POWER LAWS IN MARKET MICROSTRUCTURE

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Abstract. We develop an equilibrium model for market impact of trades when investors with private signals execute via a trading desk. Fat tails in the signal distribution lead to a power law for price impact, while the impact is logarithmic for lighter tails. Moreover, the tail distribution of the equilibrium trade volume obeys a power law. The spread decreases with the degree of noise trading and increases with the number of insiders. In case of a monopolistic insider, the last slice traded against the limit order book is priced at the fundamental value of the asset reminiscent of [17]. However, competition among insiders leads to aggressive trading, hence vanishing profit in the limit. The model also predicts that the order book flattens as the amount of noise trading increases converging to a model with proportional transactions costs with non-vanishing spread.

1. Introduction

Kyle [17] studies in a simple but remarkably powerful framework a single risk neutral informed trader and a number of non-strategic uninformed liquidity traders submitting orders to a market maker, who aggregates all the orders and clear the market at a single price. Consequently, Kyle's uniform auction model does not produce a bid-ask spread and predicts a linear price impact of trades. However, his model allows for an explicit characterization of equilibrium parameters - including the optimal strategy of the informed trader as well as the equilibrium pricing rule. This in turn allows one to analyze how the private information is disseminated to the market and gets incorporated into the prices over time. In particular, *Kyle's lambda* yields an explicit measurement of market's liquidity and the extent of linear price impact.

On the other hand, empirical data provides some clues as to the shape and scale of market impact. There is a strong consensus among the practitioners that the impact is a concave function of trade size (see [28]). But biases in the data, a low signal-to-noise ratio and other issues hinder a precise determination of the functional form of market impact, particularly for large trades where data is sparse. Impact functions used in practice range from square root to logarithm (e.g. [28], [23], [11], [1], [3], and [30]). We contribute to this debate using a rational expectations approach.

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Understanding the exact shape of the price impact is of utmost importance for optimal trade execution (see Bouchaud et al. [7]). Portfolio managers endeavour to find mispriced assets in order to beat their benchmarks. When they decide to change their holdings to take advantage of a mispricing, they create orders in an order management system and the trading desk routes these orders to broker-dealers for execution. However, this new order causes prices to move to a different level due to the price impact of the trade. The value lost by not being able to execute at the decision price is called *implementation shortfall* (see [21]) and understanding how this shortfall scales with trade size is essential to optimize the investment management process. A market impact model is needed in order to maximize net trading profits, optimize position sizes, limit liquidity risk and estimate a portfolio's capacity.

Various models yield similar results for small trade sizes but they yield very different predictions for the impact of large trades. This is unfortunate since the large trades typically dominate the aggregate implementation shortfall for most portfolios. Indeed, [8] finds that large trades (above 10,000 shares) from institutional traders account for approximately 48% of the aggregate volume and the moderate trades (500-9,999 shares) by institutions make up 38% of the trading volume. Yet, large trades are fewer in number, their implementation shortfall most exposed to market noise, and their size most subject to selection bias: a manager may execute a large trade because she can. This makes empirical testing difficult. The absence of a consensus on the shape of market impact for large trades motivates the search for a theoretical framework that would capture the essential features of trading and yet be sufficiently parsimonious to be testable.

To address these issues we propose a market microstructure model for the price impact that shares features from Kyle [17] and Glosten [13]. Our results provide the micro-foundations for a large number of empirical findings including those on price impact and volume. To the best of our knowledge, this is the first model to establish the rules governing the power laws for the price impact function in an equilibrium setting[1]. The liquidity suppliers in our model face adverse selection due to the existence of informed traders (henceforth called *insider*s). Different from [17], but consistent with the modern practice, they move first by posting limit orders as in [13]. We assume following [13] that the resulting offer curve is competitive so that the limit prices are given by 'tail expectations.' That the marginal cost of a trade depends on the trade size leads to *discriminatory pricing*.

We show that whether the market impact follows a power law or a logarithmic law depends on the distribution of the fundamental value, which is the private signal of the insiders. The price impact, and equivalently the implementation shortfall, has a power law if the market believes that the fundamental value has fat tails while lighter tails are responsible for a logarithmic behavior.

[1]See also Nadtochiy [19] for an explanation of concave price impact in a continuous-time setting. However, price is not determined in a rational expectations equilibrium setting in [19].

Our model makes sharp predictions on the shape of market impact based on the asset's return distribution. Ignoring for a moment the difference between the risk-neutral returns distribution and the physical distribution[2], the asset's conditional return distribution is predicted by options market participants who encode their views in the options data. The shape of this distribution is sensitive to the event calendar: for example, the shape and scale of the impact function for large trade in a biopharmaceutical company ahead of a major drug approval release would be quite different from the shape and scale of market impact for the same stock when no news is expected. An investment manager with an active trading strategy in this kind of company would be able to label trades as "pre-announcement" and "other", and over time, calibrate an impact model for each of these two cases. The goodness of fit measures for the predicted shapes from the present paper would provide a test of our theory[3].

Our framework also allows us to compute the tails of the probability distribution associated with the aggregate volume. For a large class of distributions that can be used to model the liquidation value we demonstrate that the tail of the trade size obeys a power law.

The power-law behavior of order sizes has been documented previously in various contexts: [15] showed that the market volume in a time interval Δt has a power-law tail with exponent 1.7; [18] consider off-book trades and find an exponent ranging from 1.59 to 1.74; [12] estimated the exponent to be in the neighbourhood of 1.5; [29] reconstructed metaorders in Spanish stock exchange using data with brokerage codes and found that the metaorder transaction size distribution has a power-law tail with exponent 1.7. And in a study performed directly on institutional trade data from Alliance Bernstein, [3] found a tail exponent 1.56 for metaorder sizes. Our model gives an equilibrium based explanation for the magnitude of these exponents.

One of the most striking features of our model is its tractability while allowing general distributions for the fundamental value. We characterize the equilibrium strategy of insiders and the corresponding market impact of trades as a solution of an integral equation. While an analytic solution to this equation can be found only in few examples, its numerical implementation is straightforward and fast.

The solution of this integral equation is given by the fixed point of a highly non-linear operator, existence of which is shown using Schauder's fixed point theorem in Sections 3 and 4. Although this integral equation arises from a one-period consideration, we obtain a priori estimates on its solution using a space-time h-transform of a standard Brownian motion on $[0, 1]$ and via comparison arguments on the solutions of the resulting h-path process. These estimates also allowed us to bound the fixed

[2]This is in fact consistent with the assumption that the market makers are risk-neutral. The more realistic case with risk-averse market makers that leads to the risk neutral measure being different than the physical one is left to future research.

[3]To formulate a test based on the options montage is more involved, in no small part because wide spreads in option markets make pricing in the tails quite challenging. Such a project lies outside the scope of this paper.

point(s) of the original integral equation by the extremal fixed points of two *better-behaved* and, most importantly, *monotone* integral operators. Such a comparison was essential in establishing the power laws considered in this paper.

Although the exact form of the market impact can only be solved numerically, we are nevertheless able to obtain the exact analytic asymptotics when the order size is large using the theory of regular variation. Moreover, due to the scaling property of the noise demand, these asymptotics still provide a good approximation for small orders if the variance of the noise demand is relatively small. The error in such an approximation appears to be still low even in the case of higher variance as confirmed visually by our numerical studies and should provide a good benchmark for empirical works.

Due to the discriminatory pricing by the liquidity suppliers, our model produces a non-zero bid-ask spread in equilibrium. We find that the minimal bid-ask spread for an *infinitesimal trade* is the same regardless of the variance of the noise trades. The shape of the order book, however, is not invariant to the changes in this variance; on the contrary, the order book flattens as it increases. In particular, the spread associated with a non-infinitesimal trade size decreases with the amount of noise trading consistent with the experimental findings of Bloomfield et al. [6]. In the limit the equilibrium converges to a model with proportional transaction costs.

Our proof of the existence of equilibrium assumes that the fundamental value of the asset is bounded. However, our numerical experiments suggest that equilibrium exists for a large class of unbounded distributions commonly found in the literature and used in practice. Moreover, the asymptotics for these solutions agree with the analytical forms that our theory predicts. Yet, unlike the bounded case, an equilibrium may not exist for all unbounded distributions. Given the premise of numerical results and our formal calculations we conjecture that existence of equilibrium requires a sufficient amount of competition among insiders if the signal distribution exhibit fat tails. This is intuitive given the fact that the liquidity suppliers post limit prices using tail expectations in equilibrium. Since the noise demand is Gaussian, the large orders are more likely to be informed and the tail risk in the total demand is completely determined by the signal distribution as the insiders' aggregate demand is a monotone function of the fundamental value. Therefore, fat tails in the signal distribution increases the liquidity suppliers' need for compensation. Moreover, sizable fat tails are typically associated with excessive variance. Our model predicts that such risks arising from the signal distribution cause the market to break down unless the risk is reduced to a manageable level by strong enough competition among informed traders.

Even though we model the interactions in a single period model, our framework also has the flavor of a continuous-time Kyle model considered by [2]. Indeed, in case of a monopolistic insider, we show that conditional on the asset value $V = v$ the last infinitesimal slice of the aggregate order traded by the desk against the order book is priced at v on average, reminiscing the convergence of the equilibrium price to the liquidation value by the end of the trading horizon in the continuous-time Kyle model. Thus, all private information gets incorporated into market prices once the last slice has been traded.

The above 'convergence' result motivates the analysis of the case of multiple insiders. In the same vein as [16], we also model and solve the imperfect competition among $N \geq 2$ insiders who observe the liquidation value V perfectly. As in [16] this competition leads to more aggressive trading. When the signal distribution is continuous, we find that (conditional on $V = v$) the average price for the last slice traded by the desk exceeds v if the insiders are buying in equilibrium, which can be attributed to more aggressive buying due to competition. An analogous observation holds when the optimal strategy in equilibrium is to sell. We obtain an explicit formula for the aggregate profit of the insiders that shows that it converges to 0 in the limit as the number of insiders increases.

This article is related to the 'econophysics' literature that uses the concepts of physics to study economic issues. Gabaix et al. [12] argue that "[econophysics] is similar in spirit to behavioral economics in that it postulates simple plausible rules of agent behavior,..." and develop a theory of large institutional trades by assuming that the liquidity providers' preferences exhibit first order risk aversion justified by prospect theory and institutional practices. While we share the same motivation of explaining the power laws observed in the data for price impact and trade size as [12], we instead follow a rational expectations approach to establish these laws. Thus, our model can be interpreted as a merger between econophysics and the market microstructure theory.

Our article also relates to the literature on limit order markets. The case of liquidity suppliers moving first and submitting limit orders to be later hit by potentially informed market order was first studied by [24] and [13]. Glosten [13] is more concerned on the equilibrium among the liquidity suppliers for a given market order and there is no general existence result for an equilibrium that incorporates the optimal strategies of both the limit order traders as well as the strategic trader submitting the market order with the exception of a special case given in an example. Moreover, studies of [4] and [26] on the limit order book and the order flow in Paris Bourse and Stockholm Stock Exchange found evidence that contradicts the predictions of [13] and support a concave price impact function. [26] also stressed the need for a richer theoretical model to replicate the empirical observations. More recent works on limit order markets are mainly set in continuous time and a short list would include [20], [10], [25], and [14]. However, the sheer complexity of equilibrium in continuous time required these works imposing very restrictive assumptions to have a tractable model such as traders being able to trade only once (essentially making it a collection of one period models) or only for one unit of the asset. We, on the other hand, establish the existence of equilibrium for general distributions and without any restriction on the trading strategy of the strategic traders, and characterize it via an integral equation that possesses an easy numerical solution.

Structure of the paper is as follows: We present the trading model and the first implications of equilibrium in the next section. In Sections 3 and 4 we demonstrate the existence of an equilibrium under mild assumptions. Section 5 considers the asymptotic behavior of market impact and the distribution of trade volume in equilibrium. Further properties of equilibrium along with some comparative statics are given in Section 6. Numerical solutions are provided in Section 7. Moreover, Section 8 explores an alternative same-price allocation model numerically and compares the results with those of the trading desk model. Section 9 contains the concluding remarks and directions for future study. The Appendix contains some auxiliary results and the proofs not presented in the main text.

2. The market structure and equilibrium

Our model is built upon Glosten [13] and the trading takes place in one-period. All random variables in this section are assumed to be defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and E is the expectation operator associated to P. There are four types of agents interacting in this market:

(1) Competitive (infinitely many) *liquidity suppliers* move first and place limit orders that give rise to an order book. The resulting *limit order book* is identified by some non-decreasing function $h : \mathbb{R} \to \mathbb{R}$ and, therefore, a market order of size x traded against the order book incurs the cost of

$$\int_0^x h(y)dy.$$

In other words, $h(y)$ corresponds to marginal price of the y-th share.

(2) A *trading desk*, which acts as a broker for its clients, does not trade for its own account, but only gathers and executes orders for its clients. At the beginning of the day, the desk does not have any orders. Any arriving order of size y is priced at

$$\int_0^y h(Y+x)dx, \tag{2.1}$$

where Y is the accumulated number of shares from prior orders. The desk does not publish Y and, thus, its clients do not observe Y prior to submitting their orders. However, they are aware of the form of the pricing rule given by (2.1).

(3) *Noise traders* whose cumulative demand for the traded asset is given by a random variable $Z \sim N(0, \sigma^2)$ are non-strategic. Z is independent of V, which is the random variable representing the liquidation value of the asset. V is assumed to be non-constant.

(4) $N \geq 1$ risk neutral *insider(s)*, who know(s) V and choose(s) a trade size to maximize their expected profit conditional on their private information. The cumulative demand of the insiders will be denoted by X. N is assumed to be deterministic, and therefore, known to the market participants.

We assume that noise traders as well as insiders are the clients of the desk. The desk aggregates all orders and trades $Z + X$ with the liquidity suppliers. Moreover, noise trades are assumed to arrive to the desk prior to the informed order(s).

Let us first consider the case $N = 1$. Since the accumulated number of shares from prior orders equals Z when the insider's order arrives at the desk, the insider is charged

$$\int_0^X h(Z+y)dy \tag{2.2}$$

for an order of X units. Observe that the insider will always prefer buying via the desk to buying from the order book since she will get a better price from the desk in case the noise traders are selling. Noise traders, on the other hand, are indifferent between two trading venues under our assumption that they arrive slightly earlier to the market than the informed trader.

The cost given by (2.1) can be justified by Bertrand competition for orders from a strategic trader (informed or uninformed) among brokerage firms each of which has already received noise orders of size Y_i. We assume the strategic trader does not split her trades among different brokers consistent with the market practice. Note that one does not have to assume that Y_is are the same, but the pairs (Y_i, V) have the same distribution over the firms.

Another, and more common, approach in industry is to charge all orders the same average price. That is, the insider's cost of trading X units via the trading desk equals $\frac{X}{X+Z}\int_0^{X+Z} h(y)dy$, while the noise trades are priced at $\frac{Z}{X+Z}\int_0^{X+Z} h(y)dy$. However, such a cost structure makes the model analytically intractable. Nevertheless we analyze this model numerically in Section 8 and find that it shares the same qualitative features and same asymptotic price impacts with the equilibrium arising from the cost structure of (2.2).

2.1. **Optimal strategies for the informed.** Let us next consider the optimization problem for the insider facing (2.2) as her trading cost. Note that the expected profit of the insider from a market order of size x is, therefore, given by

$$E^v\left[Vx - \int_0^x h(Z+y)dy\right],$$

where E^v is the expectation operator for the insider with the private information $V = v$. Since h is assumed nondecreasing, the first order condition characterizes the unique X^* achieving the maximum expected profit via $V = F(X^*)$, where

$$F(x) := \int_{-\infty}^{\infty} h(x+z)q(\sigma, z)dz \tag{2.3}$$

and $q(\sigma, \cdot)$ is the probability density function of a mean-zero Gaussian random variable with variance σ^2. Note that since h^* is non-decreasing and not constant, F is strictly increasing and one-to-one. Thus, $X^* = F^{-1}(V)$.

We shall also consider the case of multiple insiders trading via the same desk and knowing the value of V. We assume that informed orders arrive simultaneously and the desk charges each insider an amount proportional to their order size. Therefore, the expected profit of an individual insider placing an order of size x is given by

$$E^v\left[Vx - \frac{x}{U+x}\int_0^{U+x} h(Z+y)dy\right],$$

where U denotes the aggregate demand of the other insiders.

The first order condition associated with the above optimization problem of an individual insider is again given by

$$v = E^v\left[\frac{x}{U+x}h(Z+U+x) + \frac{U}{(U+x)^2}\int_0^{U+x} h(Z+u)du\right].$$

As every insider has symmetric information and is risk-neutral, the equilibrium demand x^* for each insider must be the same and satisfy

$$v = E^v\left[\frac{h(Z+Nx^*)}{N} + \frac{N-1}{N^2x^*}\int_0^{Nx^*} h(Z+u)du\right].$$

Denoting the total informed demand by X^*, the above can be rewritten as $V = F(X^*)$, where

$$F(x) := E^v\left[\frac{h(Z+x)}{N} + \frac{N-1}{Nx}\int_0^{x} h(Z+u)du\right], \tag{2.4}$$

and $F(0)$ is interpreted by continuity to be $E^v[h(Z)]$.

Note that the marginal cost function F always satisfies $F(x) \leq E^v[h(Z+x)]$ for $x > 0$ since the marginal price h is increasing. Recall that $E^v[h(Z+x)]$ is the marginal cost of buying x units for a monopolist insider. Thus, the expression of the marginal cost via (2.4) can be viewed as a weighted combination of the marginal cost for the monopolist and the average cost of buying x units. The weight placed on the average cost vanishes in case of monopolist and is increasing to 1 as the number of insiders increases.

Moreover, it follows from the monotonicity of h that F defined via (2.4) is strictly increasing. Also note that (2.4) reduces to (2.3) when $N = 1$. Thus, we shall always refer to (2.4) when discussing the optimal strategies of the insiders regardless of the value of N.

2.2. **Equilibrium and first properties.** In what follows the total informed demand will be denoted by X and we assume following [13] that limit prices are given by 'tail expectations.' That is, denoting the total demand $X+Z$ by Y,

$$h(y) = \begin{cases} E[V|Y \geq y], & \text{if } y > 0; \\ E[V|Y \leq y], & \text{if } y < 0. \end{cases} \tag{2.5}$$

The value $h(0)$ is not relevant for the subsequent computations and can be freely chosen to be any value between the best ask $h(0+) := \lim_{y\downarrow 0} h(y)$ and the best bid $h(0-) := \lim_{y\uparrow 0} h(y)$.

The definition (2.5) entails that liquidity suppliers earn zero aggregate profit on average. Indeed, the expected profit is given by

$$E\left[\int_0^Y (h(y)-V)dy\right] = \int_0^\infty E[(h(y)-V)\mathbf{1}_{[Y\geq y]}]dy + \int_{-\infty}^0 E[(V-h(y))\mathbf{1}_{[Y\leq y]}]dy = 0,$$

where the last equality is due to the definition of the conditional expectation.

Definition 2.1. *The pair (h^*, X^*) is an equilibrium if h^* is non-decreasing and non-constant, $X^* \in \mathbb{R}$ and*

i) h^* satisfies (2.5) with $Y = X^* + Z$;
ii) X^* is the profit maximising order size for the insider(s) given h^*. That is, $V = F(X^*)$, where F is given by (2.4).

The strict monotonicity of F leads to the following result that in particular yields an explicit formula for the aggregate profit of the insiders.

Proposition 2.1. *Let (X^*, h) be an equilibrium and F defined by (2.4).*
(1) For any $x \in \mathbb{R}$ we have

$$\begin{aligned} E^v[h(x+Z)] &= F(x) + \frac{N(N-1)}{x^N} \int_0^x (F(x) - F(y)) y^{N-1} dy \\ &= F(x) + N(N-1) \int_0^1 (F(x) - F(xy)) y^{N-1} dy. \qquad (2.6) \end{aligned}$$

(2) $E^v[h(X^* + Z)] = v$ when $N = 1$ and, in general

$$E^v[h(X^* + Z)] = v + (N-1) \int_0^1 (v - F(y^{\frac{1}{N}} F^{-1}(v))) dy. \qquad (2.7)$$

Thus, $E^v[h(X^* + Z)] > v$ (resp. $E^v[h(X^* + Z)] < v$) if $F^{-1}(v) \in (0, \infty)$ (resp. $F^{-1}(v) \in (-\infty, 0)$).
(3) The aggregate expected profit of the insiders conditional on $V = v$ is given by

$$\begin{aligned} \pi^*(v) &:= E^v \left[\int_0^{X^*} (v - h(y+Z)) dy \right] = N \int_0^{F^{-1}(v)} (v - F(y)) \left(\frac{y}{F^{-1}(v)} \right)^{N-1} dy \\ &= F^{-1}(v) \int_0^1 (v - F(y^{\frac{1}{N}} F^{-1}(v))) dy, \qquad (2.8) \end{aligned}$$

with the second and third equalities being valid if $F^{-1}(v)$ is finite.
(4) The expected loss of noise traders is given by

$$\int_0^\infty \frac{h(y) - h(-y)}{2} P(|Z| \geq y) dy. \qquad (2.9)$$

Moreover, if the aggregate mid-spread *function S defined as $S(x) := \int_0^x \frac{h(y)-h(-y)}{2} dy$ satisfies* $\lim_{x \to \infty} S(x) P(Z \geq x) = 0$*, then the above loss equals*

$$\int_0^\infty S(y) P(|Z| \in dy) = E[S(|Z|)]. \qquad (2.10)$$

Since the liquidity suppliers earn zero profit on average, (2.10) implies that the ex-ante profit of insiders is given by $E[S(|Z|)]$. Thus, the larger the spread, the bigger is the insiders' profit. Moreover, since S is concave and increasing, this profit (or the loss of the noise traders) is bounded by $S(E[|Z|]) = S(\sigma\sqrt{\frac{2}{\pi}}) \leq S(\sigma)$. This upper bound corresponds to the aggregate spread at the expected noise volume.

The expression (2.7) reveals an interesting feature of our model akin to Kyle's model in continuous time. Observe that $h(X^* + Z)$ can be viewed as the last slice that is traded with the limit order traders. Thus, when there is a monopolistic insider, (2.7) shows that the final slice is priced at the actual value of V similar to the convergence of the equilibrium price to the liquidation value of the asset in the continuous time version of the Kyle model studied in [2] for general payoffs.

This 'convergence' may disappear when the liquidation value of the asset is observed by more than one insider as in [16]. If the distribution of the fundamental value does not possess any atoms, the optimal strategy of the insiders will be finite for any realization v of V. In this case $E^v[h(X^* + Z)] > v$ when the optimal strategy is buy. The intuition, as observed earlier by [16], behind this overvaluation is that competition leads to more aggressive trading and results in higher market valuation.

In practice an important trading benchmark for traders is *the implementation shortfall*. [21] defines it as the difference between a 'paper trading' benchmark and the actual trading costs. Assuming that the benchmark is given by the ex-ante valuation $E[V]$, the associated shortfall in our context can be defined as follows:

Definition 2.2. *The implementation shortfall associated with trading x units is given by*

$$IS(x) := \frac{1}{x}\int_0^x E[h(Z+y)]dy.$$

Observe that $IS(x)$ is simply the expected average cost of trading x units. As the following result shows, it is smaller than the marginal cost $F(x)$, which is given by the first order condition (2.4).

Proposition 2.2. *Let h be a function that defines the order book and $F(x)$ be given by the first order condition (2.4). Then,*

$$IS(x) = N\int_0^1 F(xy)y^{N-1}dy.$$

In particular, $IS(x) < F(x)$ *for* $x > 0$ *and* $IS(x) > F(x)$ *for* $x < 0$.

3. Characterization of equilibrium

The aim of this section is to identify the integral equations whose solutions determine an equilibrium (X^*, h^*). Writing h instead of h^* to ease the exposition and using the definition of h when $y > 0$, we get

$$h(y) = E[V|X^* + Z \geq y] = E[V|F^{-1}(V) \geq y - Z] = E[V|V \geq F(y - Z)].$$

Similarly, $h(y) = E[V|V \leq F(y - Z)]$ for $y < 0$.

Next introduce the right-continuous functions $\Phi^{\pm}$ and $\Pi^{\pm}$ via

$$\begin{aligned} \Phi^+(y) &:= E[V\mathbf{1}_{[V>y]}],\ \Pi^+(y) := P(V > y) \\ \Phi^-(y) &:= E[V\mathbf{1}_{[V\leq y]}],\ \Pi^-(y) := P(V \leq y) = 1 - \Pi^+(y). \end{aligned}$$

Note that $\Phi^+(y-) = E[V\mathbf{1}_{[V\geq y]}]$ and $\Pi^+(y-) = P(V \geq y)$ are the *unnormalized* upper tail expectation and the upper tail probability, respectively.

Now let us compute $h(y)$ for $y > 0$ using the fact that $\Phi^+(x) = \Phi^+(x-)$ for almost all x:

$$h(y) = \frac{E[V\mathbf{1}_{[V \geq F(y-Z)]}]}{P(V \geq F(y-Z))} = \frac{\int_{-\infty}^{\infty} \Phi^+(F(y-z))q(\sigma,z)dz}{\int_{-\infty}^{\infty} \Pi^+(F(y-z))q(\sigma,z)dz}. \tag{3.11}$$

Similarly, for $y < 0$

$$h(y) = \frac{\int_{-\infty}^{\infty} \Phi^-(F(y-z))q(\sigma,z)dz}{\int_{-\infty}^{\infty} \Pi^-(F(y-z))q(\sigma,z)dz}. \tag{3.12}$$

In order to obtain an equation for F it will be convenient to define, for any continuous g, the mappings

$$\phi_g^+(x) := \frac{\int_{-\infty}^{\infty} \Phi^+(g(z))q(\sigma,x-z)dz}{\int_{-\infty}^{\infty} \Pi^+(g(z))q(\sigma,x-z)dz} \quad \text{and} \quad \phi_g^-(x) := \frac{\int_{-\infty}^{\infty} \Phi^-(g(z)q(\sigma,x-z)dz}{\int_{-\infty}^{\infty} \Pi^-(g(z))q(\sigma,x-z)dz}.$$

Let us also set

$$\phi_g(x) := \phi_g^+(x)\mathbf{1}_{x \geq 0} + \phi_g^-(x)\mathbf{1}_{x<0}. \tag{3.13}$$

Now, combining (2.4), (3.11) and (3.12) yields an equation for F:

$$F(x) = \frac{1}{N}\int_{-\infty}^{\infty} q(\sigma,x-z)\phi_F(z)dz + \frac{N-1}{Nx}\int_0^x dy \int_{-\infty}^{\infty} q(\sigma,y-z)\phi_F(z)dz. \tag{3.14}$$

Assuming one can change the order of integration in above[4], then (3.14) can be rewritten as

$$F(x) = \int_{-\infty}^{\infty} \left\{ \frac{1}{N} q(\sigma,x-z) + \frac{N-1}{N}\bar{q}(\sigma,x,z) \right\} \phi_F(z)dz, \tag{3.15}$$

with

$$\bar{q}(\sigma,x,z) := \mathbf{1}_{x\neq 0}\frac{1}{x}\int_0^x q(\sigma,y-z)dy + \mathbf{1}_{x=0}q(\sigma,z). \tag{3.16}$$

The preceding calculations show that the existence of a solution for the above integral equation is a necessary condition for equilibrium. Validity of the converse is obvious and we, thus, have the following:

Theorem 3.1. *Equilibrium exists if and only if there exists a function $F : \mathbb{R} \to \mathbb{R}$ that satisfies (3.14). Given such a solution F, (X^*, h^*) constitutes an equilibrium, where $X^* = F^{-1}(V)$ and h^* is defined via (3.11) and (3.12).*

In view of the above theorem finding an equilibrium boils down to finding a solution of (3.14). Moreover, equilibrium will be uniquely defined if there exists a unique solution of (3.14). As usual, solutions of (3.14) will be identified as fixed-points of a mapping.

Before analyzing in depth this fixed-point problem, we shall first consider some settings in which equilibrium is explicitly solvable. Note that since h is increasing its right and left limits exist. In particular $h(0+)$ is the lowest ask price and $h(0-)$ is the highest ask price. Thus, *the bid-ask spread* is given by $h(0+) - h(0-)$ in equilibrium.

[4]We shall see later that this is justified when V is bounded from below

Definition 3.1. *V is said to be symmetric if V and $-V$ have the same distribution. That is, $\Pi^+(y-) = \Pi^-(-y)$ for all y.*

Proposition 3.1. *Suppose that V is symmetric and there exists a unique solution F to (3.14). Then F is symmetric, i.e. $F(x) = -F(-x)$ for all x. Moreover, any symmetric solution of (3.14) is also a solution of*

$$F(x) = \frac{1}{N}\int_0^\infty q_0(\sigma, x-z)\phi_F^+(z)dz + \frac{N-1}{Nx}\int_0^x dy \int_0^\infty q_0(\sigma, y-z)\phi_F^+(z)dz, \quad (3.17)$$

where $q_0(\sigma, x, z) := q(\sigma, x-z) - q(\sigma, x+z)$.

Proof. Observe that $-\Phi^+(y-) = E[-V\mathbf{1}_{[V\geq y]}] = E[-V\mathbf{1}_{[-V\leq -y]}] = E[V\mathbf{1}_{[V\leq -y]}] = \Phi^-(-y)$. Thus, utilising the fact that Φ^+ and Π^+ differ from their left limits at most at countably many points and q is symmetric around zero, we obtain

$$\begin{aligned}
\int_{-\infty}^\infty q(\sigma, -x-z)\phi_F(z)dz &= \int_{-\infty}^0 dz\, q(\sigma, x-z)\frac{\int_{-\infty}^\infty \Phi^+(F(-u))q(\sigma, z-u)du}{\int_{-\infty}^\infty \Pi^+(F(-u))q(\sigma, z-u)du} \\
&+ \int_0^\infty dz\, q(\sigma, x-z)\frac{\int_{-\infty}^\infty \Phi^-(F(-u))q(\sigma, z-u)du}{\int_{-\infty}^\infty \Pi^-(F(-u))q(\sigma, z-u)du} \\
&= -\int_{-\infty}^0 dz\, q(\sigma, x-z)\frac{\int_{-\infty}^\infty \Phi^-(-F(-u))q(\sigma, z-u)du}{\int_{-\infty}^\infty \Pi^-(-F(-u))q(\sigma, z-u)du} \\
&- \int_0^\infty dz\, q(\sigma, x-z)\frac{\int_{-\infty}^\infty \Phi^+(-F(-u))q(\sigma, z-u)du}{\int_{-\infty}^\infty \Pi^+(-F(-u))q(\sigma, z-u)du} \\
&= -\int_{-\infty}^\infty q(\sigma, -x-z)\phi_G(z)dz,
\end{aligned}$$

where $G(x) := -F(-x)$. Moreover,

$$\begin{aligned}
-\frac{1}{x}\int_0^{-x} dy \int_{-\infty}^\infty q(\sigma, y-z)\phi_F(z)dz &= \frac{1}{x}\int_0^x dy \int_{-\infty}^\infty q(\sigma, -y-z)\phi_F(z)dz \\
&= -\frac{1}{x}\int_0^x dy \int_{-\infty}^\infty q(\sigma, y-z)\phi_G(z)dz.
\end{aligned}$$

Thus, $-G$ is also a solution of (3.14), which establishes the first assertion. The second assertion follows from a change of variable in (3.14) as above and using the assumed symmetry of F. □

Example 3.1. *Suppose that $P(V = 1) = P(V = -1) = \frac{1}{2}$. Then, the unique symmetric solution of (3.14) is defined by*

$$F(x) = \frac{1}{N}\int_0^\infty q_0(\sigma, x, z)dz + \frac{N-1}{Nx}\int_0^x dy \int_0^\infty dz q_0(\sigma, y, z), \quad x \geq 0.$$

In this case it is easily seen that in equilibrium $X^* = \infty$ (resp. $X^* = -\infty$) if $V = 1$ (resp. $V = -1$). Moreover, $h^*(y) = \mathbf{1}_{[y>0]} - \mathbf{1}_{[y<0]}$.

Although the insiders' optimal market order is to buy or sell an infinite amount cumulatively, their profit remains finite. Indeed, when $V = 1$, the aggregate expected profit is given by

$$\int_0^\infty E^1(1 - h(Z + y))dy = 2E^1 \left(\int_0^\infty \mathbf{1}_{[Z<-y]} dy \right) = \sigma \sqrt{\frac{2}{\pi}}.$$

Note that the total profit is independent of N.

Example 3.2. *Consider the case $P(V = -1) = P(V = 0) = P(V = 1) = \frac{1}{3}$. Then, similar considerations as above should yield $F(\infty) = 1$ and $F(-\infty) = -1$. Moreover, symmetry considerations lead to $F(0) = 0$, which suggests that the insider does not trade when $V = 0$. Indeed, the unique solution of (3.14) is given by*

$$F(x) = \frac{1}{N} \int_0^\infty q_0(\sigma, x, z) \frac{1}{1 + P(Z \geq z)} dz + \frac{N-1}{Nx} \int_0^x dy \int_0^\infty dz q_0(\sigma, y, z) \frac{1}{1 + P(Z \geq z)}.$$

Differently from Example 3.1, the order book will not be flat since the insider does not trade when $V = 0$. One can obtain h^* via (3.12) and (3.11). Alternatively, for $y > 0$

$$\begin{aligned} h(y) &= \frac{E[V \mathbf{1}_{[X^*(V)+Z \geq y]}]}{P(X^*(V) + Z \geq y)} = \frac{P(V = 1)}{P(V = 1) + P(V = 0, Z \geq y)} \\ &= \frac{P(V = 1)}{P(V = 1) + P(V = 0)P(Z \geq y)} = \frac{1}{1 + P(Z \geq y)}, \end{aligned}$$

where the first equality follows from the fact that $X^*(V)$ is infinite when $V = -1$ or 1. Similarly, for $y < 0$, $h(y) = -\frac{1}{1+P(Z \leq y)}$. In particular, the bid-ask spread is given by $h(0+) - h(0-) = \frac{4}{3}$, and is again independent of the noise volume and of N.

3.1. **Existence of equilibrium.** We shall denote by m (resp. M) lower (resp. upper) endpoint of the support of the random variable V. Note that, since V is assumed non-degenerate, we have $-\infty \leq m < M \leq \infty$.

Define on the support of V

$$\Psi^{\pm}(y) := \frac{\Phi^{\pm}(y)}{\Pi^{\pm}(y)} \tag{3.18}$$

so that $\Psi^+(y) = E[V|V > y]$ and $\Psi^-(y) = E[V|V \leq y]$.

We impose the following condition on the function F to ensure that the integral equation (3.14) is well-defined and changing the order of integration in (3.15) is justified.

Assumption 3.1. $\int_{-\infty}^0 \phi_F^-(z) q(\sigma, z) dz > -\infty$.

Observe that the above is automatically satisfied if V is bounded from below in view of (4.22) in the next section. Moreover, if F is a continuous function satisfying (3.14) and V is symmetric, it satisfies the above assumption, too.

One of the useful consequences of the above assumption is that the marginal tradings costs are strictly increasing.

Lemma 3.1. *Let F be a continuous non-decreasing solution of (3.14) or (6.30) satisfying Assumption 3.1. Then, $\lim_{x\to\infty} F(x) = M$ and $\lim_{x\to-\infty} F(x) = m$. Moreover, F is strictly increasing.*

This gives rise to the following sufficient condition for equilibrium. Note that since V is bounded under the hypothesis of the theorem, Assumption 3.1 is satisfied.

Theorem 3.2. *Suppose $-\infty < m < M < \infty$. Then, there exists an equilibrium.*

4. Existence of a solution for the fixed point equation

This section is devoted to the proof of Theorem 3.2 using Schauder's fixed point theorem. The next lemma provides the key foundations for the estimates that are used to prove the existence of a fixed point.

Lemma 4.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function and u^+ (resp. u^-) be the unique solution of*

$$u_t + \sigma^2 u_{xx} = 0, \qquad u(1,x) = \Pi^+(g(x)) \ (\textit{resp. } u(1,x) = \Pi^-(g(x))). \tag{4.19}$$

Then, the following hold:

(1) There exists a solution B on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ to the following SDE:

$$dB_t = \sigma dW_t + \sigma^2 \frac{u_x(t,B_t)}{u(t,B_t)} dt, \quad B_0 = x, \tag{4.20}$$

where u is either u^+ or u^- and W is a Brownian motion with $W_0 = 0$.

(2) $\phi_g^+(x) = \mathbb{E}^{\mathbb{Q}^+}[\Psi^+(g(B_1))]$ and $\phi_g^-(x) = \mathbb{E}^{\mathbb{Q}^-}[\Psi^-(g(B_1))]$, where $(B, \mathbb{Q}^+)$ (resp. $(B, \mathbb{Q}^-)$) corresponds to the solution of (4.20) if $u = u^+$ (resp $u = u^-$) and $\mathbb{E}^{\mathbb{Q}}$ stands for the expectation under $\mathbb{Q}$.

(3) $\phi_g^+(0) > \phi_g^-(0)$.

(4) Suppose further that g is non-decreasing. Then, $\phi_g^\pm$ are non-decreasing, too. Consequently, ϕ_g is non-decreasing. Moreover,

$$\phi_g^+(x) \ \leq \ \mathbb{E}^{\mathbb{Q}^+}\left[\Psi^+(g(\sigma W_1 + x))\right] \tag{4.21}$$

$$\phi_g^-(x) \ \geq \ \mathbb{E}^{\mathbb{Q}^-}\left[\Psi^-(g(\sigma W_1 + x))\right]. \tag{4.22}$$

Proof. We shall prove the claims for u^+ only, the corresponding proof for u^- being analogous.

(1) Note that

$$u^+(t,x) = \int_{-\infty}^{\infty} \Pi^+(g(z)) \frac{1}{\sqrt{2\pi\sigma^2(1-t)}} \exp\left(-\frac{(x-z)^2}{2\sigma^2(1-t)}\right) dz.$$

Then, if β is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with $\beta_0 = 0$, $u^+(t,B_t)$ is a bounded martingale with $u^+(1,B_1) = \Pi^+(g(B_1)$, where $B = \sigma\beta + x$. Thus, we can define a new measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{u(1,B_1)}{u(0,B_0)}.$$

By means of Girsanov's theorem, under $\mathbb{Q}$, B solves (4.20).

(2) Observe that

$$\phi_g^+(x) \quad = \quad \frac{\mathbb{E}\left[\Psi^+(g(B_1))\Pi^+(g(B_1))\right]}{\mathbb{E}\left[\Pi^+(g(B_1))\right]} = \frac{\mathbb{E}\left[\Psi^+(g(B_1))u^+(1,B_1)\right]}{\mathbb{E}\left[u(1,B_1)\right]} = \mathbb{E}^{\mathbb{Q}}\left[\Psi^+(g(B_1))\right].$$

(3) The claim is equivalent to

$$\int_{-\infty}^{\infty}\Phi^+(g(z))q(\sigma,z)dz\int_{-\infty}^{\infty}\Pi^-(g(z))q(\sigma,z)dz-\int_{-\infty}^{\infty}\Phi^-(g(z))q(\sigma,z)dz\int_{-\infty}^{\infty}\Pi^+(g(z))q(\sigma,z)dz>0.$$

Using $\Phi^+ = E[V] - \Phi^-$ and $\Pi^+ + \Pi^- = 1$, the above is valid if and only if

$$\begin{aligned} 0 \quad &< \quad \int_{-\infty}^{\infty}\Phi^+(g(z))q(\sigma,z)dz - E[V]\int_{-\infty}^{\infty}\Pi^+(g(z))q(\sigma,z)dz \\ &= \quad \int_{-\infty}^{\infty}\left(\Psi^+(g(z)) - E[V]\right)q(\sigma,z)\Pi^+(g(z))dz, \end{aligned}$$

which holds since for any x we have $\Psi^+(x) \geq \Psi^+(m) = E[V]$ in view of Lemma A.1 and Ψ^+ is not constant.

(4) Now, suppose g is non-decreasing, which in turn implies $u_x^+ \leq 0$ since Π^+ is non-increasing. Therefore, as $\frac{u_x(t,x)}{u(t,x)}$ is Lipschitz on $[0,t]$ for any $t < T$, the standard comparison results for SDEs applied to (4.20) in conjunction with Lemma A.1 imply

$$\mathbb{E}^{\mathbb{Q}}\left[\Psi^+(g(B_1))\middle|B_0 = y\right] \geq \mathbb{E}^{\mathbb{Q}}\left[\Psi^+(g(B_1))\middle|B_0 = x\right] \qquad \text{if } y \geq x$$

since we can construct all these solutions indexed by their starting point on the same probability space due to the local Lipschitz property of u_x^+/u^+. This shows the desired monotonicity ϕ_g^+.

Similarly, the same comparison principle yields that the solution of (4.20) is bounded from above by $\sigma W_t + x$ in case of $u = u^+$ since $u_x^+ \leq 0$. Combined with the monotonicity property of $\Psi^+(g)$, we deduce $\phi_g^+(x) \leq \mathbb{E}^{\mathbb{Q}}\left[\Psi^+(g(\sigma W_1 + x))\right]$.

□

Proof of Theorem 3.2. The proof will be an application of Schauder's fixed point theorem. Note that since V takes values in $[m, M]$, so does $\Psi^\pm$. This justifies the representation (3.15) for F. Since in equilibrium h must also be taking values in $[m, M]$, we must expect F to take values in $[m, M]$, too. Thus, we can concentrate on functions on $\mathbb{R}$ that takes values in $[m, M]$. Moreover, F will possess a derivative that is bounded by

$$K_0 := (|m| + M)\int_{-\infty}^{\infty}|z|\frac{e^{-\frac{z^2}{2\sigma^2}}}{\sigma^3\sqrt{2\pi}}dz\left(\frac{1}{N} + \frac{N-1}{2N}\right) < \infty.$$

To see this first observe that

$$\begin{aligned}\frac{\partial \bar{q}(\sigma,x,z)}{\partial x} &= \frac{q(\sigma,x-z)-\bar{q}(\sigma,x,z)}{x} = \frac{1}{x^2}\int_0^x \{q(\sigma,x-z)-q(\sigma,y-z)dy\}\\ &= \frac{1}{x^2}\int_0^x u q_x(\sigma,u-z)du.\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\frac{d}{dx}F(x)\right| &\le \int_{-\infty}^{\infty} |\phi_F(z)| \left\{\frac{|q_x(\sigma,x-z)|}{N} + \frac{N-1}{Nx^2}\int_0^{|x|} u|q_x(\sigma,u-z)|du\right\}dz\\ &\le (|m|+M)\left\{\frac{1}{N}\int_{-\infty}^{\infty}|z-x|\frac{e^{-\frac{(z-x)^2}{2\sigma^2}}}{\sigma^3\sqrt{2\pi}}dz + \frac{N-1}{Nx^2}\int_0^{|x|}duu\int_{-\infty}^{\infty}|z-u|\frac{e^{-\frac{(z-u)^2}{2\sigma^2}}}{\sigma^3\sqrt{2\pi}}dz\right\}\\ &= (|m|+M)\int_{-\infty}^{\infty}|z|\frac{e^{-\frac{z^2}{2\sigma^2}}}{\sigma^3\sqrt{2\pi}}dz\left(\frac{1}{N}+\frac{N-1}{2N}\right).\end{aligned}$$

We shall show the existence of a fixed point in the normed space $\mathcal{X} := L^2(\mathbb{R},\mu_0)$, i.e. the space of Borel measurable functions that are square integrable with respect to μ_0, where

$$\mu_0(dx) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx.$$

Note that μ_0 is equivalent to the Lebesgue measure on $\mathbb{R}$. Next define

$$D_0 : \{g|g:\mathbb{R}\mapsto[m,M] \text{ is such that } |g(x)-g(y)|\le K_0|x-y|, \forall x,y\in\mathbb{R}\}$$

and let

$$D := \{g\in\mathcal{X}|g=g_0,\ \mu_0\text{-a.e. for some } g_0\in D_0\}.$$

It is easy to see that D is a convex subset of $\mathcal{X}$.

Next define the operator T on $\mathcal{X}$ via

$$Tg(x) := \int_{-\infty}^{\infty}\left\{\frac{1}{N}q(\sigma,x-z)+\frac{N-1}{N}\bar{q}(\sigma,x,z)\right\}\phi_{\bar{g}}(z)dz,$$

where $\bar{g} := (g\vee m)\wedge M$ and ϕ_g and $\bar{q}$ are as defined in (3.13) and (3.16), respectively. Note that for each x

$$\left\{\frac{1}{N}q(\sigma,x-z)+\frac{N-1}{N}\bar{q}(\sigma,x,z)\right\}dz$$

is a probability measure on $\mathbb{R}$.

Step 1 *(T maps D into itself):* It is easy to verify that Tg is continuous and takes values in $[m,M]$ in view of Lemma 4.1 and that

$$\int_{-\infty}^{\infty}\left\{\frac{1}{N}q(\sigma,x-z)+\frac{N-1}{N}\bar{q}(\sigma,x,z)\right\}dz = 1.$$

Moreover, Tg is differentiable with a derivative bounded by K_0 by using the above computations that led to the estimate K_0. Thus, $Tg\in D_0\subset D$.

Step 2 *(D is compact):* Let $(g_n) \subset D$. Then there exists $(g_n^0) \subset D_0$ such that μ_0-a.e. we have $g_n = g_n^0$ for each $n \geq 1$. Then by Arzela-Ascoli Theorem there exists a subsequence that converges uniformly on compacts to a continuous function g^0. Without loss of generality let us assume that g_n^0 converges to g^0. Note that necessarily $|g^0(x) - g^0(y)| \leq K_0|x-y|$ for all $x, y \in \mathbb{R}$, i.e $g^0 \in D_0$. Finally, as g_ns are uniformly bounded and μ_0 is a probability measure, the dominated convergence theorem yields $g_n \to g^0$ in $L^2(\mathbb{R}, \mu_0)$.

Step 3 *($T : D \to D$ is continuous):* Suppose $g_n \to g$ in D as $n \to \infty$. In view of the definition of D we may assume without loss of generality that that g_ns are continuous since changing g_n on a Lebesgue null set does not alter the value of Tg_n. By another application of Arzela-Ascoli theorem there exists a subsequence that converges pointwise to some continuous function, which we may identify with g due to the uniqueness of L^2-limits up to a null set. Thus, we may assume g is continuous, too.

Moreover, the same argument shows that every subsequence of g_n has a further subsequence that converges to g pointwise since continuous functions that agree on Lebesgue null sets should agree at every point. Thus, $g_n \to g$ pointwise as $n \to \infty$.

Next, since ϕ_{g_n} is uniformly bounded, the dominated convergence theorem yields

$$\lim_{n\to\infty} Tg_n(x) = \int_{-\infty}^{\infty} dz \left\{ \frac{1}{N} q(\sigma, x-z) + \frac{N-1}{N} \bar{q}(\sigma, x, z) \right\} \lim_{n\to\infty} \phi_{g_n}(z).$$

On the other hand, Lemma 4.1 and Girsanov theorem imply for $z > 0$

$$\phi_{g_n}(z) = \frac{\mathbb{E}\left[\Psi^+(g_n(\sigma\beta_1 + z))\Pi^+(g_n(\sigma\beta_1 + z))\right]}{u^n(0,z)},$$

where $\mathbb{E}$ is the expectation operator for $\mathbb{P}$ under which β is a standard Brownian motion, and u^n is the function u^+ in Lemma 4.1 defined by the terminal condition $\Pi^+(g_n)$. Since Ψ^+ and Π^+ are continuous except on a Lebesgue null set, the dominated convergence theorem yields

$$\lim_{n\to\infty} \mathbb{E}\left[\Psi^+(g_n(\sigma\beta_1 + z))\Pi^+(g_n(\sigma\beta_1 + z))\right] = \mathbb{E}\left[\Psi^+(g(\sigma\beta_1 + z))\Pi^+(g(\sigma\beta_1 + z))\right].$$

Similarly,

$$\lim_{n\to\infty} u^n(0,z) = \lim_{n\to\infty} \mathbb{E}\left[\Pi^+(g_n(\sigma\beta_1 + z))\right] = \mathbb{E}\left[\Pi^+(g(\sigma\beta_1 + z))\right].$$

Thus, we have shown $\lim_{n\to\infty} \phi_{g_n}(z) = \phi_g(z)$ for $z > 0$. Analogous arguments yields the convergence for $z \leq 0$, which in turn establishes the pointwise convergence of Tg_n to T_g; i.e.

$$\lim_{n\to\infty} Tg_n(x) = Tg(x), \qquad x \in \mathbb{R}.$$

This yields the claim by an application of the dominated convergence theorem since Tg_ns are uniformly bounded.

Therefore, there exists a $g \in D$ such that $g = Tg$ by Schauder's fixed point theorem. The claim now follows from Theorem 3.1. $\square$

Under the hypotheses of Theorem 3.2 it is clear that $m < F(x) < M$ for $x \in \mathbb{R}$. However, in view of the bounds given by (4.21) and (4.22) it is possible to obtain sharper bounds on any solution of (3.14).

Theorem 4.1. *Suppose $-\infty < m < M < \infty$ and let $\Psi^{\pm}$ be as in (3.18). Then the following statements are valid.*

(1) There exists a maximal nondecreasing solution[5] *to*

$$\begin{aligned} R(x) &= \int_0^\infty dz \left\{ \frac{1}{N} q(\sigma, x-z) + \frac{N-1}{N} \bar{q}(\sigma, x, z) \right\} \int_{-\infty}^\infty \Psi^+(R(y)) q(\sigma, z-y) dy \\ & \quad + E[V] \int_{-\infty}^0 dz \left\{ \frac{1}{N} q(\sigma, x-z) + \frac{N-1}{N} \bar{q}(\sigma, x, z) \right\}. \end{aligned} \tag{4.23}$$

Moreover, this maximal solution is not constant and any solution of (3.14) is bounded from above by this maximal solution.

(2) There exists a minimal nondecreasing solution[6] *to*

$$\begin{aligned} l(x) &= E[V] \int_0^\infty dz \left\{ \frac{1}{N} q(\sigma, x-z) + \frac{N-1}{N} \bar{q}(\sigma, x, z) \right\} \qquad (4.24) \\ & \quad + \int_{-\infty}^0 dz \left\{ \frac{1}{N} q(\sigma, x-z) + \frac{N-1}{N} \bar{q}(\sigma, x, z) \int_{-\infty}^\infty \Psi^-(l(y)) q(\sigma, z-y) dy \right\}. \end{aligned}$$

Moreover, this minimal solution is not constant and any solution of (3.14) is bounded from below by this minimal solution.

Proof. (1) Let $\mathcal{D}$ be the space of nondecreasing functions on $\mathbb{R}$ that take values in $[m, M]$ and define an operator T on $\mathcal{D}$ via

$$\begin{aligned} Tr(x) &= \int_0^\infty dz \left\{ \frac{1}{N} q(\sigma, x-z) + \frac{N-1}{N} \bar{q}(\sigma, x, z) \right\} \int_{-\infty}^\infty \Psi^+(r(y)) q(\sigma, z-y) dy \\ & \quad + E[V] \int_{-\infty}^0 dz \left\{ \frac{1}{N} q(\sigma, x-z) + \frac{N-1}{N} \bar{q}(\sigma, x, z) \right\}. \end{aligned}$$

First observe that $Tr_1 \geq Tr_2$ if $r_1 \leq r_2$ since Ψ^+ is non-decreasing. Thus, setting $r_0 \equiv m$ and $r_n = Tr_{n-1}$ for $n \geq 1$, we observe that r_n is an increasing sequence of nondecreasing continuous functions taking values in $[m, M]$. Thus, the limit, r_∞ exists, is nondecreasing, and continuous by Dini's theorem. It also follows from the dominated convergence theorem and that Ψ^+ has only countably many discontinuities that r_∞ is a solution of (4.23).

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[5] R is a maximal nondecreasing solution if $R(x) \geq r(x)$ for all $x \in \mathbb{R}$, where r is any other nondecreasing solution.

[6] l is a minimal nondecreasing solution if $l(x) \leq L(x)$ for all $x \in \mathbb{R}$, where L is any other nondecreasing solution.

Now let $\mathcal{E}$ denote the set of all solutions of (4.23) in $\mathcal{D}$ and define

$$R^*(x) := \sup_{r \in \mathcal{E}} r(x).$$

Clearly, R^* takes values in $[m, M]$. Due to the aforementioned monotonicity $TR^* \geq Tr = r$ for all $r \in \mathcal{E}$. Consequently, $TR^* \geq R^*$. Setting $g_0 = R^*$ and $g_n = Tg_{n-1}$ for $n \geq 1$, we again obtain an increasing sequence which converges to some element r of $\mathcal{E}$. Moreover, $r \geq R^*$, which in turn implies $r = R^*$ by the construction of R^*.

Note that if R^* is constant, it has to equal M since $R^*(\infty) = M$. However, under this assumption, the right side of (4.23) equals

$$\int_0^\infty dz \left\{ \frac{1}{N} q(\sigma, x - z) + \frac{N-1}{N} \bar{q}(\sigma, x, z) \right\} (M \mathbf{1}_{z>0} + E[V] \mathbf{1}_{z<0}) < M$$

as $E[V] < M$. Thus, R^* cannot be constant.

To show the final assertion let F be a solution of (3.14). Since $\Psi^- \leq E[V]$ and that $\phi_F^+(z) \leq \int_{-\infty}^\infty \Psi^+(F(y)) q(\sigma, z - y) dy$ by (4.21), we deduce that

$$F(x) \leq TF(x).$$

As above, setting $g_0 = F$ and $g_n = Tg_{n-1}$ for $n \geq 1$, we again obtain an increasing sequence which converges to a member of $\mathcal{E}$, which proves the claim.

(2) Since $\Psi^- \leq E[V]$, the proof follows the similar lines as above and, hence, omitted □

5. Market impact asymptotics

The shape of market impact as a function of trade size is important to practitioners to address capacity and position sizing. Various functional forms have been explored in the literature (see, e.g., [3], [12], [23] and [28]), including $\sqrt{X}$ and $\log(1 + aX)$. Rejecting either of these forms empirically is challenging due to three practical difficulties: (1) the spread and market impact terms are collinear for small orders, (2) signal-to-noise ratios are weak for executions that take a small percentage of market volume, and (3) for very large trades, order sizes are often increased if liquidity is available, or reduced if liquidity is hard to find, leading to bias in the data. In absence of clear empirical evidence, theoretical predictions for the shape of market impact provide valuable insight into how trading costs scale with trade size.

Motivated by this need, we draw our attention in this section to the *market impact* associated with large orders. More precisely, we will be computing the asymptotics of the marginal cost of trades, which is given by the function F. As we shall see later, the asymptotic form of F will coincide (up to a scaling factor) with that of the implementation shortfall IS. We will also be able to compute the tail asymptotics of the distribution of the total demand in equilibrium using *the theory of regular variation*.

Definition 5.1. *A function* $g : (0,\infty) \to (0,\infty)$ *is said to be* regularly varying of index ρ *at* ∞ *if*

$$\lim_{\lambda\to\infty} \frac{g(\lambda x)}{g(\lambda)} = x^{\rho}, \quad \forall x > 0.$$

Analogously, a function $g : (-\infty, 0) \to (0,\infty)$ *is said to be regularly varying of index* ρ *at* $-\infty$ *if* $g(-x)$ *is regularly varying of index* ρ *at* ∞*.*

Roughly speaking, a function of regular variation behaves like a power function asymptotically. We shall obtain our asymptotic results under a mild condition on the distribution of V.

Assumption 5.1. Π^+ *has a continuous derivative on* (m, M)*. Moreover,* $\Psi^+_x(M) := \lim_{x\to M} \frac{d}{dx}\Psi^+(x)$ *and* $\Psi^-_x(m) := \lim_{x\to m} \frac{d}{dx}\Psi^-(x)$ *exist and equal the left and right derivatives, respectively, at finite end points. Moreover,* $\Psi^+_x(M)\Psi^-_x(m) \neq 0$[7]*.*

Observe that when M is finite and the left derivative of Ψ^+ at M exists,

$$\Psi^+_x(M) = \lim_{x\to M} \frac{M - \Psi^+(x)}{M - x} \le \lim_{x\to M} \frac{M - x}{M - x} = 1.$$

Similarly, $\Psi^-_x(m) \le 1$ if m is finite under an analogous condition.

In this section we will mainly focus on the asymptotics of F and IS. However, it is easy to see from (2.4) that F and h behave similarly for large values. For a more rigorous demonstration, suppose $M - F$ is regularly varying at ∞ with exponent ρ^+. Recalling the measure ν defined in (B.50), we formally obtain

$$\begin{aligned} \lim_{x\to\infty} \frac{M - h(x)}{M - F(x)} &= \lim_{x\to\infty} \int_{-\infty}^{\infty} \nu(x, 1, dy) \frac{M - \Psi^+(F(xy))}{M - F(x)} \\ &= \Psi^+_x(M) 1^{\rho^+} = \Psi^+_x(M), \end{aligned} \tag{5.25}$$

using the mean value theorem and the continuity of the derivative of Ψ^+ together with the fact that the F is regularly varying with index ρ^+ since the measure $\nu(x, 1, dy)$ converges to the point mass at 1 as $x \to \infty$. This shows that marginal price is also regularly varying with the same index ρ^+.

The next result is the first description of the asymptotics of solutions of (3.14).

Theorem 5.1. *Assume Assumption 5.1,* $N > 1$ *and* $-\infty < m < M < \infty$*. Let* F *be any solution of (3.14). Then, the following statements are valid:*

(1) $M - F$ *is regularly varying of index* ρ^+ *at* ∞*, with* $\rho^+ = \frac{\Psi^+_x(M) - 1}{1 - \frac{\Psi^+_x(M)}{N}}$*.*

Moreover, if $\Psi^+_x(M) = 1$ *and there exist an integer* $n \ge 1$ *and a real constant* $k \in (0,\infty)$ *such that*[8]

$$\lim_{x\to M} \frac{\Psi^+(x) - x}{(M - x)^{n+1}} = \frac{1}{k},$$

[7] This last condition is automatically satisfied if $M = \infty$ and $m = -\infty$ since $\Psi^+(\infty) = -\Psi^-(-\infty) = \infty$.

[8] In case $\Psi^+_x(M) \neq 1$, this limit condition is valid with $n = 0$ and $k = (1 - \Psi^+_x(M))^{-1}$. Thus, this additional regularity assumption can be viewed as an extension of the existing regularity that holds

the following asymptotics hold:

$$M - F(x) \sim \left(\frac{N}{N-1}\frac{n}{k}\right)^{-\frac{1}{n}} (\log x)^{-\frac{1}{n}}, \quad x \to \infty. \tag{5.26}$$

(2) $F - m$ *is regularly varying of index* ρ^- *at* $-\infty$*, with* $\rho^- = \frac{\Psi_x^-(m)-1}{1-\frac{\Psi_x^-(m)}{N}}$.

Moreover, if $\Psi_x^-(m) = 1$*, and there exist an integer* $n \geq 1$ *and a real constant* $k \in (0, \infty)$ *such that*

$$\lim_{x \to m} \frac{x - \Psi^-(x)}{(x-m)^{n+1}} = \frac{1}{k},$$

then

$$F(x) - m \sim \left(\frac{N}{N-1}\frac{n}{k}\right)^{-\frac{1}{n}} (\log |x|)^{-\frac{1}{n}}, \quad x \to -\infty. \tag{5.27}$$

The lengthy and technical proof of the above result is given in the Appendix. However, we can give a quick motivation of why the regular variation exponent is as given. To this end, define $\gamma(x) := \lim_{\alpha \to \infty} \frac{M-F(\alpha x)}{M-F(\alpha)}$ for $x \geq 1$. Then, using the integral equation (3.14) and taking the appropriate ratios, one can formally 'show' that

$$\gamma(x) = \frac{\Psi_x^+(M)}{N}\gamma(x) + \Psi_x^+(M)\frac{N-1}{Nx}\int_0^x \gamma(y)dy.$$

The initial condition $\gamma(1) = 1$ next reveals that $\gamma(x) = x^{\rho^+}$.

Observe that the asymptotic shape of marginal costs (or, equivalently, marginal prices) are independent of the noise variance. This stems from the tail expectation condition that defines the order book. Recall that, for $y > 0$, the marginal price of the y-th share purchased is given by $h(y) = E[V|F(y+Z)]$. Since the noise demand Z is mostly concentrated around two standard deviations in the neighborhood of 0, i.e. the large trades are most likely coming from the informed traders, the impact of the noise distribution on the asymptotic behavior of the marginal price is negligible in equilibrium. This is also responsible for the price impact depending on return distribution only by its tail.

Indeed, the power law of the price impact only depends on the number of insiders, i.e. the intensity of the competition among the informed, and the particular tail mass index $\Psi_x^+(M)$. To see why $\Psi_x^+(M) < 1$ is an indication of fat tails, consider the situation where the distribution Π^+ exhibit fat tails, i.e. power-like behavior, in the sense that for large x

$$\frac{\Pi_x^+(x)}{\Pi^+(x)} \sim -\alpha(M-x)^{-1}, \quad \alpha > 0.$$

when $\rho^+ \neq 0$. In particular, if V has a probability density p with

$$p(x) \propto (M-x)^\beta \exp\left(-\frac{\Sigma}{(M-x)^n}\right),$$

for some $\beta \in \mathbb{R}$, $\Sigma > 0$, and $n \geq 1$, the limit condition holds with $k = n\Sigma$. Analogous considerations apply in the second part.

Since $\frac{M-\Psi^+(x)}{M-x} = \frac{\int_x^M \Pi^+(y)dy}{(M-x)\Pi^+(x)} - 1$, a direct application of the L'Hôpital rule implies $\Psi_x^+(M) = \frac{\alpha}{1+\alpha} < 1$. This shows in view of Theorem 5.1 that the price impact obeys a power law when the arithmetic return of the risky asset has a fat-tailed distribution.

In the preceding example, the magnitude of ρ^+ equals $\frac{1}{1+\alpha\frac{N-1}{N}}$. That is, the impact gets bigger as the returns distribution has fatter tails (smaller α) and increased competition among insiders results in smaller impact. We shall see analogous features when we discuss unbounded return distributions in Section 7.

We have seen in Proposition 2.2 that the implementation shortfall is smaller than F. In view of Theorem 5.1 we have a more precise relationship for large x.

Corollary 5.1. *Assume Assumption 5.1, $N > 1$ and $-\infty < m < M < \infty$, and let (h^*, X^*) be an equilibrium. Suppose that F^* is given by (2.4), with h being replaced by h^*. Then,*

$$M - IS^*(x) \sim \frac{N}{N+\rho^+}(M - F^*(x)), \qquad x \to \infty,$$
$$IS^*(x) - m \sim \frac{N}{N+\rho^-}(F^*(x) - m), \qquad x \to -\infty,$$

where ρ^+ and ρ^- are as in Theorem 5.1.

Corollary 5.1 shows that for large N implementation shortfall and the marginal cost of trading are almost indistinguishable. This is even more pronounced when $M - F$ (resp. $F - m$) is slowly varying, i.e. $\rho^+ = 0$ (resp. $\rho^- = 0$). Combined with (5.25) the above yields

$$M - IS^*(x) \sim \frac{N - \Psi_x^+(M)}{\Psi_x^+(M)(N-1)+}(M - h^*(x)), \qquad x \to \infty.$$

Observe that IS can be estimated from market data using the open limit order book governed by h. Moreover, the returns distribution, hence $\Psi_x^+(M)$, can be estimated from the option data. Thus, the above relationship gives a statistic to estimate the number of informed traders in the market.

5.1. **Equilibrium distribution of the traded volume.** The above results will also allow us to compute the distribution of the traded volume in equilibrium. Although this will be achieved under an additional regularity assumption on the distribution of the asset value V, it will be satisfied for practically all distributions typically appearing in applications.

Corollary 5.2. *Assume Assumption 5.1, $N > 1$ and $-\infty < m < M < \infty$. Let F be any solution of (3.14). Then, the following statements are valid:*

(1) Suppose that there exist an integer $n \geq 0$ and a real constant $k \in (0,\infty)$ such that

$$\lim_{x \to M} \frac{\Psi^+(x) - x}{(M-x)^{n+1}} = \frac{1}{k}.$$

Then, $\Pi^+(F)$ is regularly varying at ∞ of index $-\frac{\Psi_x^+(M)}{1-\frac{\Psi_x^+(M)}{N}}$.

(2) Suppose that there exist an integer $n \geq 0$ *and a real constant* $k \in (0, \infty)$ *such that*

$$\lim_{x \to M} \frac{x - \Psi^-(x)}{(x-m)^{n+1}} = \frac{1}{k}.$$

Then, $\Pi^-(F)$ *is regularly varying at* $-\infty$ *of index* $-\frac{\Psi_x^-(m)}{1 - \frac{\Psi_x^-(m)}{N}}$.

This seemingly technical result uncovers the distribution of the total volume traded in equilibrium. Indeed, for $x > 0$

$$P(X^* > x) = P(F^{-1}(V) > x) = P(V > F(x)) = \Pi^+(F(x)).$$

Thus, under the hypothesis of Corollary 5.2, the tail distribution of equilibrium X^* is regularly varying at infinity. That is,

$$P(X^* > x) = x^{-\zeta^+} s(x),$$

where s is a slowly varying function and $\zeta^+ := \frac{\Psi_x^+(M)}{1 - \frac{\Psi_x^+(M)}{N}}$.

Moreover, since the aggregate order is given by $Y^* = X^* + Z$ and Z and V are independent, we have for $y > 0$

$$P(Y^* > y) = \int_{-\infty}^{\infty} dz P(X^* > y - z) q(\sigma, z) = \int_{-\infty}^{\infty} dz P(X^* > z) q(\sigma, y - z),$$

which can easily be shown to be regularly varying at infinity with the same index. Thus,

$$P(Y^* > y) = y^{-\zeta^+} s(y), \quad y > 0, \tag{5.28}$$

for some regularly varying s. In particular, if V has lighter tails, i.e. $\Psi_x^+(M) = 1$, $P(Y^* > y)$ is regularly varying of index $-\frac{N}{N-1}$.

Analogous computations yield for the sell side

$$P(Y^* < -y) = y^{-\zeta^-} s(y), \quad y > 0, \tag{5.29}$$

where $\zeta^- := \frac{\Psi_x^-(m)}{1 - \frac{\Psi_x^-(m)}{N}}$.

6. Further properties of equilibrium and comparative statics

The following scaling property of F is inherited from the analogous property of the Gaussian distribution.

Proposition 6.1. *Let* $F(1; \cdot)$ *be a solution of (3.14) with* $\sigma = 1$*. Then the function* $x \mapsto F(1; \frac{x}{\sigma})$ *solves (3.14).*

Proof.

$$\begin{aligned} F\Big(1;\frac{x}{\sigma}\Big) &= \frac{1}{N}\int_{-\infty}^{\infty} q\Big(1,\frac{x}{\sigma}-z\Big)\phi_F(z)dz + \sigma\frac{N-1}{Nx}\int_0^{\frac{x}{\sigma}} dy \int_{-\infty}^{\infty} q(1,y-z)\phi_F(z)dz \\ &= \frac{1}{N}\int_{-\infty}^{\infty} q\Big(\sigma,x-z\Big)\phi_F\Big(\frac{z}{\sigma}\Big)dz + \frac{N-1}{Nx}\int_0^{x} dy \int_{-\infty}^{\infty} q\Big(1,\frac{y}{\sigma}-z\Big)\phi_F(z)dz \\ &= \frac{1}{N}\int_{-\infty}^{\infty} q\Big(\sigma,x-z\Big)\phi_F\Big(\frac{z}{\sigma}\Big)dz + \frac{N-1}{Nx}\int_0^{x} dy \int_{-\infty}^{\infty} q(\sigma,y-z)\phi_F\Big(\frac{z}{y}\Big)dz. \end{aligned}$$

Similar manipulations yield

$$\phi_F^{\pm}\Big(\frac{z}{y}\Big) = \frac{\int_{-\infty}^{\infty}\Phi^{\pm}\Big(F\Big(1;\frac{u}{y}\Big)\Big)\, q\Big(\sigma,u-z\Big)du}{\int_{-\infty}^{\infty}\Pi^{\pm}\Big(F\Big(1;\frac{u}{y}\Big)\Big)\, q(\sigma,u-z)du},$$

which establishes the claim. □

The above simplifies the numerical analysis of the model considerably. A straightforward corollary to this result is the following, whose proof is left to the reader.

Corollary 6.1. *Consider the solutions of (3.14) for any σ.*

(1) If (3.14) has a unique solution for some σ, it has a unique solution for all σs.
(2) If (3.14) has a unique solution for some σ, $F(0)$ does not depend on σ.
(3) Let $h(\sigma;\cdot)$ be the function defined via (3.11) and (3.12), where F is the unique solution of (3.14) for the given σ. Then, $h(\sigma;x) = h(1;\frac{x}{\sigma})$ for all $x \neq 0$.
(4) Suppose (3.14) has a unique solution for some σ and let $X^*(\sigma)$ be the optimal order size for the given σ. Then, $X^*(\sigma) = \sigma X^*(1)$.

In particular, h inherits the same scaling property of F. A simple but striking consequence of this property concerns the equilibrium bid-ask spread. The next result is immediate, hence its proof is omitted.

Corollary 6.2. *Suppose that uniqueness holds for the solutions of (3.14). Then, $h(0+) - h(0-)$ does not depend on σ. Moreover, for fixed x, $h(x)$ is decreasing in σ for $x > 0$ and increasing in σ for $x < 0$. Consequently, the aggregate mid-spread S is decreasing in σ.*

The liquidity suppliers charge a minimal bid-ask spread that does not vanish even if the amount of 'noise' trading is excessively large. On the other hand, the spread associated with any trade size $y > 0$, i.e. $h(y) - h(-y)$, and, therefore, the aggregate mid-spread S is decreasing with the amount of noise trading, consistent with the experimental findings of [6]. Moreover, in the limit as $\sigma \to \infty$, the order book, i.e., gets flatten and converges to the one that yields transaction costs that are proportional to the order size.

Uniqueness of solutions also allow us to discuss the dependency of aggregate insider profit and the expected loss of noise traders on the noise variance. In particular, the aggregate profit of insiders is increasing in noise variance as expected. The following is a direct consequence of Proposition 2.1.

Corollary 6.3. *Suppose that uniqueness holds for the solutions of (3.14) and denote by $\pi(\sigma)$ the aggregate wealth of the insiders in equilibrium. Then, $\pi(\sigma) = \sigma\pi(1)$.*

Similarly, the expected loss of noise traders is given by $\sigma E[S(1, \sigma^{-1}|Z|)]$ where $S(1, \cdot)$ is the aggregate mid-spread function in equilibrium for $\sigma = 1$.

Another consequence of the uniqueness of solutions of (3.11) is that the aggregate expected profit of the insiders vanishes as $N \to \infty$.

Proposition 6.2. *Suppose that $-\infty < m < M < \infty$, there exists a unique solution F_N of (3.14) for each $N \geq 1$, and uniqueness holds for the solutions of*

$$F(x) = \frac{1}{x}\int_0^x dy \int_{-\infty}^{\infty} q(\sigma, y - z)\phi_F(z)dz. \tag{6.30}$$

Assume further that Π^+ is continuous. Then $F_\infty = \lim_{N\to\infty} F_N$ exists and solves (6.30). Moreover, $\lim_{N\to\infty} \pi^*(v) = 0$, where π^* is the aggregate expected profit as defined in (2.8).

Proof. First observe that ϕ_F is bounded since V is. Thus the dominated convergence theorem in conjunction with the continuity of Π^+, and consequently that of Φ^+ and Φ^-, show that both of $\liminf F_N$ and $\limsup F_N$ solve (6.30). As (6.30) can have at most one solution, $F_\infty := \lim F_N$ exists.

Also note that F_∞ is strictly increasing and continuous by Lemma 3.1. Thus, $\lim_{N\to\infty} F_N^{-1}(v) = F_\infty^{-1}(v) \in \mathbb{R}$, and

$$\begin{aligned}
\lim_{N\to\infty} \pi^*(v) &= \lim_{N\to\infty} F_N^{-1}(v) N \int_0^1 (v - F_N(F_N^{-1}(v)y)y^{N-1}dy \\
&= F_\infty^{-1}(v) \lim_{N\to\infty} N \int_0^1 (v - F_N(F_N^{-1}(v)y)y^{N-1}dy.
\end{aligned}$$

Since each F_N takes values in (m, M), $\lim_{N\to\infty} F_N(F_N^{-1}(v)y) = F_\infty(F_\infty^{-1}(v)y)$, and the measure $Ny^{N-1}dy$ on $[0, 1]$ converges weakly to the point mass at 1, we have

$$\lim_{N\to\infty} N \int_0^1 (v - F_N(F_N^{-1}(v)y)y^{N-1}dy = v - F_\infty(F_\infty^{-1}(v)) = 0.$$

□

We believe the aggregate profit of the insiders decreases and the spread increases in the number of insiders. Although we are not able to prove these intuitive properties, it is observed in all numerical studies that we perform in the next section.

7. Numerical studies

This section is devoted to description of results obtained in previous version via a naive numerical search for a fixed point: Starting with an F_0, we compute $F_{n+1} = TF_n$ until the distance between successive iterations become negligibly small, where TF corresponds to the right side of (3.14). Although the proofs of the statements concerning the existence of equilibrium and its asymptotics relied on a boundedness assumption, we shall also present the solutions of the fixed point problems associated with equilibria with unbounded signals.

Since σ can be absorbed into the units for measuring equity in view of Proposition 6.1, we will set $\sigma = 1$ in all numerical tests with no loss of generality in this section.

We consider first the case of bounded signals that was the focus of the previous section.

7.1. **Bounded signals.**

7.1.1. *Truncated Gaussian distribution.*

If signals are drawn from the truncated Gaussian distribution with density $p(v) = \frac{1}{erf(\frac{M}{\sqrt{2\Sigma}})\sqrt{2\pi\Sigma}} e^{-\frac{v^2}{2\Sigma}}$ for $v \in [-M, M]$, the numerical solution for equilibrium F converges to the upper bound as $M - F(x) \sim 1/x^{\frac{N-1}{2N}}$ in accordance with the predictions of Theorem 5.1. We show F and the theoretical prediction for its asymptotic behavior in Figure 1.

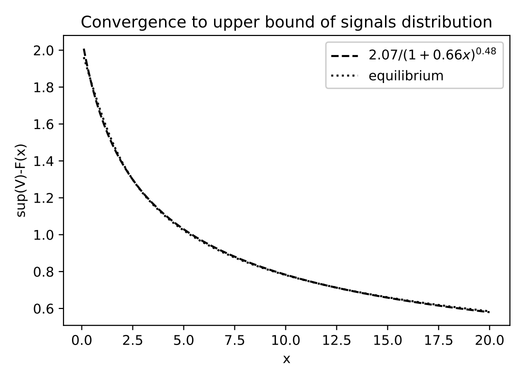

FIGURE 1. The asymptotic behavior of F is shown for the case where signals are drawn from a truncated Gaussian distribution.

7.1.2. *Logit-normal distribution.* If price is a probability-weighted average over two possible outcomes $v_{\pm} = p_0 \pm 1$, where the probability is a sigmoidal function of a Gaussian generator potential g as $p = \frac{1}{1+e^{-g}}$, the signals distribution is the logit-normal distribution with density $p(v) = \sqrt{\frac{2}{\pi\Sigma}} \frac{e^{-\frac{ln^2(\frac{1-v}{1+v})}{2\Sigma}}}{1-v^2}$. This distribution has support in $[-1, 1]$.

We show the equilibrium solution for F, the order book h and the implementation shortfall for logit-normal signals, in Figure 2.

Note that the logit normal distribution does not satisfy the hypothesis of Theorem 5.1. Thus, our theory cannot predict the asymptotics of F for this distribution. However, the following formal arguments yield the asymptotics that seem to be verified by the numerical experiments. Observe that for $x >> \sigma$, (2.4) becomes

$$F(x) \approx \frac{1}{N}h(x) + \frac{N-1}{Nx}\int_0^x h(u)du \tag{7.31}$$

It follows that $F(x) + xF'(x) \approx h(x) + \frac{1}{N}xh'(x)$. Moreover, for large values of x, we

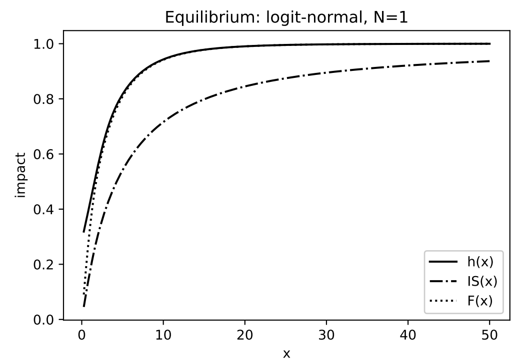

FIGURE 2. Equilibrium impact and shortfall for logit-normal signals for the case of an insider ($N = 1$).

roughly have $h(x) \approx \Psi^+(F(x))$. Let us consider the large N limit and drop the $1/N$ term. Using the approximation $erfc(x) \approx \frac{e^{-x^2}}{x\sqrt{\pi}}(1 - \frac{1}{2x^2})$ and expanding to first order in $1/\log(x)$ we find that asymptotically

$$xF' = \frac{\Sigma(1-F)}{log(1-F)},$$

which yields $F \to 1 - e^{-k\sqrt{\log(x)}}$

The numerical solution shown in Figure 3 for $N = 25$ is consistent with this asymptotic form.

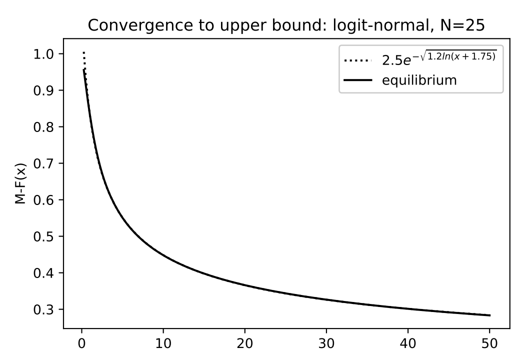

FIGURE 3. Convergence to the upper bound for $M = 1$ when signals are drawn from a logit-normal distribution and shared with $N = 25$ traders.

7.2. **Unbounded signals.** In the rest of this section we place emphasis on unbounded signals; that is, when the support of V is unbounded. Although we do not have a theoretical justification for the existence of a solution for (3.14) in the case of unbounded signals, we were able to arrive at numerical solutions via the above numerical search.

The asymptotic behavior of $F(x)$ for large signals will depend on the tail behavior of the distribution of V. Assuming the interchange of limits and integrals in the proof of Theorem 5.1, we can show that

$$\gamma(x) = \frac{\Psi_x^+(\infty)}{N}\gamma(x) + \frac{N-1}{Nx}\Psi_x^+(\infty)\int_0^x \gamma(y)dy, \quad x > 0,$$

where $\gamma(x) := \lim_{\alpha\to\infty} \frac{F(\alpha x)}{F(\alpha)}$ in case of $M = \infty$. Solution of the above equation immediately yields that, for $N > 1$, F is regularly varying at ∞ of order ρ^+, where

$$\rho^+ = \frac{\Psi_x^+(\infty) - 1}{1 - \frac{\Psi_x^+(\infty)}{N}}. \tag{7.32}$$

Observe that when $M = \infty$, $\Psi_x^+(\infty) \geq 1$ in contrast to the bounded case, where $\Psi_x^+(M) \leq 1$. Since ρ^+ must be non-negative, this places the restriction on N:

$$N > \Psi_x^+(\infty) \tag{7.33}$$

unless $\Psi_x^+(\infty) = 1$. Thus, we conjecture that (7.33) is a necessary condition for the existence of equilibrium when $M = \infty$. Observe that for a fat tailed unbounded distribution, $\Psi_x^+(\infty) > 1$. Thus, a sufficient competition among insiders is necessary for the equilibrium to exist. Such a condition is always satisfied in the bounded case since $\Psi_x^+(M) \leq 1$ for $M < \infty$.

As in the bounded case F will be slowly varying at infinity when $\Psi_x^+(\infty) = 1$. In this case, if we assume

$$\lim_{x\to\infty} (\Psi^+(x) - x)x^{n-1} = \frac{1}{k} \tag{7.34}$$

for some $k > 0$ and $n \geq 1$, formal calculations yield

$$F(x) \sim \left(\frac{N}{N-1}\frac{n}{k}\right)^{\frac{1}{n}} (\log x)^{\frac{1}{n}}, \quad x \to \infty. \tag{7.35}$$

Tables 1 and 2 summarize the predicted asymptotics for a class of distributions commonly used in the literature and practice. As can be observed by Table 1, the impact gets bigger as the return distribution has fatter tails (smaller α). At the same time, the market also needs more competition among the informed traders in order to function. Note that as $\alpha \to 1$, the mean and the variance of the distributions listed therein diverge to ∞. Thus, the liquidity suppliers are only willing to make the market if this excessive risk is shared with a large amount of informed trading.

TABLE 1. Distributions with power-law impact

Distribution	Density	ρ^+
Beta prime	$x^{\lambda-1}(1+x)^{-(\lambda+\alpha)}$	$\left(\frac{N-1}{N}\alpha - 1\right)^{-1}$
Fréchet	$(x-\beta)^{-(1+\alpha)} \exp\left\{-\left(\frac{x-\beta}{s}\right)^{-\alpha}\right\}$	$\left(\frac{N-1}{N}\alpha - 1\right)^{-1}$
Lomax	$\left(1+\frac{x}{\lambda}\right)^{-(\alpha+1)}$	$\left(\frac{N-1}{N}\alpha - 1\right)^{-1}$
Pareto	$x^{-(\alpha+1)}$	$\left(\frac{N-1}{N}\alpha - 1\right)^{-1}$
Student	$\left(1+\frac{x^2}{\alpha}\right)^{-(\alpha+1)/2}$	$\left(\frac{N-1}{N}\alpha - 1\right)^{-1}$

In above probability densities are given up to a scaling factor and implicit constraints are enforced to ensure they are well defined with finite mean. Moreover, $N > \frac{\alpha}{\alpha-1}$ in all of the above.

TABLE 2. Distributions with logarithmic impact

Distribution	Density	Asymptotics
Exponential	$\exp(-\lambda x)$	$\frac{N}{\lambda(N-1)} \log x$
Gaussian	$\exp(-(x-\mu)^2/\Sigma)$	$\sqrt{\frac{2\Sigma N}{N-1}}\sqrt{\log x}$
Inverse Gaussian	$x^{-3/2} \exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right)$	$\frac{2N\mu^2}{\lambda(N-1)} \log x$
Normal Inverse Gaussian	$\frac{K_1(\lambda\zeta(x))}{\pi\zeta(x)} \exp(\delta\gamma + \beta(x-\mu)$	$\frac{N}{(N-1)(\lambda+\beta-1)} \log x$
Weibull	$x^{d-1} \exp(-\lambda^p x^p)$	$\left(\frac{N}{\lambda^p(N-1)}\right)^{1/p} (\log x)^{1/p}$

In above probability densities are given up to a scaling factor and implicit constraints are enforced to ensure they are well defined with finite mean. Moreover, $\zeta(x) := \delta^2 + (x-\mu)^2$ for the Normal Inverse Gaussian distribution.

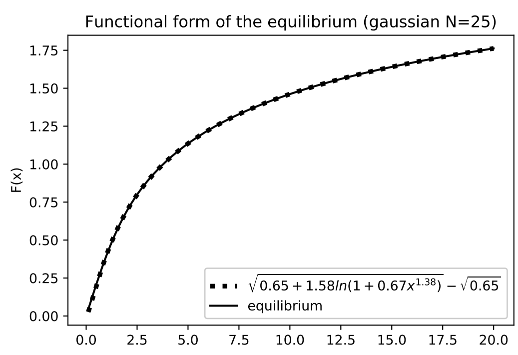

FIGURE 4. Functional form of the equilibrium for Gaussian signals, for $N = 25$.

7.2.1. *Gaussian signals.* We assume that the mean of V equals 0. The numerical solution is shown in Figure 4 together with the $\sqrt{\log x}$ asymptotic behavior.

7.2.2. *Log-normal signals.* For a log-normal distribution, the mean is an arbitrary scale factor which we set to 1 and, thus, the density is $p(v) = \frac{1}{\sqrt{2\pi\Sigma}v} e^{-\frac{(ln(v)+\Sigma/2)^2}{2\Sigma}}$. We choose a large signal variance $\sqrt{\Sigma} = 10\%$ in our numerical experiments below, illustrative of an earnings announcement for a high-volatility name. Moreover, we translate the distribution by 1 so that the mean is 0. The equilibrium solutions for h, F and IS are shown for various values of N in Figure 5.

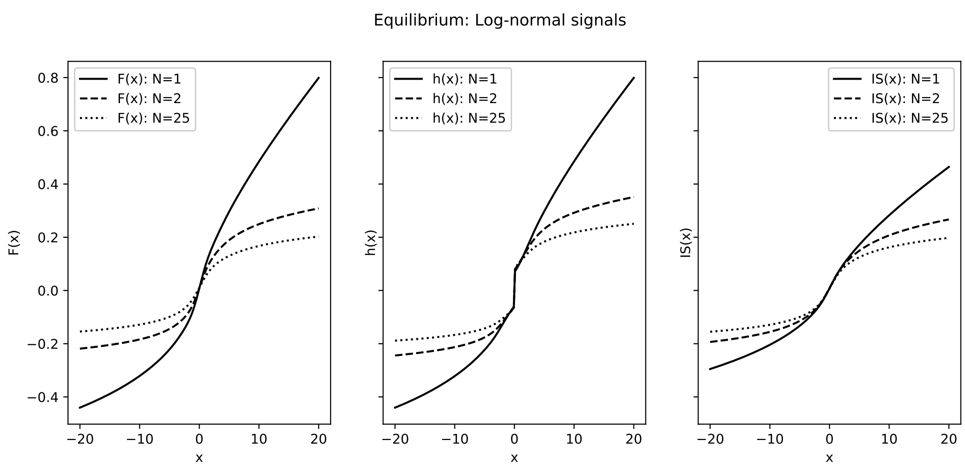

FIGURE 5. Equilibrium solutions for log-normal signals for the cases of an insider ($N = 1$), and shared signals with $N = 2$, $N = 25$. The discontinuity of $h(x)$ at the origin is the bid-ask spread.

The log-normal distribution does not satisfy the conditions of Theorem 5.1. Thus, we do not have a theoretical prediction for the asymptotic market impact. However, we find numerically that the asymptotic form is $\sqrt{\log(x)}$, in Figure 6. Evidence for more concave impact function than the square root was reported in [3]. In addition, one of us (HW) has had the opportunity to consider the shape of the impact function for asset managers with different order generation processes. We cite two relevant (unpublished) examples here. The impact function for a quantitative fund with no discernible intraday alpha is closest to a logarithmic one consistent with the findings of Bershova and Rakhlin. However, for a manager with non-discretionary executions rich in small-cap biopharmaceutical trades and other hard-to-execute orders, we found the log function to be too concave to explain the impact of large trades whereas a square root function fit the data well.

The aggregate profit is a decreasing function of the number of informed investors and we show the profit as a function of trade size below for various values of N in Figure 7. Moreover, Figure 8 shows how the spread depends on the number of informed investors.

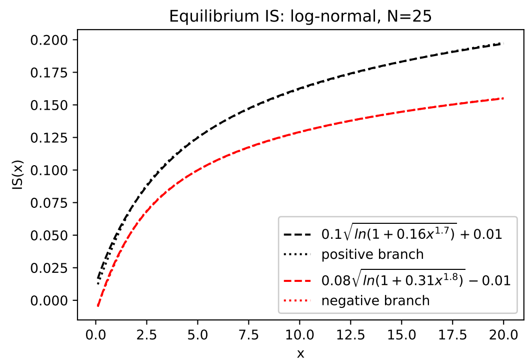

FIGURE 6. Functional form of the equilibrium for Log-normal signals, for $N = 25$. The log-normal distribution is not symmetric and this results in a notable difference between the positive and negative branches for cost as a function of trade size.

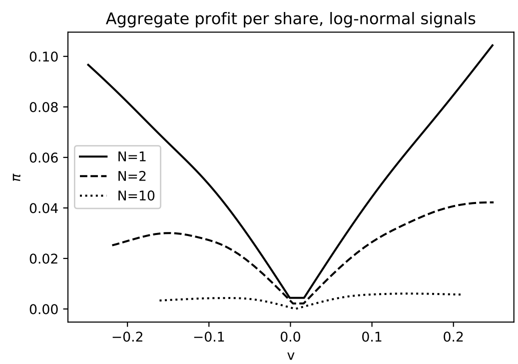

FIGURE 7. Aggregate investor profit per share for Log-normal signals.

7.2.3. *Student signals.* We explore the effect of fat tails in the signal distribution next. We consider the case where signals drawn from a Student t-distribution with $\alpha = 3$, $p(v) \sim \frac{1}{(1+\frac{v^2}{\Sigma})^2}$. This is reminiscent of some empirical studies such as [22]. However, we note that we are assuming a Student distribution of *arithmetic* returns. The power-law tail of geometric returns in Plerou's study implies an infinite expected price.

In view of Table 1 the expected asymptotics is $F(x) \sim x^{1/(\alpha-1-\frac{\alpha}{N})}$. Moreover, our conjecture predicts no equilibrium when $N \leq \frac{\alpha}{\alpha-1} = \frac{3}{2}$. Indeed, no numerical solution for F can be found when $N = 1$. The equilibrium is asymptotically parabolic for $N = 2$ (theoretical $\rho^+ = 2$), linear for $N = 3$ (theoretical $\rho^+ = 1$) and concave for $N \geq 4$. The numerical solutions are shown in Figure 9. For $N = 25$, the asymptotic exponent is $F(x) \sim x^{25/47}$ according to our theory. The numerical solution is compared to this prediction in Figure 10.

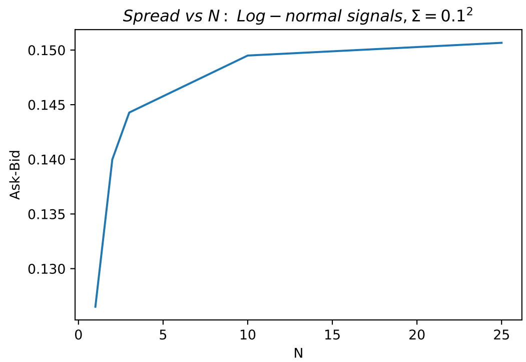

FIGURE 8. The spread is shown as a function of the number of informed investors, for log-normal signals with $\sqrt{\Sigma} = 10\%$.

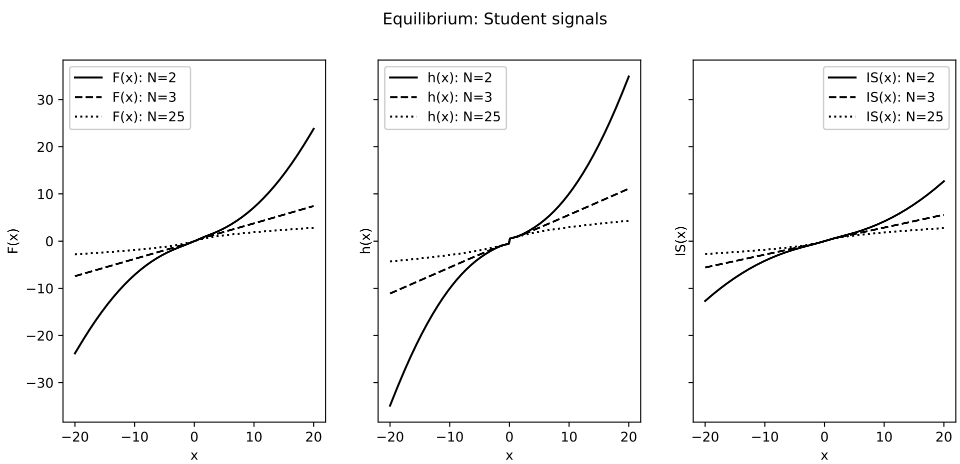

FIGURE 9. Equilibrium solutions for Student signals for the cases $N = 2$, $N = 3$ and $N = 25$.

The case of power-law tailed signal distributions was considered previously by Farmer et al. in the case of perfect competition between insiders [9]. One can view their model as the limiting case of the one considered herein as $N \to \infty$ in case of a Pareto-tailed distribution with exponent 3.

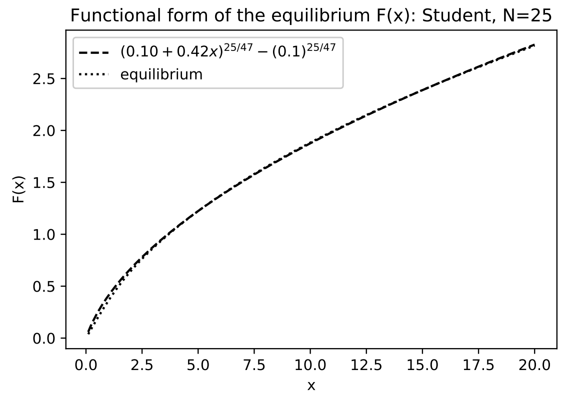

FIGURE 10. Functional form of the equilibrium for Student signals for $\alpha = 3$, $N = 25$.

8. Same-price allocation

An alternative framework also of interest to practitioners is one where portfolio managers and noise traders submit their orders to an aggregator, which merges the orders into a block ("metaorder", in the literature), liquidates $X+Z$ for some average price, and allocates shares with the same average price to noise traders and insiders. We refer to this framework as *same-price allocation*. The expected profit of the insiders in this case is

$$E^v\left[VX - \frac{X}{X+Z}\int_0^{X+Z} h(y)dy\right]. \tag{8.36}$$

In this case the corresponding first order condition for the maximization problem is given by $V = F(X^*)$, where

$$F(x) = E^v\left[\frac{x}{Z+x}\frac{h(Z+x) - \bar{h}(Z+x)}{N} + \bar{h}(Z+x)\right],$$

where $\bar{h}(x) = \frac{1}{x}\int_0^x h(y)dy$ and h is given by the tail expectation as above.

The problem with this first order condition is that it is not clear whether it yields a maximum as F defined above is not necessarily increasing. However, our numerical experiments always suggest an increasing solution yielding a 'numerical' proof of the existence of equilibrium.

The solutions are similar in form and share the same asymptotic behavior for both bounded and unbounded signal distributions. We show in Figure 11 as an example the case of log-normal signals.

Figure 12 compares the same-price allocation model to the trading desk model in the case of Gaussian signals. The dotted lines represent the same-price liquidation equilibria. Market impact is somewhat greater with same-price liquidation than for

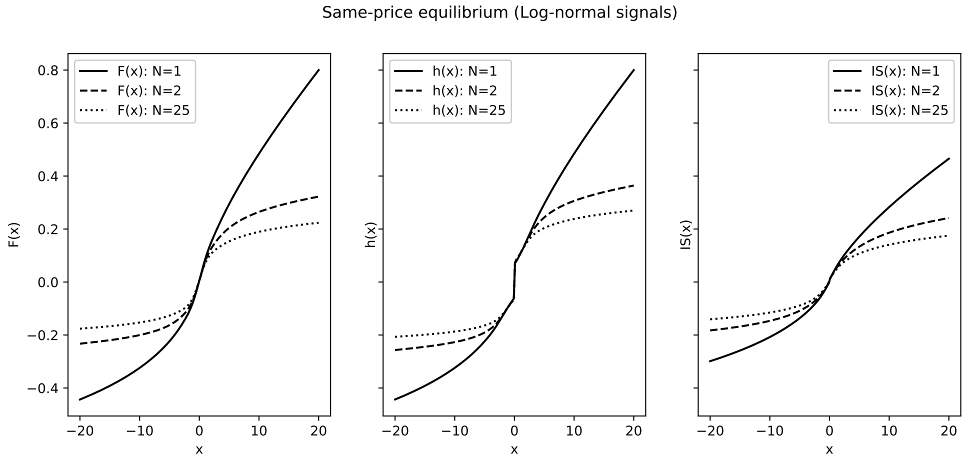

FIGURE 11. Equilibrium solutions for same-price liquidation with log-normal signals for the cases of an insider ($N = 1$), and shared signals with $N = 2$, $N = 25$.

the trading desk model, for $N > 1$. For the insider case, the two are essentially identical.

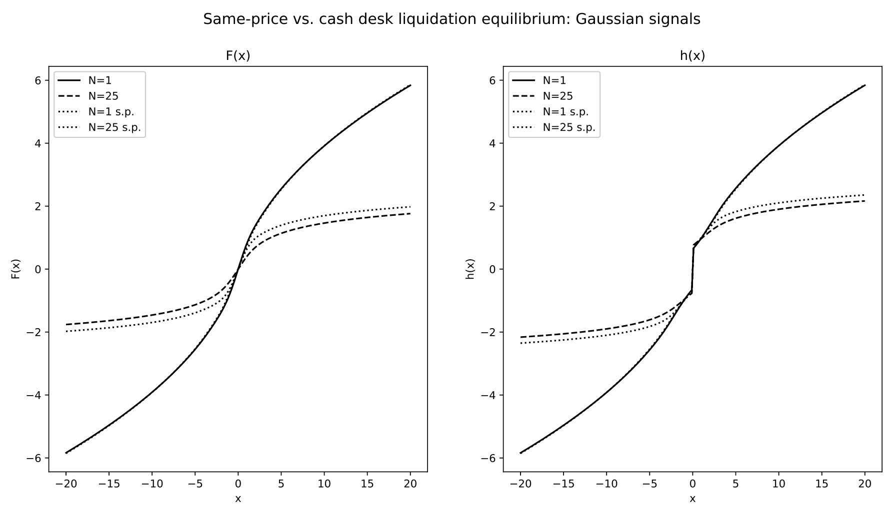

FIGURE 12. Equilibrium for Gaussian signals, comparing the same-price and trading desk liquidation models ($N = 1, 2, 25$). Dotted lines represent the same-price liquidation equilibrium.

9. Conclusion

Since empirical data on very large trades is sparse and often biased, it is important to develop a theoretical understanding of the process in order to discriminate between various proposals for the shape of the impact function. We proposed an asymmetric information based equilibrium model where informed investors draw a signal and send their orders along with noise traders to a trading desk that executes at the net cost to liquidate the aggregate amount against a limit order book. Our results provide the micro-foundations for a large number of empirical findings including those on price impact and volume.

We found that market impact is asymptotically a power of trade size if the signal has fat tails, whereas the impact becomes of the form $(\log x)^{1/p}$ for some $p > 0$ for lighter tails. Moreover, for fat-tailed signal distributions, an equilibrium only exists if there is a sufficient amount of competition. The trade volume also obeys a power law in equilibrium. The spread decreases with the amount of noise trading and, although we do not have an analytic proof, the bid-ask spread seems to be an increasing and bounded function of the number of informed investors.

A relevant and arguably more realistic extension of our framework while still remaining in a static setting is to consider the scenario in which the insiders receive different but possibly correlated signals regarding the liquidation value. On the other hand, due to our assumption that the insiders' orders arrive simultaneously to the desk, the optimization problem of each insider requires the solution of a non-linear filtering problem even in the case of Gaussian signals.

In reality limit order markets are dynamic and thus the order books change over time reflecting the changes in market parameters. The analytic characterization of the equilibrium in the current framework in terms of the fixed point of an integral operator makes us optimistic regarding an extension of the current framework to a dynamic setting in continuous time. However, continuous trading brings extra flexibilities to portfolio choice - including the option to place a market or limit order at each trade - resulting in a more complicated model. This extension, though extremely interesting, will thus be left for future research.

Appendix A. Auxiliary results and proofs

Lemma A.1. *Ψ^+ and Ψ^- are non-decreasing on the support of V.*

Proof. Suppose $x < y$. Note that Ψ^+ is non-decreasing if $E[V\mathbf{1}_{[V>y]}]P(V > x) - E[V\mathbf{1}_{[V>x]}]P(V > y) \geq 0$. Indeed, the left side of the above equals

$$\begin{aligned} & E[(V-y)\mathbf{1}_{[V>y]}]P(V>x) - E[(V-y)\mathbf{1}_{[V>x]}]P(V>y) \\ = \;& E[(V-y)\mathbf{1}_{[V>y]}]\left(P(V>x) - P(V>y)\right) - E[(V-y)\mathbf{1}_{[x<V\leq y]}]P(V>y), \end{aligned}$$

which is non-negative since $V - y \leq 0$ on the set $[x < V \leq y]$.

The second assertion is proved analogously. □

Proof of Proposition 2.1. (1) Let $g(x) := E^v[h(x+Z)]$. By direct differentiation, the expression (2.4) implies

$$xF(x) = x\frac{g(x)}{N} + \frac{N-1}{N}\int_0^x g(y)dy, \tag{A.37}$$

which is equivalent to the ODE $x\frac{g'(x)}{N} + g = F(x) + xF'(x)$.

Recall that $g(0) = E^v[h(Z)] = F(0)$ by construction. Thus, the unique solution of the above ODE with this initial condition is given by

$$g(x) = NF(x) - \frac{N(N-1)}{x^N}\int_0^x F(y)y^{N-1}dy = F(x) + \frac{N(N-1)}{x^N}\int_0^x (F(x)-F(y))y^{N-1}dy.$$

This yields (2.6) after a change of variable.

(2) The above also yields (2.7) due to the first order condition $F(X^*) = V$. The remaining assertions are direct consequences of the strict monotonicity of F.

(3) Note that the total expected profit is given by

$$\begin{aligned}
&\int_0^{X^*} (v - E^v[h(y+Z)])dy = \int_0^{X^*} (v - g(y))dy \\
= \ & vX^* - \frac{N}{N-1}\left(X^*F(X^*) - X^*\frac{g(X^*)}{N}\right) = -\frac{vX^*}{N-1} + \frac{X^*g(X^*)}{N-1} \\
= \ & \frac{X^*}{N-1}(g(X^*) - v) = N\int_0^{X^*} (v - F(y))\left(\frac{y}{X^*}\right)^{N-1} dy,
\end{aligned}$$

where the second equality follows from (A.37) and the third is due to $F(X^*) = V$.

(4) Observe that

$$\int_0^Z h(y)dy = \int_0^\infty h(y)\mathbf{1}_{[Z\geq y]}dy - \int_{-\infty}^0 h(y)\mathbf{1}_{[Z\leq y]}dy.$$

Taking expectations and using the symmetry of Z we obtain

$$E\left[\int_0^Z h(y)dy\right] = \int_0^\infty (h(y) - h(-y))P(Z \geq y)dy = \int_0^\infty \frac{h(y)-h(-y)}{2}P(|Z| \geq y)dy$$

and prove the first claim. The second claim follows directly from integration by parts. □

Proof of Proposition 2.2. Note that $E[h(y+Z)] = E^v[h(y+Z)]$ for all y. We shall show the result for $N > 1$, the remaining case is similar and easier. Using the first representation in (2.6), we obtain

$$\begin{aligned}
\int_0^x E[h(y+Z)]dy &= N\int_0^x F(y)dy - N(N-1)\int_0^x dy y^{-N}\int_0^y dz F(z)z^{N-1} \\
&= N\int_0^x F(y)dy - N(N-1)\int_0^x dz F(z)z^{N-1}\int_z^x dy y^{-N} \\
&= N\int_0^x F(y)dy + N\int_0^x dz F(z)z^{N-1}\left(x^{-N+1} - z^{-N+1}\right) \\
&= N\int_0^x dz F(z)\left(\frac{z}{x}\right)^{N-1} = xN\int_0^1 dz F(xz)z^{N-1},
\end{aligned}$$

which yields the first assertion once divided by x. The remaining claims follow from $F(xy) < F(x)$ (resp. $F(xy) > F(x)$) for $x > 0$ (resp. $x < 0$) and for all $y \in (0,1)$ since F is strictly increasing. □

Proof of Lemma 3.1. We give the proof for the solutions of (3.14), the analogous property of the solutions of (6.30) can be proven similarly.

Monotone convergence theorem in conjunction with Assumption 3.1 implies

$$\lim_{x\to\infty}\int_{-\infty}^{\infty} q(\sigma,z)\phi_F(x+z)dz = \int_{-\infty}^{\infty} q(\sigma,z)\lim_{x\to\infty}\phi_F(x+z)dz.$$

Moreover, $\lim_{x\to\infty}\phi_F(x+z)=\Psi^+(F(\infty)-)$. To see this, first note that

$$\phi_F^+(z+x)=\frac{\int_{-\infty}^{\infty}\Psi^+(F(u))\Pi^+(F(u))\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x+z-u)^2}{2\sigma^2}\right)du}{\int_{-\infty}^{\infty}\Pi^+(F(u))\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x+z-u)^2}{2\sigma^2}\right)du}.$$

Next, the measure

$$\frac{\Pi^+(F(u)q(\sigma,z+x-u)du}{\int_{-\infty}^{\infty}\Pi^+(F(u)q(\sigma,z+x-u)du}$$

converges to the point mass at ∞. Indeed, for any $0<a<\infty$, we have

$$\begin{aligned}\frac{\int_{-\infty}^{a}\Pi^+(F(u)q(\sigma,z+x-u)du}{\int_{-\infty}^{\infty}\Pi^+(F(u)q(\sigma,z+x-u)du} &= \frac{\int_{-\infty}^{a}\Pi^+(F(u)\exp\big(-\frac{u^2}{2\sigma^2}+\frac{u(z+x)}{\sigma^2}\big)du}{\int_{-\infty}^{\infty}\Pi^+(F(u)\exp\big(-\frac{u^2}{2\sigma^2}+\frac{u(z+x)}{\sigma^2}\big)du}\\ &\le \frac{\exp(\frac{a(z+x)}{\sigma^2})\int_{-\infty}^{a}\Pi^+(F(u)\exp(-\frac{u^2}{2\sigma^2})du}{\int_{2a}^{\infty}\Pi^+(F(u)\exp\big(-\frac{u^2}{2\sigma^2}+\frac{u(z+x)}{\sigma^2}\big)du}\\ &\le \frac{\exp(\frac{-a(z+x)}{\sigma^2})\int_{-\infty}^{a}\Pi^+(F(u)\exp(-\frac{u^2}{2\sigma^2})du}{\int_{2a}^{\infty}\Pi^+(F(u)\exp\big(-\frac{u^2}{2\sigma^2}\big)du},\end{aligned}$$

which converges to 0 as $x\to\infty$.

Thus, using the representation of F via (3.15), we deduce $F(\infty)=\Psi^+(F(\infty)-)$. Note that changing the order of integration is justified thanks to Assumption 3.1. On the other hand, $\Psi^+(x-)>x$ for any $x<M$ since $P(V>x)>0$ whenever $x<M$. This in turn implies $F(\infty)=M$. Similarly, $\lim_{x\to-\infty}F(x)=m$. Thus, F is strictly increasing in view of (3.15) and Lemma 4.1 since $m\neq M$ and, therefore, ϕ_F is not constant. □

Appendix B. Proof of impact asymptotics

We shall first start with the asymptotics of solutions of (4.23) and (4.24) which will later allow us to compute the asymptotics of interest.

Theorem B.1. *Suppose that $N>1$, $-\infty<m<M<\infty$, and Assumption 5.1 holds. Then, the following statements are valid:*

(1) Set $G:=M-R$, where R is any nondecreasing solution of (4.23). Then, G is regularly varying of index $\rho^+=\frac{\Psi_x^+(M)-1}{1-\frac{\Psi_x^+(M)}{N}}$.

(2) Set $G:=l-m$, where l is any nondecreasing solution of (4.24). Then, G is regularly varying of index $\rho^-=\frac{\Psi_x^-(m)-1}{1-\frac{\Psi_x^-(m)}{N}}$.

Proof. Only the proof of the first statement will be given, proof of the second being similar. We can assume without loss of generality that $E[V]=0$ since, otherwise, we can replace V by $V-E[V]$ and redefine Ψ^+ and R accordingly.

By means of a straightforward change of variable we obtain for $x>0$

$$\begin{aligned}\frac{G(\alpha x)}{G(\alpha)} &= \frac{1}{N}\int_0^\infty dzq\Big(\frac{\sigma}{\alpha},x-z\Big)\int_{-\infty}^\infty dyq\Big(\frac{\sigma}{\alpha},z-y\Big)\frac{M-\Psi^+(R(\alpha y))}{M-R(\alpha)}\\ &+\frac{N-1}{N}\int_0^\infty dz\bar{q}\Big(\frac{\sigma}{\alpha},x,z\Big)\int_{-\infty}^\infty dyq\Big(\frac{\sigma}{\alpha},y-z\Big)\frac{M-\Psi^+(R(\alpha y))}{M-R(\alpha)}\\ &+\frac{M}{M-R(\alpha)}\int_{-\infty}^0 dz\left\{\frac{1}{N}q\Big(\frac{\sigma}{\alpha},x-z\Big)+\frac{N-1}{N}\bar{q}\Big(\frac{\sigma}{\alpha},x,z\Big)\right\}\end{aligned} \tag{B.38}$$

Observe that the measure $q(\frac{\sigma}{\alpha},x-z)dz$ converges to the Dirac measure at point x as $\alpha\to\infty$.

Step 1: For $y>0$

$$\frac{(M-\Psi^+(R(\alpha y))}{M-R(\alpha)}=\frac{(M-R(\alpha y))\Psi_x^+(y^*)}{M-R(\alpha)}$$

for some $y^*\geq R(\alpha y)$ by the Mean Value Theorem, where Ψ_x^+ stands for the derivative of Ψ^+, since $\Psi^+(M)=M$. Thus,

$$\lim_{\alpha\to\infty}\frac{(M-\Psi^+(R(\alpha y))}{M-R(\alpha)}=\frac{(M-R(\alpha y))\Psi_x^+(y^*)}{M-R(\alpha)}=\Psi_x^+(M)\lim_{\alpha\to\infty}\frac{G(\alpha y)}{G(\alpha)} \tag{B.39}$$

as $R(\infty)=M$.

Step 2: Let $\gamma_*(x):=\liminf_{\alpha\to\infty}\frac{G(\alpha x)}{G(\alpha)}$. Then, in view of the first step, Fatou's lemma yields

$$\gamma_*(x)\geq\frac{\Psi_x^+(M)}{N}\gamma_*(x)+\frac{N-1}{Nx}\Psi_x^+(M)\int_0^x\gamma_*(y)dy.$$

Thus, if $\gamma^*(x)=0$ for some $x>0$, it must be 0 for almost all $x>0$. However, $\gamma_*(x)\geq 1$ for all $x\in(0,1)$ since G is decreasing. Thus, $\gamma_*>0$ for all $x>0$. In particular, γ_* is bounded away from 0 on $[0,n]$ for any $n\geq 1$.

Step 3: It follows from Step 2 and Corollary 2.0.5 in [5] that

$$\frac{x^d}{C}\leq\frac{G(\alpha x)}{G(\alpha)}\leq Cx^c\ \text{for} x\geq 1\text{ and }\alpha\geq\alpha_0$$

for some constants c,d,C and α_0. Moreover, since for $x<1$ we have

$$\frac{G(\alpha x)}{G(\alpha)}=\left(\frac{G(\alpha x x^{-1})}{G(\alpha x))}\right)^{-1},$$

and $\alpha x>\alpha_0$ for large enough α, we deduce that the mapping $(\alpha,x)\mapsto\frac{G(\alpha x)}{G(\alpha)}$ is bounded when x belongs to bounded intervals in $(0,\infty)$.

Step 4: Moreover,

$$(M-R(\alpha))\alpha\geq\frac{N-1}{N}\int_0^\alpha du\int_0^\infty dzq(\sigma,u-z)\int_{-\infty}^\infty dy(M-\Psi^+(R(y)))q(\sigma,z-y),$$

which in turn implies

$$\frac{1}{M - R(\alpha)} \leq K\alpha, \qquad \alpha > 1 \text{ for some } K < \infty. \tag{B.40}$$

Since $\frac{G(\alpha x)}{G(\alpha)}$ is bounded when x belongs to bounded intervals in $(0, \infty)$ by Step 3, we obtain, for any $\varepsilon > 0$,

$$\lim_{\alpha\to\infty} \int_{-\infty}^{\infty} dz q\Big(\frac{\sigma}{\alpha}, x-z\Big) \int_{z-\varepsilon}^{z+\varepsilon} dy q\Big(\frac{\sigma}{\alpha}, z-y\Big) \frac{M - \Psi^+(R(\alpha y))}{M - R(\alpha)} = \Psi_x^+(M)\gamma(x), \tag{B.41}$$

where $\gamma(x) := \lim_{\alpha\to\infty} \frac{G(\alpha x)}{G(\alpha)}$ in view of (B.39) provided that the limit exists.

Furthermore, in view of (B.40) we also have

$$\begin{aligned} &\int_{-\infty}^{\infty} dz q\Big(\frac{\sigma}{\alpha}, x-z\Big) \int_{\mathbb{R}\setminus(z-\varepsilon,z+\varepsilon)} dy q\Big(\frac{\sigma}{\alpha}, z-y\Big) \frac{M - \Psi^+(R(\alpha y))}{M - R(\alpha)} \\ \leq\ & KM \int_{-\infty}^{\infty} dz q\Big(\frac{\sigma}{\alpha}, x-z\Big) \int_{\mathbb{R}\setminus(z-\varepsilon,z+\varepsilon)} dy \alpha q\Big(\frac{\sigma}{\alpha}, z-y\Big) \\ \to\ & 0 \text{ as } \alpha \to \infty. \end{aligned} \tag{B.42}$$

Step 5: Applying the arguments of Step 4 to the second and the third integrals in (B.38) now shows that for $x > 0$

$$\gamma(x) = \frac{\Psi_x^+(M)}{N}\gamma(x) + \frac{N-1}{Nx}\Psi_x^+(M) \int_0^x \gamma(y)dy. \tag{B.43}$$

In particular, $\lim_{\alpha\to\infty} \frac{G(\alpha x)}{G(\alpha)}$ exists. Using the initial condition that $\gamma(1) = 1$, direct manipulations show that

$$\gamma(x) = x^{\rho^+}.$$

□

Theorem B.2. *Assume that $N > 1$, $-\infty < m < M < \infty$, Π^+ has a continuous derivative and Assumption 5.1 holds. Then, the following statements are valid:*

(1) Suppose that $\Psi_x^+(M) = 1$ and there exist an integer $n \geq 1$ and a real constant $k \in (0, \infty)$ such that

$$\lim_{x\to M} \frac{\Psi^+(x) - x}{(M - x)^{n+1}} = \frac{1}{k}.$$

Then, the following asymptotics[9] hold:

$$M - R(x) \sim \left(\frac{N}{N-1}\frac{n}{k}\right)^{-\frac{1}{n}} (\log x)^{-\frac{1}{n}}, \quad x \to \infty, \tag{B.44}$$

where R is any solution of (4.23).

[9] We write $f(x) \sim g(x)$, $x \to \pm\infty$ if $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = 1$.

(2) Suppose that $\Psi^-_x(m) = 1$ and there exist an integer $n \geq 1$ and a real constant $k \in (0, \infty)$ such that

$$\lim_{x \to m} \frac{x - \Psi^-(x)}{(x-m)^{n+1}} = \frac{1}{k}.$$

Then

$$l(x) - m \sim \left(\frac{N}{N-1}\frac{n}{k}\right)^{-\frac{1}{n}} (\log|x|)^{-\frac{1}{n}}, \quad x \to -\infty, \tag{B.45}$$

where l is any solution of (4.24).

Proof. We shall again only give the proof the first statement and assume without loss of generality that $E[V] = 0$. For any $\alpha > 0$ define

$$r(\alpha, x) := \frac{R(\alpha x) - R(\alpha)}{(M - R(\alpha))^{n+1}}.$$

Straightforward manipulations similar to the ones employed in the proof of Theorem B.1 leads to

$$\begin{aligned}
r(\alpha, x) &= -\frac{R(\alpha)}{(M-R(\alpha))^{n+1}} \int_{-\infty}^{0} dz \left\{ \frac{1}{N} q\Big(\frac{\sigma}{\alpha}, x - z\Big) + \frac{N-1}{N} \bar{q}\Big(\frac{\sigma}{\alpha}, x, z\Big) \right\} \\
&+ \int_0^\infty dz \left\{ \frac{1}{N} q\Big(\frac{\sigma}{\alpha}, x - z\Big) + \frac{N-1}{N} \bar{q}\Big(\frac{\sigma}{\alpha}, x, z\Big) \right\} \int_1^\infty dy q\Big(\frac{\sigma}{\alpha}, z - y\Big) \frac{\Psi^+(R(\alpha y)) - R(\alpha)}{(M - R(\alpha))^{n+1}} \\
&+ \int_0^\infty dz \left\{ \frac{1}{N} q\Big(\frac{\sigma}{\alpha}, x - z\Big) + \frac{N-1}{N} \bar{q}\Big(\frac{\sigma}{\alpha}, x, z\Big) \right\} \int_{-\infty}^1 dy q\Big(\frac{\sigma}{\alpha}, z - y\Big) \frac{\Psi^+(R(\alpha y)) - R(\alpha)}{(M - R(\alpha))^{n+1}} \\
&\leq \int_0^\infty dz \left\{ \frac{1}{N} q\Big(\frac{\sigma}{\alpha}, x - z\Big) + \frac{N-1}{N} \bar{q}\Big(\frac{\sigma}{\alpha}, x, z\Big) \right\} \int_1^\infty dy q\Big(\frac{\sigma}{\alpha}, z - y\Big) \frac{\Psi^+(R(\alpha y)) - R(\alpha)}{(M - R(\alpha))^{n+1}} \\
&+ \int_0^\infty dz \left\{ \frac{1}{N} q\Big(\frac{\sigma}{\alpha}, x - z\Big) + \frac{N-1}{N} \bar{q}\Big(\frac{\sigma}{\alpha}, x, z\Big) \right\} \int_{-\infty}^1 dy q\Big(\frac{\sigma}{\alpha}, z - y\Big) \frac{\Psi^+(R(\alpha)) - R(\alpha)}{(M - R(\alpha))^{n+1}} \\
&\leq K + \int_0^\infty dz \left\{ \frac{1}{N} q\Big(\frac{\sigma}{\alpha}, x - z\Big) + \frac{N-1}{N} \bar{q}\Big(\frac{\sigma}{\alpha}, x, z\Big) \right\} \int_1^\infty dy q\Big(\frac{\sigma}{\alpha}, z - y\Big) \frac{\Psi^+(R(\alpha y)) - R(\alpha)}{(M - R(\alpha))^{n+1}}
\end{aligned}$$

for some $K > 0$ independent of α, where the first inequality follows from that $R \geq E[V] = 0$ and $R(\alpha y) \leq R(\alpha)$ for $y \leq 1$ and the second is due to the boundedness of $\frac{\Psi^+(R(\alpha y)) - R(\alpha)}{(M - R(\alpha))^{n+1}}$.

Moreover, for $y > 1$

$$\begin{aligned}
\frac{\Psi^+(R(\alpha y)) - R(\alpha)}{(M - R(\alpha))^{n+1}} &= \frac{\Psi^+(R(\alpha y)) - R(\alpha y)}{(M - R(\alpha))^{n+1}} + r(\alpha, y) \\
&\leq \frac{\Psi^+(R(\alpha y)) - R(\alpha y)}{(M - R(\alpha y))^{n+1}} + r(\alpha, y)
\end{aligned}$$

due to the monotonicity of R. Thus, utilizing the boundedness of $\frac{\Psi^+(R(\alpha y))-R(\alpha)}{(M-R(\alpha))^{n+1}}$ once more, we arrive at

$$r(\alpha,x)\le\kappa+\int_0^\infty dz\left\{\frac{1}{N}q\Big(\frac{\sigma}{\alpha},x-z\Big)+\frac{N-1}{N}\bar{q}\Big(\frac{\sigma}{\alpha},x,z\Big)\right\}\int_1^\infty dyq\Big(\frac{\sigma}{\alpha},z-y\Big)r(\alpha,y). \tag{B.46}$$

for some $\kappa\in(0,\infty)$ that is independent of α.

Step 1: Let κ be as above and consider the operator $T:C([1,\infty),[0,\infty))\to C([1,\infty),[0,\infty))$ defined by

$$Tf(x)=\kappa+\int_0^\infty dz\left\{\frac{1}{N}q\Big(\frac{\sigma}{\alpha},x-z\Big)+\frac{N-1}{N}\bar{q}\Big(\frac{\sigma}{\alpha},x,z\Big)\right\}\int_1^\infty dyq\Big(\frac{\sigma}{\alpha},z-y\Big)f(y).$$

Clearly T is increasing, i.e. $Tf\ge Tg$ if $f\ge g$.

Step 2: For $z\ge 0$ and $c\in\mathbb{R}$,

$$\int_c^\infty dyq\Big(\frac{\sigma}{\alpha},z-y\Big)y=\frac{\sigma^2}{\alpha^2}q\Big(\frac{\sigma}{\alpha},z-c\Big)+z\int_{c-z}^\infty dyq\Big(\frac{\sigma}{\alpha},y\Big)\le z+\frac{\sigma}{\alpha\sqrt{2\pi}}. \tag{B.47}$$

Thus, if $f(x)=\beta x+\delta$ for some $\beta>0$ and $\delta\ge 0$, we have

$$\begin{aligned}Tf(x)&\le\delta+\kappa+\frac{\sqrt{2}\sigma\beta}{\alpha\sqrt{\pi}}+\frac{\beta x}{N}+\beta\frac{N-1}{xN}\int_0^x ydy\\&=\delta+\kappa+\frac{\sqrt{2}\sigma\beta}{\alpha\sqrt{\pi}}+\frac{\beta x(N+1)}{2N}\le\delta+\kappa+\beta\left(\frac{\sqrt{2}\sigma}{\alpha\sqrt{\pi}}+\frac{3x}{4}\right),\end{aligned}$$

where the last inequality is due to the hypothesis that $N\ge 2$.

In particular, given $\beta=\gamma\kappa$ for some $\gamma>4$, $Tf\le f$ whenever $\alpha\ge\frac{4\sqrt{2}\sigma\gamma}{\sqrt{\pi}(\gamma-4)}$.

Step 3: Define

$$\mathcal{D}(\alpha):=\{f:[1,\infty)\to[0,\infty):f\text{ is continuous and }f(x)\le 5\kappa x+\frac{1}{(M-R(\alpha))^n}\forall x\ge 1\}.$$

and observe that the restriction of $r(\alpha,\cdot)$ to $[1,\infty)$ belongs to $\mathcal{D}(\alpha)$ for all $\alpha>0$.

In view of Step 2 we have that $T\mathcal{D}(\alpha)\subset\mathcal{D}(\alpha)$ for large enough α. Thus, it admits a fixed point in $\mathcal{D}(\alpha)$ for large enough α by Tarski's theorem (Theorem 1 in [27]). In fact, it admits a unique fixed point. Indeed, if f and g are two fixed points of T in $\mathcal{D}(\alpha)$,

$$(f-g)(x)=\int_0^\infty dz\left\{\frac{1}{N}q\Big(\frac{\sigma}{\alpha},x-z\Big)+\frac{N-1}{N}\bar{q}\Big(\frac{\sigma}{\alpha},x,z\Big)\right\}\int_1^\infty dyq\Big(\frac{\sigma}{\alpha},z-y\Big)(f-g)(y).$$

Thus,

$$\begin{aligned}\inf_{x\ge1}(f-g)(x)&\ge\inf_{x\ge1}(f-g)(x)\inf_{x\ge1}\int_0^\infty dz\left\{\frac{1}{N}q\Big(\frac{\sigma}{\alpha},x-z\Big)+\frac{N-1}{N}\bar{q}\Big(\frac{\sigma}{\alpha},x,z\Big)\right\}\int_1^\infty dyq\Big(\frac{\sigma}{\alpha},z-y\Big)\\&=c\inf_{x\ge1}(f-g)(x)\end{aligned}$$

for some $c\in(0,1)$. Since $\inf_{x\ge1}(f-g)(x)<\infty$, the above implies $\inf_{x\ge1}(f-g)(x)\le 0$. Applying the same argument to $g-f$, we deduce $f=g$.

Moreover, taking $f^*(x) = 5\kappa x$ and utilizing the previous step we deduce that the unique fixed point is bounded from above by f^*. Theorem 1 in [27] now yields that $r(\alpha, \cdot) \le f^*$ for large enough α in view of (B.46).

Step 4: In view of Step 3 we have that $r(\alpha, x) \le 10\kappa$ for all $x \in [1, 2]$ for large enough α. Thus, Theorem 3.1.5 in [5] yields $r(\alpha, \cdot)$ is bounded in bounded subintervals of $(0, \infty)$ uniformly in α.

Step 5: In view of Step 4 and using arguments similar to the ones used in Steps 4 and 5 of the proof of Theorem B.1, we arrive at

$$\gamma(x) = \frac{1}{k} + \frac{\gamma(x)}{N} + \frac{N-1}{xN} \int_0^x \gamma(y)dy,$$

where $\gamma(x) := \lim_{\alpha \to \infty} r(\alpha, x)$ for $x > 0$. The unique solution of the above equation with $\gamma(1) = 0$ is given by

$$\gamma(x) = \frac{N}{k(N-1)} \log x.$$

Step 6: Let $g(x) := \exp\left((M - R(x))^{-n}\right)$ and observe that

$$\begin{aligned}
\frac{g(\alpha x)}{g(\alpha)} &= \exp\left(\frac{(M - R(\alpha))^n - (M - R(\alpha x))^n}{(M - R(\alpha))^n (M - R(\alpha x))^n}\right) \\
&= \exp\left(\frac{(R(\alpha x) - R(\alpha)) \sum_{i=0}^{n-1} (M - R(\alpha)^{n-1-i} (M - R(\alpha x)^i}{(M - R(\alpha))^n (M - R(\alpha x))^n}\right) \\
&= \exp\left(\frac{R(\alpha x) - R(\alpha)}{(M - R(\alpha))^{n+1}} \frac{M - R(\alpha)}{M - R(\alpha x)} \sum_{i=0}^{n-1} \left(\frac{M - R(\alpha)}{M - R(\alpha x)}\right)^{n-1-i}\right),
\end{aligned}$$

which converges to $\exp(n\gamma(x))$ as $\alpha \to \infty$ since $M - R$ is slowly varying at ∞. Thus,

$$g(x) = x^{\frac{N}{N-1} \frac{n}{k}} s(x), \quad x > 0.$$

where s is a slowly varying function at ∞. That is,

$$(M - R(x))^{-n} = \frac{N}{N-1} \frac{n}{k} \log x + \log s(x).$$

Since $\lim_{x \to \infty} \frac{\log s(x)}{\log x} = 0$ (cf. Proposition 1.3.6 (i) and (iii) in [5]), we have

$$M - R(x) \sim \left(\frac{N}{N-1} \frac{n}{k}\right)^{-\frac{1}{n}} (\log x)^{-\frac{1}{n}}, \quad x \to \infty.$$

□

Corollary B.1. *Assume that $N > 1$, $-\infty < m < M < \infty$, and Assumption 5.1 holds. Then, the following statements are valid:*

(1) Suppose that there exist an integer $n \ge 0$ and a real constant $k \in (0, \infty)$ such that

$$\lim_{x \to M} \frac{\Psi^+(x) - x}{(M - x)^{n+1}} = \frac{1}{k}.$$

Then, $\Pi^+(R)$ is regularly varying at ∞ of index $-\frac{\Psi_x^+(M)}{1-\frac{\Psi_x^+(M)}{N}}$, where R is any solution of (4.23).

(2) *Suppose that there exist an integer $n \geq 0$ and a real constant $k \in (0, \infty)$ such that*

$$\lim_{x \to M} \frac{x - \Psi^-(x)}{(x-m)^{n+1}} = \frac{1}{k}.$$

Then, $\Pi^-(l)$ is regularly varying at $-\infty$ of index $-\frac{\Psi_x^-(m)}{1-\frac{\Psi_x^-(m)}{N}}$, where l is any solution of (4.24).

Proof. Again we only prove the first statement. The hypothesis implies $\Psi^+(x) - x$ is regularly varying with index $n+1$ at M. Thus, $\Psi^+(R) - R$ is regularly varying of index $(n+1)\rho^+$ at ∞, which in particular implies

$$\lim_{x\to\infty} \frac{\Psi^+(R(x)) - R(x)}{\int_x^\infty \frac{\Psi^+(R(y))-R(y)}{y}} dy = -\lim_{x\to\infty} \frac{x(\Psi_x^+(R(x)R'(x) - R'(x))}{\Psi^+(R(x)) - R(x)} = -(n+1)\rho^+, \quad \text{(B.48)}$$

in view of Theorem 1.5.11(ii) in [5].

Moreover, direct manipulations yield $\Pi^+(x) = \frac{\int_x^M \Pi^+(y)dy}{\Psi^+(x)-x}$, which in turn implies $-\frac{\Pi_x^+(x)}{\Pi^+(x)} = \frac{\Psi_x^+(x)}{\Psi^+(x)-x}$. Therefore,

$$\begin{aligned}
\lim_{x\to\infty} \frac{x\Pi_x^+(R(x))R'(x)}{\Pi^+(R(x))} &= -\lim_{x\to\infty} \frac{xR'(x)\Psi_x^+(R(x))}{\Psi^+(R(x)) - R(x)} \\
&= -(n+1)\rho^+ - \lim_{x\to\infty} \frac{xR'(x)}{\Psi^+(R(x)) - R(x)} \\
&= -(n+1)\rho^+ - k \lim_{x\to\infty} \frac{xR'(x)}{(M - R(x))^{n+1}},
\end{aligned}$$

where the second equality follows from (B.48).

Note that if $\Psi_x^+(M) < 1$, $n = 0$ and $k = \frac{1}{1-\Psi^+(M)}$. In this case,

$$\lim_{x\to\infty} \frac{xR'(x)}{M - R(x)} = -\rho^+$$

by Theorem Theorem 1.5.11(ii) in [5]. Thus,

$$\lim_{x\to\infty} \frac{x\Pi_x^+(R(x))R'(x)}{\Pi^+(R(x)} = \rho^+(k-1) = -\frac{\Psi_x^+(M)}{1 - \frac{\Psi_x^+(M)}{N}}.$$

An application of Exercise 1.11.13 in [5] to $1/\Pi^+(R)$ establishes the claim.

Now, suppose $\Psi_x^+(M) = 1$ and observe that n is necessarily bigger than 0 and $\rho^+ = 0$ in this case. Recall the function g in Step 6 of the proof of Theorem B.2 and note that

$$\lim_{x\to\infty} \frac{xg'(x)}{g(x)} = \frac{Nn}{(N-1)k}$$

by another application of Theorem 1.5.11(i) in [5]. Since

$$\frac{xg'(x)}{g(x)} = \frac{xnR'(x)}{(M-R(x))^{n+1}},$$

we arrive at

$$\lim_{x\to\infty} \frac{xR'(x)}{(M-R(x))^{n+1}} = \frac{N}{k(N-1)}.$$

Therefore,

$$\lim_{x\to\infty} \frac{x\Pi_x^+(R(x))R'(x)}{\Pi^+(R(x)} = -\frac{N}{N-1} = -\frac{\Psi_x^+(M)}{1-\frac{\Psi_x^+(M)}{N}},$$

and we again conclude by means of Exercise 1.11.13 in [5]. □

We are now ready to prove the asymptotics that are of our main interest.

Proof of Theorem 5.1. We shall give a proof of the first statement as the second one can be proven along similar lines.

By means of a change of variable employed in earlier proofs we obtain for $x > 0$

$$\begin{aligned}\frac{G(\alpha x)}{G(\alpha)} \quad &= \frac{1}{N}\int_0^\infty dzq\Big(\frac{\sigma}{\alpha}, x-z\Big)\int_{-\infty}^\infty \nu(\alpha,z,dy)dy\frac{M-\Psi^+(F(\alpha y))}{M-F(\alpha)} \\ &+\frac{N-1}{N}\int_0^\infty dz\bar{q}\Big(\frac{\sigma}{\alpha}, x, z\Big)\int_{-\infty}^\infty \nu(\alpha,z,dy)dy\frac{M-\Psi^+(F(\alpha y))}{M-F(\alpha)} \qquad \text{(B.49)} \\ &+\frac{1}{M-F(\alpha)}\int_{-\infty}^0 dz\left\{\frac{1}{N}q\Big(\frac{\sigma}{\alpha}, x-z\Big)+\frac{N-1}{N}\bar{q}\Big(\frac{\sigma}{\alpha}, x, z\Big)\right\}\phi_F(\alpha z),\end{aligned}$$

where

$$\nu(\alpha,z,dy) := \frac{\Pi^+(F(\alpha y)q\Big(\frac{\sigma}{\alpha}, y-z\Big))}{\int_{-\infty}^\infty du\Pi^+(F(\alpha u)q\Big(\frac{\sigma}{\alpha}, u-z\Big))}dy. \qquad \text{(B.50)}$$

We shall first demonstrate that for $z > 0$ the measure $\nu(\alpha, z, dy)$ converges to the Dirac measure at point z as $\alpha \to \infty$. Indeed, let R^* be the maximal solution of (4.23) and observe that for any $\varepsilon > 0$ and $z > 0$

$$\nu(\alpha,z,(\mathbb{R}\backslash(z-\varepsilon,z+\varepsilon))) = \frac{\int_{\mathbb{R}\backslash(z-\varepsilon,z+\varepsilon)} dy\Pi^+(F(\alpha y)q\Big(\frac{\sigma}{\alpha}, y-z\Big)}{\int_{-\infty}^\infty du\Pi^+(F(\alpha u)q\Big(\frac{\sigma}{\alpha}, u-z\Big)} \le \frac{\int_{\mathbb{R}\backslash(z-\varepsilon,z+\varepsilon)} dyq\Big(\frac{\sigma}{\alpha}, y-z\Big)}{\int_0^\infty du\Pi^+(R(\alpha u)q\Big(\frac{\sigma}{\alpha}, u-z\Big)},$$

where the last inequality is due to the fact that $F \le R^*$ by Theorem 4.1.

Moreover, since $\Pi^+(R)$ is regularly varying at ∞ with some index $r \le 0$ by Corollary B.1, we have $\Pi^+(R(\alpha u) \ge c(1+u)^{r-\delta}\alpha^{r-\delta}$ for some $c > 0$ for all $u \ge 0$ by Proposition 1.3.6(v) in [5]. Therefore,

$$\nu(\alpha,z,(\mathbb{R}\backslash(z-\varepsilon,z+\varepsilon))) \le \frac{\alpha^{\delta-r}\int_{\mathbb{R}\backslash(z-\varepsilon,z+\varepsilon)} dyq\Big(\frac{\sigma}{\alpha}, y-z\Big)}{c\int_0^\infty du(1+u)^{r-\delta}q\Big(\frac{\sigma}{\alpha}, u-z\Big)}, \qquad \text{(B.51)}$$

the right side of which converges to 0 as $\alpha \to \infty$. Similarly, we can show

$$\lim_{\alpha\to\infty} \nu(\alpha, z, (-\infty, z-\varepsilon)) = 0.$$

Step 1: Let $\gamma_*(x) := \liminf_{\alpha\to\infty} \frac{G(\alpha x)}{G(\alpha)}$. It follows from the same argument in Step 2 of the proof of Theorem B.1 that γ_* is bounded away from 0 on $[0, n]$ for any $n \geq 1$. This in turn implies the mapping $(\alpha, x) \mapsto \frac{G(\alpha x)}{G(\alpha)}$ is bounded when x belongs to bounded intervals in $(0, \infty)$ as in Step 3 of the same proof.

Step 2: Moreover, since $F \leq R^*$, (B.40) yields

$$\frac{1}{M - R(\alpha)} \leq K\alpha, \qquad \alpha > 1 \text{ for some } K < \infty.$$

Thus, the arguments of Steps 4 and 5 of the proof of Theorem B.1 are still applicable due to (B.51) and that $\Psi_F^- \leq E[V]$. Consequently, γ still satisfies (B.43), where $\gamma(x) = \lim_{x\to\infty} \frac{G(\alpha x)}{G(\alpha)}$. In particular, $\gamma(x) = x^{\rho^+}$.

Step 3: If $\Psi_x^+(M)$, the proof of Theorem B.2 can be applied verbatim once we show that $f(\alpha, \cdot)$ is bounded in $[1, 2]$ uniformly in α, where

$$f(\alpha, x) := \frac{F(\alpha x) - F(\alpha)}{(M - F(\alpha))^{n+1}}.$$

Since $F(\alpha)$ is eventually larger than $E[V]$ and $\phi_F(z) \leq E[V]$ for $z < 0$, we have for large enough α

$$\begin{aligned} f(\alpha, x) &= \int_{-\infty}^{0} dz \left\{ \frac{1}{N} q\Big(\frac{\sigma}{\alpha}, x - z\Big) + \frac{N-1}{N} \bar{q}\Big(\frac{\sigma}{\alpha}, x, z\Big) \right\} \frac{\phi_F(\alpha z) - F(\alpha)}{(M - F(\alpha))^{n+1}} \\ &+ \int_{0}^{\infty} dz \left\{ \frac{1}{N} q\Big(\frac{\sigma}{\alpha}, x - z\Big) + \frac{N-1}{N} \bar{q}\Big(\frac{\sigma}{\alpha}, x, z\Big) \right\} \int_{-\infty}^{\infty} \nu(\alpha, z, dy) \frac{\Psi^+(F(\alpha y)) - F(\alpha)}{(M - F(\alpha))^{n+1}} \\ &\leq \int_{0}^{\infty} dz \left\{ \frac{1}{N} q\Big(\frac{\sigma}{\alpha}, x - z\Big) + \frac{N-1}{N} \bar{q}\Big(\frac{\sigma}{\alpha}, x, z\Big) \right\} \int_{-\infty}^{\infty} q\Big(\frac{\sigma}{\alpha}, z - y\Big) \frac{\Psi^+(F(\alpha y)) - F(\alpha)}{(M - F(\alpha))^{n+1}}, \end{aligned}$$

where the last inequality follows from Part (4) of Lemma 4.1 since $\Psi^+(F(\alpha y))$ is increasing in y for positive α. Thus, $f(\alpha, \cdot)$ can be shown to be bounded by the same function that bounds $r(\alpha, \cdot)$ introduces in the proof of Theorem B.2. Repeating the remaining arguments therein yields the claim.

□

Proof of Corollary 5.1. It follows from Theorem 3.1 that F^* must solve (3.14). In particular F^* is regularly varying at ∞ of order ρ^+.

In view of Proposition 2.2 we have

$$\frac{M - IS^*(x)}{M - F^*(x)} = N \int_0^1 \frac{M - F^*(xy)}{M - F^*(x)} y^{N-1} dy = \frac{N \int_0^x (M - F^*(y)) y^{N-1} dy}{x^N (M - F^*(x))}.$$

Observe that $\lim_{y\to\infty} (M - F^*(y)) y^{-\rho^+ \varepsilon} = \infty$ by Proposition 1.3.6 (v) in [5]. Thus, $\int_0^\infty (M - F^*(y)) y^{N-1} dy = \infty$. This justifies the application of the L'Hôpital rule to arrive at

$$\lim_{x\to\infty} \frac{M - IS^*(x)}{M - F^*(x)} = \lim_{x\to\infty} \frac{N(M - F^*(x))x^{N-1}}{Nx^{N-1}(M - F^*(x)) - x^N F^*_x(x)}$$
$$= \frac{1}{1 - \lim_{x\to\infty} \frac{xF^*_x(x)}{N(M-F^*(x))}} = \frac{1}{1 + \frac{\rho^+}{N}} = \frac{N}{N + \rho^+},$$

where the third equality follows from Exercise 1.11.13 applied to $1/(M - F)$.

Asymptotic relationship near $-\infty$ is proved the same way. □

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