J. Math. Anal. Appl. 533 (2024) 128000

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

On a Banach algebra of entire functions with a weighted Hadamard multiplication

Amol Sasane

Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom

A R T I C L E   I N F O

Article history:
Received 28 March 2023
Available online 1 December 2023
Submitted by J.A. Ball

Keywords:
Banach algebras
Čech cohomology
Entire functions
K-theory
Hadamard product
Corona problem

A B S T R A C T

New algebraic-analytic properties of a previously studied Banach algebra $\mathcal{A}(\boldsymbol{p})$ of entire functions are established. For a given fixed sequence $(\boldsymbol{p}(n))_{n\geq 0}$ of positive real numbers, such that $\lim_{n\to\infty} \boldsymbol{p}(n)^{\frac{1}{n}} = \infty$, the Banach algebra $\mathcal{A}(\boldsymbol{p})$ is the set of all entire functions f such that $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ $(z \in \mathbb{C})$, where the sequence $(\widehat{f}(n))_{n\geq 0}$ of Taylor coefficients of f satisfies $\widehat{f}(n) = O(\boldsymbol{p}(n)^{-1})$ for $n \to \infty$, with pointwise addition and scalar multiplication, a weighted Hadamard multiplication $*$ with weight given by $\boldsymbol{p}$ (i.e., $(f * g)(z) = \sum_{n=0}^{\infty} \boldsymbol{p}(n)\widehat{f}(n)\widehat{g}(n)z^n$ for all $z \in \mathbb{C}$), and the norm $\|f\| = \sup_{n\geq 0} \boldsymbol{p}(n)|\widehat{f}(n)|$. The following results are shown:

- The Topological stable rank of $\mathcal{A}(\boldsymbol{p})$ is 1.
- The Bass stable rank of $\mathcal{A}(\boldsymbol{p})$ is 1.
- $\mathcal{A}(\boldsymbol{p})$ is a Hermite ring.
- $\mathcal{A}(\boldsymbol{p})$ is not a projective-free ring.
- Idempotents in $\mathcal{A}(\boldsymbol{p})$ are described.
- Exponentials in $\mathcal{A}(\boldsymbol{p})$ are described, and it is shown that every invertible element of $\mathcal{A}(\boldsymbol{p})$ has a logarithm, so that the first Čech cohomology group $H^1(M(\mathcal{A}(\boldsymbol{p})), \mathbb{Z})$ with integer coefficients of the maximal ideal space $M(\mathcal{A}(\boldsymbol{p}))$ is trivial.
- A generalised necessary and sufficient 'corona-type condition' on the matricial data (A, b) with entries from $\mathcal{A}(\boldsymbol{p})$ is given for the solvability of $Ax = b$ with x also having entries from $\mathcal{A}(\boldsymbol{p})$.
- The Krull dimension of $\mathcal{A}(\boldsymbol{p})$ is infinite.
- $\mathcal{A}(\boldsymbol{p})$ is neither Artinian nor Noetherian.
- $\mathcal{A}(\boldsymbol{p})$ is coherent.
- The special linear group over $\mathcal{A}(\boldsymbol{p})$ is generated by elementary matrices.

E-mail address: A.J.Sasane@lse.ac.uk.

https://doi.org/10.1016/j.jmaa.2023.128000

1. Introduction

In [32], the following Banach algebras were introduced. Throughout the article, we will use the notation $\mathbb{N}$ for the set $\{1, 2, 3, \cdots\}$ of natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

Definition 1.1 *(The Banach algebra $\mathcal{A}(\boldsymbol{p})$).*
Let $\boldsymbol{p} : \mathbb{N}_0 \to (0, \infty)$ be such that $\lim\limits_{n\to\infty} (\boldsymbol{p}(n))^{\frac{1}{n}} = \infty$.

Define

$$\mathcal{A}(\boldsymbol{p}) = \Big\{ f : \mathbb{C} \to \mathbb{C} \,\Big|\, f(z) = \sum_{n=0}^{\infty} a_n z^n \ (z \in \mathbb{C}),\ a_n = O\Big(\frac{1}{\boldsymbol{p}(n)}\Big) \text{ for } n \to \infty \Big\}.$$

The O-notation here means as usual that there exists a constant $C > 0$ such that $\boldsymbol{p}(n)|a_n| < C$ for all $n \in \mathbb{N}_0$. For $f \in \mathcal{A}(\boldsymbol{p})$, we set

$$\widehat{f}(n) := \frac{1}{n!} \frac{d^n f}{dz^n}(0) \quad (n \in \mathbb{N}_0).$$

With pointwise addition and scalar multiplication, $\mathcal{A}(\boldsymbol{p})$ is a complex vector space. We equip $\mathcal{A}(\boldsymbol{p})$ with the weighted Hadamard multiplication $*$, given by

$$(f * g)(z) = \sum_{n=0}^{\infty} \boldsymbol{p}(n) \widehat{f}(n) \widehat{g}(n) z^n \ \ (z \in \mathbb{C}), \text{ for all } f, g \in \mathcal{A}(\boldsymbol{p}),$$

and the norm $\|\cdot\|$, defined by

$$\|f\| = \sup_{n \in \mathbb{N}_0} \boldsymbol{p}(n) |\widehat{f}(n)| \text{ for all } f \in \mathcal{A}(\boldsymbol{p}).$$

Then $\mathcal{A}(\boldsymbol{p})$ is a complex commutative unital Banach algebra with the unit element ε given by

$$\varepsilon(z) = \sum_{n=0}^{\infty} \frac{1}{\boldsymbol{p}(n)} z^n \quad (z \in \mathbb{C}). \tag{1}$$

For example, if $\boldsymbol{p}_*(n) = n!$ $(n \in \mathbb{N}_0)$, then $\lim_{n\to\infty} (n!)^{\frac{1}{n}} = \infty$, and the corresponding Banach algebra $\mathcal{A}(\boldsymbol{p}_*)$ has the identity element $\exp z$. This Banach algebra $\mathcal{A}(\boldsymbol{p}_*)$ was introduced and studied in [30]. In [32], the more general Banach algebras $\mathcal{A}(\boldsymbol{p})$ were introduced, their ideal structure was studied, and the following results were shown.

(R1) $g \in \mathcal{A}(\boldsymbol{p})$ is a divisor of $f \in \mathcal{A}(\boldsymbol{p})$ if and only if there exists a constant $C > 0$ such that $|\widehat{f}(n)| \le C|\widehat{g}(n)|$ for all $n \in \mathbb{N}_0$.
In particular, f is invertible in $\mathcal{A}(\boldsymbol{p})$ if and only if there exists a $\delta > 0$ such that $|\widehat{f}(n)| \ge \frac{\delta}{\boldsymbol{p}(n)}$ for all $n \in \mathbb{N}_0$.

(R2) Every finite collection of functions $f_1, \cdots, f_K \in \mathcal{A}(\boldsymbol{p})$ $(K \in \mathbb{N})$ has a greatest common divisor $d \in \mathcal{A}(\boldsymbol{p})$. Up to invertible elements, d is given by $\widehat{d}(n) = \max_{1\le k\le K} |\widehat{f_k}(n)|$ for all $n \in \mathbb{N}_0$.

(R3) For $f, f_1, \cdots, f_K \in \mathcal{A}(\boldsymbol{p})$, f belongs to the ideal $\langle f_1, \cdots, f_K\rangle$ in $\mathcal{A}(\boldsymbol{p})$ generated by $f_1, \cdots, f_K$ if and only if there exists a constant $C > 0$ such that $|\widehat{f}(n)| \le C \sum_{k=1}^{K} |\widehat{f_k}(n)|$ for all $n \in \mathbb{N}_0$.

(R4) Every finitely generated ideal in $\mathcal{A}(\boldsymbol{p})$ is principal.

(R5) An ideal I of $\mathcal{A}(\boldsymbol{p})$ is *fixed* if there exists an $m \in \mathbb{N}_0$ such that for all $f \in I$, $\widehat{f}(m) = 0$. Then $I_m := \{f \in \mathcal{A}(\boldsymbol{p}) : \widehat{f}(m) = 0\}$ is a fixed, maximal ideal of $\mathcal{A}(\boldsymbol{p})$. Every fixed, maximal ideal of $\mathcal{A}(\boldsymbol{p})$ is I_m for some $m \in \mathbb{N}_0$.

In the spirit of [20], where algebraic properties of the ring of entire functions with pointwise operations were investigated, in this article, we study algebraic-analytic properties of $\mathcal{A}(\boldsymbol{p})$, and show the following:

(1) The topological stable rank of $\mathcal{A}(\boldsymbol{p})$ is equal to 1.
(2) The Bass stable rank of $\mathcal{A}(\boldsymbol{p})$ is equal to 1.
(3) $\mathcal{A}(\boldsymbol{p})$ is a Hermite ring.
(4) $\mathcal{A}(\boldsymbol{p})$ is not a projective-free ring.
(5) Idempotents in $\mathcal{A}(\boldsymbol{p})$ are described.
(6) Exponentials in $\mathcal{A}(\boldsymbol{p})$ are described, and it is shown that every invertible element of $\mathcal{A}(\boldsymbol{p})$ has a logarithm, so that the first Čech cohomology group $H^1(M(\mathcal{A}(\boldsymbol{p})), \mathbb{Z})$ with integer coefficients of the maximal ideal space $M(\mathcal{A}(\boldsymbol{p}))$ is trivial.
(7) A generalised necessary and sufficient 'corona-type condition' on the matricial data (A, b) with entries from $\mathcal{A}(\boldsymbol{p})$ is given for the solvability of $Ax = b$ with x also having entries from $\mathcal{A}(\boldsymbol{p})$.
(8) The Krull dimension of $\mathcal{A}(\boldsymbol{p})$ is infinite.
(9) $\mathcal{A}(\boldsymbol{p})$ is neither Artinian nor Noetherian.
(10) $\mathcal{A}(\boldsymbol{p})$ is coherent.
(11) The special linear group over $\mathcal{A}(\boldsymbol{p})$ is generated by elementary matrices.

Motivation

Investigation of algebraic properties for rings in analysis has proven to be important for theory-building. For example, we mention the corona problem: given f, g in the Hardy algebra $H^\infty(\mathbb{D})$ of bounded holomorphic functions in the unit disk $\mathbb{D}$ in $\mathbb{C}$, Kakutani's 1941 question of whether the condition $|f(z)| + |g(z)| \geq \delta > 0$ $(z \in \mathbb{D})$ is sufficient for $H^\infty(\mathbb{D})$ to be equal to the ideal $\langle f, g\rangle$ generated by f, g, led to significant advances in complex and harmonic analysis through Carleson's 1962 solution to the problem. As another example, Kazhdan's Property (T) can be established for the special linear group over the ring $\mathcal{O}(X)$ of holomorphic functions over a finite-dimensional reduced Stein space X by investigating when the special linear group over $\mathcal{O}(X)$ can be generated by elementary matrices (Gromov's Vaserstein Problem, settled in [19]). Motivations underlying the investigation of properties (1)-(11) in this article are listed below.

Properties (1) and (2): The concept of stable rank (introduced in [4], and for topological rings in [28]) plays an important role in some stabilisation problems of K-theory. See e.g., [24] for an analysis viewpoint, and for recent work in the context of a Banach algebra of holomorphic functions, see e.g. [8].

Properties (3) and (4): Hermite and projective free rings arose in connection with Serre's problem from 1955 (whether finitely general projective modules over $k[x_1, \dots, x_n]$, k a field, are free); see e.g. [22]. They are also relevant in the stabilisation problem in control theory (see e.g., [36, Theorem 66] and [27, Theorem 6.3]).

Properties (5) and (6): These are classical topics in commutative Banach algebra theory; see [14, Chap. III, §6, §7].

Property (7): This matricial problem is generalisation of the solvability of the Bézout equation considered in the classical corona problem. For background on the corona problem, see e.g. [11].

Properties (8), (9), (10): These are natural questions from the commutative algebra viewpoint.

Property (11): This is a classical question, investigated in the context of rings of continuous functions by [23], [35], for holomorphic functions of several variables in [19], and for Banach algebras in [7], [8].

The outline of this article is as follows: In each section, we will first give the background of the property, by recalling key definitions, and then prove the property, possibly with additional commentary.

2. Topological stable rank of $\mathcal{A}(\boldsymbol{p})$ is 1

The notion of the topological stable rank for topological algebras was introduced by Rieffel in [28] analogous to the K-theoretic concept of (Bass) stable rank, as well as several related numerical invariants.

Definition 2.1. Let R be a commutative unital ring with identity element 1. We assume that $1 \neq 0$, that is, R is not the trivial ring $\{0\}$. For $n \in \mathbb{N}$, an n-tuple $(a_1, \cdots, a_n) \in R^n = R \times \cdots \times R$ (n times) is said to be *invertible* (or *unimodular*), if there exists $(b_1, \cdots, b_n) \in R^n$ such that the Bézout equation $b_1 a_1 + \cdots + b_n a_n = 1$ is satisfied. The set of all invertible n-tuples is denoted by $U_n(R)$.

Now suppose that A is a commutative unital Banach algebra. The *topological stable rank* of A is the minimum $n \in \mathbb{N}$ such that $U_n(A)$ is dense in A^n, and it is infinite if no such n exists.

Here, A^n is the normed space with the 'Euclidean norm' $\|\cdot\|_2$ given by

$$\|\boldsymbol{v}\|_2^2 := \|v_1\|^2 + \cdots + \|v_n\|^2 \tag{2}$$

for all $\boldsymbol{v}$ in A^n, where $\boldsymbol{v}$ has the components $v_1, \ldots, v_n \in A$.

Theorem 2.2. *The topological stable rank of $\mathcal{A}(\boldsymbol{p})$ is* 1.

Proof. Let $f \in \mathcal{A}(\boldsymbol{p})$, and $\epsilon > 0$. Define g by

$$\widehat{g}(n) = \begin{cases} \widehat{f}(n) & \text{if } \boldsymbol{p}(n)|\widehat{f}(n)| > \epsilon, \\ \frac{\epsilon}{\boldsymbol{p}(n)} & \text{if } \boldsymbol{p}(n)|\widehat{f}(n)| \leq \epsilon. \end{cases}$$

Then

$$|\widehat{g}(n) - \widehat{f}(n)| \begin{cases} = 0 & \text{if } \boldsymbol{p}(n)|\widehat{f}(n)| > \epsilon \\ \leq \frac{2\epsilon}{\boldsymbol{p}(n)} & \text{if } \boldsymbol{p}(n)|\widehat{f}(n)| \leq \epsilon \end{cases}$$

and so $g \in \mathcal{A}(\boldsymbol{p})$ and $\|g - f\| \leq 2\epsilon$. Also,

$$|\widehat{g}(n)| = \left\{ \begin{array}{ll} |\widehat{f}(n)| & \text{if } \boldsymbol{p}(n)|\widehat{f}(n)| > \epsilon \\ \frac{\epsilon}{\boldsymbol{p}(n)} & \text{if } \boldsymbol{p}(n)|\widehat{f}(n)| \leq \epsilon \end{array} \right\} \geq \frac{\epsilon}{\boldsymbol{p}(n)} \quad (n \in \mathbb{N}_0).$$

Consequently, g is invertible in $\mathcal{A}(\boldsymbol{p})$ by (R1) on page 2. □

3. Bass stable rank of $\mathcal{A}(\boldsymbol{p})$ is 1

The notion of the Bass stable rank was introduced in algebraic K-theory by Bass [4].

Definition 3.1. Let R be a commutative unital ring with identity 1. Let $n \in \mathbb{N}$. An $(n+1)$-tuple $(a_1, \cdots, a_{n+1}, \alpha) \in U_{n+1}(R)$ is said to be *reducible* if there exists an n-tuple $(h_1, \cdots, h_n) \in R^n$ such that $(a_1 + h_1\alpha, \cdots, a_n + h_n\alpha) \in U_n(R)$. The *Bass stable rank of* R is the smallest integer n such that every element in $U_{n+1}(R)$ is reducible. It is infinite if no such n exists.

For a commutative unital Banach algebra, the Bass stable rank is at most equal to the topological stable rank [28, Theorem 2.3]. By Theorem 2.2, it follows that the Bass stable rank of $\mathcal{A}(\boldsymbol{p})$ is 1.

Corollary 3.2. *The Bass stable rank of $\mathcal{A}(\boldsymbol{p})$ is* 1.

4. $\mathcal{A}(\boldsymbol{p})$ is a Hermite ring, but not a projective free ring

The study of Serre's conjecture naturally led to the notion of Hermite rings; see e.g., [22].

Definition 4.1. Let R be a commutative unital ring. The ring R is called *Hermite* if every finitely generated stably free R-module is free. The ring R is called *projective-free* if every finitely generated projective R-module is free.

If R-modules M, N are isomorphic, then we write $M \cong N$. Recall that if M is a finitely generated R-module, then:

- M is called *free* if $M \cong R^k$ for some integer $k \geq 0$.
- M is called *projective* if there exists an R-module N and an integer $m \geq 0$ such that $M \oplus N \cong R^m$.
- M is called *stably free* if there exist free finitely generated R-modules F and G such that $M \oplus F \cong G$.

It is clear that every projective free ring is Hermite.

For $m, n \in \mathbb{N}$, $R^{m\times n}$ denotes the set of matrices with m rows and n columns having entries from R. The identity element in $R^{k\times k}$ having diagonal elements 1, and zeroes elsewhere will be denoted by I_k.

In terms of matrices, R is Hermite if and only if left-invertible tall matrices over R can be completed to invertible ones (see e.g. [31, p.1029]): For all $k, K \in \mathbb{N}$ such that $k < K$, and for all $f \in R^{K\times k}$ such that there exists a $g \in R^{k\times K}$ so that $gf = I_k$, there exists an $f_c \in R^{K\times(K-k)}$ and there exists a $G \in R^{K\times K}$ such that $G[\, f \; f_c \,] = I_K$.

In terms of matrices (see e.g. [10, Proposition 2.6] or [2, Lemma 2.2]), the ring R is projective-free if and only if every idempotent matrix P is conjugate (by an invertible matrix S) to a diagonal matrix with elements 1 and 0 on the diagonal, that is, for every $m \in \mathbb{N}$ and every $P \in R^{m\times m}$ satisfying $P^2 = P$, there exists an $S \in R^{m\times m}$ such that S is invertible as an element of $R^{m\times m}$, and for some integer $r \geq 0$, $S^{-1}PS = \left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right]$. In 1976, it was shown independently by Quillen and Suslin, that if $\mathbb{F}$ is a field, then the polynomial ring $\mathbb{F}[x_1, \cdots, x_n]$ is projective-free, settling Serre's conjecture from 1955 (see [22]).

4.1. $\mathcal{A}(\boldsymbol{p})$ is a Hermite ring

It is known that a commutative unital ring having Bass stable rank ≤ 2 is Hermite (see e.g., [24, Corollary 36.17]). In light of this result and Corollary 3.2, we have:

Corollary 4.2. *$\mathcal{A}(\boldsymbol{p})$ is a Hermite ring.*

4.2. $\mathcal{A}(\boldsymbol{p})$ is not a projective-free ring

While every projective free ring is Hermite, the converse may not hold. In fact $\mathcal{A}(\boldsymbol{p})$ is such an example, by using the matricial characterisation of projective free rings.

Theorem 4.3. *$\mathcal{A}(\boldsymbol{p})$ is not projective free.*

Proof. Suppose $\mathcal{A}(\boldsymbol{p})$ is projective free. Let $P = \sum\limits_{m=0}^{\infty} \frac{1}{\boldsymbol{p}(2m)} z^{2m}$. Then $P \in \mathcal{A}(\boldsymbol{p})$, and

$$\widehat{P * P}(n) \;=\; \boldsymbol{p}(n)\widehat{P}(n)\widehat{P}(n) = \begin{cases} \boldsymbol{p}(2m)\frac{1}{(\boldsymbol{p}(2m))^2} = \frac{1}{\boldsymbol{p}(2m)} \text{ if } n = 2m,\ m \in \mathbb{N}_0 \\ \boldsymbol{p}(2m)\, 0^2 = 0 \text{ if } n = 2m+1,\ m \in \mathbb{N}_0. \end{cases}$$

So $P*P = P$. As we have assumed $\mathcal{A}(\boldsymbol{p})$ is projective free, there is an $f \in \{0, \epsilon\}$, $D = [\, f \,]$, and $S, S^{-1} \in \mathcal{A}(\boldsymbol{p})$ such that $P = S^{-1} * D * S$. But then $P = 0$ or $P = \varepsilon$, and either case is false. Consequently, $\mathcal{A}(\boldsymbol{p})$ is not projective free. $\square$

For a Banach space A, we denote the set of all continuous linear maps from A to $\mathbb{C}$ by A^*. Recall that for a commutative unital complex Banach algebra A, the maximal ideal space $M(A) \subset A^*$ is the set of nonzero homomorphisms $A \to \mathbb{C}$, endowed with the Gelfand topology, the weak-$*$ topology of A^*. It is a compact Hausdorff space contained in the unit sphere of A^*. Contractibility of the maximal ideal space $M(A)$ in the Gelfand topology suffices for A to be projective-free (see, e.g., [6, Corollary 1.4]). Thus the maximal ideal space $M(\mathcal{A}(\boldsymbol{p}))$ is not contractible.

5. Idempotents in $\mathcal{A}(\boldsymbol{p})$

The following result characterises idempotents in $\mathcal{A}(\boldsymbol{p})$.

Theorem 5.1. *$f \in \mathcal{A}(\boldsymbol{p})$ is an idempotent if and only if for all $n \in \mathbb{N}_0$, $\widehat{f}(n) \in \{0, \frac{1}{\boldsymbol{p}(n)}\}$.*

Proof. The 'if' part is immediate from the definition of multiplication in $\mathcal{A}(\boldsymbol{p})$. If $f \in \mathcal{A}(\boldsymbol{p})$ is an idempotent, then $f * f = f$, and so we obtain $\sum\limits_{n=0}^{\infty} (\boldsymbol{p}(n)(\widehat{f}(n))^2 - \widehat{f}(n))z^n = (f * f)(z) - f(z) = 0 \quad (z \in \mathbb{C})$. Thus $\boldsymbol{p}(n)(\widehat{f}(n))^2 - \widehat{f}(n) = 0$ for all $n \in \mathbb{N}_0$. So $\widehat{f}(n) \in \{0, \frac{1}{\boldsymbol{p}(n)}\}$. $\square$

Let A be a commutative unital complex Banach algebra. The *Gelfand transform* of $a \in A$, defined by $\hat{a}(\boldsymbol{\varphi}) := \boldsymbol{\varphi}(a)$ for $\boldsymbol{\varphi} \in M(A)$, is a nonincreasing-norm morphism from A into $C(M(A))$, the Banach algebra of complex-valued continuous functions on $M(A)$ equipped with the supremum norm $\|\cdot\|_\infty$ (given by $\|f\|_\infty := \sup_{\varphi \in M(A)} |f(\varphi)|$ for all $f \in C(M(A))$.

From the special case of (R3) on page 2 when $f = \varepsilon$ (the identity element of $\mathcal{A}(\boldsymbol{p})$), we have the following corona theorem:

Proposition 5.2. *Let $f_1, \cdots, f_n \in \mathcal{A}(\boldsymbol{p})$ $(n \in \mathbb{N})$. Then the following are equivalent:*

(1) *There exists a $\delta > 0$ such that for all $k \in \mathbb{N}_0$, $\sum\limits_{i=1}^{n} |\widehat{f}_i(k)| \geq \frac{\delta}{\boldsymbol{p}(k)}$.*

(2) *There exist $g_1, \cdots, g_n \in \mathcal{A}(\boldsymbol{p})$ such that $\sum\limits_{i=1}^{n} g_i * f_i = \varepsilon$.*

For $k \in \mathbb{N}_0$, let φ_k denote the complex homomorphism given by

$$\varphi_k(f) = \widehat{f}(k) \text{ for all } f \in \mathcal{A}(\boldsymbol{p}).$$

From (R5) (p. 2), we know that $\varphi_k \in M(\mathcal{A}(\boldsymbol{p}))$. From elementary Banach algebra theory (see e.g., [26, Lemma 9.2.6]), it follows that the set $\{\varphi_k : k \in \mathbb{N}_0\}$ is dense in $M(\mathcal{A}(\boldsymbol{p}))$.

We will now show that $\{\varphi_k : k \in \mathbb{N}_0\}$, with the topology induced from $M(\mathcal{A}(\boldsymbol{p}))$, is homeomorphic to $\mathbb{N}_0$, with the topology induced from $\mathbb{R}$. In the proof below, we will use the notation $\delta_{m,n} := 0$ if $m \neq n$ and $\delta_{m,m} := 1$ for all $m, n \in \mathbb{N}_0$.

Proposition 5.3. *For $k_0 \in \mathbb{N}_0$, a net $(\varphi_{k_i})_{i \in I}$ converges to φ_{k_0} in $M(\mathcal{A}(\boldsymbol{p}))$ if and only if $(k_i)_{i \in I}$ is eventually constant, equal to k_0.*

Proof. 'If' part: Let $(k_i)_{i \in I}$ be eventually constant, equal to k_0. There exists an $i_* \in I$ such that for all $i \geq i_*$ (where $\geq$ denotes the order on the directed set I), $k_i = k_0$, and so for all $f \in \mathcal{A}(\boldsymbol{p})$, we have that $\varphi_{k_i}(f) = \widehat{f}(k_i) = \widehat{f}(k_0) = \varphi_{k_0}(f)$. So $(\varphi_{k_i})_{i \in I}$ converges to φ_{k_0} in $M(\mathcal{A}(\boldsymbol{p}))$.
'Only if' part: Suppose that $(\varphi_{k_i})_{i \in I}$ converges to φ_{k_0} in $M(\mathcal{A}(\boldsymbol{p}))$. Then with $f = z^{k_0} \in \mathcal{A}(\boldsymbol{p})$, by the definition of the Gelfand topology, the net $(\varphi_{k_i}(z^{k_0}))_{i \in I}$ converges to $\varphi_{k_0}(z^{k_0}) = 1$ in $\mathbb{R}$. Thus for $\epsilon = \frac{1}{2} > 0$, there exists an $i_* \in I$ such that $|\delta_{k_i, k_0} - 1| = |\varphi_{k_i}(z^{k_0}) - 1| < \frac{1}{2}$ for all $i \geq i_*$, i.e., $k_i = k_0$. $\square$

As $M(\mathcal{A}(\boldsymbol{p}))$ is compact, while $\mathbb{N}_0$ with its usual topology is not compact, it follows that not all elements of $M(\mathcal{A}(\boldsymbol{p}))$ are of the form φ_k for some $k \in \mathbb{N}_0$. Explicit examples are known (see below), and these were mentioned as non-fixed maximal ideals in [32, Remark, pp. 6-7].

Example 5.4. Let $\boldsymbol{k} = (k_n)_{n\in\mathbb{N}}$ be any subsequence of the sequence of natural numbers. Define

$$I_{\boldsymbol{k}} := \{f \in \mathcal{A}(\boldsymbol{p}) : \lim_{n\to\infty} \boldsymbol{p}_{k_n} \widehat{f}(k_n) = 0\}.$$

Then $I_{\boldsymbol{k}}$ is an ideal of $\mathcal{A}(\boldsymbol{p})$. (It is clear that if $f, g \in I_{\boldsymbol{k}}$, then $f + g \in I_{\boldsymbol{k}}$. If $f \in I_{\boldsymbol{k}}$ and $g \in \mathcal{A}(\boldsymbol{p})$, then there exists a $C > 0$ such that $|\widehat{g}(k)| \le \frac{C}{\boldsymbol{p}(k)}$ for all $k \in \mathbb{N}_0$, and so

$$\begin{aligned} 0 \le \boldsymbol{p}(k_n)|\widehat{(f * g)}(k_n)| &= \boldsymbol{p}(k_n)|\boldsymbol{p}(k_n)\widehat{f}(k_n)\widehat{g}(k_n)| \\ &\le \boldsymbol{p}(k_n)\boldsymbol{p}(k_n)|\widehat{f}(k_n)|\tfrac{C}{\boldsymbol{p}(k_n)} \stackrel{n\to\infty}{\longrightarrow} 0. \end{aligned}$$

Thus $f * g \in \mathcal{A}(\boldsymbol{p})$.) Moreover, $I_{\boldsymbol{k}} \neq \mathcal{A}(\boldsymbol{p})$ since $\varepsilon \notin I_{\boldsymbol{k}}$:

$$\lim_{n\to\infty} \boldsymbol{p}(k_n)|\widehat{\varepsilon}(k_n)| = \lim_{n\to\infty} \boldsymbol{p}(k_n)\tfrac{1}{\boldsymbol{p}(k_n)} = 1 \neq 0.$$

Hence there exists a maximal ideal M in $\mathcal{A}(\boldsymbol{p})$ such that $I_{\boldsymbol{k}} \subset M$. We note that for each $m \in \mathbb{N}_0$, $M \neq \ker \varphi_m$ (since for any $m \in \mathbb{N}_0$, $f := z^m \in I_{\boldsymbol{k}} \subset M$, and then $\varphi_m(f) = 1 \neq 0$). $\diamond$

Recall the Shilov idempotent theorem (see, e.g., [14, Corollary 6.5]): If E is an open-closed subset of $M(A)$, then there is a unique element f of A such that $f^2 = f$ and $\hat{f} = \mathbf{1}_E$ (the characteristic function of E, which is identically equal to 1 on E, and 0 elsewhere on $M(A) \setminus E$). From Theorem 5.1, we get the following.

Corollary 5.5. *If $\boldsymbol{p}(n) \neq 1$ for all $n \in \mathbb{N}_0$, then $M(\mathcal{A}(\boldsymbol{p}))$ is connected.*

Proof. Let E be closed and open, and let $E \neq \emptyset$ and $E \neq M(\mathcal{A}(\boldsymbol{p}))$. There exists an idempotent f in $\mathcal{A}(\boldsymbol{p})$ such that $\hat{f} = \mathbf{1}_E$. If $n \in \mathbb{N}_0$ is such that $\varphi_n \in E$, then $\{0, \frac{1}{\boldsymbol{p}(n)}\} \ni \widehat{f}(n) = \varphi_n(f) = \hat{f}(\varphi_n) = \mathbf{1}_E(\varphi_n) = 1$, so that $\boldsymbol{p}(n) = 1$, a contradiction. Thus E does not contain any φ_n, $n \in \mathbb{N}_0$, and $M(\mathcal{A}(\boldsymbol{p})) \setminus E$ contains $\{\varphi_n : n \in \mathbb{N}_0\}$. As $\{\varphi_n : n \in \mathbb{N}_0\}$ is dense in $M(\mathcal{A}(\boldsymbol{p}))$, we conclude that $M(\mathcal{A}(\boldsymbol{p})) \setminus E = M(\mathcal{A}(\boldsymbol{p}))$, i.e., $E = \emptyset$, a contradiction. Thus $M(\mathcal{A}(\boldsymbol{p}))$ is connected. $\square$

6. Exponentials in $\mathcal{A}(\boldsymbol{p})$

We will show that every invertible element in $\mathcal{A}(\boldsymbol{p})$ has a logarithm.

Lemma 6.1. *If $f \in \mathcal{A}(\boldsymbol{p})$, then for all $z \in \mathbb{C}$, $|f(z)| \le \varepsilon(|z|)\|f\|$.*

Proof. As $|\widehat{f}(k)| = \frac{\boldsymbol{p}(k)|\widehat{f}(k)|}{\boldsymbol{p}(k)} \le \frac{\|f\|}{\boldsymbol{p}(k)}$ for all $k \in \mathbb{N}_0$, we have

$$|f(z)| = |\sum_{k=0}^{\infty} \widehat{f}(k) z^k| \le \sum_{k=0}^{\infty} \tfrac{\|f\|}{\boldsymbol{p}(k)} |z|^k = \|f\|\varepsilon(|z|). \quad \square$$

It follows that if $(f_n)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathcal{A}(\boldsymbol{p})$, then it converges pointwise.

Lemma 6.2. *If $f \in \mathcal{A}(\boldsymbol{p})$, then $(e^f)(z) = \sum_{k=0}^{\infty} \frac{e^{\boldsymbol{p}(k)\widehat{f}(k)}}{\boldsymbol{p}(k)} z^k$ for all $z \in \mathbb{C}$.*

Proof. We note that $(f^n)(z) = \sum\limits_{k=0}^{\infty} \boldsymbol{p}(k)^{n-1}(\widehat{f}(k))^n z^k$ $(z \in \mathbb{C})$. We have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \Big|\frac{\boldsymbol{p}(k)^{n-1}(\widehat{f}(k))^n}{n!} z^k\Big| \le \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{|z|^k}{\boldsymbol{p}(k)} \frac{\|f\|^n}{n!} = \sum_{k=0}^{\infty} \frac{|z|^k}{\boldsymbol{p}(k)} \sum_{n=0}^{\infty} \frac{\|f\|^n}{n!} = \sum_{k=0}^{\infty} \frac{|z|^k}{\boldsymbol{p}(k)} e^{\|f\|} = e^{\|f\|}\varepsilon(|z|) < \infty.$$

So exchange of summations below is allowed. We have

$$(e^f)(z) = \sum_{n=0}^{\infty} \frac{(f^n)(z)}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{\boldsymbol{p}(k)^{n-1}(\widehat{f}(k))^n}{n!} z^k = \sum_{k=0}^{\infty} \frac{z^k}{\boldsymbol{p}(k)} \sum_{n=0}^{\infty} \frac{(\boldsymbol{p}(k)\widehat{f}(k))^n}{n!} = \sum_{k=0}^{\infty} \frac{z^k}{\boldsymbol{p}(k)} e^{\boldsymbol{p}(k)\widehat{f}(k)}. \quad \square$$

Let $\mathcal{A}(\boldsymbol{p})^{-1}$ denote the multiplicative group of invertible elements of $\mathcal{A}(\boldsymbol{p})$, and $e^{\mathcal{A}(\boldsymbol{p})}$ the subgroup of $\mathcal{A}(\boldsymbol{p})$ consisting of all exponentials e^f, $f \in \mathcal{A}(\boldsymbol{p})$.

Theorem 6.3. $e^{\mathcal{A}(\boldsymbol{p})} = \mathcal{A}(\boldsymbol{p})^{-1}$.

Proof. Let $g \in \mathcal{A}(\boldsymbol{p})^{-1}$. By (R1) on page 2, there exists a $\delta > 0$ such that $|\widehat{g}(k)| \ge \frac{\delta}{\boldsymbol{p}(k)}$ for all $k \in \mathbb{N}_0$. In particular, $\widehat{g}(k) \ne 0$, and we define $\widehat{f}(k) := \frac{1}{\boldsymbol{p}(k)}\mathrm{Log}(\boldsymbol{p}(k)\widehat{g}(k))$, for $k \in \mathbb{N}_0$, where $\mathrm{Log} : \mathbb{C}\setminus\{0\} \to \mathbb{R}\times(-\pi,\pi]$ denotes the principal branch of the logarithm. We have $0 < \delta \le \boldsymbol{p}(k)|\widehat{g}(k)| \le \|g\|$ for all $k \in \mathbb{N}_0$, and so $\sup_{k\in\mathbb{N}_0} \boldsymbol{p}(k)|\widehat{f}(k)| \le \sqrt{(\max\{\log\delta,\, \log\|g\|\})^2 + \pi^2} =: C < \infty$, showing that $f \in \mathcal{A}(\boldsymbol{p})$. It follows from Lemma 6.2 that $e^f = g$. $\square$

For a topological space X, let $H^1(X,\mathbb{Z})$ denote the first Čech cohomology group of X with integer coefficients. For background on Čech cohomology, see [12]. For a commutative unital complex semisimple Banach algebra A, the quotient group A^{-1}/e^A is isomorphic to $H^1(M(A),\mathbb{Z})$ (see e.g. [14, Corollary 7.4]). Thus $H^1(M(\mathcal{A}(\boldsymbol{p})),\mathbb{Z}) = \{0\}$.

7. Solvability of $Ax = b$

We will show the following:

Theorem 7.1. *Let $A \in \mathcal{A}(\boldsymbol{p})^{m\times n}$, $b \in \mathcal{A}(\boldsymbol{p})^{m\times 1}$.*
Then the following are equivalent:

(1) *There exists an $x \in \mathcal{A}(\boldsymbol{p})^{n\times 1}$ such that $A * x = b$.*
(2) *There exists a $\delta > 0$ such that for all $k \in \mathbb{N}_0$ and all $y \in \mathbb{C}^{m\times 1}$, $\| (\widehat{A}(k))^* y \|_2 \ge \delta |\langle y, \widehat{b}(k)\rangle_2|$.*

Here $\langle \cdot,\cdot\rangle_2$ denotes the usual Euclidean inner product on $\mathbb{C}^{\ell\times 1}$ for $\ell \in \mathbb{N}$, and $\|\cdot\|_2$ is the corresponding induced norm. Also, if $a_{ij} \in \mathcal{A}(\boldsymbol{p})$ denotes the entry in the i^{th} row and j^{th} column of A, then $\widehat{A}(k) \in \mathbb{C}^{m\times n}$ is the matrix whose entry in the i^{th} row and j^{th} column is $\widehat{a_{ij}}(k)$, $1 \le i \le m$, $1 \le j \le n$, $k \in \mathbb{N}_0$. For a matrix $M \in \mathbb{C}^{m\times n}$, M^* denotes its Hermitian adjoint (obtained by taking the entrywise complex conjugate of M and then taking the transpose of the resulting matrix). We will use the following elementary linear algebraic result; see [29, Lemma 8.2]. We include the short proof for the sake of completeness.

Lemma 7.2. *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^{m\times 1}$ be such that*

$$\exists \delta > 0 \text{ such that for all } y \in \mathbb{C}^{m\times 1},\ \|A^* y\|_2 \ge \delta|\langle y, b\rangle_2|. \qquad (\star)$$

Then there exists an $x \in \mathbb{C}^{n\times 1}$ such that $Ax = b$ with $\|x\|_2 \le \frac{1}{\delta}$.

Proof. For $y \in \ker A^*$, ($\star$) implies $\langle y, b\rangle_2 = 0$. So $b \in (\ker A^*)^\perp = \operatorname{ran} A$. If $y \in \ker(AA^*)$, then $\|A^*y\|_2^2 = \langle A^*y, A^*y\rangle_2 = \langle AA^*y, y\rangle_2 = \langle 0, y\rangle_2 = 0$. Thus $A^*y = 0$, and so $y \in \ker A^* = (\operatorname{ran} A)^\perp$. Since we had shown above that $b \in \operatorname{ran} A$, we have $\langle b, y\rangle_2 = 0$. As $y \in \ker(AA^*)$ was arbitrary, $b \in (\ker(AA^*))^\perp = \operatorname{ran}((AA^*)^*) = \operatorname{ran}(AA^*)$. Hence there exists a $y_0 \in \mathbb{C}^{m\times 1}$ such that $AA^*y_0 = b$. Taking $x := A^*y_0 \in \mathbb{C}^{n\times 1}$, we have $Ax = b$. If $b = 0$, then we can take $x = 0$, and the estimate on $\|x\|_2$ is obvious. So we assume that $b \neq 0$ and then $A^*y_0 \neq 0$ (since we know that $AA^*y_0 = b$). We have, using the given inequality ($\star$),

$$\|A^*y_0\|_2^2 = \langle A^*y_0, A^*y_0\rangle_2 = \langle y_0, AA^*y_0\rangle_2 = \langle y_0, b\rangle_2 = |\langle y_0, b\rangle_2| \leq \frac{\|A^*y_0\|_2}{\delta}.$$

Since $A^*y_0 \neq 0$, we obtain $\|x\|_2 = \|A^*y_0\|_2 \leq \frac{1}{\delta}$. $\square$

Proof of Theorem 7.1. (1)⇒(2): As $x \in \mathcal{A}(\boldsymbol{p})^{n\times 1}$, there exists a $C > 0$ such that for all $k \in \mathbb{N}_0$, $\|\widehat{x}(k)\|_2 \leq \frac{C}{\boldsymbol{p}(k)}$. For $y \in \mathbb{C}^{m\times 1}$ and $k \in \mathbb{N}_0$,

$$\begin{aligned} |\langle y, \widehat{b}(k)\rangle_2| &= |\langle y, \boldsymbol{p}(k)\widehat{A}(k)\widehat{x}(k)\rangle_2| = \boldsymbol{p}(k)|\langle (\widehat{A}(k))^*y, \widehat{x}(k)\rangle_2| \\ &\leq \boldsymbol{p}(k)\|(\widehat{A}(k))^*y\|_2 \tfrac{C}{\boldsymbol{p}(k)} \quad \text{(Cauchy-Schwarz)} \\ &= C\|(\widehat{A}(k))^*y\|_2. \end{aligned}$$

Setting $\delta := \frac{1}{C} > 0$ and rearranging gives (2).
(2)⇒(1): Fix $k \in \mathbb{N}_0$. The condition in statement (2) and Lemma 7.2 implies the existence of an $x_k \in \mathbb{C}^{n\times 1}$ such that $\widehat{A}(k)x_k = \widehat{b}(k)$, with $\|x_k\|_2 \leq \frac{1}{\delta}$. In this way, we obtain a sequence $(x_k)_{k\geq 0}$ in $\mathbb{C}^{n\times 1}$. Let the components of x_k be denoted by $x_k^{(1)}, \cdots, x_k^{(n)} \in \mathbb{C}$. Define

$$x^{(i)}(z) = \sum_{n=0}^{\infty} \tfrac{x_k^{(i)}}{\boldsymbol{p}(k)} z^k \quad \text{for all } z \in \mathbb{C},\ 1 \leq i \leq n.$$

Then each $x^{(i)} \in \mathcal{A}(\boldsymbol{p})$ (as $|\frac{x^{(i)}}{\boldsymbol{p}(k)}| \leq \frac{1/\delta}{\boldsymbol{p}(k)}$ for all $k \in \mathbb{N}_0$), and so the column vector x having these components belongs to $\mathcal{A}(\boldsymbol{p})^{n\times 1}$. Also,

$$\widehat{(A * x)}(k) = \boldsymbol{p}(k)\widehat{A}(k)\tfrac{1}{\boldsymbol{p}(k)}x_k = \widehat{b}(k) \text{ for all } k \in \mathbb{N}_0,$$

i.e., $A * x = b$. $\square$

From Theorem 7.1, or using the density of $\{\varphi_k : k \in \mathbb{N}_0\}$ in $M(\mathcal{A}(\boldsymbol{p}))$ and the matricial corona theorem (see e.g. [36, Ch. 8, Lemma 34]), we get: For $A \in \mathcal{A}(\boldsymbol{p})^{m_1\times n}$ and $B \in \mathcal{A}(\boldsymbol{p})^{m_2\times n}$, there exist $X \in \mathcal{A}(\boldsymbol{p})^{n\times m_1}$ and $Y \in \mathcal{A}(\boldsymbol{p})^{n\times m_2}$ such that $XA + YB = I_n$ if and only if there exists a $\delta > 0$ such that for all $k \in \mathbb{N}_0$, $(\widehat{A}(k))^*\widehat{A}(k) + (\widehat{B}(k))^*\widehat{B}(k) \geq \frac{\delta^2}{\boldsymbol{p}(k)^2} I_n$. A curious application of Theorem 7.1 is the following:

Corollary 7.3. *Let $A \in \mathcal{A}(\boldsymbol{p})^{n\times k}$ and $B \in \mathcal{A}(\boldsymbol{p})^{m\times n}$. There exist $X \in \mathcal{A}(\boldsymbol{p})^{k\times n}$ and $Y \in \mathcal{A}(\boldsymbol{p})^{n\times m}$ such that $AX + YB = I_n$ if and only if there exists a $\delta > 0$ such that for all $\ell \in \mathbb{N}_0$, and all $Z \in \mathbb{C}^{n\times n}$,*

$$\mathit{trace}\big(\widehat{A}(\ell)(\widehat{A}(\ell))^*ZZ^*\big) + \mathit{trace}\big((\widehat{B}(\ell))^*\widehat{B}(\ell)Z^*Z\big) \geq \frac{\delta^2}{\boldsymbol{p}(\ell)^2}|\mathit{trace}\, Z|^2.$$

Proof. We use $\langle P, Q\rangle_2 := \operatorname{trace}(Q^*P)$ for $P, Q \in \mathbb{C}^{p\times q}$. If (P_1, Q_1), $(P_2, Q_2) \in \mathbb{C}^{k\times n} \times \mathbb{C}^{n\times m}$, then $\langle (P_1, Q_1), (P_2, Q_2)\rangle_2 := \langle P_1, P_2\rangle_2 + \langle Q_1, Q_2\rangle_2$. Define $T : \mathbb{C}^{k\times n} \times \mathbb{C}^{n\times m} \to \mathbb{C}^{n\times n}$ by $T(P, Q) := \widehat{A}(\ell)P + Q\widehat{B}(\ell)$ for $(P, Q) \in \mathbb{C}^{k\times n} \times \mathbb{C}^{n\times m}$. Then $T^*Z = \big((\widehat{A}(\ell))^*Z, Z(\widehat{B}(\ell))^*\big)$ for $Z \in \mathbb{C}^{n\times n}$. Use Theorem 7.1 with b corresponding to εI_n. $\square$

8. Krull dimension of $\mathcal{A}(\boldsymbol{p})$ is infinite

Definition 8.1. The *Krull dimension* of a commutative ring R is the supremum of the lengths of chains of distinct proper prime ideals of R.

Recall that the Hardy algebra H^∞ is the Banach algebra of bounded and holomorphic functions on the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, with pointwise operations and the supremum norm $\|\cdot\|_\infty$. In [33], von Renteln showed that the Krull dimension of H^∞ is infinite. We adapt the idea given in [33], to show that the Krull dimension of $\mathcal{A}(\boldsymbol{p})$ is infinite too. A key ingredient of the proof in [33] was the use of a canonical factorisation of H^∞ elements used to create ideals with zeroes at prescribed locations with prescribed orders. Instead of zeroes of our entire functions, we will look at the indices for the vanishing coefficients in the Taylor expansion centred at 0, and instead of orders of zeroes, we will use the notion of 'index-order' introduced below.

If $f \in \mathcal{A}(\boldsymbol{p})$ and $k \in \mathbb{N}_0$ is an index such that $\widehat{f}(k) = 0$, then we define the *index-order* $m(f,k)$ *of the index* k *for* f by

$$m(f,k) = \max\{m \in \mathbb{N} : \widehat{f}(k+\ell) = 0 \text{ whenever } 0 \le \ell \le m-1\}.$$

If $\widehat{f}(k+\ell) = 0$ for all $\ell \in \mathbb{N}_0$, then we set $m(f,k) = \infty$. If $\widehat{f}(k) \neq 0$, then we set $m(f,k) = 0$. Analogous to the order of a zero of a holomorphic function, the index-order satisfies the following property.

$$\text{(P1): If } f,g \in \mathcal{A}(\boldsymbol{p}),\ k \in \mathbb{N}_0, \text{ then } m(f+g,k) \ge \min\{m(f,k), m(g,k)\}.$$

The order of a zero ζ of the pointwise product of two holomorphic functions is the sum of the orders of ζ as a zero of each of the two holomorphic functions. For the index order, and the weighted Hadamard product, we have the following instead:

$$\text{(P2): If } f,g \in \mathcal{A}(\boldsymbol{p}),\ k \in \mathbb{N}_0, \text{ then } m(f*g,k) \ge \max\{m(f,k), m(g,k)\}.$$

We will use the following known result; see [15, Theorem, §0.16, p.6].

Proposition 8.2. *If J is an ideal in a ring R, and M is a set that is closed under multiplication and $M \cap J = \emptyset$, then there exists an ideal P such that $J \subset P$ and $P \cap M = \emptyset$, and P maximal with respect to these properties. Moreover, such an ideal P is necessarily prime.*

Theorem 8.3. *The Krull dimension of $\mathcal{A}(\boldsymbol{p})$ is infinite.*

Proof. Let $a_k = 2^k$ for all $k \in \mathbb{N}_0$. Let $n \in \mathbb{N}$. Define $f_n \in \mathcal{A}(\boldsymbol{p})$ by

$$\begin{cases} \widehat{f_n}(a_k+\ell) = 0 & \text{whenever } 0 \le \ell \le k^{n+1}, \\ \widehat{f_n}(m) = \frac{1}{\boldsymbol{p}(m)} & \text{if } m \notin \bigcup_{k\in\mathbb{N}_0}\{a_k+\ell : 0 \le \ell \le k^{n+1}\}. \end{cases}$$

Note that $m(f_n, a_k) \ge k^{n+1}$, but for each fixed $n \in \mathbb{N}$, there exists a $K_n \in \mathbb{N}_0$ such that the gap between the indices, $a_{k+1} - a_k = 2^{k+1} - 2^k = 2^k > k^{n+1}$ for all $k > K_n$, and so $m(f_n, a_k) = k^{n+1}$ for all $k > K_n$. Hence

$$\lim_{k\to\infty} \frac{m(f_n,a_k)}{k^n} = \infty \quad \text{and} \quad \lim_{k\to\infty} \frac{m(f_n,a_k)}{k^{n+1}} = 1 < \infty. \tag{3}$$

Let $I := \{f \in \mathcal{A}(\boldsymbol{p}) : \exists k_0(f) \in \mathbb{N}_0 \text{ such that } \forall k > k_0(f),\ \widehat{f}(a_k) = 0\}$. The set I is nonempty since $0 \in I$. Clearly I is closed under addition, and $f * g \in I$ whenever $f \in I$ and $g \in \mathcal{A}(\boldsymbol{p})$. So I is an ideal of $\mathcal{A}(\boldsymbol{p})$. For $n \in \mathbb{N}$, we define

$$\begin{aligned} I_n &= \{f \in I : \lim_{k\to\infty} \tfrac{m(f,a_k)}{k^n} = \infty\}, \\ M_n &= \{f \in \mathcal{A}(\boldsymbol{p}) : \sup_{k\in\mathbb{N}} \tfrac{m(f,a_k)}{k^n} < \infty\}. \end{aligned}$$

Clearly $f_n \in I_n$, and so I_n is not empty. Using (P1), we see that if $f, g \in I_n$, then $f + g \in I_n$. If $g \in \mathcal{A}(\boldsymbol{p})$ and $f \in I_n$, then (P2) implies that $f * g \in I_n$. Hence I_n is an ideal of $\mathcal{A}(\boldsymbol{p})$.

The identity element $\varepsilon \in M_n$ for all $n \in \mathbb{N}$. If $f, g \in M_n$, then it follows from (P2) that $f * g \in M_n$. Thus M_n is a nonempty multiplicatively closed subset of $\mathcal{A}(\boldsymbol{p})$.

It is easy to check that for all $n \in \mathbb{N}$, $I_{n+1} \subset I_n$ and $M_n \subset M_{n+1}$. We now prove that the inclusions are strict for each $n \in \mathbb{N}$. From (3), it follows that $f_n \in I_n$ but $f_n \notin I_{n+1}$. Also $f_n \in M_{n+1}$ and $f_n \notin M_n$.

Next we show that $I_n \cap M_n = \emptyset$. Indeed, if $f \in I_n \cap M_n$, then

$$\infty = \lim_{k\to\infty} \tfrac{m(f,a_k)}{k^n} = \limsup_{k\to\infty} \tfrac{m(f,a_k)}{k^n} \leq \sup_{k\in\mathbb{N}} \tfrac{m(f,a_k)}{k^n} < \infty,$$

a contradiction. But $I_n \cap M_{n+1} \neq \emptyset$, since $f_n \in I_n$ and $f_n \in M_{n+1}$.

We will now show that the Krull dimension of $\mathcal{A}(\boldsymbol{p})$ is infinite by showing that for all $N \in \mathbb{N}$, we can construct a chain of strictly decreasing prime ideals $P_{N+1} \subsetneq P_N \subset \cdots \subsetneq P_2 \subsetneq P_1$ in $\mathcal{A}(\boldsymbol{p})$.

Fix an $N \in \mathbb{N}$. Applying Proposition 8.2, taking $J = I_{N+1}$ and $M = M_{N+1}$, we obtain the existence of a prime ideal $P = P_{N+1}$ in $\mathcal{A}(\boldsymbol{p})$, which satisfies $I_{N+1} \subset P_{N+1}$ and $P_{N+1} \cap M_{N+1} = \emptyset$.

We claim the ideal $I_N + P_{N+1}$ of $\mathcal{A}(\boldsymbol{p})$ satisfies $(I_N + P_{N+1}) \cap M_N = \emptyset$. Let $h = f + g \in I_N + P_{N+1}$, where $f \in I_N$ and $g \in P_{N+1}$. Since $g \in P_{N+1}$, by the construction of P_{N+1} it follows that $g \notin M_{N+1}$. But $M_N \subset M_{N+1}$, and so $g \notin M_N$ as well. Thus there exists a subsequence $(k_\ell)_{\ell\in\mathbb{N}_0}$ of $(k)_{k\in\mathbb{N}_0}$ such that $\lim_{\ell\to\infty} \frac{m(g,a_{k_\ell})}{k_\ell^N} = \infty$. From (P1), we obtain $\frac{m(h,a_{k_\ell})}{k_\ell^N} \geq \min\{\frac{m(f,a_{k_\ell})}{k_\ell^N}, \frac{m(g,a_{k_\ell})}{k_\ell^N}\}$. As $f \in I_N$, it follows that

$$\sup_{\ell\in\mathbb{N}} \tfrac{m(h,a_{k_\ell})}{k_\ell^N} \geq \min\left\{ \limsup_{\ell\to\infty} \tfrac{m(f,a_{k_\ell})}{k_\ell^N}, \limsup_{\ell\to\infty} \tfrac{m(g,a_{k_\ell})}{k_\ell^N} \right\} \geq \infty.$$

Thus $h \notin M_N$. Consequently, $(I_N + P_{N+1}) \cap M_N = \emptyset$.

Clearly $I_N \subset I_N + P_{N+1}$. Applying Proposition 8.2 again, now taking $J = I_N + P_{N+1}$ and $M = M_N$, we obtain the existence of a prime ideal $P = P_N$ in $\mathcal{A}(\boldsymbol{p})$ such that $I_N + P_{N+1} \subset P_N$ and $P_N \cap M_N = \emptyset$. Thus $P_{N+1} \subset I_N + P_{N+1} \subset P_N$. The first inclusion is strict because $f_N \in I_N \subset I_N + P_{N+1}$. But $f_N \notin P_{N+1}$ (since $f_N \in M_{N+1}$ and $P_{N+1} \cap M_{N+1} = \emptyset$ by the construction of P_{N+1}). Thus $P_{N+1} \subsetneq P_N$.

Now consider the ideal $J := I_{N-1} + P_N \supset I_{N-1}$ of $\mathcal{A}(\boldsymbol{p})$ and the multiplicatively closed set $M := M_{N-1}$ of $\mathcal{A}(\boldsymbol{p})$. Similar to the argument given above, then[1] $J \cap M = (I_{N-1} + P_N) \cap M_{N-1} = \emptyset$. By Proposition 8.2, taking $J = I_{N-1} + P_N \supset I_{N-1}$ and $M = M_{N-1}$, there exists a prime ideal $P = P_{N-1}$ in $\mathcal{A}(\boldsymbol{p})$ such that $I_{N-1} + P_N \subset P_{N-1}$ and $P_{N-1} \cap M_{N-1} = \emptyset$. Thus $P_N \subset I_{N-1} + P_N \subset P_{N-1}$, and again the first inclusion is strict (because $f_{N-1} \in I_{N-1} \subset I_{N-1} + P_N$, $f_{N-1} \in M_N$ and $M_N \cap P_N = \emptyset$).

Proceeding in this manner, we obtain the chain of distinct prime ideals $P_{N+1} \subsetneq P_N \subsetneq P_{N-1} \subsetneq \cdots \subsetneq P_1$ in $\mathcal{A}(\boldsymbol{p})$. As $N \in \mathbb{N}$ was arbitrary, it follows that the Krull dimension of $\mathcal{A}(\boldsymbol{p})$ is infinite. □

[1] Let $h = f + g \in I_{N-1} + P_N$, where $f \in I_{N-1}$ and $g \in P_N$. Since $g \in P_N$, by the construction of P_N, $g \notin M_N \supset M_{N-1}$, and so $g \notin M_{N-1}$. Thus there exists a subsequence $(k_\ell)_{\ell\in\mathbb{N}_0}$ of $(k)_{k\in\mathbb{N}_0}$ such that $\lim\limits_{\ell\to\infty} \frac{m(g,a_{k_\ell})}{k_\ell^{N-1}} = \infty$. As $f \in I_{N-1}$, $\sup\limits_{\ell\in\mathbb{N}} \frac{m(h,a_{k_\ell})}{k_\ell^{N-1}} \geq \min\{\limsup\limits_{\ell\to\infty} \frac{m(f,a_{k_\ell})}{k_\ell^{N-1}}, \limsup\limits_{\ell\to\infty} \frac{m(g,a_{k_\ell})}{k_\ell^{N-1}}\} \geq \infty$. Thus $h \notin M_{N-1}$. So $(I_{N-1} + P_N) \cap M_{N-1} = \emptyset$.

9. $\mathcal{A}(\boldsymbol{p})$ is neither Artinian nor Noetherian

Even Noetherian rings can have an infinite Krull dimension (see e.g. [25, Appendix, Example E1] or [13, Exercise 9.6]). However, in our case, $\mathcal{A}(\boldsymbol{p})$ is not Noetherian.

Definition 9.1. A commutative ring R is called *Noetherian* if there is no infinite increasing sequence of ideals, that is, for every increasing sequence $I_1 \subset I_2 \subset \cdots$ of ideals of R, there exists an $N \in \mathbb{N}$ such that $I_n = I_N$ for all $n > N$. A commutative ring R is called *Artinian* if there is no infinite descending sequence of ideals, that is, for every decreasing sequence $I_1 \supset I_2 \supset \cdots$ of ideals of R, there exists an $N \in \mathbb{N}$ such that $I_n = I_N$ for all $n > N$.

$\mathcal{A}(\boldsymbol{p})$ is not Noetherian. (Let $I_n = \{f \in \mathcal{A}(\boldsymbol{p}) : \widehat{f}(k) = 0 \text{ for all } k \geq n\}$ for all $n \in \mathbb{N}$. Clearly $I_1 \subset I_2 \cdots$. Moreover, each inclusion is strict, since if $f_n := z^n \in \mathcal{A}(\boldsymbol{p})$, $n \in \mathbb{N}$, then $f_n \in I_{n+1} \setminus I_n$.)
$\mathcal{A}(\boldsymbol{p})$ is not Artinian. (Let $I_n = \{f \in \mathcal{A}(\boldsymbol{p}) : \widehat{f}(k) = 0 \text{ for all } k \leq n\}$ for all $n \in \mathbb{N}$. Clearly $I_1 \supset I_2 \supset \cdots$. Moreover, each inclusion is strict, since if $f_n := z^{n+1} \in \mathcal{A}(\boldsymbol{p})$, $n \in \mathbb{N}$, then $f_n \in I_n \setminus I_{n+1}$.)

10. $\mathcal{A}(\boldsymbol{p})$ is a coherent ring

In absence of the Noetherian 'finiteness' property, a natural finiteness question is that of coherence. We refer the reader to the article [17] and the monograph [16] for the relevance of the property of coherence in commutative holomogical algebra.

Definition 10.1. A commutative unital ring R is called *coherent* if every finitely generated ideal I is finitely presentable, that is, there exists an exact sequence $0 \to K \to F \to I \to 0$ of R-modules, where F is a finitely generated free R-module and K is a finitely generated R-module.

All Noetherian rings are coherent, but not all coherent rings are Noetherian. (For example, the polynomial ring $\mathbb{C}[x_1, x_2, x_3, \cdots]$ is not Noetherian (because the sequence of ideals $\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \cdots$ is ascending and not stationary), but $\mathbb{C}[x_1, x_2, x_3, \cdots]$ is coherent [16, Corollary 2.3.4].)
A commutative ring in which every finitely generated ideal is principal is said to be *Bézout*. By property (R4) (p. 2), $\mathcal{A}(\boldsymbol{p})$ is a Bézout ring. It is known that Bézout *domains* are coherent, but we cannot use this to conclude that $\mathcal{A}(\boldsymbol{p})$ is coherent, since $\mathcal{A}(\boldsymbol{p})$ is *not* a domain (as $\mathcal{A}(\boldsymbol{p})$ has nontrivial zero divisors: e.g. $z * z^2 = 0$).

Theorem 10.2. *$\mathcal{A}(\boldsymbol{p})$ is coherent.*

Proof. Let I be a finitely generated ideal in $\mathcal{A}(\boldsymbol{p})$. Then I is principal by the property (R4) (p. 2). So there exists an $f_I \in \mathcal{A}(\boldsymbol{p})$ such that $I = \langle f_I \rangle$. Define $\chi \in \mathcal{A}(\boldsymbol{p})$ by setting

$$\widehat{\chi}(k) = \begin{cases} \frac{1}{\boldsymbol{p}(k)} & \text{if } \widehat{f_I}(k) = 0, \\ 0 & \text{if } \widehat{f_I}(k) \neq 0. \end{cases}$$

Define $K = \langle \chi \rangle$. Then K is a finitely generated $\mathcal{A}(\boldsymbol{p})$-module. Let $F := \mathcal{A}(\boldsymbol{p}) = \langle \varepsilon \rangle$. Then F is a finitely generated free module. Consider the $\mathcal{A}(\boldsymbol{p})$-module morphism $\varphi : F \to I$ given by $\varphi(h) = f_I * h$ for all $h \in \mathcal{A}(\boldsymbol{p})$. We will show that the sequence $0 \to K \hookrightarrow F \to I \to 0$ is exact, where $K \hookrightarrow F$ denotes the inclusion map. The exactness at K and at I is clear. It remains to show $\{h \in \mathcal{A}(\boldsymbol{p}) : f_I * h = 0\} = K$. If $h \in K$, then $h = \chi * f$ for an $f \in \mathcal{A}(\boldsymbol{p})$. We have $\varphi(h) = f_I * (\chi * f)$. But $\widehat{(f_I * \chi)}(k) = 0$ for all $k \in \mathbb{N}_0$, and so $f_I * \chi = 0$, showing that $\varphi(h) = 0$, that is, $h \in \ker \varphi$. Hence $K \subset \ker \varphi$.

Now suppose $h \in \mathcal{A}(\boldsymbol{p})$ is such that $f_I * h = 0$. As $h \in \mathcal{A}(\boldsymbol{p})$, there exists a $C > 0$ such that for all $k \in \mathbb{N}_0$, $|\widehat{h}(k)| \leq \frac{C}{\boldsymbol{p}(k)}$. As $f_I * h = 0$, we have that for all $k \in \mathbb{N}_0$,

$$0 = \widehat{(f_I * h)}(k) = \boldsymbol{p}(k)\widehat{f}_I(k)\widehat{h}(k). \tag{4}$$

If $k \in \mathbb{N}_0$ is such that $\widehat{f}_I(k) \neq 0$, then by the definition of χ, we have $\widehat{\chi}(k) = 0$, and moreover, then (4) above implies that $\widehat{h}(k) = 0$, so that

$$|\widehat{h}(k)| = 0 = C \cdot 0 = C \cdot |\widehat{\chi}(k)|. \tag{5}$$

If $k \in \mathbb{N}_0$ is such that $\widehat{f}_I(k) = 0$, then we have $\chi(k) = \frac{1}{\boldsymbol{p}(k)}$, and so

$$|\widehat{h}(k)| \leq \frac{C}{\boldsymbol{p}(k)} = C \cdot \frac{1}{\boldsymbol{p}(k)} = C \cdot |\widehat{\chi}(k)|. \tag{6}$$

(5) and (6) together imply that $|\widehat{h}(k)| \leq C|\widehat{\chi}(k)|$ for all $k \in \mathbb{N}_0$, and so by the criterion (R1) (p. 2), χ divides h in $\mathcal{A}(\boldsymbol{p})$, that is, there exists some $f \in \mathcal{A}(\boldsymbol{p})$ such that $h = \chi * f$, i.e., $h \in \langle \chi \rangle = K$, as wanted. □

11. Generation of $\mathrm{SL}_n(\mathcal{A}(\boldsymbol{p}))$ by elementary matrices

Let R be a commutative unital ring with multiplicative identity 1 and additive identity element 0. Let $m \in \mathbb{N}$. The *general linear group* of invertible matrices in $R^{m\times m}$ is denoted by $\mathrm{GL}_m(R)$. The *special linear group* $\mathrm{SL}_m(R)$ is the subgroup of $\mathrm{GL}_m(R)$ of all matrices M whose determinant $\det M = 1$. An *elementary matrix* $E_{ij}(\alpha)$ is a matrix having form $E_{ij} = I_m + \alpha \boldsymbol{e}_{ij}$, where $i \neq j$, $\alpha \in R$, and $\boldsymbol{e}_{ij}$ is the $m \times m$ matrix whose entry in the i^{th} row and j^{th} column is 1, and all the other entries of $\boldsymbol{e}_{ij}$ are zeros. $\mathrm{E}_m(R)$ is the subgroup of $\mathrm{SL}_m(R)$ generated by elementary matrices. A classical question in algebra is:

$$\text{For all } m \in \mathbb{N}, \text{ is } \mathrm{SL}_m(R) = \mathrm{E}_m(R) \text{ ?}$$

The answer to this question depends on the ring R. For example, if the ring $R = \mathbb{C}$, then the answer is 'Yes', and this is an exercise in linear algebra; see for example [1, Exercise 18.(c), page 71]. If R is the polynomial ring $\mathbb{C}[z_1, \cdots, z_d]$ in the indeterminates $z_1, \cdots, z_d$ with complex coefficients, then for $d = 1$, the answer is 'Yes' (this follows from the Euclidean Division Algorithm in $\mathbb{C}[z]$), but for $d = 2$, the answer is 'No', and [9] contains the following example:

$$\begin{bmatrix} 1+z_1z_2 & z_1^2 \\ -z_2^2 & 1-z_1z_2 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{C}[z_1, z_2]) \setminus \mathrm{E}_2(\mathbb{C}[z_1, z_2]).$$

(For $d \geq 3$, the answer is 'Yes', and this is the K_1-analogue of Serre's Conjecture, which is the Suslin Stability Theorem [34].) The case of R being a ring of real/complex valued continuous functions was considered in [35]. For the ring $R = \mathcal{O}(X)$ of holomorphic functions on Stein spaces in $\mathbb{C}^d$, this was an explicit open problem [18], and was answered affirmatively in [19]. We will prove below that $\mathrm{SL}_n(\mathcal{A}(\boldsymbol{p})) = \mathrm{E}_n(\mathcal{A}(\boldsymbol{p}))$.

For Banach algebras, the following result is known [23, §7]:

Proposition 11.1. *Let A be a complex commutative unital semisimple Banach algebra, $n \in \mathbb{N}$, and $M \in SL_n(A)$. Then $M \in E_n(A)$ if and only if M is path-connected to I_n in $SL_n(A)$ (i.e., there exists a continuous map $\gamma : [0, 1] \to SL_n(A)$ such that $\gamma(0) = M$ and $\gamma(1) = I_n$).*

If $(A, \|\cdot\|)$ is a commutative unital Banach algebra, then $A^{n\times n}$ is a complex algebra with the usual matrix operations. Let A^n be the normed space of all column vectors of size n with entries from A, componentwise operations, and the Euclidean norm given by (2). For $M \in A^{n\times n}$, the multiplication map, $A^n \ni \boldsymbol{v} \mapsto M\boldsymbol{v} \in A^n$, is a continuous linear transformation, and we equip $A^{n\times n}$ with the induced operator norm, denoted by $\|\cdot\|$ again. Then $A^{n\times n}$ with this operator norm is a unital Banach algebra. Subsets of $A^{n\times n}$ are given the induced topology.

We first show some auxiliary results which we will need in the special case when the Banach algebra is $\mathcal{A}(\boldsymbol{p})$. Let the operator norm on $M \in \mathbb{C}^{n\times n}$ (when $\mathbb{C}^n$ is equipped with the Euclidean norm $\|\cdot\|_2$, and M is viewed as a map $\mathbb{C}^n \ni v \mapsto Mv \in \mathbb{C}^n$) be denoted by $\|M\|_{2,2}$. Let $\mathcal{O}(\mathbb{C})$ denote the set of all entire functions. For a matrix $A \in \mathcal{O}(\mathbb{C})^{n\times n}$, if $a_{ij} \in \mathcal{O}(\mathbb{C})$ denotes the entry in the i^{th} row and j^{th} column of A, and then $\widehat{A}(k) \in \mathbb{C}^{m\times n}$ is the matrix whose entry in the i^{th} row and j^{th} column is $\widehat{a_{ij}}(k)$, $1 \le i \le m$, $1 \le j \le n$, $k \in \mathbb{N}_0$, where $a_{ij}(z) = \sum\limits_{k=0}^{\infty} \widehat{a_{ij}}(k) z^k$ for all $z \in \mathbb{C}$.

Lemma 11.2. *$A \in \mathcal{A}(\boldsymbol{p})^{n\times n}$ if and only if $\sup_{k\in\mathbb{N}_0} \boldsymbol{p}(k)\|\widehat{A}(k)\|_{2,2} < \infty$. Moreover, then $\sup_{k\in\mathbb{N}_0} \boldsymbol{p}(k)\|\widehat{A}(k)\|_{2,2} \le n\|A\|$.*

Proof. ('Only if' part:) Suppose $A = [a_{ij}] \in \mathcal{A}(\boldsymbol{p})^{n\times n}$. Take the vector $\boldsymbol{v} \in \mathcal{A}(\boldsymbol{p})^{n\times 1}$, having only one nonzero entry, which is the j^{th} component, and is equal to ε. Then we obtain

$$\|a_{ij}\|^2 \le \sum_{i=1}^{n} \|a_{ij}\|^2 = \|A\boldsymbol{v}\|_2^2 \le \|A\|^2\|\boldsymbol{v}\|_2^2 = \|A\|^2 1 = \|A\|^2.$$

Hence $\sup_{k\in\mathbb{N}_0} \boldsymbol{p}(k)|\widehat{a_{ij}}(k)| = \|a_{ij}\| \le \|A\|$. As the i,j were arbitrary, it follows from [5, Fact 9.8.10(xii)] that $\sup_{k\in\mathbb{N}_0} \boldsymbol{p}(k)\|\widehat{A}(k)\|_{2,2} \le n\|A\|$.
('If' part:) If $C := \sup_{k\in\mathbb{N}_0} \boldsymbol{p}(k)\|\widehat{A}(k)\|_{2,2} < \infty$, then [5, Fact 9.8.10(xii)] implies $\sup_{k\in\mathbb{N}_0} \boldsymbol{p}(k)|\widehat{a_{ij}}(k)| \le C$, and so $\|a_{ij}\| \le C$ for all $1 \le i,j \le n$. Thus $A \in \mathcal{A}(\boldsymbol{p})^{n\times n}$. □

Lemma 11.3. *If $A \in \mathcal{A}(\boldsymbol{p})^{n\times n}$ then $\|A(z)\|_{2,2} \le n\|A\|\varepsilon(|z|)$ $(z \in \mathbb{C})$.*

Proof. For $z \in \mathbb{C}$, we have

$$\|A(z)\|_{2,2} = \Big\|\sum_{k=0}^{\infty} \widehat{A}(k)z^k\Big\|_{2,2} \le \sum_{k=0}^{\infty} \|\widehat{A}(k)\|_{2,2}|z|^k \le \sum_{k=0}^{\infty} \tfrac{n\|A\|}{\boldsymbol{p}(k)}|z|^k = n\|A\|\varepsilon(|z|). \quad \square$$

To apply Proposition 11.1, we will need the following result.

Theorem 11.4. *Let $A \in GL_n(\mathcal{A}(\boldsymbol{p}))$. Then there exists a $B \in \mathcal{A}(\boldsymbol{p})^{n\times n}$ such that $e^B = A$.*

Before proving this result, let us see how this gives the following.

Theorem 11.5. *For all $n \in \mathbb{N}$, $SL_n(\mathcal{A}(\boldsymbol{p})) = E_n(\mathcal{A}(\boldsymbol{p}))$.*

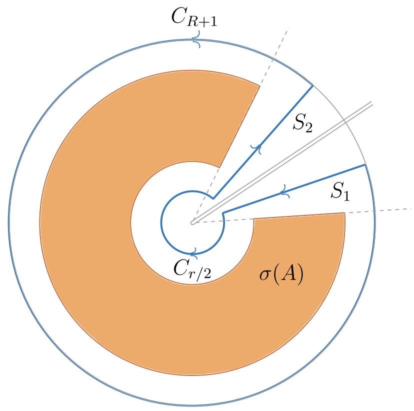
Proof. Let $A \in \mathrm{SL}_n(\mathcal{A}(\boldsymbol{p})) \subset \mathrm{GL}_n(\mathcal{A}(\boldsymbol{p}))$. Let $B \in \mathcal{A}(\boldsymbol{p})^{n\times n}$ be such that $e^B = A$. For $t \in [0,1]$, define $\gamma(t)$ to be the matrix obtained by scaling any one column, say the first one, of $e^{(1-t)B}$, by $\det(e^{-(1-t)B})$. Then $\det(\gamma(t)) = \varepsilon$, and so $\gamma(t) \in \mathrm{SL}_n(\mathcal{A}(\boldsymbol{p}))$ for all $t \in [0,1]$. We have

$$\gamma(0) = A \text{ and } \gamma(1) = e^0 = I_n = \begin{bmatrix} \varepsilon & & \\ & \ddots & \\ & & \varepsilon \end{bmatrix}.$$

Moreover, as γ is continuous, it follows that A is path connected to I_n in $\mathrm{SL}_n(\mathcal{A}(\boldsymbol{p}))$. From Proposition 11.1, we get $A \in \mathrm{E}_n(\mathcal{A}(\boldsymbol{p}))$. Thus $\mathrm{SL}_n(\mathcal{A}(\boldsymbol{p})) = \mathrm{E}_n(\mathcal{A}(\boldsymbol{p}))$. □

Recall that $A \in \mathrm{GL}_n(\mathbb{C})$ possesses a logarithm, which can be obtained as follows (see e.g. [21, Example 5.20]). We will be a bit more particular about the construction of the logarithm, since we will apply this to each $\widehat{A}(k)$, $k \in \mathbb{N}_0$, for our $A \in \mathrm{SL}_n(\mathcal{A}(\boldsymbol{p}))$, and we will then need a uniform estimate on $\log(\boldsymbol{p}(k)\widehat{A}(k))$, $k \in \mathbb{N}_0$.

Denote the spectrum (the set of eigenvalues) of A by $\sigma(A)$. There exists an open sector $\Omega_\theta = \{z \in \mathbb{C} \setminus \{0\} : |\arg z - \theta| < \frac{\pi}{n}\}$ of angular width $\frac{2\pi}{n}$ that does not intersect the spectrum of A. Moreover, we have $0 < r := \min_{\lambda \in \sigma(A)} |\lambda| \leq R := \max_{\lambda \in \sigma(A)} |\lambda|$. Let γ be the path $C_{R+1} + S_1 + C_{r/2} + S_2$ (see the following picture), where $C_{r/2}$ is a circular arc of radius $r/2$ centred at 0 traversed in the clockwise direction, C_{R+1} is a circular arc of radius $R+1$ centred at 0 traversed in the anticlockwise direction, S_1 is a radial straight line segment joining the arc C_{R+1} to the arc $C_{r/2}$ with the fixed argument $\theta - \frac{\pi}{2n}$, and S_2 is a radial straight line segment joining the arc $C_{r/2}$ to the arc C_{R+1} with the fixed argument $\theta + \frac{\pi}{2n}$.



The spectrum $\sigma(A)$ is contained in the shaded region.
The curve $\gamma = C_{R+1} + S_1 + C_{r/2} + S_2$ encloses $\sigma(A)$.

If log denotes the logarithm branch with a cut along the radial ray with fixed argument θ, then we have

$$\log A = \frac{1}{2\pi i} \int_\gamma (\log \zeta)(\zeta I_n - A)^{-1} d\zeta.$$

We will also use the following estimate[2] [3, Theorem 4.1] for the norm of the resolvent of $A \in \mathrm{GL}_n(\mathbb{C})$ at $z \in \mathbb{C} \setminus \sigma(A)$:

$$\| (zI_n - A)^{-1} \|_{2,2} \leq \frac{1}{d(z, \sigma(A))} \exp\left(c_2 \frac{2n \| A \|_{2,2}^2}{(d(z, \sigma(A)))^2} + b_2 \right) \tag{7}$$

for some universal (not depending on A or z) constants $c_2, b_2 > 0$. Here $d(z, \sigma(A)) := \inf\{|z - \lambda| : \lambda \in \sigma(A)\}$.

Proof of Theorem 11.4. Let $A \in \mathrm{GL}_n(\mathcal{A}(\boldsymbol{p}))$. Then

$$\boldsymbol{p}(k)\widehat{A^{-1}}(k)\widehat{A}(k) = \begin{bmatrix} \frac{1}{\boldsymbol{p}(k)} & & \\ & \ddots & \\ & & \frac{1}{\boldsymbol{p}(k)} \end{bmatrix}.$$

If $v \in \mathbb{C}^n \setminus \{0\}$ is an eigenvector of $\widehat{A}(k)$ of unit norm corresponding to the eigenvalue $\widetilde{\lambda}$, and $C > 0$ is such that $\| \widehat{A^{-1}}(k) \|_{2,2} \leq \frac{C}{\boldsymbol{p}(k)}$, then the above yields upon operation on v that

[2] This follows by setting $p = 2$ in the estimate given in [3, Theorem 4.1], and noting that $\nu_2(A)^2$ given there is bounded by $\|A\|_{\mathrm{F}}^2 - \sum_{k=1}^n |\lambda_k|^2$, where $\|A\|_{\mathrm{F}}$ denotes the Hilbert-Schmidt/Frobenius norm of A and $\lambda_1, \cdots, \lambda_n$ denote the n eigenvalues of A repeated with multiplicities. We have $\|A\|_{\mathrm{F}} \leq \sqrt{n} \| A \|_{2,2}$ (see e.g. [5, Fact 9.8.10(ix) and Prop. 9.4.7]) and $|\lambda_k| \leq \| A \|_{2,2}$ for all k [5, Corollary 9.4.5].

$$\boldsymbol{p}(k)\tfrac{C}{\boldsymbol{p}(k)}|\widetilde{\lambda}| \geq \|\boldsymbol{p}(k)\widehat{A^{-1}}(k)(\widetilde{\lambda}v)\|_2 = \|\tfrac{1}{\boldsymbol{p}(k)}v\|_2 = \tfrac{1}{\boldsymbol{p}(k)},$$

so that $\min\{|\lambda| : \lambda \in \sigma(\boldsymbol{p}(k)\widehat{A}(k))\} \geq \frac{1}{C} =: r > 0$ for all $k \in \mathbb{N}_0$. Also, if $R > 0$ is such that $\|\widehat{A}(k)\|_{2,2} \leq \frac{R}{\boldsymbol{p}(k)}$, then since the spectral radius is bounded by the operator norm,

$$\max\{|\lambda| : \lambda \in \sigma(\boldsymbol{p}(k)\widehat{A}(k))\} \leq \|\boldsymbol{p}(k)\widehat{A}(k)\|_{2,2} \leq R \text{ for all } k \in \mathbb{N}_0.$$

Let Ω_k denote an open sector of angular width $\frac{2\pi}{n}$ that does not intersect the spectrum of $\boldsymbol{p}(k)\widehat{A}(k)$. Since r, R do not depend on $k \in \mathbb{N}_0$, and since the angular wedge width (of $\frac{2\pi}{n}$) also does not depend on $k \in \mathbb{N}_0$, it is now clear that for any ζ lying on the image of $\gamma = C_{R+1} + S_1 + C_{r/2} + S_2$ (as in the picture above), we have that $|\log_{(k)} \zeta| \leq \widetilde{C}$ for some constant independent of $k \in \mathbb{N}_0$. (Here we use the notation $\log_{(k)}$ to emphasise the dependence of the chosen branch of the logarithm on the k at hand.) Also the length of γ can be bounded by

$$L := 2\pi\tfrac{r}{2} + 2\pi(R+1) + 2((R+1) - \tfrac{r}{2}).$$

We have

$$\begin{aligned}\|\log_{(k)}(\boldsymbol{p}(k)\widehat{A}(k))\|_{2,2} &= \|\tfrac{1}{2\pi i}\textstyle\int_\gamma (\log\zeta)(\zeta I_n - \boldsymbol{p}(k)\widehat{A}(k))^{-1}d\zeta\|_{2,2}\\ &\leq \tfrac{L}{2\pi}\widetilde{C}\max_{\zeta\in\gamma}\|(\zeta I_n - \boldsymbol{p}(k)\widehat{A}(k))^{-1}\|_{2,2}.\end{aligned}$$

To bound the final right-hand term involving the resolvent, we will use the estimate (7). First we note that if ζ lies on γ, then[3]

$$d(\zeta, \sigma(\boldsymbol{p}(k)\widehat{A}(k))) \geq \min\{r\sin\tfrac{\pi}{4n}, \tfrac{r}{2}\} =: \delta > 0.$$

Also, $\boldsymbol{p}(k)\|\widehat{A}(k)\|_{2,2} \leq R$ for all $k \in \mathbb{N}_0$. Thus

$$\max_{\zeta\in\gamma}\|(\zeta - \boldsymbol{p}(k)\widehat{A}(k))^{-1}\|_{2,2} \leq \tfrac{1}{\delta}\exp(c_2\tfrac{2nR^2}{\delta^2} + b_2) =: K.$$

This yields $\|\log_{(k)}(\boldsymbol{p}(k)\widehat{A}(k))\|_{2,2} \leq \frac{L}{2\pi}\widetilde{C}K =: \widetilde{K}$ for all $k \in \mathbb{N}_0$. Now define $\widehat{B}(k) = \frac{1}{\boldsymbol{p}(k)}\log_{(k)}(\boldsymbol{p}(k)\widehat{A}(k)) \in \mathbb{C}^{n\times n}$ for all $k \in \mathbb{N}_0$. Then

$$\|\boldsymbol{p}(k)\widehat{B}(k)\|_{2,2} \leq \widetilde{K} \text{ for all } k \in \mathbb{N}_0,$$

and so by Lemma 11.2

$$B(z) := \sum_{k=0}^{\infty}\widehat{B}(k)z^k \quad (z \in \mathbb{C})$$

is an element in $\mathcal{A}(\boldsymbol{p})^{n\times n}$. We have

$$B^m(z) = \sum_{k=0}^{\infty}\boldsymbol{p}(k)^{m-1}(\widehat{B}(k))^m z^k,$$

and thus[4]

[3] This lower bound is obtained by dropping a perpendicular from the corner of the shaded region onto S_1, which has a length $r\sin\frac{\pi}{2n}$, and the distance between $C_{r/2}$ and the shaded region is clearly $r/2$.

[4] The exchange of the two summations is justified, since:
$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\|\frac{\boldsymbol{p}(k)^{m-1}(\widehat{B}(k))^m}{m!}z^k\|_{2,2} \leq \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{|z|^k}{\boldsymbol{p}(k)}\frac{(n\|B\|)^m}{m!} = \sum_{k=0}^{\infty}\frac{|z|^k}{\boldsymbol{p}(k)}e^{n\|B\|} = e^{n\|B\|}\varepsilon(|z|) < \infty.$

$$
\begin{aligned}
(e^B)(z) &= \sum_{m=0}^{\infty} \frac{(B^m)(z)}{m!} = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\boldsymbol{p}(k)^{m-1}(\widehat{B}(k))^m}{m!} z^k \\
&= \sum_{k=0}^{\infty} \frac{z^k}{\boldsymbol{p}(k)} \sum_{m=0}^{\infty} \frac{(\boldsymbol{p}(k)\widehat{B}(k))^m}{m!} = \sum_{k=0}^{\infty} \frac{z^k}{\boldsymbol{p}(k)} e^{\boldsymbol{p}(k)\widehat{B}(k)} \\
&= \sum_{k=0}^{\infty} \frac{z^k}{\boldsymbol{p}(k)} \boldsymbol{p}(k)\widehat{A}(k) = A(z),
\end{aligned}
$$

as wanted. □

References

[1] M. Artin, Algebra, Prentice Hall, Englewood Cliffs, NJ, 1991.

[2] J. Ball, L. Rodman, I. Spitkovsky, Toeplitz corona problem for algebras of almost periodic functions, in: Toeplitz matrices and singular integral equations (Pobershau, 2001), in: Operator Theory Advances and Applications, vol. 135, Birkhäuser, 2002, pp. 25–37.

[3] O. Bandtlow, Estimates for norms of resolvents and an application to the perturbation of spectra, Math. Nachr. 267 (2004) 3–11.

[4] H. Bass, K-theory and stable algebra, Publ. Math. IHÉS 22 (1964) 5–60.

[5] D. Bernstein, Matrix Mathematics. Theory, Facts, and Formulas with Applications to Linear Systems Theory, Princeton University Press, 2005.

[6] A. Brudnyi, A. Sasane, Sufficient conditions for the projective freeness of Banach algebras, J. Funct. Anal. 257 (12) (2009) 4003–4014.

[7] A. Brudnyi, On the factorization of matrices over commutative Banach algebras, Integral Equ. Oper. Theory 90 (1) (2018) 6.

[8] A. Brudnyi, On stable rank of H^∞ on coverings of finite bordered Riemann surfaces, Rev. Mat. Iberoam. 38 (3) (2022) 1003–1012.

[9] P. Cohn, On the structure of the GL_2 of a ring, Publ. Math. IHÉS 30 (1966) 5–53.

[10] P. Cohn, Some remarks on projective-free rings, Algebra Univers. 49 (2) (2003) 159–164.

[11] R. Douglas, S. Krantz, E. Sawyer, S. Treil, B. Wick, A History of the Corona Problem. In the Corona Problem, Fields Institute Communications, vol. 72, Springer, 2014, pp. 1–29.

[12] S. Eilenberg, N. Steenrod, Foundations of Algebraic Topology, Princeton University Press, 1952.

[13] D. Eisenbud, Commutative Algebra, Graduate Texts in Mathematics, vol. 150, Springer, 1995. With a view toward algebraic geometry.

[14] T. Gamelin, Uniform Algebras, Prentice-Hall, 1969.

[15] L. Gillman, M. Jerison, Rings of Continuous Functions, D. van Nostrand Company, Princeton, New Jersey, 1960.

[16] S. Glaz, Commutative Coherent Rings, Lecture Notes in Mathematics, vol. 1371, Springer, 1989.

[17] S. Glaz, Commutative coherent rings: historical perspective and current developments, Nieuw Archief voor Wiskunde. Vierde Serie 10 (1-2) (1992) 37–56.

[18] M. Gromov, Oka's principle for holomorphic sections of elliptic bundles, J. Am. Math. Soc. 2 (4) (1989) 851–897.

[19] B. Ivarsson, F. Kutzschebauch, Holomorphic factorization of mappings into $\mathrm{SL}_n(\mathbb{C})$, Ann. Math. 175 (1) (2012) 45–69.

[20] C. Jensen, Some curiosities of rings of analytic functions, J. Pure Appl. Algebra 38 (2-3) (1985) 277–283.

[21] T. Kato, Perturbation Theory for Linear Operators, Classics in Mathematics, Springer, 1995.

[22] T. Lam, Serre's Conjecture, Lecture Notes in Mathematics, vol. 635, Springer-Verlag, 1978.

[23] J. Milnor, Introduction to Algebraic K-Theory, Annals of Mathematics Studies, vol. 72, Princeton University Press, 1971.

[24] R. Mortini, R. Rupp, Extension Problems and Stable Ranks. A Space Odyssey, Birkhäuser, 2021.

[25] M. Nagata, Local Rings, Wiley, 1962.

[26] N. Nikolski, Operators, Functions, and Systems: an Easy Reading. Vol. 1. Hardy, Hankel, and Toeplitz, Mathematical Surveys and Monographs, vol. 93, American Mathematical Society, Providence, 2002.

[27] A. Quadrat, The fractional representation approach to synthesis problems: an algebraic analysis viewpoint. II. Internal stabilization, SIAM J. Control Optim. 42 (1) (2003) 300–320.

[28] M. Rieffel, Dimension and stable rank in the K-theory of C^*-algebras, Proc. Lond. Math. Soc. 46 (2) (1983) 301–333.

[29] A. Sasane, A potpourri of algebraic properties of the ring of periodic distributions, Bull. Belg. Math. Soc. Simon Stevin 25 (5) (2018) 755–776.

[30] M. Sen, On a class of integral functions, Bull. Calcutta Math. Soc. 61 (1969) 67–74.

[31] V. Tolokonnikov, Extension problem to an invertible matrix, Proc. Am. Math. Soc. 117 (4) (1993) 1023–1030.

[32] M. von Renteln, Rings of entire functions with weighted Hadamard multiplication, Demonstr. Math. 10 (3-4) (1977) 807–813.

[33] M. von Renteln, Primeideale in der topologischen algebra $H^\infty(\beta)$, Math. Z. 157 (1977) 79–82.

[34] A. Suslin, The structure of the special linear group over rings of polynomials, Izv. Akad. Nauk SSSR, Ser. Mat. 41 (2) (1977) 235–252.

[35] L. Vaserstein, Reduction of a matrix depending on parameters to a diagonal form by addition operations, Proc. Am. Math. Soc. 103 (3) (1988) 741–746.

[36] M. Vidyasagar, Control System Synthesis: A Factorization Approach, Series in Signal Processing, Optimization, and Control, vol. 7, MIT Press, Cambridge, 1985.