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The solution to an impulse control problem motivated by optimal harvesting

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ABSTRACT

We consider a stochastic impulse control problem that is motivated by applications such as the optimal exploitation of a natural resource. In particular, we consider a stochastic system whose uncontrolled state dynamics are modelled by a non-explosive positive linear diffusion. The control that can be applied to this system takes the form of one-sided impulsive action. The objective of the control problem is to maximise a discounted performance criterion that rewards the effect of control action but involves a fixed cost at each time of a control intervention. We derive the complete solution to this problem under general assumptions. It turns out that the solution can take four qualitatively different forms, several of which have not been observed in the literature. In two of the four cases, there exist only ε-optimal control strategies. We also show that the boundary classification of 0 may play a critical role in the solution of the problem. Furthermore, we develop a way for establishing the strong solution to a stochastic impulse control problem's optimally controlled SDE.

1. Introduction

We consider a stochastic dynamical system whose controlled state process is the strong solution to the SDE

$$\mathrm{d}X_t^\zeta = b(X_t^\zeta)\,\mathrm{d}t - \mathrm{d}\zeta_t + \sigma(X_t^\zeta)\,\mathrm{d}W_t, \quad X_{0-}^\zeta = x > 0, \tag{1}$$

where W is a standard one-dimensional Brownian motion and ζ is a controlled càdlàg increasing piece-wise constant process. The objective of the optimisation problem is to maximise over all admissible processes ζ the performance criterion

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$$J_x(\zeta) = \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t + \sum_{t\geq 0} e^{-\Lambda_t^\zeta}\left(\int_0^{\Delta\zeta_t} k(X_{t-}^\zeta - u)\,\mathrm{d}u - c\mathbf{1}_{\{\Delta\zeta_t>0\}}\right)\right], \tag{2}$$

where $\Delta\zeta_t = \zeta_t - \zeta_{t-}$, with the convention that $\zeta_{0-} = 0$, and

$$\Lambda_t^\zeta = \int_0^t r(X_u^\zeta)\,\mathrm{d}u. \tag{3}$$

Throughout the paper, we write $\mathbb{E}_x$ to denote expectation so that we account for the dependence of X^ζ on its initial value x.

Stochastic impulse control problems arise in various fields. In the context of mathematical finance, economics and operations research, notable contributions include Harrison, Sellke and Tayor [20], Harrison and Taksar [21], Mundaca and Øksendal [32], Korn [24,25], Bielecki and Pliska [9], Cadenillas [11], Bar-Ilan, Sulem and Zanello [7], Bar-Ilan, Perry and Stadje [6], Ohnishi and Tsujimura [33], Cadenillas, Sarkar and Zapatero [12], LyVath, Mnif and Pham [30], and several references therein. Also, impulse control models motivated by the optimal management of a natural resource have been studied by Alvarez [1,2], Alvarez and Koskela [4] and Alvarez and Lempa [5]; singular control versions of such models have been studied by Lungu and Øksendal [29], Framstad [19], Song, Stockbridge and Zhu [40], Alvarez and Hening [3] and several references therein. In view of the wide range of applications, the general mathematical theory of stochastic impulse control is well-developed: apart from the contributions mentioned above, see also Richard [39], Stettner [41], Lepeltier and Marchal [28], Perthame [36], Egami [18], Davis, Guo and Wu [16], Djehiche, Hamadène and Hdhiri [17], Christensen [13], Helmes, Stockbridge and Zhu [22,23], Menaldi and Robin [31], Palczewski and Stettner [35], Christensen and Strauch [14], as well as the books by Bensoussan and Lions [8], Davis [15], Øksendal and Sulem [34], Pham [37], and several references therein.

The optimal management of a natural resource has motivated the problem that we study here. In this context, the state process X^ζ models the population density of a harvested species, while ζ_t is the cumulative amount of the species that has been harvested by time t. The constant $c > 0$ models a fixed cost associated with each harvesting cycle, while the function k models the marginal profit arising from each harvest. Furthermore, the function h models the utility of the harvested species, which could reflect the importance of the species to its associated ecosystem. Alternatively, the function h can be used to model the revenue or cost associated with running the ecosystem. Relative to related references, such as the ones mentioned in the previous section, we generalise by considering (a) state-space discounting, (b) a state-dependent, rather than proportional, payoff associated with each harvest size, and (c) a running payoff such as the one modelled by the function h. On the other hand, the assumptions that we make are of a rather similar nature.

In light of standard impulse control theory, a "β-γ" strategy should be a prime candidate for an optimal one in the problem that we study here. Such a strategy is characterised by two points $\gamma < \beta$ in $]0,\infty[$, which are both chosen by the controller, and can be described informally as follows. If the state process takes any value $x \geq \beta$, then it is optimal for the controller to push it in an impulsive way down to level γ. On the other hand, the controller should wait and take no action at all for as long as the state process takes values in the interval $]0,\beta[$.

We show that a β-γ strategy is indeed optimal, provided that a critical parameter $\underline{x}$ is finite and the fixed cost c is sufficiently small (see Case I of Theorem 10 in Section 5). Otherwise, we show that only ε-optimal strategies may exist (see Case II or Case IV of Theorem 10) or that never making an intervention may be optimal (see Case III of Theorem 10). The absence of an optimal strategy in Case IV of Theorem 10 is due to the relatively rapid growth of the function k at infinity. It can therefore be eliminated if we make a suitable additional growth assumption. On the other hand, the absence of an optimal strategy in Case II of Theorem 10 is due to the nature of the problem that we solve.

The family of admissible controlled strategies that we consider do not allow for the state process to hit the boundary point 0 and be absorbed by it, which would amount to "switching off" the system. If we enlarged the set of admissible controls to allow for such a possibility and 0 were a natural boundary point, then we would face only the following difference: a β-0 strategy would be optimal in Case II of Theorem 10 and we would not need to consider ε-optimal strategies. On the other hand, the situation would be radically different if 0 were an entrance boundary point: in this case, β-0 strategies would become an indispensable part of the optimal tactics. We discuss these observations more precisely in Remark 1 at the end of Section 5. To the best of our knowledge, this is the first stochastic control problem in which the boundary classification of the problem's state space has such a fundamental influence on the problem's solution. We do not investigate this issue any further because this would require substantial extra analysis that would go beyond the scope of the present article.

The evolution of an impulse control problem's state process is quite intuitive, provided that the corresponding uncontrolled dynamics are well-posed. For this reason, several references simply assume the existence of such processes. In the context of SDEs in $\mathbb{R}^d$, the state process of an impulse control problem can be derived by pasting together suitable strong solutions to the underlying uncontrolled SDE with random initial conditions (e.g., see Bensoussan and Lions [8, Section 6.1.1]). In the context of general Markov processes, the classical construction of an impulse control strategy is substantially more technical and may involve countable products of canonical spaces (e.g., see Stettner [41] and Lepeltier and Marchal [28]). If the uncontrolled state space process is a general Markov process with continuous sample paths, then comprehensive constructions of impulse control models have been derived by Helmes, Stockbridge and Zhu [23].

Impulse control problems with SDEs in $\mathbb{R}^d$ can be formulated as in (1)–(3). In itself such a formulation is straightforward. Indeed, an SDE in $\mathbb{R}^d$ such as (1) has a unique strong solution under suitable Lipschitz assumptions on b and σ for a wide class of controlled processes ζ (e.g., see Krylov [26, Theorem 2.5.7]). On the other hand, a rigorous construction of an optimally controlled process ζ, such as a β-γ strategy, is rather non-trivial. In the context of this paper, we construct a unique strong solution to the SDE (1) when the controlled process ζ is a β-γ strategy (see Theorem 5 in Section 4). Despite the central role that such strategies play in stochastic impulse control, we are not aware of any such rigorous SDE result. Furthermore, this construction allows for a probabilistic derivation of the optimal expected discounted running reward as well as the optimal expected discounted reward from control expenditure functionals (see (49) and (50) in Theorem 5). The construction that we make can most easily be adapted to derive the existence of strong solutions to optimally controlled SDEs that arise in other stochastic impulse control problems, even in dimensions higher than one.

The paper is organised as follows. Section 2 presents the precise formulation of the control problem that we solve, including all of the assumptions that we make. In Section 3, we derive several results associated with a linear ODE that we need for the solution to the stochastic control problem we consider. In Section 4, we prove that the SDE (1) has a unique strong solution when the controlled process ζ is a β-γ strategy and we derive analytic expressions for certain associated functionals using probabilistic techniques. We derive the complete solution to the control problem that we consider in Section 5. Finally, we present several examples illustrating the assumptions that we make and the results that we establish in Section 6.

2. Formulation of the stochastic control problem

Fix a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\big)$ satisfying the usual conditions and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We consider a dynamical system, the uncontrolled stochastic dynamics of which are modelled by the SDE

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \quad X_0 = x > 0, \tag{4}$$

and we make the following assumption.

Assumption 1. The functions $b, \sigma : [0, \infty[\to \mathbb{R}$ are locally Lipschitz continuous and $\sigma(x) > 0$ for all $x > 0$.

This assumption implies that the scale function p and the speed measure m of the diffusion associated with the SDE (4), which are given by

$$p(1) = 0 \quad \text{and} \quad p'(x) = \exp\left(-2\int_1^x \frac{b(s)}{\sigma^2(s)}\,\mathrm{d}s\right) \tag{5}$$

$$\text{and} \quad m(\mathrm{d}x) = \frac{2}{\sigma^2(x)p'(x)}\,\mathrm{d}x, \tag{6}$$

are well-defined. Additionally, we make the following assumption on the boundary classification of the diffusion associated with (4).

Assumption 2. The boundary point 0 is inaccessible while the boundary point ∞ is natural.

The state space of the linear diffusion associated with the SDE (4) is the interval $\mathcal{I} =]0, \infty[$. Recall that the boundary point $p \in \{0, \infty\}$ of $\mathcal{I}$ is called *inaccessible* if $\mathbb{P}_x(T_p < \infty) = 0$ for all $x \in \mathcal{I}$ and *accessible* otherwise. Furthermore, if the boundary p is inaccessible, then it is *natural* if

$$\lim_{x\in\mathcal{I},\, x\to p} \mathbb{P}_x(T_y < t) = 0 \quad \text{for all } y \in \mathcal{I} \text{ and } t > 0$$

and *entrance* otherwise, namely, if

$$\lim_{x\in\mathcal{I},\, x\to p} \mathbb{P}_x(T_y < t) > 0 \quad \text{for some } y \in \mathcal{I} \text{ and } t > 0$$

(e.g., see Revuz and Yor [38, Definition VII.3.9]). In these expressions, T_y is the first hitting time of the set $\{y\}$, which is defined by

$$T_y = \inf\{t \geq 0 \mid \ X_t = y\}, \quad \text{for } y > 0. \tag{7}$$

In Borodin and Salminen [10, II.1.6], an inaccessible boundary point is called *not-exit*, while a natural (resp., entrance) boundary point is called *natural* (resp., *entrance-not-exit*). Integral conditions for the classification of a boundary point $p \in \{0, \infty\}$ of $\mathcal{I}$ in terms of the scale function p and the speed measure m can be found in this reference.

We next consider the stochastic control problem defined by (1)–(3).

Definition 1. The family of all admissible controlled strategies is the set of all $(\mathcal{F}_t)$-adapted càdlàg processes ζ with increasing and piece-wise constant sample paths such that the SDE (1) has a unique non-explosive strong solution and

$$\mathbb{E}_x\left[\sum_{t\geq 0} e^{-\Lambda_t^\zeta}\mathbf{1}_{\{\Delta\zeta_t > 0\}}\right] < \infty. \tag{8}$$

Assumption 3. The discounting rate function r is bounded and continuous. Also, there exists $r_0 > 0$ such that $r(x) \geq r_0$ for all $x \geq 0$.

To complete the set of our assumptions, we consider the operator $\mathscr{L}$ acting on C^1 functions with absolutely continuous first-order derivatives that is defined by

$$\mathscr{L}w(x) = \frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x). \tag{9}$$

In the presence of Assumptions 1, 2 and 3, the second-order linear ODE $\mathscr{L}w(x) = 0$ has two fundamental C^2 solutions φ and ψ such that

$$0 < \varphi(x) \quad \text{and} \quad \varphi'(x) < 0 \quad \text{for all } x > 0, \tag{10}$$

$$0 < \psi(x) \quad \text{and} \quad \psi'(x) > 0 \quad \text{for all } x > 0 \tag{11}$$

$$\text{and} \quad \lim_{x\downarrow 0} \varphi(x) = \lim_{x\uparrow\infty} \psi(x) = \infty. \tag{12}$$

If 0 is a natural boundary point, then

$$\lim_{x\downarrow 0} \frac{\varphi'(x)}{p'(x)} = -\infty, \quad \lim_{x\downarrow 0} \psi(x) = 0 \quad \text{and} \quad \lim_{x\downarrow 0} \frac{\psi'(x)}{p'(x)} = 0, \tag{13}$$

while, if 0 is an entrance boundary point, then

$$\lim_{x\downarrow 0} \frac{\varphi'(x)}{p'(x)} > -\infty, \quad \lim_{x\downarrow 0} \psi(x) > 0 \quad \text{and} \quad \lim_{x\downarrow 0} \frac{\psi'(x)}{p'(x)} = 0. \tag{14}$$

Symmetric results hold for the boundary point ∞ (e.g., see Borodin and Salminen [10, II.10]).

The functions φ and ψ admit the probabilistic representations

$$\varphi(y) = \varphi(x)\,\mathbb{E}_y\left[e^{-\Lambda_{T_x}}\right] \quad \text{and} \quad \psi(x) = \psi(y)\,\mathbb{E}_x\left[e^{-\Lambda_{T_y}}\right] \quad \text{for all } x < y, \tag{15}$$

where Λ is defined by (3) with X in place of X^ζ and T_y is defined by (7).

Furthermore, φ and ψ are such that

$$\varphi(x)\psi'(x) - \varphi'(x)\psi(x) = Cp'(x) \quad \text{for all } x > 0, \tag{16}$$

where $C = \varphi(1)\psi'(1) - \varphi'(1)\psi(1)$ and p is the scale function defined by (5). To simplify the notation, we also define

$$\Psi(x) = \frac{2\psi(x)}{C\sigma^2(x)p'(x)} \quad \text{and} \quad \Phi(x) = \frac{2\varphi(x)}{C\sigma^2(x)p'(x)}. \tag{17}$$

Beyond involving standard integrability and growth assumptions, the conditions in the following assumption may appear involved. However, they are standard in the relevant literature and are satisfied by a wide range of problem data choices (see Examples 1-4 in Section 6).

Assumption 4. The following conditions hold true:

(i) The function h is continuous as well as bounded from below. Also, the limit $\lim_{x\downarrow 0} h(x)/r(x)$ exists in $\mathbb{R}$ and

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\left|h(X_t)\right| \mathrm{d}t\right] < \infty.$$

(ii) The function k is absolutely continuous,

$$\int_0^1 \big|k(s)\big|\,\mathrm{d}s < \infty \quad \text{and the function } x \mapsto \int_0^x k(s)\,\mathrm{d}s \text{ is bounded from below.} \tag{18}$$

Furthermore,

$$\mathbb{E}_x\bigg[\int_0^\infty e^{-\Lambda_t}\,\Big|\mathscr{L}\Big(\int_0^{\cdot} k(s)\,\mathrm{d}s\Big)(X_t)\Big|\,\mathrm{d}t\bigg] < \infty \quad \text{and} \quad \limsup_{x\uparrow\infty}\frac{1}{\psi(x)}\int_0^x k(s)\,\mathrm{d}s \in \mathbb{R}_+.$$

(iii) If we define

$$\Theta(x) = h(x) + \mathscr{L}\Big(\int_0^{\cdot} k(s)\,\mathrm{d}s\Big)(x), \tag{19}$$

then Θ is continuous and there exists a constant $\xi \in\,]0,\infty[$ such that the restriction of Θ/r in $]0,\xi[$ (resp., in $]\xi,\infty[$) is strictly increasing (resp., strictly decreasing).

3. Results associated with a linear ODE

Unless stated otherwise, the results in this section hold true if the coefficients of (4) satisfy the usual Engelbert and Schmidt conditions, rather than the stronger Assumption 1, and the boundary points 0, ∞ are inaccessible. We start by recalling some standard results that we will need and can be found in, e.g., Lamberton and Zervos [27, Section 4]. Consider a Borel measurable function $F :]0,\infty[\to \mathbb{R}$ such that

$$\mathbb{E}_x\bigg[\int_0^\infty e^{-\Lambda_t}\big|F(X_t)\big|\,\mathrm{d}t\bigg] < \infty \quad \text{for all } x > 0, \tag{20}$$

where Λ is defined by (3) for $X^\zeta = X$. This integrability condition is equivalent to

$$\int_0^x \big|F(s)\big|\Psi(s)\,\mathrm{d}s + \int_x^\infty \big|F(s)\big|\Phi(s)\,\mathrm{d}s < \infty \quad \text{for all } x > 0, \tag{21}$$

where Φ and Ψ are defined by (17). Given such a function F, we define

$$R_F(x) = \mathbb{E}_x\bigg[\int_0^\infty e^{-\Lambda_t}F(X_t)\,\mathrm{d}t\bigg], \quad \text{for } x > 0. \tag{22}$$

The function R_F admits the analytic presentation

$$R_F(x) = \varphi(x)\int_0^x F(s)\Psi(s)\,\mathrm{d}s + \psi(x)\int_x^\infty F(s)\Phi(s)\,\mathrm{d}s \tag{23}$$

and satisfies the ODE $\mathscr{L}R_F + F = 0$. Furthermore,

$$\lim_{x\downarrow 0}\frac{|R_F(x)|}{\varphi(x)} = 0 \quad \text{and} \quad \lim_{x\uparrow\infty}\frac{|R_F(x)|}{\psi(x)} = 0. \tag{24}$$

Conversely, consider any function $f :]0,\infty[\to \mathbb{R}$ that is C^1 with absolutely continuous first-order derivative and such that

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\big|\mathscr{L}f(X_t)\big|\,\mathrm{d}t\right] < \infty, \quad \limsup_{z\downarrow 0}\frac{|f(z)|}{\varphi(z)} < \infty \quad \text{and} \quad \limsup_{z\uparrow\infty}\frac{|f(z)|}{\psi(z)} < \infty.$$

Such a function is such that

$$\text{both of the limits } \lim_{z\downarrow 0}\frac{f(z)}{\varphi(z)} \text{ and } \lim_{z\uparrow\infty}\frac{f(z)}{\psi(z)} \text{ exist} \tag{25}$$

$$\text{and} \quad f(x) = \lim_{z\downarrow 0}\frac{f(z)}{\varphi(z)}\varphi(x) - R_{\mathscr{L}f}(x) + \lim_{z\uparrow\infty}\frac{f(z)}{\psi(z)}\psi(x) \quad \text{for all } x > 0. \tag{26}$$

Part (ii) of the following result will be important in appreciating the role that the boundary classification of 0 has on whether switching off the system might be optimal (see Remark 1 at the end of Section 5). In general, (27) is not true if 0 is an entrance boundary point (see (98) in Example 8 in Section 6).

Lemma 1. *Suppose that Assumptions 1 and 3 hold true. Also, suppose that the boundary points 0 and ∞ of the diffusion associated with the SDE (4) are both inaccessible. Let F be any Borel measurable function satisfying the equivalent integrability conditions (20) and (21), and consider the function R_F defined by (22) and (23). The following statements hold true:*

(i) *Suppose that F is bounded from below. If K is any constant such that $F(x)/r(x) \geq K$ for all $x > 0$, then $R_F(x) \geq K$ for all $x > 0$.*

(ii) *If 0 is a natural boundary point, then*

$$\liminf_{x\downarrow 0}\frac{F(x)}{r(x)} \leq \liminf_{x\downarrow 0} R_F(x) \leq \limsup_{x\downarrow 0} R_F(x) \leq \limsup_{x\downarrow 0}\frac{F(x)}{r(x)}. \tag{27}$$

Proof. Part (i) of the lemma follows immediately from the calculation

$$\inf_{x>0} R_F(x) = \inf_{x>0}\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}F(X_t)\,\mathrm{d}t\right] \geq \inf_{x>0}\frac{F(x)}{r(x)}\,\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}r(X_t)\,\mathrm{d}t\right] = \inf_{x>0}\frac{F(x)}{r(x)},$$

where we have used the definition (3) of Λ.

To establish part (ii) of the lemma suppose in what follows that 0 is a natural boundary point. Assuming that $\limsup_{x\downarrow 0} F(x)/r(x) \in \mathbb{R}$, fix any $\varepsilon > 0$ and let $x_\varepsilon > 0$ be any point such that

$$\frac{F(x)}{r(x)} \leq \limsup_{x\downarrow 0}\frac{F(x)}{r(x)} + \varepsilon \quad \text{for all } x \in\,]0, x_\varepsilon].$$

In view of (22), (23), the definition (3) of Λ and the second limit in (13), we can see that

$$\begin{aligned}
&\limsup_{x\downarrow 0} R_F(x) - \limsup_{x\downarrow 0}\frac{F(x)}{r(x)} - \varepsilon \\
&\qquad = \limsup_{x\downarrow 0}\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\left(\frac{F(X_t)}{r(X_t)} - \limsup_{x\downarrow 0}\frac{F(x)}{r(x)} - \varepsilon\right)r(X_t)\,\mathrm{d}t\right]
\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{x\downarrow 0}\Bigg(\varphi(x)\int_0^x\Big(\frac{F(s)}{r(s)} - \limsup_{x\downarrow 0}\frac{F(x)}{r(x)} - \varepsilon\Big)r(s)\Psi(s)\,\mathrm{d}s \\
&\qquad + \psi(x)\int_x^\infty\Big(\frac{F(s)}{r(s)} - \limsup_{x\downarrow 0}\frac{F(x)}{r(x)} - \varepsilon\Big)r(s)\Phi(s)\,\mathrm{d}s\Bigg) \\
&\le \lim_{x\downarrow 0}\psi(x)\int_{x_\varepsilon}^\infty\Big(\frac{F(s)}{r(s)} - \limsup_{x\downarrow 0}\frac{F(x)}{r(x)} - \varepsilon\Big)r(s)\Phi(s)\,\mathrm{d}s\Bigg) \\
&= 0,
\end{aligned}
$$

which implies that $\limsup_{x\downarrow 0} R_F(x) \le \limsup_{x\downarrow 0} F(x)/r(x)$ because ε has been arbitrary. Similarly, we can show that $\lim_{x\downarrow 0} R_F(x) = -\infty$ if $\lim_{x\downarrow 0} F(x)/r(x) = -\infty$, and the third inequality in (27) follows. Using similar arguments, we can establish the first inequality in (27). □

Lemma 2. *Suppose that Assumptions 1 and 3 hold true, suppose that the boundary points 0 and ∞ of the diffusion associated with the SDE (4) are both inaccessible and consider any Borel measurable function F satisfying the equivalent integrability conditions (20) and (21). The function $G_F :]0,\infty[\to \mathbb{R}$ defined by*

$$
G_F(x) := R_F(x) - \frac{R_F'(x)}{\psi'(x)}\psi(x) = \frac{Cp'(x)}{\psi'(x)}\int_0^x F(s)\Psi(s)\,\mathrm{d}s \tag{28}
$$

is such that

$$
\liminf_{x\downarrow 0}\frac{F(x)}{r(x)} \le \liminf_{x\downarrow 0} G_F(x) \le \limsup_{x\downarrow 0} G_F(x) \le \limsup_{x\downarrow 0}\frac{F(x)}{r(x)}. \tag{29}
$$

Furthermore, if the boundary point ∞ is natural, then

$$
\liminf_{x\uparrow\infty}\frac{F(x)}{r(x)} \le \liminf_{x\uparrow\infty} G_F(x) \le \limsup_{x\uparrow\infty} G_F(x) \le \limsup_{x\uparrow\infty}\frac{F(x)}{r(x)}. \tag{30}
$$

Proof. We first note that the equality in (28) follows immediately from the definition (23) of R_F and the identity (16). In view of (13) and (14), the assumption that the boundary point 0 is inaccessible implies that

$$
\lim_{x\downarrow 0}\frac{\psi'(x)}{p'(x)} = 0. \tag{31}
$$

This limit and the calculation

$$
\frac{\mathrm{d}}{\mathrm{d}x}\frac{\psi'(x)}{p'(x)} = \frac{2}{\sigma^2(x)p'(x)}\Big(\frac{1}{2}\sigma^2(x)\psi''(x) + b(x)\psi'(x)\Big) = \frac{2r(x)\psi(x)}{\sigma^2(x)p'(x)} = Cr(x)\Psi(x)
$$

imply that

$$
\int_0^x r(s)\Psi(s)\,\mathrm{d}s = \frac{\psi'(x)}{Cp'(x)}. \tag{32}
$$

Similarly, the calculation

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\varphi'(x)}{p'(x)} = Cr(x)\Phi(x) \tag{33}$$

and the assumption that the boundary point ∞ is inaccessible imply that

$$\int_x^\infty r(s)\Phi(s)\,\mathrm{d}s = -\frac{\varphi'(x)}{Cp'(x)}. \tag{34}$$

In view of (32) and the expression of G_F on the right-hand side of (28), we can see that

$$G_F(x) \geq \frac{Cp'(x)}{\psi'(x)} \inf_{y<x} \frac{F(y)}{r(y)} \int_0^x r(s)\Psi(s)\,\mathrm{d}s = \inf_{y<x} \frac{F(y)}{r(y)}$$

$$\text{and} \quad G_F(x) \leq \frac{Cp'(x)}{\psi'(x)} \sup_{y<x} \frac{F(y)}{r(y)} \int_0^x r(s)\Psi(s)\,\mathrm{d}s = \sup_{y<x} \frac{F(y)}{r(y)}.$$

These inequalities imply (29).

Next, we additionally assume that ∞ is a natural boundary point, which implies that $\lim_{x\uparrow\infty} \psi'(x)/p'(x) = \infty$ (e.g., see Borodin and Salminen [10, II.10]). The expression of G_F on the right-hand side of (28), the strict positivity of Ψ and the identity (32) imply that, given any $x > z > 0$,

$$\begin{aligned}
&\frac{Cp'(x)}{\psi'(x)} \int_0^z F(s)\Psi(s)\,\mathrm{d}s + \inf_{y>z} \frac{F(y)}{r(y)} \left(1 - \frac{p'(x)}{\psi'(x)}\frac{\psi'(z)}{p'(z)}\right) \\
&\quad = \frac{Cp'(x)}{\psi'(x)} \int_0^z F(s)\Psi(s)\,\mathrm{d}s + \inf_{y>z} \frac{F(y)}{r(y)} \frac{Cp'(x)}{\psi'(x)} \int_z^x r(s)\Psi(s)\,\mathrm{d}s \\
&\quad \leq G_F(x) \leq \frac{Cp'(x)}{\psi'(x)} \int_0^z F(s)\Psi(s)\,\mathrm{d}s + \sup_{y>z} \frac{F(y)}{r(y)} \frac{Cp'(x)}{\psi'(x)} \int_z^x r(s)\Psi(s)\,\mathrm{d}s \\
&\quad = \frac{Cp'(x)}{\psi'(x)} \int_0^z F(s)\Psi(s)\,\mathrm{d}s + \sup_{y>z} \frac{F(y)}{r(y)} \left(1 - \frac{p'(x)}{\psi'(x)}\frac{\psi'(z)}{p'(z)}\right).
\end{aligned}$$

Combining these observations, we can see that

$$\inf_{y>z} \frac{F(y)}{r(y)} \leq \liminf_{x\uparrow\infty} G_F(x) \leq \limsup_{x\uparrow\infty} G_F(x) \leq \sup_{y>z} \frac{F(y)}{r(y)} \quad \text{for all } z > 0,$$

and (30) follows. □

Lemma 3. *Suppose that Assumption 1 and 3 hold true. Also, suppose that the boundary points 0 and ∞ of the diffusion associated with the SDE (4) are both inaccessible. Given any Borel measurable function F satisfying the equivalent integrability conditions (20) and (21), if the boundary point 0 (resp., ∞) is inaccessible, then*

$$\liminf_{x\downarrow 0} \frac{R_F'(x)}{\varphi'(x)} \leq 0 \leq \limsup_{x\downarrow 0} \frac{R_F'(x)}{\varphi'(x)} \quad \left(\textit{resp.,}\ \liminf_{x\uparrow\infty} \frac{R_F'(x)}{\psi'(x)} \leq 0 \leq \limsup_{x\uparrow\infty} \frac{R_F'(x)}{\psi'(x)}\right). \tag{35}$$

Furthermore, if there exists $x_\dagger > 0$ *(resp.,* $x^\dagger > 0$*) such that the restriction of* F/r *in* $]0, x_\dagger[$ *(resp.,* $]x^\dagger, \infty[$*) is a monotone function, then*

$$\lim_{x \downarrow 0} \frac{R'_F(x)}{\varphi'(x)} = 0 \quad \left(\textit{resp.,} \ \lim_{x \uparrow \infty} \frac{R'_F(x)}{\psi'(x)} = 0 \right). \tag{36}$$

Proof. To establish the very first inequality in (35), we argue by contradiction. To this end, we assume that $\liminf_{x \downarrow 0} R'_F(x)/\varphi'(x) > 0$, which implies that there exist $\varepsilon > 0$ and $x_\varepsilon > 0$ such that

$$\frac{R'_F(x)}{\varphi'(x)} > \varepsilon \ \Leftrightarrow \ R'_F(x) < \varepsilon \varphi'(x) \quad \text{for all } x \in \,]0, x_\varepsilon[.$$

However, this observation and the fact that $\lim_{x \downarrow 0} \varphi(x) = \infty$ imply that $\lim_{x \downarrow 0} R_F(x)/\varphi(x) \geq \varepsilon$, which contradicts (24). The proof of the other inequalities in (35) is similar.

To proceed further, we first note that (16) and the fact that $\mathscr{L}\varphi = \mathscr{L}\psi = 0$, where $\mathscr{L}$ is the differential operator defined by (9), imply that

$$\psi'(x)\varphi''(x) - \varphi'(x)\psi''(x) = \frac{2Cr(x)}{\sigma^2(x)} p'(x). \tag{37}$$

In view of this observation and the definition (23) of R_F, we can see that the function R'_F/ψ' is absolutely continuous with derivative

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{R'_F(x)}{\psi'(x)} = \frac{2Cr(x)p'(x)}{(\sigma(x)\psi'(x))^2} \left(\int_0^x F(s)\Psi(s)\,\mathrm{d}s - \frac{F(x)}{r(x)} \frac{\psi'(x)}{Cp'(x)} \right) =: \frac{2Cr(x)p'(x)}{(\sigma(x)\psi'(x))^2} Q_F(x). \tag{38}$$

Now, suppose that there exists a point $x^\dagger > 0$ such that F/r is monotone in $[x^\dagger, \infty[$. Given any points $x_1 < x_2$ in $[x^\dagger, \infty[$, we use (32) to calculate

$$\begin{aligned} Q_F(x_2) - Q_F(x_1) &= \int_{x_1}^{x_2} F(s)\Psi(s)\,\mathrm{d}s - \frac{F(x_2)}{r(x_2)} \frac{\psi'(x_2)}{Cp'(x_2)} + \frac{F(x_1)}{r(x_1)} \frac{\psi'(x_1)}{Cp'(x_1)} \\ &= \int_{x_1}^{x_2} \left(\frac{F(s)}{r(s)} - \frac{F(x_2)}{r(x_2)} \right) r(s)\Psi(s)\,\mathrm{d}s + \frac{\psi'(x_1)}{Cp'(x_1)} \left(\frac{F(x_1)}{r(x_1)} - \frac{F(x_2)}{r(x_2)} \right) \\ &\begin{cases} \geq 0, & \text{if } F/r \text{ is decreasing in } [x^\dagger, \infty[, \\ \leq 0, & \text{if } F/r \text{ is increasing in } [x^\dagger, \infty[. \end{cases} \end{aligned} \tag{39}$$

Therefore, Q_F is monotone in $[x^\dagger, \infty[$ and the limit $\lim_{x \uparrow \infty} Q_F(x)$ exists in $[-\infty, \infty]$. However, this observation and (38) imply that there exists $\widetilde{x} \geq x^\dagger$ such that R'_F/ψ' is monotone in $[\widetilde{x}, \infty[$. Therefore, the limit $\lim_{x \uparrow \infty} R'_F(x)/\psi'(x)$ exists, which, combined with the last two inequalities in (35), implies the corresponding limit in (36).

Finally, we can establish the other limit in (36) using symmetric arguments and (33). □

The following result will play a critical role in our analysis. Example 9 in Section 6 shows that the point $\underline{x}$ introduced in part (i) of the lemma can be equal to ∞ if the sufficient conditions in (42) fail to be true. Also, in contrast to the limit in (40), Examples 5 and 6 in Section 6 show that the limit $\lim_{x \downarrow 0} R'_\Theta(x)/\psi'(x)$, which characterises part (iii) of the lemma, can take any value in $]-\infty, \infty]$.

Lemma 4. *Suppose that Assumption 1 and 3 hold true. Also, suppose that the boundary points 0 and ∞ are both inaccessible. Given a function Θ satisfying the conditions of Assumption 4.(iii), as well as the equivalent integrability conditions (20) and (21), the following statements are true:*

(i) *There exists a unique $\underline{x} \in \,]\xi, \infty]$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{R'_\Theta(x)}{\psi'(x)} \begin{cases} < 0 & \text{for all } x \in \,]0, \underline{x}[, \\ > 0 & \text{for all } x \in \,]\underline{x}, \infty[, \end{cases} \quad \text{and} \quad \lim_{x \uparrow \infty} \frac{R'_\Theta(x)}{\psi'(x)} = 0, \tag{40}$$

where we adopt the convention $]\infty, \infty[\, = \emptyset$.

(ii) *$\underline{x} < \infty$ if and only if $\lim_{x \uparrow \infty} Q_\Theta(x) > 0$, where*

$$Q_\Theta(x) = \int_0^x \Theta(s) \Psi(s) \, \mathrm{d}s - \frac{\Theta(x)}{r(x)} \frac{\psi'(x)}{Cp'(x)}. \tag{41}$$

In particular, this is the case if

$$\lim_{x \uparrow \infty} \frac{\Theta(x)}{r(x)} = -\infty \quad \text{or} \quad \lim_{x \downarrow 0} \frac{\Theta(x)}{r(x)} > \lim_{x \uparrow \infty} \frac{\Theta(x)}{r(x)}. \tag{42}$$

(iii) *If $\underline{x} < \infty$ and we define*

$$\overline{x} = \inf \left\{ s > 0 \;\middle|\; \frac{R'_\Theta(s)}{\psi'(s)} > \lim_{x \downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} \right\}, \tag{43}$$

with the usual convention that $\inf \emptyset = \infty$, then $\overline{x} > \underline{x}$,

$$\overline{x} = \infty \quad \Leftrightarrow \quad \lim_{x \downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} \geq 0 \tag{44}$$

$$\text{and} \quad \lim_{x \downarrow 0} \frac{\Theta(x)}{r(x)} = -\infty \quad \Rightarrow \quad \lim_{x \downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = \infty \quad \Rightarrow \quad \overline{x} = \infty. \tag{45}$$

Proof. The limit in (40) follows from Lemma 3 and the assumption that Θ/r is strictly decreasing in $]\xi, \infty[$. Using (32) and (38), we can see that

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{R'_\Theta(x)}{\psi'(x)} = \frac{2Cr(x)p'(x)}{(\sigma(x)\psi'(x))^2} \int_0^x \left(\frac{\Theta(s)}{r(s)} - \frac{\Theta(x)}{r(x)} \right) r(s) \Psi(s) \, \mathrm{d}s = \frac{2Cr(x)p'(x)}{(\sigma(x)\psi'(x))^2} Q_\Theta(x).$$

These expressions imply that

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{R'_\Theta(x)}{\psi'(x)} < 0 \quad \text{and} \quad Q_\Theta(x) < 0 \quad \text{for all } x \leq \xi$$

because Θ/r is strictly increasing in $]0, \xi[$. On the other hand, (39) for $F = \Theta$ implies that Q_Θ is strictly increasing in $[\xi, \infty[$ because Θ/r is strictly decreasing in $[\xi, \infty[$. It follows that there exists a unique $\underline{x} \in \,]\xi, \infty]$ such that the inequalities in (40) hold true. Furthermore, $\underline{x} < \infty$ if and only if $\lim_{x \uparrow \infty} Q_\Theta(x) > 0$.

To establish the sufficient conditions in part (ii) of the lemma, we first use the integration by parts formula and (32) to observe that

$$\begin{aligned} Q_\Theta(x) &= \int_0^\xi \Theta(s)\Psi(s)\,\mathrm{d}s - \frac{\Theta(\xi)}{r(\xi)}\frac{\psi'(\xi)}{Cp'(\xi)} - \int_\xi^x \frac{\psi'(s)}{Cp'(s)}\,\mathrm{d}\frac{\Theta(s)}{r(s)} \\ &\geq \int_0^\xi \Theta(s)\Psi(s)\,\mathrm{d}s - \frac{\Theta(x)}{r(x)}\frac{\psi'(\xi)}{Cp'(\xi)} \quad \text{for al } \xi < x. \end{aligned} \tag{46}$$

This inequality reveals that $\lim_{x\uparrow\infty} Q_\Theta(x) = \infty$ if $\lim_{x\uparrow\infty} \Theta(x)/r(x) = -\infty$.

The identity (32) implies that, given any constant K,

$$\int_0^x Kr(s)\Psi(s)\,\mathrm{d}s - \frac{Kr(x)}{r(x)}\frac{\psi'(x)}{Cp'(x)} = 0$$

Combining this observation with the definition of Q_Θ, we can see that $Q_\Theta = Q_{\Theta+Kr}$. If Θ/r satisfies the inequality in (42), then, for all K such that

$$-\lim_{x\downarrow 0}\frac{\Theta(x)}{r(x)} < K < -\lim_{x\uparrow\infty}\frac{\Theta(x)}{r(x)},$$

there exists $\eta(K) \in \,]\xi, \infty[$ such that

$$\Theta\big(\eta(K)\big) + Kr\big(\eta(K)\big) = 0 \quad \text{and} \quad Q_{\Theta+Kr}\big(\eta(K)\big) = \int_0^{\eta(K)} \big(\Theta(s) + Kr(s)\big)\Psi(s)\,\mathrm{d}s > 0.$$

It follows that

$$\lim_{x\uparrow\infty} Q_\Theta(x) = \lim_{x\uparrow\infty} Q_{\Theta+Kr}(x) > 0,$$

thanks to the fact that Q_Θ is strictly increasing in $[\xi, \infty[$.

The equivalence (44) follows immediately from (40) and the definition (43) of $\overline{x}$. To establish the implications in (45), we first note that (33) implies that the function φ'/p' is strictly increasing, so the limit $\lim_{x\downarrow 0}\varphi'(x)/p'(x)$ exists in $[-\infty, 0[$. Therefore,

$$\lim_{x\downarrow 0}\frac{\psi'(x)}{\varphi'(x)} = \lim_{x\downarrow 0}\frac{\psi'(x)}{p'(x)}\lim_{x\downarrow 0}\frac{p'(x)}{\varphi'(x)} = 0, \tag{47}$$

where we have also used (31). Using the first of these two observations, the definition (23) of R_Θ, (34), (37) and integration by parts, we can see that, if $\lim_{x\downarrow 0}\Theta(x)/r(x) = -\infty$, then

$$\begin{aligned} \lim_{x\downarrow 0}\frac{(\sigma(x)\varphi'(x))^2}{2Cr(x)p'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\frac{R'_\Theta(x)}{\varphi'(x)} &= -\lim_{x\downarrow 0}\left(\int_x^\infty \Theta(s)\Phi(s)\,\mathrm{d}s + \frac{\Theta(x)}{r(x)}\frac{\varphi'(x)}{Cp'(x)}\right) \\ &= -\int_1^\infty \Theta(s)\Phi(s)\,\mathrm{d}s - \frac{\Theta(1)}{r(1)}\frac{\varphi'(1)}{Cp'(1)} + \lim_{x\downarrow 0}\int_x^1 \frac{\varphi'(s)}{Cp'(s)}\,\mathrm{d}\frac{\Theta(s)}{r(s)} \\ &= -\infty. \end{aligned}$$

On the other hand, we use (37) to calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\psi'(x)}{\varphi'(x)} = -\frac{2Cr(x)p'(x)}{(\sigma(x)\varphi'(x))^2}.$$

In view of (36) and (47), these calculations and L' Hôpital's formula imply that

$$\lim_{x\downarrow 0}\frac{R'_\Theta(x)}{\psi'(x)} = \lim_{x\downarrow 0}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\frac{R'_\Theta(x)}{\varphi'(x)}}{\frac{\mathrm{d}}{\mathrm{d}x}\frac{\psi'(x)}{\varphi'(x)}} = \infty.$$

The implications in (45) follow from the definition (43) of $\overline{x}$. □

4. The "$\boldsymbol{\beta}$-$\boldsymbol{\gamma}$" strategy

In this section, we consider the β-γ strategy that is characterised by two points $0 < \gamma < \beta < \infty$ and takes the following form. If the state process takes any value $x \geq \beta$, the controller pushes it in an impulsive way down to the level γ. For as long as the state process takes values inside the interval $]0, \beta[$, the controller waits and takes no action. Accordingly, such a strategy is characterised by a controlled process ζ such that

$$\Delta\zeta_t = \big(X^\zeta_{t-} - \gamma\big)\mathbf{1}_{\{X^\zeta_{t-}\geq\beta\}} \quad \text{for all } t \geq 0, \tag{48}$$

where X^ζ is the associated solution to the SDE (1).

Theorem 5. *Suppose that Assumptions 1 and 3 hold true. Also, suppose that the boundary points 0 and ∞ of the diffusion associated with the uncontrolled SDE (4) are both inaccessible. Given any points $\gamma < \beta$ in $]0, \infty[$, there exists a controlled process $\zeta = \zeta(\beta, \gamma)$ that is admissible in the sense of Definition 1 and is such that (48) holds true. Furthermore, given any $x \in]0, \beta[$,*

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda^\zeta_t} h(X^\zeta_t)\,\mathrm{d}t\right] = R_h(x) + \frac{\psi(x)}{\psi(\beta) - \psi(\gamma)}\big(R_h(\gamma) - R_h(\beta)\big) \tag{49}$$

and

$$\mathbb{E}_x\left[\sum_{t\geq 0} e^{-\Lambda^\zeta_t}\mathbf{1}_{\{\Delta\zeta_t>0\}}\right] = \frac{\psi(x)}{\psi(\beta) - \psi(\gamma)}. \tag{50}$$

Proof. We start with a recursive construction of the required process ζ and its associated solution to the SDE (4). To this end, we first consider any initial state $x \in]0, \beta[$, we denote by X^1 the solution to the uncontrolled SDE (4) and we define

$$\tau_1 = \inf\big\{t \geq 0 \mid \ X^1_t \geq \beta\big\} \quad \text{and} \quad \zeta^1_t = (\beta - \gamma)\mathbf{1}_{\{\tau_1 \leq t\}}. \tag{51}$$

Given $\ell \geq 1$, suppose that we have determined X^j, τ_j and ζ^j, for $j = 1, \ldots, \ell$.

The process $\widetilde{W}^{\ell+1}$ defined by $\widetilde{W}^{\ell+1}_t = \big(W_{\tau_\ell+t} - W_{\tau_\ell}\big)\mathbf{1}_{\{\tau_\ell<\infty\}}$ is a standard $(\mathcal{F}_{\tau_\ell+t})$-Brownian motion that is independent of $\mathcal{F}_{\tau_\ell}$ under the conditional probability measure $\mathbb{P}(\cdot \mid \tau_\ell < \infty)$ (see Revuz and Yor [38, Exercise IV.3.21]). We denote by $\widetilde{X}^{\ell+1}$ the unique solution to the uncontrolled SDE (4) with $\widetilde{X}^{\ell+1}_0 = \gamma$ that is driven by the Brownian motion $\widetilde{W}^{\ell+1}$ and is defined on the probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_{\tau_\ell+t}), \mathbb{P}(\cdot \mid \tau_\ell < \infty)\big)$. Since $(t - \tau_\ell)^+$ is an $(\mathcal{F}_{\tau_\ell+t})$-stopping time for all $t \geq 0$,

$$\tau_\ell + (t - \tau_\ell)^+ = t \vee \tau_\ell \quad \text{and} \quad \widetilde{W}^{\ell+1}_{(t-\tau_\ell)^+} = \big(W_{t\vee\tau_\ell} - W_{\tau_\ell}\big)\mathbf{1}_{\{\tau_\ell<\infty\}},$$

we can see that, on the event $\{\tau_\ell < \infty\}$,

$$\widetilde{X}^{\ell+1}_{(t-\tau_\ell)^+} = \gamma + \int_0^{(t-\tau_\ell)^+} b\big(\widetilde{X}^{\ell+1}_s\big)\,\mathrm{d}s + \int_0^{(t-\tau_\ell)^+} \sigma\big(\widetilde{X}^{\ell+1}_s\big)\,\mathrm{d}\widetilde{W}^{\ell+1}_s$$
$$= \gamma + \int_0^t b\Big(\widetilde{X}^{\ell+1}_{(s-\tau_\ell)^+}\Big)\,\mathrm{d}(s-\tau_\ell)^+ + \int_0^t \sigma\Big(\widetilde{X}^{\ell+1}_{(s-\tau_\ell)^+}\Big)\,\mathrm{d}\widetilde{W}^{\ell+1}_{(s-\tau_\ell)^+}$$
$$= \gamma + \int_{\tau_\ell}^{t\vee\tau_\ell} b\Big(\widetilde{X}^{\ell+1}_{(s-\tau_\ell)^+}\Big)\,\mathrm{d}s + \int_{\tau_\ell}^{t\vee\tau_\ell} \sigma\Big(\widetilde{X}^{\ell+1}_{(s-\tau_\ell)^+}\Big)\,\mathrm{d}W_s,$$

where we have time changed the Lebesgue as well as the Itô integral (see Revuz and Yor [38, Propositions V.1.4, V.1.5]). It follows that, if we define

$$\overline{X}^{\ell+1}_t = \widetilde{X}^{\ell+1}_{(t-\tau_\ell)^+}\mathbf{1}_{\{\tau_\ell<\infty\}}, \quad \text{for } t \geq 0, \tag{52}$$

then

$$\overline{X}^{\ell+1}_t = \gamma + \int_{\tau_\ell}^{t\vee\tau_\ell} b\big(\overline{X}^{\ell+1}_s\big)\,\mathrm{d}s + \int_{\tau_\ell}^{t\vee\tau_\ell} \sigma\big(\overline{X}^{\ell+1}_s\big)\,\mathrm{d}W_s. \tag{53}$$

Furthermore, we define

$$X^{\ell+1}_t = X^\ell_t \mathbf{1}_{\{t<\tau_\ell\}} + \overline{X}^{\ell+1}_t \mathbf{1}_{\{\tau_\ell\leq t\}}, \tag{54}$$

$$\tau_{\ell+1} = \inf\big\{t > \tau_\ell \mid \ X^{\ell+1}_t \geq \beta\big\} \quad \text{and} \quad \zeta^{\ell+1}_t = \zeta^\ell_t + (\beta-\gamma)\mathbf{1}_{\{\tau_{\ell+1}\leq t\}}. \tag{55}$$

Also, we note that

$$\tau_{\ell+1} - \tau_\ell = \widetilde{T}^{\ell+1}_\beta := \inf\big\{t \geq 0 \mid \ \widetilde{X}^{\ell+1}_t \geq \beta\big\}. \tag{56}$$

Given the recursive construction we have just considered, we define

$$X^\zeta_t = \sum_{\ell=0}^{\infty} X^{\ell+1}_t \mathbf{1}_{\{\tau_\ell\leq t<\tau_{\ell+1}\}} \quad \text{and} \quad \zeta_t = \sum_{\ell=0}^{\infty} \zeta^{\ell+1}_t \mathbf{1}_{\{\tau_\ell\leq t<\tau_{\ell+1}\}}. \tag{57}$$

In view of (53)–(55), the process X^ζ given by (57) provides the unique solution to the SDE (1) for ζ being as in (57). Furthermore, these processes are such that (48) holds true. In the case that arises when the initial state $x \geq \beta$, the only modification of the arguments above involves X^1 being the solution to the uncontrolled SDE (4) for $x = \gamma$ and ζ^1 being the same as in (51) translated by adding the constant $x - \gamma$ to it.

We next establish (50), which implies the admissibility condition (8). The process $\widetilde{X}^{\ell+1}$ introduced at the beginning of the proof is independent of $\mathcal{F}_{\tau_\ell}$ under the conditional probability measure $\mathbb{P}(\cdot \mid \tau_\ell < \infty)$ and its distribution under $\mathbb{P}(\cdot \mid \tau_\ell < \infty)$ is the same as the distribution of the solution X to the uncontrolled SDE (4) with initial state $X_0 = \gamma$ under $\mathbb{P}$. In particular,

$$\mathbb{E}^{\mathbb{P}(\cdot\mid\tau_\ell<\infty)}\Big[F\big(\widetilde{X}^{\ell+1}\big)\Big] = \mathbb{E}_\gamma\big[F(X)\big]$$

for every bounded measurable functional F mapping continuous functions on $\mathbb{R}_+$ to $\mathbb{R}_+$, where we denote by $\mathbb{E}^{\mathbb{P}(\cdot|\tau_\ell<\infty)}$ expectations computed under the conditional probability measure $\mathbb{P}(\cdot \mid \tau_\ell < \infty)$. In view of these observations and the definition of conditional expectation,

$$\mathbb{E}_\gamma\Big[F\big(\widetilde{X}^{\ell+1}\big) \,\big|\, \mathcal{F}_{\tau_\ell}\Big]\mathbf{1}_{\{\tau_\ell<\infty\}} = \mathbb{E}_\gamma\big[F(X)\big]\mathbf{1}_{\{\tau_\ell<\infty\}}. \tag{58}$$

To see this claim, we first note that the Radon-Nikodym derivative of $\mathbb{P}(\cdot \mid \tau_\ell < \infty)$ with respect to $\mathbb{P}$ is given by

$$\frac{\mathrm{d}\,\mathbb{P}(\cdot \mid \tau_\ell < \infty)}{\mathrm{d}\,\mathbb{P}} = \frac{1}{\mathbb{P}(\tau_\ell < \infty)}\mathbf{1}_{\{\tau_\ell<\infty\}}.$$

Given any event $\Gamma \in \mathcal{F}_{\tau_\ell}$,

$$\begin{aligned}\frac{1}{\mathbb{P}(\tau_\ell < \infty)}\,\mathbb{E}_\gamma\Big[\mathbb{E}_\gamma\big[F(X)\big]\mathbf{1}_{\{\tau_\ell<\infty\}}\mathbf{1}_\Gamma\Big] &= \mathbb{E}_\gamma\big[F(X)\big]\frac{1}{\mathbb{P}(\tau_\ell < \infty)}\,\mathbb{E}_\gamma\big[\mathbf{1}_{\{\tau_\ell<\infty\}\cap\Gamma}\big] \\ &= \mathbb{E}^{\mathbb{P}(\cdot|\tau_\ell<\infty)}\Big[F\big(\widetilde{X}^{\ell+1}\big)\Big]\,\mathbb{E}^{\mathbb{P}(\cdot|\tau_\ell<\infty)}\big[\mathbf{1}_\Gamma\big] \\ &= \mathbb{E}^{\mathbb{P}(\cdot|\tau_\ell<\infty)}\Big[F\big(\widetilde{X}^{\ell+1}\big)\mathbf{1}_\Gamma\Big] \\ &= \frac{1}{\mathbb{P}(\tau_\ell < \infty)}\,\mathbb{E}_\gamma\Big[F\big(\widetilde{X}^{\ell+1}\big)\mathbf{1}_{\{\tau_\ell<\infty\}}\mathbf{1}_\Gamma\Big],\end{aligned}$$

and (58) follows.

In view of (52)–(58), we can see that

$$\begin{aligned}\mathbb{E}_x\big[e^{-\Lambda^\zeta_{\tau_{\ell+1}}}\big] &= \mathbb{E}_x\Bigg[e^{-\Lambda^\zeta_{\tau_\ell}}\,\mathbb{E}_\gamma\Bigg[\exp\Bigg(-\int_{\tau_\ell}^{\tau_{\ell+1}} r(X^\zeta_u)\,\mathrm{d}u\Bigg)\,\Bigg|\,\mathcal{F}_{\tau_\ell}\Bigg]\mathbf{1}_{\{\tau_\ell<\infty\}}\Bigg] \\ &= \mathbb{E}_x\Bigg[e^{-\Lambda^\zeta_{\tau_\ell}}\,\mathbb{E}_\gamma\Bigg[\exp\Bigg(-\int_{\tau_\ell}^{\tau_{\ell+1}} r(\overline{X}^{\ell+1}_u)\,\mathrm{d}u\Bigg)\,\Bigg|\,\mathcal{F}_{\tau_\ell}\Bigg]\mathbf{1}_{\{\tau_\ell<\infty\}}\Bigg] \\ &= \mathbb{E}_x\Bigg[e^{-\Lambda^\zeta_{\tau_\ell}}\,\mathbb{E}_\gamma\Bigg[\exp\Bigg(-\int_0^{\widetilde{T}^{\ell+1}_\beta} r\big(\widetilde{X}^{\ell+1}_u\big)\,\mathrm{d}u\Bigg)\,\Bigg|\,\mathcal{F}_{\tau_\ell}\Bigg]\mathbf{1}_{\{\tau_\ell<\infty\}}\Bigg] \\ &= \mathbb{E}_x\Bigg[e^{-\Lambda^\zeta_{T_\ell}}\,\mathbb{E}_\gamma\Bigg[\exp\Bigg(-\int_0^{T_\beta} r(X_u)\,\mathrm{d}u\Bigg)\Bigg]\mathbf{1}_{\{\tau_\ell<\infty\}}\Bigg] \\ &= \mathbb{E}_x\big[e^{-\Lambda^\zeta_{\tau_\ell}}\big]\,\mathbb{E}_\gamma\big[e^{-\Lambda_{T_\beta}}\big],\end{aligned}$$

where Λ is defined by (3) with X in place of X^ζ and T_β is defined as in (7). Given any $x \in \,]0, \beta[$, we iterate this result and use (15) to obtain

$$\mathbb{E}_x\big[e^{-\Lambda^\zeta_{\tau_{\ell+1}}}\big] = \mathbb{E}_x\big[e^{-\Lambda_{T_\beta}}\big]\Big(\mathbb{E}_\gamma\big[e^{-\Lambda_{T_\beta}}\big]\Big)^\ell = \frac{\psi(x)}{\psi(\beta)}\bigg(\frac{\psi(\gamma)}{\psi(\beta)}\bigg)^\ell. \tag{59}$$

It follows that

$$\mathbb{E}_x\left[\sum_{t\geq 0} e^{-\Lambda_t^\zeta}\mathbf{1}_{\{\Delta\zeta_t>0\}}\right] = \mathbb{E}_x\left[\sum_{\ell=1}^{\infty} e^{-\Lambda_{\tau_\ell}^\zeta}\right]$$
$$= \sum_{\ell=1}^{\infty}\mathbb{E}_x\big[e^{-\Lambda_{\tau_\ell}^\zeta}\big] = \frac{\psi(x)}{\psi(\beta)}\sum_{\ell=0}^{\infty}\left(\frac{\psi(\gamma)}{\psi(\beta)}\right)^\ell = \frac{\psi(x)}{\psi(\beta)-\psi(\gamma)},$$

which establishes (50).

To show (49), we consider any $x \in\,]0,\beta[$ and we use (52)–(58) as well as (59) to derive the expression

$$\mathbb{E}_x\left[\int_{\tau_\ell}^{\tau_{\ell+1}} e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t\right]$$
$$= \mathbb{E}_x\left[e^{-\Lambda_{\tau_\ell}^\zeta}\,\mathbb{E}_\gamma\left[\int_{\tau_\ell}^{\tau_{\ell+1}} \exp\left(-\int_{\tau_\ell}^{t} r\big(\overline{X}_u^{\ell+1}\big)\,\mathrm{d}u\right)h(\overline{X}_t^{\ell+1})\,\mathrm{d}t\,\Big|\,\mathcal{F}_{\tau_\ell}\right]\right]$$
$$= \mathbb{E}_x\big[e^{-\Lambda_{\tau_\ell}^\zeta}\big]\,\mathbb{E}_\gamma\left[\int_0^{T_\beta} e^{-\Lambda_t} h(X_t)\,\mathrm{d}t\right] = \mathbb{E}_x\big[e^{-\Lambda_{\tau_\ell}^\zeta}\big]\Big(R_h(\gamma) - \mathbb{E}_\gamma\big[e^{-\Lambda_{T_\beta}}\big]R_h(\beta)\Big)$$
$$= \frac{\psi(x)}{\psi(\beta)}\left(\frac{\psi(\gamma)}{\psi(\beta)}\right)^{\ell-1}\left(R_h(\gamma) - \frac{\psi(\gamma)}{\psi(\beta)}R_h(\beta)\right).$$

Similarly, we can show that

$$\mathbb{E}_x\left[\int_0^{\tau_1} e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t\right] = R_h(x) - \frac{\psi(x)}{\psi(\beta)}R_h(\beta).$$

Recalling the assumption that h is bounded from below, we can use the monotone convergence theorem and these results to obtain

$$\mathbb{E}_x\left[\int_0^{\infty} e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t\right]$$
$$= \mathbb{E}_x\left[\int_0^{\tau_1} e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t\right] + \sum_{\ell=1}^{\infty}\mathbb{E}_x\left[\int_{\tau_\ell}^{\tau_{\ell+1}} e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t\right]$$
$$= R_h(x) - \frac{\psi(x)}{\psi(\beta)}R_h(\beta) + \frac{\psi(x)}{\psi(\beta)}\left(R_h(\gamma) - \frac{\psi(\gamma)}{\psi(\beta)}R_h(\beta)\right)\sum_{\ell=1}^{\infty}\left(\frac{\psi(\gamma)}{\psi(\beta)}\right)^{\ell-1},$$

which proves (49). □

5. The solution to the control problem

We will solve the control problem we have considered by deriving a C^1 with absolutely continuous first-order derivative function $w :]0,\infty[\, \to \mathbb{R}$ that satisfies the HJB equation

$$\max\left\{\mathscr{L}w(x) + h(x),\ -c + \sup_{z\in[0,x[}\int_{x-z}^{x}\big(k(s) - w'(s)\big)\,\mathrm{d}s\right\} = 0, \tag{60}$$

Lebesgue-a.e. in $]0,\infty[$. Given such a solution, the optimal strategy can be characterised as follows. The controller should wait and take no action for as long as the state process X takes values in the interior of the set in which the ODE

$$\mathscr{L}w(x) + h(x) = 0$$

is satisfied and should take immediate action with an impulse in the negative direction if the state process takes values in the set of all points $x > 0$ such that

$$-c + \sup_{z\in[0,x[} \int_{x-z}^{x} \big(k(s) - w'(s)\big)\,\mathrm{d}s = 0.$$

We first consider the possibility for a β-γ strategy with $\gamma < \beta$ in $]0,\infty[$ to be optimal. The optimality of such a strategy is associated with a solution w to the HJB equation (60) such that

$$\mathscr{L}w(x) + h(x) = 0, \quad \text{for } x \in\,]0,\beta[, \tag{61}$$

$$\text{and} \quad w(x) = w(\gamma) + \int_{\gamma}^{x} k(s)\,\mathrm{d}s - c, \quad \text{for } x \in [\beta,\infty[. \tag{62}$$

To determine such a solution w, we first consider the so-called "principle of smooth fit", which requires that w' should be continuous, in particular, at the free-boundary point β. This condition suggests the free-boundary equation

$$\lim_{x\uparrow\beta} w'(x) = k(\beta). \tag{63}$$

Next we consider the inequality

$$-c + \sup_{z\in[0,x[} \int_{x-z}^{x} \big(k(s) - w'(s)\big)\,\mathrm{d}s \le 0,$$

which is associated with impulsive action. For $x = \beta$ and $z = \beta - u$, we can see that this implies that

$$-c + \int_{u}^{\beta} \big(k(s) - w'(s)\big)\,\mathrm{d}s \le 0 \quad \text{for all } u \in\,]0,\beta].$$

This inequality and the identity

$$-c + \int_{\gamma}^{\beta} \big(k(s) - w'(s)\big)\,\mathrm{d}s = 0, \tag{64}$$

which follows from (62), can both be true if and only if the function $u \mapsto \int_u^\beta \big(k(s) - w'(s)\big)\,\mathrm{d}s$ has a local maximum at γ. This observation gives rise to the free-boundary condition

$$w'(\gamma) = k(\gamma). \tag{65}$$

Every solution to (61) that can satisfy the so-called "transversality condition", which is required for a solution w to the HJB equation to identify with the control problem's value function, is given by

$$w(x) = R_h(x) + A\psi(x), \tag{66}$$

for some constant A, where R_h is given by (22) and (23) for $F = h$. In view of the definition (19) of Θ in Assumption 4, the expression of R_Θ as in (23) and the representation (26), we can see that

$$R_h(x) = R_\Theta(x) + \int_0^x k(s)\,\mathrm{d}s - K_\infty \psi(x), \tag{67}$$

where

$$K_\infty = \lim_{x\uparrow\infty} \frac{1}{\psi(x)} \int_0^x k(s)\,\mathrm{d}s \in \mathbb{R}_+. \tag{68}$$

Note that the limit K_∞ indeed exists in $\mathbb{R}_+$, thanks to the last condition in Assumption 4.(ii) and (25). The identity (67) implies that (66) is equivalent to

$$w(x) = R_\Theta(x) + \int_0^x k(s)\,\mathrm{d}s + (A - K_\infty)\psi(x).$$

Therefore, the solution to (61) that satisfies the boundary condition (63) is given by

$$\begin{aligned} w(x) &= \int_0^x k(s)\,\mathrm{d}s + R_\Theta(x) - \frac{R'_\Theta(\beta)}{\psi'(\beta)}\psi(x) \\ &= R_h(x) + \left(K_\infty - \frac{R'_\Theta(\beta)}{\psi'(\beta)}\right)\psi(x), \quad \text{for } x \in \,]0, \beta[. \end{aligned} \tag{69}$$

Furthermore, the boundary conditions (65) and (64) are equivalent to

$$\frac{R'_\Theta(\gamma)}{\psi'(\gamma)} = \frac{R'_\Theta(\beta)}{\psi'(\beta)} \quad \text{and} \quad F(\gamma, \beta) = -c, \tag{70}$$

respectively, where

$$F(\gamma, \beta) := G_\Theta(\beta) - G_\Theta(\gamma) = \int_\gamma^\beta \left(\frac{R'_\Theta(s)}{\psi'(s)} - \frac{R'_\Theta(\beta)}{\psi'(\beta)}\right)\psi'(s)\,\mathrm{d}s, \tag{71}$$

and G_Θ is defined by (28) in Lemma 2.

The following result is about the solvability of the system of equations given by (70) for the unknowns γ and β. Note that Lemma 4.(i) implies that a pair $0 \le \gamma < \beta < \infty$ satisfying the first equation in (70) might exist only if $\underline{x} < \infty$.

Lemma 6. *Consider the stochastic control problem formulated in Section 2 and suppose that the point* $\underline{x}$ *introduced in Lemma 4.(i) is finite. There exist a unique strictly decreasing function* $\gamma^\star : \,]0, c^\star[\, \to \,]0, \underline{x}[$ *and* *a unique strictly increasing function* $\beta^\star :]0, c^\star[\to]\underline{x}, \overline{x}[$*, where* $c^\star > 0$ *is defined by (81) in the proof below and* $\underline{x}$*,* $\overline{x}$ *are as in Lemma 4, such that*

$$\frac{R'_\Theta(x)}{\psi'(x)} - \frac{R'_\Theta\big(\beta^\star(c)\big)}{\psi'\big(\beta^\star(c)\big)} \begin{cases} > 0, & \text{if } x \in \big]0, \gamma^\star(c)\big[, \\ = 0, & \text{if } x = \gamma^\star(c), \\ < 0, & \text{if } x \in \big]\gamma^\star(c), \beta^\star(c)\big[, \end{cases} \quad \text{and} \quad F\big(\gamma^\star(c), \beta^\star(c)\big) = -c \tag{72}$$

for all $c \in]0, c^\star[$*. There exist no other points* $0 < \gamma < \beta < \infty$ *satisfying the system of equations (70). The functions* $\beta^\star$ *and* $\gamma^\star$ *are such that*

$$\lim_{c\downarrow 0} \beta^\star(c) = \lim_{c\downarrow 0} \gamma^\star(c) = \underline{x}, \tag{73}$$

$$\lim_{c\uparrow c^\star} \beta^\star(c) = \overline{x} \quad \text{and} \quad \lim_{c\uparrow c^\star} \gamma^\star(c) \begin{cases} > 0, & \text{if } \lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} > 0 \ (\overline{x} = \infty), \\ = 0, & \text{if } \lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = 0 \ (\overline{x} = \infty), \\ = 0, & \text{if } \lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} < 0 \ (\overline{x} < \infty). \end{cases} \tag{74}$$

Furthermore, $c^\star < \infty$ *if and only if*

$$\text{either} \quad \text{(I) } \overline{x} < \infty \quad \text{or} \quad \text{(II) } \overline{x} = \infty \text{ and } \lim_{x\uparrow\infty} \frac{\Theta(x)}{r(x)} > -\infty. \tag{75}$$

Proof. In view of (40) and (43) in Lemma 4, we can see that there exists a point $\gamma \in]0, \beta[$ such that the first equation in (70) holds true if and only if $\beta \in]\underline{x}, \overline{x}[$, in which case, $\gamma \in]0, \underline{x}[$. In particular, there exists a unique strictly decreasing function $\Gamma :]\underline{x}, \overline{x}[\to]0, \underline{x}[$ such that

$$\frac{R'_\Theta(x)}{\psi'(x)} - \frac{R'_\Theta(\beta)}{\psi'(\beta)} \begin{cases} > 0, & \text{if } x \in \big]0, \Gamma(\beta)\big[, \\ = 0, & \text{if } x = \Gamma(\beta), \\ < 0, & \text{if } x \in \big]\Gamma(\beta), \beta\big[, \end{cases} \tag{76}$$

$$\left(\frac{R'_\Theta}{\psi'}\right)'\big(\Gamma(\beta)\big) < 0, \qquad \left(\frac{R'_\Theta}{\psi'}\right)'(\beta) > 0, \tag{77}$$

$$\lim_{\beta\downarrow \underline{x}} \Gamma(\beta) = \underline{x} \quad \text{and} \quad \lim_{\beta\uparrow \overline{x}} \Gamma(\beta) \begin{cases} > 0, & \text{if } \lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} > 0 \ (\overline{x} = \infty), \\ = 0, & \text{if } \lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = 0 \ (\overline{x} = \infty), \\ = 0, & \text{if } \lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} < 0 \ (\overline{x} < \infty). \end{cases} \tag{78}$$

It follows that the system of equations (70) has a unique solution $\gamma < \beta$ if and only if the equation

$$F\big(\Gamma(\beta), \beta\big) = -c \tag{79}$$

has a unique solution $\beta^\star(c) \in]\underline{x}, \overline{x}[$. Using the first expression in (71), the identity in (76), the second of the inequalities in (77) and the fact that ψ is strictly increasing, we calculate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\beta} F\big(\Gamma(\beta), \beta\big) &= -\left(\frac{R'_\Theta}{\psi'}\right)'(\beta)\psi(\beta) + \left(\frac{R'_\Theta}{\psi'}\right)'\big(\Gamma(\beta)\big)\psi\big(\Gamma(\beta)\big)\Gamma'(\beta) \\ &= -\left(\frac{R'_\Theta}{\psi'}\right)'(\beta)\psi(\beta)\Big(\psi(\beta) - \psi\big(\Gamma(\beta)\big)\Big) < 0. \end{aligned}$$

Combining this result with the fact that

$$\lim_{\beta\downarrow\underline{x}} F\big(\Gamma(\beta),\beta\big) = 0, \tag{80}$$

which follows from the first limit in (78), we can see that the equation $F\big(\Gamma(\beta),\beta\big) = -c$ has a unique solution $\beta^\star(c) \in \,]\underline{x},\overline{x}[$ if and only if

$$c < -\lim_{\beta\uparrow\overline{x}} F\big(\Gamma(\beta),\beta\big) =: c^\star. \tag{81}$$

We conclude this part of the analysis by noting that the points $\beta^\star(c) \in \,]\underline{x},\overline{x}[$ and $\gamma^\star(c) := \Gamma\big(\beta^\star(c)\big) \in \,]0,\underline{x}[$ provide the unique solution to the system of equations (70) if $c \in \,]0,c^\star[$, while the system of equations (70) has no solution such that $0 < \gamma < \beta < \infty$ if $c \geq c^\star$. In particular, the inequalities in (72) follow from the corresponding ones in (76).

The fact that the function $\beta \mapsto F\big(\Gamma(\beta),\beta\big)$ is strictly decreasing, which we have established above, implies that the function $c \mapsto \beta^\star(c)$ is strictly increasing because $\beta^\star(c)$ is the unique solution to equation (79) for each $c \in \,]0,c^\star[$. In turn, this result and the fact that Γ is strictly decreasing imply that the function $\gamma^\star = \Gamma \circ \beta^\star$ is strictly decreasing. The first limit in (74) follows immediately from (81). On the other hand, the second limit in (74) follows immediately from the first limit in (74) and the second limit in (78). Furthermore, the identities in (73) follow from the first limit in (78) and (80).

To establish the equivalence of the inequality $c^\star < \infty$ with the condition in (75), we first use the first expression of F in (71) and the definition (81) of $c^\star$ to observe that

$$c^\star = -\lim_{\beta\uparrow\overline{x}} \Big(G_\Theta(\beta) - G_\Theta\big(\Gamma(\beta)\big)\Big).$$

We next use the second limit in (78) as well as Lemmas 2 and 4. If $\overline{x} < \infty$, then

$$c^\star = -G_\Theta(\overline{x}) + \lim_{x\downarrow 0} G_\Theta(x) \overset{(29)}{=} -G(\overline{x}) + \lim_{x\downarrow 0}\frac{\Theta(x)}{r(x)} < \infty,$$

the inequality following because Θ/r is strictly increasing in $]0,\xi[$. If $\overline{x} = \infty$ and $\lim_{x\downarrow 0} R'_\Theta(x)/\psi'(x) = 0$, then $\lim_{x\downarrow 0}\Theta(x)/r(x) > -\infty$ thanks to the first implication in (45). In this case,

$$c^\star \overset{(30)}{=} -\lim_{x\uparrow\infty}\frac{\Theta(x)}{r(x)} + \lim_{x\downarrow 0}\frac{\Theta(x)}{r(x)} \begin{cases} < \infty, & \text{if } \lim_{x\uparrow\infty}\frac{\Theta(x)}{r(x)} > -\infty, \\ = \infty, & \text{if } \lim_{x\uparrow\infty}\frac{\Theta(x)}{r(x)} = -\infty, \end{cases}$$

where we have also used the assumption that Θ/r is strictly decreasing in $]\xi,\infty[$. Finally, if $\overline{x} = \infty$ and $\lim_{x\downarrow 0}\frac{R'_\Theta(x)}{\psi'(x)} > 0$, then $\lim_{x\uparrow\infty}\Gamma(x) > 0$ (see (78)),

$$c^\star \overset{(30)}{=} -\lim_{x\uparrow\infty}\frac{\Theta(x)}{r(x)} + \lim_{x\uparrow\infty} G\big(\Gamma(x)\big) \begin{cases} < \infty, & \text{if } \lim_{x\uparrow\infty}\frac{\Theta(x)}{r(x)} > -\infty, \\ = \infty, & \text{if } \lim_{x\uparrow\infty}\frac{\Theta(x)}{r(x)} = -\infty, \end{cases}$$

and the proof is complete. □

In light of (62), (69) and the previous lemma, we now establish the following result, which provides the solution to the HJB equation (60) identifying with the control problem's value function when a β-γ strategy with $\gamma < \beta$ in $]0,\infty[$ is indeed optimal.

Lemma 7. *Consider the stochastic control problem formulated in Section 2 and suppose that the point $\underline{x}$ introduced in Lemma 4.(i) is finite. Also, fix any $c \in \,]0,c^\star[$, where $c^\star > 0$ is as in Lemma 6. The function w defined by*

$$w(x) = \begin{cases} R_h(x) + \Big(K_\infty - \frac{R'_\Theta(\beta^\star)}{\psi'(\beta^\star)}\Big)\psi(x), & \text{for } x \in \,]0, \beta^\star[, \\ w(\gamma^\star) + \int_{\gamma^\star}^x k(s)\,\mathrm{d}s - c, & \text{for } x \in [\beta^\star, \infty[, \end{cases} \tag{82}$$

where we write $\gamma^\star$ and $\beta^\star$ in place of the points $\gamma^\star(c)$ and $\beta^\star(c)$ given by Lemma 6, is C^1 in $]0,\infty[$ and C^2 in $]0,\infty[\setminus\{\beta^\star\}$. Furthermore, this function is a solution to the HJB equation (60) that is bounded from below.

Proof. The boundedness from below of w follows immediately from Assumption 3, the conditions in (i) and (ii) of Assumption 4 and Lemma 1.(i).

By construction, we will establish all of the lemma's other claims if we prove that

$$-c + \int_u^x \big(k(s) - w'(s)\big)\,\mathrm{d}s \le 0 \quad \text{for all } 0 < u < x < \beta^\star \tag{83}$$

$$\text{and} \quad \mathscr{L}w(x) + h(x) \le 0 \quad \text{for all } x > \beta^\star. \tag{84}$$

To this end, we use the first expression of w in (69) and (72) to note that

$$k(s) - w'(s) = \psi'(s)\left(\frac{R'_\Theta(\beta^\star)}{\psi'(\beta^\star)} - \frac{R'_\Theta(s)}{\psi'(s)}\right) \begin{cases} < 0 & \text{for all } s \in \,]0, \gamma^\star[, \\ > 0 & \text{for all } s \in \,]\gamma^\star, \beta^\star[. \end{cases}$$

The inequality (83) follows from this observation and the fact that

$$-c + \int_{\gamma^\star}^{\beta^\star} \big(k(s) - w'(s)\big)\,\mathrm{d}s = 0.$$

To show (84), we first use the expression

$$w(x) = w(\beta^\star) + \int_{\beta^\star}^x k(s)\,\mathrm{d}s, \quad \text{for } x > \beta^\star,$$

the definition (19) of Θ in Assumption 4 and the first expression in (69) to calculate

$$\begin{aligned} \mathscr{L}w(x) + h(x) &= -r(x)w(\beta^\star) + \mathscr{L}\Big(\int_0^{\cdot} k(s)\,\mathrm{d}s\Big)(x) + r(x)\int_0^{\beta^\star} k(s)\,\mathrm{d}s + h(x) \\ &= \Theta(x) - r(x)\left(R_\Theta(\beta^\star) - \frac{R'_\Theta(\beta^\star)}{\psi'(\beta^\star)}\psi(\beta^\star)\right) \\ &= \Theta(x) - r(x)G_\Theta(\beta^\star), \end{aligned} \tag{85}$$

where G_Θ is given by (28) in Lemma 2 for $F = \Theta$. In view of the calculations

$$G'_\Theta(x) = -\psi(x)\frac{\mathrm{d}}{\mathrm{d}x}\frac{R'_\Theta(x)}{\psi'(x)} = -\frac{2Cr(x)p'(x)\psi(x)}{(\sigma(x)\psi'(x))^2}\left(\int_0^x \Theta(s)\Psi(s)\,\mathrm{d}s - \frac{\Theta(x)}{r(x)}\frac{\psi'(x)}{Cp'(x)}\right),$$

the inequalities (40) in Lemma 4 and the fact that $\beta^\star > \underline{x}$, we can see that

$$G_\Theta(x) < G_\Theta(\beta^\star) \quad \text{and} \quad \int_0^x \Theta(s)\Psi(s)\,\mathrm{d}s > \frac{\Theta(x)}{r(x)}\frac{\psi'(x)}{Cp'(x)} \quad \text{for all } x > \beta^\star. \tag{86}$$

The second of these inequalities and the second expression of G_Θ in (28) imply that

$$G_\Theta(x) = \frac{Cp'(x)}{\psi'(x)} \int_0^x \Theta(s)\Psi(s)\,\mathrm{d}s > \frac{\Theta(x)}{r(x)} \quad \text{for all } x > \beta^\star.$$

However, this result, (85) and the first inequality in (86) yield

$$\mathscr{L}w(x) + h(x) < r(x)\left(\frac{\Theta(x)}{r(x)} - G_\Theta(x)\right) < 0 \quad \text{for all } x > \beta^\star,$$

and (84) follows. □

To proceed further, we assume that the problem data is such that $c^\star < \infty$, which is the case if and only if one of the two conditions of (75) in Lemma 6 holds true. In the first case, when $\overline{x} < \infty$, the limits in (74) suggest the possibility for the function w defined by (62) and (69) for $\gamma = 0$ and some $\beta > \overline{x}$ to provide a solution to the HJB equation (60) that identifies with the control problem's value function. In this case, a free-boundary condition such as (65) is not relevant anymore and we are faced with only the free-boundary condition (64) with $\gamma = 0$, which is equivalent to the equation $F(0, \beta) = -c$, where F is defined by (71).

Lemma 8. *Consider the stochastic control problem formulated in Section 2 and suppose that the point $\underline{x}$ introduced in Lemma 4.(i) is finite. Also, suppose that the problem data is such that $\overline{x} < \infty$, where $\overline{x}$ is defined by (43) in Lemma 4. The following statements hold true:*

(I) *There exists $c^\circ \in \,]c^\star, \infty]$ and a strictly increasing function $\beta^\circ : [c^\star, c^\circ[\, \to [\overline{x}, \infty[$ such that*

$$F\big(0, \beta^\circ(c)\big) = -c \text{ for all } c \in [c^\star, c^\circ[\quad \text{and} \quad \lim_{c \uparrow c^\circ} \beta^\circ(c) = \infty, \tag{87}$$

where $c^\star \in \,]0, \infty[$ is as in Lemma 6.

(II) *$c^\circ = \infty$ if and only if $\lim_{x\uparrow\infty} \Theta(x)/r(x) = -\infty$.*

(III) *Given any $c \in [c^\star, c^\circ[$, the function w defined by*

$$w(x) = \begin{cases} R_h(x) + \left(K_\infty - \frac{R'_\Theta(\beta^\circ)}{\psi'(\beta^\circ)}\right)\psi(x), & \text{for } x \in \,]0, \beta^\circ[, \\ R_h(0) + \left(K_\infty - \frac{R'_\Theta(\beta^\circ)}{\psi'(\beta^\circ)}\right)\psi(0) + \int_0^x k(s)\,\mathrm{d}s - c, & \text{for } x \in [\beta^\circ, \infty[, \end{cases} \tag{88}$$

where we write β° in place of $\beta^\circ(c)$, is C^1 in $]0, \infty[$ and C^2 in $]0, \infty[\, \setminus \{\beta^\circ\}$. Furthermore, this function is a solution to the HJB equation (60) that is bounded from below.

Proof. The definition of G_Θ as in (28), the limits (29) in Lemma 2 and the implications (45) in Lemma 4 imply that the limit $\lim_{x\downarrow 0} G_\Theta(x)$ exists in $\mathbb{R}$ thanks to Assumption 4.(iii). On the other hand, (40) and (44) in Lemma 4 imply that the limit $\lim_{x\downarrow 0} R'_\Theta(x)/\psi'(x)$ exists in $]-\infty, 0[$. In view of these observations and the definition of G_Θ as in (28), we can see that the limit $\lim_{x\downarrow 0} R_\Theta(x)$ exists in $\mathbb{R}$. Therefore, the limit $R_h(0) := \lim_{x\downarrow 0} R_h(x)$ exists in $\mathbb{R}$ thanks to (67). It follows that the function w is well-defined.

The second expression in (72) and the limits in (74) imply that

$$F(0, \overline{x}) \equiv G_\Theta(\overline{x}) - \lim_{x\downarrow 0} G_\Theta(x) = -c^\star \in \,]-\infty, 0[.$$

Part (I) of the lemma follows from this observation and the calculation

$$\frac{\mathrm{d}}{\mathrm{d}\beta}F(0,\beta) = -\left(\frac{R'_\Theta}{\psi'}\right)'(\beta)\big(\psi(\beta)-\psi(0)\big) < 0 \quad \text{for all } \beta \geq \overline{x},$$

where the inequality follows from (40) in Lemma 4 and the fact that the strictly positive function ψ is strictly increasing, for $c^\circ = -\lim_{\beta\uparrow\infty} F(0,\beta)$. Furthermore, this definition of c°, Assumption 4.(iii) and the limits (30) in Lemma 2 imply immediately part (II) of the lemma.

Finally, we can show the rest of the claims on w by using exactly the same arguments as in the proof of Lemma 7 (see (83) and (84) in particular). □

To close the "gap" in the parameter space, we still need to derive a solution to the HJB equation (60) if

$$\underline{x} < \overline{x} = \infty,\ c^\star < \infty \text{ and } c \geq c^\star, \quad \text{or} \quad \overline{x} < \infty,\ c^\circ < \infty \text{ and } c \geq c^\circ, \quad \text{or} \quad \underline{x} = \infty \text{ and } c > 0.$$

In the first case, the first limit in (74) implies that $\lim_{c\uparrow c^\star} \beta^\star(c) = \infty$. In the second case, the limit in (87) implies that $\lim_{c\uparrow c^\circ} \beta^\circ(c) = \infty$. In all cases, we are faced with the possibility for the problem's value function to identify with a solution to the ODE $\mathscr{L}w(x) + h(x) = 0$ for all $x > 0$.

Lemma 9. *Consider the stochastic control problem formulated in Section 2 and suppose that the problem data is such that one of the following cases holds true:*

(a) *The point $\underline{x}$ introduced in Lemma 4.(i) is finite,*

$$\lim_{x\uparrow\infty} \frac{\Theta(x)}{r(x)} > -\infty \quad \Leftrightarrow \quad \textit{either } (\overline{x} = \infty \textit{ and } c^\star < \infty) \textit{ or } (\overline{x} < \infty \textit{ and } c^\circ < \infty) \tag{89}$$

and $c \geq c^\star$ or $c \geq c^\circ$, depending on the case in (89).

(b) *The point $\underline{x}$ introduced in Lemma 4.(i) is equal to infinity.*

In either of these two cases, the function w defined by

$$w(x) = R_h(x) + K_\infty \psi(x), \quad \textit{for } x > 0, \tag{90}$$

is a C^2 solution to the HJB equation (60) that is bounded from below.

Proof. The equivalence (89) follows immediately from the statement related to (75) in Lemma 6 and part (II) of Lemma 8. On the other hand, the boundedness from below of w follows immediately from Assumption 3, the conditions in (i) and (ii) of Assumption 4 and Lemma 1.(i).

To establish the fact that w satisfies the HJB equation (60), we have to show that

$$\int_u^x \big(k(s) - w'(s)\big)\,\mathrm{d}s \leq c \quad \Leftrightarrow \quad R_\Theta(u) - R_\Theta(x) \leq c \quad \text{for all } 0 < u < x < \infty, \tag{91}$$

where the equivalence follows from the identity (67) and the definition (90) of w. To this end, fix any $u < x$ in $]0,\infty[$. First, suppose that $\overline{x} = \infty$ and $c^\star < \infty$. In this case, the limits in (74) imply that $x < \beta^\star(c)$ for all $c < c^\star$ sufficiently close to $c^\star$. For such a c, the identity (67) and the fact that the function w defined by (82) in Lemma 7 satisfies the HJB equation (60) imply that

$$R_\Theta(u) - R_\Theta(x) \leq c + \frac{R'_\Theta\big(\beta^\star(c)\big)}{\psi'\big(\beta^\star(c)\big)}\big(\psi(u) - \psi(x)\big).$$

Passing to the limit as $c \uparrow c^\star$ and using the fact that $\lim_{c\uparrow c^\star} \beta^\star(c) = \infty$ together with the limit in (40), we can see that $R_\Theta(u) - R_\Theta(x) \le c^\star$. It follows that (91) holds true for all $c \ge c^\star$.

If $\overline{x} < \infty$, $c^\circ < \infty$ and $c \ge c^\circ$, then we can show that the function w given by (90) satisfies the HJB equation (60) in exactly the same way using the results of Lemma 8.

Finally, suppose that the point $\underline{x}$ introduced in Lemma 4.(i) is equal to infinity and consider any points $u < x < \beta$ in $]0, \infty[$. In this case, the inequalities in (40) imply that

$$R'_\Theta(s) - \frac{R'_\Theta(\beta)}{\psi'(\beta)}\psi'(s) = \psi'(s)\left(\frac{R'_\Theta(s)}{\psi'(s)} - \frac{R'_\Theta(\beta)}{\psi'(\beta)}\right) > 0 \quad \text{for all } s < \beta.$$

In view of this observation, we can see that

$$R_\Theta(u) - R_\Theta(x) \le -\frac{R'_\Theta(\beta)}{\psi'(\beta)}\big(\psi(x) - \psi(u)\big).$$

Passing to the limit as $\beta \to \infty$, we can see that $R_\Theta(u) - R_\Theta(x) \le 0$, thanks to the limit in (40). It follows that (91) holds true for all $c > 0$. □

We conclude the section with the main result of the paper.

Theorem 10. *Consider the stochastic control problem formulated in Section 2. Depending on the problem data, the function w defined by (82), (88) or (90) in Lemmas 7, 8 or 9, respectively, identifies with the control problem's value function, namely,*

$$w(x) = \sup_{\zeta\in\mathcal{A}} J_x(\zeta). \tag{92}$$

Furthermore, the following cases hold true:

(I) *If the problem data is as in Lemma 7, then the β-γ strategy characterised by the points $\beta^\star$ and $\gamma^\star$ in Lemma 7 is optimal.*

(II) *If the problem data is as in Lemma 8, then there exists no optimal strategy. In this case, if (ε_n) is any sequence such that $\varepsilon_1 < \beta^\circ$ and $\lim_{n\uparrow\infty} \varepsilon_n = 0$, then the β-γ strategies characterised by the points $\beta = \beta^\circ$ and $\gamma = \varepsilon_n$, where β° is as in Lemma 8, provide a sequence of ε-optimal strategies.*

(III) *If the problem data is as in Lemma 9 and $K_\infty = 0$, then $\zeta^\star = 0$ is an optimal strategy.*

(IV) *If the problem data is as in Lemma 9 and $K_\infty > 0$, then there exists no optimal strategy. In this case, if γ is an arbitrary point in $]0, \infty[$ and (ε_n) is any sequence such that $\varepsilon_1^{-1} > \gamma$ and $\lim_{n\uparrow\infty} \varepsilon_n^{-1} = \infty$, then the β-γ strategies characterised by the points $\beta = \varepsilon_n^{-1}$ and γ provide a sequence of ε-optimal strategies.*

Proof. Fix any initial value $x > 0$, consider any admissible controlled process $\zeta \in \mathcal{A}$ and denote by X^ζ the associated solution to the SDE (1). Using Itô's formula, we obtain

$$e^{-\Lambda^\zeta_T} w(X^\zeta_T) = w(x) + \int_0^T e^{-\Lambda^\zeta_t} \mathscr{L} w(X^\zeta_t)\,\mathrm{d}t + \sum_{0\le t\le T} e^{-\Lambda^\zeta_t}\big(w(X^\zeta_t) - w(X^\zeta_{t-})\big)\mathbf{1}_{\{\Delta\zeta_t > 0\}} + M^\zeta_T,$$

where

$$M^\zeta_T = \int_0^T e^{-\Lambda^\zeta_t} \sigma(X^\zeta_t) w'(X^\zeta_t)\,\mathrm{d}W_t.$$

Since $\Delta X_t^\zeta \equiv X_t^\zeta - X_{t-}^\zeta = -\Delta\zeta_t \le 0$, we can see that

$$w(X_t^\zeta) - w(X_{t-}^\zeta) + \int_0^{\Delta\zeta_t} k(X_{t-}^\zeta - u)\,\mathrm{d}u - c\mathbf{1}_{\{\Delta\zeta_t>0\}}$$
$$= \left(\int_{X_{t-}^\zeta-\Delta\zeta_t}^{X_{t-}^\zeta} \big(k(u) - w'(u)\big)\,\mathrm{d}u - c\right)\mathbf{1}_{\{\Delta\zeta_t>0\}}.$$

In view of these observations and the fact that w satisfies the HJB equation (60), we derive

$$\begin{aligned}
&\int_0^T e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t + \sum_{t\in[0,T]} e^{-\Lambda_t^\zeta}\left(\int_0^{\Delta\zeta_t} k(X_{t-}^\zeta - u)\,\mathrm{d}u - c\mathbf{1}_{\{\Delta\zeta_t>0\}}\right) \\
&= w(x) - e^{-\Lambda_T^\zeta} w(X_T^\zeta) + \int_0^T e^{-\Lambda_t^\zeta}\Big(\mathscr{L}w(X_t^\zeta) + h(X_t^\zeta)\Big)\,\mathrm{d}t \\
&\quad + \sum_{0\le t\le T}\left(e^{-\Lambda_t^\zeta}\int_{X_{t-}^\zeta-\Delta\zeta_t}^{X_{t-}^\zeta} \big(k(u) - w'(u)\big)\,\mathrm{d}u - c\right)\mathbf{1}_{\{\Delta\zeta_t>0\}} + M_T^\zeta \\
&\le w(x) - e^{-\Lambda_T^\zeta} w(X_T^\zeta) + M_T^\zeta.
\end{aligned} \tag{93}$$

We next consider any sequence (τ_n) of bounded localising times for the local martingale M^ζ. Recalling Assumption 4.(ii) as well as the fact that h and w are both bounded from below, we use Fatou's lemma, the monotone convergence theorem and the admissibility condition (8) to observe that (93) implies that

$$\begin{aligned}
J_x(\zeta) &\le \liminf_{n\uparrow\infty} \mathbb{E}_x\left[\int_0^{\tau_n} e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t + \sum_{t\in[0,\tau_n]} e^{-\Lambda_t^\zeta}\left(\int_0^{\Delta\zeta_t} k(X_{t-}^\zeta - u)\,\mathrm{d}u - c\mathbf{1}_{\{\Delta\zeta_t>0\}}\right)\right] \\
&\le \lim_{n\uparrow\infty} \mathbb{E}_x\left[w(x) + e^{-\Lambda_{\tau_n}^\zeta} w^-(X_{\tau_n}^\zeta)\right] = w(x),
\end{aligned} \tag{94}$$

where $w^-(x) = -\min\{0, w(x)\}$.

Proof of (I). First, consider any $x \in \,]0,\beta[$. In view of the results in Theorem 5, the β-γ strategy $\zeta^\star$ characterised by the points $\beta^\star$ and $\gamma^\star$ is such that

$$J_x(\zeta^\star) = R_h(x) + \frac{\psi(x)}{\psi(\beta^\star) - \psi(\gamma^\star)}\left(R_h(\gamma^\star) - R_h(\beta^\star) + \int_{\gamma^\star}^{\beta^\star} k(s)\,\mathrm{d}s - c\right). \tag{95}$$

On the other hand, the identity $F(\gamma^\star, \beta^\star) = -c$ and the definition (71) of F imply that

$$\frac{R'_\Theta(\beta^\star)}{\psi'(\beta^\star)} = \frac{R_\Theta(\beta^\star) - R_\Theta(\gamma^\star) + c}{\psi(\beta^\star) - \psi(\gamma^\star)}.$$

In view of the identity (67), this expression is equivalent to

$$\frac{R'_\Theta(\beta^\star)}{\psi'(\beta^\star)} = \frac{1}{\psi(\beta^\star) - \psi(\gamma^\star)}\Biggl(R_h(\beta^\star) - R_h(\gamma^\star) - \int_{\gamma^\star}^{\beta^\star} k(s)\,\mathrm{d}s + c + K_\infty\bigl(\psi(\beta^\star) - \psi(\gamma^\star)\bigr)\Biggr).$$

However, this result, the definition (82) of w and (95) imply that $J_x(\zeta^\star) = w(x)$, which, combined with (94), establishes (92) as well as the optimality of $\zeta^\star$. The corresponding claims for $x \geq \beta$ are immediate.

Proof of (II). In this case, the identity $F(0, \beta^\circ) = -c$ implies that the sequence (c_n) defined by $c_n = -F(\varepsilon_n, \beta^\circ)$ is such that $\lim_{n\uparrow\infty} c_n = c$. By following reasoning similar to the one in the previous part of the proof, we can see that, given any $x \in \,]0, \beta[$, the β-γ strategy ζ^{ε_n} characterised by the points $\beta = \beta^\circ$ and $\gamma = \varepsilon_n$ is such that

$$J_x(\zeta^{\varepsilon_n}) = w(x) - \frac{(c - c_n)\psi(x)}{\psi(\beta^\circ) - \psi(\varepsilon_n)},$$

and the required results follow.

Proof of (III). This case follows immediately from (94) and the probabilistic expression of R_h as in (22).

Proof of (IV). In view of the results in Theorem 5, the β-γ strategy ζ^{ε_n} characterised by the points $\beta = \varepsilon_n^{-1}$ and γ is such that

$$J_x(\zeta^{\varepsilon_n}) = R_h(x) + \frac{\psi(x)}{\psi(\varepsilon_n^{-1}) - \psi(\gamma)}\Biggl(R_h(\gamma) - R_h(\varepsilon_n^{-1}) + \int_\gamma^{\varepsilon_n^{-1}} k(s)\,\mathrm{d}s - c\Biggr).$$

Combining this observation with the second limit in (24) and the definition (68) of K_∞, we can see that $\lim_{n\uparrow\infty} J_x(\zeta^{\varepsilon_n}) = R_h(x) + K_\infty\psi(x)$. However, this limit and (94) imply the required results. □

Remark 1. Suppose that we enlarged the family of admissible strategies to allow for switching the system off. In particular, suppose that we allowed for the controlled process X^ζ to hit 0 at some time and be absorbed by 0 after that time. In this context, we would face the HJB equation

$$\max\Biggl\{\mathscr{L}w(x) + h(x),\; -c + \sup_{z\in[0,x[} \int_{x-z}^{x} \bigl(k(s) - w'(s)\bigr)\,\mathrm{d}s,$$
$$-w(0) - c + \frac{h(0)}{r(0)} + \int_0^x \bigl(k(s) - w'(s)\bigr)\,\mathrm{d}s\Biggr\} = 0, \tag{96}$$

where we assume that both of the limits $h(0) := \lim_{x\downarrow 0} h(x)$ and $r(0) := \lim_{x\downarrow 0} r(x)$ exist in $\mathbb{R}$, instead of just the limit $\lim_{x\downarrow 0} h(x)/r(x)$. The third term of this HJB equation incorporates the inequality

$$w(x) \geq -c + \int_0^x k(s)\,\mathrm{d}s + \int_0^\infty e^{-r(0)s} h(0)\,\mathrm{d}s$$

that should hold with equality for those values x of the state space at which it is optimal to switch off the system.

In view of the second limit in (13) and Lemma 1.(ii), if 0 is a natural boundary point, then, in all of the cases appearing in Lemmas 7-9,

$$w(0) = R_h(0) = \frac{h(0)}{r(0)}$$

and the inequality associated with the third term of (96) follows from the one associated with the second term of (96). In view of this observation, we can see that the results of Theorem 10 hold true with the following modification: in Case II, the β-0 strategy that switches off the system as soon as the uncontrolled process X takes any value greater than or equal to $\beta = \beta^\circ$ is optimal. In Case IV of the theorem, an optimal strategy still does not exist.

The situation is entirely different if 0 is an entrance boundary point. In this case, Theorem 10 with a modification such as the one in the previous paragraph still provides the solution to the control problem if the problem data is such that the solution w to the HJB equation (60) satisfies the inequality $w(0) \geq h(0)/r(0)$. In Example 8 in the next section, we can see that this inequality may or may not be true. In particular, a β-0 strategy that may switch the system off can indeed be optimal and be associated with a payoff that is strictly greater than the value function derived in Theorem 10. Investigating the solution to the control problem if we allowed for the system to be switched off would require substantial extra analysis that goes beyond the scope of the present article. □

6. Examples

The first four examples that we consider in this section present choices for the problem data that satisfy our assumptions. In these examples, the functions r and k are strictly positive constants, so the function Θ introduced in Assumption 4 takes the form

$$\Theta(x) = h(x) + kb(x) - rkx.$$

Furthermore, $\lim_{x\uparrow\infty} \Theta(x) = -\infty$ in each of the Examples 1-4, which implies that $\underline{x} < \infty$ thanks to Lemma 4.(ii), where $\underline{x} \in]\xi, \infty]$ is as in Lemma 4.(i).

Example 1. Suppose that the uncontrolled dynamics of the state process are modelled by the SDE

$$\mathrm{d}X_t = bX_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}W_t, \quad X_0 = x > 0, \tag{97}$$

for some constants b and $\sigma > 0$. Furthermore, if $r > b$ and h is any strictly concave function such that

$$\lim_{x\downarrow 0} h'(x) > k(r - b) \quad \text{and} \quad \lim_{x\uparrow\infty} h'(x) = 0,$$

then Θ is strictly concave and satisfies the requirements of Assumption 4.

Example 2. Suppose that the uncontrolled dynamics of the state process are modelled by the SDE

$$\mathrm{d}X_t = \kappa(\gamma - X_t)X_t\,\mathrm{d}t + \sigma X_t^\ell\,\mathrm{d}W_t, \quad X_0 = x > 0,$$

for some strictly positive constants κ, γ, σ and $\ell \in [1, \frac{3}{2}]$. Note that the celebrated stochastic Verhulst-Pearl logistic model of population growth arises in the special case $\ell = 1$. Assumptions 1-3 hold true if $\ell \in]1, \frac{3}{2}]$ or if $\ell = 1$ and $k\gamma - \frac{1}{2}\sigma^2 > 0$. Furthermore, if h is any bounded from below concave function such that

$$\lim_{x\downarrow 0} h'(x) > k(r - \kappa\gamma),$$

then Θ is strictly concave and satisfies the requirements of Assumption 4. □

Example 3. Suppose that the uncontrolled dynamics of the state process are modelled by the SDE

$$\mathrm{d}X_t = \left(\kappa\gamma + \frac{1}{2}\sigma^2 - \kappa \ln(X_t)\right) X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}W_t, \quad X_0 = x > 0,$$

for some constants $\kappa, \gamma, \sigma > 0$, namely, the logarithm of the uncontrolled state process is the Ornstein-Uhlenbeck process given by

$$\mathrm{d}\ln(X_t) = \kappa\big(\gamma - \ln(X_t)\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad \ln(X_0) = \ln(x) \in \mathbb{R}.$$

Furthermore, if h is any bounded from below concave function, then Θ is strictly concave and satisfies the requirements of Assumption 4. □

Example 4. Suppose that the uncontrolled dynamics of the state process are modelled by the SDE

$$\mathrm{d}X_t = \kappa(\gamma - X_t)\,\mathrm{d}t + \sigma X_t^{\ell}\,\mathrm{d}W_t, \quad X_0 = x > 0,$$

for some strictly positive constants κ, γ, σ and $\ell \in [\frac{1}{2}, 1]$. Note that, in the special case that arises for $\ell = \frac{1}{2}$ and $\kappa\gamma - \frac{1}{2}\sigma^2 > 0$, the process X identifies with the short rate process in the Cox-Ingersoll-Ross interest rate model. Assumptions 1-3 hold true if $\ell \in]\frac{1}{2}, 1]$ or if $\ell = \frac{1}{2}$ and $k\gamma - \frac{1}{2}\sigma^2 > 0$. Furthermore, if h is any strictly concave function such that

$$\lim_{x\downarrow 0} h'(x) > k(r+\kappa) \quad \text{and} \quad \lim_{x\uparrow\infty} h'(x) = 0,$$

then Θ is strictly concave and satisfies the requirements of Assumption 4. □

The next three examples illustrate the four different cases that appear in Theorem 10, our main result. In the next three ones, X is the geometric Brownian motion that is given by (97). In this context, it is well-known that

$$\varphi(x) = x^m, \quad \psi(x) = x^n \quad \text{and} \quad p'(x) = x^{m+n-1},$$

where the constants $m < 0 < n$ are given by

$$m, n = \frac{1}{2} - \frac{b}{\sigma^2} \mp \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}},$$

while the constant C defined by (16) is equal to $n - m$. Furthermore, the identities

$$mn = -\frac{2r}{\sigma^2} \quad \text{and} \quad m + n = 1 - \frac{2b}{\sigma^2}$$

hold true, while

$$r < b \;\Leftrightarrow\; 0 < n < 1 \quad \text{and} \quad b = r \;\Leftrightarrow\; 1 = n.$$

Example 5. Suppose that $r > b$ and consider the functions

$$h(x) = x^\alpha \quad \text{and} \quad k(x) = 1, \quad \text{for } x > 0,$$

where $\alpha \in]0, 1[$ is a constant. In this case, the function Θ defined by (19) is given by

$$\Theta(x) = x^\alpha - (r-b)x,$$

and all of the conditions in Assumption 4 hold true. Furthermore,

$$R_\Theta(x) = \frac{2}{\sigma^2(\alpha-m)(n-\alpha)} x^\alpha - x,$$

which implies that

$$\lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = \lim_{x\downarrow 0} \left(\frac{2\alpha}{\sigma^2 n(\alpha-m)(n-\alpha)} x^{\alpha-n} - \frac{1}{n} x^{1-n} \right) = \infty$$

because $m < 0 < \alpha < 1 < n$. In view of Lemmas 4 and 6, we can see that

$$\underline{x} < \overline{x} = c^\star = \infty.$$

Therefore, a β-γ strategy is optimal (Case I of Theorem 10) for all $c > 0$. □

Example 6. Suppose that $r + b - \sigma^2 > 0 \Leftrightarrow m < -1$ and $b > r$. Also, consider the functions

$$h(x) = \begin{cases} -\alpha x, & \text{if } x \in \left]0,1\right[, \\ -\alpha, & \text{if } x \geq 1, \end{cases} \quad \text{and} \quad k(x) = \begin{cases} 3-2x, & \text{if } x \in \left]0,1\right[, \\ x^{-2}, & \text{if } x \geq 1, \end{cases}$$

for some constant $\alpha \in \left]-\infty, 3(b-r)\right[$. In this case,

$$\Theta(x) = \begin{cases} (r-2b-\sigma^2)x^2 + (3b-3r-\alpha)x, & \text{if } x \in \left]0,1\right[, \\ (r+b-\sigma^2)x^{-1} - 3r - \alpha, & \text{if } x \geq 1, \end{cases}$$

and all of the conditions in Assumption 4 hold true. In view of the assumption that $m < -1$ and the identity in (46), we can see that $\lim_{x\uparrow\infty} Q_\Theta(x) = \infty$, which implies that $\underline{x} < \infty$ thanks to Lemma 4.(ii). Furthermore,

$$R_\Theta(x) = R_h(x) - \int_0^x k(s)\,\mathrm{d}s = \begin{cases} x^2 + \left(\frac{\alpha}{b-r} - 3\right)x - \frac{2\alpha}{\sigma^2(n-m)n(1-n)} x^n, & \text{if } x \in \left]0,1\right[, \\ -\frac{\alpha}{r} - 3 + x^{-1} - \frac{2\alpha}{\sigma^2(n-m)m(1-m)} x^m, & \text{if } x \geq 1, \end{cases}$$

which implies that

$$\lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = -\frac{2\alpha}{\sigma^2(n-m)n(1-n)} \in \left]-\frac{6(b-r)}{\sigma^2(n-m)n(1-n)}, \infty\right[.$$

In view of Lemmas 4, 6, 8 and 9, we can see that, if $\alpha \leq 0$, then

$$\overline{x} = \infty, \quad \text{and} \quad c^\star \in \left]0,\infty\right[,$$

while, if $\alpha \in \left]0, 3(b-r)\right[$, then

$$\overline{x} < \infty \quad \text{and} \quad 0 < c^\star < c^\circ = 3 + \frac{\alpha}{r}.$$

If $\alpha \leq 0$ and $c \in \left]0, c^\star\right[$, then a β-γ strategy is optimal (Case I of Theorem 10), while, if $\alpha \leq 0$ and $c \geq c^\star$, then no intervention at all is optimal (Case III of Theorem 10). On the other hand, if $\alpha \in \left]0, 3(b-r)\right[$, then any of the Cases I, II or III of Theorem 10 arises depending on whether $0 < c < c^\star$, $c^\star \leq c < 3 + \frac{\alpha}{r}$ or $c \geq 3 + \frac{\alpha}{r}$ is the case, respectively. □

Example 7. Suppose that $b = r > \frac{1}{2}\sigma^2$, which implies that $m < -1$ and $n = 1$. Also, consider the functions

$$h(x) = \begin{cases} ax^\alpha, & \text{if } x \in \,]0,1[, \\ a, & \text{if } x \geq 1, \end{cases} \quad \text{and} \quad k(x) = \begin{cases} 4 - 2x, & \text{if } x \in \,]0,1[, \\ x^{-2} + 1, & \text{if } x \geq 1, \end{cases}$$

for some constants $\alpha \in \,]1,2]$ and $a \in \left]0, \frac{3}{2}(2b+\sigma^2)(1-\frac{1}{\alpha})\right[$. In this case, all of the conditions in Assumption 4 hold true,

$$\lim_{x\uparrow\infty} \psi^{-1}(x) \int_0^\infty k(s)\,\mathrm{d}s = 1, \quad \Theta(x) = \begin{cases} (r - 2b - \sigma^2)x^2 + ax^\alpha, & \text{if } x \in \,]0,1[, \\ (r + b - \sigma^2)x^{-1} - 3r + a, & \text{if } x \geq 1, \end{cases}$$

$$\text{and} \quad R_\Theta(x) = \begin{cases} x^2 + \frac{2a}{\sigma^2(n-\alpha)(\alpha-m)}x^\alpha - \left(3 + \frac{2a\alpha}{\sigma^2(n-m)n(n-\alpha)}\right)x, & \text{if } x \in \,]0,1[, \\ \frac{a}{r} - 3 + x^{-1} + \frac{2a\alpha}{\sigma^2(n-m)m(\alpha-m)}x^m, & \text{if } x \geq 1. \end{cases}$$

Furthermore,

$$\lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = -3 + \frac{2a\alpha}{(2b+\sigma^2)(\alpha-1)} < 0.$$

In view of Lemmas 4, 6, 8 and 9, we can see that

$$\underline{x} < \overline{x} < \infty \quad \text{and} \quad 0 < c^\star < c^\circ = 3 - \frac{a}{r}.$$

Any of the Cases I, II or IV of Theorem 10 may arise, depending on whether $0 < c < c^\star$, $c^\star \leq c < 3 - \frac{a}{r}$ or $c \geq 3 - \frac{a}{r}$ is the case, respectively. □

The next example shows that (27) in Example 1 is not necessarily true if 0 is an entrance boundary point. Furthermore, it shows that β-0 strategies would be an indispensable part of the optimal tactics if we allowed for switching off the system and 0 were an entrance boundary point (see Remark 1 at the end of the previous section).

Example 8. Suppose that X is the mean-reverting square-root process that is given by

$$\mathrm{d}X_t = \alpha(2 - X_t)\,\mathrm{d}t + \sqrt{2\alpha X_t}\,\mathrm{d}W_t, \quad X_0 = x > 0,$$

for some constant $\alpha > 0$. Also, suppose that

$$r(x) = \alpha, \quad h(x) = \begin{cases} e^x - 1, & \text{if } x \in \,]0,1[, \\ e - e^{\gamma+3} - 1 + e^{\gamma x+3}, & \text{if } x \geq 1, \end{cases} \quad \text{and} \quad k(x) = \kappa, \quad \text{for } x > 0,$$

for some constants $\gamma < 0$ and $\kappa \in \left]0, \frac{1}{2\alpha}\right[$. In this case,

$$\varphi(x) = \frac{1}{x}, \quad \psi(x) = \frac{e^x - 1}{x} \quad \text{and} \quad p'(x) = \frac{1}{x^2}e^{x-1}.$$

In particular, 0 is an entrance boundary point. The function Θ defined by (19) is given by

$$\Theta(x) = \begin{cases} 2\alpha\kappa - 1 - 2\alpha\kappa x + e^x, & \text{if } x \in \,]0,1[, \\ 2\alpha\kappa + e - e^{\gamma+3} - 1 - 2\alpha\kappa x + e^{\gamma x+3}, & \text{if } x \geq 1, \end{cases}$$

all of the conditions in Assumption 4 hold true,

$$\lim_{x\downarrow 0} R_h(x) = \frac{1}{\alpha}\left(1 + \frac{\gamma}{1-\gamma} e^{\gamma+2}\right) =: \frac{1}{\alpha} f(\gamma) \quad \text{and} \quad \lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = \frac{1}{\alpha} f(\gamma) - 2\kappa.$$

The function f is strictly decreasing in the interval $\left]-\infty, (1-\sqrt{5})/2\right[$, strictly increasing in the interval $\left](1-\sqrt{5})/2, 0\right[$,

$$\lim_{\gamma\downarrow -\infty} f(\gamma) = 1, \quad f\left(\frac{1-\sqrt{5}}{2}\right) = 1 - \frac{1}{2}(3-\sqrt{5}) e^{(5-\sqrt{5})/2} < 0 \quad \text{and} \quad f(0) = 1.$$

Therefore, there exist constants $(1-\sqrt{5})/2 < \gamma_1 < \gamma_2 < 0$ such that $f(\gamma) < 0$ for all $\gamma \in \left[(1-\sqrt{5})/2, \gamma_1\right[$ and $f(\gamma) \in \left]0, 2\alpha\kappa\right[$ for all $\gamma \in \left]\gamma_1, \gamma_2\right[$. In view of these observations, we can see that

$$\lim_{x\downarrow 0} R_h(x) \neq 0 = \lim_{x\downarrow 0} \frac{h(x)}{r(x)} \quad \text{for all } \gamma \in \left[(1-\sqrt{5})/2, \gamma_1\right[\setminus \{\gamma_2\}, \tag{98}$$

which shows that (27) in Lemma 1 is not in general true if 0 is an entrance boundary point. On the other hand, Lemma 4 implies that, if $\gamma \in \left[(1-\sqrt{5})/2, \gamma_1\right[$, then $0 < \underline{x} < \overline{x} < \infty$ and we are in the context of Lemma 8 with $c^\circ = \infty$. In this context, (88) yields the expression

$$w(0) = \lim_{x\downarrow 0} w(x) = \frac{1}{\alpha} f(\gamma) - \frac{R'_\Theta(\beta^\circ)}{\psi'(\beta^\circ)}.$$

In view of (40) in Lemma 4, (87) in Lemma 8, Remark 1 and the analysis thus far, we can see the following:

(a) If $\gamma \in \left]\gamma_1, 0\right[$, then $w(0) > 0 = h(0)/r(0)$ and a β-0 strategy would be strictly sub-optimal.

(b) If $\gamma \in \left](1-\sqrt{5})/2, \gamma_1\right[$, then $w(0) < 0 = h(0)/r(0)$ for all c sufficiently large, in which case, a β-0 strategy would be optimal. □

Our final example shows that the conditions in (42) are only sufficient for the point $\underline{x}$ introduced in part (i) of Lemma 4 to be finite.

Example 9. Suppose that X is the geometric Brownian motion that is given by (97) with $b = \frac{1}{4}$ and $\sigma = \frac{1}{\sqrt{2}}$. Also, suppose that $r = 1$, so that $m = -2$, $n = 2$ and $C \equiv n - m = 4$. The functions defined by

$$k(x) = \begin{cases} 6 - 5x, & \text{if } x \le 1, \\ x^{-5}, & \text{if } x > 1, \end{cases} \quad \text{and} \quad h(x) = \begin{cases} 7x, & \text{if } x \le 1, \\ 6 + x^{-4}, & \text{if } x > 1, \end{cases}$$

are such that

$$\Theta(x) = \begin{cases} \frac{5}{2}x, & \text{if } x \le 1, \\ \frac{9}{4} + \frac{1}{4}x^{-4}, & \text{if } x > 1, \end{cases}$$

and the function Q_Θ defined by (41) satisfies $\lim_{x\uparrow\infty} Q_\Theta(x) = -\frac{1}{6}$. In this case, the necessary and sufficient condition of Lemma 4.(ii) implies that $\underline{x} = \infty$. On the other hand, the functions defined by

$$k(x) = \begin{cases} 6 - 5x, & \text{if } x \le 1, \\ x^{-5}, & \text{if } x > 1, \end{cases} \quad \text{and} \quad h(x) = \begin{cases} 5x, & \text{if } x \le 1, \\ 4 + x^{-4}, & \text{if } x > 1, \end{cases}$$

are such that

$$\Theta(x) = \begin{cases} \frac{1}{2}x, & \text{if } x \le 1, \\ \frac{1}{4} + \frac{1}{4}x^{-4}, & \text{if } x > 1, \end{cases}$$

and the function Q_Θ defined by (41) satisfies $\lim_{x\uparrow\infty} Q_\Theta(x) = \frac{1}{6}$. In this case, the necessary and sufficient condition of Lemma 4.(ii) implies that $\underline{x} < \infty$. □

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