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Stock Market Volatility and Learning

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Abstract
We study a standard consumption based asset pricing model with rationally investing agents but allow agents' prior beliefs about price and dividend behavior to deviate slightly from rational expectations priors. Learning about stock price behavior then causes the model to become quantitatively consistent with a range of basic asset prizing 'puzzles': stock returns display momentum and mean reversion, asset prices become volatile, the price-dividend ratio displays persistence, long-horizon returns become predictable and a risk premium emerges. Comparing the moments of the model with those in the data using confidence bands from the method of simulated moments, we show that our findings are robust to different assumptions on the system of beliefs and other model features. We depart from previous studies of asset prices under learning in that agents form expectations about future stock prices using past price observations.

JEL Classifications: G12, D84
Keywords: asset pricing, learning, near-rational price forecasts

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"Investors, their confidence and expectations buoyed by past price increases, bid up speculative prices further, thereby enticing more investors to do the same, so that the cycle repeats again and again."

Irrational Exuberance, Shiller (2005, p.56)

1 Introduction

The purpose of this paper is to show that a very simple asset pricing model is able to quantitatively reproduce a variety of stylized asset pricing facts if one allows for small departures from rational expectations.

We study a simple variant of Lucas (1978) model. It is well known that the asset prices implications of this model under rational expectations (RE) are at odds with some basic facts: in the data, the price dividend ratio is too volatile and persistent, stock returns are too volatile, long horizon excess returns of stocks are negatively related to the price dividend ratio, and the risk premium is too high. We stick to Lucas' framework but allow for agents whose prior beliefs about price and dividend behavior deviate slightly from those assumed under a rational expectations (RE) analysis of the model. Our slight relaxation of prior beliefs implies that agents need to learn about stock price behavior and we show that this feature alters the asset price predictions of the model in a way to make it quantitatively consistent with all the basic facts listed above.

We consider investors who hold a consistent system of beliefs about the stochastic process for payoff-relevant variables that are beyond their control. In our model, these variables consist of the (exogenous) dividend process and the process for (competitive) stock market prices. And given these beliefs investors maximize a standard time-separable utility function subject to their budget constraints. We call such agents 'internally rational'. Just to emphasize: such agents behave as if they knew dynamic programming under uncertainty, the theory of Markov stochastic processes and Bayesian updating and filtering.

Unlike in a RE analysis, however, we do not assume that agents know perfectly how a certain dividend history maps into a market outcome for the stock price.[1] Agents express this uncertainty by specifying a subjective dis-

[1] Such uncertainty may arise from a lack of common knowledge of investors' price and dividend beliefs, as is explained in detail in Adam and Marcet (forthcoming).

tribution over stock price and dividend outcomes and optimally update their beliefs about price behavior in the light of realized market outcomes. For a general class of beliefs, we find that such learning from market outcomes imparts 'momentum' on stock prices and produces large and sustained deviations of the price dividend ratio from its mean, as can be observed in the data. Such momentum arises because if agents' *expectations* about stock price growth increase in a given period, the *actual* growth rate of prices has a tendency to increase beyond the fundamental growth rate, thereby reinforcing the initial belief of higher stock price growth. Yet, the model also displays 'mean reversion' so that so that even if expectations are very high or very low at some point, they will eventually return to fundamentals. The model thus displays something like the 'naturally occurring Ponzi schemes' described in the opening quote.

Our formulation of beliefs naturally nests RE as a special case and therefore provides a precise sense in which beliefs are close to the RE prior beliefs. To capture the distance relative to the RE prior we add just one free parameter relative to the standard model and this parameter has a natural economic and statistical interpretation capturing the strength with which agents' beliefs react to market outcomes. Under the RE analysis, market outcomes carry only redundant information, so that this parameter is set to zero. But as we document, the match with the data becomes very good for belief systems that imply that agents should place a small but strictly positive weight on market information. We find it a striking observation that the asset pricing implications of the standard model are not robust to such small departures from rational price expectations and that this non-robustness is empirically so encouraging. This suggests that allowing for such kind of departures from a strong assumption (RE) could be a promising avenue for research more generally.

We wish to emphasize that we achieve the fit with the data within a very simple setting. The asset pricing equation *derived* in the paper is the same one-step-ahead equation that is typically considered under the RE analysis of the model. Moreover, internally rational behavior implies that agents use a standard model of adaptive learning to update their subjective expectations of the one-step-ahead variables appearing in the asset pricing equation. Specifically, Bayesian belief updating implies that agents optimally use least squares learning, and as we show, agents would asymptotically converge to the RE outcome, i.e., learn to become fully rational, although this takes a long time. To avoid the issue of asymptotically vanishing volatility and to obtain an ergodic equilibrium distribution, we focus in our empirical application mainly on a constant gain learning mechanism, which places non-vanishing weight on market information.

The paper is organized as follows. In section 2 we discuss the related literature and section 3 presents the stylized asset pricing facts we seek to match. In section 4 we then outline the asset pricing model and derive a series of analytic results for the case with risk neutral investors to show how our model is able to *qualitatively* deliver the stylized asset pricing facts for a general class of belief systems. In section 5 we present the baseline asset pricing model with risk aversion used in our empirical application and the baseline calibration procedure for matching the data. Section 6 shows that the baseline model can *quantitatively* reproduce the facts discussed in section 3 and also documents the robustness of our findings to various assumptions about the model or the estimation procedure.

Readers interested in obtaining a glimpse of the quantitative performance of our one parameter extension of the RE model may directly jump to Table 2 in section 6.1.

2 Related Literature

A large body of literature has documented that the basic asset pricing model with time separable preferences and RE has great difficulties in matching the volatility and persistence of the price dividend ratio, the volatility of stock returns, the predictability of excess returns at long horizons and the risk premium.[2] Models of learning have for long been considered as a promising avenue for increasing the model implied volatility and thereby the match with the data.

A string of papers within the adaptive learning literature study agents who learn about stock prices. Bullard and Duffy (2001) and Brock and Hommes (1998) show that learning dynamics can converge to complicated attractors and that the RE equilibrium may be unstable under learning dynamics.[3] Branch and Evans (forthcoming) study a model where agents' algorithm to form expectations switches depending on which of the available forecast models is performing best. Marcet and Sargent (1992) also study convergence to RE in a model where agents use today's price to forecast the price tomorrow in a stationary environment with private information. Cárceles-Poveda and Giannitsarou (2007) assume that agents know the mean

[2] See Cambell (2003) for an overview. Cecchetti, Lam, and Mark (2000) determine the misspecification in beliefs about future consumption growth required to match the equity premium and other moments of asset prices.

[3] Stability under learning dynamics is defined in Marcet and Sargent (1989).

stock price and find that learning does then not significantly alter the behavior of asset prices. Relative to these papers we address the data more closely and we derive our model of adaptive learning and asset pricing from internally rational investor behavior.

Stock price behavior under Bayesian learning has been previously studied in Timmermann (1993, 1996), Brennan and Xia (2001), Cogley and Sargent (2008) and Pastor and Veronesi (2003) among others. Agents in these papers learn about the dividend process and then set the asset price equal to the discounted expected sum of dividends. This amounts to assuming that agents' beliefs about the joint distribution of prices and dividends contains a singularity and that market outcomes contain only redundant information. As a result, there is no feedback from market outcomes (stock prices) to beliefs (price expectations). Agents' beliefs in these settings are thus 'anchored' by the exogenous dividend process, so that the volatility effects resulting from learning are generally limited when considering standard time separable preference specifications. In contrast, we largely abstract from learning about the dividend process and consider learning about stock price behavior using past price observations. In such a setting price beliefs and actual price outcomes mutually influence each other. It is precisely this self-referential nature of the learning problem that imparts momentum to expectations and is key in explaining stock price volatility.[4]

In contrast to the RE literature, the behavioral finance literature tries to understand the decision-making process of individual investors by means of surveys, experiments and micro evidence, exploring the intersection between economics and psychology, see Shiller (2005) for a non-technical summary. We borrow from this literature an interest in deviating from RE but we are interested in small deviations from the standard approach: we assume that agents behave optimally given a consistent system of subjective beliefs that are close to RE beliefs.

3 Facts

This section describes stylized facts of U.S. stock price data that we seek to replicate in our quantitative analysis. These observations have been extensively documented in the literature. We reproduce them here as a point of

[4]Timmerman (1996) also analyzes a case with self-referential learning, assuming that agents use dividends to predict future price. He reports that this form of learning delivers even lower volatility than a settings with learning about the dividend process only. It is thus crucial for our results that agents use information on past price behavior to predict future price behavior.

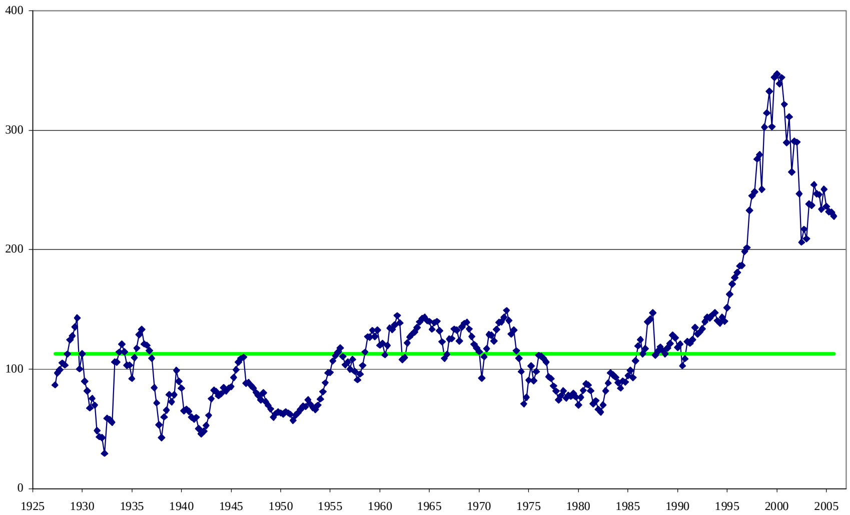

Figure 1: Quarterly U.S. price dividend ratio 1927:1-2005:4

reference using a single and updated data base.[5]

Since the work of Shiller (1981) and LeRoy and Porter (1981) it has been recognized that the volatility of stock prices in the data is much higher than standard RE asset pricing models suggest, given the available evidence on the volatility of dividends. Figure 1 shows the evolution of the quarterly price dividend (PD) ratio in the United States. The PD ratio displays very large fluctuations around its sample mean (the green horizontal line in the graph): in the year 1932, for example, the PD ratio takes on values below 30, while in the year 2000 values close to 350. The standard deviation of the PD ratio (σ_{PD}) is almost one half of its sample mean ($E(PD)$). We report this feature of the data as **Fact 1** in Table 1.

Figure 1 also shows that the deviation of the PD ratio from its sample mean are very persistent, so that the first order quarterly autocorrelation of the PD ratio ($\rho_{PD,-1}$) is very high. We report this as **Fact 2** in Table 1 below .

Related to the excessive volatility of prices is the observation that the volatility of stock returns (σ_{r^s}) in the data is almost four times the volatility of dividend growth ($\sigma_{\Delta D/D}$). We report the volatility of returns as **Fact 3** in Table 1, and the mean and standard deviation of dividend growth at the

[5] Details on the data sources are provided in Appendix 8.1.

bottom of the table.[6]

U.S. asset pricing facts, 1927:2-2005:4 (quarterly real values, growth rates & returns in percentage terms)			
Fact 1	Volatility of PD ratio	$E(PD)$ σ_{PD}	113.20 52.98
Fact 2	Persistence of PD ratio	$\rho_{PD,-1}$	0.92
Fact 3	Excessive return volatility	σ_{r^s}	11.65
Fact 4	Excess return predictability	c_5^2 R_5^2	-0.0048 0.1986
Fact 5	Equity premium	$E\left[r^s\right]$ $E\left[r^b\right]$	2.41 0.18
Dividend Behavior	Mean Growth Std of Growth	$E\left(\frac{\Delta D}{D}\right)$ $\sigma_{\frac{\Delta D}{D}}$	0.0035 2.98

Table 1: Stylized asset pricing facts

While stock returns are difficult to predict at short horizons, the PD ratio helps to predict future excess stock returns in the long run. More precisely, estimating the regression

$$X_{t,n} = c_n^1 + c_n^2 \; PD_t + u_{t,n}$$

where $X_{t,n}$ is the observed real excess return of stocks over bonds from quarter t to quarter t plus n years, and $u_{t,n}$ the regression residual, the estimate c_n^2 is found to be negative, significantly different from zero, and

[6] We treat the dividend process as exogenous in the model, therefore do not report the dividend moments as 'Facts' that have to be matched endogenously by our model.

the absolute value of c_n^2 and the R-square of this regression, denoted R_n^2, increase with n. We choose to include the OLS regression results for the 5-year horizon as **Fact 4** in Table 1.[7]

Finally, it is well known that through the lens of standard models real stock returns tend to be too high relative to short-term real bond returns. This so called equity premium puzzle is reported as **Fact 5** in Table 1, which shows that the average quarterly real return on bonds $E\left(r_t^b\right)$ is much lower than the corresponding return on stocks $E\left(r_t^s\right)$.

Table 1 reports ten statistics. As we show in section 6, once we use the evidence on dividend growth to calibrate the dividend process in the model, we can replicate the remaining 8 statistics listed as Facts 1-5 using a model that has only two free parameters.

4 The Theory

We describe in section 4.1 below a basic Lucas (1978) asset pricing model with internally rational agents and derive the resulting asset pricing equation when agents are uncertain about the price outcomes. Section 4.2 then presents - for the case with risk neutral investors - a number of analytical results regarding asset price behavior under learning. These results illustrate why our learning model can qualitatively replicate the facts mentioned in Table 1 for a large number of natural belief specifications. Finally, section 4.3 shows how internally rational behavior is compatible with well known models of adaptive learning.

4.1 The Model

The Environment: Consider an economy populated by a unit mass of infinitely-lived investors endowed with one unit of a stock that can be traded on a competitive stock market and that pays a dividend D_t, which evolves according to

$$\frac{D_t}{D_{t-1}} = a\varepsilon_t \tag{1}$$

[7] We focus on the 5-year horizon for simplicity, but obtain very similar results for other horizons. Our focus on a single horizon is justified because chapter 20 in Cochrane (2005) shows that Facts 1, 2 and 4 are closely related: up to a linear approximation, the presence of return predictability and the increase in the R_n^2 with the prediction horizon n are *qualitatively* a joint consequence of persistent PD ratios (Fact 2) and *i.i.d.* dividend growth. It is not surprising, therefore, that our model also reproduces the increasing size of c_n^2 and R_n^2 with n. We match the regression coefficients at the 5-year horizon to check the *quantitative* model implications.

for $t = 0, 1, 2, ...,$ where $\log \varepsilon_t \sim ii\mathcal{N}(-\frac{s^2}{2}, s^2)$ and $a \geq 1$. These assumptions guarantee that $E(\varepsilon_t) = 1$, $E\left(\frac{D_t}{D_{t-1}}\right) = a$ and $\sigma_{\frac{\Delta D}{D}} = s$.

Objective Function and Probability Space: Agent i has a standard time-separable expected utility function

$$E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t u(C_t^i)$$

where $u(\cdot)$ is a concave function, C_t^i denotes consumption and the expectation is with respect to a subjective probability measure $\mathcal{P}$ assigning a consistent set of probabilities to variables that are beyond the agent's control.

The competitive market assumption and the exogeneity of the dividend process imply that investors consider the process for prices and dividends as exogenous to their decision problem. The underlying sample (or state) space Ω thus consists of the space of realizations for dividends and prices. Specifically, a typical element $\omega \in \Omega$ is an infinite sequence $\omega = \{P_t, D_t\}_{t=0}^{\infty}$ where P_t is the stock price in period t. As usual, we let Ω^t denote the set of histories from period zero up to period t and ω^t its typical element. The underlying probability space is then given by $(\Omega, \mathcal{B}, \mathcal{P})$ with $\mathcal{B}$ denoting the corresponding σ-Algebra of Borel subsets of Ω, and $\mathcal{P}$ a probability measure over $(\Omega, \mathcal{B})$. Expected utility is thus defined as

$$E_0^{\mathcal{P}} \sum_{t=0}^{\infty} \delta^t u(C_t^i) \equiv \int_{\Omega} \sum_{t=0}^{\infty} \delta^t \ u(C_t^i(\omega^t)) \ d\mathcal{P}(\omega). \tag{2}$$

Importantly, our definition of the probability space is non-standard because we include price histories in the realization ω^t. Standard practice is to assume instead that agents know the mapping from history of dividends to asset prices $P_t(D^t)$, as it emerges in equilibrium, such that for each possible dividend realization, agents can exactly and without any doubt - with probability one - compute the associated price for the asset. This practice is standard in models of rational expectations, in models with rational bubbles, in Bayesian RE models, and in models of robust control.

The standard practice amounts to imposing form the start a *singularity* in the joint density over prices and dividends; and the presence of this singularity justifies that the agent's state space Ω can be defined from the outset in terms of realizations of the fundamentals (dividends) only. Without doubt this practice is a convenient modeling device, as it allows the modeler to abstract from a potential independent role for expectations and to focus on other features of the environment, as has been advocated by the RE literature over the last decades. But the assumption that agents know *exactly* the equilibrium pricing function $P_t(\cdot)$ is undoubtedly a very strong assumption and we propose to relax it here.[8] Doing so will allow us to eliminate the asymmetric treatment of learning in much of the existing literature, which assumes that agents learn about dividend behavior but know all about price behavior (conditional on dividends).

Choices and Constraints: Agents make contingent consumption and stockholding plans, i.e., choose for all t the functions

$$\left(C_t^i, S_t^i\right) : \Omega^t \to R^2 \tag{3}$$

where S_t^i denotes the stock holdings in period t. Their choices are subject to a standard budget constraint

$$C_t^i + P_t \, S_t^i \leq (P_t + D_t) \, S_{t-1}^i \tag{4}$$

for all $t \geq 0$, taking as given the initial stockholding $S_{-1}^i = 1$. In addition, the agent faces the following limit constraints on stockholdings

$$S_t^i \geq 0 \tag{5}$$

$$S_t^i \leq \overline{S} \tag{6}$$

for some $\overline{S} \in (1, \infty)$. Constraint (5) is a standard short-selling constraint and often used in the literature. The second constraint (6) is a simplified form of a leverage constraint capturing the fact that the consumer cannot buy arbitrarily large amounts of stocks.

Maximizing Behavior (Internal Rationality): The investor's problem then consists of choosing the sequence of functions $\{C_t^i, S_t^i\}_{t=0}^{\infty}$ defined in (3) to maximize (2) subject to the budget constraint (4) and the limit constraints (5) and (6), where all constraints have to hold for all $\omega^t \in \Omega^t$ and all t almost sure, taking as given the probability measure $\mathcal{P}$.

At this point we wish to emphasize that the probability measure $\mathcal{P}$ may arise from a Bayesian learning problem. For example, it may be generated by some perceived law of motion describing the evolution of prices and dividends over time containing parameters about which the agent entertains prior beliefs that are updated in the light of new information. We present

[8] Adam and Marcet (forthcoming) discuss the strong informational assumptions required for agents to be able to deduce the market equilibrium mapping $P_t(D^t)$ from the outset.

such explicit examples in the latter part of the paper. For the moment, the learning problem remains 'hidden' in the belief structure $\mathcal{P}$.

Optimality Conditions: Since the objective function is concave and the feasible set is convex and compact in S_t^i, the agent's optimal plan is characterized by the first order condition

$$\lambda_t^i + u'(C_t^i)P_t = \delta E_t^{\mathcal{P}}\left(u'(C_{t+1}^i)(P_{t+1} + D_{t+1}\right) \tag{7}$$

where $\lambda_t^i \lesseqgtr 0$ is the sum of the Lagrange multipliers associated with the constraints (5) and (6). The first order condition involves the expectation of next period's marginal utility times next period's stock payoff $P_{t+1} + D_{t+1}$. To emphasize, the optimality condition does *not* compare the discounted sum of dividends to the stock price. This is so because agents in this world engage in speculative trading in the sense of Harrison and Kreps (1978), i.e., it is optimal to 'buy low and sell high'.

To see what it would take to find a discounted sum relationship, note that one can iterate forward on (7). Assuming that subjective beliefs satisfy $\lim_{T\to\infty} \delta^T E_t^{\mathcal{P}}(P_{t+T}) = 0$ one obtains

$$P_t = E_t^{\mathcal{P}}\left[\sum_{j=1}^{\infty} \delta^j \frac{u'(C_{t+j+1}^i)}{u'(C_t^i)} D_{t+1+j} + \sum_{j=0}^{\infty} \delta^j \lambda_{t+1+j}^i\right] \tag{8}$$

Since $\lambda_t^i \gtreqless 0$ from the agents' viewpoint, the stock price can be larger, equal, or smaller than the agents' expected discounted sum of dividends. Only in the special case where the agent believes to be marginal in every period and every contingency will the agent's price expectations be implied by her dividend expectations and the first order conditions. Yet, for small deviations from RE beliefs there is no reason why the agent should think that future λ's in her maximization problem will all be zero with certainty.

In the example of this paper (with homogeneous agents), agents would know that they are marginal every period, if we assumed that it is common knowledge that all agents are alike and share the same beliefs. Such common knowledge implies

$$E_t^{\mathcal{P}}\left[\sum_{j=0}^{\infty} \delta^j \lambda_{t+1+j}^i\right] = 0, \qquad \text{for all } i \text{ and all } t. \tag{9}$$

so that agents can use equation (8) to compute exactly the relationship between dividends and prices. Their belief system then *must* exhibit a singularity to be consistent with optimal behavior. But absent such knowledge, price beliefs of internally rational agents are not implied by their dividend beliefs.

Equilibrium Asset Pricing Equation: As economic modelers we know that all agents are alike, therefore know that in equilibrium $S_t^i = 1$ and $\lambda_t^i = 0$ for all i and t. Equation (7) thus implies that the equilibrium price satisfies

$$u'(C_t^i)P_t = \delta E_t^{\mathcal{P}} \left(u'(C_{t+1}^i)(P_{t+1} + D_{t+1} \right) \tag{10}$$

This is the familiar first order condition emerging from a RE analysis of the model, but now evaluated with subjective beliefs $\mathcal{P}$ that do not necessarily contain a singularity, i.e., where price expectations can deviate from the expected discounted sum of dividends. The rest of the paper will explore the pricing implications of this equation when agents learn about price and dividend behavior.

4.2 Asset Pricing Implications: Analytical Results

We now show analytically how learning helps in qualitatively replicating Facts 1 to 4 listed in Table 1 above.[9] In the interest of deriving analytic results, we simplify the model along a number of dimensions. First, we assume investors to be risk neutral ($u(C) = C$). As argued below, the RE version of the model then misses all the facts listed in Table 1, allowing us to clearly highlight the improvements achieved from learning. Second, we assume that agents know the true dividend process, i.e., $E_t^{\mathcal{P}}(D_{t+1}) = aD_t$. This serves to emphasize that learning about prices is the crucial ingredient improving the match with the data. Third, we impose an upper bound on beliefs to insure existence of equilibrium. We relax these assumptions in our empirical section.

With risk neutrality and RE, equilibrium prices are given by

$$P_t^{RE} = \frac{\delta a}{1 - \delta a} D_t$$

The PD ratio is thus constant, implying that returns are approximately as volatile as dividend growth, that there is no return predictability, and that the risk premium is zero. Under RE the model is thus completely at odds with the evidence presented in Table 1.

[9] We explain why the model delivers Fact 5, the equity premium, in Section 6.

To analyze the model dynamics under learning, let us *define* agents' subjective expectations of stock price growth at time t:

$$\beta_t \equiv E_t^{\mathcal{P}}\left(\frac{P_{t+1}}{P_t}\right). \tag{11}$$

The equilibrium pricing equation (10) can then be written as

$$P_t = \delta\ \beta_t\ P_t + \delta a\ D_t \tag{12}$$

and provided $\beta_t \leq \delta^{-1}$, the equilibrium price under learning is

$$P_t = \frac{\delta a D_t}{1-\delta\beta_t}. \tag{13}$$

For $\beta_t = \beta^{RE} = a$ this reduces to the RE pricing outcome above. Yet, equation (13) shows that fluctuations in price expectations β_t now contribute to the fluctuations in actual prices, thereby generating 'excess volatility'. Indeed, as long as the correlation between β_t and the last dividend innovation ε_t is small (as occurs for most updating schemes for β_t), we have

$$Var\left(\ln\frac{P_t}{P_{t-1}}\right) \simeq Var\left(\ln\frac{1-\delta\beta_{t-1}}{1-\delta\beta_t}\right) + Var\left(\ln\frac{D_t}{D_{t-1}}\right), \tag{14}$$

If β_t fluctuates around values close to but below δ^{-1}, then even small fluctuations in beliefs can have large variance implications. This emerges because the pricing equation (13) has an asymptote as price growth expectations approach the inverse of the discount factor.

In general, β_t will be updated each period in the light of the newly available information, as dictated by the probability measure $\mathcal{P}$. We proceed here by assuming that β_t adjusts in the same direction as the last prediction error. This implies that agents revise price growth expectations upwards (downwards) if their expectations last period underpredicted (overpredicted) the stock price growth that was actually realized in this period. Formally, we consider measures $\mathcal{P}$ that imply updating rules of the form:[10]

$$\Delta\beta_t = f_t\left(\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}\right) \tag{15}$$

[10] Note that β_t is determined from observations up to period $t-1$ only. This simplifies the analysis and it avoids simultaneity of price and forecast determination. This lag in the information is common in the learning literature. Difficulties emerging with simultaneous information sets in models of adaptive learning are discussed in Adam (2003).

for some given functions $f_t : R \to R$ with the properties

$$\begin{aligned} & f_t(0) = 0 \\ & f_t(\cdot) \text{ increasing} \end{aligned} \tag{16}$$

The updating rule (15) is consistent with many reasonable belief systems $\mathcal{P}$ that are close to the RE beliefs and we will consider a particular example of this form in section 4.3 below. The updating rule also nests a range of standard learning rules typically considered in the literature on adaptive learning, e.g., least squares learning or constant gain learning rules.

We need to restrict the above learning scheme further so as to guarantee that expectations remain bounded below the inverse of the discount factor. In this section we impose a 'standard projection facility', which assumes that agents simply ignore observations that would push the expected price growth β_t beyond some upper bound $\beta^U < \delta^{-1}$. Formally,

$$\Delta\beta_t = \begin{cases} f_t\left(\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}\right) & \text{if } \beta_{t-1} + f_t\left(\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}\right) < \beta^U \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

This projection facility simplifies some of the proofs and has been used in many learning papers, including Timmermann (1993, 1996), Marcet and Sargent (1989), Evans and Honkapohja (2001), and Cogley and Sargent (2008). This completes the description of the learning rule.

The next section analyzes the price dividend dynamics implied by the pricing equation (13) and the postulated updating rule (15) and (17).

4.2.1 Behavior of the PD Ratio under Learning

The pricing equation (13) shows that the PD ratio is a strictly positive function of agents' conditional price growth expectations β_t. One can thus understand the qualitative dynamics of the PD ratio by studying instead the dynamics of the price growth expectations β_t. Equation (13) also implies that the realized price growth is given by

$$\frac{P_t}{P_{t-1}} = T(\beta_t, \Delta\beta_t)\ \varepsilon_t \tag{18}$$

where

$$T(\beta, \Delta\beta) \equiv a + \frac{a\delta\ \Delta\beta}{1 - \delta\beta} \tag{19}$$

The realized stock price growth is thus larger (smaller) than the fundamental growth rate $a\varepsilon_t$, whenever agents have become more (less) optimistic about stock price growth compared to the previous period, i.e., whenever $\Delta\beta > 0$ ($\Delta\beta < 0$). Substituting (18) into the updating equation (15) gives rise to a second-order stochastic difference equation for β_t:

$$\Delta\beta_{t+1} = f_{t+1} \left(T(\beta_t, \Delta\beta_t)\varepsilon_t - \beta_t \right) \tag{20}$$

This equation completely characterizes the equilibrium dynamics of β_t ($t \geq 1$), given initial conditions (D_0, P_{-1}), and initial expectations β_0. Due to non-linearities in the $T-$mapping defined in (19), this equation can not be solved analytically.

To analyze the dynamics of the price growth expectations β_t (and of the PD ratio) implied by equation (20) we restrict consideration to the deterministic case with $\varepsilon_t \equiv 1$. This allows us to focus on the endogenous stock price dynamics generated by the learning mechanism rather than the dynamics induced by exogenous dividend disturbances.

The properties of the second order difference equation (20) can then be illustrated in a 2-dimensional phase diagram for the dynamics of the expectations (β_t, β_{t-1}), as shown in Figure 2.[11] The arrows in the figure indicate the direction in which the vector (β_t, β_{t-1}) moves as it evolves according to equation (20) with $\varepsilon_t = 1$, and the solid lines indicate the boundaries of these areas.[12] Since we have a difference rather than a differential equation, we cannot plot the evolution of expectations exactly, but the arrows suggest that the expectations are likely to move in ellipses around the rational expectations equilibrium $(\beta_t, \beta_{t-1}) = (a, a)$.

Consider, for example, point A in the diagram. At this point β_t is already below its fundamental value a, but the phase diagram indicates that expectations will fall further. This shows that there is *momentum* in price changes: the fact that agents at point A have become less optimistic relative to the previous period ($\beta_t < \beta_{t-1}$) implies that price growth optimism and prices will fall further. Expectations move, for example, to point B where they will start to revert direction and move on to point C, then display upward momentum and move to point D, thereby displaying *mean reversion.* The elliptic movements imply that expectations (and thus the PD ratio) are likely to oscillate in sustained and persistent swings around the RE value a.

While under RE, the PD ratio is constant, it will tend to oscillate around the RE value under learning. Such behavior helps generating the observed volatility and serial correlation of the PD ratio (Facts 1 and 2). Also, according to our discussion around equation (14), momentum imparts variability

[11] Appendix 8.3 explains in detail the construction of the phase diagram.

[12] The vertical solid line close to δ^{-1} is meant to illustrate the restriction $\beta < \delta^{-1}$ imposed by the projection facility.

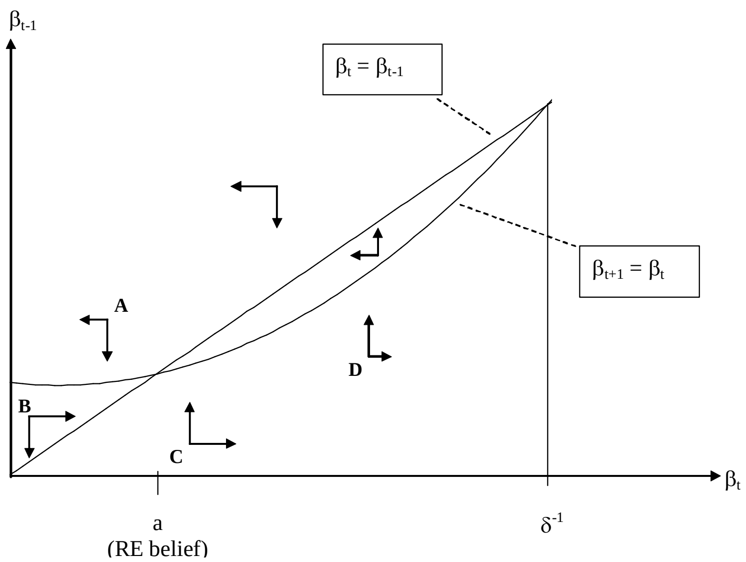

Figure 2: Phase diagram illustrating momentum and mean-reversion

to the ratio $\frac{1-\delta\beta_{t-1}}{1-\delta\beta_t}$ and it is likely to deliver more volatile stock returns (Fact 3). As discussed in Cochrane (2005), a serially correlated and mean reverting PD ratio gives rise to excess return predictability (Fact 4).

The momentum and mean reverting behavior of growth expectations (and the PD ratio) can also be stated more formally:

Momentum: *If* $\beta_t \leq \beta^{RE} \equiv a$ *and* $\Delta\beta_t > 0$, *then*

$$\Delta\beta_{t+1} > 0$$

This equally holds if all inequalities are reversed.

The previous claim follows directly from equation (19), which shows that $\Delta\beta_t > 0$ implies $T(\beta_t, \Delta\beta_t) > a$, so that β has to increase further due to the properties of the updating function f_t stated in (16).[13] The presence of $\Delta\beta_t$ in the determination of actual price growth thus imparts upward and downward momentum on stock prices around the RE value.

Regarding mean reverting the following result shows that stock prices eventually move back towards their fundamental (RE) value and do so in a monotonic way in the absence of dividend growth shocks:[14]

Mean reversion: *If in some period* t *we have* $\beta_t > a$, *then for any* $\eta > 0$ *sufficiently small, there is a finite period* $t'' > t$ *such that* $\beta_{t''} < a + \eta$.

Furthermore, oscillations are monotonic in the sense that letting t' *be the first period* $t'' \geq t' \geq t$ *such that* $\Delta\beta_{t'} < 0$, *then* β_t *is non-decreasing between* t *and* t' *and it is non-increasing between* t' *and* t''.

Symmetrically, if $\beta_t < a$ *eventually* $\beta_{t''} > a - \eta$ *and oscillations are monotonic.*

4.3 Optimal Belief Updating: Least Squares Learning

This section presents specific probability measures $\mathcal{P}$ that give rise to updating schemes satisfying equations (15) and (16). We show how the measure $\mathcal{P}$ can be chosen arbitrarily close to RE beliefs and that agents' beliefs will asymptotically converge to the RE outcome, independently of their initial beliefs.

[13] This assumes that the projection facility is suffiently 'loose' and does not bind.

[14] See Appendix 8.2 for some additional technical assumptions required for the proof.

Suppose agents believe that prices and dividends follow a random walk with drift

$$\begin{bmatrix} \log P_t/P_{t-1} \\ \log D_t/D_{t-1} \end{bmatrix} = \begin{bmatrix} \log \beta^P \\ \log \beta^D \end{bmatrix} + \begin{bmatrix} \log \varepsilon_t^P \\ \log \varepsilon_t^D \end{bmatrix} \tag{21}$$

with

$$(\log \varepsilon_t^P, \log \varepsilon_t^D)' \sim N(0, \Sigma) \tag{22}$$

The RE outcome is of this form when $\log \beta^P = \log \beta^D = (\log a) - \frac{s^2}{2}$ and $\Sigma = s^2 E$ where E is a $2x2$ matrix of with all entries equal to one. Following Adam and Marcet (forthcoming), we endow agents with a grain of doubt about the true values of $\left(\log \beta^P, \log \beta^D, \Sigma\right)$ and assume that this uncertainty can be described by a Normal-Wishart conjugate prior density over these parameters:

$$(\log \beta^P, \log \beta^D, \Sigma) \sim pri$$

Equations (21) and (22) together with the prior beliefs define a probability measure $\mathcal{P}$. As shown in Adam and Marcet (forthcoming), recursive least squares learning equations then describe (up to a log linear approximation) the evolution of the one-step-ahead price growth expectations

$$\beta_t = \beta_{t-1} + \frac{1}{\alpha_t}\left(\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}\right), \text{ all } t \geq 1 \tag{23}$$

$$\alpha_{t+1} = \alpha_t + 1 \tag{24}$$

where as in equation (15) we introduce a delay in the way information is incorporated into expectations to eliminate the simultaneity between prices and price growth expectations.[15]

The initial values β_0 and $1/\alpha_1$ in the above updating equations are thereby functions of the prior density pri. For the RE prior we have $\beta_0 = a$ and $1/\alpha_1 = 0$, so that agents believe in an expected price growth of a no matter how prices have been growing in the past. If one allows for more dispersed initial priors that remain centered on the RE outcome, then one has $\beta_0 = a$ and $1/\alpha_1 > 0$. Such priors place a grain of truth on the RE outcome but also on outcomes nearby. Price growth realizations are then used to learn about the true value of the parameters, leading to the kind of fluctuations in the conditional price growth expectations described in

[15]Similar equations describe the evolution of dividend growth expectations. Defining $\beta_t^D \equiv E_t^{\mathcal{P}}[D_{t+1}/D_t]$ and using a similar information lag, we have $\beta_t^D = \beta_{t-1}^D + \frac{1}{\alpha_t}\left(\frac{D_{t-1}}{D_{t-2}} - \beta_{t-1}^D\right)$.

equations (23) and (24). As the belief system $\mathcal{P}$ associated with such more dispersed prior densities converges *in distribution* to the RE beliefs, one has that $1/\alpha_1 \to 0$. Small values for the so-called 'gain' parameter $1/\alpha_1$ thus indicate small deviations from RE priors.

Note that the updating equations (23) and (24) deliver a pure data driven ordinary least squares (OLS) estimate for $1/\alpha_1 = 1$, which amounts to imposing an 'uninformative prior' about stock price growth. The gain parameter should thus lie in the interval $1/\alpha_1 \in [0, 1]$ with the lower bound indicating the RE prior, which places no weight on price growth data, and the upper bound indicating the pure OLS estimate, which places all weight on the data and none on prior beliefs. In our empirical applications we will consider beliefs with values for $1/\alpha_1$ very close to zero.

We now explore the asymptotic behavior of beliefs if agents use the updating scheme (23) and (24) and if equilibrium prices under learning are given by equation (13). For convenience we assume that agents know the true law of motion for the dividend process, i.e., $E_t^{\mathcal{P}}[D_{t+1}] = aD_t$ for all t.[16] We also allow here for dividend uncertainty ($\varepsilon_t \gtrless 1$):

Convergence of OLS *If $\beta^U > \beta^{RE}$, the learning scheme (23)-(24) satisfies*

$$\beta_t \to a \text{ almost surely as } t \to \infty$$

for any initial conditions $0 < 1/\alpha_1 \leq 1, \beta_0 \in (0, \beta^U)$.

The proof is stated in Appendix 8.6 and requires some mild additional technical assumptions. Note that we have a global convergence result, while many results in the literature on self-referential learning are about local convergence.

The convergence result above is useful because it shows that beliefs do not stay away from the fundamental value a forever, even in a stochastic model and even when the gain $1/\alpha_1$ is very small to start with and converges to zero. Our limiting result implies, however, that asset price volatility asymptotically decreases, which is counterfactual, see Figure 1. The empirical application in the next section, therefore, adopts a 'constant gain' learning algorithm, which assumes that $\alpha_t = \alpha$ in equation (23) for some value for α close to zero. The model then does not converge to the RE outcome asymptotically but tends to fluctuate around the limit of the least squares learning outcome a with the fluctuations been smaller the smaller is α.[17]

[16] As is well known, agents will learn all about an exogenously evolving stochastic process, if their prior beliefs contain a 'grain of truth', as is the case in the present setting.

[17] Constant gaint learning mechanism can be microfounded by slightly different systems

5 Baseline Model and Testing Procedure

The previous section explained how the introduction of learning qualitatively improves the ability of the model to match the data. This section performs a formal test of the quantitative model performance. It turns out that even under risk neutrality, learning improves the quantitative performance substantially relative to the rational expectations version, although not all facts listed in Table 1 can be replicated.[18] We therefore consider in this section a model with moderate degrees of risk aversion, which can provide a remarkably good match of *all* the facts described in Table 1. We now introduce our preferred baseline model and testing procedure.

5.1 Learning under Risk Aversion

We consider investors with a CRRA utility function

$$u(C) = \frac{C^{1-\gamma}}{1-\gamma}$$

with $\gamma \geq 0$ and assume that each investor has an additional non-tradable income stream given by ϕD_t. For $\phi > 0$ sufficiently large, subjective consumption growth is well approximated by $C_t/C_{t+1} \approx D_t/D_{t+1}$, independently of the subjective stockholding plans of the investor. This greatly facilitates the analysis because for large ϕ the asset pricing equation (10) can then be written as the familiar equation:[19]

$$P_t = \delta E_t^{\mathcal{P}} \left(\left(\frac{D_t}{D_{t+1}} \right)^{\gamma} (P_{t+1} + D_{t+1}) \right) \tag{25}$$

Under RE the equilibrium stock price is

$$P_t^{RE} = \frac{\delta \beta^{RE}}{1 - \delta \beta^{RE}} D_t \tag{26}$$

where

$$\beta^{RE} = a^{1-\gamma} e^{-\gamma(1-\gamma)\frac{s^2}{2}} \tag{27}$$

$$= E_t \left(\left(\frac{D_t}{D_{t+1}} \right)^{\gamma} \frac{P_{t+1}^{RE}}{P_t^{RE}} \right) \tag{28}$$

of beliefs $\mathcal{P}$ where agents believe the random variables $\log \beta^P$ and $\log \beta^D$ in (21) to follow a unit root process.

[18] See the working paper version for details: Adam, Marcet and Nicolini (2008).

[19] While we equalize consumption and dividend growth volatility in this section, section 6.2 considers the case where dividend volatility exceeds consumption volatility.

As in the case with risk neutrality, the price dividend ratio under RE is constant and the model predictions are at odds with the facts listed in Table 1.

When agents are only internally rational, but know the dividend process, equation (25) implies:

$$P_t = \delta\beta_t P_t + \delta E_t \left(\frac{D_t^\gamma}{D_{t+1}^{\gamma-1}} \right) \tag{29}$$

where β_t is agents' subjective conditional expectations of *risk-adjusted* stock price growth at t

$$\beta_t \equiv E_t^{\mathcal{P}} \left(\left(\frac{D_t}{D_{t+1}} \right)^\gamma \frac{P_{t+1}}{P_t} \right) \tag{30}$$

Obviously, agents have rational expectations beliefs if $\beta_t = \beta^{RE}$ for all t.

We assume agents update their estimate about (risk-adjusted) stock price growth according to

$$\beta_t = \beta_{t-1} + \frac{1}{\alpha} \left[\left(\frac{D_{t-2}}{D_{t-1}} \right)^\gamma \frac{P_{t-1}}{P_{t-2}} - \beta_{t-1} \right] \tag{31}$$

for some small constant gain $1/\alpha$, as discussed in section 4.3.

For $\gamma > 1$, the variance of realized risk-adjusted stock price growth under RE increases with γ.[20] Thus, even moderate risk aversion coefficients are likely to generate more volatility in price growth expectations and, therefore, of actual prices under learning. Risk aversion thus produces additional volatility under RE and this improves the learning model's ability to match the facts considered in Table 1.

As in the risk-neutral case we need to impose a projection facility to insure that $\beta_t < \delta^{-1}$. The projection facility described in (17) is convenient to derive analytical results but has two shortcomings. First, it introduces a discontinuity in the simulated path and thereby unnecessarily complicates numerical searches over the parameter space; second, we are not aware of a complete set of beliefs $\mathcal{P}$ for which rational updating implies that some observations are simply disregarded. Therefore, we assume instead that for high values of β_t further increases in β_t are dampened in a smooth and

[20] The variance of risk adjusted stock price growth under rational expectations is

$$VAR\left(\left(\frac{D_{t-2}}{D_{t-1}} \right)^\gamma \frac{P_{t-1}^{RE}}{P_{t-2}^{RE}} \right) = a^{2(1-\gamma)} e^{(-\gamma)(1-\gamma)\frac{s^2}{2}} (e^{(1-\gamma)^2 s^2} - 1)$$

This variance reaches a minimum for $\gamma = 1$ and it increases with γ for $\gamma \geq 1$.

continuous fashion. In other words, we assume that individuals start to *downplay* observations that would entail too high an expected growth of stock price rather than completely *ignoring* these observations above some threshold. The dampening still insures that $\beta_t < \beta^U < \delta^{-1}$ but a continuous projection facility also preserves differentiability of the solution with respect to model parameters values and thereby facilitates the estimation in section 6. Furthermore, if $\mathcal{P}$ is generated from an initial prior belief on the growth rate of stock prices that is truncated at some value β^U, rational agents would equally downplay observations of the growth rate in such a way. Details on the continuous projection facility are described in appendix 8.5.5.

5.2 Baseline Testing Procedure

This section describes and discusses our baseline procedure for fitting and testing the baseline model from the previous section. Technical details are described in appendix 8.5.

The parameter vector of the baseline model is $\theta \equiv (\delta, \gamma, \alpha, a, s)$, where δ denotes the discount factor, γ the coefficient of relative risk aversion, $1/\alpha$ the agents' gain parameter, and a the mean and s the standard deviation of dividend growth.

We fix three of these parameters up-front. To illustrate that the model can match the volatility of stock prices for levels of risk aversion that are generally considered to be 'low' within the asset pricing literature, we simply fix $\gamma = 5$. We also fix the mean and standard deviation of the dividend growth process to the U.S. values reported in Table 1.

This leaves us with *two* free parameters (δ, α) and the following *eight* remaining sample moments reported in table 1:

$$\widehat{\mathcal{S}} \equiv \left(\widehat{E}(r^s), \widehat{E}(PD), \widehat{\sigma}_{r^s}, \widehat{\sigma}_{PD}, \widehat{\rho}_{PD_t,-1}, \widehat{c}_2^5, \widehat{R}_5^2, \widehat{E}(r^b)\right)' \tag{32}$$

Our aim is to show that there are parameter values (δ, α) that make the model consistent with these eight moments.[21]

The usual practice in calibration exercises is to fix δ and/or α to match some additional moments exactly and to use the remaining moments to test the model. Yet, many of the reported asset pricing moments are estimated

[21] Strictly speaking, many elements of $\mathcal{S}$ are not sample moments but functions of sample moments. For example, the R-square coefficient is a highly non-linear function of moments. This generates some technical problems, which are discussed in appendix 8.5. It would be more precise to refer to $\widehat{\mathcal{S}}$ as 'sample statistics', as we do in the appendix. For simplicity we avoid this terminology in the main text of the paper.

very imprecisely. This is shown in Table 2 below, which reports an estimate of the standard deviation of each sample moment in the column labeled 'US Data std'. Given the substantial uncertainty about the true value of these moments, matching any of them exactly appears arbitrary because one obtains rather different parameter values depending on which moment is chosen.[22] Therefore, we use a version of the method of simulated moments (MSM) to choose values for (δ, α) that globally fit all eight moments in $\widehat{\mathcal{S}}$.

To find the best fit we proceed as follows. Let $\widetilde{\mathcal{S}}(\theta)$ denote the moments implied by the model at some parameter value θ. As in the method of simulated moments MSM the chosen parameters $\widehat{\delta}, \widehat{\alpha}$ are defined by

$$\left(\widehat{\delta}, \widehat{\alpha}\right) \equiv \arg\min_{\delta,\alpha} \left[\widehat{\mathcal{S}} - \widetilde{\mathcal{S}}(\theta)\right]' \ W \ \left[\widehat{\mathcal{S}} - \widetilde{\mathcal{S}}(\theta)\right] \tag{33}$$

for some positive definite weighting matrix W that may be a function of the data.

For our baseline calibration procedure we choose W to be the diagonal matrix with entry $1/\widehat{\sigma}^2_{\mathcal{S}_i}$ in the i-th element of the diagonal, where $\widehat{\sigma}_{\mathcal{S}_i}$ is the estimated standard deviation of the i-th element of $\widehat{\mathcal{S}}$, reported in column 3 of Table 2. Appendix 8.5 shows consistent estimates for $\widehat{\sigma}_{\mathcal{S}_i}$ and it shows that the t-ratios

$$\frac{\widehat{\mathcal{S}}_i - \widetilde{\mathcal{S}}_i(\theta_0)}{\widehat{\sigma}_{\mathcal{S}_i}} \tag{34}$$

have a standard normal asymptotic distribution if the model and the parameter values θ_0 are true. The baseline testing procedure will check whether the t-ratios are less than 2 or 3 when we compute the $t-$ratios in (34) with $\widetilde{\mathcal{S}}_i(\widehat{\theta})$ instead of $\widetilde{\mathcal{S}}_i(\theta_0)$.

This baseline testing procedure is relatively simple and lies somewhere in between standard calibration approaches and MSM estimation. It resembles MSM in that we find the parameter values that best match all remaining moments globally. But unlike MSM we test moments one by one, since the model is very simple and not designed to pass a strict test of goodness of fit. Furthermore, we use a diagonal W instead of the asymptotically efficient weighting matrix.

Our baseline approach also differs from standard calibration in that we use *data-implied* standard deviations $\widehat{\sigma}_{\mathcal{S}_i}$ in the t-ratios while standard calibration uses *model-implied* standard deviation for each moment. We find

[22] Related to this is the observation that the value of the moments varies strongly with the precise sample period used. For example, in our sample we find $\widehat{E}(PD) = 113.2$, but using data up to 1996, we find a value of 99 only.

a number of problems with using model-implied standard deviations. First, the researcher has an incentive to choose versions of her preferred models that drive up standard deviations, since this will artificially increase the denominator of the t-ratio for that model. Using data-implied $\widehat{\sigma}_{\mathcal{S}_i}$ keeps the criterion of fit constant across alternative models and thereby allows for meaningful model comparisons. Second, in our model the risk free interest rate is constant, so that the model-implied standard deviation of the mean interest rate is $\widehat{\sigma}_{E(r^b)} = 0$. Using model-implied standard deviations would then require to match the average risk free rate exactly. But the data suggests that this moment is known very imprecisely, see Table 2, so matching it exactly appears arbitrary. For these reasons we prefer to use data-implied standard deviations in the t-ratios as our baseline.

We wish to emphasize that our results do not depend on these details. In section 6.2 we consider the robustness of our findings to several variations of this baseline testing procedure.

6 Quantitative Model Performance

6.1 Baseline Results

We now discuss the quantitative model performance for the baseline approach described in the previous section.

Results are summarized in Table 2 below. The second column reports the asset pricing moments ($\widehat{\mathcal{S}}_i$) from Table 1 that we seek to match. The third column lists the estimated standard deviation ($\widehat{\sigma}_{\mathcal{S}_i}$) for each of the sample moments. Clearly, some of these moments can be estimated only imprecisely due to the large volatility present in the data. The fourth column reports the moments implied by our estimated model and the last column reports the t-ratios.

The empirical performance of our estimated learning model is remarkable. For all moments, the t-ratios are well below 2. Clearly, the point estimates of some data moments are not exactly in line with the model implied moments, but this tends to occur for moments that, in the short sample, have a large variance and on which the estimation procedure appropriately places little weight. For example, the model implies an average risk-free interest rate that is more than twice as large as the point estimate in the data. Yet, with the standard deviation of $\widehat{E}(r^b)$ being fairly large, one nevertheless obtains a low t-ratio.

The bottom of the table reports the estimated parameter values of the learning model. The estimated gain parameter $1/\alpha$ is small, reflecting the tendency of the data to give large (but less than full) weight to the RE prior mean about stock price growth. The estimated gain value implies that in the initial period the RE prior receives a weight of approximately 99.5% and the first quarterly price growth observation a weight of about 0.5% only. The data prefer such a small gain value because higher gain values cause beliefs and the model-implied prices to become much more volatile than in the data.

The estimate for the time discount factor δ is slightly larger than one. Economic growth and risk aversion nevertheless cause agents to discount the future, so that the real interest rate remains positive and the consumers' problem well defined.[23] We impose the restriction $\delta \leq 1$ in the robustness section below.

Quarterly Statistics		US Data		Model	
			std		t-ratio
Mean real stock return	$E(r^s)$	2.41	0.45	2.26	0.34
Mean real bond return	$E(r^b)$	0.18	0.23	0.40	-0.95
Mean PD ratio	$E(PD)$	113.20	15.15	110.46	0.18
Std. dev. stock return	σ_{r^s}	11.65	2.88	14.77	-1.08
Std. dev. PD ratio	σ_{PD}	52.98	16.53	75.41	-1.36
Autocorrel. PD ratio	$\rho_{PD,-1}$	0.92	0.02	0.94	-0.84
Coeff. excess ret. regression	c_5^2	-0.0048	0.002	-0.0059	0.5622
R^2 excess ret. regression	R_5^2	0.1986	0.0828	0.2413	-0.5151
	Parameters:	$\widehat{\delta} = 1.000375$, $1/\widehat{\alpha} = 0.00633$			

Table 2: Moments and parameters.
Baseline model and baseline calibration

To show that our model also passes an 'eyeball test', we report in Figure 3 three 'typical' realizations of the PD ratio from the estimated baseline model for the same number of quarters as shown in Figure 1 for the data. The stock PD ratio in our simulations has the tendency to display sustained price increases that are followed by rather sharp price reductions and prolonged periods of low prices, similar to the behavior of the data shown in Figure

[23] Equations (26) and (27) show that a finite asset price under RE is obtained whenever $\delta^{-1} > \beta^{RE}$. Since risk aversion and growth can bring β^{RE} below 1, this allows for $\delta > 1$. For a discussion, see Kocherlakota (1990).

1. The figure illustrates that the oscillations around the RE, suggested by the phase diagram shown in Figure 2, are very persistent and take the form of rather low-frequency movements with occasional sharp outbreaks and reversals to the top.

We now briefly discuss why our model is also able to generate a sizable risk premium for stocks. Surprisingly, the model generates an ex-post risk premium for stocks even when investors are risk neutral ($\gamma = 0$). To understand this feature, note that the realized gross stock return between period 0 and period N can be written as the product of three terms

$$\prod_{t=1}^{N} \frac{P_t + D_t}{P_{t-1}} = \underbrace{\prod_{t=1}^{N} \frac{D_t}{D_{t-1}}}_{=R_1} \cdot \underbrace{\left(\frac{PD_N + 1}{PD_0}\right)}_{=R_2} \cdot \underbrace{\prod_{t=1}^{N-1} \frac{PD_t + 1}{PD_t}}_{=R_3}.$$

The first term (R_1) is independent of the way prices are formed, thus cannot contribute to explaining the emergence of an equity premium. The second term (R_2) could potentially generate an equity premium but is on average below one in our simulations, while it is slightly larger than one under RE. The equity premium in the learning model must thus be due the last component (R_3). This term is convex in the PD ratio, so that a model that generates higher volatility of the PD ratio (but the same mean value) will also give rise to a higher equity premium. Therefore, because our learning model generates a considerably more volatile PD ratio, it also gives rise to a larger ex-post risk premium.

In summary, the learning model fits the data very well: all eight moments from Table 1 can be matched by just two free parameters and the simulated PD series are comparable to those of Figure 1. We find this result remarkable. We employed one of the simplest versions of the asset pricing model and combined with one of the simplest available learning mechanisms which adds only one free parameter, namely the gain parameter $1/\alpha$. Furthermore, agents' beliefs are found to be close to RE belief. The strict RE version of the model, however, is far from matching any of the facts discussed here.

6.2 Robustness

This section shows that the quantitative performance of the model is robust to a number of deviations from the assumptions about the baseline model and testing procedure imposed in section 5. We consider one deviation at a time.

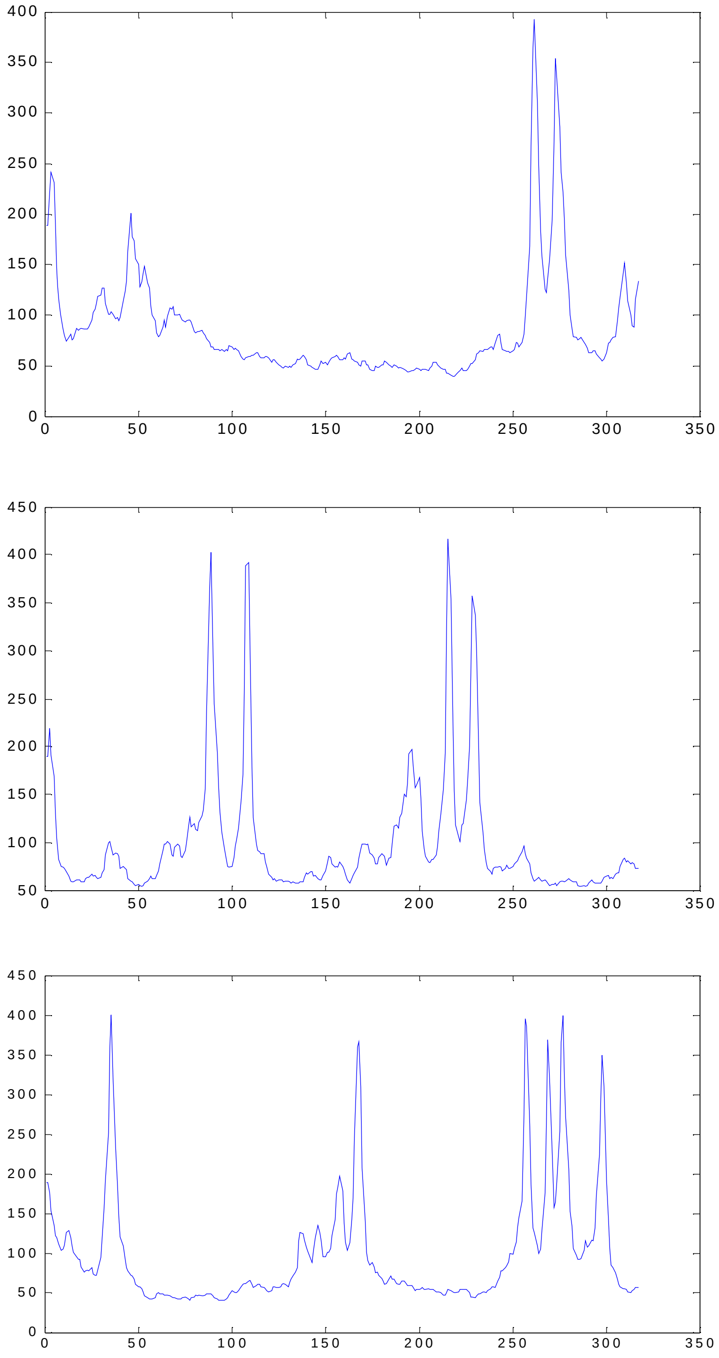

Figure 3: Simulated PD ratio, estimated constant gain model (Table 2)

Learning about dividends. For simplicity we assumed that agents know the true process for dividends. Also, learning about dividends has been considered previously in other papers. In appendix 8.4 we describe a model with learning about prices *and* dividends. Although the analysis is slightly more involved, the basic properties of the model do not change. Table 3 below shows the results and confirms that the parameter estimates are largely unchanged and that the model fit remains very good.

Consumption data. Throughout the paper we made the simplifying assumption $C_t/C_{t-1} = D_t/D_{t-1}$. We calibrated this process to dividend data since the variance of dividends has to be brought out when studying stock price volatility. Since actual consumption growth is much less volatile than dividend growth, this is a somewhat unsatisfactory aspect. We now calibrate the volatility of the consumption and dividend processes separately to the data.[24] While the dividend process remains as before, we set

$$\frac{C_{t+1}}{C_t} = a\varepsilon^c_{t+1} \quad \text{for} \quad \ln \varepsilon^c_t \sim iiN(-\frac{s_c^2}{2}; s_c^2)$$

The presence of two shocks modifies the equations for the RE version of the model in a well known way and we do not describe it in detail here. We calibrate the consumption process following Campbell and Cochrane (1999), i.e., set $s_c = \frac{s}{7}$ and $\rho(\varepsilon^c_t, \varepsilon_t) = .2$.[25]

The quantitative results are reported in table 3. The match of the risk-free rate now worsens and the corresponding t-ratio falls outside the 95% confidence interval. Equivalently, one could say that the model now marginally fails in matching the risk premium puzzle. Clearly, this occurs because the agents' stochastic discount factor is now less volatile. For a number of reasons, we do not wish to over-interpret this deterioration in the model fit. First, the rejection is marginal, the highest t-ratio is still remains below 3. Second, the equity premium is not the main focus of this paper and the model continues to perform well along the other dimensions of the data. Finally, to a RE fundamentalist, who might dismiss models of learning from the outset as being 'non rigorous because they can always match any data', this finding illustrates that models of learning can be rejected by the data in the same way as RE models.

[24] For the learning model to be consistent with internal rationality this requires assuming that the process for income follows a different process than that for dividends, that agents have rational expectations about this additional process, and that dividend income is again a minor part of agents' total income.

[25] We take these ratios and values from table 1 in Campbell and Cochrane (1999), which is based on a slightly shorter sample than the one used in this paper.

Restricting $\delta \leq 1$ As explained before, the estimated value of the discount factor is larger than one in the baseline case. Although this does not generate any inconsistency in our model or even under RE, some economists may feel uncomfortable with discount factors larger than one. Table 3 shows that the model behaves almost as well when the constraint $\delta \leq 1$ is imposed.

Statistic	US Data	Learning on Div.	t-ratio	$C_t/C_{t-1} \neq D_t/D_{t-1}$	t-ratio	$\delta \leq 1$	t-ratio
$E(r^s)$	2.41	2.20	0.48	2.01	0.89	2.26	0.34
$E(r^b)$	0.18	0.43	-1.06	0.74	-2.45	0.44	-1.11
$E(PD)$	113.20	111.58	0.11	111.43	0.12	109.82	0.22
σ_{r^s}	11.65	14.22	-0.89	13.63	-0.69	14.55	-1.00
σ_{PD}	52.98	74.16	-1.28	73.92	-1.27	74.60	-1.31
$\rho_{PD,-1}$	0.92	0.94	-0.95	0.94	-0.83	0.94	-0.81
c_5^2	-0.0048	-0.0054	0.2844	-0.0068	0.9950	-0.0059	0.5344
R_5^2	0.1986	0.2133	-0.1777	0.1588	0.4804	0.2443	-0.5516
Parameters:							
$\widehat{\delta}$		1.000075		1.010051		1	
$1/\widehat{\alpha}$		0.0061		0.0082		0.0063	

Table 3: Robustness to model choice

Model-generated standard deviations We now consider deviations from the baseline testing procedure. We first use t-ratios using a model-implied standard deviation of the moments $\widehat{\sigma}_{\mathcal{S}_i}$, closer to standard practice in calibration exercises. For this case δ is chosen to match $\widehat{E}\left(r^b\right)$ exactly since the model implied standard deviation is $\widehat{\sigma}_{E(r^b)} = 0$. Clearly, the corresponding $t-$ratio is undefined. Table 4 below reports the fit of the model. Although there is some worsening relative to the baseline findings, the overall fit remains very good.

Full Weighting matrix A classical econometrician might complain that using a diagonal weighting matrix W as in (33) yields consistent but inefficient estimates of δ, α. We now use an efficient weighting matrix W derived from standard MSM results. The procedure is described in appendix 8.5.2. Table 4 below shows that the fit of the model worsens, some of the t-ratios approach the rejection area, but the overall fit remains surprisingly good

Overall, the results in this section show that our quantitative results are robust to many deviations from the baseline model and baseline testing procedure.

Statistic	US Data	Model $\hat{\sigma}_{\mathcal{S}_i}$	t-ratio	Full matrix	t-ratio
$E(r^s)$	2.41	2.25	0.66	1.61	1.77
$E(r^b)$	0.18	0.18	—	0.08	0.47
$E(PD)$	113.20	114.34	-0.06	137.58	-1.61
σ_{r^s}	11.65	15.36	-1.41	10.99	0.23
σ_{PD}	52.98	76.24	-2.02	67.19	-0.86
$\rho_{PD,-1}$	0.92	0.94	-0.69	0.96	-1.97
c_5^2	-0.0048	-0.0062	1.6977	-0.0056	0.4006
R_5^2	0.1986	0.2376	-0.7887	0.3646	-2.0047
Parameters:					
δ		1.002526		1.003587	
$1/\alpha_1$		0.0063		0.0045	

Table 4: Robustness to testing procedure

7 Conclusions and Outlook

A very simple consumption based asset pricing model is able to quantitatively replicate a number of important asset pricing facts, provided one slightly relaxes the assumption that agents perfectly know how stock prices are formed in the market. We assume that agents formulate their doubts about market outcomes using a consistent set of subjective beliefs about prices which is close to, but not equal to, the RE prior beliefs typically assumed in the literature. Agents then optimally learn about the equilibrium price process using past price observations and this gives rise to a self-referential model of learning that imparts momentum and mean reversion behavior into the price dividend ratio. As a result, sustained departures of asset prices from their fundamental value emerge, even though all agents act rationally in the light of their beliefs.

Given the difficulties documented in the empirical asset pricing literature in accounting for these facts under RE, our results suggest that models of learning may be economically more relevant than previously thought.

Indeed, the most convincing case for models of learning can be made by explaining facts that appear 'puzzling' from the RE viewpoint, as we attempt to do in this paper.

The fact that stock prices can deviate for a long time from their fundamental value in a way that matches the data, even in a model with rational behavior, near-RE expectations and no frictions, is likely to have a wide range of practical implications that are worth exploring in future work. As a result of our findings, asset market and banking regulations could be seen in a rather different light. Accounting rules requiring that institutions mark assets to market appear arbitrary, if prices do not reflect the true asset value but rather the optimism or pessimism of investors about future capital gains. The effects of shortselling constraints on market outcomes might equally have to be reassessed: on the one hand such constraints could prevent agents from leveraging their (optimistic or pessimistic) expectations, on the other hand they could prevent useful speculation against the prevailing market sentiment. Finally, the theory of portfolio choice would have to be generalized to take into account the state of market expectations in addition to the state of fundamentals, something that appears familiar to practitioners for quite some time already.

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8 Appendix

8.1 Data Sources

Our data is for the United States and has been downloaded from 'The Global Financial Database' (http://www.globalfinancialdata.com). The period covered is 1925:4-2005:4. For the subperiod 1925:4-1998:4 our data set corresponds very closely to Campbell's (2003) handbook data set available at http://kuznets.fas.harvard.edu/~campbell/ data.html.

In the calibration part of the paper we use moments that are based on the same number of observations. Since we seek to match the return predictability evidence at the five year horizon (c_5^2 and R_5^2) we can only use data points up to 2000:4. For consistency the effective sample end for all other moments reported in table 1 has been shortened by five years to 2000:4. In addition, due to the seasonal adjustment procedure for dividends described below and the way we compute the standard errors for the moments described in appendix 8.5, the effective starting date was 1927:2.

To obtain real values, nominal variables have been deflated using the 'USA BLS Consumer Price Index' (Global Fin code 'CPUSAM'). The monthly price series has been transformed into a quarterly series by taking the index value of the last month of the considered quarter.

The nominal stock price series is the 'SP 500 Composite Price Index (w/GFD extension)' (Global Fin code '_SPXD'). The weekly (up to the end of 1927) and daily series has been transformed into quarterly data by taking the index value of the last week/day of the considered quarter. Moreover, the series has been normalized to 100 in 1925:4.

As nominal interest rate we use the '90 Days T-Bills Secondary Market' (Global Fin code 'ITUSA3SD'). The monthly (up to the end of 1933), weekly (1934-end of 1953), and daily series has been transformed into a quarterly series using the interest rate corresponding to the last month/week/day of the considered quarter and is expressed in quarterly rates, i.e., not annualized.

Nominal dividends have been computed as follows

$$D_t = \left(\frac{I^D(t)/I^D(t-1)}{I^{ND}(t)/I^{ND}(t-1)} - 1 \right) I^{ND}(t)$$

where I^{ND} denotes the 'SP 500 Composite Price Index (w/GFD extension)' described above and I^D is the 'SP 500 Total Return Index (w/GFD extension)' (Global Fin code '_SPXTRD '). We first computed monthly dividends and then quarterly dividends by adding up the monthly series. Following

Campbell (2003), dividends have been deseasonalized by taking averages of the actual dividend payments over the current and preceding three quarters.

8.2 Proof of mean reversion

To prove mean reversion for the general learning scheme of (17) we need the following additional technical assumptions on the updating function f_t:

Assumption 1 *There is a $\overline{\eta} > 0$ such that f_t is differentiable in the interval $(-\overline{\eta}, \overline{\eta})$ for all t and, letting*

$$\mathcal{D}_t \equiv \inf_{\overline{\Delta} \in (-\overline{\eta}, \overline{\eta})} \frac{\partial f_t(\overline{\Delta})}{\partial \Delta} \quad ,$$

we have

$$\sum_{t=0}^{\infty} \mathcal{D}_t = \infty$$

This is satisfied by all the updating rules considered in this paper and by most algorithms used in the stochastic control literature. For example, it is guaranteed in the OLS case where $\mathcal{D}_t = 1/(t + \alpha_1)$ and in the constant gain where $\mathcal{D}_t = 1/\alpha$ for all t. If the assumption would fail and $\sum \mathcal{D}_t < \infty$, then beliefs would get 'stuck' away from the fundamental value simply because updating of beliefs ceases to incorporate new information for t large enough. In this case, the growth rate is a certain constant but agents believe it is a different constant and agents make systematic mistakes forever. It is unlikely that such a scheme with $\sum \mathcal{D}_t < \infty$ would arise as optimal behavior from any system of beliefs that puts a grain of truth on the actual value of the growth rate for stock prices.

Assumption 2 *The learning rule satisfies $f_t(-z) \geq -z$ for all $z \in [0, \beta^U]$* and all t.

Since $\beta_t \geq \beta_{t-1} + f_t\ (-\beta_{t-1})$ Assumption 2 insures that $\beta_t, P_t \geq 0$ for all t, i.e., that agents predict future stock prices to be non-negative. Again, OLS and constant gain learning satisfy this assumption for $\alpha_1 \geq 1$, as is assumed throughout the paper. Therefore we have $0 \leq \beta_t \leq \beta^U$ for all t.

We start proving mean reversion for the case $\beta_t > a$. Fix $\eta > 0$ small enough that $\eta < \min(\overline{\eta}, (\beta_t - a)/2)$ where $\overline{\eta}$ is as in assumption 1.

We first prove that there exists a finite $t' \geq t$ such that

$$\Delta\beta_{\tilde{t}} \geq 0 \text{ for all } \tilde{t} \text{ such that } t < \tilde{t} < t', \text{ and} \quad (35)$$

$$\Delta\beta_{t'} < 0 \quad (36)$$

To prove this, choose $\boldsymbol{\epsilon} = \eta\left(1 - \delta\beta^U\right)$. It cannot be that $\Delta\beta_{\tilde{t}} \geq \boldsymbol{\epsilon}$ for all $\tilde{t} > t$, since $\epsilon > 0$ and this would contradict the bound $\beta_{\tilde{t}} \leq \beta^U$. Therefore $\Delta\beta_{\bar{t}} < \boldsymbol{\epsilon}$ for some finite $\bar{t} \geq t$. Take $\bar{t} \geq t$ to be the *first* period where $\Delta\beta_{\bar{t}} < \boldsymbol{\epsilon}$.

There are two possible cases: either *i)* $\Delta\beta_{\bar{t}} < 0$ or *ii)* $\Delta\beta_{\bar{t}} \geq 0$.

In case *i)* we have (35) and (36) hold if we take $t' = \bar{t}$.

In case *ii)* β_t can not decrease between t and $\bar{t}$ so that

$$\beta_{\bar{t}} \geq \beta_t > a + \eta$$

Furthermore, we have

$$\begin{aligned} T(\beta_{\bar{t}}, \Delta\beta_{\bar{t}}) &= a + \frac{\Delta\beta_{\bar{t}}}{1 - \delta\beta_{\bar{t}}} < a + \frac{\boldsymbol{\epsilon}}{1 - \delta\beta_{\bar{t}}} \\ &< a + \frac{\boldsymbol{\epsilon}}{1 - \delta\beta^U} = a + \eta \end{aligned}$$

where the first equality follows from the definition of T, the first inequality uses $\Delta\beta_{\bar{t}} < \boldsymbol{\epsilon}$ and the second inequality that $\beta_t < \beta^U$ and the last equality follows from the choice for $\boldsymbol{\epsilon}$. The previous two relations imply

$$\beta_{\bar{t}} > T(\beta_{\bar{t}}, \Delta\beta_{\bar{t}})$$

This together with (20), $\varepsilon_t \equiv 1$ and the fact that f_t is increasing gives

$$\beta_{\bar{t}} + f_{\bar{t}+1}\left(T(\beta_{\bar{t}}, \Delta\beta_{\bar{t}}) - \beta_{\bar{t}}\right) < \beta_{\bar{t}} < \beta_t < \beta^U$$

so that the projection facility does not apply at $\bar{t} + 1$. Therefore

$$\Delta\beta_{\bar{t}+1} = f_{\bar{t}+1}\left(T(\beta_{\bar{t}}, \Delta\beta_{\bar{t}}) - \beta_{\bar{t}}\right) < 0$$

and in case *ii)* we have that (35) and (36) hold for $t' = \bar{t} + 1$.

This shows that (35) and (36) hold for a finite t'. Now we need to show that from then on beliefs decrease and, eventually, they go below $a + \eta$.

Consider η as defined above. First, notice that given any $j \geq 0$, if

$$\Delta\beta_{t'+j} < 0 \quad \text{and} \quad (37)$$

$$\beta_{t'+j} > a + \eta \quad (38)$$

then

$$\Delta\beta_{t'+j+1} = f_{t'+j+1}\left(a + \frac{\Delta\beta_{t'+j}}{1-\delta\beta_{t'+j}} - \beta_{t'+j}\right) < f_{t'+j+1}\left(a - \beta_{t'+j}\right) \quad (39)$$

$$< f_{t'+j+1}(-\eta) \leq -\eta\mathcal{D}_{t'+j+1} \leq 0 \quad (40)$$

where the first inequality follows from (37), the second inequality from (38) and the third from the mean value theorem, $\eta > 0$ and $\mathcal{D}_{t'+j+1} \geq 0$. Assume, towards a contradiction, that (38) holds for all $j \geq 0$. Since (37) holds for $j = 0$, it follows by induction that $\Delta\beta_{t'+j} \leq 0$ for all $j \geq 0$ and, therefore, that (40) would hold for all $j \geq 0$ hence

$$\beta_{t'+j} = \sum_{i=1}^{j} \Delta\beta_{t'+i} + \beta_{t'} \leq -\eta \sum_{i=1}^{j} \mathcal{D}_{t'+i} + \beta_{t'}$$

for all $j > 0$. Assumption 1 above would then imply $\beta_t \to -\infty$ showing that (38) can not hold for all j. Therefore there is a finite j such that $\beta_{t'+j}$ will go below $a + \eta$ and β is decreasing from t' until it goes below $a + \eta$.

For the case $\beta_t < a - \eta$ we need to make the additional assumption that $\beta^U > a$. Then, choosing $\boldsymbol{\epsilon} = \eta$ we can use a symmetric argument to construct the proof.

8.3 Details on the phase diagram

The second order difference equation (20) with $\varepsilon_t \equiv 1$, which describes the deterministic evolution of beliefs, allows to construct the directional dynamics in the $\left(\beta_t, \beta_{t-1}\right)$ plane shown in Figure 2. Here we show the algebra leading to the arrows displayed in this figure. For clarity, we define $x_t' \equiv (x_{1,t}, x_{2,t}) \equiv \left(\beta_t, \beta_{t-1}\right)$, whose dynamics are given by

$$x_{t+1} = \begin{pmatrix} x_{1,t} + f_{t+1}\left(a + \frac{a\delta(x_{1,t}-x_{2,t})}{1-\delta x_{1,t}} - x_{1,t}\right) \\ x_{1,t} \end{pmatrix}$$

The points in Figure 2 where there is no change in each of the elements of x are the following: we have $\Delta x_2 = 0$ at points $x_1 = x_2$, so that the 45^o line gives the point of no change in x_2, and $\Delta x_2 > 0$ above this line. We have $\Delta x_1 = 0$ for $x_2 = \frac{1}{\delta} - \frac{x_1(1-\delta x_1)}{a\delta}$, this is the curve labelled "$\beta_{t+1} = \beta_t$" in Figure 2 and we have $\Delta x_1 > 0$ below this curve. So the zeroes for Δx_1 and Δx_2 intersect are at $x_1 = x_2 = a$ which is the REE and, interestingly, at $x_1 = x_2 = \delta^{-1}$ which is the limit of rational bubble equilibria. These results give rise to the directional dynamics shown in figure 2.

8.4 Model with learning about dividends

This section considers agents who learn to forecast future dividends in addition to forecast future price. We make the arguments directly for the general model with risk aversion from section 5. Equation (25) then becomes

$$P_t = \delta E_t^{\mathcal{P}}\left(\left(\frac{D_t}{D_{t+1}}\right)^{\gamma} P_{t+1}\right) + \delta E_t^{\mathcal{P}}\left(\frac{D_t^{\gamma}}{D_{t+1}^{\gamma-1}}\right) \tag{41}$$

Under RE one has

$$\begin{aligned} E_t\left(\frac{D_t^{\gamma}}{D_{t+1}^{\gamma-1}}\right) &= E_t\left(\left(\frac{D_{t+1}}{D_t}\right)^{1-\gamma}\right) D_t \\ &= E_t\left((a\varepsilon)^{1-\gamma}\right) D_t \\ &= \beta^{RE} D_t \end{aligned}$$

Definining the (risk-adjusted) dividend growth forecast as

$$\beta_t^D \equiv E_t^{\mathcal{P}}\left(\left(\frac{D_{t+1}}{D_t}\right)^{1-\gamma}\right)$$

agents' forecast of the latter term in (41) is given by

$$E_t^{\mathcal{P}}\left(\frac{D_t^{\gamma}}{D_{t+1}^{\gamma-1}}\right) = \beta_t^D D_t$$

Under the belief specification described in section 4.3, agents' conditional expectations for dividend growth evolve according to

$$\beta_t^D = \beta_{t-1}^D + \frac{1}{\alpha_t}\left(\left(\frac{D_{t-1}}{D_{t-2}}\right)^{1-\gamma} - \beta_{t-1}^D\right) \tag{42}$$

where the gain sequence $1/\alpha_t$ is the same as the one used for updating the estimate for risk-adjusted stock price growth β_t, which continues to evolve according to equation (31). As with stock price growth expectations, we center initial beliefs for dividend growth at the RE outcome, which implies

$$\beta_t^D = \beta^{RE}.$$

With these assumptions, equation (41) implies that the equilibrium price is given by

$$P_t = \frac{\delta \beta_t^D}{1 - \delta \beta_t} D_t$$

Since $\beta_t^D \to a$ for the decreasing gain case, and since β_t^D will fluctuate closely around a for small but constant gain values, the pricing implications with dividend learning are very similar those derived for the case where agents have RE about the dividend process.

8.5 Details of the testing procedure

This appendix describes details of our baseline testing approach and gives estimators of the standard deviation of the sample statistics reported in table 2.

8.5.1 Baseline estimation and testing

Here we show that the estimator defined in (33) is consistent and we derive the asymptotic distribution for the moments of the model we report.

Let N be the sample size, $(\mathbf{y}_1, ..., \mathbf{y}_N)$ the observed data sample, with $\mathbf{y}_t$ containing m variables. In the text we talked about "moments" as describing all statistics to be matched in (32) even though some of these statistics are not proper moments, they are only functions of moments. In this appendix we properly use the term "statistic" as possibly different from "moment".

We consider sample statistics $\mathcal{S}(\widehat{M}_N)$ where $\mathcal{S} : R^q \to R^s$ is a *statistic function* that maps sample moments $\widehat{M}_N$ into the considered statistics. The moments are defined by $\widehat{M}_N \equiv \frac{1}{N}\sum_{t=1}^{N} h(\mathbf{y}_t)$ for a given function $h : R^m \to R^q$. The explicit expressions for $h(\cdot)$ and $\mathcal{S}(\cdot)$ for our particular application are stated in 8.5.3 below.

Since we match these statistics instead of proper moments this is not an immediate application of standard MSM, so we adapt standard proofs to derive the asymptotic theory results that are needed.

In the main text we have denoted the observed sample statistics as $\widehat{\mathcal{S}} \equiv \mathcal{S}(\widehat{M}_N)$. Let $y_t(\theta)$ be the series generated by the model for parameter values θ and some realization of the underlying shocks, let θ_0 be the true parameter value, $M_0 \equiv E\,[\ h(y_t(\theta_0))\ ]$ are the true moments, $M(\theta) \equiv E\,[\ h(y_t(\theta))\ ]$ are the true moments for parameter values θ at the stationary distribution for $y_t(\theta)$ and $\widetilde{\mathcal{S}}(\theta) \equiv \mathcal{S}(M(\theta))$ are the true statistics when the model parameter is θ.

Using standard results from MSM, the estimates $\widehat{\delta}, \widehat{\alpha}$ defined by (33) converge asymptotically to their true values if the following conditions hold: y is stationary and ergodic, the effect of initial conditions dies down sufficiently quickly, W converges almost surely to a positive definite matrix $\widetilde{W}$

as the sample grows, and

$$\left[\widetilde{\mathcal{S}}(\theta_0) - \widetilde{\mathcal{S}}(\theta)\right]' \ \widetilde{W} \ \left[\widetilde{\mathcal{S}}(\theta_0) - \widetilde{\mathcal{S}}(\theta)\right] = 0$$

holds *only* for $\theta = \theta_0$, i.e., the set of statistics uniquely identifies θ_0. The latter identification assumption typically requires the number of parameters to be less than the number of statistics s.

Denote by M_0^j the j-th autocovariance of the moment function at the true parameter, that is

$$M_0^j \equiv E[\,[h(y_t(\theta_0)) - E(h(y_t(\theta_0))]\,[h(y_{t-j}(\theta_0)) - E(h(y_{t-j}(\theta_0))]'\,]$$

Define

$$S_w \equiv \sum_{j=-\infty}^{\infty} M_0^j \tag{43}$$

we have the following

Result 1: Suppose that in addition to the assumptions required for consistency we have

- $S_w < \infty$,
- a consistent estimator $\widehat{S}_{w,N}$ such that $\widehat{S}_{w,N} \to S_w$ a.s. as $N \to \infty$, and
- $\mathcal{S}$ is continuously differentiable at M_0.

Then, defining

$$\widehat{\Sigma}_{\mathcal{S},N} \equiv \frac{\partial \mathcal{S}(M_N)}{\partial M'} \widehat{S}_{w,N} \frac{\partial \mathcal{S}(M_N)'}{\partial M}$$

and letting $\widehat{\sigma}_{\mathcal{S}_i}$ denote the square root of the i-th diagonal element of $\frac{1}{N}\widehat{\Sigma}_{\mathcal{S},N}$, we have that

$$\frac{\widehat{\mathcal{S}}_i - \mathcal{S}_i(M_0)}{\widehat{\sigma}_{\mathcal{S}_i}} \to \mathcal{N}(0,1) \quad \text{in distribution as } N \to \infty. \tag{44}$$

for each $i = 1, ..., s$ whenever the model is true.

Proof. The proof uses standard arguments, we just give an outline. The central limit theorem implies

$$\sqrt{N}\left[\widehat{M}_N - M_0\right] \to \mathcal{N}(0, S_w) \quad \text{in distribution}$$

The mean value theorem implies

$$\sqrt{N}\left[\mathcal{S}(M_N)-\mathcal{S}(M_0)\right] \rightarrow \mathcal{N}(0,\Sigma_{\mathcal{S}}) \text{ in distribution for} \quad (45)$$

$$\Sigma_{\mathcal{S}} = \frac{\partial\mathcal{S}(M_0)}{\partial M'} S_w \frac{\partial\mathcal{S}'(M_0)}{\partial M} \quad (46)$$

Since $\widehat{M}_N \rightarrow M_0$ a.s. we have $\widehat{\Sigma}_{\mathcal{S},N} \rightarrow \Sigma_{\mathcal{S}}$ a.s. which implies (44). ■

Therefore, the denominator of (34) is found by combining an estimator of S_w with $\frac{\partial\mathcal{S}(\widehat{M}_N)}{\partial M'}$ to obtain the variance-covariance matrix of the statistics $\widehat{\Sigma}_{\mathcal{S},N}$. Consistent estimates $\widehat{S}_{w,N}$ can be obtained from the data by using the Newey West estimator, which substitutes the expectations in (43) by the sample means, truncates the infinite sum and gives decreasing weight to the sample autocovariances at longer lags. This is standard and we do not describe the details here. An explicit expression for $\partial\mathcal{S}/\partial M'$ for the statistics (32) that we match is given in appendix 8.5.4. Notice that $\widehat{\Sigma}_{\mathcal{S},N}$ can be formed using data only, the model or its parameter estimates do not enter the computation.

There are various ways to compute the moments of the model $\widetilde{\mathcal{S}}(\theta)$ for a given $\theta \in R^n$.We use the following Monte-Carlo procedure. Let ω^i denote a realization of shocks drawn randomly from the known distribution that the underlying shocks are assumed to have and $\left(y_1(\theta,\omega^i),...y_N(\theta,\omega^i)\right)$ the random variables corresponding to a history of length N generated by the model for shock realization ω^i and parameter values θ. Furthermore, let

$$M_N(\theta,\omega^i) \equiv \frac{1}{N}\sum_{t=1}^{N} h(y_t(\theta,\omega^i))$$

denote the model moment for realization ω^i. We set the model statistics $\widetilde{\mathcal{S}}(\theta)$ equal to

$$\frac{1}{K}\sum_{i=1}^{K} \mathcal{S}(M_N(\theta,\omega^i))$$

for large K. In other words, $\widetilde{\mathcal{S}}(\theta)$ is an average across a large number of simulations of length N of the statistics $\mathcal{S}(M_N(\theta,\omega^i))$ implied by each simulation. We use K of the order of 1000, so any error introduced by the simulation is very small. These are the averages reported as model moments in tables 2 to 4 of the main text.

Our baseline procedure uses a diagonal weighting matrix W where the diagonal contains the inverse of the diagonal elements of $\widehat{S}_{w,N}$. This procedure thus finds parameters that match the model statistics as closely as possible to the data statistics, but gives less weight to statistics with a larger standard deviation. Notice that the calibration result is invariant to a rescaling of the variables of interest.

8.5.2 Full matrix estimation

We chose a diagonal W in the baseline calibration for simplicity. As is well known an efficient matrix that minimizes asymptotic variance of the estimators given the statistics to be matched amounts to setting $W = \widehat{\Sigma}_{\mathcal{S},N}^{-1}$. The resulting estimates and tests are the ones described in the last columns of table 4. For $\widehat{\Sigma}_{\mathcal{S},N}$ to be invertible one requires that $s \leq q$. In our case we have $s = 8$ and $q = 9$.

8.5.3 The statistic and moment functions

This section gives explicit expressions for the statistic function $\mathcal{S}(\cdot)$ and the moment functions $h(\cdot)$ that map our estimates into the framework just discussed in this appendix.

The underlying sample moments are

$$M_N \equiv \frac{1}{N}\sum_{t=1}^{N} h(\mathbf{y}_t)$$

where $h(\cdot) : R^{42} \rightarrow R^9$ and $\mathbf{y}_t$ are defined as

$$h(\mathbf{y}_t) \equiv \begin{bmatrix} r_t^s \\ PD_t \\ (r_t^s)^2 \\ (PD_t)^2 \\ PD_t \; PD_{t-1} \\ r_{t-20}^{s,20} \\ \left(r_{t-20}^{s,20}\right)^2 \\ r_{t-20}^{s,20} PD_{t-20} \\ r_t^b \end{bmatrix}, \qquad \mathbf{y}_t \equiv \begin{bmatrix} PD_t \\ D_t/D_{t-1} \\ PD_{t-1} \\ D_{t-1}/D_{t-2} \\ \vdots \\ PD_{t-19} \\ D_{t-19}/D_{t-20} \\ PD_{t-20} \\ r_t^b \end{bmatrix}$$

where $r_t^{s,20}$ denotes the stock return over 20 quarters, which can be computed using from $\mathbf{y}_t$ using $(PD_t, D_t/D_{t-1}, ..., PD_{t-19}, D_{t-19}/D_{t-20})$.

The eight statistics we consider can be expressed as function of the moments as follows:

$$\mathcal{S}(M) \equiv \begin{bmatrix} E(r_t^s) \\ E(PD_t) \\ \sigma_{r_t^s} \\ \sigma_{PD_t} \\ \rho_{PD_t,-1} \\ c_2^5 \\ R_5^2 \\ E(r_t^b) \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \\ \sqrt{M_3 - (M_1)^2} \\ \sqrt{M_4 - (M_2)^2} \\ \frac{M_5 - (M_2)^2}{M_4 - (M_2)^2} \\ c_2^5(M) \\ R_5^2(M) \\ M_9 \end{bmatrix}$$

where M_i denotes the $i-$th element of M and the functions $c_2^5(M)$ and $R_5^2(M)$ define the OLS and R^2 coefficients of the excess returns regressions, more precisely

$$c^5(M) \equiv \begin{bmatrix} 1 & M_2 \\ M_2 & M_4 \end{bmatrix}^{-1} \begin{bmatrix} M_6 \\ M_8 \end{bmatrix}$$
$$R_5^2(M) \equiv 1 - \frac{M_7 - [M_6, M_8]\, c^5(M)}{M_7 - (M_6)^2}$$

8.5.4 Derivatives of the statistic function

This appendix gives explicit expressions for $\partial\mathcal{S}/\partial M'$ using the statistic function stated in appendix 8.5.3. Straightforward but tedious algebra shows

$$\frac{\partial\mathcal{S}_i}{\partial M_j} = 1 \qquad\qquad \text{for } (i,j) = (1,1),(2,2),(8,9)$$

$$\frac{\partial\mathcal{S}_i}{\partial M_i} = \frac{1}{2\mathcal{S}_i(M)} \quad \text{for} \quad i = 3,4$$

$$\frac{\partial\mathcal{S}_i}{\partial M_j} = \frac{-M_j}{\mathcal{S}_i(M)} \quad \text{for } (i,j) = (3,1),(4,2)$$

$$\frac{\partial\mathcal{S}_5}{\partial M_2} = \frac{2M_2(M_5 - M_4)}{(M_4 - M_2^2)^2}, \qquad \frac{\partial\mathcal{S}_5}{\partial M_5} = \frac{1}{M_4 - M_2^2}, \qquad \frac{\partial\mathcal{S}_5}{\partial M_4} = -\frac{M_5 - M_2^2}{\left(M_4 - M_2^2\right)^2}$$

$$\frac{\partial\mathcal{S}_6}{\partial M_j} = \frac{\partial c_2^5(M)}{\partial M_j} \qquad \text{for } i = 2,4,6,8$$

$$\frac{\partial\mathcal{S}_7}{\partial M_j} = \frac{[M_6, M_8]\,\frac{\partial c^5(M)}{\partial M_j}}{M_7 - M_6^2} \qquad \text{for } j = 2,4$$

$$\frac{\partial\mathcal{S}_7}{\partial M_6} = \frac{\left[c_1^5(M) + [M_6, M_8]\,\frac{\partial c^5(M)}{\partial M_6}\right]\left(M_7 - M_6^2\right) - 2M_6\,[M_6, M_8]\,c^5(M)}{\left(M_7 - M_6^2\right)^2}$$

$$\frac{\partial\mathcal{S}_7}{\partial M_7} = \frac{M_6^2 - [M_6, M_8]\,c^5(M)}{\left(M_7 - M_6^2\right)^2}$$

$$\frac{\partial\mathcal{S}_7}{\partial M_8} = \frac{c_2^5(M) + [M_6, M_8]\,\frac{\partial c^5(M)}{\partial M_8}}{M_7 - M_6^2}$$

Using the formula for the inverse of a 2x2 matrix

$$c^5(M) = \frac{1}{M_4 - M_2^2}\begin{bmatrix} M_4 M_6 - M_2 M_8 \\ M_8 - M_2 M_6 \end{bmatrix}$$

we have

$$\frac{\partial c^5(M)}{\partial M_2} = \frac{1}{M_4 - M_2^2}\left(2M_2 c^5(M) - \begin{bmatrix} M_8 \\ M_6 \end{bmatrix}\right)$$
$$\frac{\partial c^5(M)}{\partial M_4} = \frac{1}{M_4 - M_2^2}\left(-c^5(M) + \begin{bmatrix} M_6 \\ 0 \end{bmatrix}\right)$$
$$\frac{\partial c^5(M)}{\partial M_6} \equiv \frac{1}{M_4 - M_2^2}\begin{bmatrix} M_4 \\ -M_2 \end{bmatrix}$$
$$\frac{\partial c^5(M)}{\partial M_8} \equiv \frac{1}{M_4 - M_2^2}\begin{bmatrix} -M_2 \\ 1 \end{bmatrix}$$

All remaining terms $\partial \mathcal{S}_i / \partial M_j$ not listed above are equal to zero.

8.5.5 Differentiable projection facility

As discussed in the main text, we introduce a projection facilitye that prevents perceived stock price growth β_t from being higher than δ^{-1}, so as to insure a finite stock price. In addition, it is convenient for our calibration exercises if the learning scheme is a continuous and differentiable function, see the discussion in appendix 8.5. The standard projection facility described in (17) causes the simulated equilibrium price series $P_t(\theta, \omega^i)$ for a given shock realization ω^i to be discontinuous as a function of the parameters θ. This is because the price will jump at a parameter value where the facility is exactly binding.

We thus introduce a projection facility that 'phases in' more gradually. We define

$$\beta_t^* = \beta_{t-1} + \frac{1}{\alpha_t}\left[\left(\frac{D_{t-1}}{D_{t-2}}\right)^{-\gamma} \frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}\right] \tag{47}$$

and modify the updating scheme (31) to

$$\beta_t = \begin{cases} \beta_t^* & \text{if } \beta_t^* \leq \beta^L \\ \beta^L + w(\beta_t^* - \beta^L)(\beta^U - \beta^L) & \text{otherwise} \end{cases} \tag{48}$$

for some weighting function w, and constants β^U, β^L. Here $\beta^U < \delta^{-1}$ is the upper bound on beliefs, chosen to insure that the implied PD ratio is always less than a certain upper bound $U^{PD} \equiv \frac{\delta a}{1-\delta\beta^U}$, the constant $\beta^L < \beta^U$ is some arbitrary level of beliefs above which the projection facility starts to operate and $w(\cdot): R^{+} \rightarrow [0,1]$ is a weighting function. Since w takes values between zero and one this formula insures that the beliefs are below β^U. We further require that w is increasing, $w(0) = 0$ and $w(\infty) = 1$, and we want to insure that the resulting beliefs β_t are continuously differentiable w.r.t. β_t^* at the point β^L.

In particular, we use

$$w(x) = 1 - \frac{\beta^U - \beta^L}{x + \beta^U - \beta^L}.$$

With this weighting function

$$\lim_{\beta_t^* \nearrow \beta^L} \beta_t = \lim_{\beta_t^* \searrow \beta^L} \beta_t = \beta^L$$
$$\lim_{\beta_t^* \nearrow \beta^L} \frac{\partial \beta_t}{\partial \beta_t^*} = \lim_{\beta_t^* \searrow \beta^L} \frac{\partial \beta_t}{\partial \beta_t^*} = 1$$
$$\lim_{\beta_t^* \to \infty} \beta_t = \beta^U$$

In our numerical applications we choose β^U so that the implied PD ratio never exceeds $U^{PD} = 500$ and $\beta^L = \delta^{-1} - 2(\delta^{-1} - \beta^U)$, which implies that the dampening effect of the projection facility starts to come into effect for values of the PD ratio above 250.

The figure below shows how the standard projection facility in (17) operates versus the continuous projection facility proposed in this appendix. It displays the discontinuity introduced by the standard projection facility and that for most β^* the projection facility is irrelevant. For this graph $\beta^{RE} = 1.0035$.

The continuous dampening of observations for high stock price growth that is implied by this continuous projection facility is analogous to the one that would arise from optimally incorporating new information given a consistent beliefs and a prior about β^P that is bounded above by β^U.

8.6 Convergence of least squares to RE

We show global convergence when agents use least squares learning and they have risk aversion as in section 5. The proof shows global convergence, that is, it obtains a stronger result than is usually found in many applications of least squares learning using the associated o.d.e. approach. The proof below is for the standard projection facility.

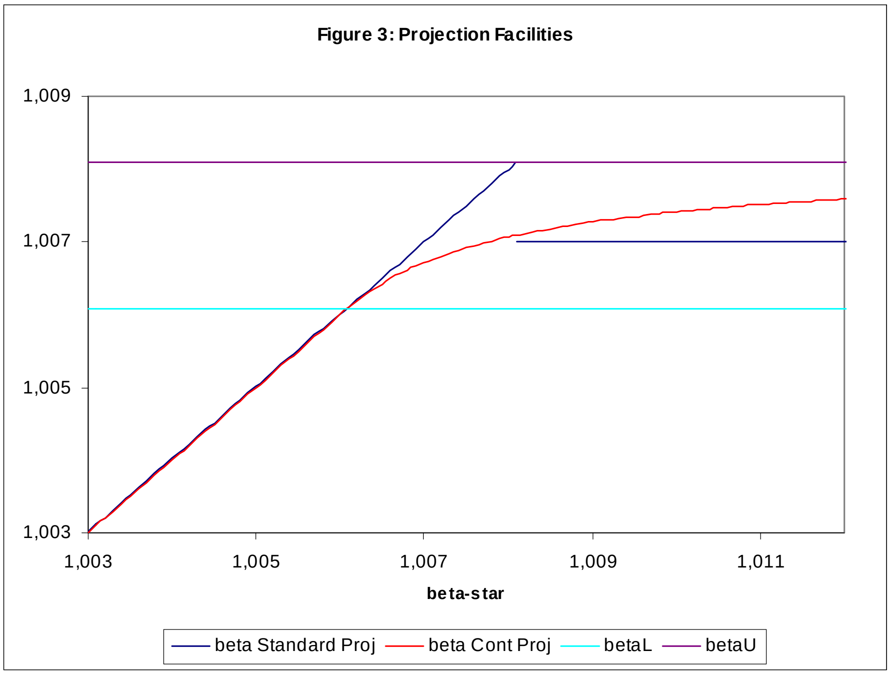
Figure 3: Projection Facilities
1,009
1,007
1,005
1,003
1,003
1,005
1,007
1,009
1,011
beta-star
beta Standard Proj
beta Cont Proj
betaL
betaU

Figure 4:

We remove the assumption of log-normality on the ε's but assume $D_t \geq 0$ with probability one, which requires that $\varepsilon_t \geq 0$. We also need $\varepsilon_t^{1-\gamma}$ to be bounded above, formally we assume existence of some positive $U^{\varepsilon} < \infty$ such that

$$\text{Prob}(\varepsilon_t^{1-\gamma} < U^{\varepsilon}) = 1$$

This excludes log-normality (except for the case of log utility) but it still allows for a rather general distribution for ε_t. Obviously, if $\gamma < 1$ this is satisfied if ε_t is bounded above a.s. by a finite constant, and if $\gamma > 1$ this is satisfied if ε_t is bounded away from zero. We also denote $U^{PD} \equiv \frac{\delta\beta^U}{1-\delta\beta^U} < \infty$, this is the highest PD ratio that can be achieved given that the projection facility insures $\beta_t < \beta^U$.

We first show that the projection facility will almost surely cease to be binding after some finite time. In a second step, we prove that β_t converges to β^{RE} from that time onwards.

The standard projection facility implies

$$\Delta\beta_t = \begin{cases} \alpha_t^{-1}\left((a\varepsilon_{t-1})^{-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}\right) & \text{if } \beta_{t-1} + \alpha_t^{-1}\left((a\varepsilon_{t-1})^{-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}\right) < \beta^U \\ 0 & \text{otherwise} \end{cases} \tag{49}$$

If the lower equality applies one has $\alpha_t^{-1}(a\varepsilon_{t-1})^{-\gamma}\frac{P_{t-1}}{P_{t-2}} \geq \beta_{t-1} \geq 0$ and this shows the following inequality

$$\beta_t \leq \beta_{t-1} + \alpha_t^{-1}\left((a\varepsilon_{t-1})^{-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}\right) \tag{50}$$

holds for all t a.s., whether or not the projection facility is binding at t. We also have that

$$\left|\beta_t - \beta_{t-1}\right| \leq \alpha_t^{-1}\left|(a\varepsilon_{t-1})^{-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}\right| \tag{51}$$

holds for all t a.s., because if $\beta_t < \beta^U$ this holds with equality and if $\beta_{t-1} + \alpha_t^{-1}\left((a\varepsilon_{t-1})^{-\gamma}\frac{P_{t-1}}{P_{t-2}} - \beta_{t-1}\right) \geq \beta^U$ then $\left|\beta_t - \beta_{t-1}\right| = 0$.

Substituting recursively backwards in (50) for past β's delivers the first line in

$$\beta_t \leq \frac{1}{t-1+\alpha_1}\left((\alpha_1-1)\ \beta_0 + \sum_{j=0}^{t-1}(a\varepsilon_j)^{-\gamma}\,\frac{P_j}{P_{j-1}}\right)$$
$$= \underbrace{\frac{t}{t-1+\alpha_1}\left(\frac{(\alpha_1-1)\ \beta_0}{t} + \frac{1}{t}\sum_{j=0}^{t-1}(a\ \varepsilon_j)^{1-\gamma}\right)}_{=T_1} + \underbrace{\frac{1}{t-1+\alpha_1}\left(\sum_{j=0}^{t-1}\frac{\delta\ \Delta\beta_j}{1-\delta\beta_j}\,(a\varepsilon_j)^{1-\gamma}\right)}_{=T_2} \tag{52}$$

a.s., where the second line follows from

$$\frac{P_t}{P_{t-1}} = \left(1+\frac{\delta\ \Delta\beta_t}{1-\delta\beta_t}\right)\ a\varepsilon_t$$

Clearly, $T_1 \to 1(0 + E((a\ \varepsilon_j)^{1-\gamma})) = \beta^{RE}$ as $t \to \infty$ a.s. Also, if we can establish $|T_2| \to 0$ a.s. this will show that β_t will eventually be bounded away from its upper bound or, more formally, that $\limsup_{t\to\infty}\beta_t \leq \beta^{RE} < \beta^U$. This is achieved by noting that

$$\begin{aligned}
|T_2| &\leq \frac{1}{t-1+\alpha_1}\sum_{j=0}^{t-1}\frac{\delta\,(a\ \varepsilon_j)^{1-\gamma}}{1-\delta\beta_j}\left|\Delta\beta_j\right| \\
&\leq \frac{U^\varepsilon}{t-1+\alpha_1}\sum_{j=0}^{t-1}\frac{a^{1-\gamma}\delta\ \left|\Delta\beta_j\right|}{1-\delta\beta_j} \\
&\leq \frac{U^\varepsilon}{t-1+\alpha_1}\,\frac{\delta a^{1-\gamma}}{1-\delta\beta^U}\sum_{j=0}^{t-1}\left|\Delta\beta_j\right|
\end{aligned} \tag{53}$$

a.s., where the first inequality results from the triangle inequality and the fact that both ε_j and $\frac{1}{1-\delta\beta_j}$ are positive, the second inequality follows from the a.s. bound on ε_j, and the third inequality from $\beta_t \leq \beta^U$. Next, observe that a.s. for all t

$$(a\varepsilon_t)^{-\gamma}\,\frac{P_t}{P_{t-1}} = \frac{1-\delta\beta_{t-1}}{1-\delta\beta_t}\,(a\varepsilon_t)^{1-\gamma} < \frac{(a\varepsilon_t)^{1-\gamma}}{1-\delta\beta_t} < \frac{a^{1-\gamma}U^\varepsilon}{1-\delta\beta^U} \tag{54}$$

where the equality follows from

$$P_t = \frac{\delta\beta^{RE}}{1-\delta\beta_t}D_t,$$

the first inequality from $\beta_{t-1} > 0$, and the second inequality from the bounds on ε and β_t. Applying the triangle inequality in the right side of equation (51), using (54) and $\beta_{t-1} < \beta^U$ gives the inequality in

$$\begin{aligned}\frac{1}{t-1+\alpha_1}\sum_{j=0}^{t-1}\left|\Delta\beta_j\right| &\leq \frac{1}{t-1+\alpha_1}\sum_{j=0}^{t-1}\alpha_j^{-1}\left(\frac{a^{1-\gamma}U^{\varepsilon}}{1-\delta\beta^U}+\beta^U\right)\\ &= \left(\frac{a^{1-\gamma}U^{\varepsilon}}{1-\delta\beta^U}+\beta^U\right)\frac{1}{t-1+\alpha_1}\sum_{j=0}^{t-1}\frac{1}{j-1+\alpha_1}\end{aligned} \tag{55}$$

the equality follows from simple algebra. Now, for any $\zeta > 0$

$$t^{-1}\sum_{i=0}^{t}i^{-1} = t^{\zeta-1}\sum_{i=0}^{t}t^{-\zeta}i^{-1} \leq t^{-1+\zeta}\sum_{i=0}^{t}i^{-(1+\zeta)} \to 0 \;\text{ as } t\to\infty$$

where the convergence follows from the well known fact that the over-harmonic series $\sum_{i=0}^{t} i^{-(1+\zeta)}$ is convergent. This and the fact that the large parenthesis in (55) is finite implies

$$\frac{1}{t-1+\alpha_1}\sum_{j=0}^{t-1}\left|\Delta\beta_j\right| \to 0 \;\text{ for all } t \text{ a.s.}$$

Then (53) implies that $|T_2| \to 0$ a.s. as $t\to\infty$. Taking the lim sup on both sides of (52), it follows from $T_1 \to \beta^{RE}$ and $|T_2| \to 0$ that

$$\limsup_{t\to\infty}\beta_t \leq \beta^{RE} < \beta^U$$

a.s. The projection facility is thus binding finitely many periods with probability one.

We now proceed with the second step of the proof. Consider for a given realization a finite period $\bar{t}$ where the projection facility is not binding for all $t > \bar{t}$. Then the upper equality in (49) holds for all $t > \bar{t}$ and simple algebra gives

$$\begin{aligned}\beta_t &= \frac{1}{t-\bar{t}+\alpha_{\bar{t}}}\left(\sum_{j=\bar{t}}^{t-1}(a\varepsilon_j)^{-\gamma}\frac{P_j}{P_{j-1}}+\alpha_{\bar{t}}\,\beta_{\bar{t}}\right)\\ &= \frac{t-\bar{t}}{t-\bar{t}+\alpha_{\bar{t}}}\left(\frac{1}{t-\bar{t}}\sum_{j=\bar{t}}^{t-1}(a\varepsilon_j)^{1-\gamma}+\frac{1}{t-\bar{t}}\sum_{j=\bar{t}}^{t-1}\frac{\delta\,\Delta\beta_j}{1-\delta\beta_j}(a\varepsilon_j)^{1-\gamma}+\frac{\alpha_{\bar{t}}}{t-\bar{t}}\,\beta_{\bar{t}}\right)\end{aligned} \tag{56}$$

for all $t > \bar{t}$. Equations (50) and (51) now hold with equality for all $t > \bar{t}$. Similar operations as before then deliver

$$\frac{1}{t-\bar{t}} \sum_{j=\bar{t}}^{t-1} \frac{\delta \, \Delta\beta_j}{1-\delta\beta_j} (a\varepsilon_j)^{1-\gamma} \to 0$$

a.s. for $t \to \infty$. Finally, taking the limit on both sides of (56) establishes

$$\beta_t \to a^{1-\gamma} E(\varepsilon_t^{1-\gamma}) = \beta^{RE}$$

a.s. as $t \to \infty$.■

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