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Projective freeness and Hermiteness of complex function algebras

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The paper studies projective freeness and Hermiteness of algebras of complex-valued continuous functions on topological spaces, Stein algebras, and commutative unital Banach algebras. New sufficient cohomology conditions on the maximal ideal spaces of the algebras are given that guarantee the fulfilment of these properties. The results are illustrated by nontrivial examples. Based on the Borsuk theory of shapes, a new class $\mathscr{C}$ of commutative unital complex Banach algebras is introduced (an analog of the class of local rings in commutative algebra) such that the projective tensor product with algebras in $\mathscr{C}$ preserves projective freeness and Hermiteness. Some examples of algebras of class $\mathscr{C}$ and of other projective free and Hermite function algebras are assembled. These include, e.g., Douglas algebras, finitely generated algebras of symmetric functions, Bohr-Wiener algebras, algebras of holomorphic semi-almost periodic functions, and algebras of bounded holomorphic functions on Riemann surfaces.

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1. Introduction

In this article, we study projective freeness and Hermiteness of certain function algebras. Recall that a unital commutative ring R is said to be *projective free* if every finitely generated projective R-module is free (i.e., if M is an R-module such that $M \oplus N \cong R^n$ for an R-module N and $n \in \mathbb{Z}_+$ $(:= \mathbb{N} \cup \{0\})$, then $M \cong R^m$ for some $m \in \mathbb{Z}_+$). In terms of matrices, the ring R is projective free if and only if for each $n \in \mathbb{N}$ every $n \times n$ matrix $X \notin \{0_n, I_n\}$ over R such that $X^2 = X$ (i.e., an idempotent) has a form $X = S(I_r \oplus 0_{n-r})S^{-1}$ for some $S \in GL_n(R)$, $r \in \{1, \dots, n-1\}$; here $GL_n(R)$ denotes the group of invertible $n \times n$ matrices over R and 0_k and I_k are zero and identity $k \times k$ matrices; see [13, Proposition 2.6].

Quillen and Suslin (see, e.g., [14]) proved, independently, that the polynomial ring over a projective free ring is again projective free; in particular settling affirmatively Serre's problem from 1955, which asked if the polynomial ring $\mathbb{F}[x_1, \dots, x_n]$ is projective free for any field $\mathbb{F}$ (see [32]). Also, if R is any projective free ring, then the formal power series ring $R[\![x]\!]$ is again projective free [14, Theorem 7], and so the formal power series ring $\mathbb{F}[\![x_1, \dots, x_n]\!]$ is projective free. In the context of rings arising in analysis, it is known that for example the algebra of complex continuous functions on a contractible topological space is projective free [56]. According to the Grauert theorem [24] and the Novodvorski-Taylor theory (see, e.g., [39], [54, §7.5]), the same holds for the algebra of holomorphic functions on a contractible reduced Stein space and a commutative unital complex Banach algebra with a contractible maximal ideal space.

In control theory, projective freeness of rings of stable transfer functions plays an important role in the stabilisation problem, since if the underlying ring of stable transfer functions is projective free, then the stabilisability of an unstable system is equivalent to the existence of a doubly coprime factorisation, see [40, Theorem 6.3].

The concept of a Hermite ring is a weaker notion than that of a projective free ring. A unital commutative ring R is *Hermite* if every finitely generated stably free R-module is free. An R-module M is *stably free* if it is stably isomorphic to a free module, i.e., $M \oplus R^k$ is free for some $k \in \mathbb{Z}_+$. Since every stably free module is projective, every projective free ring is Hermite. In terms of matrices, R is Hermite if and only if every left-invertible $n \times k$ matrix, $k, n \in \mathbb{N}$, $k < n$, over R can be completed to an invertible one (see, e.g., [32, p.VIII], [55, p.1029]).

In control theory, Hermiteness of the underlying ring of stable transfer functions implies that if the transfer function of an unstable system has a right (or left) coprime factorisation, then it has a doubly coprime factorisation; see [57, Theorem 66].

In the article, we study successively projective freeness and Hermiteness of algebras of complex-valued continuous functions on topological spaces (§2), Stein algebras (§3) and commutative unital complex Banach algebras (§4). We begin with a survey of some results describing the structure of finitely generated projective modules over these algebras in terms of complex vector bundles over their maximal ideal spaces using the Swan and Vaserstein theorems [52] and [56], the Grauert Oka principle [23], [24] and the Novodvorski-Taylor theory [39], [54]. Then, based on this description, we give new sufficient cohomology conditions for projective freeness and Hermiteness of the considered algebras and illustrate our results by nontrivial examples. In §5, we introduce and study a new class $\mathscr{C}$ of commutative unital complex Banach algebras A (namely those whose maximal ideal spaces are of trivial shape) such that for every commutative unital complex Banach algebra B the isomorphism classes of finitely generated projective modules over B are in a one-to-one correspondence with those over the projective tensor product $A\widehat{\otimes}_\pi B$. In particular, the projective tensor product with algebras in $\mathscr{C}$ preserves projective freeness and Hermiteness. Finally, in §6 several examples of algebras of class $\mathscr{C}$ and of projective free function algebras are assembled. These include finitely generated algebras of symmetric functions (§6.1), Bohr-Wiener algebras (§6.2), algebras of holomorphic semi-almost periodic functions (§6.3), and algebras of bounded holomorphic functions on Riemann surfaces (§6.4). The Appendix contains auxiliary results used in §6.

2. Algebras of continuous functions

For a topological space X, we denote by $C(X)$ the algebra (with pointwise operations) of complex-valued continuous functions on X. Also, by $C_b(X) \subset C(X)$ we denote the subalgebra of bounded functions. $C_b(X)$ equipped with the supremum norm, $\|f\|_\infty = \sup_{x\in X} |f(x)|$, is a complex commutative unital Banach algebra.

We recall that a bundle over X is of *finite type* if there is a finite set S of nonnegative continuous functions on X whose sum is 1 such that the restriction of the bundle to the set $\{x \in X \,:\, f(x) \neq 0\}$ is trivial for each f in S.

Every bundle over a compact Hausdorff space X is of finite type. In turn, a bundle over a normal space X is of finite type if and only if there is a finite open covering $\mathfrak{U}$ of X such that the restriction of the bundle to each $U \in \mathfrak{U}$ is trivial.

Let $\mathbf{Gr}(\mathbb{C}^n)$ be the set of all subspaces of $\mathbb{C}^n$ equipped with the topology of the disjoint union of the Grassmanian manifolds $\mathbf{Gr}_m(\mathbb{C}^n)$, $0 \leqslant m \leqslant n$. For each $n \in \mathbb{Z}_+$ there is a natural embedding $\mathbf{Gr}(\mathbb{C}^n) \subset \mathbf{Gr}(\mathbb{C}^{n+1})$ so that the space $\mathbf{Gr}(\mathbb{C}^\infty) = \bigcup_{n\in\mathbb{Z}_+} \mathbf{Gr}(\mathbb{C}^n)$, equipped with the inductive limit topology, is well-defined.

The next result follows from Vaserstein's extension [56, Theorem 2] of the Swan theorem [52].

Proposition 2.1. *The following statements are equivalent:*

(1) *The algebra $C(X)$ is projective free.*
(2) *The algebra $C_b(X)$ is projective free.*
(3) *The space X is connected and each finite type complex vector bundle over X is trivial.*
(4) *Each continuous map $X \to \mathbf{Gr}(\mathbb{C}^\infty)$ with a relatively compact image is homotopic, via a homotopy with relatively compact image, to a constant map.*

In what follows, for a topological space Y and an abelian group G, the n^{th} Čech cohomology group of Y with values in G is denoted by $H^n(Y, G)$, $n \in \mathbb{Z}_+$.

Let βX be the Stone-Čech compactification of X. Then βX is naturally homeomorphic to the maximal ideal space of $C_b(X)$, and $C(\beta X)$ is isometrically isomorphic to $C_b(X)$. Proposition 2.1 implies that if $C(X)$ is projective free, then $C(\beta X)$ is projective free, and so βX is connected and all finite rank complex vector bundles over βX are trivial.

Since isomorphism classes of rank one complex vector bundles over βX are in one-to-one correspondence (determined by assigning to a bundle its first Chern class) with elements of $H^2(\beta X, \mathbb{Z})$, projective freeness of $C(X)$ implies that $H^2(\beta X, \mathbb{Z}) = 0$. Also, in this case Proposition 2.1 implies that the Grothendieck group $K_0(C_b(X))$ of the algebraic K-theory for $C_b(X)$ (isomorphic to the group $K^0(\beta X)$ of the Atiyah-Hirzebruch theory) is $\mathbb{Z}$. Since the Chern character ch determines a ring isomorphism $K_0(C_b(X)) \otimes \mathbb{Q} \to H^{\text{even}}(\beta X, \mathbb{Q})$, see, e.g., [31, Theorem 3, p.16-09], we obtain the following result.

Corollary 2.2. *If the algebra $C(X)$ is projective free, then βX is connected, $H^2(\beta X, \mathbb{Z}) = 0$, and $H^{2n}(\beta X, \mathbb{Q}) = 0$ for all $n \geqslant 2$.*

For instance, if X is a connected closed orientable manifold such that the algebra $C(X)$ is projective free, then the corollary along with the Poincaré duality imply that X has an odd dimension n and $H^k(X, \mathbb{Q}) = 0$ for all $k \notin \{0, n\}$. Note that even if $H^k(X, \mathbb{Z}) = 0$ for all $k \notin \{0, n\}$, $C(X)$ is not necessarily projective free. In fact, it is not even necessarily Hermite; see Example 2.5 below.

Let $\Omega^n = X \times \mathbb{C}^n$ denote the standard trivial rank n complex vector bundle over a topological space X. A complex vector bundle E over X is *stably trivial* if there exist $m, n \in \mathbb{Z}_+$ such that the bundle $E \oplus \Omega^m$ is isomorphic to Ω^n.

Our next result can be easily deduced from [56, Theorem 2] as well.

Proposition 2.3. *The following statements are equivalent:*

(1) *The algebra $C(X)$ is Hermite.*
(2) *The algebra $C_b(X)$ is Hermite.*
(3) *Each stably trivial finite type complex vector bundle over X is trivial.*

Propositions 2.1 and 2.3 imply the following result.

Corollary 2.4. *Projective freeness and Hermiteness of the algebra $C(X)$ depend only on the homotopy type of X.*

Example 2.5. Let S^n denote the n-dimensional unit sphere. Cutting the sphere S^n into two 'bowls' (each homeomorphic to an n-dimensional ball) such that their intersection is a 'collar' C containing the subsphere S^{n-1}, it can be seen that the isomorphism classes $\text{Vect}_k(S^n)$ of rank k complex vector bundles on S^n are in a one-to-one correspondence with the elements of the $(n-1)^{\text{st}}$ homotopy group $\pi_{n-1}(U(k))$ of the unitary group $U(k)$. As the homotopy group $\pi_4(U(2)) \cong \pi_3(S^4) \cong \mathbb{Z}_2$, it follows that there exists a nontrivial rank 2 complex vector bundle E over S^5. Hence, $C(S^5)$ is not projective free, although $H^k(S^5, \mathbb{Z}) = 0$ for all $k \notin \{0, 5\}$. Moreover, since $K_0(C(S^5)) = K^0(S^5) = \mathbb{Z}$ (see, e.g., [30, Corollary 5.2 on p.121 and p.150 of Chap. 11]), every finite rank complex vector bundle on S^5 is stably trivial. Hence, $C(S^5)$ is not Hermite, as E is nontrivial.

Next, we formulate an auxiliary result used in our proofs which follows easily, e.g., from [16, Theorem 11.9, p.287], [33, Lemmas 1, 2], and [16, Theorem 3.1, p.261].

For preliminaries on projective and injective limits, see, e.g., [16, Chap. VIII, §2]. Recall that the projective limit of a projective system $((X_i)_{i\in I}, (f_{ij})_{i\leqslant j\in I})$ of nonempty compact Hausdorff spaces is a nonempty compact Hausdorff space in the limit topology. Henceforth, the projective limit will be denoted by $\varprojlim X_i$, with the projective system $((X_i), (f_{ij}))$ being understood. As usual, the abbreviation ANR stands for absolute neighbourhood retract (for the corresponding definition, see, e.g., [29, p.80]). Throughout the article, $\cong$ denotes an isomorphism of objects of a category in question, and $\underset{h}{\sim}$ a homotopy of maps or a homotopy equivalence of topological spaces.

Lemma 2.6. *Suppose $X = \varprojlim X_i$ and $\pi_i : X \to X_i$ are canonical continuous projections, where the X_i are nonempty compact Hausdorff spaces, and Y is an ANR.*

(1) *Given a continuous map $f : X \to Y$, there exists an index i and a continuous map $f_i : X_i \to Y$ such that $f_i \circ \pi_i \underset{h}{\sim} f$.*
(2) *Given a finite rank complex vector bundle E over X, there exists an index i and a complex vector bundle E_i over X_i such that the pullback bundle $\pi_i^*(E_i) \cong E$. Moreover, if E is stably trivial, then E_i is stably trivial as well.*
(3) *For each $k \in \mathbb{Z}_+$, the Čech cohomology group $H^k(X, \mathbb{Z}) = \cup_i \pi_i^*(H^k(X_i, \mathbb{Z}))$.*[1]

The following result expresses continuity of projective freeness and Hermiteness for algebras $C(X)$ on compact Hausdorff spaces X.

Proposition 2.7. *Suppose $X \underset{h}{\sim} \varprojlim X_i$, where the X_i are nonempty compact Hausdorff spaces such that either all algebras $C(X_i)$ are projective free, or all algebras $C(X_i)$ are Hermite. Then $C(X)$ is projective free or Hermite, respectively.*

Proof. Due to Corollary 2.4, we can assume $X = \varprojlim X_i$. Then the required result follows from Propositions 2.1, 2.3 (applied to the algebras $C(X_i)$) and Lemma 2.6. □

To formulate our next result, we recall some definitions.

[1] In other words, $H^k(X, \mathbb{Z}) \cong \varinjlim H^k(X_i, \mathbb{Z})$, the injective limit of the related injective system of groups $(H^k(X_i, \mathbb{Z}), f_{ij}^*)$.

The *covering dimension* of a topological space X, denoted by $\dim X$, is the smallest integer d such that every open covering of X has an open refinement of order at most $d+1$. If no such integer exists, then X is said to have *infinite covering dimension*.

Given a finite open cover $\alpha = \{U_1, \ldots, U_r\}$ of a compact topological space X, its *nerve* is the abstract simplicial complex A_α, whose vertex set is α and such that the simplex $\sigma = [U_{i_0}, U_{i_1}, \ldots, U_{i_n}] \subset A_\alpha$ if and only if $\cap_{j=0}^n U_{i_j} \neq \emptyset$.

We also recall that a path-connected topological space X is n-*simple* if for each $x \in X$, the fundamental group $\pi_1(X, x_0)$ acts trivially on the n^{th} homotopy group $\pi_n(X, x_0)$, see, e.g., [28, Chap. IV, §16] for details. For instance, every path-connected topological group is n-simple for all $n \in \mathbb{N}$.

The following result, generalising [8, Corollary 1.4] and [55, Theorem 5], gives a sufficient condition for $C(X)$ to be projective free or Hermite.

Theorem 2.8. *If $X \underset{h}{\sim} \varprojlim X_i$, where the $X_i\,(\neq \emptyset)$ are finite-dimensional compact Hausdorff spaces such that $H^n(X_i, \mathbb{Z}) = 0$ for all $n \geqslant 5$, then $C(X)$ is Hermite. In addition, if all spaces X_i are connected and satisfy $H^2(X_i, \mathbb{Z}) = H^4(X_i, \mathbb{Z}) = 0$, then $C(X)$ is projective free.*

Proof. In the proof, we use the fact that the rank of a stably trivial complex vector bundle or a complex vector bundle over a connected space is well-defined. Due to Proposition 2.7, it suffices to prove the result for each space X_i.

In what follows, we use the terminology from [27]. So A_α denotes the nerve of a finite open covering $\alpha = \{U_1, U_2, \ldots, U_r\}$ of X_i and $\phi_\alpha : X_i \to A_\alpha$ denotes a canonical map of α, i.e., for each point $x \in X_i$, $\phi_\alpha(x)$ is contained in the closure of the simplex $[U_{i_0}, U_{i_1}, \ldots, U_{i_n}]$ of A_α, where $U_{i_0}, U_{i_1}, \ldots, U_{i_n}$ denote the members of α containing x.

Given a continuous map $f : X_i \to Y$, a continuous map $\psi_\alpha : A_\alpha \to Y$ is called a *bridge* if $\psi_\alpha \circ \phi_\alpha \underset{h}{\sim} f$ for each canonical map $\phi_\alpha : X_i \to A_\alpha$ of the covering α. The family $(A_\alpha)_\alpha$, together with the natural simplicial projection maps $p_\alpha^\beta : A_\beta \to A_\alpha$, where β is a finite refinement of α, forms a projective system with limit X_i.

Let E_i be a stably trivial complex vector bundle over X_i. Applying Proposition 2.6(2), we obtain for some α, a stably trivial bundle E_α over A_α such that $\phi_\alpha^*(E_\alpha) \cong E_i$; in particular its Chern classes $c_k(E_\alpha) = 0\,(\in H^{2k}(A_\alpha, \mathbb{Z}))$, $k \in \mathbb{N}$. If E_α is of complex rank one, then the latter implies that E_i is trivial. Next, suppose E_α has complex rank $n \geqslant 2$. Let A_α^4 be the 4^{th} skeleton of A_α, that is, the totality of the simplexes of A_α with dimensions not exceeding 4. Consider the bundle $E' := E_\alpha|_{A_\alpha^4}$. Since the rank n of E_i is $\geqslant 2$, E' is isomorphic to the Whitney sum $E'' \oplus \Omega^{n-2}$, where $\Omega^{n-2} := A_\alpha^4 \times \mathbb{C}^{n-2}$, and E'' is a rank 2 complex vector bundle over A_α^4 (see [30, Part II, Chap. 9, Theorem 1.2]). Since the Chern classes of E' are zeros, the latter implies that Chern classes of E'' are zeros as well. In turn, A_α^4 is four-dimensional, and therefore the vanishing of Chern classes of E'' implies that E'' is trivial (it follows, e.g., from [36, Problem 14-C, p.171]). Hence, the bundle $E_\alpha|_{A_\alpha^4} = E' = E'' \oplus \Omega^{n-2}$ is trivial as well. In particular, due to the construction of [30, Part I, Theorem 3.5.5], there is a continuous map into a complex Grassmanian,

$h : A_\alpha \to \mathbf{Gr}_n(\mathbb{C}^{n+k})$, constant in a neighbourhood of A_α^4, such that $h^*(\gamma_{n,n+k}) \cong E_\alpha$, where $\gamma_{n,n+k}$ is the tautological bundle over $\mathbf{Gr}_n(\mathbb{C}^{n+k})$. Let the value of h on A_α^4 be x. Consider the continuous maps $f_0 := h \circ \phi_\alpha : X_i \to \mathbf{Gr}_n(\mathbb{C}^{n+k})$ and $f_1 : X_i \to \mathbf{Gr}_n(\mathbb{C}^{n+k})$ of constant value x. According to the definition of n-homotopy[2] in [27, §6], f_0 and f_1 are 4-homotopic. Moreover, by the hypotheses we have $H^s(X_i, \mathbb{Z}) = 0$ for all $s \geqslant 5$ which implies that $H^s(X, G) = 0$ for all $s \geqslant 5$, for any abelian group G (by the Universal Coefficient Theorem for Čech cohomology [48, Chap. 6, §8, Theorem 10]). In particular, $H^s(X, \pi_s) = 0$ for all $s \geqslant 5$ where π_s is the s^{th} homotopy group of $\mathbf{Gr}_n(\mathbb{C}^{k+n})$. Since the latter is a simply connected manifold, it is r-simple for all $r \geqslant 1$. Hence, the previous facts based on [29, Theorem 8.1] imply that f_0 and f_1 are homotopic maps. Thus $f_0^*(\gamma_{n,n+k})$ and $f_1^*(\gamma_{n,n+k})$ are isomorphic bundles over X_i. But the former is isomorphic to the original bundle E_i by our construction, and the latter is trivial. This proves that E_i is trivial. Thus every stably trivial complex vector bundle over $C(X_i)$ is trivial. Hence, all $C(X_i)$, and therefore $C(X)$, are Hermite due to Proposition 2.7.

Now, assume in addition that X_i is connected and $H^2(X_i, \mathbb{Z}) = H^4(X_i, \mathbb{Z}) = 0$. Let E_i be a finite rank complex vector bundle over X_i. Then as above, there exist α and a complex vector bundle E_α over A_α, such that $\phi_\alpha^*(E_\alpha) \cong E_i$. Due to Lemma 2.6 (3), we can choose the α such that the Chern classes $c_i(E_\alpha) = 0\,(\in H^{2i}(A_\alpha, \mathbb{Z}))$, $i = 1, 2$. Hence, the complex vector bundle $E' := E_\alpha|_{A_\alpha^4}$ is trivial. So arguing as above, we obtain that the bundle E_i is trivial. Thus every finite rank complex vector bundle over $C(X_i)$ is trivial. Hence, all $C(X_i)$, and therefore $C(X)$, are projective free by Proposition 2.7. $\square$

We deduce from the theorem the following result.

Theorem 2.9. *Let X be a Hausdorff paracompact space of finite covering dimension such that $H^n(X, \mathbb{Z}) = 0$ for all $n \geqslant 5$. Then the algebra $C(X)$ is Hermite. If, in addition, X is connected and $H^2(X, \mathbb{Z}) = H^4(X, \mathbb{Z}) = 0$, then $C(X)$ is projective free.*

Proof. Due to the Dieudonné theorem, a Hausdorff paracompact space is normal; see, e.g., [17, Theorem 5.1.5]. So the embedding $X \hookrightarrow \beta X$ induces a natural isomorphism $H_f^*(X, \mathbb{Z}) \cong H^*(\beta X, \mathbb{Z})$; here $H_f^*(X, \mathbb{Z})$ are Čech cohomology groups of X defined with respect to finite open coverings, see [16, p. 282]. Further, for a paracompact space X, the groups $H^*(X, \mathbb{Z})$ coincide with the Čech cohomology groups of X defined with respect to numerable open coverings. Thus, according to [11, Corollary (6.3)], the group $H_f^n(X, \mathbb{Z})$ coincides with the Čech cohomology group $H^n(X, \mathbb{Z})$ for all $n \geqslant 2$. These facts and our hypotheses imply that $H^n(\beta X, \mathbb{Z}) = 0$ for all $n \geqslant 5$. Moreover, since X is normal, $\dim \beta X = \dim X$, and in particular, βX is finite-dimensional; see, e.g., [17, Theorem 7.1.17, p.390]. Thus, applying Theorem 2.8 to βX, we obtain that the algebra

[2] Let X be a normal space, Y be a connected ANR, and either X or Y be compact. Continuous maps $f_0, f_1 : X \to Y$ are *n-homotopic* if there exists a bridge α for the pair (f_0, f_1) with bridge map $\psi_i : A_\alpha \to Y$ for f_i $(i = 0, 1)$ such that $\psi_0 = \psi_1$ on the n-dimensional skeleton A_α^n of the nerve A_α of the covering α of X.

$C(\beta X)$ is Hermite. Then Proposition 2.3 implies that the algebra $C(X)$ is Hermite as well.

In the second case, the previous argument implies that $H^2(\beta X, \mathbb{Z}) = H^4(\beta X, \mathbb{Z}) = 0$ also. Moreover, if X is connected, then βX is connected as well. Then, under these additional assumptions, Theorem 2.8 implies that the algebra $C(\beta X)$ is projective free. Therefore due to Proposition 2.1, the algebra $C(X)$ is projective free as well, as required. $\square$

Example 2.10. Let $X \subset \mathbb{R}^5$ be a closed subset. Then $H^n(X, \mathbb{Z}) = 0$ for all $n \geqslant 5$. (In fact, the Čech cohomology groups of a closed subset of $\mathbb{R}^m$ are isomorphic to the injective limit of Čech cohomology groups of its open neighbourhoods; see, e.g., [48, Chap. 6, §1, Theorem 12, §8, Corollary 8]. Also, by a result due to Whitehead [58, Theorem 3.2], an open subset U of $\mathbb{R}^m$ is homotopy equivalent to an $m-1$ dimensional simplicial complex $\Gamma \subset U$. As the dimension of Γ is $m-1$, $H^n(U, \mathbb{Z}) = H^n(\Gamma, \mathbb{Z}) = 0$ for all $n > m-1$.) Thus, due to Theorem 2.9, the algebra $C(X)$ is Hermite.

In connection with Theorems 2.8, 2.9 the following question seems quite natural:

Question. *Is there a Hausdorff topological space X with $H^n(X, \mathbb{Z}) \neq 0$ for some $n \geqslant 5$ such that the algebra $C(X)$ is Hermite?*

3. Stein algebras

For basic facts about complex analytic spaces and Stein spaces we refer the readers to the book [25].

Let $\Gamma(X, \mathcal{O}_X)$ be the ring of global sections of the structure sheaf $\mathcal{O}_X$ on a finite-dimensional complex analytic space $(X, \mathcal{O}_X)$. There is a natural algebra homomorphism $\hat{}\, : \Gamma(X, \mathcal{O}_X) \to C(X)$ with image $\mathcal{O}(X)$, the ring of holomorphic functions on X, injective if $(X, \mathcal{O}_X)$ is reduced. A space $(X, \mathcal{O}_X)$ is said to be *Stein* if it is *holomorphically convex* (i.e., for each infinite discrete set $D \subset X$ there exists an $f \in \mathcal{O}(X)$ which is unbounded on D) and *holomorphic separable* (i.e., for all $x, y \in X$, $x \neq y$, there exists an $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$).

By the Cartan and Oka theorem, the nilradical $\mathfrak{n}(\mathcal{O}_X)$ of $\mathcal{O}_X$ (i.e., the union of nilradicals of stalks $\mathcal{O}_x$, $x \in X$) is a coherent sheaf of ideals on X and so if $(X, \mathcal{O}_X)$ is Stein, then by Cartan's Theorem B we have the following exact sequence of global sections of sheaves

$$0 \to \Gamma(X, \mathfrak{n}(\mathcal{O}_X)) \to \Gamma(X, \mathcal{O}_X) \xrightarrow{r^*} \Gamma(X, \mathcal{O}_{\mathrm{red}\, X}) \to 0, \tag{3.1}$$

where $\mathcal{O}_{\mathrm{red}\, X} := \mathcal{O}_X / \mathfrak{n}(\mathcal{O}_X)$ is the structure sheaf on the reduction of X. It easily seen that $\Gamma(X, \mathfrak{n}(\mathcal{O}_X))$ is the *Jacobson radical* of $\Gamma(X, \mathcal{O}_X)$, i.e., the intersection of all maximal ideals of $\Gamma(X, \mathcal{O}_X)$; see, e.g., [19, §1.4]. Moreover, the algebra $\Gamma(X, \mathcal{O}_X)$ is

$\Gamma(X, \mathfrak{n}(\mathcal{O}_X))$-*complete*, i.e., the natural homomorphism from $\Gamma(X, \mathcal{O}_X)$ to the projective limit of quotient algebras $\varprojlim \Gamma(X, \mathcal{O}_X)/\Gamma(X, \mathfrak{n}(\mathcal{O}_X))^N$ is an isomorphism,[3] see, e.g., [25, Ch. V, §4.3].

Theorem 3.1. *Let $(X, \mathcal{O}_X)$ be a finite-dimensional Stein space. The homomorphism $\hat{}$: $\Gamma(X, \mathcal{O}_X) \to C(X)$ induces a bijection between isomorphism classes of finitely generated projective $\Gamma(X, \mathcal{O}_X)$ and $C(X)$ modules.*

Proof. Since the algebra $\Gamma(X, \mathcal{O}_X)$ is $\Gamma(X, \mathfrak{n}(\mathcal{O}_X))$-*complete*, the correspondence

$$P \cong \Gamma(X, \mathcal{O}_X) \otimes_{\Gamma(X,\mathcal{O}_X)} P \xrightarrow{r^* \otimes \mathrm{id}_P} \Gamma(X, \mathcal{O}_{\mathrm{red}\, X}) \otimes_{\Gamma(X,\mathcal{O}_X)} P \tag{3.2}$$

determines a bijection between isomorphism classes of finitely generated projective $\Gamma(X, \mathcal{O}_X)$ modules and finitely generated projective $\Gamma(X, \mathcal{O}_{\mathrm{red}\, X})$ modules, see, e.g., [53, Theorem 2.26].

For the reduced Stein space $(X, \mathcal{O}_{\mathrm{red}\, X})$, the algebra $\Gamma(X, \mathcal{O}_{\mathrm{red}\, X})$ can be naturally identified with $\mathcal{O}(X)$. Then it follows from [20, Sätze 6.7, 6.8] (see also [38, Theorem 2.1]) that there is a bijection between isomorphism classes of finitely generated projective $\mathcal{O}(X)$ modules and isomorphism classes of holomorphic vector bundles over X of bounded rank.[4] Moreover, according to the Grauert theorem (see [23], [24] and [12]) the inclusion of sheaves $i : \mathcal{O}_{\mathrm{red}\, X} \hookrightarrow C_X$ (the sheaf of germs of continuous functions on X) induces a bijection between isomorphism classes of holomorphic and continuous complex vector bundles over X of bounded rank. Next, since X is a Hausdorff paracompact of finite covering dimension (by the definition of a finite-dimensional complex analytic space), each continuous complex vector bundle over X is of bounded rank if and only it is of finite type (see, e.g., [30, Ch. 3, Proposition 5.4]) and, hence, by Swan's theorem (see [56, Theorem 2]), there is a bijection between isomorphism classes of continuous complex vector bundles over X of bounded rank and of finitely generated projective $C(X)$ modules. This implies that the correspondence

$$P \cong \mathcal{O}(X) \otimes_{\mathcal{O}(X)} P \xrightarrow{i \otimes \mathrm{id}_P} C(X) \otimes_{\mathcal{O}(X)} P \tag{3.3}$$

determines a bijection between isomorphism classes of finitely generated projective $\mathcal{O}(X)$ modules and finitely generated projective $C(X)$ modules.

The composition of the bijections in (3.2) and (3.3) gives the required statement: the correspondence

[3] Equivalently, in the topology on $\Gamma(X, \mathcal{O}_X)$ determined by letting the family of ideals $\{\Gamma(X, \mathfrak{n}(\mathcal{O}_X))^N\}_{N\in\mathbb{N}}$ be a base of open neighbourhoods of 0, every Cauchy sequence converges to a unique limit.

[4] I.e., the complex ranks of restrictions of such bundles to connected components of X are uniformly bounded from above.

$$P \cong \Gamma(X, \mathcal{O}_X) \otimes_{\Gamma(X,\mathcal{O}_X)} P \xrightarrow{\hat{}\otimes \mathrm{id}_P} C(X) \otimes_{\Gamma(X,\mathcal{O}_X)} P \tag{3.4}$$

determines a bijection between isomorphism classes of finitely generated projective $\Gamma(X, \mathcal{O}_X)$ modules and finitely generated projective $C(X)$ modules. $\square$

Theorems 3.1 and 2.9 imply the following:

Theorem 3.2. *The algebra $\Gamma(X, \mathcal{O}_X)$ is projective free or Hermite if and only if the algebra $C(X)$ is projective free or Hermite.*
In particular, if $H^n(X, \mathbb{Z}) = 0$ for all $n \geqslant 5$, then $\Gamma(X, \mathcal{O}_X)$ is Hermite, and if, in addition, X is connected and $H^2(X, \mathbb{Z}) = H^4(X, \mathbb{Z}) = 0$, then it is projective free.

Example 3.3. (1) According to [26], a reduced Stein space X of (complex) dimension k is homotopy equivalent to a k-dimensional CW complex. Hence, $H^n(X, \mathbb{Z}) = 0$ for all $n > k$. Thus, due to Theorem 3.2 if X is of dimension $\leqslant 4$, then $\mathcal{O}(X)$ is Hermite, and if X is one-dimensional and connected, then $\mathcal{O}(X)$ is projective free.

(2) Let U be an open subset of a Stein manifold X. Equipped with the topology of uniform convergence on compact subsets of U, the algebra $\mathcal{O}(U)$ becomes a complex Fréchet space. Each nonzero homomorphism $\mathcal{O}(U) \to \mathbb{C}$ is an element of the dual space $\mathcal{O}(U)^*$ (see, e.g., [25, Chap. 5, §7.1]). The space of such homomorphisms equipped with the weak-$*$ topology of $\mathcal{O}(U)^*$ is denoted by $M(\mathcal{O}(U))$. If $f \in \mathcal{O}(U)$, then $\hat{f} \in C(M(\mathcal{O}(U)))$ is defined by $\hat{f}(\alpha) = \alpha(f)$ for each $\alpha \in M(\mathcal{O}(U))$.

Since X is Stein, $M(\mathcal{O}(X)) = X$ (see, e.g., [25, Chap. 5, §7]), so we have the natural restriction map $\pi_U : M(\mathcal{O}(U)) \to X$ given by $\pi_U(\alpha)(f) = \alpha(f|_U)$. Rossi [41] has shown that $M(\mathcal{O}(U))$ admits the structure of a Stein manifold in such a way that: (i) the map $U \to M(\mathcal{O}(U))$ sending $z \in U$ to the evaluation homomorphism at z is a biholomorphism of U with an open subset of $M(\mathcal{O}(U))$ (we will regard U as an open subset of $M(\mathcal{O}(U))$); (ii) if $f \in \mathcal{O}(U)$, then $\hat{f}$ is the unique holomorphic extension of f to $M(\mathcal{O}(U))$ (so that $\mathcal{O}(M(\mathcal{O}(U))) \cong \mathcal{O}(U)$); (iii) π_U is locally a biholomorphism. This and Theorem 3.2 imply that

- *if X is of dimension $\leqslant 4$, then the algebra $\mathcal{O}(U)$ is Hermite.*

Assume, in addition, that the set U is holomorphically contractible (e.g., $X = \mathbb{C}^k$ and $U \subset X$ is a star-shaped domain), then

- *the algebra $\mathcal{O}(U)$ is projective free.*

Indeed, let the holomorphic contraction be given by a continuous map $H : U \times [0, 1] \to U$, so that $H(\cdot, 1) = \mathrm{id}_U$, $H(\cdot, 0) = z_o \in U$, and $H(\cdot, t) : U \to U$ is holomorphic for all $t \in [0, 1]$. Then H determines the map H^* from $[0, 1]$ to the set of homomorphisms $\mathcal{O}(U) \to \mathcal{O}(U)$ given by $(H^*(t))(f) = f(H(\cdot, t))$. The transpose of each $H^*(t)$ induces a holomorphic map $\hat{H}(\cdot, t) : M(\mathcal{O}(U)) \to M(\mathcal{O}(U))$ such that $\hat{H}(\cdot, t)|_U = H(\cdot, t)$. Let us show that $\hat{H} : M(\mathcal{O}(U)) \times [0, 1] \to M(\mathcal{O}(U))$ is continuous. To this end, let $\{(z_n, t_n)\}_{n\in\mathbb{N}} \subset M(\mathcal{O}(U)) \times [0, 1]$ be a sequence converging to $(z, t) \in M(\mathcal{O}(U)) \times [0, 1]$. For each $f \in \mathcal{O}(U)$,

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} |(\hat{H}(z_n,t_n)-\hat{H}(z,t))(f)| \\
&\quad \leqslant \overline{\lim_{n\to\infty}} |(\hat{H}(z_n,t_n)-\hat{H}(z_n,t))(f)| + \overline{\lim_{n\to\infty}} |(\hat{H}(z_n,t)-\hat{H}(z,t))(f)| \\
&\quad = \overline{\lim_{n\to\infty}} |z_n(f(H(\cdot,t_n))-f(H(\cdot,t)))| + \overline{\lim_{n\to\infty}} |(z_n-z)(f(H(\cdot,t)))| =: I+II.
\end{aligned}$$

By the definition of convergence in the weak-* topology, the limit II equals 0. Similarly, by continuity of H, the sequence of functions $\{f(H(\cdot,t_n))\}_{n\in\mathbb{N}} \subset \mathcal{O}(U)$ converges uniformly on compact subsets of U to the function $f(H(\cdot,t)) \in \mathcal{O}(U)$. Since $z_n \in \mathcal{O}(U)^*$, the latter implies that the limit I equals 0 as well. Hence, $\lim_{n\to\infty} \hat{H}(z_n,t_n) = \hat{H}(w,t)$ in the topology of $M(\mathcal{O}(U))$, as required.

Thus, $\hat{H} : M(\mathcal{O}(U)) \times [0,1] \to M(\mathcal{O}(U))$ is a homotopy between $\hat{H}(\cdot,1) = \mathrm{id}_{M(\mathcal{O}(U))}$ and $\hat{H}(\cdot,0) = z_o$, i.e., the Stein manifold $M(\mathcal{O}(U))$ is holomorphically contractible. From here and Theorem 3.2, it follows that the algebra $\mathcal{O}(U) \cong \mathcal{O}(M(\mathcal{O}(U)))$ is projective free.

4. Commutative unital complex Banach algebras

Recall that for a commutative unital complex Banach algebra A, the maximal ideal space $M(A) \subset A^*$ is the set of nonzero homomorphisms $A \to \mathbb{C}$[5] endowed with the Gelfand topology, the weak-$*$ topology of A^*. It is a compact Hausdorff space contained in the unit sphere of A^*. The Gelfand transform $\hat{}\, : A \to C(M(A))$, defined by $\hat{a}(\varphi) := \varphi(a)$ for $a \in A$ and $\varphi \in M(A)$, is a nonincreasing-norm morphism of Banach algebras.

Theorem 4.1. *Let A be a commutative unital complex Banach algebra. Then:*

(1) *A is projective free if and only if $C(M(A))$ is projective free.*
(2) *A is Hermite if and only if $C(M(A))$ is Hermite.*

In particular, if $M(A) \underset{h}{\sim} \varprojlim X_i$, where the X_i are finite-dimensional compact Hausdorff spaces such that $H^n(X_i,\mathbb{Z}) = 0$ for all $n \geqslant 5$, then A is Hermite. If, in addition, each space X_i is connected and $H^2(X_i,\mathbb{Z}) = H^4(X_i,\mathbb{Z}) = 0$, then A is projective free.

Parts (1) and (2) of the theorem follow from a one-to-one correspondence (determined via the Gelfand transform) between the isomorphism classes of finitely generated projective A modules and the isomorphism classes of complex vector bundles over $M(A)$ (see [39], and also [54, §7.5, Theorem on p.199]) along with the Swan theorem [52, Theorem 2]. In turn, the last statement follows from Theorem 2.8.

Let $\bar{A}$ be the uniform closure in $C(M(A))$ of the image under the Gelfand transform of algebra A. It is known that $M(\bar{A}) = M(A)$, see, e.g., [42, Proposition 3]. Then we obtain from Theorem 4.1:

[5] Every such homomorphism is continuous, see, e.g., [34, §23(A), Theorem].

Corollary 4.2. *A is projective free or Hermite if and only if $\bar{A}$ is projective free or Hermite.*

Example 4.3. (1) Let $\mathrm{L}^1[0,1]$ be the Banach space of complex-valued Lebesgue integrable functions on $[0,1]$ with the norm $\|f\|_1 := \int_0^1 |f(t)|dt$. The space $\mathrm{L}^1[0,1]$ equipped with the multiplication given by truncated convolution $(f*g)(t) := \int_0^t f(\tau)g(t-\tau)d\tau$ becomes a complex commutative Banach algebra V called the *Volterra algebra*. The algebra V is non-unital without maximal ideals, see, e.g., [37, Example 9.82]. Let V_1 denote the algebra of pairs (f,c), where $f \in V$ and $c \in \mathbb{C}$ with addition and multiplication given by

$$(f,c)+(f',c') := (f+f', c+c'), \quad (f,c)\cdot(f',c') := (f*f' + c\cdot f' + c'\cdot f, c\cdot c').$$

We equip V_1 with the norm $\|(f,c)\| := \|f\|_1 + |c|$. Then V_1 becomes a commutative unital complex Banach algebra. Since V is without maximal ideals, $V\times\{0\}$ is the only maximal ideal of V_1. Thus due to Theorem 4.1 the algebra V_1 is projective free.

(2) Let A be a commutative unital complex Banach algebra such that the algebra $\bar{A}\subset C(M(A))$ is generated by k elements (this is true, e.g., if A itself is generated by k elements). Then A is Hermite if $k\leqslant 5$, and projective free if $k\leqslant 2$ and A does not contain nontrivial idempotent elements. Indeed, in this case $M(A)$ is homeomorphic to a polynomially convex subset of $\mathbb{C}^k$ (see, e.g., [21, Chap. III, Theorem 1.4]). Recall that a compact set $K\subset\mathbb{C}^k$ is *polynomially convex* if for every $\mathbf{z}\notin K$ there is a polynomial $p\in\mathbb{C}[z_1,\dots,z_k]$ such that $|p(\mathbf{z})| > \sup_{\mathbf{w}\in K}|p(\mathbf{w})|$. It is known, see, e.g., [50, Corollary 2.3.6], that if $K\subset\mathbb{C}^k$ is a compact polynomially convex set, then $H^n(K,\mathbb{Z})=0$ for all $n\geqslant k$. This and Theorem 4.1 imply that A is Hermite if $k\leqslant 5$. If $k\leqslant 2$ and A does not contain nontrivial idempotent elements, then due to the Shilov idempotent theorem (see, e.g., [21, Chap. III, Corollary 6.5]) $M(A)$ is connected. Hence, in this case Theorem 4.1 implies that A is projective free.

(3) Let $L^\infty(S)$ be the Banach algebra of essentially bounded measurable complex-valued functions on a measure space S, with pointwise operations and the supremum norm. Then the maximal ideal space $M(L^\infty(S))$ is totally disconnected (see, e.g., [21, Chap. I, Lemma 9.1]) and, hence, $\dim M(L^\infty(S))=0$. Thus, Theorem 4.1 implies that the algebra $L^\infty(S)$ is Hermite.

(4) Let $\mathbb{D} := \{z\in\mathbb{C} : |z|<1\}$ and $\mathbb{T} := \{z\in\mathbb{C} : |z|=1\}$. With pointwise operations and the supremum norm, L^∞ denotes the Banach algebra of essentially bounded Lebesgue measurable functions on $\mathbb{T}$, and H^∞ the Banach algebra of all bounded holomorphic functions in $\mathbb{D}$. Via identification with boundary values, H^∞ is a uniformly closed subalgebra of L^∞. According to the Chang-Marshall theorem, see, e.g., [22, Chap. IX, §3], any uniformly closed subalgebra A between H^∞ and L^∞ is a *Douglas algebra* generated by H^∞ and a family $\mathscr{B}_A\subset\overline{H^\infty}$ of functions conjugate to some inner functions of H^∞ (written $A=[H^\infty,\mathscr{B}_A]$). If $H^\infty\subsetneq A$, then the maximal ideal space $M(A)$ is a closed subset of $M(H^\infty)\setminus\mathbb{D}$ of the form (see, e.g., [22, Chap. IX, Theorem 1.3]):

$$M(A) = \bigcap_{\bar{u} \in \mathscr{B}_A} \{x \in M(H^\infty) \,:\, |x(u)| = 1\}.$$

According to the results of Suárez [51], for each closed set $K \subset M(H^\infty)$ $\dim K \leqslant 2$ and $H^2(K, \mathbb{Z}) = 0$. This and Theorem 4.1 imply that A is Hermite and it is projective free if $M(A)$ is connected. Next, due to the Shilov idempotent theorem, $M(A)$ is connected if and only if A does not contain nontrivial idempotents in L^∞. For instance, $M(A)$ is connected if A is one of the algebras: H^∞, $H^\infty + C$, where $C := C(\mathbb{T})$, or $B_1 = [H^\infty, C_1]$ (the closed subalgebra generated by H^∞ and the Banach algebra C_1 of all complex-valued functions on $\mathbb{T}$ which are continuous except possibly at $z = 1$ but which have one-sided limits at $z = 1$; for details, see e.g., [45] or [22, Chap. IX, Exercise 7]). Thus, in these special cases, A is projective free.

5. The class $\mathscr{C}$

One way to construct new Banach algebras from known ones is to take their projective tensor product. In general, the projective tensor product of projective free or Hermite Banach algebras does not inherit the property. In this section we introduce a new class of projective free Banach algebras such that their projective tensor product with projective free or Hermite Banach algebras continues to be projective free or Hermite, respectively.

A topological space X is said to be of *trivial shape* if every continuous map from X to an ANR is homotopic to a constant map; see, e.g., [35, p.248]. A space of trivial shape generalises the notion of a contractible space, and, in particular, if a space of trivial shape is homotopy equivalent to an ANR, then it is contractible. If X is a compact Hausdorff space of trivial shape, then it is connected and Čech cohomology groups $H^k(X, \mathbb{Z}) = 0$ for all $k \geqslant 1$.

We say that a commutative unital complex Banach algebra A belongs to the class $\mathscr{C}$ if $M(A)$ is a space of trivial shape. In this section, we study some properties of class $\mathscr{C}$.

Let B, C be unital closed subalgebras of a commutative unital complex Banach algebra $\mathfrak{A}$, and let $B \widehat{\otimes}_{\mathfrak{A}} C \subset \mathfrak{A}$ be the closure of the subalgebra $\langle B, C \rangle$ generated by B and C. Following [1], we assume that the following property is satisfied:

There exists a constant c such that for all $\xi \in M(B)$ and every $n \in \mathbb{N}$ and $b_i \in B$, $c_i \in C$, $1 \leqslant i \leqslant n$,

$$\left\| \sum_{k=1}^{n} \xi(b_k) c_k \right\|_C \leqslant c \left\| \sum_{k=1}^{n} b_k c_k \right\|_{\mathfrak{A}}. \tag{5.1}$$

Example 5.1. (For basic definitions and results on topological tensor products, see, e.g., [44].)

Let B, C be commutative unital complex Banach algebras and let $B \widehat{\otimes}_\alpha C$ be the completion of the algebraic tensor product $B \otimes C$ equipped with a *reasonable crossnorm*

$\|\cdot\|_\alpha$, i.e., such that $\|v\|_\varepsilon \leqslant \|v\|_\alpha \leqslant \|v\|_\pi$ for all $v \in B \otimes C$. Here $\|\cdot\|_\pi$ and $\|\cdot\|_\varepsilon$ denote the projective and injective tensor norms on $B \otimes C$, given for $v \in B \otimes C$ by

$$\|v\|_\pi := \inf\Big\{ \sum_{i=1}^n \|b_i\|_B \, \|c_i\|_C \, : \, v = \sum_{i=1}^n b_i \otimes c_i, \, n \in \mathbb{N} \Big\},$$
$$\|v\|_\varepsilon := \sup\{|(\xi \otimes \eta)(v)| \, : \, \xi \in B^*, \ \|\xi\|_{B^*} \leqslant 1, \ \eta \in C^*, \ \|\eta\|_{C^*} \leqslant 1\}.$$

(As usual, B^* and C^* stand for duals of B and C, respectively.)

Suppose that $B\widehat{\otimes}_\alpha C$ is a Banach algebra with operations compatible with operations on $B \otimes C$. Then (5.1) is satisfied with $c = 1$ when $\mathfrak{A} := B\widehat{\otimes}_\alpha C$. This is the case, e.g., if $\mathfrak{A}$ is (a) the projective tensor product $B\widehat{\otimes}_\pi C$; (b) the injective tensor product $B\widehat{\otimes}_\varepsilon C$ where either B or C is a uniform algebra; see, e.g., [15, §1.3] for the references. (For other examples see, e.g., [47, Theorem 4].)

Let i_B denote the embedding $B \hookrightarrow B\widehat{\otimes}_{\mathfrak{A}} C$.

Theorem 5.2. *If $C \in \mathscr{C}$, then the correspondence*

$$P \cong B \otimes_B P \xrightarrow{i_B \otimes \mathrm{id}_P} (B\widehat{\otimes}_{\mathfrak{A}} C) \otimes_B P \tag{5.2}$$

determines a bijection between isomorphism classes of finitely generated projective B modules and finitely generated projective $B\widehat{\otimes}_{\mathfrak{A}} C$ modules.

Remark 5.3. The result shows that the class $\mathscr{C}$ is an analog of the class of local rings (i.e., those with unique maximal ideals) in commutative algebra (see, e.g., [2] for the corresponding definitions and results). Indeed, if S is a local commutative ring with the maximal ideal $\mathfrak{m}$ and R is a commutative ring, then for the completion in the Krull topology $\widehat{R \otimes S}_I := \varprojlim (R \otimes S)/I^n$ with respect to the ideal $I = R \otimes \mathfrak{m}$, we obtain an analog of Theorem 5.2, i.e., the correspondence $R \mapsto \widehat{R \otimes S}_I$ induces a bijection between isomorphism classes of finitely generated projective R modules and finitely generated projective $\widehat{R \otimes S}_I$ modules. Note that the canonical map π from $R \otimes S$ to $\widehat{R \otimes S}_I$ is injective on $R \otimes 1_S$ ($\cong R$). If, in addition, S is Noetherian, then by the Krull intersection theorem, $\cap_{n=1}^\infty \mathfrak{m}^n = 0$ and, hence, π is injective on $1_R \times S$ ($\cong S$) (here 1_S and 1_R are units in S and R). Moreover, the subalgebra generated by $\pi(R \otimes 1_S)$ and $\pi(1_R \otimes S)$ is dense in $\widehat{R \otimes S}_I$. Thus, in this case the completion $\widehat{R \otimes S}_I$ is an analog of $B\widehat{\otimes}_{\mathfrak{A}} C$ in Theorem 5.2.

To prove Theorem 5.2, first, we prove the following general result.

Lemma 5.4. *Under condition (5.1), $M(B\widehat{\otimes}_{\mathfrak{A}} C)$ is homeomorphic to $M(B) \times M(C)$.*

Proof. Condition (5.1) implies that for $\xi \in M(B)$ the map $\langle B, C\rangle \to C$,

$$\left(\sum_{k=1}^{n} b_k c_k\right) \mapsto \sum_{k=1}^{n} \xi(b_k) c_k \in C, \tag{5.3}$$

extends by continuity to a bounded multiplicative projection

$$P_\xi : B\widehat{\otimes}_{\mathfrak{A}} C \to C.$$

In particular, $\eta \circ P_\xi \in M(B\widehat{\otimes}_{\mathfrak{A}} C)$ for each $\eta \in M(C)$. It is easily seen that the map

$$F : M(B) \times M(C) \to M(B\widehat{\otimes}_{\mathfrak{A}} C), \quad F(\xi, \eta) := \eta \circ P_\xi,$$

is continuous. Also, F is injective, as if $\eta_1 \circ P_{\xi_1} = \eta_2 \circ P_{\xi_2}$ for some $(\xi_i, \eta_i) \in M(B) \times M(C)$, $i = 1, 2$, then for all $b \in B$, $c \in C$,

$$\xi_1(b)\eta_1(c) = \xi_2(b)\eta_2(c). \tag{5.4}$$

Applying (5.4) with $c = 1$ (the unit of $\mathfrak{A}$) and then again with $b = 1$, we get $\xi_1 = \xi_2$ and $\eta_1 = \eta_2$, as required.

Further, the map F is surjective, as if $\varphi \in M(B\widehat{\otimes}_{\mathfrak{A}} C)$, then clearly $\xi := \varphi|_B \in M(B)$ and $\eta := \varphi|_C \in M(C)$ and due to (5.3)

$$\begin{aligned} F(\xi, \eta)\left(\sum_{k=1}^{n} b_k c_k\right) &= (\eta \circ P_\xi)\left(\sum_{k=1}^{n} b_k c_k\right) = \eta\left(\sum_{k=1}^{n} \varphi(b_k) c_k\right) \\ &= \sum_{k=1}^{n} \varphi(b_k)\varphi(c_k) = \varphi\left(\sum_{k=1}^{n} b_k c_k\right), \end{aligned}$$

i.e., $F(\xi, \eta) = \varphi$.

This completes the proof of the lemma. $\square$

Proof of Theorem 5.2. Due to Lemma 5.4 without loss of generality we will identify $M(B\widehat{\otimes}_{\mathfrak{A}} C)$ with $M(B) \times M(C)$. Then the transpose of i_B restricted to $M(B\widehat{\otimes}_{\mathfrak{A}} C)$ is the map $p_B : M(B) \times M(C) \to M(B)$, $p_B(x, y) = x$ for all $(x, y) \in M(B) \times M(C)$.

According to the Novodvorski-Taylor theorem ([39], [54, §7.5]) and the Swan theorem [52], to prove the result we must show that the pullback by p_B determines a bijection between isomorphism classes of complex vector bundles over $M(B)$ and $M(B) \times M(C)$. In turn, it suffices to prove the same for complex vector bundles of constant rank over clopen subsets $U \subset M(B)$ and $U \times M(C) \subset M(B) \times M(C)$.

To this end, we present U as $\varprojlim U_i$, where all U_i are finite-dimensional compact simplicial complexes (cf. the argument of the proof of Theorem 2.8). Then $U \times M(C) = \varprojlim (U_i \times M(C))$. Moreover, if $\pi_i : U \to U_i$ are canonical projections for the first limit, then $\tilde{\pi}_i := (\pi_i, \mathrm{id}_{M(C)}) : U \times M(C) \to U_i \times M(C)$ are canonical projections for the second one.

Suppose E is a complex vector bundle of rank n over $U \times M(C)$. Due to Lemma 2.6(2) there is an index i and a complex vector bundle E_i over $U_i \times M(C)$ such that the pullback bundle $\tilde{\pi}_i^*(E_i) \cong E$. Then there is a map $h \in C(U_i, \mathbf{Gr}_n(\mathbb{C}^m))$[6] from U_i into a complex Grassmanian such that $h^*(\gamma_{n,m}) \cong E_i$, where $\gamma_{n,m}$ is the tautological bundle over $\mathbf{Gr}_n(\mathbb{C}^m)$, see, e.g., [30, Part I, Theorem 3.5.5]. Consider the map $H : M(C) \to C(U_i, \mathbf{Gr}_n(\mathbb{C}^m))$,

$$H(y)(x) := h(x, y), \quad y \in M(C), \quad x \in U_i.$$

Since U_i is an ANR (as it is a compact simplicial complex; see, e.g., [29, Chap. III, Corollary 8.4]), and also $\mathbf{Gr}_n(\mathbb{C}^m)$ is an ANR (as it is a compact complex manifold [29, Chap. III, Corollary 8.3]), the space $C(U_i, \mathbf{Gr}_n(\mathbb{C}^m))$ of continuous maps from U_i to $\mathbf{Gr}_n(\mathbb{C}^m)$ equipped with the topology of uniform convergence is an ANR as well; see, e.g. [29, Chapter VI, Theorem 2.4]. Moreover, since every continuous map between compact metrisable spaces is uniformly continuous, the map H is continuous. Thus by the definition of a space of trivial shape, H is homotopic to a constant map $M(C) \to C(U_i, \mathbf{Gr}_n(\mathbb{C}^m))$. This homotopy gives rise to a homotopy between h and a map $h_o : U_i \times M(C) \to \mathbf{Gr}_n(\mathbb{C}^m)$, $h_o(x, y) := h(x, o)$ for all $(x, y) \in U_i \times M(C)$, where $o \in M(C)$ is a fixed point. In particular, we have the following isomorphisms of bundles

$$E_i \cong h^*(\gamma_{n,m}) \cong h_o^*(\gamma_{n,m}) \cong p_i^*(E_i), \tag{5.5}$$

where $p_i : U_i \times M(C) \to U_i \times \{o\}$ is defined by the formula $p_i(x, y) = (x, o)$ for all $x \in U_i$. Applying to (5.5) the pullback map $\tilde{\pi}_i^*$ we obtain that the bundle E is isomorphic to the pullback by the natural projection $U \times M(C) \to U \times \{o\}$ of the restriction of E to $U \times \{o\}$. Since p_B maps $U_i \times \{o\}$ homeomorphically onto U, this shows that E is isomorphic to a bundle pulled back by p_B from U. Thus the map p_B^* determines a surjection between isomorphism classes of complex vector bundles over $M(B)$ and $M(B) \times M(C)$. Clearly, it determines an injection between these sets as well, since if E_1, E_2 are bundles over $M(B)$ such that the pullback bundles $p_B^*(E_1)$ and $p_B^*(E_2)$ are isomorphic, then their restrictions to $M(B) \times \{o\}$ are isomorphic and so E_1 and E_2 must be isomorphic.

The proof of the theorem is complete. □

Theorem 5.2 leads straightforwardly to the following:

Corollary 5.5. *Let $B, C \subset \mathfrak{A}$ be Banach algebras satisfying conditions of Theorem 5.2. If $C \in \mathscr{C}$ and B is Hermite, then $B \widehat{\otimes}_{\mathfrak{A}} C$ is Hermite. If $C \in \mathscr{C}$ and B is projective free, then $B \widehat{\otimes}_{\mathfrak{A}} C$ is projective free.*

[6] For topological spaces X, Y we denote by $C(X, Y)$ the set of continuous maps from X to Y.

Let B, C be commutative unital complex Banach algebras and $\|\cdot\|_\alpha$ be a norm on the algebraic tensor product $B \otimes C$. Let $\mathfrak{A} := B \widehat{\otimes}_\alpha C$ be the completion of $B \otimes C$ with respect to $\|\cdot\|_\alpha$. Suppose that $\mathfrak{A}$ is a Banach algebra. Identifying B and C with subalgebras $B \otimes 1_C$ and $1_B \otimes C$ of $\mathfrak{A}$ (here 1_B and 1_C are units of B and C), we assume that the triple $B, C, \mathfrak{A}$ satisfies condition (5.1). Then we have:

Proposition 5.6. *If $B, C \in \mathscr{C}$, then $\mathfrak{A} \in \mathscr{C}$.*

Proof. According to Lemma 5.4, $M(\mathfrak{A})$ is homeomorphic to the direct product $M(B) \times M(C)$. Let $f : M(B) \times M(C) \to X$ be a continuous map to an ANR X. Since by the hypotheses $M(C)$ is of trivial shape, repeating the arguments of the proof of Theorem 5.2, we obtain that f is homotopic to the continuous map $f_o : M(B) \times M(C) \to X$, where $f_o(x, y) := f(x, o)$ for all $(x, y) \in M(B) \times M(C)$; here $o \in M(C)$ is a fixed point. In turn, since $M(B)$ is of trivial shape, the restriction $f_o|_{M(B)}$ is homotopic to a constant map. This implies that f_o (and, hence, f) is homotopic to a constant map. Thus, $M(B) \times M(C)$ is of trivial shape, as required. $\square$

6. Examples

In Sections 6.1 and 6.2 we present some examples of algebras of class $\mathscr{C}$. We restrict ourselves to the case of semisimple algebras only. The choice of examples of Sections 6.3 and 6.4 is based on the research interests of the authors.

6.1. Banach algebras of symmetric functions

Recall that the *polynomial convex hull* of a bounded set $K \subset \mathbb{C}^n$ is the minimal polynomially convex set $\widehat{K} \Subset \mathbb{C}^n$ containing K, i.e.,

$$\widehat{K} := \Big\{\mathbf{z} \in \mathbb{C}^n : |p(\mathbf{z})| \leqslant \sup_{\mathbf{w} \in K} |p(\mathbf{w})| \text{ for all } p \in \mathbb{C}[z_1, \ldots, z_n]\Big\}.$$

If A is a finitely generated semisimple commutative unital complex Banach algebra with generators $f_1, \ldots, f_n$, then the map

$$F(x) := (x(f_1), \ldots, x(f_n)), \quad x \in M(A), \tag{6.1}$$

is an embedding with image a polynomially convex subset of $\mathbb{C}^n$ (see, e.g., [21, Chap. III, Theorem 1.4]). If we identify $M(A)$ with $F(M(A))$, then A becomes a (not necessarily closed) subalgebra of $C(F(M(A)))$ such that the restriction of polynomial algebra $\mathbb{C}[z_1, \ldots, z_n]$ to $F(M(A))$ is dense in A. In what follows, we consider a more general situation of a unital complex Banach algebra $A \subset C_b(K)$ on a subset $K \Subset \mathbb{C}^n$ such that $\mathbb{C}[z_1, \ldots, z_n]|_K$ *is dense in* A. We also assume that A is *weakly inverse closed*, that is, it possesses the following property:

(wi) *If* $f \in A$, *and* $\sup_K |f(x)| < 1$, *then* $\frac{1}{1-f} \in A$.

Under these conditions, $M(A)$ is naturally identified with $\widehat{K}$, see, e.g., [42, Proposition 1] and [21, Chap. III, Theorem 1.4].

Let $G \subset GL_n(\mathbb{C})$ be a finite group of order $|G|$. We say that the Banach algebra $A \subset C(K)$ is *G-invariant* if $G(K) = K$ and A is invariant with respect to the action of G on $C(K)$: $g^*(f) = f \circ g$ for all $f \in C(K)$, $g \in G$. (Since $\mathbb{C}[z_1, \dots, z_n]$ is invariant with respect to the action of G, this implies that $G(\widehat{K}) = \widehat{K}$ as well.) In this case, each $g^* : A \to A$, $g \in G$, is an automorphism of A, and since A is semisimple, each g^* is continuous (see, e.g., [34, §24B, Theorem]).

For a G-invariant algebra A, the subalgebra $A_G \subset A$ of elements invariant with respect to the action of G on A (i.e., such $f \in A$ that $g^*(f) = f$ for all $g \in G$) is said to be *G-symmetric*.

Let $\mathcal{P}_G \subset \mathbb{C}[z_1, \dots, z_n]$ be the subalgebra of polynomials invariant with respect to the action of G. If $f \in A_G$ and $(p_j)_{j\in\mathbb{N}} \subset \mathbb{C}[z_1, \dots, z_n]$ is a sequence such that $f = \lim_{j\to\infty} p_j|_K$, then all $\tilde{p}_j := \frac{1}{|G|}\sum_{g\in G} g^*(p_j) \in \mathcal{P}_G$ and $f = \lim_{j\to\infty} \tilde{p}_j|_K$ (because all $g^* : A \to A$ are continuous). Hence, $\mathcal{P}_G|_K$ is a dense subalgebra of A_G.

By the Hilbert basis theorem, see, e.g., [2, Theorem 7.5], there exist homogeneous polynomials $h_1, \dots, h_m \in \mathbb{C}[z_1, \dots, z_n]$, $m \geqslant n$, invariant with respect to the action of G which generate $\mathcal{P}_G$.[7] Hence, A_G is generated by elements $h_1|_K, \dots, h_m|_K$. In particular, the continuous map $F_G : M(A_G) \to \mathbb{C}^m$,

$$F_G(x) := (x(h_1|_K), \dots, x(h_m|_K)), \quad x \in M(A_G), \tag{6.2}$$

embeds $M(A_G)$ into $\mathbb{C}^m$ as a polynomially convex subset. On the other hand, by Lemma 7.1 we obtain that $M(A_G)$ can be identified with the quotient space $\widehat{K}/G$. In this identification, if $\pi : \widehat{K} \to \widehat{K}/G$ is the quotient map, then A_G is isomorphic to a subalgebra $\tilde{A}_G \subset C(\widehat{K}/G)$ such that $\pi^*(f)|_K \in A_G$ for all $f \in \tilde{A}_G$. Let $\tilde{h}_i \in \tilde{A}_G$ be such that $\pi^*(\tilde{h}_i) = h_i|_{\widehat{K}}$, $1 \leqslant i \leqslant m$. Then the map F_G becomes

$$F_G(x) := (\tilde{h}_1(x)), \dots, \tilde{h}_m(x)), \quad x \in \widehat{K}/G.$$

This implies that the map $H := (h_1, \dots, h_m) : \mathbb{C}^n \to \mathbb{C}^m$ maps $\widehat{K}$ onto $F_G(\widehat{K}/G)$.

Remark 6.1. The image $H(\mathbb{C}^n)$ is an n-dimensional complex algebraic subvariety of $\mathbb{C}^m$. For each polynomially convex set $S \subset \mathbb{C}^n$ invariant with respect to the action of G on $\mathbb{C}^n$, the uniform algebra $P(S)$ (defined as the closure in $C(S)$ of the algebra $\mathbb{C}[z_1, \dots, z_n]|_S$) is n-generated, G-invariant, and $M(P(S)) = S$. Hence, the map H determines a one-to-one correspondence $S \mapsto H(S)$ between G-invariant polynomially convex subsets of $\mathbb{C}^n$ and polynomially convex subsets of $\mathbb{C}^m$ lying in $H(\mathbb{C}^n)$.

[7] For instance, if $G = S_n$ acts by permutation of coordinates, then we can choose h_i to be the elementary symmetric polynomial of degree i, $1 \leqslant i \leqslant n$.

Proposition 6.2. *Suppose $A \subset C(K)$ is G-invariant for some finite group $G \subset GL_n(\mathbb{C})$ and $K \Subset \mathbb{C}^n$ star-shaped with respect to the origin. Then $A_G \in \mathscr{C}$. In particular, A_G is projective free.*

Proof. Since K is star-shaped with respect to the origin and all h_i are homogeneous polynomials, we have $t \cdot \mathbf{z} \in K$ for all $\mathbf{z} \in K$, $t \in [0,1]$ and

$$H(t \cdot \mathbf{z}) = \big(t^{\deg h_1} \cdot h_1(\mathbf{z}), \dots, t^{\deg h_m} \cdot h_m(\mathbf{z})\big).$$

We set for $\mathbf{w} = (w_1, \dots, w_m) \in \mathbb{C}^m$ and $t \in [0,1]$,

$$D(\mathbf{w}, t) := \big(t^{\deg h_1} \cdot w_1, \dots, t^{\deg h_m} \cdot w_m\big).$$

Then $D : \mathbb{C}^m \times [0,1] \to \mathbb{C}^m$ is continuous, maps $F_G(\widehat{K}/G) \times [0,1]$ into $F_G(\widehat{K}/G)$ and is such that $D(\cdot, 1) = \mathrm{id}_{F_G(\widehat{K}/G)}$ and $D(\cdot, 0) = \mathbf{0}$. Hence, $F_G(\widehat{K}/G)$ is contractible and therefore $A_G \in \mathscr{C}$, as required. □

Let us consider several explicit examples of nonuniform algebras A.

(1) Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing concave function, not identically zero, and such that $\omega(0) = 0$. We set

$$d(z_1, z_2) := \omega(\|z_1 - z_2\|), \quad z_1, z_2 \in \mathbb{C}^n,$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{C}^n$ ($\cong \mathbb{R}^{2n}$).

Then d is a metric on $\mathbb{C}^n$ compatible with its topology.

For a bounded set $K \subset \mathbb{C}^n$, let $\mathrm{Lip}_d(K) \subset C(K)$ be the Banach algebra of complex-valued Lipschitz functions on K with respect to d, equipped with norm

$$\|f\|_{\mathrm{Lip}} := \sup_K |f| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

In this case, A is defined to be the completion in $\mathrm{Lip}_d(K)$ of the algebra $\mathbb{C}[z_1, \dots, z_n]|_K$. Clearly, A is weakly inverse closed, see (wi).

(2) Let $C^p(K) \subset C(K)$ be the restriction to K of the algebra $C^p(\mathbb{C}^n)$ of bounded complex-valued functions f on $\mathbb{C}^n$ having bounded continuous partial derivatives up to order p with the norm the sum of supremum norms of f and of all its partial derivatives. Equipped with the quotient norm, $C^p(K)$ becomes a unital commutative Banach algebra. In this case, A is defined to be the completion in $C^p(K)$ of the algebra $\mathbb{C}[z_1, \dots, z_n]|_K$. Clearly, A is weakly inverse closed, see (wi).

If $G \subset GL_n(\mathbb{C})$ is a finite group and K is invariant with respect to the action of G on $\mathbb{C}^n$, then it is easy to check that algebras $\mathrm{Lip}_d(K)$ and $C^p(K)$ are G-invariant. Since $\mathbb{C}[z_1, \dots, z_n]|_K$ is invariant with respect to the corresponding action of G, this implies that the algebras A_G of G-symmetric $\mathrm{Lip}_d(K)$ and $C^p(K)$ functions are G-invariant as well. If, in addition K is star-shaped with respect to the origin, then by Proposition 6.2, these algebras are projective free.

6.2. Bohr-Wiener algebras

Let G be a connected compact abelian group and let Γ be its (multiplicative) character group. Thus Γ consists of continuous homomorphisms of G into the group $\mathbb{T}$ of unimodular complex numbers and separates points of G. As G is connected, Γ can be made into a linearly ordered group (see e.g. [43, 8.1.8]). Let $\preccurlyeq$ be a fixed linear order such that $(\Gamma, \preccurlyeq)$ is an ordered group. We henceforth write Γ additively and denote its identity element by 0.

Standard widely used examples of Γ are $\mathbb{Z}^k$ and $\mathbb{R}^k$ ($k \in \mathbb{Z}_+$) with a lexicographic ordering; here we use usual addition in $\mathbb{Z}$ and in $\mathbb{R}$, and the discrete topology in both cases.

For a nonempty set I, we denote by $\ell^1(I)$ the complex Banach space of complex-valued sequences $\mathbf{a} = (a_i)_{i\in I}$ with pointwise operations and the norm

$$\|\mathbf{a}\|_1 := \sum_{i\in I} |a_i|.$$

If $\emptyset \neq J \subset I$, then we view $\ell^1(J)$ as a subset of $\ell^1(I)$.

If $I = \Gamma$, then $\ell^1(\Gamma)$ is a commutative unital complex Banach algebra with multiplication given by convolution:

$$(\mathbf{a} * \mathbf{b})_j = \sum_{k\in\Gamma} a_k b_{j-k}, \quad \mathbf{a} = (a_j)_{j\in\Gamma},\ \mathbf{b} = (b_j)_{j\in\Gamma} \in \ell^1(\Gamma).$$

The algebra $\ell^1(\Gamma)$ is semisimple and its maximal ideal space is G. The Gelfand transform $\hat{}\ : \ell^1(\Gamma) \to C(G)$ is given by the formula

$$\hat{\mathbf{a}}(g) := \sum_{j\in\Gamma} a_j \mathbf{e}_j(g), \quad g \in G, \quad \mathbf{a} = (a_j)_{j\in\Gamma} \in \ell^1(\Gamma). \tag{6.3}$$

Here $\mathbf{e}_j(g) := \langle j, g\rangle \in \mathbb{T}$ is the action of $j \in \Gamma$ on $g \in G$.

The function $\hat{\mathbf{a}}$ is called the *symbol* of $\mathbf{a}$ with *Bohr-Fourier coefficients* a_j and with the *Bohr-Fourier spectrum* $\{j \in \Gamma : a_j \neq 0\}$. The image of $\ell^1(\Gamma)$ under $\hat{}$ is denoted by $W(G)$. For a subsemigroup Σ of Γ, we denote by $W(G)_\Sigma$ the algebra of symbols of elements in $\ell^1(\Sigma)$. We let $C(G)_\Sigma$ be the closure of $W(G)_\Sigma$ in $C(G)$ (so that $C(G)_\Gamma = C(G)$). The notions of Bohr-Fourier coefficients and spectrum are extended from functions in $W(G)$ to $C(G)$ by continuity. The Bohr-Fourier spectrum of an element of $C(G)$ is at most countable, and $C(G)_\Sigma$ coincides with the set of functions having the Bohr-Fourier spectra in Σ.

Assume that $\Sigma \subset \Gamma$ is *pointed*, i.e., such that

$$0 \in \Sigma, \quad \text{and} \quad \Sigma \cap (-\Sigma) = \{0\}. \tag{6.4}$$

(E.g., if $\Gamma_+ = \{j \in \Gamma : 0 \preccurlyeq j\}$, $\Gamma_- = \{j \in \Gamma : j \preccurlyeq 0\}$, then Γ_+ and Γ_- are pointed subsemigroups.)

Theorem *([9, Theorem 1.2]). If Σ is a pointed subsemigroup, then the algebras $W(G)_\Sigma$ and $C(G)_\Sigma$ belong to the class $\mathscr{C}$.*

Next, for a commutative unital complex Banach algebra B with norm $\|\cdot\|_B$, we let $\ell^1(\Sigma, B)$ denote the complex Banach space of B-valued sequences $\mathbf{b} = (b_j)_{j\in\Sigma}$ with pointwise operations and the norm

$$\|\mathbf{b}\|_{1,B} := \sum_{j\in\Sigma} \|b_j\|_B.$$

Then the projective tensor product $B\widehat{\otimes}_\pi W(G)_\Sigma$ consists of B-valued continuous functions on G

$$\widetilde{\mathbf{b}}(g) := \sum_{j\in\Sigma} b_j \mathbf{e}_j(g), \quad g \in G, \quad \mathbf{b} = (b_j)_{j\in\Sigma} \in \ell^1(\Sigma, B),$$

with norm $\|\widetilde{\mathbf{b}}\| := \|\mathbf{b}\|_{1,B}$.
In turn, the injective tensor product $B\widehat{\otimes}_\varepsilon C(G)_\Sigma$ is the closure of the commutative algebra $B\widehat{\otimes}_\pi W(G)_\Sigma$ in $C(G, B)$ equipped with norm $\|f\| := \max_{g\in G} \|f(g)\|_B$.

Now Theorem 5.2 implies the following:

Corollary. *If Σ is a pointed subsemigroup, then the correspondences*

$$P \cong B \otimes_B P \xrightarrow{(\mathrm{id}_B \otimes 1)\otimes \mathrm{id}_P} (B\widehat{\otimes}_\pi W(G)_\Sigma) \otimes_B P$$

$$P \cong B \otimes_B P \xrightarrow{(\mathrm{id}_B \otimes 1)\otimes \mathrm{id}_P} (B\widehat{\otimes}_\varepsilon C(G)_\Sigma) \otimes_B P$$

determine bijections between isomorphism classes of finitely generated projective B modules and finitely generated projective $B\widehat{\otimes}_\pi W(G)_\Sigma$ and $B\widehat{\otimes}_\varepsilon C(G)_\Sigma$ modules, respectively.

In particular, if B is projective free (or Hermite), then the algebras $B\widehat{\otimes}_\pi W(G)_\Sigma$ and $B\widehat{\otimes}_\varepsilon C(G)_\Sigma$ are projective free (respectively, Hermite).

6.3. Algebras of holomorphic semi-almost periodic functions

Recall that $f \in C_b(\mathbb{R})$ is *almost periodic* if the family $\{S_\tau f : \tau \in \mathbb{R}\}$ of its translates, where $(S_\tau f)(x) := f(x+\tau)$ $(x \in \mathbb{R})$, is relatively compact in $C_b(\mathbb{R})$. Let $AP(\mathbb{R})$ be the Banach algebra of almost periodic functions endowed with the supremum norm.

Let $\mathbb{T}$ be the boundary of the unit disc $\mathbb{D}$, with the counterclockwise orientation. For $s := e^{it}$, $t \in [0, 2\pi)$, let $\gamma_s^k(\delta) := \{se^{ikx} : 0 \leqslant x < \delta < 2\pi\}$, $k \in \{-1, 1\}$, be two open arcs having s as the right and the left endpoints (with respect to the orientation), respectively.

A function $f \in \mathrm{L}^\infty (:= \mathrm{L}^\infty(\mathbb{T}))$ is *semi-almost periodic* if for any $s \in \mathbb{T}$, and any $\varepsilon > 0$ there exist a number $\delta = \delta(s, \varepsilon) \in (0, \pi)$ and functions $f_k : \gamma_s^k(\delta) \to \mathbb{C}$, $k \in \{-1, 1\}$, such that functions

$$\widetilde{f_k}(x) := f_k(se^{ik\delta e^x}), \quad -\infty < x < 0, \quad k \in \{-1, 1\},$$

are restrictions of some almost periodic functions from $AP(\mathbb{R})$, and

$$\sup_{z \in \gamma_s^k(\delta)} |f(z) - f_k(z)| < \varepsilon, \quad k \in \{-1, 1\}.$$

By SAP we denote the Banach algebra of semi-almost periodic functions on $\mathbb{T}$ endowed with the supremum norm. The algebra SAP contains as a special case an algebra introduced by Sarason [46] in connection with some problems in the theory of Toeplitz operators.

It is easy to see that the set of points of discontinuity of a function in SAP is at most countable. For a closed subset S of $\mathbb{T}$, we denote by $SAP(S)$ the Banach algebra of semi-almost periodic functions on $\mathbb{T}$ that are continuous on $\mathbb{T} \setminus S$.

Next, for a semi-almost periodic function $f \in SAP$, $k \in \{-1, 1\}$, and a point $s \in S$, the *left* ($k = -1$) and the *right* ($k = 1$) *mean values of f over s* are given by the formulas

$$M_s^k(f) := \lim_{n \to \infty} \frac{1}{b_n - a_n} \int_{a_n}^{b_n} f(se^{ike^t})dt, \tag{6.5}$$

where $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are arbitrary sequences of real numbers converging to $-\infty$ such that $\lim\limits_{n \to \infty} (b_n - a_n) = +\infty$.

The *Bohr-Fourier coefficients of f over s* can be then defined by the formulas

$$a_\lambda^k(f, s) := M_s^k(fe^{-i\lambda \log_s^k}), \tag{6.6}$$

where

$$\log_s^k(se^{ikx}) := \log x, \quad 0 < x < 2\pi, \quad k \in \{-1, 1\}.$$

The *spectrum of f over s* is

$$\operatorname{spec}_s^k(f) := \{\lambda \in \mathbb{R} : a_\lambda^k(f, s) \neq 0\}. \tag{6.7}$$

Let $\Sigma : S \times \{-1, 1\} \to 2^{\mathbb{R}}$ be a set-valued map which associates with each $s \in S$, $k \in \{-1, 1\}$, a unital semi-group $\Sigma(s, k) \subset \mathbb{R}$. By $SAP_\Sigma(S) \subset SAP(S)$, we denote the

Banach algebra of semi-almost periodic functions f with $\mathrm{spec}_s^k(f) \subset \Sigma(s,k)$ for all $s \in S$, $k \in \{-1, 1\}$.

Let H^∞ be the Banach algebra of bounded holomorphic functions in $\mathbb{D}$ with the supremum norm. Then $SAP_\Sigma(S) \cap H^\infty$ is called the *algebra of holomorphic semi-almost period functions with spectrum* Σ.

If $f \in SAP_\Sigma(S) \cap H^\infty$, then

$$\mathrm{spec}_s^{-1}(f) = \mathrm{spec}_s^1(f) =: \mathrm{spec}_s(f) \tag{6.8}$$

and, moreover,

$$a_\lambda^{-1}(f,s) = e^{\lambda\pi} a_\lambda^1(f,s) \quad \text{for each } \; \lambda \in \mathrm{spec}_s(f), \tag{6.9}$$

see [10, Proposition 3.3]. Thus, $SAP_\Sigma(S) \cap H^\infty = SAP_{\Sigma_{\mathrm{sym}}}(S) \cap H^\infty$, where $\Sigma_{\mathrm{sym}}(s) := \Sigma_{\mathrm{sym}}(s, \pm 1) = \Sigma(s,-1) \cap \Sigma(s,1)$ for all $s \in S$.

Let A_Σ^S be the closed subalgebra of H^∞ generated by the disk-algebra $A(\mathbb{D})$ and the functions of the form $ge^{\lambda h}$, where $\mathrm{Re}(h)|_{\mathbb{T}}$ is the characteristic function of the closed arc going in the counterclockwise direction from the initial point at s to the endpoint at $-s$ such that $s \in S$, $\frac{\lambda}{\pi} \in \Sigma_{\mathrm{sym}}(s)$ and $g(z) := z + s$, $z \in \mathbb{D}$ (in particular, $ge^{\lambda h}$ has discontinuity at s only). It is shown in [10, Corollary 3.7] that

$$SAP_\Sigma(S) \cap H^\infty = A_\Sigma^S.$$

Let b_Σ^S denote the maximal ideal space of the algebra $SAP_\Sigma(S) \cap H^\infty$. The structure of b_Σ^S is described in [10, Theorem 3.13]. Specifically, the transpose of the embedding $A(\mathbb{D}) \hookrightarrow SAP_\Sigma(S) \cap H^\infty$ determines a continuous epimorphism $a_\Sigma^S : \bar{\mathbb{D}} \setminus S \to \bar{\mathbb{D}}$ one-to-one over $\bar{\mathbb{D}} \setminus S$ and such that for each $s \in S$ the fibre of a_Σ^S over s is homeomorphic to the maximal ideal space $b_{\Sigma_{\mathrm{sym}}(s)}(T)$ of the algebra $APH_{\Sigma_{\mathrm{sym}}(s)}(T)$ of holomorphic almost periodic functions on the closed strip $T = \{z \in \mathbb{C} \, : \, \mathrm{Im}(z) \in [0, \pi]\}$ with the spectrum in $\Sigma_{\mathrm{sym}}(s)$.

Now, we have (cf. [9, Theorem 1.2]):

Theorem *([10, Theorem 3.19]).*

(1) *The map of cohomology groups induced by embeddings $(a_\Sigma^S)^{-1}(s) \hookrightarrow b_\Sigma^S$, $s \in S$, produces an isomorphism*

$$H^k(b_\Sigma^S, \mathbb{Z}) \cong \bigoplus_{s \in S} H^k(b_{\Sigma_{\mathrm{sym}}(s)}(T), \mathbb{Z}), \quad k \geqslant 1.$$

(2) *Suppose that each $\Sigma_{\mathrm{sym}}(s)$ is a subset of $\mathbb{R}_+$ or $\mathbb{R}_-$. Then $H^k(b_\Sigma^S, \mathbb{Z}) = 0$ for all $k \geqslant 1$ and the algebra $SAP_\Sigma(S) \cap H^\infty$ is projective free.*

6.4. Algebras of bounded holomorphic functions

Let X be a connected Riemann surface such that the Banach algebra $H^\infty(X)$ of bounded holomorphic functions on X separates points of X. Since X is homotopy equivalent to a one-dimensional CW-complex, Theorem 2.8 and Proposition 2.1 imply that the algebras $C(X)$ and $C_b(X)$ are projective free. The transpose of the isometric embedding $H^\infty(X) \hookrightarrow C_b(X)$ induces a continuous map $p : \beta(X) \to M(H^\infty(X))$ of the maximal ideal spaces, with image $\mathrm{cl}\, X$, the closure of X (identified with the set of evaluations at points of X) in $M(H^\infty(X))$. Since $\dim X = 2$ and $H^2(X, \mathbb{Z}) = 0$, $\dim \beta X = 2$ and $H^2(\beta X, \mathbb{Z}) = 0$ as well (see the proof of Theorem 2.9 for the references). Moreover, the map p is identical on X. These make the following conjecture plausible (cf. the Question in [7]):

Conjecture 6.3. *The covering dimension of* $\mathrm{cl}\, X$ *is* 2 *and the Čech cohomology group* $H^2(\mathrm{cl}\, X, \mathbb{Z}) = 0$.

If the conjecture is true, it implies the validity of the following:

Conjecture 6.4. *If* $\mathrm{cl}\, X = M(H^\infty(X))$,[8] *then* $H^\infty(X)$ *is a projective free algebra.*

Let us describe some classes of Riemann surfaces X for which one of the conjectures is valid.

(1) Let R be an unbranched covering of a bordered Riemann surface and X be a domain in R such that inclusion $i : X \hookrightarrow R$ induces a monomorphism $i_* : \pi_1(X) \to \pi_1(R)$ of fundamental groups. The corona theorem, $\mathrm{cl}\, X = M(H^\infty(X))$, was proved in [5, Corollary 1.6] and projective freeness of $H^\infty(X)$ was established in [8, Theorem 1.5] using an analog of the classical Lax-Halmos theorem proved in [6, Theorem 1.7]. Thus Conjecture 6.4 is valid for such X.

In the special case $X = R$, it was proved in [7, Theorem 1.3] that Conjecture 6.3 is valid as well.

Remark 6.5. Conjecture 6.4 is false if X has a nonempty corona, i.e., if $M(H^\infty(X)) \setminus \mathrm{cl}\, X \neq \emptyset$. Indeed, using [4, Theorem 1.2], given an integer $n \geqslant 2$, one can construct an unbranched covering of a compact Riemann surface X for which $H^\infty(X)$ separates points, $\dim M(H^\infty(X)) \geqslant n$, and $H^2(M(H^\infty(X)), \mathbb{Z}) \neq 0$. The latter implies that $H^\infty(X)$ is not projective free.

(2) A wide class of planar domains, the so-called $\mathscr{B}$-domains, was introduced and studied by Behrens [3]. A set $X \subset \mathbb{C}$ is called a *$\mathscr{B}$-domain* if it is obtained from a domain $Y \subset \mathbb{C}$ by deleting a (possibly finite) hyperbolically-rare sequence of closed discs (Δ_n) contained

[8] I.e., that the corona theorem is valid for $H^\infty(X)$.

in Y with centres α_n, i.e., such that there are disjoint closed discs D_n with centres α_n satisfying $\Delta_n \Subset D_n \subset Y$ and $\sum \frac{\operatorname{rad}\Delta_n}{\operatorname{rad} D_n} < \infty$. It was shown in [7, Theorem 1.1] that if X is obtained from a domain $Y \subset \mathbb{C}$ by deleting a (possibly finite) hyperbolically-rare sequence of closed discs such that (i) $\operatorname{cl} Y = M(H^\infty(Y))$ and (ii) Conjecture 6.3 is valid for Y, then $\operatorname{cl} X = M(H^\infty(X))$ and Conjecture 6.3 is valid for X as well. (In particular, algebras $H^\infty(Y)$ and $H^\infty(X)$ are projective free.)

The class of domains Y satisfying (i), (ii) includes planar unbranched coverings of bordered Riemann surfaces (see part (1)) and domains obtained from them by deleting compact subsets of analytic capacity zero (e.g., totally disconnected compact subsets). Starting from such a Y one can construct a descending chain $Y := Y_0 \supset Y_1 \supset Y_2 \supset \cdots \supset Y_n$, $n \in \mathbb{N}$, of $\mathscr{B}$-domains, such that each Y_i is defined by deleting a hyperbolically-rare sequence of closed discs and then a compact subset of analytic capacity zero from Y_{i-1}. Then all Y_i satisfy assumptions (i), (ii), and, in particular, all algebras $H^\infty(Y_i)$ are projective free.

Recall that every bordered Riemann surface S is a domain in a compact Riemann surface $\widetilde{S}$ such that $\widetilde{S} \setminus S$ is the disjoint union of finitely many discs with analytic boundaries (see [49]). Here are some examples of planar unbranched coverings of S:

(a) It is known (see, e.g., [18, Chap. X]) that each $\widetilde{S}$ is the quotient of a planar domain Ω by the discrete action of a Schottky group G (the free group with g generators, where g is the genus of $\widetilde{S}$) by Möbius transformations. The corresponding quotient map $r : \Omega \to \widetilde{S}$ determines the regular covering of $\widetilde{S}$ with the deck transformation group G. Then $Y := r^{-1}(S) \subset \Omega$ is a regular covering of S satisfying conditions (i), (ii). By definition, Y is the complement in Ω of the finite disjoint union of G-orbits of compact simply connected domains with analytic boundaries biholomorphic by r to the connected components of $\widetilde{S} \setminus S$.

(b) Consider the universal covering $r_u : \widetilde{S}_u \to \widetilde{S}$ of $\widetilde{S}$ (where $\widetilde{S}_u = \mathbb{D}$ if $g \geqslant 2$, $\widetilde{S}_u = \mathbb{C}$ if $g = 1$, and $\widetilde{S}_u = \mathbb{CP}$ if $g = 0$), then $Y := r_u^{-1}(S) \subset \widetilde{S}_u$ satisfies conditions (i), (ii) as well. Here Y is the complement in $\widetilde{S}_u$ of the finite disjoint union of orbits under the action by Möbius transformations of the fundamental group $\pi_1(\widetilde{S})$ of $\widetilde{S}$ of compact simply connected domains with analytic boundaries biholomorphic by r_u to the connected components of $\widetilde{S} \setminus S$.

Finally, let us mention that if a connected Riemann surface X is such that $H^\infty(X)$ separates its points, $\operatorname{cl} X = M(H^\infty(X))$, $\dim \operatorname{cl} X = 2$ and $H^2(\operatorname{cl} X, \mathbb{Z}) = 0$, then by the Künneth formula (see, e.g., [48, Chap. 6, Exercise E]) for $M(H^\infty(X))^i$, $2 \leqslant i \leqslant 4$, the Čech cohomology groups $H^n(M(H^\infty(X))^i, \mathbb{Z}) = 0$ for all $n \geqslant 5$ and $\dim M(H^\infty(X))^i = 2i$. Hence, due to Theorem 4.1, for all $2 \leqslant i \leqslant 4$, the uniform algebra $(H^\infty(X))^{\widehat{\otimes}_\varepsilon i}$ (the i^{th} injective tensor power of $H^\infty(X)$) is Hermite.

7. Appendix

Let A be a commutative unital complex Banach algebra and G be a finite subgroup of automorphisms of A. By $A_G \subset A$ we denote the Banach subalgebra of elements invariant with respect to the action of G, i.e.,

$$A_G := \{a \in A \,:\, g(a) = a \quad \forall g \in G\}.$$

There is a natural projection $P_G : A \to A_G$ given by the formula

$$P_G(a) := \frac{1}{|G|} \sum_{g \in G} g(a) \quad \text{for all } a \in A. \tag{7.1}$$

Here $|G|$ is the cardinality of the set G.

For each $g \in G$, the transpose of the map $a \mapsto g(a) : A \to A$ induces a homeomorphism g^* of the maximal ideal space $M(A)$ of A. Thus we obtain an action of G on $M(A)$. Let $M(A)/G$ be the quotient space by this action and $\pi : M(A) \to M(A)/G$ be the quotient map. We equip $M(A)/G$ with the smallest topology in which the map π is continuous. Then $M(A)/G$ becomes a compact Hausdorff space homeomorphic to the maximal ideal space of the subalgebra $C(M(A))_G \subset C(M(A))$ of continuous functions invariant with respect to the action of G on $M(A)$. We have:

Lemma 7.1. *$M(A)/G$ is homeomorphic to the maximal ideal space of the algebra A_G.*

Proof. Let $\{g^*(x) \,:\, g \in G\} \subset M(A)$ be the equivalence class representing $\pi(x) \in M(A)/G$ for $x \in M(A)$. Since for each $a \in A_G$,

$$(g^*(x))(a) = x(g(a)) = x(a)$$

(as elements of A_G are invariant with respect to the action of G on A), the functional φ_z, $z \in M(A)/G$,

$$\varphi_z(a) := x(a), \quad x \in \pi^{-1}(z), \quad \forall a \in A_G,$$

is well-defined and is an element of $M(A_G)$. Thus, we have a map $\Phi : M(A)/G \to M(A_G)$, $\Phi(z) := \varphi_z$. Let us show that Φ is continuous. Indeed, by the definition of the Gelfand topology on $M(A_G)$, it suffices to show that for an open set $U_a := \{y \in M(A_G) \,:\, |y(a)| < 1\} \subset M(A_G)$ with $a \in A_G$, the set $\Phi^{-1}(U_a)$ is open in $M(A)/G$. In turn, by the definition of topology on $M(A)/G$, the previous set is open if and only if its preimage under π is open in $M(A)$. We have

$$\begin{aligned}\pi^{-1}(\Phi^{-1}(U_a)) &= \{x \in M(A) \,:\, |\varphi_{\pi(x)}(a)| < 1\}\\ &= \{x \in M(A) \,:\, |x(a)| < 1\}.\end{aligned}$$

The latter is an open subset of $M(A)$ by the definition of the Gelfand topology.

Next, let us show that the map Φ is injective.

Let z_1, z_2 be distinct points of $M(A)/G$. Since every finite subset of $M(A)$ is interpolating for A, there is $a \in A$ such that

$$x(a) = 0 \text{ for all } x \in \pi^{-1}(z_1), \quad \text{and} \quad y(a) = 1 \text{ for all } y \in \pi^{-1}(z_2).$$

Then, for $P_G(a) \in A_G$ (see (7.1)) we obtain

$$\varphi_{z_1}(P_G(a)) = 0, \quad \text{and} \quad \varphi_{z_2}(P_G(a)) = 1,$$

i.e., $\Phi(z_1) \neq \Phi(z_2)$. Therefore Φ is injective and, hence, embeds $M(A)/G$ as a compact subset of $M(A_G)$.

Finally, let us show that the map Φ is onto. If, on the contrary, there is $y \in M(A_G) \setminus \Phi(M(A)/G)$, then by the definition of the Gelfand topology on $M(A_G)$, there exist $a_1, \dots, a_k \in A_G$ such that

$$y(a_1) = \cdots = y(a_k) = 0 \quad \text{and} \quad \max_{1 \leqslant i \leqslant k} \min_{z \in M(A)/G} |\varphi_z(a_i)| \geqslant \delta > 0. \tag{7.2}$$

This implies that

$$\max_{1 \leqslant i \leqslant k} \min_{x \in M(A)} |x(a_i)| \geqslant \delta > 0.$$

Equivalently, the family $a_1, \dots, a_n$ does not belong to a maximal ideal of A, i.e., there exist $b_1, \dots, b_n \in A$ such that

$$b_1 a_1 + \cdots + b_n a_n = 1. \tag{7.3}$$

We set

$$\tilde{b}_i := P_G(b_i), \quad 1 \leqslant i \leqslant k.$$

Then each $\tilde{b}_i \in A_G$, and since 1 and all $a_i \in A_G$, equation (7.3) implies that

$$\tilde{b}_1 a_1 + \cdots + \tilde{b}_n a_n = 1. \tag{7.4}$$

Applying y to (7.4), we get due to (7.2),

$$1 = y(1) = y(\tilde{b}_1) y(a_1) + \cdots + y(\tilde{b}_n) y(a_n) = 0,$$

a contradiction which shows that Φ is a surjection. Therefore $\Phi : M(A)/G \to M(A_G)$ is a homeomorphism, as required. $\square$

References

[1] G. Allan, On one-sided inverses in Banach algebras of holomorphic vector-valued functions, J. Lond. Math. Soc. 42 (1967) 463–470.
[2] M. Atiyah, I. Macdonald, Introduction to Commutative Algebra, Addison-Wesley, 1969.
[3] M. Behrens, The maximal ideal space of algebras of bounded analytic functions on infinitely connected domains, Trans. Am. Math. Soc. 161 (1971) 359–379.
[4] A. Brudnyi, A uniqueness property for H^∞ on coverings of projective manifolds, Mich. Math. J. 51 (2003) 503–507.
[5] A. Brudnyi, Projections in the space H^∞ and the corona theorem for subdomains of coverings of finite bordered Riemann surfaces, Ark. Mat. 42 (2004) 31–59.
[6] A. Brudnyi, Grauert- and Lax-Halmos-type theorems and extension of matrices with entries in H^∞, J. Funct. Anal. 206 (2004) 87–108.
[7] A. Brudnyi, Projective free algebras of bounded holomorphic functions on infinitely connected domains, Algebra Anal. 33 (2021) 49–65.
[8] A. Brudnyi, A. Sasane, Sufficient conditions for the projective freeness of Banach algebras, J. Funct. Anal. 257 (2009) 4003–4014.
[9] A. Brudnyi, L. Rodman, I. Spitkovsky, Projective free algebras of continuous functions on compact abelian groups, J. Funct. Anal. 259 (2010) 918–932.
[10] A. Brudnyi, D. Kinzebulatov, Holomorphic semi-almost periodic functions, Integral Equ. Oper. Theory 66 (2010) 293–325.
[11] A. Calder, J. Siegel, Homotopy and uniform homotopy, Trans. Am. Math. Soc. 235 (1978) 245–270.
[12] H. Cartan, Espaces fibrés analytiques, in: International Symposium on Algebraic Topology, Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958, pp. 97–121.
[13] P. Cohn, Free Rings and Their Relations, second edition, London Mathematical Society Monographs, vol. 19, Academic Press, 1985.
[14] P. Cohn, Some remarks on projective-free rings, Algebra Univers. 49 (2003) 159–164.
[15] S. Dineen, R. Harte, C. Taylor, Spectra of tensor product elements. I. Basic theory, Math. Proc. R. Ir. Acad. 101A (2) (2001) 177–196.
[16] S. Eilenberg, N. Steenrod, Foundations of Algebraic Topology, Princeton University Press, 1952.
[17] R. Engelking, General Topology, second edition, Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, 1989.
[18] L. Ford, Automorphic Functions, McGraw Hill, 1929.
[19] O. Forster, Primärzerlegung in Steinschen Algebren, Math. Ann. 154 (1964) 307–329.
[20] O. Forster, Zur Theorie der Steinschen Algebren und Moduln, Math. Z. 97 (1967) 376–405.
[21] T. Gamelin, Uniform Algebras, Prentice-Hall, 1969.
[22] J. Garnett, Bounded Analytic Functions, Pure and Applied Mathematics, vol. 96, Academic Press, 1981.
[23] H. Grauert, Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen, Math. Ann. 133 (1957) 450–472.
[24] H. Grauert, Analytische Faserungen über holomorph-vollständigen Räumen, Math. Ann. 135 (1958) 263–273.
[25] H. Grauert, R. Remmert, Theory of Stein Spaces, Classics in Mathematics, Springer, 2004.
[26] H. Hamm, Zum Homotopietyp q-vollständiger Räume, J. Reine Angew. Math. 364 (1986) 1–9.
[27] S-T. Hu, Mappings of a normal space into an absolute neighborhood retract, Trans. Am. Math. Soc. 64 (1948) 336–358.
[28] S-T. Hu, Homotopy Theory, Pure and Applied Mathematics, vol. VIII, Academic Press, 1959.
[29] S-T. Hu, Theory of Retracts, Wayne State University Press, 1965.
[30] D. Husemoller, Fibre Bundles, third edition, Graduate Texts in Mathematics, vol. 20, Springer, 1994.
[31] M. Karoubi, Les isomorphismes de Chern et de Thom-Gysin en K-théorie Séminaire Cartan–Schwartz 1963/64, Benjamin, 1967.
[32] T. Lam, Serre's Conjecture, Lecture Notes in Mathematics, vol. 635, Springer-Verlag, Berlin-New York, 1978.
[33] V. Lin, Holomorphic fiberings and multivalued functions of elements of a Banach algebra, Akad. Nauk SSSR, Funkc. Anal. Pril. 7 (1973) 43–51.
[34] L. Loomis, An Introduction to Abstract Harmonic Analysis, Van Nostrand, 1953.
[35] S. Mardešić, Absolute neighborhood retracts and shape theory, in: History of Topology, North-Holland, 1999, pp. 241–269.

[36] J. Milnor, J. Stasheff, Characteristic Classes, Annals of Mathematics Studies, vol. 76, Princeton University Press, 1974.
[37] R. Mortini, R. Rupp, Extension Problems and Stable Ranks - a Space Odyssey, Birkhäuser/Springer, 2021.
[38] A. Morye, Note on the Serre-Swan theorem, Math. Nachr. 286 (2-3) (2013) 272–278.
[39] M. Novodvorskiĭ, Certain homotopic invariants of the space of maximal ideals, Akad. Nauk SSSR, Mat. Zametki 1 (1967) 487–494.
[40] A. Quadrat, The fractional representation approach to synthesis problems: an algebraic analysis viewpoint. II. Internal stabilization, SIAM J. Control Optim. 42 (2003) 300–320.
[41] H. Rossi, On envelopes of holomorphy, Commun. Pure Appl. Math. 16 (1963) 9–17.
[42] H. Royden, Function algebras, Bull. Am. Math. Soc. 69 (1963) 281–298.
[43] W. Rudin, Fourier Analysis on Groups, Wiley Classics Library, Wiley, 1990.
[44] R. Ryan, Introduction to Tensor Products of Banach Spaces, Springer Monographs in Mathematics, Springer, 2002.
[45] D. Sarason, Approximation of piecewise continuous functions by quotients of bounded analytic functions, Can. J. Math. 24 (1972) 642–657.
[46] D. Sarason, Toeplitz operators with semi-almost periodic symbols, Duke Math. J. 44 (1977) 357–364.
[47] B. Simon, Uniform crossnorms, Pac. J. Math. 46 (1973) 555–560.
[48] E. Spanier, Algebraic Topology, McGraw-Hill, 1966.
[49] E. Stout, Bounded holomorphic functions on finite Riemann surfaces, Trans. Am. Math. Soc. 120 (1965) 255–285.
[50] E. Stout, Polynomial Convexity, Progress in Mathematics, vol. 261, Birkhäuser, 2007.
[51] D. Suárez, Čech cohomology and covering dimension for the H^∞ maximal ideal space, J. Funct. Anal. 123 (1994) 233–263.
[52] R. Swan, Vector bundles and projective modules, Trans. Am. Math. Soc. 105 (1962) 264–277.
[53] R. Swan, Algebraic K-Theory, Lecture Notes in Mathematics, vol. 76, Springer, 1968.
[54] J. Taylor, Topological invariants of the maximal ideal space of a Banach algebra, Adv. Math. 19 (1976) 149–206.
[55] V. Tolokonnikov, Extension problem to an invertible matrix, Proc. Am. Math. Soc. 117 (1993) 1023–1030.
[56] L. Vaserstein, Vector bundles and projective modules, Trans. Am. Math. Soc. 294 (1986) 749–755.
[57] M. Vidyasagar, Control System Synthesis: A Factorization Approach, Series in Signal Processing, Optimization, and Control, vol. 7, MIT Press, Cambridge, 1985.
[58] J. Whitehead, The immersion of an open 3-manifold in Euclidean 3-space, Proc. Lond. Math. Soc. (3) 11 (1961) 81–90.