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Homomorphisms from Functional Equations: The Goldie Equation, II

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Abstract. This first of three sequels to *Homomorphisms from Functional equations: The Goldie equation* (Ostaszewski in Aequationes Math 90:427–448, 2016) by the second author—the second of the resulting quartet—starts from the Goldie functional equation arising in the general regular variation of our joint paper (Bingham et al. in J Math Anal Appl 483:123610, 2020). We extend the work there in two directions. First, we algebraicize the theory, by systematic use of certain groups—the *Popa groups* arising in earlier work by Popa, and their relatives the *Javor groups* . Secondly, we extend from the original context on the real line to multi-dimensional (or infinite-dimensional) settings.

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1. Introduction

The *Goldie functional equation* (GFE) in its simplest form, involving as unknowns a primary function K called a *kernel* and an *auxiliary* g, both *continuous*, reads

$$K(x+y)=K(x)+g(x)K(y). \qquad (GFE)$$

We encounter a more general version of (GFE) below, a special case of a *Levi–Civita equation.* The real-valued version above is closely related to the better-known *Gołąb-Schinzel functional equation*

$$\eta(x+y\eta(x))=\eta(x)\eta(y), \qquad (GS)$$

It emerged most clearly in [10] in the investigation of functions of regular variation, where (GFE) is key—see §2 below, that that equation is best studied by

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reference to *Popa groups*. These involve a group structure on $\mathbb{R}$ first introduced by Popa [28], defined by the binary operation

$$x \circ y := x + y\eta(x),$$

which enables (GS) to be restated as homomorphy of $\mathbb{G}_\eta^+(\mathbb{R}) := \{x : \eta(x) > 0\}$ with the multiplicative group of positive reals. Its generalization below to $\mathbb{R}^d$, has

$$\eta(x) \equiv 1 + \rho(x)$$

with $\rho(.)$ linear on $\mathbb{R}^d$. With the induced Euclidean topology, $\mathbb{G}_\rho(\mathbb{R}^d) = \mathbb{G}^+_{1+\rho(.)}(\mathbb{R}^d)$ is an open subspace of $\mathbb{R}^d$, so by the argument in Hewitt and Ross ([21, 15.18]), for λ_d Lebesgue measure, the Popa Haar-measure on $\mathbb{G}_\rho(\mathbb{R}^d)$ is (as in [10]) proportional to

$$\frac{\lambda_d(\mathrm{d}x)}{1 + \rho(x)}.$$

This enables the identification of Fourier transforms, for instance for $\mathbb{G}_\rho(\mathbb{R})$ with $\rho \in (0, \infty)$,

$$\hat{f}(\gamma) = \int_{\mathbb{G}_\rho} f(u)\gamma(u_\rho^{-1})(1 + \rho)\frac{du}{1 + \rho u} \qquad (\gamma \in \mathbb{R}),$$

where the characters take the form $u \mapsto e^{i\gamma \log(1+\rho u)}$ with $\gamma \in \mathbb{R}$ and u_ρ^{-1} denotes inversion in the group $\mathbb{G}_\rho(\mathbb{R})$.

It was noticed in [27], again in the context of $\mathbb{R}$, that (GFE) itself can be equivalently formulated as a homomorphy between a pair of Popa groups on $\mathbb{R}$.

In this paper we develop radial properties of multivariate Popa groups in order to characterize *Popa homomorphisms*—homomorphisms between Popa groups.

Regular variation in one dimension (widely used in analysis, probability and elsewhere—cf. [3]) explores the ramifications of limiting relations such as

$$f(\lambda x)/f(x) \to K(\lambda) \equiv \lambda^\gamma \qquad (Kar_\times)$$

or its additive variant, more thematic here:

$$f(x + u) - f(x) \to K(u) \equiv \kappa u \qquad (Kar_+)$$

([5, Ch. 1]), and

$$[f(x + u) - f(x)]/h(x) \to K(u) \equiv (u^\gamma - 1)/\gamma \qquad (BKdH)$$

(Bojanić & Karamata, de Haan, ([5, Ch. 3])). Beurling regular variation similarly explores the ramifications of relations such as

$$\varphi(x + t\varphi(x))/\varphi(x) \to 1 \text{ or } \eta(t) \qquad (Beu)$$

([5, § 2.11]) and [26]. The underlying Popa structure lies disguised in the limit function $\eta(t)$, which takes the form $1+\gamma t$ for $t>-1/\gamma$.

For background and applications, see the standard work [5] and e.g. [6–11], [2–4]. Both theory and applications prompt the need to work in higher dimensions, finite or infinite. This is the ultimate motivation for the present paper.

2. The multivariate Goldie functional equation

For X a real topological vector space, write $\langle u\rangle_X$ for the *linear span* of $u\in X$ (to be differentiated from the use of $\langle u\rangle_\rho$ below for ρ in the dual of X). Following [26] call a function $\varphi: X\to\mathbb{R}$ *self-equivarying* over X, $\varphi\in SE_X$, if for each $u\in X$ both $\varphi(tu)=O(t)$ and

$$\varphi(tu+v\varphi(tu))/\varphi(tu)\to\eta_u^\varphi(v)\qquad(v\in\langle u\rangle_X, t\to\infty)$$

locally uniformly in v. This appeals to the underlying uniformity structure on X generated by the neighbourhoods of the origin. As in [26] (by restriction to the linear span $\langle u\rangle_X$) the limit function $\eta=\eta_u^\varphi$ satisfies (GS) for $x,y\in\langle u\rangle_X$. When the limit function η_u is continuous, one of the forms it may take is

$$\eta_u(x)=1+\rho_u x\qquad(x\in\langle u\rangle_X)$$

for some $\rho_u\in\mathbb{R}$, the alternative form being $\eta(x)=\max\{1+\rho_u x,0\}$. A closer inspection of the proof in [26] shows that in fact the restriction $x,y\in\langle u\rangle_X$ placed on (GS) above is unneccessary. Consequently, one may apply the Brillouët–Dhombres–Brzdęk theorem ([14, Prop. 3]), ([15, Th. 4]), on the continuous solutions of (GS) with $\eta: X\to\mathbb{R}$, to infer that η here takes the form

$$\eta(x)=1+\rho(x)\qquad(x\in X),$$

for some continuous linear functional $\rho: X\to\mathbb{R}$, the alternative form being $\eta(x)=\max\{1+\rho(x),0\}$. On this matter, see also [1,14,15]; cf. [18,19], the former cited in detail below. (For the same conclusion under assumptions such as radial continuity, or Christensen measurability, see [16,22,23] under boundedness on a non-meagre set.)

Below we study the implications of replacing ρ_u in η_u by a continuous linear function $\rho(x)$. For this we now need to extend the definition of *general regular variation* [10] from the real line to a multivariate setting. For real topological vector spaces X,Y, a function $f: X\to Y$ is *φ-regularly varying* for $\varphi\in SE_X$ relative to the (auxiliary) *norming* function $h: X\to\mathbb{R}$ if the *kernel* function K below is well defined for all $x\in X$ by

$$K(x):=\lim_{t\to\infty}[f(tx+x\varphi(tx))-f(tx)]/h(tx)\qquad(x\in X).\qquad(GRV)$$

For later use, we note the underlying *radial dependence*: for $u \in X$ put

$$K_u(x) := \lim_{s\to\infty}[f(su + x\varphi(su)) - f(su)]/h(su) \qquad (x \in \langle u\rangle_X).$$

Writing $x = \xi u$ with $\xi > 0$ and $s := t\xi > 0$,

$$\begin{aligned} K(x) = K(\xi u) &= \lim_{t\to\infty} f(t\xi u + x\varphi(t\xi u)) - f(t\xi u)]/h(t\xi u) \\ &= \lim_{s\to\infty} f(su + x\varphi(su)) - f(su)]/h(su) = K_u(x). \end{aligned}$$

So here $K_u = K|\langle u\rangle_X$, as $K(\xi u) = K_u(\xi u)$.

We work radially: above with half-lines $(0, \infty)$ and below with those of the form $(-1/\rho, \infty)$ for $\rho > 0$ (on $\langle u\rangle_X$ with context determining u) and $(-\infty, \infty)$ when $\rho = 0$, see [10]. Proposition 1 below identifies the emergence of functional equations satisfied by the kernel function $K : X \to Y$ and by its other auxiliary g defined below. The latter, once η^φ is identified in the continuous context (for which see again [26]), as above, yields a multivariate form of (GS). Given the natural association of the auxiliary to the Goldie equation, its defining multiplicative equation has 'dual citizenship', being both a special case of GFE (take logarithms!) and a partially pexiderized variant of (GS), for which see [17,22].

Proposition 1. *Let h and $\varphi \in SE_X$ be such that the limit*

$$g(x) := \lim_{t\to\infty} h(tx + x\varphi(tx))/h(tx) \qquad (x \in X)$$

exists. Then the kernel $K : X \to Y$ in (GRV) satisfies the Goldie functional equation:

$$K(x + \eta^\varphi(x)y) = K(x) + g(x)K(y) \qquad (GFE)$$

for $y \in \langle x\rangle_X$. Furthermore, g satisfies (GFE) in the alternative form

$$g(x + \eta^\varphi(x)y) = g(x)g(y) \qquad (y \in \langle x\rangle_X). \qquad (GS/GFE_\times)$$

Proof. Fix x and y. Writing $s = s_x := t + \varphi(tx)$, so that $sx = tx + x\varphi(tx)$,

$$\begin{aligned} &\frac{f(tx + (x + y)\varphi(tx)) - f(tx)}{h(tx)} \\ &= \frac{f(sx + y[\varphi(tx)/\varphi(sx)]\varphi(sx)) - f(sx)}{h(sx)} \cdot \frac{h(tx + x\varphi(tx))}{h(tx)} \\ &\quad + \frac{f(tx + x\varphi(tx)) - f(tx)}{h(tx)}. \end{aligned}$$

Here $\varphi(sx)/\varphi(tx) = \varphi(tx + x\varphi(tx))/\varphi(tx) \to \eta(x)$. Passage to the limit yields (GFE), since $\varphi(tx) = O(t)$. The final assertion is similar but simpler. □

We will achieve a characterization of the kernel function K by identifying the dependence between the different *radial restrictions* $K|\langle u\rangle_X$.

3. Popa–Javor circle groups and their radial subgroups

For a real topological vector space X and a continuous linear function $\rho : X \to \mathbb{R}$, the associated function

$$\varphi(x) = \eta_\rho(x) := 1 + \rho(x)$$

satisfies (GS), as may be routinely checked. The associated circle operation $\circ_\rho$:

$$x \circ_\rho y = x + y\varphi(x) = x + y + \rho(x)y$$

(which gives for $\rho(x) = I(x) \equiv x$ and $X = \mathbb{R}$ the *circle operation* of ring theory: cf. ([24, II.3]), ([20, 3.1]), and ([27, §2.1]) for the historical background) is due to Popa in 1965 on the line and by Javor in 1968 in a vector space ([25, 28], cf. [9]). It is associative, as noted in [25]. As in [10] we need the open sets

$$\mathbb{G}_\rho = \mathbb{G}_\rho(X) := \{x \in X : \eta_\rho(x) = 1 + \rho(x) > 0\}.$$

Note that if $x, y \in \mathbb{G}_\rho$, then $x \circ_\rho y \in \mathbb{G}_\rho$, as

$$\eta_\rho(x \circ_\rho y) = \eta_\rho(x)\eta_\rho(y) > 0.$$

Definition. We refer to

$$\mathbb{G}^*_\rho = \mathbb{G}^*_\rho(X) := \{x \in X : \eta_\rho(x) \neq 0\}$$

as the *Javor group* since, as Javor [25] shows, the set is a group under $\circ_\rho$. The Javor result remains true under the additional restriction $\eta_\rho(y) > 0$, as we are about to verify in Theorem J below. Thus, likewise, we refer to

$$\mathbb{G}_\rho = \mathbb{G}_\rho(X) := \{x \in X : \eta_\rho(x) > 0\}$$

as a *Popa group* under $\circ_\rho$.

Theorem J (after Javor [25])**.** *For X a topological vector space and $\rho : X \to \mathbb{R}$ a continuous linear function, $(\mathbb{G}_\rho(X), \circ_\rho)$ is a group.*

Proof. This is routine, and one argues just as in [25], but must additionally check preservation of the positivity of η_ρ on $\mathbb{G}_\rho$. Here $0 \in \mathbb{G}_\rho$ and is the neutral element; the inverse of $x \in \mathbb{G}_\rho$ is $x^{-1}_\rho := -x/(1 + \rho(x))$, which is in $\mathbb{G}_\rho$ since $1 = \eta_\rho(0) = \eta_\rho(x)\eta_\rho(x^{-1}_\rho)$, so that $\eta_\rho(x^{-1}_\rho) > 0$. □

Definitions. 1. For $u \in \mathbb{G}_\rho(X)$, put

$$\langle u\rangle_\rho := \langle u\rangle_X \cap \mathbb{G}_\rho(X) = \{tu : \eta_\rho(tu) = 1 + t\rho(u) > 0, t \in \mathbb{R}\}.$$

(If $\rho(u) \neq 0$, then $\langle u\rangle_\rho = \{tu : t > -1/\rho(u)\}$, which is a half-line in $\langle u\rangle_X$; otherwise $\langle u\rangle_\rho = \langle u\rangle_X$. Note that $\mathbb{G}_\rho(X)$ is an affine half-space in X.)

Given the context, the notation $\langle u\rangle_\rho$ will not clash with that of $\langle u\rangle_X$.

2. For K with domain $\mathbb{G}_\rho(X)$ we will write $K_u = K|\langle u\rangle_\rho$. (This too will not clash with the radial notation of §2.)

Lemma. *The one-dimensional subgroup* $\langle u\rangle_\rho$ *is an abelian subgroup of* $\mathbb{G}_\rho(X)$ *isomorphic with* $\mathbb{G}_{\rho(u)}(\mathbb{R})$.

Proof. We check closure under multiplication and inversion. For $s,t\in\mathbb{R}$, as before $\varphi(su\circ_\rho tu)=\varphi(su)\varphi(tu)>0$; also, writing $r(tu)$ for the ρ -inverse, $\varphi(r(tu))>0$ for $\varphi(tu)>0$, as $1=\varphi(0)=\varphi(tu\circ_\rho r(tu))=\varphi(tu)\varphi(r(tu))$. Further, since

$$su\circ_\rho tu=su+tu+st\rho(u)u=(s\circ_{\rho(u)}t)u,$$

the operation $\circ_\rho$ is abelian on $\langle u\rangle_\rho$. □

Remark. Despite the lemma above, unless $\rho\equiv 0$ or $X=\mathbb{R}$, the group $\mathbb{G}_\rho(X)$ itself is non-abelian. (In the commutative case, except when $X=\mathbb{R}$, one may select $x\neq 0$ with $\rho(x)=0$; then $x\rho(y)=y\rho(x)=0$ and so $\rho(y)=0$ for all y.) We return to this matter in detail in Theorem 2 below.

Definition. Say that a subgroup H of $\mathbb{G}_\rho(X)$ is *radial* if $H\subseteq\langle u\rangle_\rho$ for some $u\in H$.

Theorem 1 below concerns radial subgroups. The assumption there on Σ is effectively that all its radial subgroups are closed and dense in themselves. Key to the proof is the observation that if $1+\rho(u)<0$, then a fortiori $1+\rho(-u)=1-\rho(u)>0$, i.e. if $u\notin\langle u\rangle_\rho$, then its negative $-u\in\langle u\rangle_\rho$ and likewise its $\mathbb{G}_\rho(X)$-inverse $(-u)_\rho^{-1}\in\langle u\rangle_\rho$.

Theorem 1. *Radial subgroups of Popa groups are Popa. That is, for* Σ *a subgroup of* $\mathbb{G}_\rho(X)$ *with* $\langle u\rangle_\rho\subseteq\Sigma$ *for each* $u\in\Sigma$:

$$\Sigma=\mathbb{G}_\rho(\langle\Sigma\rangle_X).$$

Proof. With $\langle\Sigma\rangle$ the linear span, $\Sigma\subseteq\mathbb{G}_\rho(\langle\Sigma\rangle_X)$ follows from $\Sigma\subseteq\langle\Sigma\rangle_X$, as Σ and $\mathbb{G}_\rho(\langle\Sigma\rangle_X)$ are subgroups of $\mathbb{G}_\rho(X)$.

For the converse, we first show that $\alpha x+\beta y\in\Sigma$ for $x,y\in\Sigma$ and scalars α,β whenever $\alpha x+\beta y\in\mathbb{G}_\rho(\langle\Sigma\rangle_X)$. First, notice that one at least of $\alpha x,\beta y$ is in Σ. Otherwise, $1+\rho(\alpha x)<0$, as $x\in\Sigma$ and $\alpha x\in\langle x\rangle_X\backslash\Sigma$, and likewise $1+\rho(\beta y)<0$. Summing,

$$2+\rho(\alpha x)+\rho(\beta y)<0.$$

But $\alpha x+\beta y\in\mathbb{G}_\rho(X)$, so

$$0<1+\rho(\alpha x+\beta y)=1+\rho(\alpha x)+\rho(\beta y)<-1,$$

a contradiction. We proceed by cases.

Case 1. Both $u:=\alpha x$ *and* $v:=\beta y$ *are in* Σ. Here

$$\alpha x+\beta y=u+v=u\circ_\rho[v/(1+\rho(u))]\in\Sigma;$$

indeed, by assumption $1+\rho(u))>0$ and $1+\rho(u+v)>0$, so by linearity

$$1+\rho(v/(1+\rho(u)))=[1+\rho(u+v)]/(1+\rho(u))>0,$$

and so $v/(1+\rho(u)) \in \langle v\rangle_\rho \subseteq \Sigma$.
Case 2. One of $u := \alpha x, v =: \beta y$ is not in Σ ('off the half-line $\langle x\rangle_\rho$ or $\langle y\rangle_\rho$').

By commutativity of addition, without loss of generality (briefly: w.l.o.g.) $v \notin \Sigma$. Then $-v \in \Sigma$. As Σ is a subgroup, $(-v)^{-1} = v/(1-\rho(v)) \in \Sigma$ and, setting

$$\delta := (1-\rho(v))/[1+\rho(u)],$$
$$\alpha x + \beta y = u + v = u \circ_\rho \delta(-v)^{-1} = u + \delta v[1+\rho(u)]/(1-\rho(v)) \in \Sigma.$$

Indeed, $\delta(-v)^{-1} = \delta v/(1-\rho(v)) \in \langle v\rangle_\rho \subseteq \Sigma$, since by assumption $1+\rho(u)) > 0$ and $1+\rho(u+v) > 0$, so

$$1+\rho(\delta(-v)^{-1}) = 1+\rho\left(\frac{v}{1+\rho(u)}\right) = \frac{1+\rho(u+v)}{1+\rho(u)} > 0.$$

Thus in all the possible cases $\alpha x + \beta y \in \Sigma$ for $x, y \in \Sigma$ with $\alpha x + \beta y \in \mathbb{G}_\rho(\langle\Sigma\rangle_X)$.

Next we proceed by induction, with what has just been established as the base step, to show that for all $n \geq 2$, if $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \in \mathbb{G}_\rho(\langle\Sigma\rangle_X)$, for $u_1, u_2, \ldots, u_n \in \Sigma$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, then $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \in \Sigma$.

Assuming the above for n, we pass to the case of $u_1, u_2, \ldots, u_{n+1} \in \Sigma$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ with $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_{n+1} u_{n+1} \in \mathbb{G}_\rho(\langle\Sigma\rangle_X)$.

Again as a preliminary, notice that, by permuting the subscripts as necessary, w.l.o.g. $x := \alpha_1 u_1 + \cdots + \alpha_n u_n \in \mathbb{G}_\rho(\langle\Sigma\rangle_X)$; otherwise, for $j = 1, \ldots, n+1$

$$1+\rho\left(\sum\nolimits_{i\neq j} \alpha_i u_i\right) < 0,$$

and again as above, on summing, this leads to the contradiction

$$0 < n[1+\rho(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_{n+1} u_{n+1})] < -1.$$

So we suppose w.l.o.g. that $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \in \mathbb{G}_\rho(\langle\Sigma\rangle_X)$; by the inductive hypothesis, $x := \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \in \Sigma$. Take $y := u_{n+1} \in \Sigma$ and apply the base case $n = 2$ to x and y. Then, since $w := \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_{n+1} u_{n+1} = x + \alpha_{n+1} y \in \mathbb{G}_\rho(\langle\Sigma\rangle_X)$, $w \in \Sigma$. This completes the induction, showing $\mathbb{G}_\rho(\langle\Sigma\rangle_X) \subseteq \Sigma$. □

In view of the role in quantifier weakening of countable subgroups dense in themselves [8, 10], we note in passing that the proof above may be relativized to the subfield of *rational* scalars to give (with $\langle\cdot\rangle^{\mathbb{Q}}$ below the rational linear span):

Theorem 1Q. *For Σ a countable subgroup of $\mathbb{G}_\rho(X)$ with $\langle u\rangle_\rho^{\mathbb{Q}} \subseteq \Sigma$ for each $u \in \Sigma$, if $\rho(\Sigma) \subseteq \mathbb{Q}$:*

$$\Sigma = \mathbb{G}_\rho(\langle\Sigma\rangle^{\mathbb{Q}}).$$

4. Abelian dichotomy and homomorphisms

Our first result here, Theorem 2, allows us to characterize in Theorems 4A and 4B homomorphisms between Popa groups in vector spaces. We recall that

$$\eta_1(t) := 1 + t \qquad (t \in \mathbb{R}_+ := (0, \infty))$$

takes $\mathbb{G}_1(\mathbb{R}) \xrightarrow{\eta_1} (\mathbb{R}_+, \times)$, isomorphically. For the next result note that

$$\eta_\rho(x) = \eta_1(\rho(x)) = 1 + \rho(x).$$

In the case of $X = \mathbb{R}$, where $\rho(x) \equiv \rho x$, this reduces to

$$1 + \rho x.$$

We think of our first result here as expressing an *abelian dichotomy.* Below $\circ_I$ refers to $\circ_\rho$ when $\rho = I$, the identity map on $\mathbb{R}$, as in the 'circle operation' (above).

Theorem 2. *A commutative subgroup Σ of $\mathbb{G}_\rho(X)$ is either*

(i) a subspace of the null space $\mathcal{N}(\rho)$, so a subgroup of $(X, +)$, or

(ii) for some $u \in \Sigma$ a subgroup of $\langle u \rangle_\rho$ isomorphic under ρ to a subgroup of $\mathbb{G}_1(\mathbb{R})$:

$$\rho(x \circ_\rho y) = \rho(x) \circ_I \rho(y).$$

Thus the image of Σ under η_ρ is a subgroup of $(\mathbb{R}_+, \times)$.

Proof. Either $\rho(z) = 0$ for each $z \in \Sigma$, in which case Σ is a subgroup of $(X, +)$, or else there is $z \in \Sigma \backslash \{0\}$ with $\rho(z) \neq 0$ (since $\rho(0) = 0$). In this case take $u = u_\rho(z) := z/\rho(z) \neq 0$. Then $\rho(u) = 1$ so $u \in \Sigma$, and for all $x \in \Sigma$ by commutativity $x = \rho(u)x = \rho(x)u$, i.e. Σ is contained in the linear span $\langle u \rangle_X$ and so in $\langle u \rangle_\rho$. So the operation $\circ_\rho$ on Σ takes the form

$$x \circ_\rho y = \rho(x)u + \rho(y)u + \rho(\rho(x)u)\rho(y)u.$$

But $x \circ_\rho y = \rho(x \circ_\rho y)u$, so as $u \neq 0$ the asserted isomorphism follows from

$$\rho(x \circ_\rho y)u = [\rho(x) + \rho(y) + \rho(x)\rho(y)]u.$$

In turn this implies

$$\eta_\rho(x \circ_\rho y) = 1 + \rho(x \circ_\rho y) = (1 + \rho(x))(1 + \rho(y)) = \eta_\rho(x)\eta_\rho(y),$$

i.e. η_ρ is a homomorphism into $(\mathbb{R}_+, \times)$. □

Before we pass to a study of radial behaviours in §5, we recall the following result ([27, Prop. A]), [17] (cf. ([10, Th. 3])) for the context $\mathbb{G}_\rho(\mathbb{R})$ with $\rho(x) = \rho x$. To accommodate alternative forms of (GFE), the matrix includes the multiplicative group $(\mathbb{R}_+, \times)$ as $\rho = \infty$; for a derivation via a passage to the limit see [10], but note that

$$\rho x + \rho y + \rho x \rho y = [\rho x \cdot \rho y](1 + o(\rho)) \qquad (x, y \in \mathbb{R}_+, \rho \to \infty).$$

Theorem BO. *Take* $\psi : \mathbb{G}_\rho(\mathbb{R}_+) \to \mathbb{G}_\sigma(\mathbb{R})$ *a homomorphism with* $\rho, \sigma \in [0, \infty]$. *Then the lifting* $\Psi : \mathbb{R} \to \mathbb{R}$ *of* ψ *to* $\mathbb{R}$ *defined by the canonical isomorphisms* $\log, \exp$, $\{\eta_\rho : \rho > 0\}$ *is bounded above on* $\mathbb{G}_\rho$ *iff* Ψ *is bounded above on* $\mathbb{R}$, *in which case* Ψ *and* ψ *are continuous. Then for some* $\kappa \in \mathbb{R}$ *one has* $\psi(t)$ *as below:*

Popa parameter	$\sigma = 0$	$\sigma \in (0, \infty)$	$\sigma = \infty$
$\rho = 0$	κt	$\eta_\sigma^{-1}(e^{\sigma\kappa t})$	$e^{\kappa t}$
$\rho \in (0, \infty)$	$\log \eta_\rho(t)^{\kappa/\rho}$	$\eta_\sigma^{-1}(\eta_\rho(t)^{\sigma\kappa/\rho})$	$\eta_\rho(t)^{\kappa/\rho}$
$\rho = \infty$	$\log t^\kappa$	$\eta_\sigma^{-1}(t^{\sigma\kappa})$	t^κ

After linear transformation, all the cases reduce to some variant (mixing additive or multiplicative structures) of the Cauchy functional equation. (The parameters are devised to achieve continuity across cells, see [10].)

We next show how this theorem is related to the current context of (GFE). As a preliminary we note a result of Chudziak in which $\circ_\rho$ is applied to all of X, so in practice to Javor groups—i.e. without restriction to $\mathbb{G}_\rho(X)$. We thus think of this as a Javor Homomorphism Theorem. We repeat Chudziak's proof, amending it to the range context of $\mathbb{G}_\sigma(Y)$.

Theorem Ch ([18, Th. 1]). *Let* X, Y *be real topological vector spaces and* $K : X \to \mathbb{G}_\sigma(Y)$ *a continuous function satisfying*

$$K(x \circ_\rho y) = K(x) \circ_\sigma K(y) \qquad (x, y \in X)$$

with $\rho \neq 0$. *Then for any* u *with* $\rho(u) = 1$ *there are constants* $\kappa = \kappa(u), \tau = \sigma(K(u))$, *and continuous* $A_u : X \to \mathbb{G}_\sigma(Y)$ *satisfying*

$$A_u(x + y) = A_u(x) \circ_\sigma A_u(y) \qquad (x, y \in X) \tag{A}$$

(so with abelian range) such that

$$K(x) = \begin{cases} A_u(x) + [1 + \sigma(A_u(x))][(1 + \rho(x))^{\tau\kappa} - 1]K(u)/\tau, & \tau \neq 0, \\ K(u)\log(1 + \rho(x))/\log 2, & \tau = 0. \end{cases}$$

Proof. Take any $u \in X$ with $\rho(u) = 1$ and set

$$A_u(x) := K\left(x - \rho(x)u\right), \qquad \mu_u(t) := K((t - 1)u).$$

The former is continuous and satisfies (A). To see this, take $v_i = x_i - \rho(x_i)u$; then $v_1 + v_2 = v_1 \circ_\rho v_2$, since $\rho(v_i) = \rho(x_i) - \rho(x_i)\rho(u) = 0$ and $\circ_\rho$ reduces to addition on the kernel of ρ. Now, by linearity of ρ,

$$v_1 \circ_\rho v_2 = v_1 + v_2 = x_1 + x_2 - \rho(x_1 + x_2)u.$$

So

$$\begin{aligned} A_u(x_1+x_2) &= K\left(x_1+x_2-\rho(x_1+x_2)u\right) = K(v_1 \circ_\rho v_2) \\ &= K(v_1) \circ_\sigma K(v_2) \\ &= K\left(x_1-\rho(x_1)u\right) \circ_\sigma K\left(x_2-\rho(x_2)u\right) \\ &= A_u(x_1) \circ_\sigma A_u(x_2). \end{aligned}$$

Hence A_u has image an abelian subgroup of $\mathbb{G}_\sigma(Y)$.

The other mapping is an isomorphism between $(\mathbb{R}_+,\times)$ and a subgroup of $\mathbb{G}_\sigma(Y)$ with

$$\mu_u(st) = \mu_u(s) \circ_\sigma \mu_u(t).$$

This last follows via $\rho(u)=1$ from the identity

$$(st-1)u = (s-1)u + [1+\rho((s-1)u)](t-1)u.$$

Now the image subgroup under μ_u, being abelian, is a subgroup of $\langle K(u)\rangle_\sigma$ by Theorem 2, so isomorphic to a subgroup of $\mathbb{G}_\tau(\mathbb{R})$ for $\tau := \sigma(K(u)) \in \mathbb{R}$. Thus μ_u is an isomorphism from $(\mathbb{R}_+,\times) = \mathbb{G}_\infty(\mathbb{R})$ to $\mathbb{G}_\tau(\mathbb{R})$, for $\tau = \sigma(K(u))$, and by Theorem BO for some $\kappa = \kappa(u)$

$$\mu_u(t) = \eta^{-1}_{\sigma(K(u))}(t^{\sigma(K(u))\kappa(u)})K(u).$$

So, as $\rho([x-\rho(x)u]) = 0$,

$$\begin{aligned} K(x) &= K([x-\rho(x)u]) \circ_\rho \rho(x)u) = A_u(x) \circ_\sigma K(\rho(x)u) \\ &= A_u(x) \circ_\sigma \mu_u(1+\rho(x)). \end{aligned}$$

For $\sigma(K(u)) = 0$ the above result should be amended to its limiting value as $\tau \to 0$, namely $K([x-\rho(x)u]) + K(u)\log(1+\rho(x))/\log 2$ (since $\kappa(u) = 1/\log 2$). □

Remark. As the proof shows, in Theorem Ch. one fixes u with $\rho(u)=1$, obtaining constants $\kappa = \kappa(u)$, and $\tau = \tau(u) := \sigma(K(u))$. The case $\tau = 0$ is then best approached using L'Hospital's rule so that, for $x = u$, identity of both sides of the representation of K yields

$$1 = \lim_{\tau\to 0} \frac{2^{\tau\kappa(u)}-1}{\tau} = \kappa(u)\log 2.$$

5. Radial behaviours

Our next two results help establish in §6 Theorems 4A and 4B two not entirely dissimilar representations for the Popa groups, including the case $\rho \equiv 0$, from which the form of A_u above may be deduced in view of equation (A) in Th. Ch. Our first result concerns radial behaviour *outside* $\mathcal{N}(\rho)$.

Theorem 3A. *For real topological vector spaces X, Y, if $K : \mathbb{G}_\rho(X) \to \mathbb{G}_\sigma(Y)$ is continuous and satisfies*

$$K(x \circ_\rho y) = K(x) \circ_\sigma K(y) \qquad (x, y \in \mathbb{G}_\rho(X)), \tag{K}$$

then, for x with $\rho(x) \neq 0$ and $\sigma(K(x)) \neq 0$, there is $\kappa = \kappa(x) \in \mathbb{R} \setminus \{0\}$ with

$$K(z) = \eta_\sigma^{-1}(\eta_\rho(z)^{\sigma(K(x))\kappa}) \qquad (z \in \langle x \rangle_\rho).$$

Moreover, the index $\gamma(x) := \sigma(K(x))\kappa(x)$ is then continuous and extends to satisfy the equation

$$\gamma(a \circ_\rho b) = \gamma(a) + \gamma(b) \qquad (a, b \in \mathbb{G}_\rho(X)).$$

Proof. For x as above, take $u = u_\rho(x) \neq 0$ and $v = u_\sigma(K(x)) \neq 0$, both well-defined as $\rho(x)$ and $\sigma(K(x))$ are non-zero (also $u \in \langle x \rangle_\rho$ and $v \in \langle K(x) \rangle_\sigma$, as $\rho(u) = \sigma(v) = 1$). The restriction $K_u = K|\langle u \rangle_\rho$ yields a continuous homomorphism into $\mathbb{G}_\sigma(Y)$. As $\langle u \rangle_\rho$ is an abelian group under $\circ_\rho$, its image under K_u is an abelian subgroup of $\mathbb{G}_\sigma(Y)$. So, as in Theorem 2, it is a *non-trivial* subgroup of $\langle v \rangle_\sigma$. As noted, $\rho(u) = \sigma(v) = 1$, so we have the following *isomorphisms*:

$$\langle u \rangle_\rho \xrightarrow{\rho} \mathbb{G}_1(\mathbb{R}) \xrightarrow{\eta_1} (\mathbb{R}_+, \times),$$
$$\langle v \rangle_\sigma \xrightarrow{\sigma} \mathbb{G}_1(\mathbb{R}) \xrightarrow{\eta_1} (\mathbb{R}_+, \times)$$

(writing $\rho, \sigma =$ for $\rho|_{\langle u \rangle}$ and $\sigma|_{\langle v \rangle}$) with $\langle . \rangle$ here short for $\langle . \rangle_\mathbb{R}$), which combine to give

$$k(t) := \eta_1 \sigma K_u \rho^{-1} \eta_1^{-1}(t) = \eta_\sigma K_u \eta_\rho^{-1}(t)$$

as a *non-trivial* homomorphism of $(\mathbb{R}_+, \times)$ into itself:

$$k(st) = k(s)k(t).$$

Solving this Cauchy equation for a non-constant continuous k yields

$$k(t) \equiv t^\gamma \qquad (t \in \mathbb{R}_+),$$

for some $\gamma = \gamma(u) \in \mathbb{R}\backslash\{0\}$; so k is bijective. Write $\gamma = \gamma(u) = \sigma(K(u))\kappa(u)$, then, as asserted (abbreviating the symbols),

$$\begin{aligned} K_u(z) &= \eta_\sigma^{-1} k \eta_\rho(z) = \eta_\sigma^{-1}(\eta_\rho(z)^{\sigma\kappa}) \\ &= \sigma^{-1}(\eta_1^{-1}(1 + \rho(z))^{\sigma\kappa})) \qquad (z \in \langle u \rangle_\rho). \end{aligned}$$

In particular, K_u is injective. As $u \neq 0$, $0 \neq K(u) \in \langle v \rangle_\sigma$, so $K(u) = sv$ for some $s \neq 0$. Hence $\sigma(K(u)) = s\sigma(v) = s \neq 0$. Since $\sigma(tK(u)) = t\sigma(K(u))$,

$$K(z) = K_u(z) = [((1 + \rho(z))^{\sigma(K(u))\kappa(u)} - 1)/\sigma(K(u))]K(u) \qquad (z \in \langle u \rangle_\rho).$$

Here $\rho(z) = t$ for $z = tu$, as $\rho(u) = 1$ by choice. Taking $z = u$ gives

$$(2^{\sigma(K(u))\kappa(u)} - 1)/\sigma(K(u)) = 1 : \qquad \kappa(u) = \log(1 + \sigma(K(u))/[\sigma(K(u)) \log 2],$$

and so $\gamma(u) := \sigma(K(u))\kappa(u)$ is continuous and satisfies the equation

$$\gamma(a \circ_\rho b) = \gamma(a) + \gamma(b) \qquad (a, b \in \mathbb{G}_\rho(X)).$$

Indeed, write $\alpha = K(a), \beta = K(b)$; then as $K(a \circ_\rho b) = \alpha \circ_\sigma \beta$, by linearity of σ

$$\begin{aligned}\log(1+\sigma(K(a\circ_\rho b)) &= \log(1+\sigma(\alpha+\beta+\sigma(\alpha)\beta)) \\ &= \log(1+\sigma(\alpha)+\sigma(\beta)+\sigma(a)\sigma(\beta)) \\ &= \log(1+\sigma(\alpha)) + \log(1+\sigma(\beta)).\end{aligned}$$

For $\sigma(K(x)) = 0$, the map $\langle v\rangle_{\sigma(v)} \stackrel{\eta_\sigma}{\to} (\mathbb{R}_+, \times)$ above must be interpreted as exponential. A routine adjustment of the argument yields

$$K(z) = K_u(z) = K(u)\log(1+\rho(z))/\log 2 \qquad (z \in \langle u\rangle_\rho),$$

justifying hereafter a *L'Hospital convention* (of taking limits $\sigma(K(u)) \to 0$ in the 'generic' formula). □

We consider now the case $\rho(x) = 0$, which turns out as expected, despite Theorem 2 being of no help here. This complement to Theorem 3A thus describes radial behaviour *inside* $\mathcal{N}(\rho)$. A more detailed analysis, including the case $\rho(u) = 1$, along the lines followed here, is to be found in ([12, Th. 3.1]) and again in a Banach algebra context in [BinO8].

Theorem 3B. *Let X, Y be real topological vector spaces. If $K : \mathbb{G}_\rho(X) \to \mathbb{G}_\sigma(Y)$ is continuous and satisfies (K), then for any $u \neq 0$ with $\rho(u) = 0$*

$$K(\langle u\rangle_\rho) \subseteq \langle K(u)\rangle_\sigma \sim \mathbb{G}_{\sigma(K(u))}(\mathbb{R}),$$

with $\sim$ denoting isomorphism,*and there is a function $\lambda_u : (\mathbb{R}, +) \to \mathbb{G}_{\sigma(K(u))}(\mathbb{R})$ satisfying*

$$K(\xi u) = \lambda_u(\xi)K(u) \qquad (\xi \in \mathbb{R}).$$

Moreover, if $K(u) \neq 0$, then for some constant $\kappa = \kappa(u)$

$$\lambda_u(t) = \begin{cases} (e^{\sigma(K(u))\kappa(u)t} - 1)/\sigma(K(u)), & \sigma(K(u)) \neq 0, \\ t, & \sigma(K(u)) = 0, \end{cases}$$

for $t \in \mathbb{R}$, so that λ_u is an isomorphism.

Proof. As $\rho(u) = 0$, $\xi u + \xi u = \xi u \circ_\rho \xi u$. Notice that

$$K(2u) = K(u) + K(u) + \sigma(K(u))K(u) = (2 + \sigma(K(u))K(u).$$

By induction,

$$K(nu) = a_n(u)K(u) \in \langle K(u)\rangle_Y,$$

where $a_1 = 1$ and $a_n = a_n(u)$, for $n = 1, 2, \ldots$, solves

$$a_{n+1} = 1 + (1 + \sigma(K(u))a_n,$$

since

$$K(u + nu) = K(u) + a_n K(u)(1 + \sigma(K(u)).$$

Suppose w.l.o.g. $\sigma(K(u)) \neq 0$, the case $\sigma(K(u)) = 0$ being similar, but simpler (with $a_n = n$). So

$$a_n = ((1 + \sigma(K(u))^n - 1)/\sigma(K(u)) \neq 0 \qquad (n = 1, 2, \ldots).$$

Replacing u by u/n and then rearranging gives

$$K(u) = K(nu/n) = a_n(u/n)K(u/n) : \qquad K(u/n) = a_n(u/n)^{-1}K(u) \in \langle K(u)\rangle_Y.$$

So

$$\begin{aligned} K(mu/n) &= a_m(u/n)K(u/n) = a_m(u/n)a_n(u/n)^{-1}K(u) \\ &= \frac{((1+\sigma(K(u/n))^m - 1)/\sigma(K(u/n)}{((1+\sigma(K(u/n))^n - 1)/\sigma(K(u/n)}K(u) \\ &= \frac{((1+\sigma(K(u/n))^m - 1)}{((1+\sigma(K(u/n))^n - 1)}K(u) \in \langle K(u)\rangle_Y. \end{aligned}$$

By continuity of K (and of scalar multiplication), this implies that $K(\xi u) \in \langle K(u)\rangle_Y$ for any $\xi \in \mathbb{R}$. So we may uniquely define $\lambda(s) = \lambda_u(s)$ via

$$K(su) = \lambda_u(s)K(u).$$

(In the case $\sigma(K(u)) = 0$ with $a_n = n$, $K(mu/n) = (m/n)K(u)$, so that $K(su) = sK(u)$.) Then, as $\rho(u) = 0$,

$$\begin{aligned} \lambda(\xi+\eta)K(u) &= K((\xi+\eta)u) = K(\xi u \circ_\rho \eta u) = K(\xi u) + K(\eta u) + \sigma(K(\xi u))K(\eta u) \\ &= K(\xi u) + K(\eta u) + \sigma(\lambda(\xi)K(u))\lambda(\eta)K(u) \\ &= \lambda(\xi)K(u) + \lambda(\eta)K(u) + \lambda(\xi)\lambda(\eta)\sigma(K(u))K(u) \\ &= [\lambda(\xi) + \lambda(\eta) + \lambda(\xi)\lambda(\eta)\sigma(K(u))]K(u). \end{aligned}$$

So if $K(u) \neq 0$

$$\lambda_u(\xi+\eta) = \lambda_u(\xi) + \lambda_u(\eta) + \lambda_u(\xi)\lambda_u(\eta)\sigma(K(u)) = \lambda_u(\xi) \circ_{\sigma(K(u))} \lambda_u(\eta).$$

Thus $\lambda_u : (\mathbb{R}, +) \to \mathbb{G}_{\sigma(K(u))}(\mathbb{R})$. By Theorem BO, with $\tau = \sigma(K(u))$ for some $\kappa = \kappa(u)$

$$\lambda_u(t) = \begin{cases} (e^{\tau\kappa(u)t} - 1)/\tau, & \text{if } \sigma(K(u)) \neq 0, \\ \kappa(u)t = t, & \text{if } \sigma(K(u)) = 0. \end{cases}$$

□

Corollary 1. *In Theorem 3B, if* $\rho(u) = 0$ *and* $K(u) \neq 0$*, then either*

(i) $\sigma(K(u)) = 0$ *and* $\kappa(u) = 1$, *or*

(ii) $\sigma(K(u)) > 0$, $\kappa(u) = \log[1 + \sigma(K(u))]/\sigma(K(u))$ *and the index* $\gamma(u) := \sigma(K(u))\kappa(u)$ *is additive on* $\mathcal{N}(\rho)$*:*

$$\gamma(u+v) = \gamma(u) + \gamma(v) \qquad (u, v \in \mathcal{N}(\rho)).$$

Proof. As $\rho(u) = 0$, the notation in the proof above is valid, so $\lambda_u(1) = 1$, as $0 \neq K(u) = \lambda_u(1)K(u)$. If $\sigma(K(u)) = 0$, then $\kappa(u) = 1$, by Theorem 3B. Otherwise,

$$(e^{\sigma(K(u))\kappa(u)} - 1)/\sigma(K(u)) = 1 : \qquad \kappa(u) = \log(1 + \sigma(K(u)))/\sigma(K(u)),$$

and, as $\gamma(u) = \log(1 + \sigma(K(u)))$, the concluding argument is as in Theorem 3A (with $\circ_\rho = +$ on $\mathcal{N}(\rho)$). □

6. Homomorphism dichotomy

The paired Theorems 4A and 4B below, our main contribution, amalgamate the earlier radial results according to the two forms identified by Theorem 2 that an *abelian* Popa subgroup may take (see below). Theorem 4A covers $\sigma \equiv 0$ as $\mathcal{N}(\sigma) = \mathbb{G}_\sigma(Y) = Y$, whereas $\rho \equiv 0$ may occur in the context of either theorem. Relative to Theorem Ch., new here is Theorem 4B exhibiting an additional source of regular variation.

We begin by noting that, since $\circ_\rho$ on $\mathcal{N}(\rho)$ is addition, $\mathcal{N}(\rho)$ is an abelian subgroup of $\mathbb{G}_\rho(X)$ and so

$$\Sigma := K(\mathcal{N}(\rho)),$$

as a homeomorph, is also an abelian subgroup of $\mathbb{G}_\sigma(Y)$. By Theorem 2 there are now two cases to consider, differing only in their treatment of radial behaviour (in or out of $\mathcal{N}(\rho)$). The former is our First Popa Homomorphism Theorem which follows.

Theorem 4A. *Let X, Y be real topological vector spaces and $K : \mathbb{G}_\rho(X) \to \mathbb{G}_\sigma(Y)$ a continuous function K satisfying* (*K*) *with*

$$K(\mathcal{N}(\rho)) \subseteq \mathcal{N}(\sigma).$$

Then:

$K|\mathcal{N}(\rho)$ is linear, and either

(i) *K is linear, or*

(ii) *for any u with $\rho(u) = 1, \pi_u(x) := x - \rho(x)u$ is the projection onto $\mathcal{N}(\rho)$ parallel to u and*

$$K(x) = \begin{cases} K(\pi_u(x)) + K(u)[(1+\rho(x))^{\log(1+\tau)/\log 2} - 1]/\tau, & \tau \neq 0, \\ K(\pi_u(x)) + K(u)\log(1+\rho(x))/\log 2, & \tau = 0, \end{cases}$$

for $\tau = \sigma(K(u))$. In particular, $x \mapsto K(\pi_u(x))$ is linear.

Proof. If $\rho \equiv 0$, then $K(X) = K(\mathcal{N}(\rho)) \subseteq \mathcal{N}(\sigma)$. Here $\sigma(K(x)) = 0$ for all x so, since $\circ_\sigma = +$ on $\mathcal{N}(\sigma)$, K is linear.

Otherwise, fix $u \in X$ with $\rho(u) = 1$. Then $x \mapsto \pi_u(x) = x - \rho(x)u$ is a (linear) projection onto $\mathcal{N}(\rho)$ and, since $\rho(x - \rho(x)u) = 0$,

$$x = (x - \rho(x)u) \circ_\rho \rho(x)u.$$

(So $\mathbb{G}_\rho(X)$ is generated by $\mathcal{N}(\rho)$ and any $u \notin \mathcal{N}(\rho)$.)

By assumption $\sigma(\pi_u(x)) = 0$ and as $K|\mathcal{N}(\rho)$ is linear

$$K(x) = K(\pi_u(x)) \circ_\sigma K(\rho(x)u) = K(\pi_u(x)) + K(\rho(x)u).$$

If $\tau := \sigma(K(u)) \neq 0$, then by Theorem 3A

$$K(\rho(x)u) = [(1+\rho(x))^{\log(1+\tau)/\log 2} - 1]K(u)/\tau.$$

Now consider $u, v \in \mathbb{G}_\rho(X)$ with $\rho(u) = 1 = \rho(v)$. As $v - u \in \mathcal{N}(\rho)$, also $\sigma(K(v - u)) = 0$ and also

$$v = (v - u) + u = (v - u) \circ_\rho u.$$

Moreover, as $\sigma(K(v - u)) = 0$,

$$K(v) = K(v - u) \circ_\sigma K(u) = K(v - u) + K(u) : \qquad K(v - u) = K(v) - K(u).$$

So, by linearity of σ,

$$0 = \sigma(K(v - u)) = \sigma(K(v)) - \sigma(K(u)) : \qquad \sigma(K(v)) = \sigma(K(u)) = \tau.$$

Thus also

$$K(\rho(x)v) = [(1 + \rho(x))^{\log(1+\tau)/\log 2} - 1]K(v)/\tau.$$

If $\tau := \sigma(K(u)) = 0$, then as in Theorem 3A,

$$K(\rho(x)u) = K(u) \log(1 + \rho(x))/\log 2,$$

again justifying the L'Hospital convention in force (the formula follows from the main case taking limits as $\tau \to 0$). □

We pass to the remaining case, our Second Popa Homomorphism Theorem.

Theorem 4B. *Let X, Y be real topological vector spaces and $K : \mathbb{G}_\rho(X) \to \mathbb{G}_\sigma(Y)$ a continuous function K satisfying (K) with*

$$K(\mathcal{N}(\rho)) = \langle K(w) \rangle_\sigma$$

for some w with $\rho(w) = 0$ and $\sigma(K(w)) = 1$. Then
(i) *$V_0 := \mathcal{N}(\rho) \cap K^{-1}(\mathcal{N}(\sigma))$ is a vector subspace and $K_0 = K|V_0 = 0$;*
(ii) *for any subspace $V_1 \ni w$ complementary to V_0 in $\mathcal{N}(\rho)$, and any $u \in X$ with $\rho(u) = 1$, there is a linear map $\kappa_w : V_1 \to \mathbb{R}$ such that for $\tau = \sigma(K(u))$*

$$K(x) = \begin{cases} [e^{\kappa_w(\pi_1(x))} - 1]K(w) + \\ e^{\kappa_w(\pi_1(x))}[(1 + \rho(x))^{\log(1+\tau)/\log 2} - 1]K(u)/\tau, & \tau \neq 0, \\ [e^{\kappa_w(\pi_1(x))} - 1]K(w) + \\ e^{\kappa_w(\pi_1(x))}K(u) \log(1 + \rho(x))/\log 2, & \tau = 0, \end{cases}$$

where π_i denotes projection from X onto V_i, and $\sigma(K(\pi_1(x))) \neq 0$ unless $\pi_1(x) = 0$.

(The final term in each case is excluded when there are no $u \in X$ with $\rho(u) = 1$.)

Proof. The assumption on K here is taken in an initially more convenient form: $K(\mathcal{N}(\rho)) \subseteq \langle w \rangle_\sigma$, *for some $w \in \Sigma = K(\mathcal{N}(\rho))$*, and of course w.l.o.g. $\sigma(w) \neq 0$, as otherwise this case is covered by Theorem 4A.

To begin with $V_0 := \mathcal{N}(\rho) \cap K^{-1}(\mathcal{N}(\sigma))$ is a subgroup of $\mathbb{G}_\rho(X)$, as K is a homomorphism. Similarly as in Theorem 4A, we work with a linear map, namely $K_0 := K|V_0$, as we claim V_0 to be a subspace of $\mathcal{N}(\rho)$. (Then $V_0 = \mathbb{G}_0(V_0)$.)

The claim follows by linearity of σ and Theorem 3B. Indeed, if $\rho(x) = \rho(y) = 0$ and $\sigma(K(x)) = \sigma(K(y)) = 0$, then $K(\alpha x) = \lambda_x(\alpha)K(x)$ and $K(\beta y) = \lambda_y(\beta)K(y)$, and since $\mathcal{N}(\rho)$ is a vector subspace on which $+$ agrees with $\circ_\rho$:

$$\begin{aligned} K(\alpha x + \beta y) &= K(\alpha x \circ_\rho \beta y) \\ &= \lambda_x(\alpha)K(x) + \lambda_y(\beta)K(y) + \lambda_x(\alpha)\lambda_y(\beta)\sigma(K(x))K(y) \\ &= \lambda_x(\alpha)K(x) + \lambda_y(\beta)K(y), \end{aligned}$$

as $\sigma(K(x)) = 0$. So

$$\sigma(K(\alpha x + \beta y)) = \lambda_x(\alpha)\sigma(K(x)) + \lambda_y(\beta)\sigma(K(y)) = 0.$$

Hence V_0 is a subspace of $\mathcal{N}(\rho)$ and $K_0 : V_0 \to \mathcal{N}(\sigma)$ is linear with $K_0(V_0) \subseteq \mathcal{N}(\sigma)$, as in Theorem 4A. Hence $K_0 = 0$; indeed, for $v_0 \in V_0$, as $V_0 \subseteq \mathcal{N}(\rho)$ there is λ_0 with $K(v_0) = \lambda_0 w \in \mathcal{N}(\sigma)$ and so $0 = \sigma(\lambda_0 w) = \lambda_0\sigma(w)$ and as $\sigma(w) \neq 0$ we have $\lambda_0 = 0$. That is, $K_0 = 0$.

Since $K(\mathcal{N}(\rho)) \subseteq \mathcal{N}(\sigma)$ does not hold, choose in $\mathcal{N}(\rho)$ a subspace V_1 complementary to V_0, and let $\pi_i : X \to V_i$ denote projection onto V_i. For $v \in \mathcal{N}(\rho)$ and $v_i = \pi_i(v) \in V_i$, as $K(v_0) \in \mathcal{N}(\sigma)$,

$$K(v) = K(\pi_0(v) \circ_\rho \pi_1(v)) = K(\pi_0(v)) \circ_\sigma K(\pi_1(v)) = K_0(\pi_0(v)) + K(\pi_1(v)). \tag{$V0$}$$

Here $K_0 \circ \pi_0$ is linear and $\sigma(K(v_1)) \neq 0$ unless $v_1 = 0$. Recalling that V_1 is a subgroup of $\mathbb{G}_\rho(X)$, re-write the result of Theorem 3B as $K(v_1) = \lambda_w(v_1)w$ with $\lambda_w : V_1 \to \mathbb{G}_{\sigma(w)}(\mathbb{R})$ and

$$\lambda_w(v_1 + v_1') = \lambda_w(v_1) \circ_{\sigma(w)} \lambda_w(v_1') \qquad (v_1, v_1' \in V_1).$$

With w fixed, λ_w is continuous (as K is), with $1 + \sigma(w)\lambda_w(v_1) > 0$.

So, as in Theorem 4A, for $v \in V_1$ and some $\kappa = \kappa_w(v)$

$$K(tv) = \lambda_w(tv)w = \sigma(w)^{-1}[e^{\sigma(w)\kappa_w(v)t} - 1]w \qquad (t \in \mathbb{R}).$$

Taking $t = 1$ gives

$$\sigma(w)\kappa_w(v) = \log[1 + \sigma(w)\lambda_w(v)].$$

As λ_w is continuous, so is $\kappa_w : V_1 \to \mathbb{R}$ (as $\sigma(w) \neq 0$). However, as in Theorem 2 but with $\sigma(w)$ fixed, κ_w is additive and so by continuity linear on V_1. So, as $t\kappa_w(v) = \kappa_w(tv)$,

$$K(v) = \sigma(w)^{-1}[e^{\sigma(w)\kappa_w(v)} - 1]w \qquad (v \in V_1). \tag{$V1$}$$

For $x \in X$ take $v_i := \pi_i(x) \in V_i$ and $v := v_0 + v_1$. If ρ is not identically zero, again fix $u \in X$ with $\rho(u) = 1$, and then $x \mapsto \pi_u(x) = x - \rho(x)u$ is again (linear) projection onto $\mathcal{N}(\rho)$. If $\rho \equiv 0$, set u below to 0. Then, whether or not $\rho \equiv 0$, as $\rho(x - \rho(x)u) = 0$,

$$x = v_0 + v_1 + \rho(x)u = v \circ_\rho \rho(x)u.$$

So, as $\sigma(K(v_0)) = 0$ and $\rho(\rho(x)u) = \rho(x)\rho(u) = \rho(x)$,with $\tau = \sigma(K(u)) \neq 0$

$$K(x) = K(v) \circ_\sigma K(\rho(x)u) = K(v) \circ_\sigma \eta^{-1}_{\sigma(K(u))}(\eta_\rho(\rho(x)u)^\kappa),$$

here with $\kappa = \log(1+\tau)/\log 2$, which we expand as

$$\begin{aligned}
&K(v_0) + K(v_1) + [1 + \sigma(K(v_0 + v_1))][(1 + \rho(x))^{\log(1+\tau)/\log 2} - 1]K(u)/\tau \\
&\quad = K_0(\pi_0(v)) + K(\pi_1(v)) + [1 + \sigma(K(v_1))][(1 + \rho(x))^{\log(1+\tau)/\log 2} - 1]K(u)/\tau : \\
&K(x) = K_0(\pi_0(x)) + [e^{\sigma(w)\kappa_w(\pi_1(x))} - 1]w/\sigma(w) \\
&\quad + [1 + \sigma(K(\pi_1(x)))][(1 + \rho(x))^{\log(1+\sigma(K(u)))/\log 2} - 1]K(u)/\sigma(K(u)).
\end{aligned}$$

Here $K_0 = 0$, as above. Finally, $(V1)$ and linearity of σ yields via $(V0)$ that

$$1 + \sigma(K(\pi_1(x))) = e^{\sigma(w)\kappa_w(\pi_1(x))}.$$

For $v_1 \neq 0$, $\sigma(K(v_1)) \neq 0$, as otherwise $v_1 \in \mathcal{N}(\rho) \cap K^{-1}(\mathcal{N}(\sigma)) = V_0$, contradicting complementarity of V_1.

Here $\sigma(w/\sigma(w)) = 1$. Finally, as $w \in \Sigma = K(\mathcal{N}(\rho))$, we replace w by $K(w)$ with $\rho(w) = 0$ and $\sigma(K(w)) = 1$. If $\tau = \sigma(K(u)) = 0$, then, as in Theorem 4A, the final term is to be interpreted by the L'Hospital convention (limiting value as $\tau = \sigma(K(u)) \to 0$). We thus have:

$$\begin{aligned}
K(x) &= K_0(\pi_0(x)) + [e^{\kappa_w(\pi_1(x))} - 1]K(w) \\
&\quad + e^{\kappa_w(\pi_1(x))}K(u)[(1 + \rho(x))^{\log(1+\tau)/\log 2} - 1]/\tau,
\end{aligned}$$

where $\tau = \sigma(K(u))$ *and again* $K_0 = 0$.

If $\rho \equiv 0$, then $u = 0$ so that $K(u) = 0$, and the final term vanishes. □

Remark. We close with the final dénoument, which is the connection between (GFE) and Popa groups. Theorems 4A and 4B are used in [12] to characterize, for X, Y real topological vector spaces, the continuous solutions $K : \mathbb{G}_\rho(X) \to Y$ of (GFE) as homomorphisms between Popa groups $\mathbb{G}_\rho(X)$ and $\mathbb{G}_\sigma(Y)$ for some σ. For an inkling of the context, notice that for $K : \mathbb{G}_\rho(X) \to Y$ as in Prop. 2.1, under the strong assumption that K is injective, a linear $\sigma : Y \to \mathbb{R}$ can readily be deduced yielding

$$K(u \circ_\rho v) = K(u) + g(u)K(v) = K(u) \circ_\sigma K(v),$$

so that K is a Popa homomorphism (cf. [27, Th. 1]). We relax the strong assumption in [12].

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References

[1] Baron, K.: On the continuous solutions of the Gołąb–Schinzel equation. Aequationes Math. **38**(2–3), 155–162 (1989)
[2] Bingham, N.H.: Tauberian theorems and the central limit theorem. Ann. Prob. **9**, 221–231 (1981)
[3] Bingham, N.H.: Scaling and regular variation. Publ. Inst. Math. Beograd **97**(111), 161–174 (2015)
[4] Bingham, N.H.: Riesz means and Beurling moving averages. Risk and Stochastics (Ragnar Norberg Memorial volume, ed. P. M. Barrieu), Imperial College Press, 2019, Ch. 8, pp. 159–172. arXiv:1502.07494
[5] Bingham, N.H., Goldie, C.M., Teugels, J.L.: Regular Variation, 2nd ed. Cambridge University Press (1989) (1st ed. 1987)
[6] Bingham, N.H., Ostaszewski, A.J.: Homotopy and the Kestelman–Borwein–Ditor theorem. Canad. Math. Bull. **54**, 12–20 (2011)
[7] Bingham, N.H., Ostaszewski, A.J.: Beurling slow and regular variation. Trans. Lond. Math. Soc. **1**, 29–56 (2014)
[8] Bingham, N.H., Ostaszewski, A.J.: Cauchy's functional equation and extensions: Goldie's equation and inequality, the Gołąb–Schinzel equation and Beurling's equation. Aequationes Math. **89**, 1293–1310 (2015)
[9] Bingham, N.H., Ostaszewski, A.J.: Beurling moving averages and approximate homomorphisms. Indag. Math. **27**, 601–633 (2016)
[10] Bingham, N.H., Ostaszewski, A.J.: General regular variation, Popa groups and quantifier weakening. J. Math. Anal. Appl. **483**, 123610 (2020)
[11] Bingham,N. H., Ostaszewski,A. J.: Extremes and regular variation. A lifetime of excursions through random walks and Lévy processes, pp. 121–137, Progr. Probab., 78. Birkhäuser/Springer, Cham (2021). arXiv: 2001.05420
[12] Bingham,N. H., Ostaszewski,A. J.: The Goldie Equation: III. Homomorphisms from Functional Equations (initially titled: Multivariate Popa groups and the Goldie Equation). arXiv:1910.05817
[13] Bingham, N.H., Ostaszewski, A.J.: The Gołąb–Schinzel and Goldie functional equations in Banach algebras. arXiv:2105.07794

[14] Brillouët, N., Dhombres, J.: Équations fonctionnelles et recherche de sous-groupes. Aequationes Math. **31**(2–3), 253–293 (1986)
[15] Brzdęk, J.: Subgroups of the group Z_n and a generalization of the Gołąb–Schinzel functional equation. Aequationes Math. **43**, 59–71 (1992)
[16] Brzdęk, J.: Bounded solutions of the Gołąb–Schinzel equation. Aequationes Math. **59**(3), 248–254 (2000)
[17] Chudziak, J.: Semigroup-valued solutions of the Gołąb–Schinzel type functional equation. Abh. Math. Sem. Univ. Hamburg **76**, 91–98 (2006)
[18] Chudziak, J.: Semigroup-valued solutions of some composite equations. Aequationes Math. **88**, 183–198 (2014)
[19] Chudziak, J.: Continuous on rays solutions of a Gołąb–Schinzel type equation. Bull. Aust. Math. Soc. **91**, 273–277 (2015)
[20] Cohn, P.M.: Algebra, vol. 1, 2nd ed. Wiley, New York (1982) (1st ed. 1974)
[21] Hewitt,E., Ross,K. A.: Abstract Harmonic Analysis. Vol. I, Grundl. math. Wiss. **115**. Springer (1963) [Vol. II, Grundl. **152**, 1970]
[22] Jabłońska, E.: Continuous on rays solutions of an equation of the Gołąb–Schinzel type. J. Math. Anal. Appl. **375**, 223–229 (2011)
[23] Jabłońska, E.: Christensen measurability and some functional equation. Aequationes Math. **81**, 155–165 (2011)
[24] Jacobson, N.: Lectures in Abstract Algebra, vol. I. Van Nostrand, New York (1951)
[25] Javor, P.: On the general solution of the functional equation $f(x + yf(x)) = f(x)f(y)$. Aequationes Math. **1**, 235–238 (1968)
[26] Ostaszewski, A.J.: Beurling regular variation, Bloom dichotomy, and the Gołąb–Schinzel functional equation. Aequationes Math. **89**, 725–744 (2015)
[27] Ostaszewski, A.J.: Homomorphisms from functional equations: the Goldie Equation. Aequationes Math. **90**, 427–448 (2016)
[28] Popa, C.G.: Sur l'équation fonctionelle $f[x+yf(x)] = f(x)f(y)$. Ann. Polon. Math. **17**, 193–198 (1965)

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