The Competition Complexity of Dynamic Pricing

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We study the competition complexity of dynamic pricing relative to the optimal auction in the fundamental single-item setting. In prophet inequality terminology, we compare the expected reward $A_m(F)$ achievable by the optimal online policy on m i.i.d. random variables distributed according to F to the expected maximum $M_n(F)$ of n i.i.d. draws from F. We ask how big m have to be to ensure that $(1+\varepsilon)A_m(F) \geq M_n(F)$ for all F.

We resolve this question and characterize the competition complexity as a function of ε. When $\varepsilon = 0$, the competition complexity is unbounded. That is, for any n and any m there is a distribution F such that $A_m(F) < M_n(F)$. In contrast, for any $\varepsilon > 0$, it is sufficient and necessary to have $m = \phi(\varepsilon)n$ where $\phi(\varepsilon) = \Theta(\log\log 1/\varepsilon)$. Therefore, the competition complexity not only drops from unbounded to linear, it is actually linear with a very small constant.

The technical core of our analysis is a loss-less reduction to an infinite dimensional and non-linear optimization problem that we solve optimally. A corollary of this reduction is a novel proof of the factor ≈ 0.745 i.i.d. prophet inequality, which simultaneously establishes matching upper and lower bounds.

1. Introduction.

An important line of work at the intersection of Economics and Computation concerns the *competition complexity* of auctions [4, 14, 15, 3]. The basic idea is to examine how many bidders need to be added to a simple, suboptimal auction mechanism so that its performance is guaranteed to match that of the optimal but more complicated auction mechanism.

This competition complexity approach originates in a seminal paper by Bulow and Klemperer [4], who asked this question for the revenue achievable by the simple but suboptimal second-price auction and Myerson's optimal auction. They showed that for i.i.d. bidders whose valuations are drawn from a regular distribution F, the second-price auction with $n+1$ bidders is guaranteed to achieve at least the expected revenue of the optimal auction with n bidders. They concluded that rather than going for the more complicated auction mechanism, one could simply attract one more buyer to the simpler auction mechanism.

Subsequent work has extended this basic result to a variety of more complex auction settings [14, 29, 3], and also introduced the idea of approximate competition complexity where instead of shooting for optimality, one aims at 99% or 99.9% of optimal [15].

1.1. Our Question. In this work, we initiate the study of the competition complexity of posted pricing. We focus on the fundamental single-item case and compare optimal dynamic pricing versus the optimal auction. While we study the social welfare case, all our results translate to revenue maximization under the standard regularity assumption (see Section 2 for a detailed discussion).

Since we are focusing on social welfare, the simplest way to state our question is in prophet inequality terminology. Our goal is to compare the expected reward $A_m(F)$ achievable by the optimal policy found by backward induction on $m \geq n$ i.i.d. draws from a distribution F, to the expected maximum $M_n(F)$ of n i.i.d. draws from F. For fixed $\varepsilon \geq 0$ and fixed n, we want to find the smallest $m \geq n$ such that for every F we have

$$(1+\varepsilon) \cdot A_m(F) \geq M_n(F).$$

We refer to the functional dependence of m on n and ε as the *competition complexity of dynamic pricing*. We sometimes refer to the case $\varepsilon = 0$ as *exact* competition complexity and to the case $\varepsilon > 0$ as the *approximate* version.

1.2. Warm-Up: The Uniform Case. As a warm-up and to illustrate some of the key ideas in our general competition complexity analysis, consider the case where $F = U[0,1]$ is a uniform distribution over $[0,1]$, and convince ourselves that in this case $A_{2n} \geq M_n$ for all n, so the exact competition complexity is linear. We have that M_n is just the maximum of n i.i.d. draws from a uniform distribution over $[0,1]$, and therefore $M_n = n/(n+1)$. On the other hand, we can compute A_n through the usual backward induction: The recursion is $A_{n+1} = \mathbb{E}(\max\{X, A_n\})$ for $n \geq 1$ and $A_1 = \mathbb{E}(X)$ where $X \sim U[0,1]$. That is, $A_1 = 1/2$, and for $n \geq 1$,

$$\begin{aligned} A_{n+1} &= \mathbb{E}(\max\{X, A_n\}) \\ &= A_n \Pr(X < A_n) + \mathbb{E}(X \mid X \geq A_n) \Pr(X \geq A_n) \\ &= A_n^2 + \frac{(1+A_n)}{2}(1-A_n) = \frac{1}{2}(1+A_n^2). \end{aligned}$$

Observe that apart from getting an exact formula for the recurrence, we get a simple expression for $A_{n+1} - A_n$, that is, the marginal gain of the optimal algorithm when we add one more buyer: $A_{n+1} - A_n = (1-A_n)^2/2$ for $n \geq 1$. In particular, this idea will be further exploited to understand the competition complexity of general distributions.

To analyze the competition complexity for the uniform case, we proceed by induction. It is easy to verify that the claim holds for $n = 1$ since $A_2 = 5/8 > 1/2 = M_1$. So we assume $A_{2n} \geq M_n$, and we want to show $A_{2n+2} \geq M_{n+1}$. Note that if $A_{2n+1} \geq M_{n+1}$ then also $A_{2n+2} \geq A_{2n+1} \geq M_{n+1}$, and there we are done, so we consider the case $A_{2n+1} < M_{n+1}$. We have

$$\begin{aligned} A_{2n+2} &= A_{2n} + (A_{2n+2} - A_{2n+1}) + (A_{2n+1} - A_{2n}) \\ &= A_{2n} + \frac{1}{2}(1 - A_{2n+1})^2 + \frac{1}{2}(1 - A_{2n})^2. \end{aligned}$$

Since the function $f(x) = x + \frac{1}{2}(1-x)^2$ is increasing in $\mathbb{R}_+$, and given that $A_{2n} \geq M_n$, we obtain a lower bound that together with $A_{2n+1} < (n+1)/(n+2)$ yields

$$A_{2n+2} \geq M_n + \frac{1}{2}\left(\left(\frac{1}{n+1}\right)^2 + \left(\frac{1}{n+2}\right)^2\right).$$

The argument is completed by observing that what we add to M_n on the right-hand side is at least $M_{n+1} - M_n = 1/((n+1)(n+2))$. We conclude that for the uniform distribution, it suffices to choose $m \geq 2n$. A closer examination of the asymptotic behavior of A_m and M_n shows that this analysis is in fact tight. Indeed for large m and n, $A_m \approx 1 - 2/(m + \log(m) + 1.76799)$ [17, 30] while $M_n \approx 1 - 1/n$ which roughly shows that we need $m = 2n + o(n)$.

1.3. Our Contribution. The above analysis of the uniform case already rules out a "plus constant" result as in Bulow and Klemperer [4]. It leaves some hope that the exact competition complexity of dynamic pricing may be linear or, if not, then at least polynomial with a small polynomial. Our first main result shows that this hope is unfounded. Indeed, the exact competition complexity is not only "large," it is in fact unbounded.

Main Result 1 (exact competition complexity): For any $m \geq n$, there exists a distribution F such that $A_m(F) < M_n(F)$.

In light of this strong impossibility, a natural question is whether this impossibility persists if we relax our goals and aim for 99% or 99.99% of optimal. It turns out that things change, and quite drastically so. This is formalized by our second main result, which nails down the approximate competition complexity in terms of function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$\phi(\varepsilon) = \int_0^1 \frac{1}{y(1 - \log(y)) + \varepsilon} \mathrm{d}y.$$

Main Result 2 (approximate competition complexity): Consider $\varepsilon > 0$ and any n. Then, we have $(1+\varepsilon)A_m(F) \geq M_n(F)$ for every F if $m \geq \phi(\varepsilon)n$, and for large n this is tight.

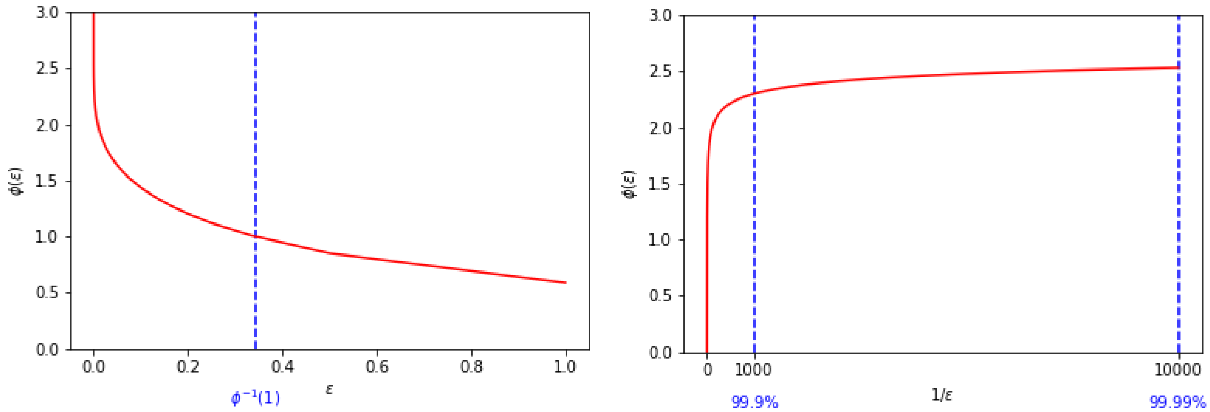

FIGURE 1. Plot of $\phi(\varepsilon)$ as a function of ε on the left, and as a function of $1/\varepsilon$ on the right. Plotting $\phi(\varepsilon)$ as a function of $1/\varepsilon$ serves to illustrate the very slow growth of $\phi(\varepsilon)$ as $\Theta(\log\log 1/\varepsilon)$ when $\varepsilon \to 0$. The dashed blue line in the left plot is at $\varepsilon = \phi^{-1}(1) \approx 0.342$ which implies the optimal factor $1/(1+\phi^{-1}(1)) \approx 0.745$ for the i.i.d. prophet inequality. In the other plot the two blue dashed lines are at $1/\varepsilon = 100$ and $1/\varepsilon = 1000$ which correspond to approximation ratios of 99.9% and 99.99%. The value of $\phi(\varepsilon)$ at these points is the constant required to obtain these approximation ratios.

While our first main result shows that the exact competition complexity of dynamic pricing is unbounded, our second main result shows that if we aim for approximate optimality, then the competition complexity not only drops from being unbounded to being linear, it is actually *linear with a very small constant.*

We illustrate this in Figure 1. In the technical part of the paper, we show that the function $\phi(\epsilon)$ grows as $\Theta(\log\log 1/\varepsilon)$ as $\varepsilon \to 0$, with very small constants hidden in the big-O notation. For example, to obtain 99% of optimal it is sufficient to have $m \geq 2.30 \cdot n$, and to obtain 99.99% of optimal it is sufficient to have $m \geq 2.53 \cdot n$.

An interesting implication of our analysis is that it yields the factor 0.745 i.i.d. prophet inequality [8, 26, 33, 28] and its tightness [21] as a special case. Here is how: Rather than fixing ε and finding $m(n,\varepsilon)$, we may fix $m(n,\varepsilon) = n$ and find ε. The equality $m(n,\varepsilon) = \phi(\varepsilon)n$ corresponds to solving $\phi(\varepsilon) = 1$. This yields $\varepsilon = \phi^{-1}(1)$ and corresponds to an approximation guarantee of $1/(1+\phi^{-1}(1)) \approx 0.745$.

1.4. Our Techniques. Our argument for the uniform distribution $F = U[0,1]$ that we presented above relied on a formula for the differences between two consecutive terms A_{n+1} and A_n, and at its core compared $A_{2(n+1)} - A_{2n}$ to $M_{n+1} - M_n$. Intuitively, we explored properties of the rate of growth and curvature of the two sequences $A_1, A_2, \ldots, A_m$ and $M_1, M_2, \ldots, M_n$.

Our general argument builds on this intuition. Our first key observation characterizes the sequences $A_1, A_2, \ldots, A_m$ that can arise. Namely, we show that for any distribution F, the corresponding infinite sequence $(A_i(F))_{i\in\mathbb{N}}$ satisfies the following three properties. Moreover, for any infinite sequence $(A_i)_{i\in\mathbb{N}}$ satisfying these properties there is a distribution F that leads to this sequence. The three properties are:

(1) The sequence $(A_i)_{i\in\mathbb{N}}$ is non-decreasing,

(2) The sequence $(A_{i+1}-A_i)_{i\in\mathbb{N}}$ is non-increasing, and

(3) The sequence $((A_{i+2}-A_{i+1})/(A_{i+1}-A_i))_{i\in\mathbb{N}}$ is non-decreasing.

Our second key observation is that given a *fixed* infinite sequence $(A_i)_{i\in\mathbb{N}}$ with these properties, we can identify the compatible distribution F that maximizes M_n. This worst-case distribution is a simple piece-wise constant distribution, and allows us to express the largest possible M_n as a function of the $(A_i)_{i\in\mathbb{N}}$. We thus reduce the problem of checking whether for a fixed n and m, $(1+\varepsilon)A_m(F)-M_n(F)\geq 0$ for all F, to an infinite dimensional optimization problem that seeks to minimize $(1+\varepsilon)A_m(F)-M_n(F)$ over all infinite sequences satisfying properties (1)–(3): The inequality is satisfied by all F if and only if the objective value of this infinite-dimensional optimization problem is non-negative. To show our two main results, we then solve this infinite-dimensional optimization problem optimally. This reduces the problem to the analysis of a recursion, which can be pointwise bounded by a differential equation, which, by a careful analysis, leads to the function $\phi(\varepsilon)$.

1.5. Other Gaps and Future Work. An additional set of questions that fits the wider theme of this paper concerns the competition complexity of *static pricing*. Here—unlike in the case of dynamic pricing—there are two questions we could ask. The first comparison is between static pricing A'_m and the optimal auction M_n; the other is between static pricing A'_m and dynamic pricing A_n.

For the first comparison between A'_m and M_n, we observe the following. First, since $A'_m\leq A_m$ for all m, our impossibility (Main Result 1) implies that the exact competition complexity of static pricing is unbounded. Moreover, while the approximate competition complexity of static pricing may be linear (similar to our Main Result 2 for dynamic pricing), the dependence on ε certainly has to be worse. This follows from considering the uniform case: For m sufficiently large, we have that $1-2\log(m)/m\leq A'_m\leq 1-\log(m)/(3m)$ (see Appendix 5 for a derivation of these inequalities). Since $M_n\approx 1-1/n$, for large m and n, this means that to ensure that $(1+\varepsilon)A'_m\geq M_n$, we approximately need that $(1+\varepsilon)(1-\log(m)/(3m))\geq 1-1/n$. Then, for ε small with respect to n, say $\varepsilon=1/n^2$, we can approximate by subtracting ε from the left-hand side. We get $1-(1+\varepsilon)\log(m)/(3m)\geq 1-1/n$, which happens if and only if $3m/\log(m)\geq n(1+\varepsilon)$, which for ε of this order implies that we need at least $m=cn$ with $c=\Omega(\log(1/\varepsilon))$.

For the other comparison, between A'_m and A_n, observe that for the exact version, we need $m=\Omega(n\log(n))$, even for the uniform distribution. This again follows from the asymptotic formulas for $A'_m \approx 1 - 2\log(m)/m$ and $A_n \approx 1 - 2/(n + \log(n) + 1.76799)$, which show that roughly what we need is that $m/\log(m) \geq n$ and therefore $m = \Omega(n\log(n))$. We leave the full resolution of these gaps, which will shed additional light on the relative power of static and dynamic pricing, to future work.

1.6. Further Related Work. Our work examines the relative power of a simple mechanism (dynamic pricing) to that of an optimal mechanism (the optimal auction) and thus fits under the broader umbrella of *simple vs. optimal mechanisms* (e.g., [20, 19]).

At the technical core of our work, we rely on a connection between posted-price mechanisms and prophet inequalities that was pioneered and explored in the last fifteen years [18, 5, 6, 9]. This line of work motivated work on prophet inequalities more generally. Most relevant for us is the work on the i.i.d. single-item prophet inequality [1, 8, 26, 33, 7, 28], but there is also exciting work on combinatorial extensions such as [27, 16, 11, 13]. A closely related line of work has examined the gap between various simple mechanisms including posted-price mechanisms and the optimal mechanism on the same number of bidders [2, 12, 23, 24, 22].

2. Formal Statement of our Results. For our analysis, it will be convenient to consider $\mathbb{N} = \{0, 1, 2, \ldots\}$, the natural numbers including zero. We consider distributions F over the non-negative reals with finite expectation. For a distribution F, we let $M_0(F) = 0$ and for $n \geq 1$ we let $M_n(F) = \mathbb{E}(\max\{X_1, X_2, \ldots, X_n\})$, where $X_1, \ldots, X_n$ is an i.i.d. sample distributed according to F. We denote by $A_n(F)$ the value of the optimal policy and the sequence $(A_n(F))_{n\in\mathbb{N}}$ satisfies the following recurrence: $A_0(F) = 0$, $A_1(F) = \mathbb{E}(X)$ and $A_{n+1}(F) = \mathbb{E}(\max\{X, A_n(F)\})$, where X is a random variable distributed according to F. We now formally state our main results.

THEOREM 1. *For every positive integer $n > 1$, and every positive integer $m \geq n$, there exists a distribution F such that $A_m(F) < M_n(F)$.*

THEOREM 2. *Let $\varepsilon > 0$ and let n be a positive integer. Then, for every $m \geq \phi(\varepsilon)n = \Theta(\log\log 1/\varepsilon)n$, and every distribution F we have $(1+\varepsilon)A_m(F) \geq M_n(F)$. Conversely, for any $\delta > 0$, there exists a distribution G such that for n sufficiently large and $m < (\phi(\varepsilon) - \delta)n$, we have $(1+\varepsilon)A_m(G) < M_n(G)$.*

While Theorem 1 shows that the exact competition complexity of dynamic pricing is unbounded, Theorem 2 shows that the approximate competition complexity not only drops from being unbounded to being linear, it is actually linear with a very small constant (see Figure 1).

As mentioned in the introduction, Theorems 1 and 2 translate to the case of revenue by using standard reductions between social welfare and revenue optimization for the i.i.d. case [9, 6, 18].

Given a distribution F, the *virtual valuation* of F is the function $\phi_F(x) = x - (1 - F(x))/f(x)$, where f is the probability density function of F. To construct an algorithm for the revenue setting in the i.i.d. case with distribution F and n buyers, we reduce to the social welfare case as follows: We run the optimal dynamic welfare policy for an instance with n buyers identically and independently distributed according to F^ϕ, where F^ϕ is the distribution of the random variable $\tilde{\phi}_F(X) = \max(0, \phi_F(X))$ when X is distributed according to F. By doing so, the optimal dynamic welfare policy is defined by thresholds $\tau_1, \ldots, \tau_n$, which can be converted into optimal dynamic revenue prices (a posted price mechanism) with $p_i = \tilde{\phi}_F^{-1}(\tau_i)$, for every $i \in \{1, \ldots, n\}\}$, when F is *regular*, i.e., ϕ_F is monotone non-decreasing [18]. We remark that this reduction is based in the classic result of Myerson for revenue maximizing single-item auctions [31].

3. An Equivalent Optimization Problem. In this section, we develop the main building block of our analysis. The key result of this section, Theorem 3, shows that the question of whether for a given $\varepsilon \geq 0$, $n \geq 1$, and $m \geq 1$ it holds that $(1+\varepsilon)A_m(F) \geq M_n(F)$ for all F reduces to showing whether the following infinite-dimensional, non-linear optimization problem has a non-negative objective.

$$\text{minimize} \quad (1+\varepsilon)\sum_{i=0}^{m-1}\delta_i - \sum_{i=0}^{\infty}\left(1 - \left(\frac{\delta_{i+1}}{\delta_i}\right)^n\right)\delta_i \tag{1}$$

$$\text{subject to} \quad \delta_{j+1} \leq \delta_j \quad \text{for every integer } j \geq 0, \tag{2}$$

$$\delta_j^2 \leq \delta_{j-1}\delta_{j+1} \quad \text{for every integer } j \geq 1, \tag{3}$$

$$\delta_0 = 1 \text{ and } \delta_j > 0 \ \text{ for every integer } j \geq 1. \tag{4}$$

THEOREM 3. *Let $\varepsilon \geq 0$, and let n and m be two positive integers. Then, we have $(1+\varepsilon)A_m(F) \geq M_n(F)$ for every distribution F if and only if the optimal value of the optimization problem (1)-(4) is non-negative.*

We prove this theorem by characterizing the sequences $(A_j(F))_{j\in\mathbb{N}}$ that can result from distributions F and by relating the value of $M_n(F)$ to the values of the sequence $(A_j(F))_{j\in\mathbb{N}}$. The characterization uncovers the properties of the sequences that can arise. Given a sequence of non-negative real values $(S_n)_{n\in\mathbb{N}}$, we denote by $(\partial S_n)_{n\in\mathbb{N}}$ the sequence such that $\partial S_n = S_{n+1} - S_n$ for every non-negative integer n. Consider the following properties:

(a) The sequence $(S_n)_{n\in\mathbb{N}}$ is strictly increasing.

(b) The sequence $(\partial S_n)_{n\in\mathbb{N}}$ is non-increasing.

(c) The sequence $(\partial S_{n+1}/\partial S_n)_{n\in\mathbb{N}}$ is non-decreasing.

Observe that the properties (b)-(c) imply that the sequence $(\partial S_{n+1}/\partial S_n)_{n\in\mathbb{N}}$ is not only non-decreasing, but also bounded with $\partial S_{n+1}/\partial S_n \le 1$ for every $n\in\mathbb{N}$, and therefore it is convergent to a limit value of at most one. In what follows, given a distribution F, let $\omega_0(F)=\inf\{y\in\mathbb{R}:F(y)>0\}$ and $\omega_1(F)=\sup\{y\in\mathbb{R}:F(y)<1\}$ be the left and right endpoints of the support of F.

We need a few lemmas to prove Theorem 3. We also use the following proposition about the optimal policy.

Proposition 1. *For every distribution F the following holds:*

(i) $A_{n+1}(F)=A_n(F)+\int_{A_n(F)}^{\infty}(1-F(y))\mathrm{d}y$ *for every* $n\in\mathbb{N}$.

(ii) $A_{n+2}(F)=A_{n+1}(F)+\int_{A_n(F)}^{A_{n+1}(F)}F(y)\mathrm{d}y$ *for every* $n\in\mathbb{N}$.

(iii) $\lim_{n\to\infty}A_n(F)=\omega_1(F)$.

(iv) If $\omega_0(F)<\omega_1(F)$ *and* F *has finite expectation, then* $A_n(F)<A_{n+1}(F)$ *for every* $n\in\mathbb{N}$.

Proof. Since $A_{n+1}(F)=\mathbb{E}(\max\{A_n(F),X\})$, where X is distributed according to F, we get

$$A_{n+1}(F)=A_n(F)F(A_n(F))+\int_{A_n(F)}^{\infty}sf(s)\mathrm{d}s.$$

By integrating by parts, we have

$$\int_{A_n(F)}^{\infty}sf(s)\mathrm{d}s=(1-F(A_n(F)))A_n(F)+\int_{A_n(F)}^{\infty}(1-F(s))\mathrm{d}s,$$

and therefore (i) holds since we have

$$\begin{aligned}A_{n+1}(F)&=A_n(F)F(A_n(F))+(1-F(A_n(F))A_n(F)+\int_{A_n(F)}^{\infty}(1-F(s))\mathrm{d}s\\&=A_n(F)+\int_{A_n(F)}^{\infty}(1-F(s))\mathrm{d}s.\end{aligned}$$

To prove (ii), observe that

$$\begin{aligned}\int_{A_n(F)}^{\infty}(1-F(s))\mathrm{d}s&=\int_{A_n(F)}^{A_{n+1}(F)}(1-F(s))\mathrm{d}s+\int_{A_{n+1}}^{\infty}(1-F(s))\mathrm{d}s\\&=A_{n+1}(F)-A_n(F)-\int_{A_n(F)}^{A_{n+1}(F)}F(s)\mathrm{d}s+\int_{A_{n+1}(F)}^{\infty}(1-F(s))\mathrm{d}s,\end{aligned}$$

and therefore, (i) implies that

$$\int_{A_n(F)}^{A_{n+1}(F)}F(s)\mathrm{d}s=\int_{A_{n+1}(F)}^{\infty}(1-F(s))\mathrm{d}s=A_{n+2}(F)-A_{n+1}(F),$$

where the last equality holds also by (i).

We now show (iii). Let $L=\lim_{n\to\infty}A_n(F)$ and assume, for the sake of contradiction, that $L<\omega_1(F)$. Since $(A_n(F))_{n\in\mathbb{N}}$ is non-decreasing, we have $A_n(F)\le L$ for every n. Let $U=\min\{L+1,(L+\omega_1(F))/2\}$. From (i), we have

$$\begin{aligned} A_{n+1}(F) - A_n(F) &= \int_{A_n(F)}^{\infty} (1 - F(y))dy \\ &= \int_{A_n(F)}^{\omega_1(F)} (1 - F(y))dy \\ &\geq \int_{L}^{U} (1 - F(y))dy \geq (U - L)(1 - F(U)) > 0, \end{aligned}$$

where the first inequality holds since $A_n(F) \leq L < U \leq \omega_1(F)$, and the second inequality holds since F is non-decreasing. The last inequality follows by the definition of $\omega_1(F)$ and using that $L < U < \omega_1(F)$. Since this inequality holds for all $n \in \mathbb{N}$, it implies that

$$A_{n+1}(F) = \sum_{j=0}^{n} (A_{j+1}(F) - A_j(F)) \geq \frac{n+1}{2}(U - L)(1 - F(U)) \to \infty$$

as $n \to \infty$, which contradicts that $L < \omega_1(F) \leq \infty$. Finally, we show (iv). Since F has a finite expectation, $\omega_0(F) < \omega_1(F)$ and the support is contained in the non-negative reals, we have that $A_1(F) = \mathbb{E}(X) > 0 = A_0(F)$. Then, the property holds by induction on n and property (ii). $\square$

An important implication of Proposition 1(iv) is that the sequence $(A_j(F))_{j \in \mathbb{N}}$ is strictly increasing unless F is a distribution that puts probability one on a single value. For these distributions F, however, $A_m(F) = M_n(F)$ for all $m, n \geq 1$, so they trivially satisfy $(1 + \varepsilon)A_m(F) \geq M_n(F)$.

In the remainder, we will consider distributions F with $\omega_0(F) < \omega_1(F)$. We begin by showing that for such distributions F the sequence $(A_j(F))_{j \in \mathbb{N}}$ satisfies properties (a)-(c).

LEMMA 1. *For every distribution F with $\omega_0(F) < \omega_1(F)$, the sequence $(A_n(F))_{n \in \mathbb{N}}$ satisfies properties (a)-(c).*

Proof. Consider a distribution F with $\omega_0(F) < \omega_1(F)$ and a non-negative integer n. Observe that property (a) holds directly for the sequence $(A_n(F))_{n \in \mathbb{N}}$ from Proposition 1(iv). By Proposition 1(ii), it holds that

$$A_{n+2}(F) - A_{n+1}(F) = \int_{A_n(F)}^{A_{n+1}(F)} F(y)\mathrm{d}y \leq A_{n+1}(F) - A_n(F),$$

where the inequality holds since $F(y) \leq 1$ for every $y \in \mathbb{R}$. Therefore, property (b) holds. Observe that thanks to Proposition 1(ii) again, we have

$$\frac{A_{n+2}(F) - A_{n+1}(F)}{A_{n+1}(F) - A_n(F)} = \frac{1}{A_{n+1}(F) - A_n(F)} \int_{A_n(F)}^{A_{n+1}(F)} F(y)\mathrm{d}y,$$

and since F is monotone non-decreasing, we therefore have

$$F(A_n(F)) \leq \frac{A_{n+2}(F) - A_{n+1}(F)}{A_{n+1}(F) - A_n(F)} \leq F(A_{n+1}(F)),$$

from where we conclude that that $(A_n(F))_{n \in \mathbb{N}}$ satisfies property (c). $\square$

Next we show that for the type for distributions we are interested in, it is possible to prove an upper bound on the value of $M_n(F)$ in terms of the values of the sequence $(A_j(F))_{j\in\mathbb{N}}$.

LEMMA 2. *For every distribution F with $\omega_0(F) < \omega_1(F)$, we have*

$$M_n(F) \le \sum_{j=0}^{\infty} \left(1 - \left(\frac{\partial A_{j+1}(F)}{\partial A_j(F)}\right)^n\right) \partial A_j(F).$$

Proof. Consider the concave function $\varphi : \mathbb{R} \to \mathbb{R}$ given by $\varphi(x) = 1 - x^n$, and for every non-negative integer j let $\mu_j(y) = 1/\partial A_j(F)$ for every $y \in [A_j(F), A_{j+1}(F))$ and zero elsewhere. In particular, μ_j is a probability density function over $[A_j(F), A_{j+1}(F))$. Then, by Jensen's inequality, we have

$$\begin{aligned}
\frac{1}{\partial A_j(F)} \int_{A_j(F)}^{A_{j+1}(F)} (1 - F(y)^n)\mathrm{d}y &= \int_{\mathbb{R}} \varphi(F(y))\mu_j(y)\mathrm{d}y \\
&\le \varphi\left(\int_{\mathbb{R}} F(y)\mu_j(y)\mathrm{d}y\right) \\
&= 1 - \left(\frac{1}{\partial A_j(F)} \int_{A_j(F)}^{A_{j+1}(F)} F(y)\mathrm{d}y\right)^n = 1 - \left(\frac{\partial A_{j+1}(F)}{\partial A_j(F)}\right)^n,
\end{aligned}$$

where the last equality holds by Proposition 1(ii). In particular, for every non-negative integer j we have

$$\int_{A_j(F)}^{A_{j+1}(F)} (1 - F(y)^n)\mathrm{d}y \le \left(1 - \left(\frac{\partial A_{j+1}(F)}{\partial A_j(F)}\right)^n\right) \partial A_j(F). \tag{5}$$

Therefore, we have

$$\begin{aligned}
M_n(F) = \int_0^{\infty} (1 - F(y)^n)\mathrm{d}y &= \sum_{j=0}^{\infty} \int_{A_j(F)}^{A_{j+1}(F)} (1 - F(y)^n)\mathrm{d}y \\
&\le \sum_{j=0}^{\infty} \left(1 - \left(\frac{\partial A_{j+1}(F)}{\partial A_j(F)}\right)^n\right) \partial A_j(F),
\end{aligned}$$

where the second equality holds by Proposition 1(iii) and the inequality comes from (5). □

Our final ingredient is a reverse to the previous two lemmas. It shows that for any sequence satisfying properties (a)-(c) we can construct a distribution G for which $(A_j(G))_{j\in\mathbb{N}}$ matches the values of the sequence and $M_n(G)$ matches the upper bound on $M_n(G)$ in terms of the values of the sequence.

LEMMA 3. *For every $(B_n)_{n\in\mathbb{N}}$ with $B_0 = 0$, and satisfying (a)-(c), there exists a distribution G such that $A_n(G) = B_n$ for every non-negative integer n. Furthermore, we have*

$$M_n(G) = \sum_{j=0}^{\infty} \left(1 - \left(\frac{\partial B_{j+1}}{\partial B_j}\right)^n\right) \partial B_j.$$

Proof. We construct explicitly the distribution G satisfying the statement of the lemma. Recall that since $(B_n)_{n\in\mathbb{N}}$ satisfies properties (b)-(c) the sequence $(\partial B_{n+1}/\partial B_n)_{n\in\mathbb{N}}$ converges to a value $\rho\in(0,1]$. We prove the following claim.

Claim 1. *Suppose that $\rho<1$. Then, there exists a value $\mathcal{B}>0$ such that $\lim_{n\to\infty} B_n=\mathcal{B}$.*

Since the sequence $(B_n)_{n\in\mathbb{N}}$ satisfies property (c), we have that $\partial B_n\le\rho\partial B_{n-1}$, and therefore $\partial B_n\le\rho^n\partial B_0=\rho^n B_1$ for every $n\in\mathbb{N}$. On the other hand, we have

$$B_n=\sum_{j=0}^{n-1}(B_{j+1}-B_j)=\sum_{j=0}^{n-1}\partial B_j\le B_1\sum_{j=0}^{n-1}\rho^n\le\frac{B_1}{1-\rho},$$

which implies that the sequence $(B_n)_{n\in\mathbb{N}}$ is upper bounded. Since by property (a) the sequence $(B_n)_{n\in\mathbb{N}}$ is strictly increasing, we conclude that $(B_n)_{n\in\mathbb{N}}$ is a convergent sequence and we call $\mathcal{B}$ the value of this limit. This establishes the claim.

We now construct the distribution G satisfying the conditions of the statement. Consider $G:\mathbb{R}\to\mathbb{R}$ defined as follows: $G(x)=0$ for every $x\in(-\infty,0)$, for every non-negative integer j and every $x\in[B_j,B_{j+1})$ we have $G(x)=\partial B_{j+1}/\partial B_j$, and let $G(x)=1$ for every $x\ge\lim_{n\to\infty}B_n$. Since the sequence $(B_n)_{n\in\mathbb{N}}$ satisfies property (a), the function G is well defined for every non-negative integer n. Furthermore, since the sequence $(B_n)_{n\in\mathbb{N}}$ satisfies (c), we have that G is non-decreasing, and property (b) implies that $G(x)\le 1$ for every $x\in\mathbb{R}_+$. If $\rho=1$ then $\lim_{x\to\infty}G(x)=1$. Otherwise, if $\rho<1$, by Claim 4 there exists a value $\mathcal{B}>0$ such that $\lim_{n\to\infty}B_n=\mathcal{B}$, and therefore $G(x)=1$ for every $x\ge\mathcal{B}$. Therefore, we conclude that G is a distribution.

In what follows, we show that $A_n(G)=B_n$ for every non-negative integer n. We proceed by induction. By construction, we have $A_0(G)=0=B_0$. Suppose that $B_i=A_i(G)$ for every $i\in\{0,1,\dots,n\}$. By Proposition 1, for every positive integer n it holds that

$$\int_{A_{n-1}(G)}^{A_n(G)}G(y)\mathrm{d}y=\int_{A_n(G)}^{\infty}(1-G(y))\mathrm{d}y=A_{n+1}(G)-A_n(G),$$

and therefore the inductive step implies that

$$\int_{B_{n-1}}^{B_n}G(y)\mathrm{d}y=A_{n+1}(G)-B_n. \tag{6}$$

On the other hand, by construction of G it holds that

$$\int_{B_{n-1}}^{B_n}G(y)\mathrm{d}y=\frac{B_{n+1}-B_n}{B_n-B_{n-1}}\cdot(B_n-B_{n-1})=B_{n+1}-B_n=\partial B_n,$$

and therefore together with (6) we conclude that $A_{n+1}(G)=B_{n+1}$. Finally, we have

$$M_n(G)=\int_0^{\infty}(1-G(y)^n)\mathrm{d}y=\sum_{j=0}^{\infty}\int_{A_j(G)}^{A_{j+1}(G)}(1-G(y)^n)\mathrm{d}y=\sum_{j=0}^{\infty}\left(1-\left(\frac{\partial B_{j+1}}{\partial B_j}\right)^n\right)\partial B_j,$$

where the second equality holds since $\lim_{j\to\infty}A_j(G)=\omega_1(G)$, by Proposition 1(iii). □

We are now ready to prove Theorem 3.

Proof of Theorem 3. We start by showing that if for some $\varepsilon \geq 0$, $n \geq 1$, and $m \geq 1$, there exists a distribution F such that $(1+\varepsilon)A_m(F) < M_n(F)$ then the objective value of the optimization problem (1)-(4) must be negative. Note that for this distribution F it must hold that $\omega_0(F) < \omega_1(F)$ because otherwise $A_m(F) = M_n(F)$, and so we must have $A_{j+1}(F) > A_j(F)$ for all $j \in \mathbb{N}$ by Proposition 1(iv).

We construct a solution $(\delta_j)_{j\in\mathbb{N}}$ for the optimization problem as follows. For every non-negative integer j, let $\delta_j(F) = \partial A_j(F)/\partial A_0(F)$. We begin by showing that the sequence $(\delta_j)_{j\in\mathbb{N}}$ satisfies (2)-(4). By construction we have $\delta_0(F) = \partial A_0(F)/\partial A_0(F) = 1$, that is, (4) holds. By Lemma 1, the sequence $(A_j(F))_{j\in\mathbb{N}}$ satisfies properties (a)-(c). In particular, the sequence $(\partial A_j(F))_{j\in\mathbb{N}}$ is non-increasing and therefore $\delta_{j+1}(F) \leq \delta_j(F)$ for every integer $j \geq 0$, that is, (2) is satisfied. The sequence $(\partial A_{j+1}(F)/\partial A_j(F))_{j\in\mathbb{N}}$ is non-decreasing, and therefore $\delta_{j+1}(F)/\delta_j(F) \geq \delta_j(F)/\delta_{j-1}(F)$ for every integer $j \geq 1$, that is, $\delta_j(F)^2 \leq \delta_{j-1}(F)\delta_{j+1}(F)$, and therefore (3) is satisfied. Finally, observe that

$$
\begin{aligned}
0 > \frac{1}{\partial A_0(F)}\Big((1+\varepsilon)A_m(F) - M_n(F)\Big) &= (1+\varepsilon)\sum_{i=0}^{m-1}\delta_i(F) - \frac{M_n(F)}{\partial A_0(F)} \\
&\geq (1+\varepsilon)\sum_{i=0}^{m-1}\delta_i(F) - \sum_{j=0}^{\infty}\left(1 - \left(\frac{\partial A_{j+1}(F)}{\partial A_j(F)}\right)^n\right)\frac{\partial A_j(F)}{\partial A_0(F)} \\
&= (1+\varepsilon)\sum_{i=0}^{m-1}\delta_i(F) - \sum_{j=0}^{\infty}\left(1 - \left(\frac{\delta_{j+1}(F)}{\delta_j(F)}\right)^n\right)\delta_j(F),
\end{aligned}
$$

where the first inequality holds by assumption and the second inequality comes from Lemma 2. So, in particular, the last expression of the above chain, which coincides with the objective in (1) must be negative.

Conversely, suppose that the value of the optimization problem (1)-(4) is negative. That is, there exists a sequence $(\delta_j^\star)_{j\in\mathbb{N}}$ satisfying (2)-(4) such that

$$
(1+\varepsilon)\sum_{i=0}^{m-1}\delta_i^\star - \sum_{i=0}^{\infty}\left(1 - \left(\frac{\delta_{i+1}^\star}{\delta_i^\star}\right)^n\right)\delta_i^\star < 0. \tag{7}
$$

Consider the sequence $(B_j)_{j\in\mathbb{N}}$ defined as follows: $B_0 = 0$ and $B_j = \sum_{i=0}^{j-1}\delta_i^\star$ for every $j \geq 1$. In particular, we have

$$
B_{j+1} = \sum_{i=0}^{j}\delta_i^\star > \sum_{i=0}^{j-1}\delta_i^\star = B_j
$$

for every integer $j \geq 1$, and therefore the sequence $(B_j)_{j\in\mathbb{N}}$ satisfies (a). Since the sequence $(\delta_j^\star)_{j\in\mathbb{N}}$ satisfies (2)-(3), by construction it holds directly that $(B_j)_{j\in\mathbb{N}}$ satisfies (b)-(c), and therefore by

Lemma 3 there exists a distribution G such that $A_j(G) = B_j$ for every non-negative integer j, and we have

$$\begin{aligned}(1+\varepsilon)A_m(G) &= (1+\varepsilon)B_m \\ &= (1+\varepsilon)\sum_{i=0}^{m-1}\delta_i^\star \\ &< \sum_{i=0}^{\infty}\left(1-\left(\frac{\delta_{i+1}^\star}{\delta_i^\star}\right)^n\right)\delta_i^\star = \sum_{i=0}^{\infty}\left(1-\left(\frac{\partial B_{i+1}}{\partial B_i}\right)^n\right)\partial B_i = M_n(G),\end{aligned}$$

where the last equality also holds by Lemma 1. This finishes the proof of the theorem. □

4. Exact Competition Complexity: Proof of Theorem 1. We show next how to use Theorem 3 to prove the impossibility result in Theorem 1 about the exact competition complexity.

Proof of Theorem 1. Letting $\varepsilon = 0$ in Theorem 3, it suffices to show that the value of the optimization problem (1)-(4) is strictly negative. Consider the sequence $(b_i)_{i\in\mathbb{N}}$ defined as follows: $b_0 = 1$, and $b_1 \in (0,1)$ to be specified later. For every $i \in \{1,\dots,m-1\}$ let

$$b_{i+1} = b_i\left(\frac{n}{n-1}\right)^{\frac{1}{n}}\left(\frac{b_i}{b_{i-1}}\right)^{\frac{n-1}{n}}, \tag{8}$$

and for every $i \geq m$ let $b_{i+1} = b_i^2/b_{i-1}$. We first show that $(b_i)_{i\in\mathbb{N}}$ is feasible for the optimization problem (1)-(4). By construction the sequence satisfies (4). We start with the monotonicity property (2). Consider the function $h(x) = (n/(n-1))^{1/n}x^{(n-1)/n}$ and let $h^{(i)}$ be the function obtained from the composition of h with itself i times. From (8), we get $b_{i+1}/b_i = h^{(i)}(b_1/b_0) = h^{(i)}(b_1)$ for every $i \in \{0,1,\dots,m-1\}$. Observe that $h(x)$ is monotone increasing and continuous on $x \in [0,1]$, with $h(0) = 0$, and therefore $h^{(i)}$ is also monotone increasing, continuous and $h^{(i)}(0) = 0$, for every $i \in \{0,1,\dots,m-1\}$. Since we also know $b_{j+1}/b_j = b_m/b_{m-1}$ for every $j \geq m$, it suffices to prove $b_i/b_{i-1} \leq 1$ for every $i \in \{1,\dots,m\}$ in order to show that the sequence $(b_i)_{i\in\mathbb{N}}$ satisfies (2). To this end, we make any choice of b_1 in a way that $\max_{i\in\{0,1,\dots,m-1\}} h^{(i)}(b_1) \leq 1$. This implies that property (2) is satisfied.

Claim 2. *For every $x \in [0,1]$ we have $\left(\frac{n}{n-1}\right)^{\frac{1}{n}} x^{\frac{n-1}{n}} \geq x$.*

To see this, consider the function $g:\mathbb{R}\to\mathbb{R}$ given by $g(x) = \left(\frac{n}{n-1}\right)^{\frac{1}{n}} x^{\frac{n-1}{n}} - x$. This function is concave in the interval $[0,1]$ and therefore the minimum is attained in either zero or one. Since $g(0) = 0$ and $g(1) = (n/(n-1))^{1/n} - 1 > 0$, we conclude that $g(x) \geq 0$ for every $x \in [0,1]$, proving the claim.

In particular, for every $i \in \{1, \ldots, m-1\}$ we have

$$\frac{b_{i+1}}{b_i} = \left(\frac{n}{n-1}\right)^{\frac{1}{n}} \left(\frac{b_i}{b_{i-1}}\right)^{\frac{n-1}{n}} = g\left(\frac{b_i}{b_{i-1}}\right) + \frac{b_i}{b_{i-1}} \geq \frac{b_i}{b_{i-1}},$$

where we used the fact that $0 \leq b_i/b_{i-1} \leq 1$ by the monotonicity property (2). Since $b_{i+1}/b_i = b_m/b_{m-1} \leq 1$ for every $i \geq m$, we conclude that (3) is also satisfied, and therefore the sequence $(b_i)_{i\in\mathbb{N}}$ is feasible for the optimization problem (1)-(4). We now show that the objective value of the sequence $(b_i)_{i\in\mathbb{N}}$ is strictly negative. We first observe that the objective value is equal to

$$\begin{aligned}&\sum_{i=0}^{m-1} b_i - \sum_{i=0}^{m-1}\left(1-\left(\frac{b_{i+1}}{b_i}\right)^n\right)b_i - \sum_{i=m}^{\infty}\left(1-\left(\frac{b_{i+1}}{b_i}\right)^n\right)b_i \\ &= \sum_{i=0}^{m-1}\left(\frac{b_{i+1}}{b_i}\right)^n b_i - \sum_{i=m}^{\infty}\left(1-\left(\frac{b_{i+1}}{b_i}\right)^n\right)b_i\end{aligned}$$

By construction of the sequence we have

$$\begin{aligned}\sum_{i=0}^{m-1} b_i\left(\frac{b_{i+1}}{b_i}\right)^n &= b_1^n + \frac{n}{n-1}\sum_{i=1}^{m-1} b_i\left(\frac{b_i}{b_{i-1}}\right)^{n-1} \\ &= b_1^n + \frac{n}{n-1}\sum_{i=1}^{m-1} b_{i-1}\left(\frac{b_i}{b_{i-1}}\right)^n = b_1^n + \frac{n}{n-1}\sum_{i=0}^{m-2} b_i\left(\frac{b_{i+1}}{b_i}\right)^n,\end{aligned}$$

and therefore

$$\begin{aligned}b_1^n &= \sum_{i=0}^{m-1} b_i\left(\frac{b_{i+1}}{b_i}\right)^n - \frac{n}{n-1}\sum_{i=0}^{m-2} b_i\left(\frac{b_{i+1}}{b_i}\right)^n \\ &= b_{m-1}\left(\frac{b_m}{b_{m-1}}\right)^n + \sum_{i=0}^{m-2} b_i\left(\frac{b_{i+1}}{b_i}\right)^n - \frac{n}{n-1}\sum_{i=0}^{m-2} b_i\left(\frac{b_{i+1}}{b_i}\right)^n \\ &= b_{m-1}\left(\frac{b_m}{b_{m-1}}\right)^n - \frac{1}{n-1}\sum_{i=0}^{m-2} b_i\left(\frac{b_{i+1}}{b_i}\right)^n.\end{aligned}$$

By rearranging terms we conclude that

$$\begin{aligned}\sum_{i=0}^{m-1} b_i\left(\frac{b_{i+1}}{b_i}\right)^n &= b_{m-1}\left(\frac{b_m}{b_{m-1}}\right)^n + \sum_{i=0}^{m-2} b_i\left(\frac{b_{i+1}}{b_i}\right)^n \\ &= b_{m-1}\left(\frac{b_m}{b_{m-1}}\right)^n + (n-1)\left(b_{m-1}\left(\frac{b_m}{b_{m-1}}\right)^n - b_1^n\right) \\ &= nb_{m-1}\left(\frac{b_m}{b_{m-1}}\right)^n - (n-1)b_1^n.\end{aligned}$$

Let $\gamma = b_m/b_{m-1}$. We have $\gamma < 1$, $b_m = \gamma b_{m-1}$ and inductively $b_{m+i} = \gamma^{i+1} b_{m-1}$ for every non-negative i. Therefore, overall, the objective value of the sequence is equal to

$$\sum_{i=0}^{m-1}\left(\frac{b_{i+1}}{b_i}\right)^n b_i - \sum_{i=m}^{\infty}\left(1-\left(\frac{b_{i+1}}{b_i}\right)^n\right)b_i$$

$$
\begin{aligned}
&= nb_{m-1}\gamma^n - (n-1)b_1^n - (1-\gamma^n)\sum_{i=0}^{\infty}\gamma^{i+1}b_{m-1} \\
&= nb_{m-1}\gamma^n - (n-1)b_1^n - \frac{(1-\gamma^n)\gamma}{1-\gamma}b_{m-1} \\
&= nb_{m-1}\gamma^n - (n-1)b_1^n - b_{m-1}\sum_{i=1}^{n}\gamma^i \\
&= -(n-1)b_1^n - b_{m-1}\left(\sum_{i=1}^{n}\gamma^i - n\gamma^n\right) < 0,
\end{aligned}
$$

which concludes the proof. □

We note that the sequence $(b_n)_{n\in\mathbb{N}}$ defined in the proof of Theorem 1 gives one possible construction of a distribution such that $(1+\varepsilon)A_m(F)\geq M_n(F)$. More precisely, $(b_n)_{n\in\mathbb{N}}$ is a sequence such that the value of the optimization problem (1)-(4) is negative. In other words, it satisfies the properties of $(\delta_j^\star)_{j\in\mathbb{N}}$ (7) as defined in (the converse direction of) the proof of Theorem 3.

5. Approximate Competition Complexity: Proof of Theorem 2. In this section we show how to use Theorem 3 to derive Theorem 2 about the approximate competition complexity. In particular, we show how to optimally solve the optimization problem (1)-(4). For every m,n and $\varepsilon>0$, we show how to reduce the task to finding the minimum of a real convex function in finite dimension. Then, using this reduction, we show that the optimal value of (1)-(4) is obtained by a recursive formula. As a final step, we analyze this recurrence by considering a continuous counterpart defined by a differential equation.

Consider the function $\Gamma^\varepsilon_{n,m}:\mathbb{R}_+^{m-1}\to\mathbb{R}$ defined by

$$\Gamma^\varepsilon_{n,m}(x)=\varepsilon+x_1^n+\sum_{i=1}^{m-2}x_i\left(\varepsilon+\left(\frac{x_{i+1}}{x_i}\right)^n\right)-x_{m-1}(n-1-\varepsilon).$$

Given $\varepsilon>0$ and positive integer $n\geq 2$, let $(\rho_{\varepsilon,j})_{j\in\mathbb{N}}$ be the sequence defined by the following recurrence:

$$\rho_{\varepsilon,1}=1,\ \text{and}\ (n-1)\rho^n_{\varepsilon,j-1}-\varepsilon=n\rho^{n-1}_{\varepsilon,j}\ \text{for every}\ j\geq 2\ \text{such that}\ (n-1)\rho^n_{\varepsilon,j-1}-\varepsilon>0. \qquad (9)$$

For fixed ε and n we say that $\rho_{\varepsilon,j}$ is well defined if $(n-1)\rho^n_{\varepsilon,j-1}-\varepsilon>0$. Observe that by letting $x=\rho_{\varepsilon,j}$ in Claim 2, we get that $\rho_{\varepsilon,j}$ is decreasing in j. It follows that if $\rho_{\varepsilon,m}$ is well defined, then so is $\rho_{\varepsilon,j}$ for $j\leq m$. As a first step, we will show that the optimal value of (1)-(4) can be obtained in terms of the sequence $(\rho_{\varepsilon,j})_{j\in\mathbb{N}}$. To prove this result we require a few propositions.

Proposition 2. *Let $\varepsilon>0$, and let $n\geq 2$ and $m\geq 3$ be two positive integers such that $\rho_{\varepsilon,m}$ is well defined. Then, $\Gamma^\varepsilon_{n,m}$ is convex over $\mathbb{R}_+^{m-1}$ and it has a unique minimizer Y in this region, given by*

$$Y_1=\rho_{\varepsilon,m}\ \text{ and }\ Y_j=\prod_{k=0}^{j-1}\rho_{\varepsilon,m-k}\ \text{ for every}\ j\in\{2,\dots,m-1\}. \qquad (10)$$

Furthermore, $\Gamma^{\varepsilon}_{n,m}(Y) = \varepsilon - (n-1)\rho^n_{\varepsilon,m}$.

Proof. We begin by proving (strict) convexity of $\Gamma^{\varepsilon}_{n,m}$. We proceed by induction on m. Observe first that when $m=3$, we have that $\Gamma^{\varepsilon}_{n,3}(x_1,x_2) = \varepsilon + x_1^n + p(x_1,x_2) - x_2(n-1-\varepsilon)$, where $p(y,z) = y(\varepsilon + (z/y)^n)$. The Hessian of p is

$$\nabla^2 p(y,z) = n(n-1)z^{n-2}y^{1-n}\begin{pmatrix} z^2/y^2 & -z/y \\ -z/y & 1 \end{pmatrix},$$

and this is a positive semidefinite matrix for every $(y,z)\in\mathbb{R}^2_+$, since one eigenvalue is equal to zero, and the other is $n(n-1)z^{n-2}y^{1-n}((z/y)^2+1) > 0$. In particular, p is convex over $\mathbb{R}^2_+$. Since the function $\varepsilon + x_1^n - x_2(n-1-\varepsilon)$ is also convex over $\mathbb{R}^2_+$, we conclude that $\Gamma^{\varepsilon}_{n,3}$ is convex over $\mathbb{R}^2_+$. Now consider an integer value $m>3$, and observe that

$$\Gamma^{\varepsilon}_{n,m+1}(x_1,\dots,x_m) = p(x_{m-1},x_m) - (x_m - x_{m-1})(n-1-\varepsilon) + \Gamma^{\varepsilon}_{n,m}(x_1,\dots,x_{m-1}),$$

and therefore the convexity follows by the inductive step, that is, $\Gamma^{\varepsilon}_{n,m}$ convex over $\mathbb{R}^{m-1}_+$, together with p convex over $\mathbb{R}^2_+$. Every minimizer y of $\Gamma^{\varepsilon}_{n,m}$ over $\mathbb{R}^{m-1}_+$ is a solution to the system given by $\nabla\Gamma^{\varepsilon}_{n,m} = 0$, that is,

$$(n-1)\left(\frac{y_2}{y_1}\right)^n - \varepsilon = ny_1^{n-1}, \tag{11}$$

$$(n-1)\left(\frac{y_{i+1}}{y_i}\right)^n - \varepsilon = n\left(\frac{y_i}{y_{i-1}}\right)^{n-1} \quad \text{for every } i\in\{2,\dots,m-2\}, \tag{12}$$

$$n-1-\varepsilon = n\left(\frac{y_{m-1}}{y_{m-2}}\right)^{n-1}, \text{ and } y\in\mathbb{R}^{m-1}_+. \tag{13}$$

The above system has a unique solution and therefore this proves the first part.

To finish the proof, we show that Y defined in (10) is strictly positive, satisfies the system (11)-(13), and $\Gamma^{\varepsilon}_{n,m}(Y) = \varepsilon - (n-1)\rho^n_{\varepsilon,m}$. Since $\rho_{\varepsilon,j}$ is well-defined for all $j\le m$, we have $\rho_{\varepsilon,j} = ((n-1)\rho^n_{\varepsilon,j-1} - \varepsilon)^{1/(n-1)} > 0$. This implies that $Y\in\mathbb{R}^{m-1}_+$. Next observe that $Y_2 = \rho_{\varepsilon,m}\rho_{\varepsilon,m-1}$ and therefore $Y_2/Y_1 = \rho_{\varepsilon,m-1}$. Then, we have

$$(n-1)(Y_2/Y_1)^n - \varepsilon = (n-1)\rho^n_{\varepsilon,m-1} - \varepsilon = n\rho^{n-1}_{\varepsilon,m} = nY_1^{n-1},$$

and therefore (11) is satisfied. Similarly, for every $j\in\{2,\dots,m-2\}$, we have $Y_j/Y_{j-1} = \rho_{m-j+1}$ and $Y_{j+1}/Y_j = \rho_{m-j}$. Then, we have

$$(n-1)(Y_{j+1}/Y_j)^n - \varepsilon = (n-1)\rho^n_{\varepsilon,m-j} - \varepsilon = n\rho^{n-1}_{\varepsilon,m-j+1} = n(Y_j/Y_{j-1})^{n-1},$$

and therefore (12) is satisfied. Finally, since $Y_{m-1}/Y_{m-2} = \rho_{\varepsilon,2}$, we have

$$n-1-\varepsilon = (n-1)\rho^n_{\varepsilon,1} - \varepsilon = n\rho^{n-1}_{\varepsilon,2} = n(Y_{m-1}/Y_{m-2})^{n-1},$$

and therefore (13) is satisfied. We now evaluate $\Gamma^{\varepsilon}_{n,m}(Y)$. The vector Y satisfies (11)-(13) and therefore

$$\begin{aligned}(n-1)\sum_{i=1}^{m-2} Y_i\left(\frac{Y_{i+1}}{Y_i}\right)^n + (n-1)Y_{m-1} - \varepsilon\sum_{i=1}^{m-1} Y_i &= nY_1^n + n\sum_{i=2}^{m-1} Y_i\left(\frac{Y_i}{Y_{i-1}}\right)^{n-1}\\ &= nY_1^n + n\sum_{i=2}^{m-1} Y_{i-1}\left(\frac{Y_i}{Y_{i-1}}\right)^{n}\\ &= nY_1^n + n\sum_{i=1}^{m-2} Y_i\left(\frac{Y_{i+1}}{Y_i}\right)^{n}.\end{aligned}$$

By subtracting the first term of the left hand side we get

$$\sum_{i=1}^{m-2} Y_i\left(\frac{Y_{i+1}}{Y_i}\right)^n = (n-1)Y_{m-1} - \varepsilon\sum_{i=1}^{m-1} Y_i - nY_1^n,$$

and by rearranging terms we obtain that

$$\sum_{i=1}^{m-2} Y_i\left(\varepsilon + \left(\frac{Y_{i+1}}{Y_i}\right)^n\right) = (n-1-\varepsilon)Y_{m-1} - nY_1^n.$$

Therefore, the minimum of $\Gamma^{\varepsilon}_{n,m}$ over $\mathbb{R}^{m-1}_{+}$ is equal to

$$\varepsilon + Y_1^n + (n-1-\varepsilon)Y_{m-1} - nY_1^n - (n-1-\varepsilon)Y_{m-1} = \varepsilon - (n-1)Y_1^n.$$

The proof follows since we have $Y_1 = \rho_{\varepsilon,m}$. $\square$

Proposition 3. *Let $\varepsilon > 0$, let $n \geq 2$ and $m \geq 3$ be two positive integers such that $\rho_{\varepsilon,m}$ is well defined, and let Y be as defined in (10). Then, the following holds:*

(a) For every $j \in \{1,\dots,m-1\}$ we have that $Y_{j+1} \leq Y_j$.

(b) For every $j \in \{2,\dots,m-1\}$ we have that $Y_j^2 \leq Y_{j-1}Y_{j+1}$.

Proof. Observe that for every $k \in \{1,\dots,m-1\}$, we have $Y_{m-k+1}/Y_{m-k} = \rho_{\varepsilon,k}$. For $k=1$ we have $Y_m/Y_{m-1} = \rho_{\varepsilon} = 1$. From the definition of the recurrence, we have

$$(n-1)\rho^n_{\varepsilon,k-1} \geq (n-1)\rho^n_{\varepsilon,k-1} - \varepsilon = n\rho^{n-1}_{\varepsilon,k}$$

for every $k \in \{2,\dots,m-1\}$. By induction, if $\rho_{\varepsilon,k-1} \leq 1$, we have $\rho^{n-1}_{\varepsilon,k} \leq (n-1)/n$ and therefore $\rho_{\varepsilon,k} \leq 1$. This concludes part (a). Since for every $j \in \{1,\dots,m-1\}$ we have $Y_{j+1}/Y_j = \rho_{\varepsilon,m-j}$, to prove part (b) it suffices to show $\rho_{\varepsilon,k+1} \leq \rho_{\varepsilon,k}$ for every $k \in \{1,\dots,m-2\}$. From the construction of the recurrence, for every $k \in \{1,\dots,m-2\}$ it holds that

$$\rho_{\varepsilon,k} \geq \left(\frac{n}{n-1}\right)^{\frac{1}{n}} \rho^{(n-1)/n}_{\varepsilon,k+1}.$$

By (a) we have $\rho_{\varepsilon,k+1} \in [0,1]$, which together with Claim 2 implies that

$$\left(\frac{n}{n-1}\right)^{\frac{1}{n}} \rho_{\varepsilon,k+1}^{(n-1)/n} \geq \rho_{\varepsilon,k+1}.$$

Therefore we conclude that $\rho_{\varepsilon,k+1} \leq \rho_{\varepsilon,k}$. This proves part (b). □

PROPOSITION 4. *For every sequence $(\delta_j)_{j\in\mathbb{N}}$ satisfying (2)-(4) for which $\delta_m/\delta_{m-1} < 1$, there exists a sequence $(\beta_j)_{j\in\mathbb{N}}$ satisfying (2)-(4), and such that the following holds:*

(a) For every $j \in \{0,1,\ldots,m-1\}$ we have $\delta_j = \beta_j$, and $\beta_m/\beta_{m-1} < 1$.

(b) $\displaystyle\sum_{i=m-1}^{\infty}\left(1-\left(\frac{\delta_{i+1}}{\delta_i}\right)^n\right)\delta_i \leq \beta_{m-1}\sum_{i=0}^{n-1}\left(\frac{\beta_m}{\beta_{m-1}}\right)^i.$

Proof. Suppose we are given $(\delta_j)_{j\in\mathbb{N}}$ satisfying (2)-(4) for which $\delta_m/\delta_{m-1} < 1$. We claim that then there exists a sequence $(\beta_j)_{j\in\mathbb{N}}$ satisfying (2)-(4) such that (a) holds and furthermore (i) $\beta_j \geq \delta_j$ for all $j \geq m$ and (ii) $\beta_j/\beta_{j-1} = \beta_m/\beta_{m-1}$ for all $j \geq m$.

If $(\delta_j)_{j\in\mathbb{N}}$ does not already satisfy these properties, then it must be because of (ii). In particular, there must be a smallest index $j \geq m$ such that $\delta_{j+1}/\delta_j > \delta_m/\delta_{m-1}$. We next describe a procedure that maintains all properties, but extends (ii) so that it holds for one more index. Applying this procedure iteratively, we obtain $(\beta_j)_{j\in\mathbb{N}}$.

Given $(\delta_j)_{j\in\mathbb{N}}$ satisfying (2)-(4), let $k(\delta) \geq m$ be the first value j such that $\delta_{j+1}/\delta_j > \delta_m/\delta_{m-1}$. In particular, we have $\delta_j/\delta_{j-1} = \delta_m/\delta_{m-1}$ for every $j \in \{m,\ldots,k(\delta)\}$. Consider the sequence $(D_j)_{j\in\mathbb{N}}$ defined as follows: $D_j = \delta_j$ for every $j \in \{0,1,\ldots,m-1\}$,

$$D_m = \delta_{m-1}\left(\frac{\delta_{k(\delta)+1}}{\delta_{m-1}}\right)^{\frac{1}{k(\delta)-m+2}},$$

$D_j = D_m(D_m/\delta_{m-1})^{j-m}$ for every $j \in \{m+1,\ldots,k(\delta)\}$, and $D_j = \delta_j$ for every $j \geq k(\delta)+1$. Observe that from the construction it holds directly that $D_{j+1}/D_j = D_m/\delta_{m-1}$ for every $j \in \{m,\ldots,k(\delta)-1\}$. Furthermore, we have

$$\frac{\delta_{k(\delta)+1}}{D_{k(\delta)}} = D_m\left(\frac{D_m}{\delta_{m-1}}\right)^{k(\delta)-m+1}\cdot\frac{1}{D_m}\left(\frac{\delta_{m-1}}{D_m}\right)^{k(\delta)-m} = \frac{D_m}{\delta_{m-1}},$$

and therefore, we have $D_{j+1}/D_j = D_m/D_{m-1}$ for every $j \in \{m-1,\ldots,k(\delta)\}$. By construction, the sequence $(D_j)_{j\in\mathbb{N}}$ satisfies (2)-(4) and $D_m/D_{m-1} < 1$. We show next that $D_j \geq \delta_j$ for every $j \in \{m,\ldots,k(\delta)\}$. Since $\delta_{j+1}/\delta_j \geq \delta_m/\delta_{m-1}$ for every $j \in \{m,\ldots,k(\delta)\}$, we have

$$\left(\frac{\delta_m}{\delta_{m-1}}\right)^{k(\delta)-m+1} \leq \prod_{j=m}^{k(\delta)}\frac{\delta_{j+1}}{\delta_j} = \frac{\delta_{k(\delta)+1}}{\delta_m},$$

which implies that $\delta_m \leq \delta_{m-1}(\delta_{k(\delta)+1}/\delta_{m-1})^{\frac{1}{k(\delta)-m+2}} = D_m$. For $j \in \{m+1,\ldots,k(\delta)\}$ we proceed by induction:

$$D_j = D_{j-1}\frac{D_m}{\delta_{m-1}} \geq \delta_{j-1}\frac{D_m}{\delta_{m-1}} \geq \delta_{j-1}\frac{\delta_m}{\delta_{m-1}} = \delta_j\cdot\frac{\delta_{j-1}}{\delta_j}\cdot\frac{\delta_m}{\delta_{m-1}} = \delta_j,$$

where the first equality holds by construction of the sequence, the first inequality holds by the inductive hypothesis, the second inequality holds since $D_m \geq \delta_m$, and the last equality follows since $\delta_j/\delta_{j-1} = \delta_m/\delta_{m-1}$ for every $j \in \{m, \ldots, k(\delta)\}$. This finishes the proof of part (a).

In the remainder we will prove part (b) using the existence of a sequence $(\beta_j)_{j\in\mathbb{N}}$ for which (a) holds as well as (i) and (ii). To this end we need the following definition and claim. For every sequence $(\delta_j)_{j\in\mathbb{N}}$ let

$$\mathcal{R}(\delta) = \sum_{i=m-1}^{\infty} \left(1 - \left(\frac{\delta_{i+1}}{\delta_i}\right)^n\right) \delta_i.$$

Claim 3. *$\mathcal{R}$ is non-decreasing in δ_i for every $i \geq m$.*

Before proving Claim 3, we show how together with the properties of the sequence $(\beta_j)_{j\in\mathbb{N}}$ it implies property (b). Namely,

$$\begin{aligned}
\sum_{i=m-1}^{\infty} \left(1 - \left(\frac{\delta_{i+1}}{\delta_i}\right)^n\right) \delta_i &\leq \sum_{i=m-1}^{\infty} \left(1 - \left(\frac{\beta_{i+1}}{\beta_i}\right)^n\right) \beta_i \\
&= \left(1 - \left(\frac{\beta_m}{\beta_{m-1}}\right)^n\right) \sum_{i=m-1}^{\infty} \beta_i \\
&= \beta_{m-1} \left(1 - \left(\frac{\beta_m}{\beta_{m-1}}\right)^n\right) \sum_{i=0}^{\infty} \left(\frac{\beta_m}{\beta_{m-1}}\right)^i = \beta_{m-1} \sum_{i=0}^{n-1} \left(\frac{\beta_m}{\beta_{m-1}}\right)^i,
\end{aligned}$$

where the inequality holds by Claim 3 and (i), the first equality holds by (ii), and the second equality holds because (ii) implies $\beta_i = \beta_{m-1} (\beta_m/\beta_{m-1})^{i-m+1}$ for $i \geq m-1$.

It remains to prove Claim 3. Consider the function $\varphi : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that $\varphi(x,y) = (1-(y/x)^n)x$. In particular, the derivative of $\mathcal{R}$ with respect to δ_i, with $i \geq m$, is equal to

$$\begin{aligned}
\frac{\partial \varphi}{\partial y}(\delta_{i-1}, \delta_i) + \frac{\partial \varphi}{\partial x}(\delta_i, \delta_{i+1}) &= -n \left(\frac{\delta_i}{\delta_{i-1}}\right)^{n-1} + 1 + (n-1)\left(\frac{\delta_{i+1}}{\delta_i}\right)^n \\
&= n \left(\frac{\delta_{i+1}}{\delta_i}\right)^{n-1} \left(\frac{1}{n}\left(\frac{\delta_i}{\delta_{i+1}}\right)^{n-1} + \left(1 - \frac{1}{n}\right)\frac{\delta_{i+1}}{\delta_i} - \left(\frac{\delta_i^2}{\delta_{i+1}\delta_{i-1}}\right)^{n-1}\right) \\
&\geq n \left(\frac{\delta_{i+1}}{\delta_i}\right)^{n-1} \left(1 - \left(\frac{\delta_i^2}{\delta_{i+1}\delta_{i-1}}\right)^{n-1}\right) \geq 0,
\end{aligned}$$

The first inequality holds because for any $p \in [0,1]$ we have that $(1-1/n)p + 1/(np^{n-1}) \geq 1$, and $\delta_{i-1} \leq \delta_i \leq \delta_{i+1}$ for every $i \geq m$, and the second holds since $(\delta_j)_{j\in\mathbb{N}}$ satisfies constraint (3). This concludes the proof of the claim. ☐

The following lemma relates the optimal value of the optimization problem (1)-(4) with the sequence $(\rho_{\varepsilon,j})_{j\in\mathbb{N}}$. Using Lemma 4 and Theorem 3 we can numerically find the competition complexity by computing the recurrence (9) (see Figure 2). More specifically, given n and ε, we just have to find the last value m for which the value of the optimization problem is non-negative, and this can be found by numerically computing the values of the recurrence (9).

Lemma 4. *Let $\varepsilon > 0$, and let $n \geq 2$ and $m \geq 3$ be two positive integers such that $\rho_{\varepsilon,m}$ is well defined. Then, the value of the optimization problem (1)-(4) is equal to $\varepsilon - (n-1)\rho^n_{\varepsilon,m}$.*

Proof. Consider $Y \in \mathbb{R}^{m-1}_+$ as defined in (10). For every $\alpha \in (0,1)$, consider the sequence $(\mathcal{Y}_j(\alpha))_{j\in\mathbb{N}}$ defined as follows: $\mathcal{Y}_0(\alpha) = 1$, $\mathcal{Y}_j(\alpha) = Y_j$ for every $j \in \{1,\dots,m-1\}$ and $\mathcal{Y}_j(\alpha) = \alpha^{m-j+1}Y_{m-1}$ for every $j \geq m$. Thanks to Proposition 2 and Proposition 3, for every $\alpha \in (0,1)$ the sequence $(\mathcal{Y}_j(\alpha))_{j\in\mathbb{N}}$ satisfies (2)-(4). The objective value (1) of the sequence is equal to

$$\begin{aligned}
&(1+\varepsilon)\sum_{i=0}^{m-1}\mathcal{Y}_i(\alpha) - \sum_{i=0}^{\infty}\left(1-\left(\frac{\mathcal{Y}_{i+1}(\alpha)}{\mathcal{Y}_i(\alpha)}\right)^n\right)\mathcal{Y}_i(\alpha)\\
&= \varepsilon + Y_1^n + (1+\varepsilon)Y_{m-1} + \sum_{i=1}^{m-2}\left(\varepsilon + \left(\frac{Y_{i+1}}{Y_i}\right)^n\right)Y_i - \sum_{i=m-1}^{\infty}\left(1-\left(\frac{\mathcal{Y}_{i+1}(\alpha)}{\mathcal{Y}_i(\alpha)}\right)^n\right)\mathcal{Y}_i(\alpha)\\
&= \varepsilon + Y_1^n + (1+\varepsilon)Y_{m-1} + \sum_{i=1}^{m-2}\left(\varepsilon + \left(\frac{Y_{i+1}}{Y_i}\right)^n\right)Y_i - (1-\alpha^n)\sum_{i=0}^{\infty}\mathcal{Y}_{m+i-1}(\alpha)\\
&= \varepsilon + Y_1^n + \sum_{i=1}^{m-2}\left(\varepsilon + \left(\frac{Y_{i+1}}{Y_i}\right)^n\right)Y_i - Y_{m-1}\left(\left((1-\alpha^n)\sum_{i=0}^{\infty}\alpha^i\right) - 1 - \varepsilon\right)\\
&= \Gamma^\varepsilon_{n,m}(Y) + Y_{m-1}\left(n - (1-\alpha^n)\sum_{i=0}^{\infty}\alpha^i\right)\\
&= \varepsilon - (n-1)\rho^n_{\varepsilon,m} + Y_{m-1}\left(n - (1-\alpha^n)\sum_{i=0}^{\infty}\alpha^i\right),
\end{aligned}$$

where the last equality holds by Proposition 2. In particular, the feasibility of $(\mathcal{Y}_j(\alpha))_{j\in\mathbb{N}}$ for every $\alpha \in (0,1)$ implies that the value of the optimization problem (1)-(4) is upper bounded by

$$\varepsilon - (n-1)\rho^n_{\varepsilon,m} + Y_{m-1}\inf_{\alpha\in(0,1)}\left\{n - (1-\alpha^n)\sum_{i=0}^{\infty}\alpha^i\right\} = \varepsilon - (n-1)\rho^n_{\varepsilon,m}. \tag{14}$$

Let $(\delta_j)_{j\in\mathbb{N}}$ be any sequence satisfying (2)-(4). We denote by $\mathcal{V}(\delta)$ the objective value (1), which by rearranging terms, is equal to

$$\mathcal{V}(\delta) = \Gamma^\varepsilon_{n,m}(\delta_1,\dots,\delta_{m-1}) + n\delta_{m-1} - \sum_{i=m-1}^{\infty}\left(1-\left(\frac{\delta_{i+1}}{\delta_i}\right)^n\right)\delta_i.$$

Now either $\delta_{i+1}/\delta_i = 1$ for all $i \geq m-1$ in which case $\mathcal{V}(\delta) = \Gamma^\varepsilon_{n,m}(\delta_1,\dots,\delta_{m-1}) + n\delta_{m-1} \geq \min_{x\in\mathbb{R}^{n-1}_+}\Gamma^\varepsilon_{n,m}(x) = \varepsilon - (n-1)\rho^n_{\varepsilon,m}$, where the last inequality holds by Proposition 2. Otherwise, by Proposition 4, there exists a sequence $(\beta_j)_{j\in\mathbb{N}}$ satisfying (2)-(4) for which the following holds:

$$\begin{aligned}
\mathcal{V}(\delta) &= \Gamma^\varepsilon_{n,m}(\delta_1,\dots,\delta_{m-1}) + n\delta_{m-1} - \sum_{i=m-1}^{\infty}\left(1-\left(\frac{\delta_{i+1}}{\delta_i}\right)^n\right)\delta_i\\
&= \Gamma^\varepsilon_{n,m}(\beta_1,\dots,\beta_{m-1}) + n\beta_{m-1} - \sum_{i=m-1}^{\infty}\left(1-\left(\frac{\delta_{i+1}}{\delta_i}\right)^n\right)\delta_i
\end{aligned}$$

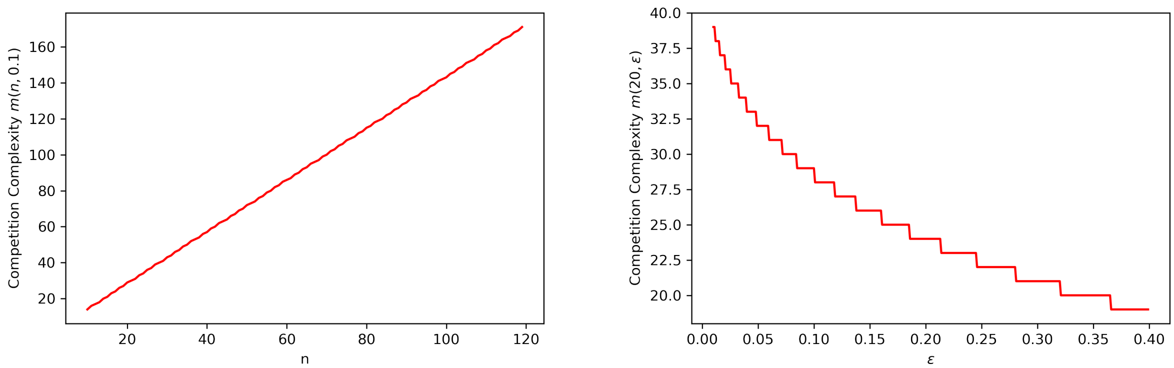

FIGURE 2. On the left, we have a plot of the competition complexity as a function of n when $\varepsilon = 0.1$. On the right, we have a plot of the competition complexity as a function of ε when $n = 20$.

$$\geq \Gamma^{\varepsilon}_{n,m}(\beta_1, \ldots, \beta_{m-1}) + n\beta_{m-1} - \beta_{m-1} \sum_{i=0}^{n-1} \left(\frac{\beta_m}{\beta_{m-1}} \right)^i$$

$$\geq \min_{x \in \mathbb{R}^{m-1}_+} \Gamma^{\varepsilon}_{n,m}(x) + \beta_{m-1} \left(n - \sum_{i=0}^{n-1} \left(\frac{\beta_m}{\beta_{m-1}} \right)^i \right)$$

$$\geq \varepsilon - (n-1)\rho^n_{\varepsilon,m} + \beta_{m-1} \left(n - \sum_{i=0}^{n-1} \left(\frac{\beta_m}{\beta_{m-1}} \right)^i \right),$$

where the second equality holds by property (a) in Proposition 4, the first inequality holds by property (b) in Proposition 4, and the last inequality again holds by Proposition 2. Observe that for every $(\beta_j)_{j \in \mathbb{N}}$, the last term of the above inequality can be lower bounded by zero, and therefore, we get that $\mathcal{V}(\delta) \geq \varepsilon - (n-1)\rho^n_{\varepsilon,m}$ also in this case. This, together with the upper bound in (14), concludes the proof of the lemma. ☐

As a second step, we study the recurrence $(\rho_{\varepsilon,j})_{j \in \mathbb{N}}$ to find the point in which it becomes non-positive. More specifically, by Theorem 3 and Lemma 4, our aim is to find the greatest index m for which $\rho_{\varepsilon,m}$ is well defined, or equivalently the unique m for which $(n-1)\rho^n_{\varepsilon,m} - \varepsilon \leq 0$. To understand this problem we consider a differential equation that will serve as an upper bound to our recurrence relation. Recall the definition of $\phi(\varepsilon) = \int_0^1 1/(y(1-\log(y)) + \varepsilon)\mathrm{d}y$. Given a value $\varepsilon > 0$, consider the following ordinary differential equation:

$$y'(t) = y(t)(\log(y(t)) - 1) - \varepsilon \text{ for every } t \in (0, \phi(\varepsilon)), \tag{15}$$

$$y(0) = 1. \tag{16}$$

We define $y(\phi(\varepsilon)) = \lim_{t \uparrow \phi(\varepsilon)} y(t)$ as the continuous extension of y in $\phi(\varepsilon)$. The following lemma summarizes our results for the differential equation and $\phi(\varepsilon)$.

LEMMA 5. *For every $\varepsilon > 0$, the differential equation (15)-(16) has a unique solution y_ε. Furthermore, the following holds:*

(a) *For every* $t \in [0, \phi(\varepsilon))$ *we have* $y_\varepsilon'(t) < 0$. *In particular,* y_ε *is decreasing and invertible on* $[0, \phi(\varepsilon))$ *and* $y_\varepsilon(\phi(\varepsilon)) = 0$.

(b) *For every integer* $n \geq 2$, *and every* $j \in \mathbb{N}$ *for which* $\rho_{\varepsilon,j}$ *is well-defined, we have*

$$\frac{n-1}{n}\rho_{\varepsilon,j}^n - \frac{\varepsilon}{n} \leq y_\varepsilon\left(\frac{j}{n}\right).$$

(c) *For every* $\delta \in (0, \phi(\varepsilon))$, *there exists* n_0 *such that for every* $n \geq n_0$ *we have* $(n-1)\rho_{\varepsilon,k}^n - \varepsilon > 0$, *where* $k = \lfloor (\phi(\varepsilon) - \delta)n \rfloor$.

(d) *We have* $\phi(\varepsilon) = \Theta(\log\log 1/\varepsilon)$.

We prove Lemma 5. For (a), (b) and (c) we first need a few propositions.

Proposition 5. *For every* $\varepsilon > 0$, *there exists a unique solution of the differential equation* (15)-(16), *that we denote* y_ε. *Furthermore, for every* $t \in [0, \phi(\varepsilon))$ *we have* $y_\varepsilon'(t) < 0$. *In particular,* y_ε *is decreasing and invertible in* $[0, \phi(\varepsilon)]$, *and* $y_\varepsilon(\phi(\varepsilon)) = 0$.

Proof. Observe that for any solution y of the differential equation (15)-(16), we have $y'(0) = -\varepsilon < 0$. Furthermore, for every $y \in (0,1]$, since $\log(y) \leq 0$ and $\varepsilon > 0$, we have $y_\varepsilon'(t) < 0$ for every $t \in [0, \phi(\varepsilon))$. We also know the second derivative

$$y''(t) = y'(t)(\log(y(t)) - 1) + y(t)\frac{y'(t)}{y(t)} = y'(t)\log(y(t)) > 0.$$

In particular, if $y \in (0,1)$, then $|y''|$ is bounded, implying that y' is Lipschitz continuous. Therefore, by the Picard-Lindelöf theorem [32], there is a unique solution on $(0, \phi(\varepsilon))$. As $y(0)$ is given, and we defined $y(\phi(\varepsilon))$ as the continuous extension of y, the solution of the ODE is unique on $[0, \phi(\varepsilon)]$ and we denote it by y_ε. In particular, the function y_ε is invertible and with a differentiable inverse in $[0,1]$. Let $T = y_\varepsilon^{-1}(0)$. Then, we have

$$\begin{aligned} y_\varepsilon^{-1}(1) &= T + \int_0^1 \frac{1}{y_\varepsilon'(y_\varepsilon^{-1}(s))}\mathrm{d}s \\ &= T - \int_0^1 \frac{1}{s(1 - \log(s)) + \varepsilon}\mathrm{d}s = T - \phi(\varepsilon), \end{aligned}$$

and since $y_\varepsilon^{-1}(1) = 0$ we conclude that $y_\varepsilon^{-1}(0) = T = \phi(\varepsilon)$. $\square$

Given $\varepsilon > 0$, consider the function $M_\varepsilon : \mathbb{R} \to \mathbb{R}$ given by

$$M_\varepsilon(x) = \left(\log(x) - 1 - \frac{\varepsilon}{x}\right)\left(x\log^2(x) + x\log(x) - x - \varepsilon\right).$$

Proposition 6. *Let* $\varepsilon > 0$ *and let* $\alpha_\varepsilon = y_\varepsilon^{-1}(\exp(-\frac{1}{2}(1+\sqrt{5})))$. *Then, the following holds:*

(a) *For every* $t \in [0, \phi(\varepsilon)]$ *we have* $y_\varepsilon'''(t) = M_\varepsilon(y_\varepsilon(t))$.

(b) *For every* $t \in [0, \alpha_\varepsilon]$ *we have* $y_\varepsilon'''(t) \geq 0$.

(c) *When* $\varepsilon \leq 0.25$, *we have* $y_\varepsilon'''(t) \geq -1.173$ *for every* $t \in [\alpha_\varepsilon, \phi(\varepsilon)]$.

(d) When $\varepsilon \le 0.25$, there exists $x_\varepsilon \in (0.01, 0.067)$ such that y_ε''' is increasing in $[y_\varepsilon^{-1}(x_\varepsilon), \phi(\varepsilon)]$.

(e) When $\varepsilon \ge 0.25$, we have $y_\varepsilon'''(t) \ge 0$ for every $t \in [0, \phi(\varepsilon)]$.

Proof. By a direct computation, we have that

$$y_\varepsilon''(t) = y_\varepsilon'(t)(\log(y_\varepsilon(t)) - 1) + y_\varepsilon(t) \cdot y_\varepsilon'(t)/y_\varepsilon(t) = y_\varepsilon'(t)\log(y_\varepsilon(t)),$$

and therefore,

$$\begin{aligned} y_\varepsilon'''(t) &= y_\varepsilon''(t)\log(y_\varepsilon(t)) + y_\varepsilon'(t) \cdot \frac{y_\varepsilon'(t)}{y_\varepsilon(t)} \\ &= y_\varepsilon'(t)\log^2(y_\varepsilon(t)) + y_\varepsilon'(t) \cdot \frac{y_\varepsilon'(t)}{y_\varepsilon(t)} \\ &= y_\varepsilon'(t)\left(\log^2(y_\varepsilon(t)) + \log(y_\varepsilon(t)) - 1 - \frac{\varepsilon}{y_\varepsilon(t)}\right) \\ &= \Big(y_\varepsilon(t)(\log(y_\varepsilon(t)) - 1) - \varepsilon\Big)\left(\log^2(y_\varepsilon(t)) + \log(y_\varepsilon(t)) - 1 - \frac{\varepsilon}{y_\varepsilon(t)}\right) \\ &= M_\varepsilon(y_\varepsilon(t)), \end{aligned}$$

which proves (a). Consider the function $g(x) = x\log^2(x) + x\log(x) - x$. We have that $g(x) \le 0$ for every $\exp(-\frac{1}{2}(1+\sqrt{5})) \le x \le 1$, and together with Proposition 5, implies that $g(y_\varepsilon(t)) - \varepsilon \le 0$ for every $t \in [0, \alpha_\varepsilon]$. Furthermore, by Proposition 5 we have that $y_\varepsilon'(t)/y_\varepsilon(t) \le 0$ for every $t \in [0, \alpha_\varepsilon]$ and therefore

$$y_\varepsilon'''(t) = M_\varepsilon(y_\varepsilon(t)) = \frac{y_\varepsilon'(t)}{y_\varepsilon(t)}(g(y_\varepsilon(t)) - \varepsilon) \ge 0,$$

which proves (b). To prove (c), observe that by Proposition 5 we have that $y_\varepsilon(\alpha_\varepsilon) \ge y_\varepsilon(t) \ge 0$ for every $t \in [\alpha_\varepsilon, \phi(\varepsilon)]$, and since $0.199 > y_\varepsilon(\alpha_\varepsilon) = \exp(-(1+\sqrt{5})/2) > 0.198$, we have that

$$\min_{\varepsilon \in (0,0.25)} \min_{t \in [\alpha_\varepsilon, \phi(\varepsilon)]} y_\varepsilon'''(t) = \min_{\varepsilon \in (0,0.25)} \min_{t \in [\alpha_\varepsilon, \phi(\varepsilon)]} M_\varepsilon(y_\varepsilon(t)) \ge \min_{\substack{\varepsilon \in [0,0.25], \\ x \in [0,0.199]}} M_\varepsilon(x) \approx -1.1722,$$

where the first equality comes from part (a). We now prove (d). By a direct computation, we have

$$\begin{aligned} M_\varepsilon'(x) &= -\frac{\varepsilon^2}{x^2} - \frac{2\varepsilon}{x} - \frac{2\varepsilon\log(x)}{x} + \log^3(x) + 3\log^2(x) - 2\log(x) - 1, \\ M_\varepsilon''(x) &= \frac{2\varepsilon^2}{x^3} + \frac{2\varepsilon\log(x)}{x^2} - \frac{2}{x} + \frac{3\log^2(x)}{x} + \frac{6\log(x)}{x}. \end{aligned}$$

Furthermore, we have

$$\min_{\substack{\varepsilon \in [0,0.25], \\ x \in [0,067]}} M_\varepsilon''(x) \approx 0.716,$$

and therefore the function M_ε' is increasing in $(0, 0.067]$ for every $\varepsilon \in (0, 0.25]$. On the other hand, we have

$$M_\varepsilon'(0.067) > -\frac{\varepsilon^2}{(0.067)^2} - \frac{2\varepsilon}{0.067} - \frac{2\varepsilon\log(0.067)}{x} + 6.57,$$

and this is a quadratic concave function over $[0,0.25]$ that attains the minimum at $\varepsilon = 0.25$ with a value of ≈ 5.36. Furthermore, we have

$$M_\varepsilon'(0.01) < -\frac{\varepsilon^2}{(0.01)^2} - \frac{2\varepsilon}{0.01} - \frac{2\varepsilon\log(0.01)}{x} - 25.83,$$

and this is a quadratic concave function over $[0,0.25]$ that attains the maximum at ≈ 0.036 with a value of ≈ -12.83. Therefore, for every $\varepsilon \in (0,0.25]$, the continuity of M_ε' implies the existence of a value $x_\varepsilon \in (0.01, 0.0067)$ such that $M_\varepsilon'(x_\varepsilon) = 0$. Since the function M_ε' is increasing in $[0,0.067]$, we have $M_\varepsilon'(x) \leq M_\varepsilon(x_\varepsilon) = 0$ for every $x \in [0,x_\varepsilon]$, and therefore the function M_ε is decreasing in the interval $[0,x_\varepsilon]$. By Proposition 5 we have that y_ε is decreasing in $[0,\phi(\varepsilon)]$, and therefore we conclude that $y_\varepsilon''' = M_\varepsilon \circ y_\varepsilon$ is increasing in the interval $[y_\varepsilon^{-1}(x_\varepsilon), \phi(\varepsilon)]$.

Finally, we prove (e). Recall that $g(x) = x\log^2(x) + x\log(x) - x$. It is sufficient to verify that $g(x) \leq \varepsilon$ for every $x \in (0,1]$ when $\varepsilon \geq 0.25$, since we have $y_\varepsilon'''(t) = y_\varepsilon'(t)(g(y_\varepsilon(t)) - \varepsilon)/y_\varepsilon(t)$, and $y_\varepsilon' \leq 0$ in $[0,\phi(\varepsilon)]$. We have

$$g'(x) = \log^2(x) + 2x\log(x)\cdot\frac{1}{x} + \log(x) + x\cdot\frac{1}{x} - 1 = \log(x)(\log(x)+3).$$

We have $g'(x) \geq 0$ when $x \in (0, e^{-3}]$ and $g'(x) \leq 0$ when $x \in [e^{-3}, 1]$. Therefore, the maximum of g in $(0,1]$ is attained at e^{-3} and we conclude that $g(x) \leq 5e^{-3} - \varepsilon \leq 5e^{-3} - 0.25 < 0$ for every $x \in (0,1]$. This concludes the proof of the proposition. $\square$

Given $\varepsilon > 0$ and a positive integer $n \geq 2$, consider the function $F_{n,\varepsilon} : \mathbb{R} \to \mathbb{R}$ given by

$$F_{n,\varepsilon}(x) = x + \frac{x(\log(x)-1)}{n} + \frac{\log(x)(x(\log(x)-1) - \varepsilon)}{2n^2}.$$

PROPOSITION 7. *Let $n \geq 2$ be an integer value and let $\varepsilon \in (0, 0.25]$. Then, the following holds:*

(a) For every $x \in (0,1]$ we have $F_{n,\varepsilon}(x) \geq \left(\frac{n-1}{n}\right) x^{\frac{n}{n-1}}$.

(b) For every $x \in [0.01, 0.199]$ we have $F_{n,\varepsilon}(x) \geq \left(\frac{n-1}{n}\right) x^{\frac{n}{n-1}} + \frac{1.173}{6n^6}$.

(c) For every $x \in [0, 0.07]$ we have $F_{n,\varepsilon}(x) + \frac{M_\varepsilon(x)}{6n^6} \geq \left(\frac{n-1}{n}\right) x^{\frac{n}{n-1}}$.

Proof. The inequality in (a) holds by [8, Proposition D.1.]. Consider the function $G_n : \mathbb{R} \to \mathbb{R}$ given by

$$G_n(x) = 1 + \frac{\log(x)-1}{n} + \frac{\log(x)(\log(x)-1)}{2n^2} - \left(\frac{n-1}{n}\right) x^{\frac{1}{n-1}} - \frac{1.173}{6xn^6}.$$

To prove (b) it suffices to show that $G_n(x) \geq 0$ for every $x \in [0.01, 0.199]$, since $-\varepsilon\log(x) \geq 0$ for every $x \in [0.01, 0.199]$, and therefore

$$F_{n,\varepsilon}(x) - \left(\frac{n-1}{n}\right) x^{\frac{n}{n-1}} - \frac{1.173}{6n^6} \geq x \cdot G_n(x) \geq 0.$$

We have that $\{G_n(0.199)\}_{n\in\mathbb{N}}$ is a strictly positive and decreasing sequence, and therefore it is sufficient to show that G_n is decreasing in the interval $[0.01, 0.199]$. We have

$$\begin{aligned} G_n'(x) &= \frac{1}{nx} + \frac{\log(x)}{n^2 x} - \frac{1}{2n^2 x} - \frac{1}{n} x^{\frac{1}{n-1}-1} + \frac{1.173}{6n^6 x^2} \\ &= \frac{1}{nx^2}\left(x + \frac{x\log(x) - x/2}{n} - x^{\frac{n}{n-1}} + \frac{1.173}{6n^5}\right), \end{aligned}$$

and let

$$h_n(x) = x + \frac{x\log(x) - x/2}{n} - x^{\frac{n}{n-1}} + \frac{1.173}{6n^5}.$$

It is sufficient to show that h_n is non-positive in $[0.01, 0.199]$. We have

$$\begin{aligned} h_n'(x) &= 1 + \frac{\log(x) + 1/2}{n} - \left(1 - \frac{1}{n}\right) x^{\frac{1}{n-1}}, \\ h_n''(x) &= \frac{1}{nx} - \frac{1}{n} x^{\frac{1}{n-1}-1} = \frac{1}{nx}\left(1 - x^{\frac{1}{n-1}}\right), \end{aligned}$$

and therefore $h_n''(x) > 0$ for every $x \in [0.01, 0.199]$. This implies that h_n is convex in the interval $[0.01, 0.199]$, and therefore it is sufficient to verify that $h_n(0.01) < 0$ and $h_n(0.199) < 0$. In fact, we have

$$\begin{aligned} h_n(0.01) &= 0.01 + \frac{0.01\log(0.01) - 0.005}{n} - 0.01^{\frac{n}{n-1}} + \frac{1.173}{6n^5} \\ &\leq \frac{0.01\log(0.01) - 0.005}{2} + \frac{1.173}{6\cdot 2^5} < -0.019, \\ h_n(0.199) &= 0.199 + \frac{0.199\log(0.199) - 0.0995}{n} - 0.199^{\frac{n}{n-1}} + \frac{1.173}{6n^5} \\ &\leq \frac{0.199\log(0.199) - 0.0995}{2} + \frac{1.173}{6\cdot 2^5} < -0.2, \end{aligned}$$

and therefore we conclude that h_n is non-positive in $[0.01, 0.199]$, which implies that G_n is positive in $[0.01, 0.199]$. This proves (b). Finally, to prove (c), consider the function $\Psi : \mathbb{R}^3_+ \to \mathbb{R}$ given by

$$\Psi(x, y, \varepsilon) = x + \frac{x(\log(x) - 1)}{y} + \frac{\log(x)(x(\log(x) - 1) - \varepsilon)}{2y^2} + \frac{M_\varepsilon(x)}{6y^6} - \left(1 - \frac{1}{y}\right) x^{\frac{y}{y-1}}.$$

Then, we have

$$\inf_{\substack{n\geq 2,\\ \varepsilon\in(0,0.25],\\ x\in[0,0.07]}} \left(F_{n,\varepsilon}(x) + \frac{M_\varepsilon(x)}{6n^6} - \left(\frac{n-1}{n}\right) x^{\frac{n}{n-1}}\right) \geq \min_{\substack{y\geq 2,\\ \varepsilon\in(0,0.25],\\ x\in[0,0.07]}} \Psi(x, y, \varepsilon) \geq 0,$$

which concludes the proof. □

Proof of Lemma 5. Part (a) holds by Proposition 5. To prove (b) we proceed by induction. When $j = 1$, we have $\rho_{\varepsilon,1}^{n-1} = 1 = y_\varepsilon(0)$. For every $j \geq 1$, Taylor's theorem implies that

$$y_\varepsilon\left(\frac{j}{n}\right) = y_\varepsilon\left(\frac{j-1}{n}\right) + \frac{1}{n} y_\varepsilon'\left(\frac{j}{n}\right) + \frac{1}{2n^2} y_\varepsilon''\left(\frac{j}{n}\right) + \frac{1}{6n^6} y_\varepsilon'''(\xi)$$

$$
\begin{aligned}
&= y_\varepsilon\left(\frac{j-1}{n}\right) + \frac{1}{n}y_\varepsilon'\left(\frac{j}{n}\right)\left(1+\frac{1}{2n}\log\left(y_\varepsilon\left(\frac{j-1}{n}\right)\right)\right) + \frac{1}{6n^6}y_\varepsilon'''(\xi) \\
&= y_\varepsilon\left(\frac{j-1}{n}\right) + \frac{y_\varepsilon(\frac{j-1}{n})(\log(y_\varepsilon(\frac{j-1}{n}))-1)-\varepsilon}{n}\left(1+\frac{1}{2n}\log\left(y_\varepsilon\left(\frac{j-1}{n}\right)\right)\right) + \frac{1}{6n^6}y_\varepsilon'''(\xi) \\
&= F_{n,\varepsilon}\left(y_\varepsilon\left(\frac{j-1}{n}\right)\right) - \frac{\varepsilon}{n} + \frac{1}{6n^6}y_\varepsilon'''(\xi)
\end{aligned}
$$

where $\xi \in ((j-1)/n, j/n)$, and the second and third equalities come from the ODE definition. We consider four different cases.

Case 1: Suppose that $\varepsilon \geq 0.25$. By Proposition 6(e) we have $y_\varepsilon'''(\xi) \geq 0$, and therefore

$$
\begin{aligned}
y_\varepsilon\left(\frac{j}{n}\right) &= F_{n,\varepsilon}\left(y_\varepsilon\left(\frac{j-1}{n}\right)\right) - \frac{\varepsilon}{n} + \frac{1}{6n^6}y_\varepsilon'''(\xi) \\
&\geq F_{n,\varepsilon}\left(y_\varepsilon\left(\frac{j-1}{n}\right)\right) - \frac{\varepsilon}{n} \\
&\geq \left(\frac{n-1}{n}\right) y_\varepsilon\left(\frac{j-1}{n}\right)^{\frac{n}{n-1}} - \frac{\varepsilon}{n} \geq \left(\frac{n-1}{n}\right)\rho^n_{\varepsilon,j} - \frac{\varepsilon}{n} = \rho^{n-1}_{\varepsilon,j+1},
\end{aligned}
$$

where the second inequality holds from Proposition 7(a), and in the third inequality we used the inductive step.

Case 2: Suppose that $\varepsilon \leq 0.25$ and $2 \leq j \leq \alpha_\varepsilon n + 1$. In particular, we have $(j-1)/n \in [0, \alpha_\varepsilon]$. By Proposition 6(b) we have $y_\varepsilon'''(\xi) \geq 0$, and therefore

$$
\begin{aligned}
y_\varepsilon\left(\frac{j}{n}\right) &= F_{n,\varepsilon}\left(y_\varepsilon\left(\frac{j-1}{n}\right)\right) - \frac{\varepsilon}{n} + \frac{1}{6n^6}y_\varepsilon'''(\xi) \\
&\geq F_{n,\varepsilon}\left(y_\varepsilon\left(\frac{j-1}{n}\right)\right) - \frac{\varepsilon}{n} \\
&\geq \left(\frac{n-1}{n}\right) y_\varepsilon\left(\frac{j-1}{n}\right)^{\frac{n}{n-1}} - \frac{\varepsilon}{n} \geq \left(\frac{n-1}{n}\right)\rho^n_{\varepsilon,j} - \frac{\varepsilon}{n} = \rho^{n-1}_{\varepsilon,j+1},
\end{aligned}
$$

where the second inequality holds from Proposition 7(a), and in the third inequality we used the inductive step.

Case 3: Suppose that $\varepsilon \leq 0.25$ and $\alpha_\varepsilon n + 1 \leq j \leq y_\varepsilon^{-1}(x_\varepsilon)n + 1$, where x_ε is the value guaranteed by Proposition 6(d). In particular, we have $(j-1)/n \in [\alpha_\varepsilon, y_\varepsilon^{-1}(x_\varepsilon)]$, and by Proposition 6(d) we have $0.01 < x_\varepsilon$, which implies that $0.01 < y_\varepsilon((j-1)/n) \leq 0.199$. By Proposition 6(c) we have $y_\varepsilon'''(\xi) \geq -1.173$, and therefore

$$
\begin{aligned}
y_\varepsilon\left(\frac{j}{n}\right) &= F_{n,\varepsilon}\left(y_\varepsilon\left(\frac{j-1}{n}\right)\right) - \frac{\varepsilon}{n} + \frac{1}{6n^6}y_\varepsilon'''(\xi) \\
&\geq F_{n,\varepsilon}\left(y_\varepsilon\left(\frac{j-1}{n}\right)\right) - \frac{\varepsilon}{n} - \frac{1.173}{6n^6} \\
&\geq \left(\frac{n-1}{n}\right) y_\varepsilon\left(\frac{j-1}{n}\right)^{\frac{n}{n-1}} - \frac{\varepsilon}{n} \geq \left(\frac{n-1}{n}\right)\rho^n_{\varepsilon,j} - \frac{\varepsilon}{n} = \rho^{n-1}_{\varepsilon,j+1},
\end{aligned}
$$

where the second inequality holds from Proposition 7(b), and in the third inequality we used the inductive step.

Case 4: Suppose that $\varepsilon \le 0.25$ and $j \ge y_\varepsilon^{-1}(x_\varepsilon)n+1$. In particular, we have $(j-1)/n \ge y_\varepsilon^{-1}(x_\varepsilon)$ and $y_\varepsilon((j-1)/n) \le x_\varepsilon < 0.067$. By Proposition 6(d), y_ε''' is increasing in $[y_\varepsilon^{-1}(x_\varepsilon), \phi(\varepsilon)]$, and therefore $y_\varepsilon'''(\xi) \ge y_\varepsilon'''((j-1)/n)$. Then, we have

$$\begin{aligned}
y_\varepsilon\Big(\frac{j}{n}\Big) &= F_{n,\varepsilon}\Big(y_\varepsilon\Big(\frac{j-1}{n}\Big)\Big) - \frac{\varepsilon}{n} + \frac{1}{6n^6} y_\varepsilon'''(\xi)\\
&\ge F_{n,\varepsilon}\Big(y_\varepsilon\Big(\frac{j-1}{n}\Big)\Big) - \frac{\varepsilon}{n} + \frac{1}{6n^6} y_\varepsilon'''\Big(\frac{j-1}{n}\Big)\\
&= F_{n,\varepsilon}\Big(y_\varepsilon\Big(\frac{j-1}{n}\Big)\Big) + \frac{1}{6n^6} M_\varepsilon\Big(y_\varepsilon\Big(\frac{j-1}{n}\Big)\Big) - \frac{\varepsilon}{n}\\
&\ge \left(\frac{n-1}{n}\right) y_\varepsilon\Big(\frac{j-1}{n}\Big)^{\frac{n}{n-1}} - \frac{\varepsilon}{n} \ge \left(\frac{n-1}{n}\right)\rho_{\varepsilon,j}^n - \frac{\varepsilon}{n} = \rho_{\varepsilon,j+1}^{n-1},
\end{aligned}$$

where the second inequality holds from Proposition 7(c), and in the third inequality we used the inductive step.

Part (c) is a direct extension of [25, Corollary 6.9]. Finally we prove (d). By definition, recall that

$$\phi(\varepsilon) = \int_0^1 \frac{1}{y(1-\log(y))+\varepsilon}\mathrm{d}y.$$

We apply the change of variables $x = -\log(y)$ to get that

$$\phi(\varepsilon) = \int_0^\infty \frac{1}{1+x+\varepsilon e^x}\mathrm{d}x.$$

Note that the function $f(x) = 1+x-\varepsilon e^x$ has a unique root in $x \in [0,\infty)$, that we denote r_ε (i.e., $f(r_\varepsilon)=0$). In particular, we have $1+x \ge \varepsilon e^x$ for every $x \le r_\varepsilon$, and $1+x \le \varepsilon e^x$ for every $x \ge r_\varepsilon$. Then, we have

$$\int_0^{r_\varepsilon} \frac{1}{2(1+x)}\mathrm{d}x \le \int_0^{r_\varepsilon} \frac{1}{1+x+\varepsilon e^x}\mathrm{d}x$$

and

$$\int_{r_\varepsilon}^\infty \frac{1}{2\varepsilon e^x}\mathrm{d}x \le \int_{r_\varepsilon}^\infty \frac{1}{1+x+\varepsilon e^x}\mathrm{d}x.$$

By adding both inequalities we get

$$\frac{1}{2}\left(\int_0^{r_\varepsilon} \frac{1}{1+x}\mathrm{d}x + \int_{r_\varepsilon}^\infty \frac{1}{\varepsilon e^x}\mathrm{d}x\right) \le \phi(\varepsilon).$$

On the other hand, we have

$$\phi(\varepsilon) \le \int_0^{r_\varepsilon} \frac{1}{1+x}\mathrm{d}x + \int_{r_\varepsilon}^\infty \frac{1}{\varepsilon e^x}\mathrm{d}x,$$

and therefore, by evaluating the integrals, we have

$$\frac{1}{2}\left(\log(1+r_\varepsilon)+\frac{\exp(-r_\varepsilon)}{\varepsilon}\right)\le\phi(\varepsilon)\le\log(1+r_\varepsilon)+\frac{\exp(-r_\varepsilon)}{\varepsilon}.$$

Observe that $r_\varepsilon=\log(1+r_\varepsilon)+\log(1/\varepsilon)$, and therefore $r_\varepsilon\ge\log(1/\varepsilon)$. Furthermore, when ε is sufficiently small, we have $f(2\log(1/\varepsilon))=1+2\log(1/\varepsilon)-1/\varepsilon<0$, and therefore $r_\varepsilon\le 2\log(1/\varepsilon)$. Then, for ε sufficiently small we have $\log(1/\varepsilon)\le r_\varepsilon\le 2\log(1/\varepsilon)$, which implies that

$$\begin{aligned}\log\left(1+\log\left(\frac{1}{\varepsilon}\right)\right)+\frac{\exp(-2\log(\frac{1}{\varepsilon}))}{\varepsilon}&\le\log(1+r_\varepsilon)+\frac{\exp(-r_\varepsilon)}{\varepsilon}\\&\le\log\left(1+2\log\left(\frac{1}{\varepsilon}\right)\right)+\frac{\exp(-\log(\frac{1}{\varepsilon}))}{\varepsilon}.\end{aligned}$$

The result now follows from the fact that the leftmost expression is lower bounded as

$$\log\log\left(\frac{1}{\varepsilon}\right)\le\log\left(1+\log\left(\frac{1}{\varepsilon}\right)\right)+\frac{\exp(-2\log(\frac{1}{\varepsilon}))}{\varepsilon},$$

and the rightmost is upper bounded as $\log\left(1+2\log\left(\frac{1}{\varepsilon}\right)\right)+\frac{\exp(-\log(\frac{1}{\varepsilon}))}{\varepsilon}\le 2\log\log\left(\frac{1}{\varepsilon}\right)$.

We are now ready to prove Theorem 2. □

Proof of Theorem 2. Fix $\varepsilon>0$ and consider the non-trivial case where $n\ge 2$. We begin with the first part of the theorem. By Lemma 4 it suffices to find the largest index j for which $\rho_{\varepsilon,j}$ is well defined. Suppose for a contradiction that for some $m\ge\phi(\varepsilon)n$, $\rho_{\varepsilon,m}$ is well defined but $(n-1)\rho^n_{\varepsilon,m}-\varepsilon>0$. Define $\varepsilon'>0$ such that $m/n=\phi(\varepsilon')$. Note that such an ε' exists and $\varepsilon'\le\varepsilon$ because ϕ is monotone and continuous.

Claim 4. *For every positive integer j, $\rho_{\varepsilon',j}$ is well defined when $\rho_{\varepsilon,j}$ is well defined, and $\rho_{\varepsilon',j}\ge\rho_{\varepsilon,j}$.*

Using Claim 4, we have

$$\frac{n-1}{n}\rho^n_{\varepsilon,m}-\frac{\varepsilon}{n}\le\frac{n-1}{n}\rho^n_{\varepsilon',m}-\frac{\varepsilon'}{n}\le y_{\varepsilon'}\left(\phi(\varepsilon')\right)=0,$$

where the second inequality holds by Lemma 5(b) and the final equality holds by Lemma 5(a). This yields a contradiction. To prove the claim, we consider an inductive argument. The claim clearly holds for $j=1$, and assume that it holds for every $k\le j-1$. If $\rho_{\varepsilon,j}$ is well defined, that is $(n-1)\rho^n_{\varepsilon,j-1}-\varepsilon>0$, by the inductive step we have $\rho_{\varepsilon',j-1}\ge\rho_{\varepsilon,j-1}$ and therefore,

$$(n-1)\rho^n_{\varepsilon',j-1}-\varepsilon'\ge(n-1)\rho^n_{\varepsilon,j-1}-\varepsilon>0,$$

meaning $\rho_{\varepsilon',j}$ is also well defined. Furthermore, in this case we have

$$n\rho^{n-1}_{\varepsilon,j}=(n-1)\rho^n_{\varepsilon,j-1}-\varepsilon\le(n-1)\rho^n_{\varepsilon',j-1}-\varepsilon'=n\rho^{n-1}_{\varepsilon',j},$$

which implies that $\rho_{\varepsilon',j}\ge\rho_{\varepsilon,j}$.

By Lemma 5(d) we have $\phi(\varepsilon)=\Theta(\log\log 1/\varepsilon)$. The second part of the theorem holds by Lemma 4 and Lemma 5(c). This finishes the proof of the theorem. □

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Appendix.

In this appendix, we show that for sufficiently large values of m, we have $1-2\log(m)/m \le A'_m \le 1-\log(m)/(3m)$. First, observe that the expected welfare with m buyers, obtained by a static price of T, can be lower bounded by the expected revenue, which is equal to $T \cdot \mathbb{P}(\max_{j\in\{1\dots,m\}} X_j > T) = T(1-T^m)$. Then, the optimal welfare with static prices can be lower bounded by the revenue of the static price $T^\star_m$ that maximizes the expected revenue $R(T) = T - T^{m+1}$, which is equal to $T^\star_m = (\frac{1}{m+1})^{1/m}$. In particular, the expected revenue with price $T^\star_m$ satisfies that

$$R(T^\star_m) = T^\star_m - (T^\star_m)^{m+1} = \left(\frac{1}{m+1}\right)^{1/m} - \left(\frac{1}{m+1}\right)^{(m+1)/m}.$$

For every $m \ge 1$ we have $R(T^\star_m) \ge R((1/m)^{1/m}) = (1/m)^{1/m} - (1/m)^{1+1/m}$, and we have $(1/m)^{1+1/m} \le 2/m$. Therefore, $R(T^\star_m) \ge (1/m)^{1/m} - 2/m$. Furthermore, observe that $(1/m)^{1/m} = \exp(-\log(m)/m) \ge 1-\log(m)/m$, where the last inequality holds since $\exp(-x) \ge 1-x$ for every $x \ge 0$. Hence, we conclude that the optimal revenue is at least $1-\log(m)/m-2/m \ge 1-2\log(m)/m$. For a given static price $T > 0$, the expected welfare with m buyers is equal to

$$W(T) = (1-T^m)T + (1-T^m)(1-T)/2 = \frac{1}{2}(1+T)(1-T^m) = \frac{1}{2}\left(\frac{1}{T}+1\right)R(T),$$

If $T > T^\star_m$, we have $R(T) < R(T^\star_m)$, and therefore

$$W(T) = \frac{1}{2}\left(\frac{1}{T}+1\right)R(T) < \frac{1}{2}\left(\frac{1}{T^\star_m}+1\right)R(T^\star_m) = W(T^\star_m).$$

Let $\overline{T}_m$ be the maximizer of the welfare W. The previous inequality implies that $\overline{T}_m \le T^\star_m$, and therefore

$$\begin{aligned} w(\overline{T}) &= \frac{1}{2}(1+T^\star_m)(1-T^\star_m) \\ &\le \frac{1}{2}(1+T^\star_m) = \frac{1}{2}\left(1+\left(\frac{1}{m+1}\right)^{1/m}\right) \le 1-\frac{\log(m)}{3m}, \end{aligned}$$

where the last inequality holds since the function $f(x) = 1-\frac{\log(x)}{3x} - \frac{1}{2}(1+(\frac{1}{x+1})^{1/x})$ is strictly decreasing in $[1,\infty)]$ and $\lim_{x\to\infty} f(x) = 0$.