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Long running times for hypergraph bootstrap percolation☆

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ABSTRACT

Consider the hypergraph bootstrap percolation process in which, given a fixed r-uniform hypergraph H and starting with a given hypergraph G_0, at each step we add to G_0 all edges that create a new copy of H. We are interested in maximising the number of steps that this process takes before it stabilises. For the case where $H = K_{r+1}^{(r)}$ with $r \geq 3$, we provide a new construction for G_0 that shows that the number of steps of this process can be of order $\Theta(n^r)$. This answers a recent question of Noel and Ranganathan. To demonstrate that different running times can occur, we also prove that, if H is $K_4^{(3)}$ minus an edge, then the maximum possible running time is $2n - \lfloor \log_2(n-2) \rfloor - 6$. However, if H is $K_5^{(3)}$ minus an edge, then the process can run for $\Theta(n^3)$ steps.

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1. Introduction

The *hypergraph bootstrap percolation process* is an infection process on hypergraphs which was introduced by Bollobás in 1968 under the name of *weak saturation* [6]. For an integer $r \geq 2$ and a set S, denote by $\binom{S}{r}$ the set of all subsets of S of size r. Given an r-uniform hypergraph H and a positive integer n, the *H-bootstrap percolation process* is a deterministic process defined as follows. We start with a given r-uniform hypergraph G_0 on vertex set $[n] := \{1, \ldots, n\}$. For each time step $t \geq 1$, we define the hypergraph G_t on the same vertex set $[n]$ by letting

$$E(G_t) := E(G_{t-1}) \cup \left\{ e \in \binom{[n]}{r} : \exists \text{ an } H\text{-copy } H' \text{ s.t. } E(H') \nsubseteq E(G_{t-1}) \text{ and } E(H') \subseteq E(G_{t-1}) \cup \{e\} \right\},$$

that is, G_t is an r-uniform hypergraph on $[n]$ defined by including all edges of G_{t-1} together with all edges $e \in \binom{[n]}{r}$ which create a new copy of H with the edges of G_{t-1}. The hypergraph G_0 is called the *initial infection*, and the edges $E(G_t) \setminus E(G_{t-1})$ are said to be *infected* at time t. If there exists some $T \geq 0$ such that $G_T = K_n^{(r)}$, we say that G_0 *percolates* under this process. The hypergraph G_0 is said to be weakly H-saturated if G_0 is H-free and percolates under H-bootstrap percolation, that is, if there exists an ordering of $E(K_n^{(r)}) \setminus E(G_0) = \{e_1, \ldots, e_t\}$ such that the addition of e_i to $G_0 \cup \{e_1, \ldots, e_{i-1}\}$ will create a new copy of H, for every $i \in [t]$.

Given a fixed hypergraph H, one of the most studied extremal problems in this setting is establishing the minimum size of an n-vertex hypergraph which is weakly H-saturated. For the most basic case, where $r = 2$ and $H = K_k$, it was conjectured by Bollobás [6] that the minimum size of a weakly K_k-saturated n-vertex graph is $(k-2)n - \binom{k-1}{2}$. About a decade after, Lovász [15] was the first to confirm this conjecture (using a generalisation of the Bollobás Two Families Theorem [5]). This was later independently reproved by Alon [1], Frankl [9] and Kalai [12,13] using methods from linear algebra. For the hypergraph case, the work of Frankl [9] proving the Skew Two Families Theorem (strengthening the Bollobás Two Families Theorem) also settles this problem for $K_k^{(r)}$ with $r \geq 3$. This problem has also been studied for other graphs H, and for host graphs other than the complete graph, and other related settings; see, e.g., [1,8,14,17,19,20,22].

Even though the initial infection graphs which are solutions to the weak saturation problem have the minimum possible number of edges, it is interesting to note that, in many of the known examples, they require only very few steps until the infection process stabilises. For example, the standard construction of a weakly K_k-saturated graph achieving the minimum size is given by removing the edges of a clique of size $n-k+2$ from K_n, which means that only one step is needed in order to complete the infection process. In this direction, Bollobás raised the problem of finding the initial infection for which the running time of the H-bootstrap percolation process is maximised. This was previously studied in the related setting of neighbourhood percolation by Benevides and Przykucki [3,4,21], and for a random initial infection by Gunderson, Koch and Przykucki [10].

Here we consider this problem in the hypergraph bootstrap percolation setting. Given a fixed r-uniform hypergraph H and an r-uniform initial infection G_0, we define the *running time* of the H-bootstrap percolation process on G_0 to be

$$M_H(G_0) := \min\{t \geq 0 : G_t = G_{t+1}\}.$$

We denote the *maximum running time* over all r-uniform hypergraphs G_0 on n vertices as $M_H(n)$. We shall simplify these notations to $M_k^r(G_0)$ and $M_k^r(n)$ when $H = K_k^{(r)}$ is the complete r-uniform hypergraph on k vertices, and drop the superscript to $M_k(n)$ in the graph setting ($r = 2$). Note that a trivial upper bound for $M_H(n)$ is given by $\binom{n}{r}$, the total number of edges of $K_n^{(r)}$.

The simplest setting to consider is for graph bootstrap percolation and $H = K_k$. For $k = 3$, it is not hard to see that $M_3(n) = \lceil \log_2(n-1) \rceil$, where an extremal example is given by an n-path (see, e.g., [7] for the details). Bollobás, Przykucki, Riordan and Sahasrabudhe [7] and independently Matzke [16] considered this problem for higher values of k. By carefully analysing the growth of cliques during the percolation process, both groups of authors showed that $M_4(n) = n-3$. Moreover, for $k \geq 5$, Bollobás, Przykucki, Riordan and Sahasrabudhe [7] obtained the lower bound $M_k(n) \geq n^{2-\alpha_k-o(1)}$, where $\alpha_k = (k-2)/(\binom{k}{2} - 2)$, using a probabilistic argument. The authors of [7] conjectured that $M_k(n) = o(n^2)$ for all $k \geq 5$. However, in a subsequent paper, Balogh, Kronenberg, Pokrovskiy and Szabó [2] disproved this conjecture for $k \geq 6$, showing that the natural upper bound is tight up to a constant factor. The authors of [2] also improved the lower bound for $k = 5$ to $M_5(n) \geq n^{2-O(1/\sqrt{\log n})}$, using Behrend's construction of 'dense' 3-AP-free sets, and conjectured that $M_5(n) = o(n^2)$. It remains an open problem to determine whether this is the case.

In this paper we consider the question of the maximum running time when H is an r-uniform hypergraph with $r \geq 3$. This was recently investigated by Noel and Ranganathan [18]. By providing an explicit construction to establish the lower bound (noting the trivial upper bound of $\binom{n}{r}$), they proved the following theorem for the case $k \geq r + 2$.

Theorem 1 (*Noel and Ranganathan [18]*)**.** *Let $r \geq 3$. If $k \geq r + 2$, then $M_k^r(n) = \Theta(n^r)$.*

For the case $k = r + 1$, they established the following lower bound.

Theorem 2 (*Noel and Ranganathan [18]*)**.** *Let $r \geq 3$. If $k = r + 1$, then $M_k^r(n) = \Omega(n^{r-1})$.*

This theorem leaves a gap between the lower bound and the trivial upper bound $M_{r+1}^r(n) = O(n^r)$. Noel and Ranganathan conjectured that $M_4^3(n) = O(n^2)$ [18, Conjecture 5.1], but suggested that, for sufficiently large r, it is indeed true that the maximum running time achieves $M_{r+1}^r(n) = \Theta(n^r)$ [18, Question 5.2].

In this paper, we show the conjecture to be false and prove that the trivial upper bound is in fact tight, up to a constant factor, for all $r \geq 3$. This also gives a positive answer to their question, in a strong sense.

Theorem 3. *For any fixed integer $r \geq 3$, we have $M_{r+1}^r(n) = \Theta(n^r)$.*

Another proof for Theorem 3 was independently announced by Hartarsky and Lichev [11].

We note that Theorem 3 establishes a clear difference with respect to the graph case $r = 2$, where $M_k(n) = o(n^r)$ for $k \in \{r+1, r+2\}$ (and possibly also $r+3$). It may therefore seem that the behaviour of hypergraph bootstrap percolation is less rich than its graph counterpart. We propose a modification of the problem above that shows this is not the case, and that different (and very interesting) asymptotic running times may still occur in the hypergraph setting.

Indeed, recall that we may think of H-bootstrap percolation as an infection process where the infection spreads to a new copy of H if only one edge of said copy was not infected in the previous step. It is reasonable then to consider models where the infection is more powerful, in the sense that it will extend to copies of H which are missing at most m edges, for some fixed integer m. We consider here in particular the case $m = 2$. Note that if $m = 2$ and H is a complete hypergraph (which is the case we will focus on), then this modified model is equivalent to the original hypergraph percolation process for the hypergraph H' obtained by deleting an arbitrary edge from H.

Formally, let H be a given r-uniform hypergraph, and let G be an r-uniform hypergraph on $[n]$. For each copy H' of H on $[n]$, if $|E(H') \setminus E(G)| \leq m$, we say that H' is *m-completable* in G. We define the (H, m)-bootstrap percolation process on an initial infection G_0 on $[n]$ to be the sequence of hypergraphs $G_0, G_1, \ldots$ on $[n]$ given by setting, for each $t \geq 1$,

$$E(G_t) := E(G_{t-1}) \cup \bigcup_{\substack{H' \text{ copy of } H \text{ on } [n] \\ H' \text{ } m\text{-completable in } G_{t-1}}} E(H').$$

Note that the $(H, 1)$-bootstrap percolation process simply corresponds to the usual H-bootstrap percolation process. Let us denote the running time of this hypergraph percolation process as $M_{(H,m)}(G_0) := \min\{t \geq 0 : G_t = G_{t+1}\}$, and the maximum running time over all r-uniform n-vertex hypergraphs G_0 as $M_{(H,m)}(n)$. The next result shows that we get interesting new behaviour when $m = 2$ and $H = K_4^{(3)}$ (which is probably the most natural first case to consider).

Theorem 4. *For all $n \geq 4$, we have $M_{(K_4^{(3)},2)}(n) = 2n - \lfloor \log_2(n-2) \rfloor - 6$.*

It is worth remarking here that this is the first nontrivial exact result about running times of hypergraph bootstrap percolation. The only nontrivial exact results in graph bootstrap percolation are those for K_3- and K_4-bootstrap percolation [7].

We also prove that in the next case, $H = K_5^{(3)}$, the running time can once again be cubic (i.e., as large as possible).

Theorem 5. *We have* $M_{(K_5^{(3)},2)}(n) = \Theta(n^3)$.

Let $K_s^{(r)} - e$ denote the hypergraph obtained by deleting an edge from $K_s^{(r)}$. As mentioned above, the $(K_s^{(r)}, 2)$-process is the same as the usual bootstrap percolation process for $K_s^{(r)} - e$, so the results above can be reformulated as follows.

Theorem 4′. *For all* $n \geq 4$, *we have* $M_{K_4^{(3)}-e}(n) = 2n - \lfloor \log_2(n-2) \rfloor - 6$.

Theorem 5′. *We have* $M_{K_5^{(3)}-e}(n) = \Theta(n^3)$.

We present our proof of Theorem 3 in Section 2. We defer the proofs of Theorems 4 and 5 to Section 3. We also propose some open problems in our concluding remarks.

2. Long running times for simple infections

In order to prove Theorem 3, we will use a result of Noel and Ranganathan [18] that allows us to focus on the case $r = 3$. To state their result, we need to recall some definitions from [18]. Let G_0 be an r-uniform hypergraph, let G_t be the hypergraph at time t for the $K_{r+1}^{(r)}$-bootstrap process starting with G_0 as initial infection, and let $T = M_{r+1}^r(G_0)$ be the time the process stabilises. We say that G_0 is $K_{r+1}^{(r)}$-*civilised* if the following conditions are satisfied for some edge e_0 of G_0.

(1) For each $t \in [T]$, G_t contains only one more edge e_t than G_{t-1}, and one more copy H_t of $K_{r+1}^{(r)}$.
(2) For all $t \in [T]$ we have $E(H_t) \cap \{e_0, e_1, \ldots, e_T\} = \{e_{t-1}, e_t\}$.
(3) The $K_{r+1}^{(r)}$-bootstrap percolation process starting with $G_0 - e_0$ infects no edge.

Lemma 6 (*Noel and Ranganathan [18, Lemma 2.11]*)**.** *If for all n there exists a $K_4^{(3)}$-civilised hypergraph G_0 on $\Theta(n)$ vertices such that $M_4^3(G_0) = \Theta(n^3)$, then for all $r \geq 3$ we have $M_{r+1}^r(n) = \Theta(n^r)$.*

Before we give the formal proof of Theorem 3, let us give an informal description of the construction that gives a lower bound for the number of steps of the percolation process. As noted above, by Lemma 6 it is enough to consider the case $r = 3$. The main part of the construction consists of three layers of vertices: 'top' vertices labelled t_i, 'bottom' vertices labelled b_j, and 'middle' vertices labelled m_ℓ. In each time step, just one new edge will become infected. That infection will happen because one copy of $K_4^{(3)}$, which had only two edges present in the initial infection, has a third edge infected in the previous step of the process.

The process will consist mainly of chains of infections, where we move from one chain to another by using special gadgets. The chains will have the format of the so-called 'beachball hypergraph'. The vertex set of this hypergraph consists of one top and one bottom vertex, and some ordered vertices in the middle; the edges are the triples consisting of two consecutive middle vertices, and either the top or the bottom vertex. See Fig. 1 for an illustration.

It will be convenient to think of the process as having n phases, each phase having $\Theta(n)$ stages, and each stage having $\Theta(n)$ infection steps. A phase will represent the infection process that occurs when we fix a top vertex t_i. In each phase, we have $\Theta(n)$ stages, where each stage is the process that occurs when we fix b_j (for the fixed t_i of this phase). At a specified phase and stage, the initial infected set will be the above mentioned beachball hypergraph, and the infection will spread through the middle vertices. This gives $\Theta(n)$ infection steps for each stage.

The challenge will then be to move to another top or bottom vertex without infecting more than one edge in each step of the process. For this purpose, we will introduce, at the end of each stage,

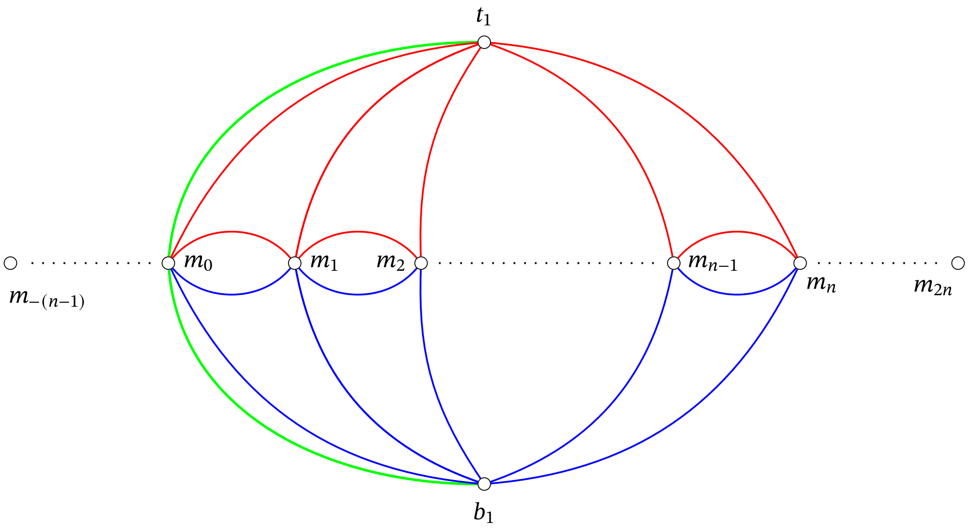

Fig. 1. Initial infection G_0 showing only the first top and bottom vertices, t_1 and b_1. Each red or blue triangle represents an edge of G_0, and together they form the first beachball hypergraph in our process. The green arc represents an edge containing the vertices it passes through. To form G_1, the edge $t_1b_1m_1$ is added, as this completes a copy of $K_4^{(3)}$ on $\{t_1, m_0, m_1, b_1\}$. It is clear to see that subsequently all edges of the form $t_1b_1m_\ell$ for ℓ increasing from 2 to n are added in turn. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

a new middle vertex and a special 'switching' gadget. Each stage of the process will be represented by a tuple of a top vertex t_i, a bottom vertex b_j, and consecutive middle vertices starting from m_s and ending in m_ℓ, where $-(n-1) \le s \le 0$ and $n \le \ell \le 2n$. For moving between phases, we will introduce a different type of gadget.

Let us first describe the first few stages of the process to give a better intuition. The first n infection steps will come from a 'path' on the middle layer. The edges $t_1m_\ell m_{\ell+1}$ and $b_1m_\ell m_{\ell+1}$ will be present at time zero for all $0 \le \ell \le n-1$, as well as the edge $t_1b_1m_0$. Once the edge $t_1b_1m_\ell$ becomes infected, it propagates the infection in the next step to $t_1b_1m_{\ell+1}$. See Fig. 1 for an illustration.

After $\Theta(n)$ such infections, we want to swap out b_1 to another bottom vertex (labelled b_{-1}). We do this by making sure that the last infected edge using b_1 (namely, the edge $t_1b_1m_n$) makes the middle path longer, that is, it makes the edge $t_1m_nm_{n+1}$ infected in the next step. To achieve this, we will have $b_1m_nm_{n+1}$ and $t_1b_1m_{n+1}$ present in the original hypergraph G_0; see Fig. 2.

Once $t_1m_nm_{n+1}$ is infected, it can start a chain of infections using the new bottom vertex b_{-1}. However, this time the chain of infections will go in the opposite direction on the middle path: we will first infect $t_1b_{-1}m_n$ (for this we will need the edges $t_1b_{-1}m_{n+1}$ and $b_{-1}m_nm_{n+1}$ to be present initially, as in Fig. 2), then we infect $t_1b_{-1}m_{n-1}$, and so on, until $t_1b_{-1}m_0$.

At this point we again swap out the bottom vertex to a different one (labelled b_2) — we do this using the same trick as above, i.e., making the middle path one longer, and then changing the direction we traverse the path. We keep repeating the steps above for $\Theta(n)$ bottom vertices to get $\Theta(n^2)$ infections which all use the same top vertex t_1.

Once we have the $\Theta(n^2)$ infections using t_1, we wish to swap out the top vertex t_1 to a different one (labelled t_2). We could do this similarly to how we swapped the b_j's, but it is more convenient to simply introduce a gadget using three 'dummy' vertices d_1, d_2, d_3 to do this swap. The last infection using t_1 (namely, $t_1m_{2n-1}m_{2n}$) will start a short chain of infections using the $K_4^{(3)}$'s given by $t_1m_{2n-1}m_{2n}d_1$, $m_{2n-1}m_{2n}d_1d_2$, $m_{2n}d_1d_2d_3$, $d_1d_2d_3t_2$, $d_2d_3t_2m_0$, and $d_3t_2m_0m_1$. The last one of these allows us to start a repeat of the previous infection process, using t_2 instead of t_1. We will use three

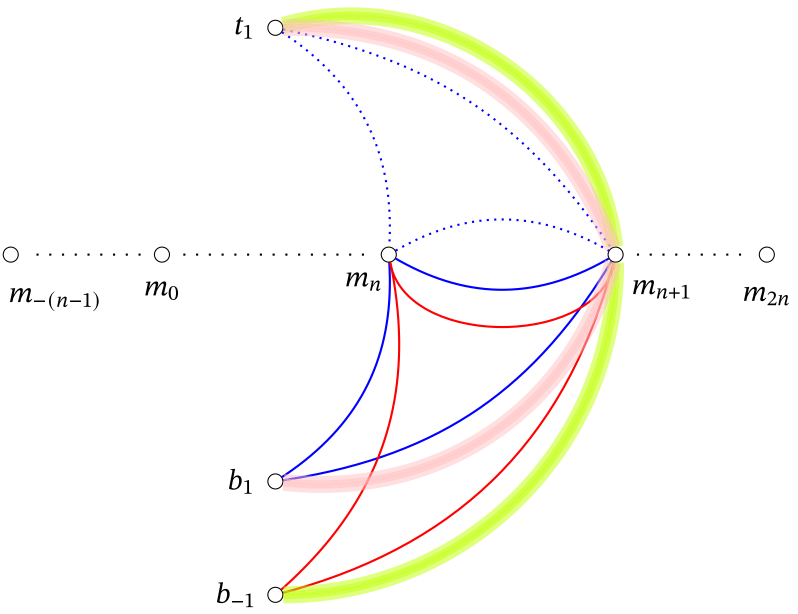

Fig. 2. Switching gadget to change from $K_4^{(3)}$ copies containing b_1 to those containing b_{-1}. The edges $b_1m_nm_{n+1}$ and $b_{-1}m_nm_{n+1}$ are present in the initial infection G_0. After the edge $t_1b_1m_n$ is created by the percolation process, the copy of $K_4^{(3)}$ induced by the vertices $\{t_1, m_n, m_{n+1}, b_1\}$ is present, except for the missing edge $t_1m_nm_{n+1}$ shown in the dotted blue line. Thus this edge is added, followed by $t_1b_{-1}m_n$. This triggers the process to run backwards and create all edges of form $t_1b_{-1}m_i$, for i decreasing from $i = n-1$ to $i = 0$, in turn. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

such dummy vertices d_i for each of the $n-1$ swaps at the top — so only $3n-3 = \Theta(n)$ dummy vertices in total. See Fig. 3 for an illustration.

Let us now turn to the formal proof of our theorem.

Proof of Theorem 3. By Lemma 6, it suffices to consider the case $r = 3$ and show that there are $K_4^{(3)}$-civilised 3-uniform hypergraphs on $\Theta(n)$ vertices such that the $K_4^{(3)}$-bootstrap process takes $\Theta(n^3)$ steps to stabilise. We now describe a construction achieving this.

The initial infection hypergraph G_0 has $9n-4 = \Theta(n)$ vertices, which are labelled as follows: $t_1, \ldots, t_n, b_1, \ldots, b_n, b_{-1}, \ldots, b_{-(n-1)}, m_{-(n-1)}, m_{-(n-2)} \ldots, m_{2n}$, and $d_{i,1}, d_{i,2}, d_{i,3}$ for $i \in [n-1]$. The edges of G_0 are given below:

(a) $t_1m_0m_1$;
(b) $t_im_\ell m_{\ell+1}$ for all $i \in [n]$ and $\ell \in [n-1]$;
(c) $b_jm_\ell m_{\ell+1}$ for all $j \in [n]$ and $\ell \in [-(j-1), n+j-1]$;
(d) $b_{-j}m_\ell m_{\ell+1}$ for all $j \in [n-1]$ and $\ell \in [-j, n+j-1]$;
(e) $t_ib_jm_{-(j-1)}$ and $t_ib_jm_{n+j}$ for all $i, j \in [n]$;
(f) $t_ib_{-j}m_{n+j}$ and $t_ib_{-j}m_{-j}$ for all $i \in [n]$ and $j \in [n-1]$;
(g) $t_im_{2n-1}d_{i,1}$, $t_im_{2n}d_{i,1}$, $m_{2n-1}m_{2n}d_{i,2}$, $m_{2n-1}d_{i,1}d_{i,2}$, $m_{2n}d_{i,1}d_{i,3}$, $m_{2n}d_{i,2}d_{i,3}$, $d_{i,1}d_{i,2}t_{i+1}$, $d_{i,1}d_{i,3}t_{i+1}$, $d_{i,2}d_{i,3}m_0$, $d_{i,2}t_{i+1}m_0$, $d_{i,3}t_{i+1}m_1$, and $d_{i,3}m_0m_1$, for all $i \in [n-1]$.

As mentioned in the informal discussion, it will be easier to think about the initial infected hypergraph as a set of beachball hypergraphs, and gadgets connecting between them. For this purpose, we note the following.

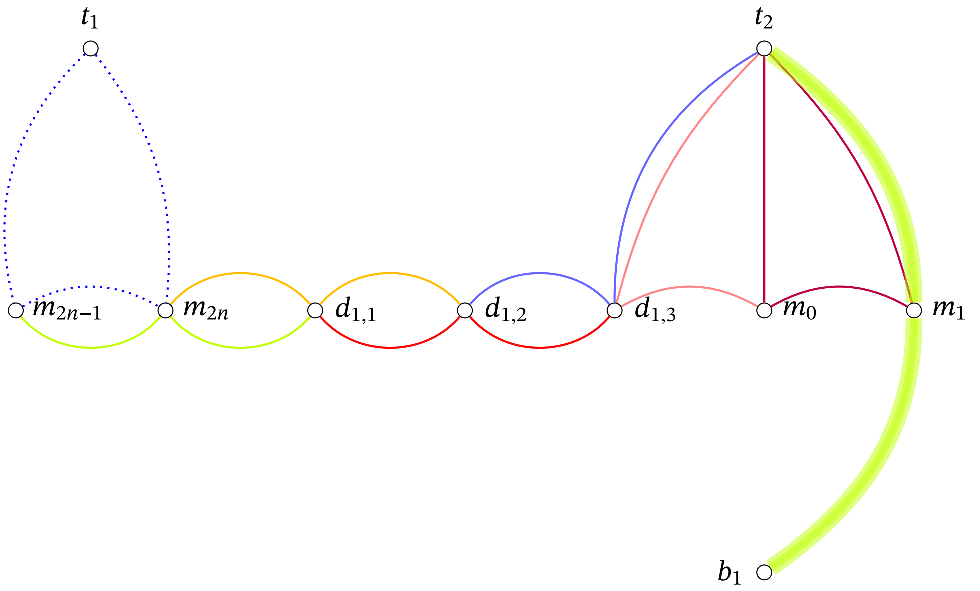

Fig. 3. Switching gadget for changing the top vertex t_1 to t_2. Edges present in the initial infection G_0 are omitted for clarity. When the dotted blue edge $t_1m_{2n-1}m_{2n}$ is infected, this causes the edges along the chain to become infected, ending in $t_2m_0m_1$. This triggers the infection of $t_2b_1m_1$, and in turn the process from the stage as shown in Fig. 1, with t_1 replaced with t_2. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

- The edges from (b) and (c), as well as those from (b) and (d), (nearly) form beachball hypergraphs. For the beachballs with edges from (b) and (c) the infection process increases with the indices of the middle vertices, whereas for those from (b) and (d) it decreases with the indices of the middle vertices. These hypergraphs are used as the main ingredients of the infection process.
- The second type of edges from (e) together with the first type of edges from (f) form the gadgets that help us swap from b_j to b_{-j}, where t_i is fixed; that is, they help us move between the beachball with t_i, b_j as top and bottom and the beachball with t_i, b_{-j}. Fig. 2 illustrates the gadget swapping from b_1 to b_{-1}.
- The first type of edges from (e) and second type of edges from (f) create the gadgets that help us swap from b_{-j} to b_{j+1}, where t_i is fixed; that is, these help us move from the beachball with t_i, b_{-j} as top and bottom to the beachball with t_i, b_{j+1}.
- The edges in (g) form the gadgets swapping between top vertices, from t_i to t_{i+1}, using the dummy vertices $d_{i,s}$. In other words, these gadgets move us from the beachball with t_i, b_n as top and bottom to the beachball with t_{i+1}, b_1. Fig. 3 illustrates the gadget swapping from t_1 to t_2.

We will show that there are three types of edges that are being infected during the process:

(I) missing edges of the beachballs, that is, edges of the form $t_ib_jm_\ell$;
(II) edges from the gadgets swapping bottom vertices, of the form $t_im_\ell m_{\ell+1}$, and
(III) edges from the gadgets swapping top vertices (these have several different forms).

We will now name the edges being infected during the process. For each $i, j \in [n]$, let $A_{i,j}$ denote the following sequence of edges:

$$A_{i,j} := (t_ib_jm_{-(j-2)},\ t_ib_jm_{-(j-3)}, \ldots,\ t_ib_jm_{n+j-1},\ t_im_{n+j-1}m_{n+j}). \tag{2.1}$$

These will be the edges infected in the stage of phase i corresponding to the bottom vertex b_j. Similarly, for each $i \in [n]$ and $j \in [n-1]$, let

$$A_{i,-j} := (t_i b_{-j} m_{n+j-1},\ t_i b_{-j} m_{n+j-2}, \ldots,\ t_i b_{-j} m_{-(j-1)},\ t_i m_{-(j-1)} m_{-j}). \tag{2.2}$$

These will correspond to the stage with b_{-j} as bottom vertex. Concatenating these, we get the sequence A_i of edges corresponding to phase i (these are edges of types (I) and (II) above):

$$A_i := A_{i,1} A_{i,-1} A_{i,2} A_{i,-2} \ldots A_{i,n-1} A_{i,-(n-1)} A_{i,n}.$$

For the phase change using the dummy vertices $d_{i,j}$, let us write, for each $i \in [n-1]$,

$$D_i := (m_{2n-1} m_{2n} d_{i,1},\ m_{2n} d_{i,1} d_{i,2},\ d_{i,1} d_{i,2} d_{i,3},\ d_{i,2} d_{i,3} t_{i+1},\ d_{i,3} t_{i+1} m_0,\ t_{i+1} m_0 m_1). \tag{2.3}$$

These are the edges of type (III). Finally, let us write A for the concatenation

$$A := A_1 D_1 A_2 D_2 \ldots A_{n-1} D_{n-1} A_n.$$

We will show that during the infection process, edges become infected one-by-one, according to the sequence A.

Let T be the number of triples in A, and let $A = (e_1, e_2, \ldots, e_T)$. Note that $T = \Theta(n^3)$. Let us also write e_0 for the edge $t_1 m_0 m_1$, and for all $a \in [T-1]$ let H_a be the copy of $K_4^{(3)}$ with vertex set $e_{a-1} \cup e_a$ (note that $|e_{a-1} \cap e_a| = 2$ for all a). For each $s \in [T]$, let us write G_s for the hypergraph with edge set $E(G_0) \cup \{e_1, e_2, \ldots, e_s\}$. (Note that we do not yet know that these coincide with the hypergraphs obtained during the $K_4^{(3)}$-bootstrap process, but we will see that they do.) Let us also write $G_{-1} := G_0 - e_0$.

Claim 1. *Assume that $s \in [-1, T]$ is an integer and $e = x_1 x_2 x_3$ is a triple not contained in $E(G_s)$. Suppose that adding e to G_s completes a copy of $K_4^{(3)}$ whose fourth vertex is x_4. Then, $s \in [0, T-1]$, $e = e_{s+1}$ and $\{x_1, x_2, x_3, x_4\} = V(H_{s+1})$.*

We note here that the case $s = -1$ is needed to formally justify that G_0 is $K_4^{(3)}$-civilised below.

Proof. We consider the following two cases.

Case 1: a vertex $d_{i,c}$ appears among $x_1, \ldots, x_4$ (for some $i \in [n]$ and $c \in [3]$). Let us temporarily write $d_{i,-2} := t_i$, $d_{i,-1} := m_{2n-1}$, $d_{i,0} := m_{2n}$, $d_{i,4} := t_{i+1}$, $d_{i,5} := m_0$ and $d_{i,6} := m_1$, so the edges $d_{i,a} d_{i,a+1} d_{i,a+3}$ and $d_{i,a} d_{i,a+2} d_{i,a+3}$ are present in G_0 for all $-2 \le a \le 3$. Observe that the only vertices appearing in an edge of G_s together with $d_{i,c}$ (recall $1 \le c \le 3$) are of the form $d_{i,a}$ with $|c - a| \le 3$ (see (g) as well as (2.3)). Hence, $x_1, \ldots, x_4$ are all of the form $d_{i,a}$ for some $-2 \le a \le 6$. Observe furthermore that every edge of G_s of the form $d_{i,p} d_{i,q} d_{i,r}$ $(-2 \le p < q < r \le 6)$ satisfies $|r - p| \le 3$, or $(p, q, r) = (-2, 5, 6)$ or $(p, q, r) = (-1, 0, 4)$. It is easy to deduce that the only possible quadruples of vertices $d_{i,a}$ forming a $K_4^{(3)}$ minus an edge are of the form $\{x_1, x_2, x_3, x_4\} = \{d_{i,a}, d_{i,a+1}, d_{i,a+2}, d_{i,a+3}\}$ (for some $-2 \le a \le 3$). So exactly one of $d_{i,a} d_{i,a+1} d_{i,a+2}$ and $d_{i,a+1} d_{i,a+2} d_{i,a+3}$ appears in G_s, as the other two triples appear in G_0 (recall (g)). Since these are the edges e_N and e_{N+1}, respectively, for some $N \in [T-1]$, we must have $e = e_{N+1}$ and $e_N \in G_s$. So $N = s$, $e = e_{s+1}$, and $\{x_1, x_2, x_3, x_4\} = \{d_{i,a}, d_{i,a+1}, d_{i,a+2}, d_{i,a+3}\} = e_s \cup e_{s+1} = V(H_{s+1})$, as claimed.

Case 2: no vertex of the form $d_{i,c}$ $(c \in [3])$ appears among $x_1, \ldots, x_4$. Then, $x_1, \ldots, x_4$ are all of the form t_i, b_j or m_ℓ. Observe that no pair of the form $t_i t_{i'}$ or $b_j b_{j'}$ appears simultaneously in an edge of G_s $(i \ne i', j \ne j')$, so $X = \{x_1, \ldots, x_4\}$ contains at most one vertex of the form t_i and at most one vertex of the form b_j. So it must contain at least two vertices of the form m_ℓ. But m_ℓ and $m_{\ell'}$ appear simultaneously in an edge only if $|\ell - \ell'| \le 1$. It follows that X must be of the form $\{t_i, b_j, m_\ell, m_{\ell+1}\}$ for some i, j, ℓ. Assume that $j > 0$ (the case $j < 0$ is similar). If $\ell \le -j$, then neither $t_i b_j m_\ell$ nor $b_j m_\ell m_{\ell+1}$ appear in G_s (see (c), (e) and (2.1)), giving a contradiction. Similarly, if $\ell \ge n + j$, then neither $t_i b_j m_{\ell+1}$ nor $b_j m_\ell m_{\ell+1}$ appear in G_s, again giving a contradiction. Hence, we have $-(j-1) \le \ell \le n + j - 1$. It follows that $b_j m_\ell m_{\ell+1}$ is an edge of $G_0 - e_0$. So e is one of $t_i b_j m_\ell$, $t_i b_j m_{\ell+1}$ and $t_i m_\ell m_{\ell+1}$.

First, consider the case $e = t_i b_j m_\ell$. Since $t_i b_j m_{\ell+1}$ is already present, we must have $\ell = n+j-1$ (see (e)). But if $t_i m_\ell m_{\ell+1} = t_i m_{n+j-1} m_{n+j} = e_N$ appears in G_s, then so does its preceding edge $e_{N-1} = t_i b_j m_{n+j-1} = t_i b_j m_\ell$ (see (2.1)), giving a contradiction.

If the new edge is $e = t_i b_j m_{\ell+1}$, then $-(j-1) \le \ell \le n+j-2$. So we have $e = e_N$ for some $N \in [T]$, and the edge e_{N-1} is either $t_i b_j m_\ell$ (if $\ell \ne -(j-1)$) or $t_i m_{-(j-2)} m_{-(j-1)} = t_i m_\ell m_{\ell+1}$ (if $\ell = -(j-1)$). In either case, we have $e_{N-1} \in E(G_s)$ and $e_N \notin E(G_s)$, giving $s = N-1$, $e = e_{s+1}$, $\{x_1, x_2, x_3, x_4\} = e_s \cup e_{s+1} = V(H_{s+1})$, as claimed.

Finally, consider the case when the new edge is $e = t_i m_\ell m_{\ell+1}$. So $t_i b_j m_\ell$ and $t_i b_j m_{\ell+1}$ are edges of G_s. Note that $t_i b_j m_\ell$ or $t_i b_j m_{\ell+1}$ is of the form e_N for some $N \in [T]$. It follows that all edges $e_{N'}$ with $N' < N$ appear in G_s, so, in particular, $t_i m_{\ell'} m_{\ell'+1}$ is in G_s for all $-(j-1) \le \ell' \le n+j-2$. Hence $\ell = n+j-1$. But then there is some $M \in [T]$ such that $e_M = t_i m_\ell m_{\ell+1}$, and we have $e_{M-1} = t_i b_j m_{n+j-1} = t_i b_j m_\ell$, which appears in G_s. If follows that $s = M-1$, $e = e_{s+1}$ and $\{x_1, x_2, x_3, x_4\} = e_s \cup e_{s+1} = V(H_{s+1})$, as claimed. ◀

It is straightforward to check that for all $s \in [T]$ we have $E(H_s) \setminus E(G_{s-1}) = \{e_s\}$ and $E(H_s) \cap \{e_0, e_1, \dots, e_s\} = \{e_{s-1}, e_s\}$. Using these observations and the claim above, we see that all conditions of being $K_4^{(3)}$-civilised are satisfied for G_0, and the result follows from Lemma 6. □

Remark 7. It immediately follows from the construction and the proof above that our proposed initial infection has $9n + O(1)$ vertices and that the infection process takes $4n^3 + O(n^2)$ steps. It therefore follows that $M_4^3(n) \ge 4n^3/9^3 + O(n^2)$. We note that we have made no effort to optimise the leading constant.

3. Long running times for double infections

3.1. Double infections for $K_4^{(3)}$

We now move on to the proof of our results about the variant where we allow two edges to be infected at the same time if they together complete a copy of H. We begin with Theorem 4, giving tight bounds in the case $H = K_4^{(3)}$. Our approach is motivated by the proof of Bollobás, Przykucki, Riordan and Sahasrabudhe [7] of the fact that $M_4(3) = n-3$, but both the construction and the proof of the upper bound are significantly more complicated here. We start with an informal description of the infection process for the extremal construction.

We will construct the initially infected hypergraph inductively. Assume that for some n we have already constructed a hypergraph G_0 on n vertices $\{x_1, \dots, x_n\}$ for which the process runs for T steps. Furthermore, assume that G_T is complete, but there exist two vertices $u, v \in \{x_1, \dots, x_n\}$ such that no edge of G_{T-1} contains u and v simultaneously. (These conditions might at first seem arbitrary, but they are satisfied in the obvious construction when $n = 4$.) Then, we can add another vertex x_{n+1} and another edge $x_{n+1}uv$ to G_0 without changing the first T steps of the infection process. (Indeed, the process will only be altered if we create a new 2-completable copy of $K_4^{(3)}$, and this requires having two edges sharing two vertices.) Moreover, at time $T+1$, the new edges that become infected are all those of the form $x_{n+1}ws$ with $w \in W_1 := \{u, v\}$ and $s \in \{x_1, \dots, x_n\} \setminus W_1$. Finally, at time $T+2$, the vertices $x_1, \dots, x_{n+1}$ will form a complete hypergraph. This gives a construction on $n+1$ vertices with running time $T+2$.

To obtain our construction for $n+2$ vertices, notice that if we pick some $u_2 \in \{x_1, \dots, x_n\} \setminus W_1$, then no edge of G_T contains both x_{n+1} and u_2. So if we add a new vertex x_{n+2} and a new edge $x_{n+2}x_{n+1}u_2$, then the first $T+1$ steps of the infection will remain unaffected by this change. Furthermore, one can check that at time $T+2$ the edges containing x_{n+2} are given as $x_{n+2}x_{n+1}w$ and $x_{n+2}u_2w$ with $w \in W_1$. Moreover, at time $T+3$ the edges $x_{n+2}ws$ with $w \in W_2 := W_1 \cup \{u_2, x_{n+1}\}$ and $s \in \{x_1, \dots, x_{n+1}\} \setminus W_2$ will become infected, and at time $T+4$ we get a complete hypergraph.

We can keep repeating these steps: take some $u_j \in \{x_1, \dots, x_n\} \setminus W_{j-1}$, add a new vertex x_{n+j} and a new edge $x_{n+j}x_{n+j-1}u_j$. This will extend the process by 2 steps, and at time $T+2j-1$ the edges containing the new vertex x_{n+j} will be of the form $x_{n+j}ws$ with $w \in W_j := W_{j-1} \cup \{u_j, x_{n+j-1}\}$ and $s \in \{x_1, \dots, x_{n+j-1}\}$. Moreover, at time $T+2j$ our hypergraph will contain all edges on $\{x_1, \dots, x_{n+j}\}$.

This means that we can keep adding a vertex and extending the process by 2 steps each time. However, the set W_j is growing, and at some point it will contain all of our vertices. When this happens, we will no longer be able to pick an appropriate u_j, and we will 'lose' 1 step of the infection process (i.e., by adding a new vertex we can only extend the process by 1 step at this point). So 'usually' adding a vertex extends the infection by 2 steps, giving the leading term $2n$ for the running time, but sometimes (when W becomes everything) we only gain one extra step, and this will contribute the term $-\lfloor \log_2(n-2)\rfloor$.

Let us now start the formal construction. Let G_0 be any 3-uniform hypergraph on some vertex set V, and let $G_0, G_1, \dots$ be the corresponding $(K_4^{(3)}, 2)$-process. Let T be the running time of this process. We say that G_0 is *nice* if $T \neq 0$, G_T is complete, and there exist distinct vertices $u, v \in V$ such that no edge of G_{T-1} contains both u and v. The following lemma will be used to obtain the lower bound.

Lemma 8. *Suppose that there is a nice hypergraph on $k \geq 4$ vertices such that the corresponding $(K_4^{(3)}, 2)$-process has running time T. Then, for all $\ell \in [k+1, 2k-3]$ we have*

$$M_{(K_4^{(3)},2)}(\ell) \geq T + 2(\ell - k).$$

Furthermore,

$$M_{(K_4^{(3)},2)}(2k-2) \geq T + 2k - 5,$$

and there exists a nice hypergraph on $2k-2$ vertices whose corresponding $(K_4^{(3)}, 2)$-process has running time $T + 2k - 5$.

Proof. Let G_0 be a nice 3-uniform hypergraph on k vertices $x_1, \dots, x_k$ such that the corresponding $(K_4^{(3)}, 2)$-process $G_0, G_1, \dots$ has running time T, G_T is complete, and x_1, x_k do not appear in any edge of G_{T-1} simultaneously. Let $\ell \in [k+1, 2k-2]$ be arbitrary. We define a hypergraph G_0' on a vertex set $\{x_1, \dots, x_\ell\}$ of size ℓ as follows. For any $i \in [\ell - k]$, let

$$e_i := x_{k+i}x_{k+i-1}x_i,$$

and let $\mathcal{E} := \{e_i : i \in [\ell - k]\}$. Then, set

$$E(G_0') := E(G_0) \cup \mathcal{E}.$$

Let $G_0', G_1', \dots$ be the corresponding $(K_4^{(3)}, 2)$-bootstrap percolation process with initial infection G_0', and let us write $W_j := \{x_1, \dots, x_j, x_k, x_{k+1}, \dots, x_{k+j-1}\}$ for all $j \in [\ell - k]$.

Claim 2. *We have*

$$E(G_t') = \begin{cases} E(G_t) \cup \mathcal{E} & \text{if } t \in [T], \\ \binom{\{x_1, \dots, x_{k+j-1}\}}{3} \cup \mathcal{E} \cup F_j^{\text{odd}} & \text{if } t = T + 2j - 1 \text{ with } j \in [\ell - k], \\ \binom{\{x_1, \dots, x_{k+j}\}}{3} \cup \mathcal{E} \cup F_j^{\text{even}} & \text{if } t = T + 2j \text{ with } j \in [0, \ell - k], \end{cases}$$

where

$$F_j^{\text{odd}} := \{x_{k+j}wx_a : w \in W_j, a \in [k+j-1], x_a \neq w\}$$

and

$$F_j^{\text{even}} := \{x_{k+j+1}wx_a : w \in W_j, a \in \{j+1, k+j\}\},$$

unless $j \in \{0, \ell - k\}$, in which case $F_j^{\text{even}} := \varnothing$.

Proof. We show this statement by induction on t. The case $t \in [T]$ is straightforward. Indeed, note that $|e_i \cap e_j| < 2$ for all distinct $i, j \in [\ell - k]$, and $|e_i \cap f| < 2$ whenever $f \in E(G_{T-1})$ (by our assumption that G_{T-1} does not contain any triple containing both x_1 and x_k). Recall that, if a copy H of $K_4^{(3)}$ is 2-completable in a hypergraph G, then G contains two edges of H, which must share two vertices. Thus, in the first T steps of the process, the addition of the edges in $\mathcal{E}$ does not result in any infections that did not occur for G_0. Now assume that $t > T$ and the statement above holds for $t-1$. We split the analysis into two cases.

Case 1. Consider first the case $t = T + 2j - 1$ (with $j \in [\ell - k]$). By the induction hypothesis,

$$E(G'_{t-1}) = \binom{\{x_1, \dots, x_{k+j-1}\}}{3} \cup \mathcal{E} \cup F^{\mathrm{even}}_{j-1}.$$

We first verify that $F^{\mathrm{odd}}_j \subseteq E(G'_t)$. Let $w \in W_j$ and $a \in [k+j-1]$ with $x_a \neq w$, so that $x_{k+j}wx_a \in F^{\mathrm{odd}}_j$. Then, there exists some $w' \in W_j$ with $x_{k+j}ww' \in E(G'_{t-1})$. (Indeed, if we pick w' such that $w' \in \{x_j, x_{k+j-1}\}$ and $w' \neq w$, then $x_{k+j}ww' \in F^{\mathrm{even}}_{j-1} \cup \{e_j\}$.) If $x_a = w'$, then trivially $x_{k+j}wx_a \in E(G'_t)$. If $x_a \neq w'$, then we also have $ww'x_a \in E(G'_{t-1})$. It follows that the copy of $K_4^{(3)}$ with vertex set $\{x_{k+j}, w, w', x_a\}$ is 2-completable in G'_{t-1}, and so $x_{k+j}wx_a \in E(G'_t)$.

We next show that any edge infected at time t must belong to F^{odd}_j. Indeed, if $h \in [j+1, \ell-k]$ then $|e_h \cap f| < 2$ for all $f \in E(G'_{t-1}) \setminus \{e_h\}$, so e_h cannot appear in a copy of $K_4^{(3)}$ completed in this step. It follows that any added edge must be of the form $e = x_{k+j}x_ax_b$ with $a, b \in [k+j-1]$ distinct. Furthermore, either x_a or x_b must appear together with x_{k+j} in an edge of $E(G'_{t-1}) \setminus \{e_i : i \in [j+1, \ell-k]\}$, so one of x_a, x_b must belong to $W_{j-1} \cup \{x_j, x_{k+j-1}\} = W_j$. But then $e \in F^{\mathrm{odd}}_j$, as claimed.

Case 2. Consider now the case $t = T + 2j$ (with $j \in [\ell - k]$). By induction, we know

$$E(G'_{t-1}) = \binom{\{x_1, \dots, x_{k+j-1}\}}{3} \cup \mathcal{E} \cup F^{\mathrm{odd}}_{j}.$$

Observe first that, whenever $a \in [k+j-2]$, we have $x_{k+j}x_{k+j-1}x_a \in F^{\mathrm{odd}}_j \subseteq E(G'_{t-1})$. It follows that, whenever $a, b \in [k+j-2]$ are distinct, the copy of $K_4^{(3)}$ with vertex set $\{x_{k+j}, x_{k+j-1}, x_a, x_b\}$ is 2-completable in G'_{t-1}. Hence, $x_{k+j}x_cx_d \in E(G'_t)$ whenever $c, d \in [k+j-1]$ (distinct). Thus, $\binom{\{x_1,\dots,x_{k+j}\}}{3} \subseteq E(G'_t)$. Furthermore, assume that $j \neq \ell - k$ and $e \in F^{\mathrm{even}}_j$, so $e = x_{k+j+1}wx_a$ with $w \in W_j$ and $a \in \{j+1, k+j\}$. Then, $e_{j+1} = x_{k+j+1}x_{k+j}x_{j+1} \in E(G'_{t-1})$ and $x_{k+j}wx_{j+1} \in F^{\mathrm{odd}}_j \subseteq E(G'_{t-1})$, so the copy of $K_4^{(3)}$ with vertex set $\{x_{k+j+1}, x_{k+j}, x_{j+1}, w\}$ is 2-completable in G'_{t-1}, which implies $x_{k+j+1}wx_a \in E(G'_t)$. So $F^{\mathrm{even}}_j \subseteq E(G'_t)$.

It remains to show that any edge added in this step must belong to $\binom{\{x_1,\dots,x_{k+j}\}}{3} \cup F^{\mathrm{even}}_j$. Indeed, as in the previous case, we see that any copy of $K_4^{(3)}$ which is 2-completable in G'_{t-1} must have vertex set $\{x_a, x_b, x_c, x_d\}$ with $a, b, c, d \in [k+j+1]$ (distinct). So any edge which is infected at time t is of the form $e = x_ax_bx_c$ with $a, b, c \in [k+j+1]$. If $a, b, c \in [k+j]$, the containment holds trivially, so we may assume that $c = k+j+1$. In order for e to become infected at time t, we must have that $x_{k+j+1}x_ax_d$ or $x_{k+j+1}x_bx_d$ appears in $E(G'_{t-1})$; we may assume that it is the former. But this edge must be $e_{j+1} = x_{k+j+1}x_{k+j}x_{j+1}$, and hence $\{a, d\} = \{j+1, k+j\}$. It also follows that $x_{k+j+1}x_bx_d \notin E(G'_{t-1})$, and hence $x_ax_bx_d \in E(G'_{t-1})$, i.e., $x_{k+j}x_bx_{j+1} \in E(G'_{t-1})$. This implies $x_{k+j}x_bx_{j+1} \in F^{\mathrm{odd}}_j$ and, therefore, $x_b \in W_j$. So $c = k+j+1$, $a \in \{j+1, k+j\}$ and $x_b \in W_j$, hence $x_ax_bx_c \in F^{\mathrm{even}}_j$, as claimed. ◀

By the claim above, $G'_{T+2(\ell-k)}$ is complete, but $G'_{T+2(\ell-k)-1}$ is not unless $\ell = 2k-2$ (indeed, if $\ell \neq 2k-2$, then $x_{k-2}x_{k-1}x_\ell \notin E(G'_{T+2(\ell-k)-1})$). Moreover, if $\ell = 2k-2$, then $G'_{T+2(\ell-k-1)} = G'_{T+2k-6}$ does not contain an edge in which both x_{2k-2} and x_{k-1} appear. The statement of the lemma follows. □

We are ready to deduce the lower bound.

Lemma 9. *For all $n \geq 4$ we have $M_{(K_4^{(3)},2)}(n) \geq 2n - \lfloor \log_2(n-2) \rfloor - 6$.*

Proof. Observe first that there is a nice 3-uniform hypergraph G_0 on 4 vertices $\{x_1, x_2, x_3, x_4\}$, given by $E(G_0) = \{x_1x_2x_3, x_2x_3x_4\}$, for which the running time of the $(K_4^{(3)}, 2)$-process is $T = 1$.

A straightforward induction using Lemma 8 shows that for all $m \geq 1$ there exists a nice hypergraph on 2^m+2 vertices for which the running time of the $(K_4^{(3)}, 2)$-process is $2^{m+1}-(m+2)$. Furthermore, also by Lemma 8, whenever $2^m + 2 \leq n < 2^{m+1} + 2$ we have

$$M_{(K_4^{(3)},2)}(n) \geq 2^{m+1} - (m+2) + 2(n - 2^m - 2) = 2n - m - 6 = 2n - \lfloor \log_2(n-2) \rfloor - 6,$$

as claimed. □

We now turn to the proof of the upper bound. For any $t \geq 1$, let $m := m(t)$ denote the unique positive integer which satisfies

$$2^{m+1} - (m+2) \leq t < 2^{m+2} - (m+3).$$

The following key lemma essentially shows that the infections must contain a substructure similar to the one in our construction.

Lemma 10. *Let G_0 be a 3-uniform hypergraph on $n \geq 4$ vertices, and consider the $(K_4^{(3)}, 2)$-process $G_0, G_1, \ldots$ with G_0 as the initial infection. Assume that $a \geq 1$ and $e \in E(G_a) \setminus E(G_{a-1})$. Then, there exist some $t_e \geq a$, $S_e \subseteq V(G_0)$, $v_e \in S_e$ and $W_e \subseteq S_e \setminus \{v_e\}$ such that, for $m = m(t_e)$,*

(P1) $e \subseteq S_e$,
(P2) $|S_e| = (t_e + m + 6)/2$,
(P3) $|W_e| = t_e - (2^{m+1} - (m+2))$,
(P4) $G_{t_e}[S_e]$ *is complete, and*
(P5) $\binom{S_e \setminus \{v_e\}}{3} \cup \{v_e w s : w \in W_e, s \in S_e \setminus \{v_e, w\}\} \subseteq E(G_{t_e - 1})$.

Proof. We prove the statement by induction on a. If $a = 1$, then we know e is in some copy H of $K_4^{(3)}$ in G_1. We can set $t_e = 1$ (so $m = 1$), $S_e = V(H)$, $v_e \in V(H)$ such that $V(H) \setminus \{v_e\} \in E(G_0)$, and $W_e = \varnothing$; the properties (P1)–(P5) are then satisfied.

Now assume that $a \geq 2$ and the statement holds for smaller values of a. We know there is some copy H of $K_4^{(3)}$ in G_a such that $e \in E(H)$, $|E(H) \cap E(G_{a-1})| \geq 2$ and $|E(H) \cap E(G_{a-2})| < 2$. It follows that there is some $f \in E(H)$ such that $f \in E(G_{a-1}) \setminus E(G_{a-2})$, i.e., f is infected at time $a-1$. Furthermore, there is another edge $f' \in E(H) \cap E(G_{a-1})$ (with $f' \neq f$). Let us write $t := t_f$, $S := S_f$, $W := W_f$ and $v := v_f$, and let $m = m(t)$. We consider several cases according to how e, f and f' overlap with S (and v, W). Note that, if $e \not\subseteq S$, then there is some $p \in V(G_0)$ such that $e \setminus f = e \setminus S = \{p\}$ and $p \in f'$.

Case 1: $e \subseteq S$. Since $e \notin E(G_{a-1})$ and $G_t[S]$ is complete, we have $t \geq a$. It follows that $t_e = t$, $S_e = S$, $v_e = v$, $W_e = W$ satisfy properties (P1)–(P5).

Case 2: $e \not\subseteq S$, $f' \in E(G_{t-1})$, and $f' = pss'$ for some $s, s' \in S \setminus W$ (recall $\{p\} = e \setminus S$). By (P5) for f, we know that, whenever $w \in W$, we have $wss' \in E(G_{t-1})$. This, together with the fact that $f' = pss' \in E(G_{t-1})$, guarantees that

$$pws, pws' \in E(G_t). \tag{3.1}$$

Let $u \in W \cup \{s, s'\}$ and $u' \in \{s, s'\}$ with $u' \neq u$, and let $z \in S \setminus \{s, s'\}$ with $z \neq u$. Then, (3.1) and our assumption on f' tell us that $puu' \in E(G_t)$, and by (P4) for f we have that $uu'z \in E(G_t)$. This implies $puz \in E(G_{t+1})$. That is, for all $u \in W \cup \{s, s'\}$ and all $z \in S \setminus \{u\}$ we have $puz \in E(G_{t+1})$. This in turn implies that $G_{t+2}[S \cup \{p\}]$ is complete (as $psz, psz' \in E(G_{t+1})$ implies $pzz' \in E(G_{t+2})$).

If $t = 2^{m+2} - (m+4)$, then $|W| = 2^{m+1} - 2$ and $|S| = 2^{m+1} + 1$, so we have $|W \cup \{s, s'\}| = |S| - 1$ and hence $G_{t+1}[S \cup \{p\}]$ is complete by the observation above that $puz \in E(G_{t+1})$ for all $u \in W \cup \{s, s'\}$ and $z \in S \setminus \{u\}$. Hence, $t_e = t + 1$, $S_e = S \cup \{p\}$, $v_e = p$, $W_e = \varnothing$ satisfy properties (P1)–(P5) (note that in this case $m(t_e) = m + 1$).

On the other hand, if $t \neq 2^{m+2} - (m+4)$, then $t \leq 2^{m+2} - (m+6)$ (since $t + m = 2|S| - 6$ is even by (P2)). So $m(t+2) = m(t)$. It follows that $t_e = t + 2$, $S_e = S \cup \{p\}$, $v_e = p$, $W_e = W \cup \{s, s'\}$ satisfy the properties.

Case 3: $e \not\subseteq S$, $f' \in E(G_{t-1})$, and $f' = pws$ for some $w \in W$, $s \in S$, $\{p\} = e \setminus S$. Then, by (P5) for f, whenever $z \in S \setminus \{w, s\}$ we have $wsz \in E(G_{t-1})$. Since $f' = pws \in E(G_{t-1})$, it follows that $pwz \in E(G_t)$ for all $z \in S \setminus \{w\}$. Therefore, whenever $z, z' \in S \setminus \{w\}$ are distinct, we have $pwz, pwz' \in E(G_t)$, and hence $pzz' \in E(G_{t+1})$. Thus, $G_{t+1}[S \cup \{p\}]$ is complete.

If $t = 2^{m+2} - (m+4)$, then $t_e = t+1$, $S_e = S \cup \{p\}$, $v_e = p$, $W_e = \varnothing$ satisfy properties (P1)–(P5).

On the other hand, assume that $t < 2^{m+2} - (m+4)$ (as in case 2, we then have $m(t+2) = m(t)$). Then, (P2) and (P3) imply that $|W| < |S| - 3$. Let W' be an arbitrary subset of S of size $|W| + 2$. Then, $t_e = t+2$, $S_e = S \cup \{p\}$, $v_e = p$, $W_e = W'$ satisfy properties (P1)–(P5).

Case 4: $e \not\subseteq S$, $f' \notin E(G_{t-1})$. Then, we must have $t = a - 1$ and $f' \in E(G_t) \setminus E(G_{t-1})$. We may assume that $t_{f'} = t$, since otherwise we can swap the roles of f and f' and we are done by the previous cases. Let us write S' for $S_{f'}$. Note that $|S| = |S'|$, and $S \cap S' \supseteq f \cap f'$ has size at least 2.

Assume first that $S' \setminus S = \{p\}$ for some $p \in V(G_0)$ (where necessarily $\{p\} = e \setminus f$). Then, $S \setminus S' = \{q\}$ for some $q \in V(G_0)$ (with $\{q\} = e \setminus f'$). Observe that, by (P4), whenever $s, s' \in S \cap S'$ are distinct, we have $pss' \in E(G_t)$ and $qss' \in E(G_t)$. This implies that $pqs \in E(G_{t+1})$ for every $s \in S \cap S'$. Hence, $G_{t+1}[S \cup S']$ is complete, where $|S \cup S'| = |S| + 1$. If $t = 2^{m+2} - (m+4)$, then properties (P1)–(P5) are satisfied for $t_e = t+1$, $S_e = S \cup S'$, $v_e = p$ and $W_e = \varnothing$. Otherwise, $t \leq 2^{m+2} - (m+6)$ (as $t+m$ is even), so (P1)–(P5) are satisfied for $t_e = t+2$, $S_e = S \cup S'$, $v_e = p$, and W_e an arbitrary subset of S of size $|W| + 2$.

Now assume that $|S' \setminus S| \geq 2$. Observe that, whenever $x, y \in S \cap S'$ (distinct), $s \in S \setminus S'$ and $s' \in S' \setminus S$, by (P4) we know $xys, xys' \in E(G_t)$, and therefore $xss' \in E(G_{t+1})$. By the same argument, if $\bar{s}' \in S' \setminus S$ (with $\bar{s}' \neq s'$), then $xs\bar{s}' \in E(G_{t+1})$. This then implies that $ss'\bar{s}' \in E(G_{t+2})$. Similarly, if $s' \in S' \setminus S$ and $s, \bar{s} \in S \setminus S'$ (distinct), then $s\bar{s}s' \in E(G_{t+2})$. Hence, $G_{t+2}[S \cup S']$ is complete, where $|S \cup S'| \geq |S| + 2$. Now pick any two vertices $p, p' \in S' \setminus S$ with $e \subseteq S \cup \{p, p'\}$. If $t = 2^{m+2} - (m+6)$, then let $t_e = t+3$, $S_e = S \cup \{p, p'\}$, $v_e = p$ and $W_e = \varnothing$. If $t = 2^{m+2} - (m+4)$, then let $t_e = t+3$, $S_e = S \cup \{p, p'\}$, $v_e = p$ and W_e an arbitrary subset of $S \cap S'$ of size 2. Finally, assume $t < 2^{m+2} - (m+6)$. Since $t+m$ is even, we have $t \leq 2^{m+2} - (m+8)$. Then, let $t_e = t+4$, $S_e = S \cup \{p, p'\}$, $v_e = p$ and W_e an arbitrary subset of S of size $|W| + 4$. These choices satisfy properties (P1)–(P5). This finishes the proof of the lemma. □

Proof of Theorem 4. The lower bound follows from Lemma 9. For the upper bound, given an arbitrary hypergraph G_0 on a vertex set V of size $n \geq 4$, consider the $(K_4^{(3)}, 2)$-process $G_0, G_1, \ldots$ with G_0 as initial infection. Let $T := M_{(K_4^{(3)},2)}(G_0)$ be the running time of the process, and let $e \in E(G_T) \setminus E(G_{T-1})$ be arbitrary. By Lemma 10, there is some $t \geq T$ such that G_t contains a clique of size $\frac{t+m(t)+6}{2} \geq \frac{T+m(T)+6}{2}$. Hence,

$$\frac{T+m(T)+6}{2} \leq n.$$

It follows that

$$T \leq 2n - \lfloor \log_2(n-2) \rfloor - 6,$$

as we wanted to prove. Indeed, if $\lfloor \log_2(n-2) \rfloor = \alpha$, then $2^\alpha + 2 \leq n \leq 2^{\alpha+1} + 1$ and hence $2^{\alpha+1} - (\alpha+2) \leq 2n - \alpha - 6 < 2^{\alpha+2} - (\alpha+3)$, giving $m(2n-\alpha-6) = \alpha$ and $\frac{(2n-\alpha-6)+m(2n-\alpha-6)+6}{2} = n$. □

3.2. Double infections for $K_5^{(3)}$

We now turn our attention to Theorem 5. In order to prove it, observe that, for an r-uniform hypergraph H, the trivial upper bound $M_{(H,k)}(n) \leq \binom{n}{r}$ still holds, so it suffices to provide a lower bound. We will proceed by constructing an initial infection for which the $(K_5^{(3)}, 2)$-bootstrap percolation process runs for a cubic number of steps. At most two edges will become infected in each step of the infection process, which will make it easier to analyse the number of steps. Our construction is intuitively similar to the one we constructed for the proof of Theorem 3, albeit a bit more convoluted. Let us begin with an intuitive description.

Our vertex set will again be split into three layers, with vertices t_i playing the role of 'top' vertices, vertices b_j playing the role of 'bottom' vertices, and vertices m_ℓ conforming the 'middle' layer. We will also have a number of 'dummy' vertices. For fixed top and bottom vertices, the vertices in the middle layer will allow us to infect two edges at a time, while traversing this layer, for a linear total number of steps. For each fixed bottom vertex, there will be some extra edges at the end of the middle layer which will allow us to swap the bottom vertex for the next one and continue the process. Finally, the dummy vertices will allow us to swap the top vertex and start the process anew.

To be more precise, let us describe the first few stages of the infection process. For each $0 \leq \ell \leq 2n-2$, our initial infection will contain all edges of the copy of $K_5^{(3)}$ with vertex set $t_1, b_1, m_\ell, m_{\ell+1}, m_{\ell+2}$ except for $t_1m_\ell b_1$, $t_1m_{\ell+1}b_1$, and $t_1m_{\ell+2}b_1$. It will also contain $t_1m_0b_1$. This edge will trigger the infection of $t_1m_1b_1$ and $t_1m_2b_1$ in the first step of the process, then of $t_1m_3b_1$ and $t_1m_4b_1$, and the infection will keep propagating towards higher values of ℓ, until finally, after n steps, the edges $t_1m_{2n-1}b_1$ and $t_1m_{2n}b_1$ become infected.

At this point, we will swap out b_1 to b_{-1}. This can be achieved in two steps of the infection process. Our initial infection will already contain all edges of the copy of $K_5^{(3)}$ with vertex set $t_1, b_1, m_{2n}, m_{2n+1}, m_{2n+2}$ except for $t_1m_{2n}m_{2n+1}$, $t_1m_{2n+1}m_{2n+2}$, and the edge $t_1m_{2n}b_1$ which was just added in the previous step; $t_1m_{2n}m_{2n+1}$ and $t_1m_{2n+1}m_{2n+2}$ will therefore become infected in the next step. The initial infection also contains all the edges of the copy of $K_5^{(3)}$ with vertex set $t_1, b_{-1}, m_{2n}, m_{2n+1}, m_{2n+2}$ except for $t_1m_{2n}b_{-1}$, $t_1m_{2n+1}b_{-1}$, and the two that were just added. These two edges now become infected as well, and start a new infection process where now the indices decrease through the middle layer.

Finally, suppose we have reached a point where the copy of $K_5^{(3)}$ defined on the vertices t_1, b_n, m_{4n-4}, m_{4n-3}, m_{4n-2} has been completely infected, with the edges infected in the last step of the process being $t_1m_{4n-3}b_n$ and $t_1m_{4n-2}b_n$. We now want to swap out the top vertex to t_2, using for this purpose four dummy vertices. Similarly as in the proof of Theorem 3, these will simply create a short chain of infections that allows us to restart the process.

We now give a formal proof.

Proof of Theorem 5. Consider an initial infection hypergraph G_0 whose vertex set consists of $12n-5$ vertices, labelled as $t_1, \ldots, t_n$, $b_1, \ldots, b_n$, $b_{-1}, \ldots, b_{-(n-1)}$, $m_{-2(n-1)}, \ldots, m_{4n}$, and $d_{i,1}, d_{i,2}, d_{i,3}$ for $i \in [n-1]$. For notational purposes, for each $i \in [n]$ let $d_{i,-2} := t_i$, $d_{i,-1} := b_n$, $d_{i,0} := m_{4n-2}$, $d_{i,4} := m_0$, $d_{i,5} := b_1$, and $d_{i,6} := t_{i+1}$. The edges of G_0 appear in the following list:

(a) $t_1m_0b_1$;
(b) $m_\ell m_{\ell+1}m_{\ell+2}$, for all $-2(n-1) \leq \ell \leq 4n-4$;
(c) $t_im_\ell m_{\ell+2}$, for all $i \in [n]$ and $-2(n-1) \leq \ell \leq 4n-4$;
(d) $b_jm_\ell m_{\ell+2}$, for all $j \in [n]$ and $-2(j-1) \leq \ell \leq 2(n+j-1)$;
(e) $b_{-j}m_\ell m_{\ell+2}$, for all $j \in [n-1]$ and $-2j \leq \ell \leq 2(n+j-1)$;
(f) $t_im_\ell m_{\ell+1}$, for all $i \in [n]$ and $0 \leq \ell \leq 2n-1$;
(g) $b_jm_\ell m_{\ell+1}$, for all $j \in [n]$ and $-2(j-1) \leq \ell \leq 2(n+j)-1$;
(h) $b_{-j}m_\ell m_{\ell+1}$, for all $j \in [n-1]$ and $-2j \leq \ell \leq 2(n+j)-1$;
(i) $t_im_{2(n+j)-1}b_j$, $t_im_{2(n+j)}b_j$ and $t_im_{2(n+j)}b_{-j}$, for all $i \in [n]$ and $j \in [n-1]$;
(j) $t_im_{-2j+1}b_{-j}$, $t_im_{-2j}b_{-j}$ and $t_im_{-2j}b_{j+1}$, for all $i \in [n]$ and $j \in [n-1]$;
(k) $d_{i,j}d_{i,j+1}d_{i,j+3}$, $d_{i,j}d_{i,j+1}d_{i,j+4}$, $d_{i,j}d_{i,j+2}d_{i,j+3}$, $d_{i,j}d_{i,j+2}d_{i,j+4}$, $d_{i,j}d_{i,j+3}d_{i,j+4}$, $d_{i,j+1}d_{i,j+2}d_{i,j+3}$, and $d_{i,j+1}d_{i,j+3}d_{i,j+4}$, for all $i \in [n-1]$ and $j \in \{-2, 0, 2\}$.

To compare this with the construction hinted at before the proof, consider the following. The edge in (a) is an edge e_0 which starts the whole infection process. The edges in (b)–(h) are there to ensure the correct propagation of the infection through the middle layer, where the edges in (d) and (g) will be used to propagate the infection towards larger values of ℓ, using some bottom vertex of the form b_j, and those in (e) and (h) will be used to propagate the infection towards smaller values of ℓ, using some bottom vertex of the form b_{-j}. The edges in (i) and (j) are needed for swapping the bottom vertices. Finally, the edges which appear in (k) are used to swap the top vertices.

For each pair (i,j) with $i,j \in [n]$, let $A_{i,j}$ be the sequence of edges

$$A_{i,j} := (t_i m_{-2(j-1)+\ell} b_j)_{\ell=1}^{2n+4(j-1)}. \tag{3.2}$$

Similarly, for each pair (i,j) with $i \in [n]$ and $j \in [n-1]$, we define

$$A_{i,-j} := (t_i m_{2(n+j)-\ell} b_{-j})_{\ell=1}^{2n+4(j-1)+2}. \tag{3.3}$$

Note that each of these has an even number of elements. Using $\bigtimes$ to denote concatenation, for each phase $i \in [n]$ we define the sequence

$$A_i := A_{i,1} \bigtimes_{j=1}^{n-1} (t_i m_{2(n+j-1)} m_{2(n+j)-1}, t_i m_{2(n+j)-1} m_{2(n+j)}) A_{i,-j} (t_i m_{-2(j-1)} m_{-2j+1}, t_i m_{-2j+1} m_{-2j}) A_{i,j+1}. \tag{3.4}$$

Finally, we set

$$\begin{aligned} A &:= A_1 \bigtimes_{i=1}^{n-1} (d_{i,-1}d_{i,0}d_{i,2}, d_{i,0}d_{i,1}d_{i,2}, d_{i,1}d_{i,2}d_{i,4}, d_{i,2}d_{i,3}d_{i,4}, d_{i,3}d_{i,4}d_{i,6}, d_{i,4}d_{i,5}d_{i,6}) A_{i+1} \\ &= A_1 \bigtimes_{i=1}^{n-1} (b_n m_{4n-2} d_{i,2}, m_{4n-2} d_{i,1} d_{i,2}, d_{i,1} d_{i,2} m_0, d_{i,2} d_{i,3} m_0, d_{i,3} m_0 t_{i+1}, m_0 b_1 t_{i+1}) A_{i+1}. \end{aligned} \tag{3.5}$$

We will sometimes abuse notation and treat each of the above sequences as sets. Observe that A has an even number of elements and that none appear repeatedly. We may label these elements as $(e_1, e'_1, e_2, e'_2, \ldots, e_T, e'_T)$, for some $T > 0$. Note that $T = 4n^3 + O(n^2)$ by construction. Note, moreover, that by construction we are guaranteed that $|\{e_t, e'_t\} \cap A_{i,j}| \in \{0, 2\}$ for all i and j. Additionally, any two consecutive triples in A share exactly two vertices, thus, it is easy to check that any three consecutive triples span five vertices.

Let $e'_0 := t_1 m_0 b_1$. For each $t \in [T-1]$, let H_t denote the copy of $K_5^{(3)}$ with vertex set $e'_{t-1} \cup e_t \cup e'_t$, and let H'_t denote the copy of $K_5^{(3)}$ with vertex set $e_{t-1} \cup e'_{t-1} \cup e_t$ (if $t > 1$). For each $t \in [T]$, let G_t be the hypergraph with edge-set $E(G_{t-1}) \cup \{e_t, e'_t\}$. We will show that these hypergraphs indeed coincide with those obtained by the $K_5^{(3)}$-bootstrap percolation process with initial infection G_0.

Claim 3. *Let H be a copy of $K_5^{(3)}$ on the vertex set of G_0. Assume that, for some $t \in [0, T-1]$, we have that $H \nsubseteq G_t$ but H is 2-completable in G_t. Then, the following hold.*

- *If H is 1-completable in G_t, suppose that adding e to G_t completes H. Then, $t \geq 1$, $e = e_{t+1}$ and $H = H'_{t+1}$.*
- *If H is not 1-completable in G_t, suppose that adding e and e' to G_t completes H. Then, $\{e, e'\} = \{e_{t+1}, e'_{t+1}\}$ and $H = H_{t+1}$.*

Proof. Consider any copy H of $K_5^{(3)}$ on $V(G_0)$. If H contains two vertices of the form t_i and $t_{i'}$ with $1 \leq i < i' \leq n$, since G_T does not contain any edge with two 'top' vertices (see (a)–(k) as well as (3.2)–(3.5)), H must be missing at least three edges in G_T. As $G_0 \subseteq G_1 \subseteq \ldots \subseteq G_T$, it follows that H is not 2-completable in G_t for any $t \in [0, T-1]$. The same argument holds if H contains two vertices of the form b_j and $b_{j'}$. Hence, we may assume that H contains at most one vertex t_i and one vertex b_j. Similarly, if H contains two vertices m_ℓ and $m_{\ell'}$ with $|\ell - \ell'| \geq 3$, then G_T does not contain any edge containing both m_ℓ and $m_{\ell'}$, so H is not 2-completable in G_T. Therefore, H contains at most three vertices of the form m_ℓ, and their indices must be within distance two of each other.

Assume first that H contains some vertex of the form $d_{i,c}$ with $i \in [n-1]$ and $c \in [3]$. All triples in G_T containing one such vertex are of the form $d_{i,r}d_{i,p}d_{i,q}$ with $-2 \leq r < p < q \leq 6$ and $q - r \leq 4$ (see (k) and (3.5)). It follows easily that, if $V(H)$ does not consist of five consecutive vertices $d_{i,p}, d_{i,p+1}, \ldots, d_{i,p+4}$ with $-2 \leq p \leq 2$, then H cannot be 2-completable in G_T. Moreover, if we assume $V(H) = \{d_{i,p+h} : 0 \leq h \leq 4\}$ for some $i \in [n-1]$ and $p \in \{-1, 1\}$, then we also know that the triples $d_{i,p}d_{i,p+1}d_{i,p+4}$, $d_{i,p}d_{i,p+2}d_{i,p+4}$ and $d_{i,p}d_{i,p+3}d_{i,p+4}$ do not appear in G_T, so again H cannot be 2-completable. So we must have $V(H) = \{d_{i,p+h} : 0 \leq h \leq 4\}$ with $p \in \{-2, 0, 2\}$. But then, by (k) and (3.5), the only three edges missing from H in G_0 are e'_{N-1}, e_N and e'_N, for some $N \in [T]$. Let $t \in [0, T-1]$ be such that H is 2-completable in G_t but $E(H) \not\subseteq E(G_t)$. It is easy to see that we must have $t = N-1$; furthermore, $H = H_{t+1}$, H is not 1-completable in G_t, and $E(G_{t+1}) \setminus E(H) = \{e_N, e'_N\}$, as desired.

Assume next that H does not contain any vertex of the form $d_{i,c}$ with $i \in [n-1]$ and $c \in [3]$, so it must contain one top vertex t_i, one bottom vertex b_j, and three consecutive middle vertices m_ℓ, $m_{\ell+1}$ and $m_{\ell+2}$. Assume $j > 0$ (the other case can be argued analogously). If $\ell \geq 2(n+j)-1$, then G_T is missing the edges $t_im_{\ell+2}b_j$, $b_jm_{\ell+1}m_{\ell+2}$ and $b_jm_\ell m_{\ell+2}$ (see (d), (g), (i), (3.2) and (3.4)), so H cannot be 2-completable at any stage of the process. Similarly, if $\ell < -2(j-1)$, then G_T is missing the edges $t_im_\ell b_j$, $b_jm_\ell m_{\ell+1}$ and $b_jm_\ell m_{\ell+2}$ (see (d), (g), (j), (3.2) and (3.4)), hence H is not 2-completable. Thus, we must have $-2(j-1) \leq \ell \leq 2(n+j-1)$. However, for the case when $j = n$ and $\ell \in \{4n-3, 4n-2\}$, it follows from (b), (c), (f) and (3.4) that G_T is missing the triples $m_\ell m_{\ell+1}m_{\ell+2}$, $t_im_\ell m_{\ell+2}$ and $t_im_{\ell+1}m_{\ell+2}$, hence H cannot be 2-completable. So we must have $-2(j-1) \leq \ell \leq 2(n+j-1)$ when $j \in [n-1]$ and $-2(j-1) \leq \ell \leq 2(n+j-2)$ when $j = n$. Let $t \in [0, T-1]$ be such that H is 2-completable in G_t but $E(H_t) \not\subseteq E(G_t)$. We now split the analysis into further cases.

Assume first that $j < n$ and $\ell = 2(n+j-1)$. It follows from (b)–(j) that the only triples of H missing from G_0 are $t_im_\ell b_j$, $t_im_\ell m_{\ell+1}$ and $t_im_{\ell+1}m_{\ell+2}$. These three are added throughout the sequence of edges defined above, as follows from (3.2) and (3.4), as e'_{N-1}, e_N and e'_N, respectively, for some $N \in [T]$. Then, in order for H to be 2-completable in G_t, we must have $e'_{N-1} \in E(G_t)$, and hence $t = N$, $H = H_{t+1}$, H is not 1-completable, and $E(G_{t+1}) \setminus E(H) = \{e_N, e'_N\}$, as desired.

Consider next the case that $j < n$ and $\ell = 2(n+j-1)-1$. The triples of H missing from G_0 are $t_im_\ell b_j$, $t_im_{\ell+1}b_j$ and $t_im_{\ell+1}m_{\ell+2}$ (see (b)–(j)), which are e_{N-1}, e'_{N-1} and e_N, respectively, for some $N \in [T]$ (see (3.2) and (3.4)). Thus, in order for H to be 2-completable in G_t, this hypergraph must contain at least one of the missing triples; however, since e_{N-1} and e'_{N-1} are added simultaneously in the sequence of hypergraphs, we must have $e_{N-1}, e'_{N-1} \in E(G_t)$, and so H is 1-completable. Then, the only edge that can complete H is $e = e_N$, and so it follows that $N = t+1$ and $H = H'_{t+1}$.

Assume now that $\ell = -2(j-1)$. Here we have two further subcases. If $j = 1$, then the only triples of H missing in G_0 are precisely e_1 and e'_1 (see (a)–(j) as well as (3.2)). Therefore, we have $\{e, e'\} = \{e_1, e'_1\}$ and $H = H_1$. So suppose that $j \geq 2$. Then, the triples of H missing in G_0 are $t_im_\ell m_{\ell+1}$, $t_im_{\ell+1}m_{\ell+2}$, $t_im_{\ell+1}b_j$ and $t_im_{\ell+2}b_j$ (see (b)–(j)). But then it follows from (3.2) and (3.4) that these triples take the form e_{N-1}, e'_{N-1}, e_N, e'_N, for some $N \in [T]$. In order for H to be 2-completable in G_t, we must have $e_{N-1}, e'_{N-1} \in E(G_t)$. Then, it follows that $t = N-1$, $H = H_{t+1}$, H is not 1-completable, and $E(G_{t+1}) \setminus E(H) = \{e_N, e'_N\}$.

Suppose now that $\ell = -2(j-1)+1$. Again, we must consider two subcases. If $j = 1$, the edges of H missing in G_0 are $t_im_\ell b_j$, $t_im_{\ell+1}b_j$ and $t_im_{\ell+2}b_j$, which correspond to e_1, e'_1 and e_2 (see (b)–(j) as well as (3.2)). Thus, in order for H to be 2-completable in G_t we must have $e_1, e'_1 \in E(G_t)$, so $t = 1$ and H is 1-completable in G_1. It then follows that $e = e_2$, and $H = H'_2$. So suppose $j \geq 2$. Then, the triples of H missing in G_0 are $t_im_\ell m_{\ell+1}$, $t_im_\ell b_j$, $t_im_{\ell+1}b_j$ and $t_im_{\ell+2}b_j$ (see (b)–(j)). By (3.2) and (3.4), these triples take the form e'_{N-2}, e_{N-1}, e'_{N-1}, e_N, for some $N \in [2, T]$. In order for H to be 2-completable in G_t, at least two of these edges must be added. But e_{N-1} and e'_{N-1} are added simultaneously, so we conclude that $e'_{N-2}, e_{N-1}, e'_{N-1} \in E(G_t)$ and H is 1-completable in G_t. Therefore, $E(G_{t+1}) \setminus E(H) = \{e_N\}$, $N = t+1$ and $H = H'_{t+1}$.

Suppose finally that $-2(j-2) \leq \ell \leq 2(n+j-2)$. Then, the only edges of H missing in G_0 are $t_im_\ell b_j$, $t_im_{\ell+1}b_j$ and $t_im_{\ell+2}b_j$ (see (b)–(j)). If ℓ is even, it follows from (3.2) that these edges take the form e'_{N-1}, e_N and e'_N, respectively, for some $N \in [T]$; on the contrary, if ℓ is odd, then they take the form e_{N-1}, e'_{N-1} and e_N. In the former case, in order for H to be 2-completable in G_t, we must have $e'_{N-1} \in E(G_t)$, and it follows that $E(G_{t+1}) \setminus E(H) = \{e_N, e'_N\}$, $N = t+1$ and $H = H_{t+1}$. In the latter case, we must have $e_{N-1}, e'_{N-1} \in E(G_t)$, so H is 1-completable, and it follows that $E(G_{t+1}) \setminus E(H) = \{e_N\}$, $N = t+1$ and $H = H'_{t+1}$. ◂

By applying Claim 3 iteratively, we conclude that the $(K_5^{(3)}, 2)$-bootstrap percolation process with initial infection G_0 indeed generates the sequence of hypergraphs $G_0, G_1, \ldots, G_T, \ldots$, so its running time is at least $T = 4n^3 + O(n^2)$. By taking into account the number of vertices of the hypergraphs we are considering, we conclude that $M_{(K_5^{(3)},2)}(n) \geq 4n^3/12^3 + O(n^2)$. $\square$

4. Concluding remarks

Graph and hypergraph bootstrap percolation have seen a lot of research in recent years, with many intriguing questions remaining open and many possible avenues for further research. We have focused particularly on understanding the maximum running time of these processes. Our first main result, Theorem 3, building on the previous work of Noel and Ranganathan [18], has allowed us to conclude that the maximum running time of $K_k^{(r)}$-bootstrap percolation is of order $\Theta(n^r)$ for any $k > r \geq 3$. A first very natural problem is to determine the leading constant in this asymptotic behaviour.

Problem 11. For each $k > r \geq 2$, does $\lim_{n\to\infty} M_k^r(n)/n^r$ exist? If so, what is the value of the limit?

In particular, all results in this hypergraph context have relied on the trivial upper bound that $M_H^r(n) \leq \binom{n}{r}$; obtaining better upper bounds should be the first step towards this problem. We also note that the lower bound arising from our construction (see Remark 7) is not tight.

Another very natural direction is to study the asymptotic growth of $M_H(n)$ when H is an r-uniform hypergraph which is not complete. We have made the first progress in this direction by addressing two particular cases, see Theorems 4′ and 5′. A more general study of this problem for different instances of H is crucial towards a unified understanding of hypergraph bootstrap percolation.

More generally, the notion of more 'powerful' infections that we proposed when considering (H, m)-bootstrap percolation leads to many new open problems. Here we have only addressed two particular instances to showcase that this notion leads to interesting results. In the case of $(K_5^{(3)}, 2)$-bootstrap percolation, Theorem 5 shows that the maximum running time is cubic, that is, as large as it could possibly be (up to constant factors). In the case of $(K_4^{(3)}, 2)$-bootstrap percolation, however, the maximum running time is only linear, and in Theorem 4 we have determined the exact value of this maximum running time for all values of n. Remarkably, this is the only nontrivial exact result in the area other than those for K_3- and K_4-bootstrap percolation [7]. It would certainly be desirable to understand the behaviour of the maximum running time of (H, m)-bootstrap percolation more generally. To begin, we propose the following problem.

Problem 12. Given $k > r \geq 2$ and $m \in [\binom{k}{r}]$, determine the asymptotic behaviour of the maximum running time of the $(K_k^{(r)}, m)$-bootstrap percolation process.

It would also be interesting to consider this more general (H, m)-bootstrap percolation process in other contexts where (hyper)graph bootstrap percolation has been studied. In particular, one may consider the extremal problem, i.e., what is the minimum number of edges an initial r-uniform infection G_0 on n vertices can have if we know the (H, m)-percolation process $G_0, \ldots, G_T$ satisfies $G_T = K_n^{(r)}$?

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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