}$ for τ, one has

$$\mathbb{1}_A \notin \bigcap_{n\in\mathbb{N}} \operatorname{cl}\mathcal{G}_{\tau_n} \qquad \text{for all } A \in \mathscr{F}_{\tau-} \text{ with } P[A] > 0, \tag{3.6}$$

which then yields

$$\bigcap_{n\in\mathbb{N}} \overline{\mathcal{G}}_{\tau_n} = \{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}_{\tau-}\} =: \overline{\mathcal{G}}_{\tau-}, \tag{3.7}$$

and hence $L_- > 0$. The novel condition (3.6) may be interpreted as the LOP for wealth transfers between $\tau-$ and T for predictable $\tau \in \mathcal{T}$. Without the LOP condition (3.6), the first equality in (3.7) may fail and this can occur even for *continuous* price processes. Both conditions $L > 0$ and $L_- > 0$ can therefore be seen as dynamic versions of Schweizer's [32, Sect. 4] requirement

no approximate profits in L^2 *for* $\mathcal{G}$,

which yields the absence of a restricted set of "L^2 free lunches" tailored to quadratic optimisation criteria.

With these observations in mind, let us revisit the notion of a family of state price densities satisfying the law of one price as per Definition 2.6. For a fixed $\tau \in \mathcal{T}$, this family, too, defines a set of terminal wealths available at zero cost at time τ, namely

$$\widehat{\mathcal{G}}_\tau := \{W \in L^2(P|\mathscr{F}_\tau) : E[W^\tau Z_T|\mathscr{F}_\tau] = 0\}.$$

The set $\widehat{\mathcal{G}}_\tau$ is trivially closed in $L^2(P|\mathscr{F}_\tau)$. Furthermore, one has

$$\mathbb{1}_A \notin \widehat{\mathcal{G}}_\tau \qquad \text{for all } A \in \mathscr{F}_\tau \text{ with } P[A] > 0 \tag{3.8}$$

since none of the payoffs $\mathbb{1}_A$ have zero cost in view of

$$E[\mathbb{1}_A{}^\tau Z_T|\mathscr{F}_\tau] = \mathbb{1}_A E[{}^\tau Z_T|\mathscr{F}_\tau] = \mathbb{1}_A \neq 0.$$

Hence ${}^\tau Z_T$ yields the law of one price for wealth transfers between τ and T in the statically complete market characterised by $W \in L^2(P|\mathscr{F}_\tau)$ being available at the price $E[W{}^\tau Z_T|\mathscr{F}_\tau]$ at time τ. Observe that the corresponding opportunity process $\hat{L}$ reads

$$\hat{L}_\tau = \min_{W \in \widehat{\mathcal{G}}_\tau} E[(1-W)^2|\mathscr{F}_\tau] = \frac{1}{E[{}^\tau Z_T^2|\mathscr{F}_\tau]}.$$

So far, we have only exploited properties (1) and (3) of Definition 2.6. The time-consistency condition (2) additionally ensures that $\widehat{\mathcal{G}}_\sigma \supseteq \widehat{\mathcal{G}}_\tau$ for $\sigma, \tau \in \mathcal{T}$ with $\sigma \leq \tau$. Let now τ be a predictable stopping time in $\mathcal{T}$ and (τ_n) an announcing sequence. In principle, it may happen that

$$\mathbb{1}_A \in \bigcap_{n\in\mathbb{N}} \widehat{\mathcal{G}}_{\tau_n} \text{ for some } A \in \bigcap_{n\in\mathbb{N}} \mathscr{F}_{\tau_n} \text{ with } P[A] > 0, \tag{3.9}$$

which is somewhat surprising given that (3.8) holds for all $\tau \in \mathcal{T}$. Condition (4) of Definition 2.6 is needed to prevent (3.9).

We may now interpret Theorem 3.2 as a market extension theorem: S satisfies the LOP if and only if trading on S can be embedded in a statically complete market that satisfies the LOP.

3.4 L^2-projections under the law of one price

In Duffie and Richardson [17], the *mean–variance hedging problem* seeks the mean–variance frontier of wealth distributions of the form $H + \vartheta \cdot S_T$ over admissible ϑ with initial wealth 1 and a fixed contingent claim $H \in L^2$. As in [17] and Schweizer [30], it is convenient to approach mean–variance hedging by first minimising the L^2-distance between an arbitrary contingent claim and the terminal wealth of a self-financing trading strategy, i.e.,

$$\min_{\vartheta \in \overline{\Theta}_\tau} E[(v + \vartheta \cdot S_T - H)^2|\mathscr{F}_\tau] \qquad \text{for } v \in L^2(\mathscr{F}_\tau, P),\ \tau \in \mathcal{T}. \tag{3.10}$$

In solving (3.10), Theorem 3.6 recovers and extends the main results of Černý and Kallsen [4] "on the general structure of mean–variance hedging" in two ways. First, the L^2-projection is obtained without assuming the existence of an equivalent local martingale measure for S, but under the milder LOP assumption. The second novelty of Theorem 3.6 is an expression for the conditional hedging error in (3.14) and (3.15), and the orthogonality statement (3.16), which allow us in Sect. 3.6 to formulate a *conditional version* of the efficient frontier in the spirit of Hansen and Richard [19].

To prepare for the statement of Theorem 3.6, let us recall some notation. Let X be a special $\mathbb{R}^d$-valued semimartingale with predictable characteristics (B^X, C^X, ν^X); see [21, II.2.6]. Denote by $\mathscr{B}^d$ the Borel σ-algebra on $\mathbb{R}^d$. By [21, II.2.9], there exist some increasing predictable process of integrable variation, some predictable $\mathbb{R}^{d\times d}$-valued process c^X whose values are nonnegative symmetric matrices, and some transition kernel F^X from $(\Omega \times \mathbb{R}_+, P)$ into $(\mathbb{R}^d, \mathscr{B}^d)$ such that

$$B^X = b^X \bullet A,\ C^X = c^X \bullet A,\ \nu^X([0,t]\times G) = F^X(G) \bullet A_t \text{ for } t \in [0,T], G \in \mathscr{B}^d.$$

We call (b^X, c^X, F^X, A) *differential characteristics* of X.

Especially when one can take $A_t = t$, one can interpret b^X_t as a drift rate, c^X_t as a diffusion coefficient, and F^X_t as a jump arrival intensity. The differential characteristics are typically derived from other "local" representations of the process, e.g. in terms of a stochastic differential equation. From now on, we choose the same fixed process A for all the (finitely many) semimartingales in this paper. The results do not depend on the particular choice of A.

If $[X, X]$ is special (i.e., X is locally square-integrable), then

$$\tilde{c}^X = c^X + \int xx^\top F^X(\mathrm{d}x) = b^{[X,X]}$$

stands for the modified second characteristic of X. If they refer to some probability measure $P^\star$ rather than P, we write instead $(b^{X\star}, c^{X\star}, F^{X\star}, A)$ and $\tilde{c}^{X\star}$, respectively. We denote the joint characteristics of two special vector-valued semimartingales X, Y by

$$(b^{X,Y}, c^{X,Y}, F^{X,Y}, A) = \left(\begin{pmatrix} b^X \\ b^Y \end{pmatrix}, \begin{pmatrix} c^X & c^{XY} \\ c^{YX} & c^Y \end{pmatrix}, F^{X,Y}, A\right).$$

Furthermore, for locally square-integrable X and Y, we let

$$\tilde{c}^{XY} = c^{XY} + \int xy^\top F^{X,Y}(\mathrm{d}x, \mathrm{d}y).$$

We write c^{-1} for the Moore–Penrose pseudoinverse of a matrix c.

Theorem 3.6 *Suppose that S is locally square-integrable and satisfies the law of one price (or, equivalently, any of the conditions (ii)–(v) in Theorem 3.2). Then:*

1) *The opportunity process $L = (L_t)_{0\le t\le T}$ is the unique bounded semimartingale such that*

(a) $L > 0$, $L_- > 0$ *and* $L_T = 1$;

(b) $\frac{L}{\mathscr{E}(B^{\mathcal{L}(L)})} > 0$ *is a martingale on* $[0, T]$;

(c) *S and $[S, S]$ are $P^\star$-special for the opportunity-neutral measure $P^\star \approx P$ defined by*

$$\frac{dP^\star}{dP} = \frac{L_T}{E[L_0]\mathscr{E}(B^{\mathcal{L}(L)})_T} > 0,$$

which implies

$$b^{S\star} = \frac{b^S + c^{S\,\mathcal{L}(L)} + \int xy F^{S,\mathcal{L}(L)}(\mathrm{d}x, \mathrm{d}y)}{1 + \Delta B^{\mathcal{L}(L)}},$$
$$\tilde{c}^{S\star} = \frac{c^S + \int xx^\top (1+y) F^{S,\mathcal{L}(L)}(\mathrm{d}x, \mathrm{d}y)}{1 + \Delta B^{\mathcal{L}(L)}};$$

(d) *for* $a = (\tilde{c}^{S\star})^{-1} b^{S\star}$, *one has for all* $\tau \in \mathcal{T}$ *that*

$$\frac{b^{\mathcal{L}(L)}}{1 + \Delta B^{\mathcal{L}(L)}} = -\min_{\vartheta \in \mathbb{R}^d} (\vartheta \tilde{c}^{S\star} \vartheta^\top - 2\vartheta b^{S\star}) = -a\tilde{c}^{S\star} a^\top + 2ab^{S\star},$$
$$-a \mathbb{1}_{]\!]\tau,T]\!]} {}^{\tau}\mathscr{E}(-a \cdot S)_- \in \overline{\Theta}_\tau. \tag{3.11}$$

2) *The optimal strategy* $\varphi^{(\tau)} = \varphi^{(\tau)}(v, H) \in \overline{\Theta}_\tau$ *for the conditional quadratic hedging problem* (3.10) *exists and is given in feedback form by*

$$\varphi^{(\tau)}(v, H) = \mathbb{1}_{]\!]\tau,T]\!]} \Big(\xi(H) + a\big(V_-(H) - v - \varphi^{(\tau)}(v, H) \cdot S_-\big)\Big), \tag{3.12}$$

where the mean value process $V = (V_t)_{0 \le t \le T} = (V_t(H))_{0 \le t \le T}$ *is given by*

$$V_t = V_t(H) = \frac{1}{L_t} E[{}^t\mathscr{E}(-a \cdot S)_T H | \mathscr{F}_t], \tag{3.13}$$

and the pure hedging coefficient $\xi = \xi(H) = (\tilde{c}^{S\star})^{-1} \tilde{c}^{SV\star}$ *satisfies*

$$\min_{\vartheta \in \mathbb{R}^d} (\vartheta \tilde{c}^{S\star} \vartheta^\top - 2\vartheta \tilde{c}^{SV\star}) = \xi \tilde{c}^{S\star} \xi^\top - 2\xi \tilde{c}^{SV\star}$$

with

$$\tilde{c}^{SV\star} = \frac{c^{SV} + \int xz(1+y) F^{S,\mathcal{L}(L),V}(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)}{1 + \Delta B^{\mathcal{L}(L)}}.$$

3) *For* $t \in [0, T]$, *let*

$$\varepsilon_t^2(H) := E\big[\big(\mathbb{1}_{]\!]t,T]\!]} L(\tilde{c}^{V\star} - 2\xi \tilde{c}^{SV\star} + \xi \tilde{c}^{S\star} \xi^\top)\big) \cdot A_T \big| \mathscr{F}_t\big] \tag{3.14}$$

with

$$\tilde{c}^{V\star} = \frac{c^V + \int z^2 (1+y) F^{\mathcal{L}(L),V}(\mathrm{d}y, \mathrm{d}z)}{1 + \Delta B^{\mathcal{L}(L)}}.$$

Then $\varepsilon(H)$ *is a semimartingale and for* $\tau \in \mathcal{T}$, *the hedging error of the optimal strategy satisfies*

$$E\big[\big(v + \varphi^{(\tau)}(v, H) \cdot S_T - H\big)^2 \big| \mathscr{F}_\tau\big] = L_\tau\big(v - V_\tau(H)\big)^2 + \varepsilon_\tau^2(H). \tag{3.15}$$

We further have that $\varphi^{(\tau)}(H) := \varphi^{(\tau)}(V_\tau(H), H) \in \overline{\Theta}_\tau$ *is the unique strategy (in the sense of the wealth process* $\varphi^{(\tau)}(H) \cdot S$ *being unique up to* P*-indistinguishability) such that*

$$E\big[\big(V_\tau(H) + \varphi^{(\tau)}(H) \cdot S_T - H\big)(1 + \vartheta \cdot S_T)\big|\mathscr{F}_\tau\big] = 0 \qquad \textit{for all } \vartheta \in \overline{\Theta}_\tau. \quad (3.16)$$

Conversely, if there exists a bounded semimartingale $L = (L_t)_{0\le t\le T}$ *satisfying* 1)(a)–(d), *then* L *is the opportunity process,* 2) *and* 3) *hold, and* S *satisfies the LOP.*

Proof Fix $\tau \in \mathcal{T}$ and $v \in L^2(\mathscr{F}_\tau, P)$. Recall from the proof of Theorem 3.2(vi) that $\overline{\Theta}(\tilde{S}) = \overline{\Theta}_\tau(S)$ for $\tilde{S} = \mathbb{1}_{]\!]\tau,T]\!]} \cdot S$, that $\vartheta \cdot \tilde{S} = \vartheta \cdot S$ for all $\vartheta \in \overline{\Theta}(\tilde{S}) = \overline{\Theta}_\tau(S)$, and that the conditions (i)–(v) of Theorem 3.2 for S imply that the conditions also hold for $\tilde{S}$. Therefore the solution $\tilde{\varphi} \in \overline{\Theta}(\tilde{S})$ to the *unconditional* L^2-approximation

$$\min_{\tilde{\vartheta} \in \overline{\Theta}(\tilde{S})} E[(v + \tilde{\vartheta} \cdot \tilde{S}_T - H)^2] \quad (3.17)$$

exists since $\{\tilde{\vartheta} \cdot \tilde{S}_T : \tilde{\vartheta} \in \overline{\Theta}(\tilde{S})\}$ is closed in L^2 by Theorem 3.2 applied to $\tilde{S}$. By the strict convexity of the square, the terminal wealth $\tilde{\varphi} \cdot \tilde{S}_T$ of the optimal strategy is unique. The LOP then yields uniqueness of the wealth process $\tilde{\varphi} \cdot \tilde{S} = (\tilde{\varphi} \cdot \tilde{S}_t)_{0\le t\le T}$ by Remark 2.4.

Next, we show that $\tilde{\varphi} \in \overline{\Theta}(\tilde{S}) = \overline{\Theta}_\tau(S)$ also optimises (3.10). To this end, note that the solution of (3.17) is uniquely characterised by the first-order condition

$$E[(v + \tilde{\varphi} \cdot \tilde{S}_T - H)(\tilde{\vartheta} \cdot \tilde{S}_T)] = 0 \qquad \text{for all } \tilde{\vartheta} \in \overline{\Theta}(\tilde{S}),$$

which yields $E[(v + \tilde{\varphi} \cdot \tilde{S}_T - H)(\tilde{\vartheta} \cdot \tilde{S}_T)|\mathscr{F}_\tau] = 0$ for all $\tilde{\vartheta} \in \overline{\Theta}(\tilde{S})$ by the definition of the conditional expectation, since

$$\tilde{\vartheta} \in \overline{\Theta}_\tau(\tilde{S}) \qquad \Longleftrightarrow \qquad \mathbb{1}_F \tilde{\vartheta} \in \overline{\Theta}_\tau(\tilde{S}) \text{ for all } F \in \mathscr{F}_\tau.$$

In view of $\overline{\Theta}(\tilde{S}) = \overline{\Theta}_\tau(S)$ and $\vartheta \cdot S_T = \vartheta \cdot \tilde{S}_T$ for all $\vartheta \in \overline{\Theta}_\tau(S) = \overline{\Theta}(\tilde{S})$, the strategy $\tilde{\varphi}$ indeed optimises (3.10), i.e.,

$$E[(v + \tilde{\varphi} \cdot S_T - H)\vartheta \cdot S_T|\mathscr{F}_\tau] = 0 \qquad \text{for all } \vartheta \in \overline{\Theta}_\tau(S).$$

1) and the converse: As shown above, the optimiser $\varphi^{(\tau)}(1, 0) \in \overline{\Theta}_\tau$ exists. Parts (1) and (2) of Czichowsky and Schweizer [11, Proposition 6.1] then yield that the opportunity process is the unique semimartingale $L = (L_t)_{0\le t\le T}$ satisfying 1)(a)–(d). The converse implication that a bounded semimartingale $L = (L_t)_{0\le t\le T}$ satisfying 1)(a)–(d) is the opportunity process follows from [11, Proposition 6.1(3)].

2) and 3): This follows by observing that one only needs to solve the unconditional L^2-approximation problem (3.17) and that the arguments of [4, Sect. 4] only need the properties 1)(a)–(d). As shown above, the latter follow from the LOP and do not require the existence of an equivalent local martingale measure with square-integrable density as assumed in [4]. □

Remark 3.7 All statements about trading from τ onwards in Theorem 3.6 have a natural counterpart for trading from $\tau-$ for predictable $\tau \in \mathcal{T}$, using strategies in $\overline{\Theta}_{\tau-}$. For example, for $v \in L^2(\mathscr{F}_{\tau-}, P)$ and all $\vartheta \in \overline{\Theta}_{\tau-}$, one obtains

$$E\big[\big(v + \varphi^{(\tau-)}(v, H) \cdot S_T - H\big)^2 \big| \mathscr{F}_{\tau-}\big] = L_{\tau-}\big(v - V_{\tau-}(H)\big)^2 + \varepsilon^2_{\tau-}(H),$$
$$E\big[\big(V_{\tau-}(H) + \varphi^{(\tau-)}(H) \cdot S_T - H\big)(1 + \vartheta \cdot S_T)\big| \mathscr{F}_{\tau-}\big] = 0,$$

thus increasing the scope of Theorem 3.6. The proofs for $\tau-$ are completely analogous and therefore omitted.

Remark 3.8 When the law of one price for S fails, the L^2-approximation problem (1.1) becomes degenerate in the following sense. Let

$$\sigma := \inf\{t > 0 : L_{t-} = 0\} \qquad \text{and} \qquad \tau := \inf\{t > 0 : L_t = 0\}.$$

Then at least one of the events $\{\sigma \leq T\}$ and $\{\tau < T\}$ has positive probability, and *any* random variable in L^2 supported on these events can be approximated with arbitrary precision in L^2 by zero-cost strategies in $\overline{\Theta}$. In that case, $\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}\}$ may or may not be closed, as illustrated by Delbaen et al. [12, Examples 3.10–3.12 and Theorem 5.3].

Combining Theorems 3.2 and 3.6 allows us to give a simplified verification theorem for the law of one price that only involves finding a candidate $\hat{L} = (\hat{L}_t)_{0 \leq t \leq T}$ for the opportunity process and not the actual opportunity process $L = (L_t)_{0 \leq t \leq T}$. Its significance compared to earlier results based on the converse implication of Theorem 3.6 is that we do not need to verify the admissibility of the so-called adjustment process a in (3.11), which is usually a difficult task as illustrated in the counterexample of Černý and Kallsen [5].

Theorem 3.9 *Let S be locally square-integrable. Suppose that there exists a bounded semimartingale $\hat{L} = (\hat{L}_t)_{0 \leq t \leq T}$ such that*

(a) $\hat{L} > 0$, $\hat{L}_- > 0$, *and* $\hat{L}_T = 1$;

(b) *for*

$$\hat{b} = \frac{b^S + c^{S\mathcal{L}(\hat{L})} + \int xy F^{S,\mathcal{L}(\hat{L})}(\mathrm{d}x, \mathrm{d}y)}{1 + \Delta B^{\mathcal{L}(\hat{L})}},$$
$$\hat{c} = \frac{c^S + \int xx^\top (1+y) F^{S,\mathcal{L}(\hat{L})}(\mathrm{d}x, \mathrm{d}y)}{1 + \Delta B^{\mathcal{L}(\hat{L})}},$$

one has

$$\frac{b^{\mathcal{L}(\hat{L})}}{1 + \Delta B^{\mathcal{L}(\hat{L})}} = -\min_{\vartheta \in \mathbb{R}^d} (\vartheta \hat{c} \vartheta^\top - 2\vartheta \hat{b}).$$

Then S satisfies the law of one price (or, equivalently, any of the conditions (ii)–(v) in Theorem 3.2). In particular, the opportunity process $L = (L_t)_{0 \leq t \leq T}$ is the maximal bounded semimartingale satisfying (a) and (b).

Proof By properties (a) and (b), the semimartingale $\hat{L}$ is a solution to the BSDE [11, Eq. (4.18)] that takes the form [11, Eq. (6.8)] without cone constraints. Because the opportunity process L is the maximal solution to this BSDE by [11, Lemma 4.17], it follows that the opportunity process L satisfies $L \geq \hat{L}$. Therefore $L > 0$ and $L_- > 0$ by (a) so that (v) and hence any of the conditions (i)–(iv) in Theorem 3.2 holds. □

3.5 Hansen and Richard (1987) framework

The results in Sects. 3.1 and 3.4 have natural counterparts for trading in a wider class of admissible strategies $\widetilde{\Theta}_\tau$, where one only requires *conditional* square-integrability of the terminal wealth. This will allow us to study the Hansen and Richard [19] framework enhanced by dynamic trading in Sect. 3.6.

Definition 3.10 For $\tau \in \mathcal{T}$, a trading strategy $\vartheta \in L(S)$ is in $\widetilde{\Theta}_\tau$ if $\vartheta = 0$ on $[\![0, \tau]\!]$ and there exists a sequence $(\vartheta^n)_{n\in\mathbb{N}}$ of simple strategies $\vartheta^n = (\vartheta^n_t)_{0\leq t\leq T}$ in Θ_τ such that

(1) $\vartheta^n \cdot S_T \to \vartheta \cdot S_T$ in $L^2(P|\mathscr{F}_\tau)$, i.e., $E[(\vartheta^n \cdot S_T - \vartheta \cdot S_T)^2|\mathscr{F}_\tau] \to 0$ in L^0.

(2) $\vartheta^n \cdot S_\sigma \to \vartheta \cdot S_\sigma$ in L^0 for all $\sigma \in \mathcal{T}$.

For predictable $\tau \in \mathcal{T}$, the set $\widetilde{\Theta}_{\tau-}$ is defined analogously by replacing τ with $\tau-$ above and requiring $\vartheta = 0$ only on $[\![0, \tau[\![$.

Remark 3.11 The convergence in Definition 3.10 (1) has the equivalent form

(1′) There is a bounded $0 < K \in L^0(\mathscr{F}_\tau, P)$ with $(K\vartheta^n \cdot S_T) \to (K\vartheta \cdot S_T)$ in L^2,

from which it immediately follows that for all $\tau \in \mathcal{T}$, the set $\widetilde{\Theta}_\tau$ satisfies

$$\widetilde{\Theta}_\tau = L^0(\mathscr{F}_\tau, P)\overline{\Theta}_\tau. \tag{3.18}$$

Indeed, assume that (1) holds. To see that K in (1′) exists, consider the sequence $(X_n)_{n\in\mathbb{N}}$ given by $X_n = \vartheta^n \cdot S_T$ with $\vartheta^n \in \widetilde{\Theta}_\tau$ such that $X_n \to X := \vartheta \cdot S_T$ in $L^2(P|\mathscr{F}_\tau)$. By passing to a subsequence, we may assume that $E[(X_n - X)^2|\mathscr{F}_\tau] \to 0$ P-almost surely. Since $E[X_n^2|\mathscr{F}_\tau] < \infty$, this yields $E[X^2|\mathscr{F}_\tau] < \infty$ as well as

$$\sup_{n\in\mathbb{N}} E[X_n^2|\mathscr{F}_\tau] < \infty.$$

Hence (1′) holds with

$$K = \frac{1}{\sqrt{1 + \sup\{E[X_n^2|\mathscr{F}_\tau] + E[X^2|\mathscr{F}_\tau] : n \in \mathbb{N}\}}} \in (0, 1).$$

Conversely, if (1′) holds, then (1) follows from the properties of conditional expectations in He et al. [20, Theorem I.1.21].

By Theorem 3.2(vi), the law of one price for the price process S yields L^2-closedness of admissible terminal wealths $\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}_\tau\}$ for all $\tau \in \mathcal{T}$. The next result translates the L^2-closedness of $\{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}_\tau\}$ into the conditional closedness of terminal wealths generated by the wider class $\widetilde{\Theta}_\tau$. To avoid repetition, we do not state an analogous result for $\widetilde{\Theta}_{\tau-}$ for predictable $\tau \in \mathcal{T}$.

Theorem 3.12 *Assume the law of one price for S. Then the following statements hold for any $\tau \in \mathcal{T}$, $H \in L^2(P|\mathscr{F}_\tau)$, and $v \in L^0(\mathscr{F}_\tau, P)$:*

1) *The set of terminal wealths $\{\vartheta \cdot S_T : \vartheta \in \widetilde{\Theta}_\tau\}$ is closed in $L^2(P|\mathscr{F}_\tau)$.*

2) *The optimiser in $\min_{\vartheta \in \widetilde{\Theta}_\tau} E[(v + \vartheta \cdot S_T - H)^2 | \mathscr{F}_\tau]$ is the unique (up to a null strategy) process $\varphi^{(\tau)}(v, H)$ satisfying*

$$E\big[\big(v + \varphi^{(\tau)}(v, H) \cdot S_T - H\big)(\vartheta \cdot S_T)\big|\mathscr{F}_\tau\big] = 0 \qquad \text{for all } \vartheta \in \widetilde{\Theta}_\tau.$$

3) *For $H \in L^2(\mathscr{F}_T, P)$ and $L_\tau v^2 \in L^1(\mathscr{F}_\tau, P)$, the optimal hedges in $\overline{\Theta}_\tau$ and $\widetilde{\Theta}_\tau$ coincide. In particular, the opportunity process over $\overline{\Theta}$ coincides with that over $\widetilde{\Theta}$.*

4) *The mean value process $V(H)$ is well defined by (3.13) on $[\![\tau, T]\!]$. The feedback formula (3.12) for the optimal strategy remains valid on $[\![\tau, T]\!]$.*

5) *The minimal hedging error in (3.14) is well defined on $[\![\tau, T]\!]$. Furthermore, (3.15) continues to hold.*

Proof 1): Consider the sets $\mathcal{G}_\tau := \{\vartheta \cdot S_T : \vartheta \in \Theta_\tau\}$, $\overline{\mathcal{G}}_\tau := \{\vartheta \cdot S_T : \vartheta \in \overline{\Theta}_\tau\}$, and $\widetilde{\mathcal{G}}_\tau := \{\vartheta \cdot S_T : \vartheta \in \widetilde{\Theta}_\tau\}$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence that converges in $L^2(P|\mathscr{F}_\tau)$ to some X. By (3.18), we have $\widetilde{\mathcal{G}}_\tau = L^0(\mathscr{F}_\tau, P)\overline{\mathcal{G}}_\tau$. By Remark 3.11, there is a bounded positive $K \in L^0(\mathscr{F}_\tau, P)$ such that $KX_n \in (L^0(\mathscr{F}_\tau, P)\mathcal{G}_\tau) \cap L^2 = \mathcal{G}_\tau$ converges to KX in L^2. Hence $KX \in \operatorname{cl} \mathcal{G}_\tau = \overline{\mathcal{G}}_\tau$ and $X \in \frac{1}{K}\overline{\mathcal{G}}_\tau \subseteq \widetilde{\mathcal{G}}_\tau$.

2)–5): There is a bounded positive $K \in L^0(\mathscr{F}_\tau, P)$ such that Kv and KH are in L^2. It is then immediate from (3.18) that $\varphi^{(\tau)}(Kv, KH) = K\varphi^{(\tau)}(v, H)$, and the formulae follow by applying Theorem 3.6 to the pair (Kv, KH). □

3.6 Mean–variance hedging

The next theorem extends the mean–variance hedging results of Duffie and Richardson [17, Sect. 4.3] to general contingent claims and a conditional mean–variance frontier. Observe that even in the classical case with trivial $\mathscr{F}_0$ and square-integrable H, our characterisation of the mean–variance frontier simplifies that of [17]. By not requiring $L_\tau < 1$, we also expand the conditional mean–variance analysis of Hansen and Richard [19] to a setting without their Assumption 3.1. Observe that dynamic trading has not previously been considered in the setting of [19]. All statements below have natural counterparts for trading from $\tau-$ for predictable $\tau \in \mathcal{T}$ in the spirit of Remark 3.7.

Theorem 3.13 *Assume that S satisfies the law of one price. Then for $\tau \in \mathcal{T}$ and $H \in L^2(P|\mathscr{F}_\tau)$, the following are equivalent:*

(i) *The random variable $W \in H + \{\vartheta \cdot S_T : \vartheta \in \widetilde{\Theta}_\tau\}$ has the smallest conditional variance for a given conditional mean, i.e., W lies on the efficient frontier.*

(ii) *One has $W = H - \varphi^{(\tau)}(H) \cdot S_T + (\lambda - V_\tau(H))(1 - {}^\tau\mathscr{E}(-a \cdot S)_T)$ for some λ in $L^0(\mathscr{F}_\tau, P)$.*

(iii) *One has*

$$\operatorname{Var}[W|\mathscr{F}_\tau] = \varepsilon_\tau^2(H) + \frac{L_\tau}{1 - L_\tau}\big(E[W|\mathscr{F}_\tau] - V_\tau(H)\big)^2,$$

with the convention $0/0 = 0$. *In particular, on the set* $\{L_\tau = 1\}$, *the efficient frontier collapses to a single point with* $E[W|\mathscr{F}_\tau] = V_\tau(H)$ *and* $\mathrm{Var}[W|\mathscr{F}_\tau] = \varepsilon_\tau^2(H)$.

Proof Since $\varphi^{(\tau)}(0,1)$ maximises the conditional expected quadratic utility over $\overline{\Theta}_\tau$, it is mean–variance efficient in $\overline{\Theta}_\tau$. Letting $X_\tau := \varphi^{(\tau)}(0,1) \bullet S_T$, one has

$$E[X_\tau|\mathscr{F}_\tau] = E[X_\tau^2|\mathscr{F}_\tau] = 1 - L_\tau$$

in view of the orthogonality (3.16) and the identity

$$X_\tau = 1 - {}^{\tau}\mathscr{E}(-a \bullet S)_T. \tag{3.19}$$

By (3.16), $\widetilde{\mathscr{G}}_\tau = \{\vartheta \bullet S_T : \vartheta \in \widetilde{\Theta}_\tau\}$ decomposes orthogonally into a subspace whose elements have conditional mean zero and $L^0(\mathscr{F}_\tau, P)X_\tau$. This shows that all efficient payoffs in $\widetilde{\mathscr{G}}_\tau$ are of the form λX_τ for $\lambda \in L^0(\mathscr{F}_\tau, P)$. Writing

$$H + \widetilde{\mathscr{G}}_\tau = H - V_\tau(H) - \varphi^{(\tau)}(H) \bullet S_T + V_\tau(H)(1 - X_\tau) + \widetilde{\mathscr{G}}_\tau,$$

the equivalence of (i) and (ii) now follows from (3.16) and (3.19) in view of the mutual orthogonality of $H - V_\tau(H) - \varphi^{(\tau)}(H) \bullet S_T$, $V_\tau(H)(1 - X_\tau)$ and $\widetilde{\mathscr{G}}_\tau$. The orthogonality yields moment expressions for W in (ii), namely

$$\begin{aligned} E[W|\mathscr{F}_\tau] &= \lambda(1 - L_\tau) + V_\tau(H)L_\tau, \\ E[W^2|\mathscr{F}_\tau] &= \varepsilon_\tau^2(H) + \lambda^2(1 - L_\tau) + V_\tau^2(H)L_\tau, \end{aligned}$$

which after algebraic manipulations shows the equivalence of (ii) and (iii). □

We next examine the mean–variance frontier when the contingent claim is not part of the endowment, but can instead be purchased at time τ for the price $\pi \in L^0(\mathscr{F}_\tau, P)$. The question is then how to select an amount of the contingent claim to be held from τ to maturity T so as to maximise the Sharpe ratio of a zero-cost position that involves dynamic trading in the underlying assets $(1, S)$ and a static trade in the contingent claim.

Definition 3.14 For $\tau \in \mathcal{T}$, $\pi \in L^0(\mathscr{F}_\tau, P)$ and $H \in L^2(P|\mathscr{F}_\tau)$, we call

$$\rho_\tau := \sup\left\{\frac{E[\vartheta \bullet S_T + \eta(\pi - H)|\mathscr{F}_\tau]}{\sqrt{\mathrm{Var}[\vartheta \bullet S_T + \eta(\pi - H)|\mathscr{F}_\tau]}} : \vartheta \in \Theta_\tau, \eta \in L^0(\mathscr{F}_\tau, P)\right\}$$

the *maximal Sharpe ratio on* $]\!]\tau, T]\!]$, with the convention $0/0 = 0$.

Theorem 3.15 *Assume* S *satisfies the law of one price. Fix a stopping time* $\tau \in \mathcal{T}$ *and a contingent claim* $H \in L^2(P|\mathscr{F}_\tau)$ *and suppose further that at time* τ, *the contingent claim* H *delivered at* T *is available at the price* $\pi \in L^0(\mathscr{F}_\tau, P)$, *to be held to maturity. Assume this* π *satisfies* $\mathbb{1}_{\{\varepsilon_\tau(H)=0\}}(\pi - V_\tau(H)) = 0$. *Then the maximal conditional Sharpe ratio* ρ_τ *is given by*

$$\rho_\tau^2 = L_\tau^{-1} - 1 + \frac{(\pi - V_\tau(H))^2}{\varepsilon_\tau^2(H)}, \tag{3.20}$$

with the convention $0/0 = 0$. *Furthermore, this Sharpe ratio is attained by the terminal wealth*

$$\hat{\eta}(\pi - H) + \varphi^{(\tau)}\big(0, 1 - \hat{\eta}(\pi - H)\big) \cdot S_T, \tag{3.21}$$

with the contingent claim position

$$\hat{\eta} = \frac{\pi - V_\tau(H)}{\varepsilon_\tau^2(H)} \frac{1}{1 + \rho_\tau^2}. \tag{3.22}$$

Proof The proof mirrors Černý and Kallsen [6, Lemma 5.1]. For $\vartheta \in \overline{\Theta}_\tau$ and $\mathscr{F}_\tau$-measurable η and π, let $X^{\eta,\vartheta} := \eta(\pi - H) + \vartheta \cdot S_T$. For conditionally square-integrable X, one easily obtains that

$$\frac{(E[X|\mathscr{F}_\tau])^2}{\mathrm{Var}[X|\mathscr{F}_\tau]} = \frac{1}{\inf_{\alpha \in L^0(\mathscr{F}_\tau, P)} E[(1 - \alpha X)^2 |\mathscr{F}_\tau]} - 1.$$

Then

$$\begin{aligned}
\rho_\tau^2 &= \sup_{\eta \in L^0(\mathscr{F}_\tau, P), \vartheta \in \overline{\Theta}_\tau} \frac{(E[X^{\eta,\vartheta}|\mathscr{F}_\tau])^2}{\mathrm{Var}[X^{\eta,\vartheta}|\mathscr{F}_\tau]} \\
&= \sup_{\alpha, \eta \in L^0(\mathscr{F}_\tau, P), \vartheta \in \overline{\Theta}_\tau} \frac{1}{E[(1 - \alpha X^{\eta,\vartheta})^2|\mathscr{F}_\tau]} - 1 \\
&= \frac{1}{\inf_{\eta \in L^0(\mathscr{F}_\tau, P)} \inf_{\vartheta \in \overline{\Theta}_\tau} E[(1 - X^{\eta,\vartheta})^2|\mathscr{F}_\tau]} - 1 \\
&= \frac{1}{\inf\{L_\tau(1 - \eta(\pi - V_\tau(H)))^2 + \eta^2 \varepsilon_\tau^2(H) : \eta \in L^0(\mathscr{F}_\tau, P)\}} - 1,
\end{aligned}$$

where the last equality follows from (3.15) with the contingent claim $1 - \eta(\pi - H)$. Straightforward calculations yield the optimal volume sold in (3.22) and the maximal conditional Sharpe ratio (3.20). By (3.12) with the contingent claim $1 - \hat{\eta}(\pi - H)$, the optimal investment-cum-hedging wealth is given by (3.21). □

Remark 3.16 A square-integrable $\mathscr{E}$-density compatible with S defines an extended market that embeds admissible trading in S, i.e., for $\vartheta \in \overline{\Theta}$ and $t \in [0, T]$, one has

$$\vartheta \cdot S_t = E[(\vartheta \cdot S_T)\, {}^t\mathscr{E}(N)_T |\mathscr{F}_t].$$

The extended market also yields a price process $(\tilde{S}_t)_{0 \le t \le T}$ for every contingent claim $H \in L^2(\mathscr{F}_T, P)$ via the formula

$$\tilde{S}_t := E[H\, {}^t\mathscr{E}(N)_T |\mathscr{F}_t].$$

If $\mathscr{E}(N) > 0$, then $\mathscr{E}(N)$ is the density process of an equivalent local martingale measure for both S and $\tilde{S}$; hence the extended market $(S_t, \tilde{S}_t)_{0 \le t \le T}$ is arbitrage-free over admissible strategies. However, as soon as $\mathscr{E}(N) \le 0$ with positive P-probability, the extended market will contain arbitrage opportunities since the contingent claim $H = \mathbb{1}_{\{\tau \le T\}}$, where τ is the first time $\mathscr{E}(N)$ is less than or equal to zero, trades at a non-positive price $\tilde{S}_0 = E[\mathscr{E}(N)_\tau \mathbb{1}_{\{\tau \le T\}}|\mathscr{F}_0]$ at time 0.

Remark 3.17 Fix $\tau \in \mathcal{T}$ and consider a statically complete market for wealth transfers between τ and T, where every terminal wealth distribution $W \in L^2(P|\mathscr{F}_\tau)$ is available to purchase at time τ at the price

$$p_\tau(W) = E[{}^\tau\mathscr{E}(N)_T W|\mathscr{F}_\tau]$$

with $N = -a \cdot S + \mathcal{L}(L) - [a \cdot S, \mathcal{L}(L)]$. Such a market subsumes static positions in any contingent claim $H \in L^2(P|\mathscr{F}_\tau)$ at the price $V_\tau(H)$ as well as dynamic trading in S using strategies in $\widetilde{\Theta}_\tau$. It is well known (see e.g. Hansen and Jagannathan [18, Eq. (17)]) that the highest Sharpe ratio attainable in such a complete market takes the value

$$\mathrm{Var}[{}^\tau\mathscr{E}(N)_T|\mathscr{F}_\tau] = E\left[\frac{{}^\tau\mathscr{E}(-a \cdot S)_T^2}{L_\tau^2}\middle|\mathscr{F}_\tau\right] - 1 = L_\tau^{-1} - 1.$$

Thus the variance-optimal $\mathscr{E}$-density $({}^\tau\mathscr{E}(N))_{\tau\in\mathcal{T}}$ is the only family of state price densities compatible with S whose complete market does not expand the conditional efficient frontiers generated by trading in S alone; cf. Theorem 3.15.

4 Proofs

4.1 Relation between state price densities and $\mathscr{E}$-densities

We begin with two propositions that are of more general interest. Recall that by a conditional version of the Riesz representation theorem for Hilbert spaces in Hansen and Richard [19, Theorem 2.1], a conditionally linear and continuous operator can be represented as a conditional expectation involving a conditionally square-integrable random variable. More precisely, for $\tau \in \mathcal{T}$ and an $\mathscr{F}_\tau$-conditionally linear and continuous operator $p_\tau : L^2(\mathscr{F}_T, P) \to L^0(\mathscr{F}_\tau, P)$, there is an $\mathscr{F}_T$-measurable random variable ${}^\tau Z_T$ such that

$$\begin{aligned} &E[{}^\tau Z_T^2|\mathscr{F}_\tau] < \infty \quad \text{and} \\ &p_\tau(H) = E[{}^\tau Z_T H|\mathscr{F}_\tau] \quad \text{for all } H \in L^2(\mathscr{F}_T, P). \end{aligned} \tag{4.1}$$

The conditional operator norm of p_τ then satisfies $\|p_\tau\| = (E[{}^\tau Z_T^2|\mathscr{F}_\tau])^{1/2}$. We refer to Cerreia-Vioglio et al. [7] for more details about conditional L^p-spaces and an analysis of conditionally linear operators on them via L^0 modules.

The next result shows that the existence of a price system $(p_\tau)_{\tau\in\mathcal{T}}$ and a family of state price densities $({}^\tau Z_T)_{\tau\in\mathcal{T}}$, both satisfying the law of one price, are equivalent. Comparing Definitions 2.5 and 2.6, the equivalence of properties (1)–(3b) of Definition 2.5 and (1)–(3) of Definition 2.6 directly follows from using (4.1). Therefore we only need to show the equivalence of properties (4) of Definition 2.5 and (4) of Definition 2.6, which is a consequence of the conditional version of the uniform boundedness principle below.

Proposition 4.1 *For a predictable stopping time $\sigma \in \mathcal{T}$ with an announcing sequence $(\sigma_m)_{m\in\mathbb{N}}$ in $\mathcal{T}$, the following are equivalent:*

(i) *There exist $\mathscr{F}_{\sigma_m}$-conditionally linear and continuous mappings*

$$p_{\sigma_m} : L^2(\mathscr{F}_T, P) \to L^0(\mathscr{F}_{\sigma_m}, P)$$

that are pointwise convergent, that is, $(p_{\sigma_m}(H))_{m\in\mathbb{N}}$ is a convergent sequence in L^0 for all $H \in L^2(\mathscr{F}_T, P)$.

(ii) *There exists an $\mathscr{F}_{\sigma-}$-conditionally linear and continuous mapping*

$$p_{\sigma-} : L^2(\mathscr{F}_T, P) \to L^0(\mathscr{F}_{\sigma-}, P)$$

such that $p_{\sigma_m}(H) \to p_{\sigma-}(H)$ in L^0 as $m \to \infty$, for all $H \in L^2(\mathscr{F}_T, P)$.

(iii) *There is a sequence $({}^{\sigma_m}Z_T)_{m\in\mathbb{N}}$ of random variables such that for the p_{σ_m} given by (4.1), the "conditional operator norms" $\|p_{\sigma_m}\| := (E[({}^{\sigma_m}Z_T)^2|\mathscr{F}_{\sigma_m}])^{1/2}$ are bounded, that is,*

$$C := \sup_{m\in\mathbb{N}} \big(E[({}^{\sigma_m}Z_T)^2|\mathscr{F}_{\sigma_m}]\big)^{1/2} < \infty \qquad P\text{-a.s.}, \tag{4.2}$$

and there is a dense subset D of $L^2(\mathscr{F}_T, P)$ such that for all $H \in D$, the sequence $(p_{\sigma_m}(H))_{m\in\mathbb{N}}$ converges in L^0 to some random variable.

Proof (ii) $\Rightarrow$ (i) is clear.

(iii) $\Rightarrow$ (ii): Set $p_{\sigma-}(H) := \lim_{m\to\infty} p_{\sigma_m}(H)$ for all $H \in D$. Then the mapping $p_{\sigma-} : D \to L^0(\mathscr{F}_{\sigma-}, P)$ is continuous and $\mathscr{F}_{\sigma-}$-conditionally linear, where $D \subseteq L^2(\mathscr{F}_T, P)$ is equipped with the topology of $L^2(\mathscr{F}_T, P)$ and $L^0(\mathscr{F}_{\sigma-}, P)$ with the (completely metrisable) topology of convergence in probability. Indeed, for $H_1, H_2 \in D$, we have that

$$\begin{aligned}
|p_{\sigma-}(H_1) - p_{\sigma-}(H_2)| &= \Big|\lim_{m\to\infty} p_{\sigma_m}(H_1) - \lim_{m\to\infty} p_{\sigma_m}(H_2)\Big| \\
&= \lim_{m\to\infty} |p_{\sigma_m}(H_1 - H_2)| \\
&= \lim_{m\to\infty} \big|E[{}^{\sigma_m}Z_T(H_1 - H_2)|\mathscr{F}_{\sigma_m}]\big| \\
&\le \lim_{m\to\infty} \big(E[({}^{\sigma_m}Z_T)^2|\mathscr{F}_{\sigma_m}]\big)^{1/2}\big(E[(H_1 - H_2)^2|\mathscr{F}_{\sigma_m}]\big)^{1/2} \\
&\le C\big(E[(H_1 - H_2)^2|\mathscr{F}_{\sigma-}]\big)^{1/2},
\end{aligned}$$

where we use that $(\mathscr{F}_{\sigma_m})_{m\in\mathbb{N}}$ is increasing and $\bigvee_{m\in\mathbb{N}} \mathscr{F}_{\sigma_m} = \mathscr{F}_{\sigma-}$, which also implies that the range of $p_{\sigma-}$ is $L^0(\mathscr{F}_{\sigma-}, P)$. Since D is dense in $L^2(\mathscr{F}_T, P)$ and $L^0(\mathscr{F}_{\sigma-}, P)$ is complete, we can therefore extend $p_{\sigma-}$ from D to $L^2(\mathscr{F}_T, P)$ by continuity. Note that this implies that $p_{\sigma-}(H) = \lim_{m\to\infty} p_{\sigma_m}(H)$ for all $H \in L^2(\mathscr{F}_T, P)$ and hence that $p_{\sigma-}$ is $\mathscr{F}_{\sigma_m}$-linear for all $m \in \mathbb{N}$. The $\mathscr{F}_{\sigma-}$-linearity then follows from the continuity of $p_{\sigma-}$ and the fact that the indicator function $\mathbb{1}_A$ of every set $A \in \mathscr{F}_{\sigma-}$ can be approximated by the indicator functions $\mathbb{1}_{A_m}$ of sets $A_m \in \mathscr{F}_{\sigma_m}$ in L^0 by Caratheodory's extension theorem, since $\bigvee_{m\in\mathbb{N}} \mathscr{F}_{\sigma_m} = \mathscr{F}_{\sigma-}$.

(i) $\Rightarrow$ (iii): This is a modification of a direct proof of the uniform boundedness principle in Tao [35, Remark 1.7.6]. By way of contradiction, we suppose that (4.2) fails. Then there is a subsequence $(m_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \big(E[({}^{\sigma_{m_k}} Z_T)^2|\mathscr{F}_{\sigma_{m_k}}]\big)^{1/2} = \infty$$

on $A := \{\sup_{m\in\mathbb{N}}(E[({}^{\sigma_m} Z_T)^2|\mathscr{F}_{\sigma_m}])^{1/2} = \infty\}$ with $P[A] > 0$. By passing to a further subsequence still denoted by $(m_k)_{k\in\mathbb{N}}$, we can assume that

$$\lim_{k\to\infty} \big(E[({}^{\sigma_{m_k}} Z_T)^2|\mathscr{F}_{\sigma_{m_k}}]\big)^{1/2}\mathbb{1}_{A_k} = \infty \tag{4.3}$$

on A for $A_k := \{(E[({}^{\sigma_{m_k}} Z_T)^2|\mathscr{F}_{\sigma_{m_k}}])^{1/2} \geq 100^k\} \in \mathscr{F}_{\sigma_{m_k}}$. Set

$$\tilde{H}_k := \frac{{}^{\sigma_{m_k}} Z_T}{(E[({}^{\sigma_{m_k}} Z_T)^2|\mathscr{F}_{\sigma_{m_k}}])^{1/2}}.$$

Then $\tilde{H}_k$ is in $L^2(\mathscr{F}_T, P)$ with $(E[(\tilde{H}_k)^2|\mathscr{F}_{\sigma_{m_k}}])^{1/2} = 1$. Therefore, we have that

$$H = \sum_{k=1}^{\infty} \varepsilon_k 10^{-k}\tilde{H}_k \in L^2(\mathscr{F}_T, P)$$

for all choices $\varepsilon_k \in \{-1, +1\}$. Moreover, we can choose the signs $\varepsilon_k \in \{-1, +1\}$ successively in the following way. After $\varepsilon_1, \dots, \varepsilon_{k-1}$ have been determined, we select $\varepsilon_k \in \{-1, +1\}$ such that

$$E\left[{}^{\sigma_{m_k}} Z_T\left(\sum_{n=1}^{k} \varepsilon_n 10^{-n}\tilde{H}_n\right)\middle|\mathscr{F}_{\sigma_{m_k}}\right] \geq 10^{-k}\big(E[({}^{\sigma_{m_k}} Z_T)^2|\mathscr{F}_{\sigma_{m_k}}]\big)^{1/2}.$$

By the Cauchy–Schwarz and triangle inequalities, this implies

$$\begin{aligned} E[{}^{\sigma_{m_k}} Z_T H|\mathscr{F}_{\sigma_{m_k}}] &= E\left[{}^{\sigma_{m_k}} Z_T\left(\sum_{n=1}^{k} \varepsilon_n 10^{-n}\tilde{H}_n + \sum_{n=k+1}^{\infty} \varepsilon_n 10^{-n}\tilde{H}_n\right)\middle|\mathscr{F}_{\sigma_{m_k}}\right] \\ &\geq 10^{-k}\big(E[({}^{\sigma_{m_k}} Z_T)^2|\mathscr{F}_{\sigma_{m_k}}]\big)^{1/2} \\ &\quad - \big(E[({}^{\sigma_{m_k}} Z_T)^2|\mathscr{F}_{\sigma_{m_k}}]\big)^{1/2}\left(\sum_{n=k+1}^{\infty} 10^{-n}\right) \\ &= 10^{-k}(1 - 1/9)\big(E[({}^{\sigma_{m_k}} Z_T)^2|\mathscr{F}_{\sigma_{m_k}}]\big)^{1/2}. \end{aligned} \tag{4.4}$$

But since $10^{-k}(E[({}^{\sigma_{m_k}} Z_T)^2|\mathscr{F}_{\sigma_{m_k}}])^{1/2} \geq 10^k$ on A_k, this contradicts the convergence of $p_{\sigma_{m_k}}(H) = E[{}^{\sigma_{m_k}} Z_T H|\mathscr{F}_{\sigma_{m_k}}]$ in L^0 in view of (4.3) and (4.4), which completes the proof. □

The next result shows that a family $({}^\tau Z_T)_{\tau\in\mathcal{T}}$ of state price densities satisfying the law of one price can be represented in terms of stochastic exponentials of a local martingale, more precisely, as a square-integrable $\mathscr{E}$-density.

Proposition 4.2 *For a family $({}^\tau Z_T)_{\tau\in\mathcal{T}}$ of random variables, the following statements are equivalent*:

(i) *The collection $({}^\tau Z_T)_{\tau\in\mathcal{T}}$ of random variables is a family of state price densities satisfying the law of one price (i.e., properties (1)–(4) of Definition 2.6 hold).*

(ii) *There is a locally square-integrable martingale N such that $({}^\tau\mathscr{E}(N))_{\tau\in\mathcal{T}}$ is a square-integrable $\mathscr{E}$-density and for all $\tau\in\mathcal{T}$, one has ${}^\tau Z_T = {}^\tau\mathscr{E}(N)_T$.*

Proof (i) $\Rightarrow$ (ii): We adapt the proof technique of Delbaen and Schachermayer [14] as in [11, Lemma 3.5] to our situation. To this end, fix $\tau\in\mathcal{T}$ and define the process ${}^\tau Z = ({}^\tau Z_t)_{0\le t\le T}$ by ${}^\tau Z_t = E[{}^\tau Z_T|\mathscr{F}_t]$. Furthermore define the stopping times $(\sigma_n)_{n\in\mathbb{N}}$ and σ by setting

$$\sigma = \inf\{t>0 : {}^\tau Z_t = 0\}, \qquad \sigma_n = \inf\left\{t>0 : |{}^\tau Z_t| \le \frac{1}{n+1}\right\}\wedge T.$$

Next let $F = \{{}^\tau Z_{\sigma-} = 0\}$, observing that ${}^\tau Z$ is a locally square-integrable local martingale by property (3) of Definition 2.6 and hence ${}^\tau Z_{\sigma-}$ is well defined. Note that ${}^\tau Z_t(\omega) = 1$ for $0\le t\le\tau(\omega)$ by property (1) of Definition 2.6. The Cauchy–Schwarz inequality yields

$$\begin{aligned}
\mathbb{1}_{\{\sigma_n<\sigma\}} &= E[{}^{\sigma_n} Z_T \mathbb{1}_{\{\sigma_n<\sigma\}}|\mathscr{F}_{\sigma_n}] \\
&= E[{}^{\sigma_n} Z_T \mathbb{1}_{\{\sigma_n<\sigma\}}\mathbb{1}_{F^c}|\mathscr{F}_{\sigma_n}] \\
&\le (E[{}^{\sigma_n} Z_T^2 \mathbb{1}_{\{\sigma_n<\sigma\}}|\mathscr{F}_{\sigma_n}])^{\frac{1}{2}}(P[F^c|\mathscr{F}_{\sigma_n}])^{\frac{1}{2}} \\
&= (E[{}^{\sigma_n} Z_T^2|\mathscr{F}_{\sigma_n}])^{\frac{1}{2}}(P[F^c|\mathscr{F}_{\sigma_n}])^{\frac{1}{2}}.
\end{aligned} \tag{4.5}$$

On sending $n\to\infty$, one obtains in view of (4.5) and the bounded conditional second moments at predictable stopping times (by property (4) of Definition 2.6) that

$$\begin{aligned}
\mathbb{1}_F &= \lim_{n\to\infty}\mathbb{1}_{\{\sigma_n<\sigma\}}\mathbb{1}_F \\
&\le \limsup_{n\to\infty}\Big((E[{}^{\sigma_n} Z_T^2|\mathscr{F}_{\sigma_n}])^{\frac{1}{2}}\mathbb{1}_F\mathbb{1}_{\{\sigma_n<\sigma\}}(P[F^c|\mathscr{F}_{\sigma_n}]^{\frac{1}{2}})\Big) \\
&= \limsup_{n\to\infty}(E[{}^{\sigma_n} Z_T^2|\mathscr{F}_{\sigma_n}])^{\frac{1}{2}}\mathbb{1}_F\mathbb{1}_{F^c} \\
&\le C\mathbb{1}_F\mathbb{1}_{F^c} = 0,
\end{aligned}$$

where $C^2 = \sup_{n\in\mathbb{N}} E[{}^{\sigma_n} Z_T^2|\mathscr{F}_{\sigma_n}]$. This yields $P[F]=0$ and hence ${}^\tau Z_{\sigma-}\neq 0$, which means that each ${}^\tau Z$ can only jump to zero. Therefore ${}^\tau Z_- \neq 0$ on $[\![0,\sigma]\!]$ and ${}^\tau Z = 0$ on $[\![\sigma, T]\!]$ so that its stochastic logarithm $\mathcal{L}({}^\tau Z) = \frac{\mathbb{1}_{[\![0,\sigma]\!]}}{{}^\tau Z_-}\bullet{}^\tau Z$ is well defined and gives ${}^\tau Z = \mathscr{E}(\mathcal{L}({}^\tau Z))$; see Choulli et al. [8, Proposition 2.2]. Note that since ${}^\tau Z$ is a locally square-integrable martingale, so is $\mathcal{L}({}^\tau Z)$.

Let $(T_n)_{n\in\mathbb{N}_0}$ be the sequence of stopping times given by $T_0 = 0$ and

$$T_{n+1} = \inf\{t > T_n : {}^{T_n}Z_t = 0\} \wedge T \qquad \text{for } n \in \mathbb{N}_0.$$

Then because each ${}^{T_n}Z$ can only jump to zero, the sequence $(T_n)_{n\in\mathbb{N}_0}$ converges stationarily to T. Since $\mathcal{L}({}^{T_n}Z)_t(\omega) = 0$ for $t \notin (T_n(\omega), T_{n+1}(\omega)]$ and $(T_n)_{n\in\mathbb{N}_0}$ converges stationarily to T, setting $N := \sum_{n=0}^{\infty} \mathcal{L}({}^{T_n}Z)$ yields a locally square-integrable martingale. Fix $\tau \in \mathcal{T}$. Then for each $\omega \in \Omega$, there is only one index $n(\omega) \in \mathbb{N}_0$ such that $\tau(\omega) \in [T_{n(\omega)}(\omega), T_{n(\omega)+1}(\omega))$. Observe that the mapping $\omega \mapsto n(\omega)$ is $\mathscr{F}_\tau$-measurable since $\{n(\omega) = k\} = \{T_k \le \tau < T_{k+1}\} \in \mathscr{F}_\tau$ for all $k \in \mathbb{N}_0$. Combining the time-consistency of ${}^\tau Z$ (property (2) of Definition 2.6) with Yor's formula for stochastic exponentials and the definition of N yields

$${}^\tau Z_T(\omega) = \frac{{}^{T_{n(\omega)}}Z_T(\omega)}{{}^{T_{n(\omega)}}Z_\tau(\omega)} = \frac{\mathscr{E}({}^{T_{n(\omega)}}N)_T(\omega)}{\mathscr{E}({}^{T_{n(\omega)}}N)_\tau)(\omega)} = \mathscr{E}({}^\tau N)_T(\omega) = {}^\tau\mathscr{E}(N)_T(\omega)$$

and hence ${}^\tau Z_T = {}^\tau\mathscr{E}(N)_T$. By the square-integrability of $({}^\tau Z_T)_{\tau\in\mathcal{T}}$ (property (3) of Definition 2.6), the latter also implies that $E[{}^\tau\mathscr{E}(N)_T^2|\mathscr{F}_\tau] = E[({}^\tau Z_T)^2|\mathscr{F}_\tau] < \infty$ for all $\tau \in \mathcal{T}$ so that $({}^\tau\mathscr{E}(N))_{\tau\in\mathcal{T}}$ is a square-integrable $\mathscr{E}$-density (Definition 2.8).

(ii) $\Rightarrow$ (i): We only need to verify that setting ${}^\tau Z_T := {}^\tau\mathscr{E}(N)_T$ for $\tau \in \mathcal{T}$ defines a family $({}^\tau Z_T)_{\tau\in\mathcal{T}}$ of state price densities satisfying the law of one price. Indeed, the family $({}^\tau\mathscr{E}(N))_{\tau\in\mathcal{T}}$ being an $\mathscr{E}$-density (as in Definition 2.8) implies the correct pricing of the risk-free asset (property (1) of Definition 2.6) as well as the time-consistency (property (2) of Definition 2.6) by applying Yor's formula for stochastic exponentials. The conditional square-integrability (property (3) of Definition 2.6) follows directly from the fact that the $\mathscr{E}$-density $({}^\tau\mathscr{E}(N))_{\tau\in\mathcal{T}}$ is square-integrable (as in Definition 2.8). We prove the bounded conditional second moments at predictable stopping times (property (4) of Definition 2.6) by contradiction. For this, let $\tau \in \mathcal{T}$ be a predictable stopping time and $(\tau_n)_{n\in\mathbb{N}}$ an announcing sequence for τ such that $P[F_1] > 0$ for $F_1 := \{\sup_{n\in\mathbb{N}} E[{}^{\tau_n}\mathscr{E}(N)_T^2|\mathscr{F}_{\tau_n}] = \infty\}$. Note that $F_1 \subseteq \{\tau > 0\}$ by the conditional square-integrability (property (3) of Definition 2.6). Let $(T_m)_{m\in\mathbb{N}_0}$ be the sequence of stopping times given by $T_0 = 0$ and

$$T_{m+1} = \inf\{t > T_m : {}^{T_m}\mathscr{E}(N)_t = 0\} \wedge T \qquad \text{for } m \in \mathbb{N}_0.$$

Because each ${}^{T_m}\mathscr{E}(N)$ can only jump to zero, the sequence $(T_m)_{m\in\mathbb{N}_0}$ converges stationarily to T. Hence there is $k \in \mathbb{N}$ such that $F_2 := \{T_k < \tau \le T_{k+1}\} \cap F_1$ has $P[F_2] > 0$. On F_2, we have ${}^{T_k}\mathscr{E}(N)_T = {}^{T_k}\mathscr{E}(N)_{\tau_n}{}^{\tau_n}\mathscr{E}(N)_T$ for sufficiently large n. Since $E[{}^{T_k}\mathscr{E}(N)_T^2|\mathscr{F}_{T_k}] < \infty$, this yields $\lim_{n\to\infty} {}^{T_k}\mathscr{E}(N)_{\tau_n} = {}^{T_k}\mathscr{E}(N)_{\tau-} = 0$ on F_2. The latter contradicts the fact that ${}^{T_k}\mathscr{E}(N)_{\tau-} \ne 0$ on $\{T_k < \tau \le T_{k+1}\}$ by the definition of T_{k+1}, which completes the proof. □

Remark 4.3 The proof of Proposition 4.2 shows that condition (i) can be written in the following equivalent form:

(i′) The family $({}^\tau Z_T)_{\tau\in\mathcal{T}}$ of state price densities satisfies properties (1)–(3) of Definition 2.6 and ${}^\tau Z = ({}^\tau Z_t)_{0\le t\le T}$ does not reach zero continuously and is absorbed in zero in the sense of (2.1), for each $\tau \in \mathcal{T}$.

4.2 Proof of Theorem 3.2

We now construct a specific family $({}^{\tau}\hat{Z}_T)_{\tau\in\mathcal{T}}$ of variance-optimal state price densities as discussed in Remark 3.3.

Lemma 4.4 *For any stopping time $\tau \in \mathcal{T}$, let $\hat{G}_T^{(\tau)}$ be the orthogonal projection of* 1 *onto* $\mathrm{cl}\{\vartheta \cdot S_T : \vartheta \in \Theta_\tau\}$. *Furthermore, for such τ, define the square-integrable martingale $\hat{M}^{(\tau)} = (\hat{M}_t^{(\tau)})_{0\le t\le T}$ by*

$$\hat{M}_t^{(\tau)} := E[(1 - \hat{G}_T^{(\tau)})|\mathcal{F}_t]. \tag{4.6}$$

Then the process

$$\hat{M}^{(\tau)}(x + \vartheta \cdot S) = \big(\hat{M}_t^{(\tau)}(x + \vartheta \cdot S_t)\big)_{0\le t\le T} \tag{4.7}$$

is a martingale for any $x \in L^2(\mathscr{F}_0, P)$ and $\vartheta \in \overline{\Theta}_\tau(0)$, and

$$L_\tau = E[(1 - \hat{G}_T^{(\tau)})^2|\mathcal{F}_\tau] = \hat{M}_\tau^{(\tau)}. \tag{4.8}$$

Proof The martingale property of (4.7) follows from the first-order condition of optimality. Indeed, $\vartheta \in \Theta_\tau$ implies that $\mathbb{1}_{F\times(s,t]}\vartheta$ is in Θ_τ for all $s \le t$ and arbitrary $F \in \mathscr{F}_s$. Therefore by the definition of $\hat{G}_T^{(\tau)}$, we have that

$$E[\mathbb{1}_F(\vartheta \cdot S_t - \vartheta \cdot S_s)(1 - \hat{G}_T^{(\tau)})] = E\big[\big((\mathbb{1}_{F\times(s,t]}\vartheta) \cdot S_T\big)(1 - \hat{G}_T^{(\tau)})\big] = 0.$$

As $F \in \mathscr{F}_s$ was arbitrary, the tower property of conditional expectations yields

$$\begin{aligned} E[(\vartheta \cdot S_t)\hat{M}_t^{(\tau)}|\mathscr{F}_s] &= E[(\vartheta \cdot S_t)(1 - \hat{G}_T^{(\tau)})|\mathscr{F}_s] \\ &= E[(\vartheta \cdot S_s)(1 - \hat{G}_T^{(\tau)})|\mathscr{F}_s] = (\vartheta \cdot S_s)\hat{M}_s^{(\tau)}. \end{aligned}$$

The case $\vartheta \in \overline{\Theta}_\tau$ then follows by approximating $\vartheta \in \overline{\Theta}_\tau$ by a sequence $(\vartheta^n)_{n\in\mathbb{N}}$ of strategies $\vartheta^n = (\vartheta_t^n)_{0\le t\le T}$ in Θ_τ. This yields

$$\begin{aligned} E[(\vartheta \cdot S_T)\hat{M}_T^{(\tau)}|\mathscr{F}_t] &= E\Big[\Big(\lim_{n\to\infty} \vartheta^n \cdot S_T\Big)\hat{M}_T^{(\tau)}\Big|\mathscr{F}_t\Big] \\ &= \lim_{n\to\infty} E[(\vartheta^n \cdot S_T)\hat{M}_T^{(\tau)}|\mathscr{F}_t] \\ &= \lim_{n\to\infty} (\vartheta^n \cdot S_t)\hat{M}_t^{(\tau)} = (\vartheta \cdot S_t)\hat{M}_t^{(\tau)}, \end{aligned}$$

where we have used $\lim_{n\to\infty} \vartheta^n \cdot S_T \to \vartheta \cdot S_T$ in L^2 in the first and second equality and $\lim_{n\to\infty} \vartheta^n \cdot S_t \to \vartheta \cdot S_t$ in probability in the last equality.

Finally, let $(\varphi^n)_{n\in\mathbb{N}}$ be a sequence of trading strategies $\varphi^n = (\varphi^n_t)_{0\le t\le T}$ in $\overline{\Theta}_\tau$ with $\varphi^n \cdot S_T \to \hat{G}^{(\tau)}_T$ in L^2. Then

$$\begin{aligned} L_\tau &= E[(1-\hat{G}^{(\tau)}_T)^2|\mathcal{F}_\tau] \\ &= E[1-\hat{G}^{(\tau)}_T|\mathcal{F}_\tau] - E\Big[(1-\hat{G}^{(\tau)}_T)\Big(\lim_{n\to\infty}\varphi^n\cdot S_T\Big)\Big|\mathcal{F}_\tau\Big] \\ &= E[1-\hat{G}^{(\tau)}_T|\mathcal{F}_\tau] - \lim_{n\to\infty} E[(1-\hat{G}^{(\tau)}_T)(\varphi^n\cdot S_T)|\mathcal{F}_\tau] \\ &= E[1-\hat{G}^{(\tau)}_T|\mathcal{F}_\tau]. \end{aligned}$$

□

Using (4.8), we next obtain a process $\hat{G}^{(\tau)} = (\hat{G}^{(\tau)}_t)_{0\le t\le T}$ acting as a substitute for the unavailable stochastic integral $(\varphi^{(\tau)}(1,0)\cdot S_t)_{0\le t\le T}$ that leads to the optimal terminal wealth $\hat{G}^{(\tau)}_T$ defined in Lemma 4.4. Observe that the construction in (4.9) only needs the weak LOP condition $L > 0$.

Lemma 4.5 *Suppose $L > 0$. For each $\tau \in \mathcal{T}$, define the process $\hat{G}^{(\tau)} = (\hat{G}^{(\tau)}_t)_{0\le t\le T}$ by*

$$\hat{G}^{(\tau)}_t = \begin{cases} 0, & 0 \le t < \tau, \\ 1 - \frac{\hat{M}^{(\tau)}_t}{L_t}, & \tau \le t \le T. \end{cases} \tag{4.9}$$

Then for all stopping times $\sigma \ge \tau$ in $\mathcal{T}$, we have that

$$1-\hat{G}^{(\tau)}_T = (1-\hat{G}^{(\tau)}_\sigma)(1-\hat{G}^{(\sigma)}_T). \tag{4.10}$$

Proof Let $(\vartheta^n)_{n\in\mathbb{N}}$ be a sequence of trading strategies $\vartheta^n = (\vartheta^n_t)_{0\le t\le T}$ in Θ_τ such that $\vartheta^n\cdot S_T \to \hat{G}^{(\tau)}_T$ in L^2. Then

$$(1-\vartheta^n\cdot S_\sigma)^2 L_\sigma \le E[(1-\vartheta^n\cdot S_T)^2|\mathscr{F}_\sigma] \tag{4.11}$$

by [11, Proposition 3.1]. Since $(\vartheta^n\cdot S_T)_{n\in\mathbb{N}}$ is convergent in $L^2(P)$, it is bounded in $L^2(P)$ and hence $(\vartheta^n\cdot S_\sigma)_{n\in\mathbb{N}}$ is bounded in $L^2(P^\sigma)$ by (4.11), where $P^\sigma \approx P$ is defined by $\frac{dP^\sigma}{dP} = \frac{L_\sigma}{E[L_\sigma]} > 0$. By Mazur's lemma, see Brezis [3, Corollary 3.8], there exist a sequence $(\varphi^n)_{n\in\mathbb{N}}$ of trading strategies $\varphi^n \in \mathrm{conv}(\vartheta^n, \vartheta^{n+1}, \dots) \subseteq \Theta_\tau$ and a random variable $X_\sigma \in L^2(\mathscr{F}_\sigma, P^\sigma)$ such that $\varphi^n\cdot S_\sigma \to X_\sigma$ in $L^2(P^\sigma)$ and hence

$$(1-X_\sigma)^2 L_\sigma \le E[(1-\hat{G}^{(\tau)}_T)^2|\mathscr{F}_\sigma]. \tag{4.12}$$

Note that (4.12) implies that

$$1-\big(X_\sigma + (1-X_\sigma)\hat{G}^{(\sigma)}_T\big) = (1-X_\sigma)(1-\hat{G}^{(\sigma)}_T) \in L^2(P)$$

and hence $Y_\sigma := X_\sigma + (1-X_\sigma)\hat{G}^{(\sigma)}_T \in L^2(P)$. If we can show that

$$Y_\sigma = X_\sigma + (1-X_\sigma)\hat{G}^{(\sigma)}_T \in \mathcal{G}_\tau, \tag{4.13}$$

then it follows from (4.12) that Y_σ is optimal for (3.2) and hence

$$Y_\sigma = \hat{G}_T^{(\tau)}. \tag{4.14}$$

For the proof of (4.13), let $(\psi^n)_{n\in\mathbb{N}}$ be a sequence of simple trading strategies $\psi^n = (\psi^n_t)_{0\le t\le T}$ in Θ_σ such that $\psi^n \boldsymbol{\cdot} S_T \to \hat{G}_T^{(\sigma)}$ in $L^2(P)$. By Egorov's theorem, see [1, 10.38], there exists a sequence $(F_m)_{m\in\mathbb{N}}$ of sets $F_m \in \mathscr{F}_\sigma$ with $P[F_m] \ge 1 - 1/m$ such that for each $m \in \mathbb{N}$, we have that $(\varphi^n \boldsymbol{\cdot} S_\sigma)_{n\in\mathbb{N}}$ and X_σ are uniformly bounded on F_m and $\varphi^n \boldsymbol{\cdot} S_\sigma \to X_\sigma$ uniformly on F_m. Then $(\xi^{m,n})_{n\in\mathbb{N}}$ given by

$$\xi^{m,n} = \varphi^n \mathbb{1}_{F_m^c} + \big(\varphi^n \mathbb{1}_{[\![0,\sigma]\!]} + (1 - \varphi^n \boldsymbol{\cdot} S_\sigma)\psi^n \mathbb{1}_{]\!]\sigma,T]\!]}\big)\mathbb{1}_{F_m}$$

is a sequence of trading strategies $\xi^{m,n} = (\xi^{m,n}_t)_{0\le t\le T}$ in Θ_τ such that

$$\xi^{n,m} \boldsymbol{\cdot} S_T = (\varphi^n \boldsymbol{\cdot} S_T)\mathbb{1}_{F_m^c} + (1 - \varphi^n \boldsymbol{\cdot} S_\sigma)(\psi^n \boldsymbol{\cdot} S_T)\mathbb{1}_{F_m}$$

by the local character of the stochastic integral. Hence for each $m \in \mathbb{N}$,

$$\xi^{m,n} \boldsymbol{\cdot} S_T \longrightarrow \hat{G}_T^{(\tau)}\mathbb{1}_{F_m^c} + (1 - X_\sigma)\hat{G}_T^{(\sigma)}\mathbb{1}_{F_m} \text{ in } L^2(P) \qquad \text{as } n \to \infty.$$

Because $\hat{G}_T^{(\tau)}\mathbb{1}_{F_m^c} + (1 - X_\sigma)\hat{G}_T^{(\sigma)}\mathbb{1}_{F_m} \to (1 - X_\sigma)\hat{G}_T^{(\sigma)}$ in $L^2(P)$ as $m \to \infty$, we can select a diagonal sequence $(\xi^{m,n_m})_{m\in\mathbb{N}}$ with $\xi^{m,n_m} = (\xi^{m,n_m}_t)_{0\le t\le T} \in \Theta_\tau$ and

$$\xi^{m,n_m} \boldsymbol{\cdot} S_T \longrightarrow Y_\sigma = X_\sigma + (1 - X_\sigma)\hat{G}_T^{(\sigma)} \text{ in } L^2(P) \qquad \text{as } m \to \infty,$$

and hence (4.13) holds.

From (4.13), we obtain

$$\begin{aligned}\hat{G}_\sigma^{(\tau)} &= 1 - \frac{\hat{M}_\sigma^{(\tau)}}{L_\sigma} \\ &= 1 - E[(1 - X_\sigma)(1 - \hat{G}_T^{(\sigma)})|\mathscr{F}_\sigma] \\ &= 1 - (1 - X_\sigma)E[(1 - \hat{G}_T^{(\sigma)})|\mathscr{F}_\sigma] = X_\sigma\end{aligned}$$

so that X_σ is uniquely determined as $X_\sigma = \hat{G}_\sigma^{(\tau)}$, and we have (4.10) by (4.14). □

Lemma 4.6 *Suppose that $L > 0$. For each $\tau \in \mathcal{T}$, let*

$${}^\tau\hat{Z}_t = \begin{cases} 1, & 0 \le t < \tau, \\ \frac{\hat{M}_t^{(\tau)}}{L_\tau} = \frac{L_t(1-\hat{G}_t^{(\tau)})}{L_\tau}, & \tau \le t \le T. \end{cases} \tag{4.15}$$

Then $({}^\tau\hat{Z}_T)_{\tau\in\mathcal{T}}$ is a family of state price densities satisfying properties (1)–(3) of Definition 2.6, compatible with S, and such that for all $\tau \in \mathcal{T}$,

$$E[({}^\tau\hat{Z}_T)^2|\mathscr{F}_\tau] = \frac{1}{L_\tau}.$$

If in addition $L_- > 0$, then $({}^\tau \hat{Z}_T)_{\tau \in \mathcal{T}}$ satisfies the law of one price, that is, it also fulfils property (4) of Definition 2.6.

Proof We begin by verifying that $({}^\tau \hat{Z}_T)_{\tau \in \mathcal{T}}$ is a family of state price densities satisfying properties (1)–(3) of Definition 2.6. The definition (4.6) of $\hat{M}^{(\tau)}$ together with (4.8) implies that for all $\tau \in \mathcal{T}$, one has

$$E[{}^\tau \hat{Z}_T | \mathscr{F}_\tau] = E\left[\frac{\hat{M}_T^{(\tau)}}{L_\tau}\middle| \mathscr{F}_\tau\right] = \frac{\hat{M}_\tau^{(\tau)}}{\hat{M}_\tau^{(\tau)}} = 1$$

and hence the correct pricing of the risk-free asset (property (1) of Definition 2.6). The time-consistency (Definition 2.6(2)) follows from ${}^\tau \hat{Z}_t = (1 - \hat{G}_t^{(\tau)})/L_\tau$, which is an easy consequence of (4.15), together with (4.10). On combining the identity ${}^\tau \hat{Z}_T = (1 - \hat{G}_T^{(\tau)})/L_\tau$ with $E[(1 - \hat{G}_T^{(\tau)})^2 | \mathcal{F}_\tau] = L_\tau < \infty$ for all $\tau \in \mathcal{T}$ by (4.8), we obtain that ${}^\tau \hat{Z}_T$ is conditionally square-integrable (property (3) of Definition 2.6) and $E[({}^\tau \hat{Z}_T)^2 | \mathscr{F}_\tau] = \frac{1}{L_\tau}$ for all $\tau \in \mathcal{T}$. Because $\mathbb{1}_{]\!]0, \sigma_n \wedge \sigma]\!]} \in \Theta \subseteq \overline{\Theta}$ is a simple trading strategy for any localising sequence $(\sigma_n)_{n \in \mathbb{N}}$ of stopping times with $S^*_{\sigma_n} \in L^2$, we have that S is compatible with $({}^\tau Z_T)_{\tau \in \mathcal{T}}$ in the sense of Definition 2.6 by choosing $\tau = 0$, $x = S_0$, and $\vartheta = \mathbb{1}_{]\!]0, \sigma_n \wedge \sigma]\!]}$ in (4.7).

For the proof of the bounded conditional expectations of squares at predictable stopping times (property (4) of Definition 2.6), suppose that $L_- > 0$. Let $\sigma \in \mathcal{T}$ be a predictable stopping time and $(\sigma_n)_{n \in \mathbb{N}}$ an announcing sequence for σ. Because

$$E[({}^{\sigma_n} \hat{Z}_T)^2 | \mathscr{F}_{\sigma_n}] = E\left[\left(\frac{1 - \hat{G}_T^{(\sigma_n)}}{L_{\sigma_n}}\right)^2 \middle| \mathscr{F}_{\sigma_n}\right] = \frac{1}{L_{\sigma_n}},$$

we obtain that $\sup_{n \in \mathbb{N}} E[({}^{\sigma_n} \hat{Z}_T)^2 | \mathscr{F}_{\sigma_n}] = \sup_{n \in \mathbb{N}} \frac{1}{L_{\sigma_n}} < \infty$, since L_{σ_n} converges to $L_{\sigma-}$ P-a.s. and we have both $L > 0$ and $L_- > 0$. □

Proof of Theorem 3.2 The most efficient way to prove the theorem is to show (i) ⇒ (v), (v) ⇒ (iv), (iv) ⇒ (iii), (iii) ⇒ (ii), and (ii) ⇒ (i).

(i) ⇒ (v): For a proof by contradiction, we first suppose that $L > 0$ fails. Then for the stopping time $\tau := \inf\{t > 0 : L_t = 0\} \wedge T$ and $F := \{L_\tau = 0\}$, we have $P[F] > 0$ and therefore $0 = \mathbb{1}_F L_\tau = \mathbb{1}_F (\operatorname{ess\,inf}_{\vartheta \in \overline{\Theta}_\tau} E[(1 - \vartheta \cdot S_T)^2 | \mathscr{F}_\tau])$. Because the family

$$\Gamma = \{E[(1 - \vartheta \cdot S_T)^2 | \mathscr{F}_\tau] : \vartheta \in \overline{\Theta}_\tau\}$$

of random variables is stable under taking minima by [11, Lemma 2.18(1)], there exists a sequence $(\vartheta^n)_{n \in \mathbb{N}}$ of trading strategies $\vartheta^n = (\vartheta^n_t)_{0 \le t \le T}$ in $\overline{\Theta}_\tau$ such that

$$E\big[\big(\mathbb{1}_F - \mathbb{1}_F(\vartheta^n \cdot S_T)\big)^2\big] = E\big[\mathbb{1}_F E[(1 - \vartheta^n \cdot S_T)^2 | \mathscr{F}_\tau]\big] \longrightarrow 0.$$

As 0 and ϑ^n are in $\overline{\Theta}_\tau$ for all $n \in \mathbb{N}$, we have by [11, Lemma 2.18(1)] that

$$\psi^n := (\mathbb{1}_F \vartheta^n) \in \overline{\Theta}_\tau \qquad \text{for all } n \in \mathbb{N}.$$

Therefore we have a sequence $(\psi^n)_{n\in\mathbb{N}}$ of trading strategies in $\overline{\Theta}_\tau$ and $F \in \mathscr{F}_\tau$ such that $\mathbb{1}_F(1-\psi^n \cdot S_T) \to 0$ in L^2, but $\lim_{n\to\infty} \mathbb{1}_F(1-\psi^n \cdot S_\tau) = \mathbb{1}_F \neq 0$. Approximating the strategies $\psi^n \in \overline{\Theta}_\tau$ by simple trading strategies $\varphi^n \in \overline{\Theta}_\tau$, this contradicts the LOP for S, namely, item (1) of Definition 2.3.

If $L_- > 0$ fails, one has $P[F] > 0$ for $F := \{L_{\sigma-} = 0\}$ with the predictable stopping time $\sigma = \inf\{t > 0 : L_{t-} = 0\} \wedge T$. Define a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of stopping times by $\sigma_n = \inf\{t > 0 : L_{t-} \leq \frac{1}{n}\} \wedge T$ and set $F_n = \{L_{\sigma_n-} \leq \frac{1}{n}\}$. Then $(\sigma_n)_{n\in\mathbb{N}}$ is an announcing sequence for σ and one has $\mathbb{1}_{F_n} \to \mathbb{1}_F$ in L^0 and $\mathbb{1}_{F_n} L_{\sigma_n} \to \mathbb{1}_F L_{\sigma-}$ in L^2. Since for each $n \in \mathbb{N}$, the family of random variables

$$\Gamma_n = \{E[(1-\vartheta \cdot S_T)^2 | \mathscr{F}_{\sigma_n}] : \vartheta \in \overline{\Theta}_{\sigma_n}\}$$

is stable under taking minima by [11, Lemma 2.18(1)], there exists a diagonal sequence $(\vartheta^n)_{n\in\mathbb{N}}$ of trading strategies $\vartheta^n = (\vartheta^n_t)_{0\leq t\leq T}$ in $\overline{\Theta}_{\sigma_n}$ such that

$$E\big[\big(\mathbb{1}_{F_n} - \mathbb{1}_{F_n}(\vartheta^n \cdot S_T)\big)^2\big] = E\big[\mathbb{1}_{F_n} E[(1-\vartheta^n \cdot S_T)^2 | \mathscr{F}_{\sigma_n}]\big] \longrightarrow 0.$$

Because 0 and ϑ^n are in $\overline{\Theta}_{\sigma_n}$ for all $n \in \mathbb{N}$, we have again by [11, Lemma 2.18(1)] that $\psi^n := (\mathbb{1}_{F_n}\vartheta^n) \in \overline{\Theta}_{\sigma_n}$ for all $n \in \mathbb{N}$. This gives a sequence $(\psi^n)_{n\in\mathbb{N}}$ of trading strategies with $\psi^n \in \overline{\Theta}_{\sigma_n}$ and $F_n \in \mathscr{F}_{\sigma_n}$ such that $\mathbb{1}_{F_n}(1-\psi^n \cdot S_T) \to 0$ in L^2, while at the same time, one has $\lim_{n\to\infty} \mathbb{1}_{F_n}(1-\psi^n \cdot S_\tau) = \mathbb{1}_F \neq 0$. This yields a contradiction to property (2) of Definition 2.3 of the LOP for S since one can approximate each $\psi^n \in \overline{\Theta}_{\sigma_n}$ by simple trading strategies in Θ_{σ_n} and then extract a diagonal sequence.

(v) ⇒ (iv): This follows directly from Proposition 4.2 and Lemma 4.6.

(iv) ⇒ (iii): Since $({}^\tau\mathscr{E}(N))_{\tau\in\mathcal{T}}$ is a square-integrable $\mathscr{E}$-density (Definition 2.8), Proposition 4.2 yields that setting ${}^\tau Z_T := {}^\tau\mathscr{E}(N)_T$ for $\tau \in \mathcal{T}$ gives a family $({}^\tau Z_T)_{\tau\in\mathcal{T}}$ of state price densities satisfying the LOP (Definition 2.6). Moreover, the property that S is an $\mathscr{E}(N)$-local martingale (Definition 2.9) directly implies that $({}^\tau Z_T)_{\tau\in\mathcal{T}}$ is compatible with S (Definition 2.6).

(iii) ⇒ (ii): Given a compatible family $({}^\tau Z_T)_{\tau\in\mathcal{T}}$ of state price densities satisfying the LOP, we define a family $(p_\tau)_{\tau\in\mathcal{T}}$ of operators $p_\tau : L^2(\mathscr{F}_T, P) \to L^0(\mathscr{F}_\tau, P)$ by

$$p_\tau(H) = E[{}^\tau Z_T H | \mathscr{F}_\tau] \qquad \text{for all } H \in L^2(\mathscr{F}_T, P).$$

Since $({}^\tau Z_T)_{\tau\in\mathcal{T}}$ satisfies the LOP and is compatible with S (Definition 2.6), it is straightforward to check that $(p_\tau)_{\tau\in\mathcal{T}}$ is a compatible price system satisfying the LOP (Definition 2.5) by comparing the definitions and using the implication (iii) ⇒ (i) of Proposition 4.1.

(ii) ⇒ (i): For a proof by way of contradiction, suppose that (i) and therefore either condition (1) or condition (2) of Definition 2.3 fail.

We begin with condition (1). If that fails, there exist a stopping time $\tau \in \mathcal{T}$, an $\mathscr{F}_\tau$-measurable endowment x_τ and a sequence $(\vartheta^n)_{n\in\mathbb{N}}$ of simple trading strategies such that $x_\tau + \vartheta^n \mathbb{1}_{]\!]\tau,T]\!]} \cdot S_T \to 0$ in L^2 and $x_\tau \neq 0$. By the conditional linearity (property (3a) of Definition 2.5) and the time-consistency for simple strategies, we have $p_\tau(x_\tau + \vartheta^n \mathbb{1}_{]\!]\tau,T]\!]} \cdot S_T) = x_\tau$ and hence $x_\tau = p_\tau(x_\tau + \vartheta^n \mathbb{1}_{]\!]\tau,T]\!]} \cdot S_T) \to 0$

in L^0 by the conditional linearity and continuity of p_τ (properties (3a) and (3b) of Definition 2.5). This, however, is a contradiction to $x_\tau \neq 0$.

Similarly, if condition (2) fails, there exist a predictable stopping time $\sigma \in \mathcal{T}$, an announcing sequence $(\sigma_n)_{n\in\mathbb{N}}$ for σ, and sequences $(x^n_{\sigma_n})_{n\in\mathbb{N}}$ of $\mathscr{F}_{\sigma_n}$-measurable random variables and $(\vartheta^n)_{n\in\mathbb{N}}$ of simple trading strategies with $x^n_{\sigma_n}+\vartheta^n \mathbb{1}_{]\!]\sigma_n,T]\!]}\boldsymbol{\cdot} S_T \to 0$ in L^2 and $x^n_{\sigma_n} \to x_{\sigma-}$ in L^0 for some random variable $x_{\sigma-}$ with $x_{\sigma-}\neq 0$. Then again by the conditional linearity (property (3a) of Definition 2.5) and the time-consistency for simple strategies, we have $p_{\sigma_n}(x^n_{\sigma_n} + \vartheta^n \mathbb{1}_{]\!]\sigma_n,T]\!]}\boldsymbol{\cdot} S_T) = x^n_{\sigma_n}$. By Definition 2.5, $(p_{\sigma_n})_{n\in\mathbb{N}}$ is a sequence of continuous mappings $p_{\sigma_n} : L^2(\mathscr{F}_T, P) \to L^0(\mathscr{F}_{\sigma_n}, P)$ that are $\mathscr{F}_{\sigma_n}$-linear and pointwise convergent, that is, $(p_{\sigma_n}(H))_{n\in\mathbb{N}}$ is a convergent sequence in L^0 for all $H \in L^2(\mathscr{F}_T, P)$. Therefore it follows from the implication (i) $\Rightarrow$ (iii) of Proposition 4.1 that the conditional operator norms $\|p_{\sigma_n}\|$ are uniformly bounded, that is,

$$C := \sup_{n\in\mathbb{N}} \|p_{\sigma_n}\| < \infty \qquad P\text{-a.s.}$$

Combining the latter with $x^n_{\sigma_n} + \vartheta^n \mathbb{1}_{]\!]\sigma_n,T]\!]}\boldsymbol{\cdot} S_T \to 0$ in L^2 gives that

$$\begin{aligned}|x_{\sigma_n}| &= |p_{\sigma_n}(x^n_\tau + \vartheta^n \mathbb{1}_{]\!]\sigma_n,T]\!]}\boldsymbol{\cdot} S_T)|\\ &\le \sup_{n\in\mathbb{N}} \Big(\|p_{\sigma_n}\| \big(E[(x^n_{\sigma_n} + \vartheta^n \mathbb{1}_{]\!]\sigma_n,T]\!]}\boldsymbol{\cdot} S_T)^2|\mathscr{F}_{\sigma_n}]\big)^{\frac{1}{2}}\Big) \longrightarrow 0 \qquad \text{in } L^0,\end{aligned}$$

which contradicts $x_{\sigma-}\neq 0$ with $x^n_{\sigma_n} \to x_{\sigma-}$ in L^0.

(vi) By (iv), there exists a semimartingale N such that $({}^\tau\mathscr{E}(N))_{\tau\in\mathcal{T}}$ is a square-integrable $\mathscr{E}$-density and S is an $\mathscr{E}(N)$-local martingale. Therefore the closedness of the set $\{\vartheta\boldsymbol{\cdot} S_T : \vartheta \in \overline{\Theta}\}$ follows by applying [11, Theorem 2.16].

(vii) Fix $\tau \in \mathcal{T}$ and let $\tilde{S} = \mathbb{1}_{]\!]\tau,T]\!]}\boldsymbol{\cdot} S$. Observe that $\overline{\Theta}(\tilde{S}) = \overline{\Theta}_\tau(S)$ and $\vartheta\boldsymbol{\cdot}\tilde{S} = \vartheta\boldsymbol{\cdot} S$ for all $\vartheta \in \overline{\Theta}(\tilde{S}) = \overline{\Theta}_\tau(S)$ by the associativity of stochastic integrals. Moreover, the properties (i)–(v) imposed on S in Theorem 3.2 are inherited by $\tilde{S}$. By (iv), there exists a semimartingale N such that $({}^\tau\mathscr{E}(N))_{\tau\in\mathcal{T}}$ is a square-integrable $\mathscr{E}$-density and S, hence also $\tilde{S}$, is an $\mathscr{E}(N)$-local martingale. Therefore the closedness of

$$\{\vartheta\boldsymbol{\cdot} S_T : \vartheta \in \overline{\Theta}_\tau\} = \{\tilde{\vartheta}\boldsymbol{\cdot}\tilde{S}_T : \tilde{\vartheta} \in \overline{\Theta}(\tilde{S})\}$$

follows again by applying [11, Theorem 2.16] to $\tilde{S}$.

(viii) Fix $\vartheta \in \overline{\Theta}_{\sigma-}$. Because $\mathbb{1}_{[\![0,\sigma_n]\!]}\vartheta = 0$ for all $n \in \mathbb{N}$ by the definition of $\overline{\Theta}_{\sigma-}$, we have $\vartheta \in \overline{\Theta}_{\sigma_n}$ for all $n \in \mathbb{N}$ and hence

$$\{\vartheta\boldsymbol{\cdot} S_T : \vartheta \in \overline{\Theta}_{\sigma-}\} \subseteq \bigcap_{n\in\mathbb{N}} \{\vartheta\boldsymbol{\cdot} S_T : \vartheta \in \overline{\Theta}_{\sigma_n}\}.$$

Conversely, suppose that $\vartheta \in \overline{\Theta}_{\sigma_n}$ for all $n \in \mathbb{N}$. Then $\mathbb{1}_{[\![0,\sigma_n]\!]}\vartheta = 0$ for all $n \in \mathbb{N}$ by the definition of $\overline{\Theta}_{\sigma_n}$. Therefore $(\mathbb{1}_{[\![0,\sigma[\![}\vartheta)\boldsymbol{\cdot} S = \lim_{n\to\infty}(\mathbb{1}_{[\![0,\sigma_n]\!]}\vartheta)\boldsymbol{\cdot} S = 0$ in the semimartingale topology and hence $\vartheta \in \overline{\Theta}_{\sigma-}$ so that

$$\bigcap_{n\in\mathbb{N}} \{\vartheta\boldsymbol{\cdot} S_T : \vartheta \in \overline{\Theta}_{\sigma_n}\} \subseteq \{\vartheta\boldsymbol{\cdot} S_T : \vartheta \in \overline{\Theta}_{\sigma-}\}.$$

□

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Competing interests The authors declare no competing interests.

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