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Mean–variance hedging of contingent claims with random maturity

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Abstract

We study the mean–variance hedging of an American-type contingent claim that is exercised at a random time in a Markovian setting. This problem is motivated by applications in the areas of employee stock option valuation, credit risk, or equity-linked life insurance policies with an underlying risky asset value guarantee. Our analysis is based on dynamic programming and uses PDE techniques. In particular, we prove that the complete solution to the problem can be expressed in terms of the solution to a system of one quasi-linear parabolic PDE and two linear parabolic PDEs. Using a suitable iterative scheme involving linear parabolic PDEs and Schauder's interior estimates for parabolic PDEs, we show that each of these PDEs has a classical $C^{1,2}$ solution. Using these results, we express the claim's mean–variance hedging value that we derive as its expected discounted payoff with respect to an equivalent martingale measure that does not coincide with the minimal martingale measure, which, in the context that we consider, identifies with the minimum entropy martingale measure as well as the variance-optimal martingale measure. Furthermore, we present a numerical study that illustrates aspects of our theoretical results.

KEYWORDS
classical solutions to PDEs, credit risk, employee stock options, life insurance, mean–variance hedging, quasi-linear parabolic PDEs, random time horizon

1 | INTRODUCTION

We consider a frictionless financial market consisting of two primary assets. The first one is risk-free and its unit initialized price is given by

$$\mathrm{d}B_t = r(t)B_t\,\mathrm{d}t, \quad B_0 = 1. \tag{1}$$

The second one is risky and has price process S that is modeled by the solution to the SDE

$$\mathrm{d}S_t = \mu(t, S_t)S_t\,\mathrm{d}t + \sigma(t, S_t)S_t\,\mathrm{d}W_t, \quad S_0 = s > 0. \tag{2}$$

In this market, we consider an American-type contingent claim with maturity time $T > 0$ that is exercised at time $\eta \wedge T$, where η is a random time that is characterized by a hazard rate process $\big(\ell(t, S_t),\ t \geq 0\big)$ (see Equation 12 for the precise definition). Once exercised, the claim yields the payoff

$$F_{\mathrm{E}}(\eta, S_\eta)\mathbf{1}_{\{\eta<T\}} + F_{\mathrm{T}}(S_T)\mathbf{1}_{\{\eta\geq T\}} \tag{3}$$

to its holder.

The study of such contingent claims has been motivated by three types of applications. The first one is in the context of employee stock options (ESOs). To fix ideas, consider a firm that issues an ESO that expires at time T and is vested at time $T_{\mathrm{v}} \in [0, T[$, meaning that the ESO can be exercised at any time between T_{v} and T. The firm estimate that the holder of the ESO will either exercise it or have their job terminated at time $\eta \wedge T$. If we denote by F the payoff function of the ESO, then the functions F_{E} and F_{T} in Equation (3) are given by

$$F_{\mathrm{E}}(t, s) = F(s)\mathbf{1}_{[T_{\mathrm{v}},T]}(t) \quad \text{and} \quad F_{\mathrm{T}}(s) = F(s), \quad \text{for } t \in [0, T[\text{ and } s > 0. \tag{4}$$

The use of a random time η to model the exercise time of an ESO arises from endogenous considerations and has become popular in the literature. Indeed, an ESO's holder faces restrictions in trading the option and might exercise the option earlier than dictated by risk-neutrality if in need of liquidity.

Credit risk is the second area of application, in which, the random time η models an exogenous shock. In this context, suppose that the underlying risky asset is default-free and consider a European option's writer who faces the risk of default at a random time η. If we denote by F the payoff function of the European option that such a writer sells, then the functions F_{E} and F_{T} in Equation (3) that represent the options holder's payoff are given by

$$F_{\mathrm{E}}(t, s) = \delta(t)F(s) \quad \text{and} \quad F_{\mathrm{T}}(s) = F(s), \quad \text{for } t \in [0, T[\text{ and } s > 0,$$

where the function δ models the recovery rate.

The third area of application arises in the context of equity-linked life insurance policies with an underlying risky asset value guarantee. Such a policy payoff depends on the performance of a risky asset, such as the value of a reference portfolio, subject to a minimum guaranteed benefit. The policy matures at time T if the insured is still alive and has not withdrawn from the policy. On the other hand, it expires early if the insured dies or withdraws from the policy at a random time η prior to T. A standard example of a policy payoff function F is given by $F(t,s) = \max\{Ke^{\delta t}, s\}$, where K is a fixed guarantee (sometimes $K = S_0$) and δ is a fixed guarantee rate. In terms of the functions F_{E} and F_{T} in Equation (3), we have the following examples:

$$\begin{aligned}
\text{Pure endowment policy:} \quad & F_{\mathrm{E}}(t,s) = 0 \quad \text{and} \quad F_{\mathrm{T}}(s) = F(T,s); \\
\text{Term insurance policy:} \quad & F_{\mathrm{E}}(t,s) = F(t,s) \quad \text{and} \quad F_{\mathrm{T}}(s) = 0; \\
\text{Endowment policy:} \quad & F_{\mathrm{E}}(t,s) = F(t,s) \quad \text{and} \quad F_{\mathrm{T}}(s) = F(T,s).
\end{aligned}$$

The time $\eta \wedge T$ at which the claim is liquidated introduces market incompleteness in the model because η is not a stopping time but a random time. As a result, perfect hedging of the claim's payoff is not possible. Therefore, one has to rely on an incomplete market methodology (see Rheinlander & Sexton, 2011 for a textbook).

The super-replication value of the claim that we consider is obtained by viewing the liquidation time $\eta \wedge T$ as a discretionary stopping time and then treating the claim as a standard American option. Such a value of the claim is unrealistically high because it ignores all of the modeling issues that give rise to the consideration of the random time η. Indeed, it is this observation that has given rise to most of the relevant research literature.

Another approach is to assign a value to the claim by computing its expected discounted payoff with respect to a martingale measure. For instance, we can assign the risk-neutral value

$$x^{\mathrm{rn}} = \mathbb{E}^{\mathbb{Q}_1}\left[e^{-\int_0^{\eta\wedge T} r(u)\,\mathrm{d}u}\left(F_{\mathrm{E}}(\eta, S_\eta)\mathbf{1}_{\{\eta<T\}} + F_{\mathrm{T}}(S_T)\mathbf{1}_{\{\eta\geq T\}}\right)\right] \tag{5}$$

to the claim, where $\mathbb{Q}_1$ is the minimal martingale measure, which, in the context that we consider here, coincides with the variance-optimal martingale measure as well as the minimum entropy measure (see Remarks 2.5 and 4.4). In the context of ESO valuation, such a choice was proposed by Jennergren and Näslund (1993) and Carr and Linetsky (2000) by appealing to a heuristic diversification argument, which amounts to assuming that the jump risk is not priced. In the context of pricing unit-linked life insurance policies, the same choice was proposed by Aase and Persson (1994) and features in part of the analysis by Møller (1998). Furthermore, the minimal martingale measure is a standard choice in the intensity-based credit risk theory (e.g., see Bielecki & Rutkowski, 2002, Chapter 13).

Here, we establish the existence of a different martingale measure that arises from the mean–variance hedging of the claim's payoff. To this end, we consider the optimization problem

$$\text{minimize } \mathbb{E}^{\mathbb{P}}\left[\left(e^{-\int_0^{\eta\wedge T} r(u)\,\mathrm{d}u}\left(X^{x,\pi}_{\eta\wedge T} - F_{\mathrm{E}}(\eta, S_\eta)\mathbf{1}_{\{\eta<T\}} - F_{\mathrm{T}}(S_T)\mathbf{1}_{\{\eta\geq T\}}\right)\right)^2\right] \text{ over } (x,\pi), \tag{6}$$

where $X^{x,\pi}$ is the value process of an admissible self-financing portfolio strategy π that starts with initial endowment x and $\mathbb{P}$ is the natural probability measure. We solve this problem by first using the properties of the random time η to integrate it out of the performance criterion in Equation (6) and obtain a performance index involving stochastic optimal control over a finite time horizon. We then use dynamic programming and PDE techniques. In particular, we prove that the complete solution to the problem can be expressed in terms of the solution to a system of one quasi-linear parabolic PDE and two linear parabolic PDEs. Using a suitable iterative scheme involving linear parabolic PDEs and Schauder's interior estimates for parabolic PDEs, we show that each of these PDEs has a classical $C^{1,2}$ solution. Furthermore, we identify a martingale measure $\mathbb{Q}$ such that the portfolio's initial endowment x^{mvh} arising from the solution to the problem in Equation (6) admits the expression

$$x^{\mathrm{mvh}} = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{\eta\wedge T} r(u)\,\mathrm{d}u}\Big(F_{\mathrm{E}}(\eta, S_\eta)\mathbf{1}_{\{\eta<T\}} + F_{\mathrm{T}}(S_T)\mathbf{1}_{\{\eta\geq T\}}\Big)\right] \tag{7}$$

(see Remark 4.2). In full generality, this martingale measure $\mathbb{Q}$ is different from the minimal martingale measure $\mathbb{Q}_1$, which, in the context that we consider here, identifies with the minimum entropy martingale measure as well as the variance-optimal martingale measure, because the mean–variance hedging is implemented in the random time interval $[0, \eta\wedge T]$ rather than in the deterministic time interval $[0, T]$ (see Remarks 2.5 and 4.3).

In the context of the credit risk applications including the one discussed above, Bielecki et al. (2004, Section 9), Biagini and Cretarola (2007, Section 6), and several references therein study the mean–variance hedging of a payoff delivered at a given time T (see Remark 4.3 for more context). Using ideas from Jeanblanc et al. (2012), Kharroubi et al. (2013) study a problem of mean–variance hedging of a contingent claim's payoff over a random time horizon, which has several similarities to the problem that we study here. However, the techniques that these authors use and the nature of the results that they obtain are different from the ones in this paper. In particular, they show that the problem admits an optimal strategy that is described by the solution to a system of BSDEs with random time horizon and prove that this system of BSDEs does have a solution.

The paper is organized as follows. In Section 2, we formulate the mean–variance hedging problem that we study. In Section 3, we derive the classical solution to the problem's HJB equation, which reduces to a system of one quasi-linear parabolic PDE and two linear parabolic PDEs. We establish the main results on the mean–variance hedging of the claim's payoff in Section 4. Finally, we present a numerical investigation of the theory that we develop in Section 5. In this last section, we focus on ESO valuation because there is a rather large body of relevant literature, particularly, on the more applied side.

2 | THE SETTING

We build the model that we study on a complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$ carrying a standard one-dimensional Brownian motion W as well as an independent random variable U that has the uniform distribution on [0,1]. We denote by $(\mathcal{F}_t)$ the natural filtration of W, augmented by the $\mathbb{P}$-negligible sets in $\mathcal{G}$.

We fix a time horizon $T > 0$ and we consider a frictionless market consisting of two primary assets with price processes given by Equations (1) and (2) in the introduction. The value process $X^{x,\pi}$ of a self-financing portfolio with a position in these two assets that starts with initial endowment x has dynamics given by

$$\mathrm{d}X_t^{x,\pi} = \big(r(t)X_t^{x,\pi} + \sigma(t,S_t)\vartheta(t,S_t)\pi_t\big)\,\mathrm{d}t + \sigma(t,S_t)\pi_t\,\mathrm{d}W_t, \quad X_0^{x,\pi} = x, \tag{8}$$

where π_t is the amount of money invested in the risky asset at time t and

$$\vartheta(t, S_t) = \frac{\mu(t, S_t) - r(t)}{\sigma(t, S_t)}, \quad t \geq 0,$$

defines the market price of risk process. We make the following assumption.

Assumption 2.1. The functions $r : [0, T] \to \mathbb{R}$ and $\sigma, \vartheta : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ are C^2. Furthermore, there exists a constant $\kappa \geq 1$ such that

$$0 < \sigma(t, s), \quad \kappa^{-1} \leq \sigma^2(t, s) \leq \kappa, \quad |r(t)| \leq \kappa \tag{9}$$

and

$$\vartheta^2(t, s) + |\sigma_t(t, s)| + (s + 1)|\sigma_s(t, s)| + s^2|\sigma_{ss}(t, s)| + (s + 1)|\vartheta_s(t, s)| + s^2|\vartheta_{ss}(t, s)| \leq \kappa \tag{10}$$

for all $t \in [0, T]$ and $s > 0$.[1]

Apart from ensuring that the SDE (2) has a unique strong solution, this assumption includes several conditions that we will need in the analysis of the stochastic control problem that we solve.

We restrict our attention to admissible portfolio strategies that are introduced by the following definition.

Definition 2.2. A portfolio process π is admissible if it is $(\mathcal{F}_t)$-progressively measurable and

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \pi_t^2 \, \mathrm{d}t\right] < \infty. \tag{11}$$

We denote by $\mathcal{A}$ the family of all such portfolio processes.

In this context, we consider an American-type contingent claim with maturity time $T > 0$ that may be liquidated at a random time $\eta \wedge T$. On the event of early exercise, namely, on the event $\{\eta < T\}$, the claim yields a payoff $F_{\mathrm{E}}(\eta, S_\eta)$. On the event $\{\eta \geq T\}$, the claim yields a payoff $F_{\mathrm{T}}(S_T)$. Furthermore, we model the random time η by

$$\eta = \inf\left\{ t \geq 0 \,\middle|\, \exp\left(-\int_0^t \ell(u, S_u) \, \mathrm{d}u\right) \leq U \right\}. \tag{12}$$

We make the following additional assumption.

Assumption 2.3. The functions $F_{\mathrm{T}} : \mathbb{R}_+ \to \mathbb{R}$ and $\ell, F_{\mathrm{E}} : [0, T] \times \mathbb{R}_+ \to \mathbb{R}$ are C^1. Furthermore, there exist constants $\kappa \geq 1$[2] and $\xi \geq 1$ such that

$$0 \leq \ell(t, s), \quad (s + 1)\ell(t, s) + s|\ell_s(t, s)| \leq \kappa, \quad 0 \leq F_{\mathrm{E}}(t, s), \quad 0 \leq F_{\mathrm{T}}(s) \tag{13}$$

$$\text{and} \quad F_{\mathrm{E}}(t, s) + s|(F_{\mathrm{E}})_s(t, s)| + F_{\mathrm{T}}(s) + s|F_{\mathrm{T}}'(s)| \leq \kappa(1 + s^{\xi}) \tag{14}$$

for all $t \in [0, T]$ and $s > 0$.

Remark 2.4. The independence of U and $\mathcal{F}_\infty$ imply that

$$\mathbb{P}(\eta > t \mid \mathcal{F}_t) = \mathbb{P}\left(U < \exp\left(-\int_0^t \ell(u, S_u)\, \mathrm{d}u\right) \,\Big|\, \mathcal{F}_t\right) = \exp\left(-\int_0^t \ell(u, S_u)\, \mathrm{d}u\right). \tag{15}$$

We denote by $(\mathcal{G}_t)$ the filtration derived by rendering right-continuous the filtration defined by $\mathcal{F}_t \vee \sigma\big(\{\eta \le s\},\ s \le t\big)$, for $t \ge 0$. It is a standard exercise of the credit risk theory to show that the process M defined by

$$M_t = \mathbf{1}_{\{\eta \le t\}} - \int_0^{t\wedge\eta} \ell(u, S_u)\, \mathrm{d}u \tag{16}$$

is a $(\mathcal{G}_t)$-martingale.

Remark 2.5. In the context that we consider here, the so-called (H) hypothesis, namely, every square integrable $(\mathcal{F}_t)$-martingale is a square integrable $(\mathcal{G}_t)$-martingale, which is equivalent to

$$\mathbb{P}(\eta > t \mid \mathcal{F}_t) = \mathbb{P}(\eta > t \mid \mathcal{F}_\infty) \quad \text{for all } t \ge 0,$$

is satisfied (see Blanchet-Scalliet & Jeanblanc, 2004, Section 3.2).

The family of all probability measures that are equivalent to $\mathbb{P}$ is characterized by $(\mathcal{G}_t)$-predictable processes $\gamma > 0$ satisfying suitable integrability conditions. Given such a process, the solution to the SDE

$$\mathrm{d}L_t^\gamma = (\gamma_{t-} - 1) L_{t-}^\gamma\, \mathrm{d}M_t - \vartheta(t, S_t) L_t^\gamma\, \mathrm{d}W_t,$$

where M is the $(\mathcal{G}_t)$-martingale defined by Equation (16), which is given by

$$L_t^\gamma = \exp\left(\mathbf{1}_{\{\eta \le t\}} \ln \gamma_\eta - \int_0^{t\wedge\eta} \ell(u, S_u)\big(\gamma_u - 1\big)\, \mathrm{d}u - \frac{1}{2}\int_0^t \vartheta^2(u, S_u)\, \mathrm{d}u - \int_0^t \vartheta(u, S_u)\, \mathrm{d}W_u\right),$$

defines an exponential martingale. If we denote by $\mathbb{Q}_\gamma$ the probability measure on $(\Omega, \mathcal{G}_T)$ that has Radon–Nikodym derivative with respect to $\mathbb{P}$ given by $\frac{\mathrm{d}\mathbb{Q}_\gamma}{\mathrm{d}\mathbb{P}}|_{\mathcal{G}_T} = L_T^\gamma$, then Girsanov's theorem implies that the process $\big(\tilde{W}_t,\ t \in [0, T]\big)$ is a standard Brownian motion under $\mathbb{Q}_\gamma$, while the process $\big(\tilde{M}_t,\ t \in [0, T]\big)$ is a martingale under $\mathbb{Q}_\gamma$, where

$$\tilde{W}_t = \int_0^t \vartheta(u, S_u)\, \mathrm{d}u + W_t \quad \text{and} \quad \tilde{M}_t = \mathbf{1}_{\{\eta \le t\}} - \int_0^{t\wedge\eta} \ell(u, S_u)\gamma_u\, \mathrm{d}u, \quad \text{for } t \in [0, T].$$

Furthermore, the price process of the risky asset satisfies the SDE

$$\mathrm{d}S_t = r(t) S_t\, \mathrm{d}t + \sigma(t, S_t) S_t\, \mathrm{d}\tilde{W}_t, \quad S_0 = s > 0,$$

in the time interval $[0, T]$, while the conditional distribution of η is given by

$$\mathbb{Q}_\gamma(\eta > t \mid \mathcal{F}_t) = \exp\left(-\int_0^t \ell(u, S_u)\gamma_u\, \mathrm{d}u\right), \quad \text{for } t \in [0, T]. \tag{17}$$

The claims in this paragraph can be found in Blanchet-Scalliet et al. (2005, Proposition 1) and Björk et al. (1997, Theorem 3.12).

In the context of contingent claims with random maturity, Biagini and Cretarola (2007, (36) on p. 441 and Lemma 5.7) show that the minimal martingale measure (Biagini & Cretarola, 2007, Definition 5.5; Blanchet-Scalliet et al., 2005, Definition 3) coincides with the variance-optimal martingale measure (Biagini & Cretarola, 2007, p. 440; Szimayer, 2004, Definition 3.(a)) if ℓ is a deterministic function of time. Blanchet-Scalliet et al. (2005, Propositions 6 and 7) show that the minimum entropy martingale measure (Blanchet-Scalliet et al., 2005, Definition 4; Szimayer, 2004, Definition 3.(b)) coincides with the minimal martingale measure if σ and ϑ are deterministic functions of time. Furthermore, Szimayer (2004, Theorem 4) shows that the minimum entropy measure coincides with the variance-optimal measure if σ and ϑ are constants. The identity of the three martingale measures remains valid in the setting of stochastic σ, ϑ, and ℓ that we consider here thanks to a proof that was communicated to us by Professor Tahir Choulli. These three martingale measures correspond to the choice $\gamma = 1$.

In the setting we have developed thus far, we consider the problem of investing an initial amount x in a self-financing portfolio with a view to hedging the claim's payoff. To this end, we are faced with the market's incompleteness. We, therefore, consider minimizing the expected squared hedging error, which gives rise to the stochastic control problem that aims at minimizing the performance criterion

$$J_{T,x,s}(\pi) = \mathbb{E}^{\mathbb{P}}\left[e^{-2\int_0^{\eta\wedge T} r(u)\,\mathrm{d}u}\left(X^{x,\pi}_{\eta\wedge T} - F_{\mathrm{E}}(\eta, S_\eta)\mathbf{1}_{\{\eta<T\}} - F_{\mathrm{T}}(S_T)\mathbf{1}_{\{\eta\geq T\}}\right)^2\right] \tag{18}$$

over all admissible self-financing portfolio strategies. In view of the underlying probabilistic setting, this performance index admits the expression

$$J_{T,x,s}(\pi) = \mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{-\Lambda_t}\ell(t,S_t)\big(X^{x,\pi}_t - F_{\mathrm{E}}(t,S_t)\big)^2\,\mathrm{d}t + e^{-\Lambda_T}\left(X^{x,\pi}_T - F_{\mathrm{T}}(S_T)\right)^2\right], \tag{19}$$

where

$$\Lambda_t = \int_0^t \big(2r(u) + \ell(u,S_u)\big)\,\mathrm{d}u.$$

The value function of the resulting optimization problem is defined by

$$v(T,x,s) = \inf_{\pi\in\mathcal{A}} J_{T,x,s}(\pi). \tag{20}$$

3 | THE CLASSICAL SOLUTION TO THE HJB EQUATION

In view of standard stochastic control theory, the value function v should identify with a solution w to the HJB PDE

$$w_t(t,x,s) + \inf_{\pi}\left\{\frac{1}{2}\sigma^2(t,s)\pi^2 w_{xx}(t,x,s) + \sigma^2(t,s)s\pi w_{xs}(t,x,s) + \big(r(t)x + \sigma(t,s)\vartheta(t,s)\pi\big)w_x(t,x,s)\right\}$$

$$+\frac{1}{2}\sigma^2(t,s)s^2 w_{ss}(t,x,s) + \mu(t,s)s w_s(t,x,s) - \big(2r(t) + \ell(t,s)\big)w(t,x,s)$$

$$+\ell(t,s)\big(x - F_{\mathrm{E}}(t,s)\big)^2 = 0 \tag{21}$$

that satisfies the boundary condition

$$w(T,x,s) = \big(x - F_{\mathrm{T}}(s)\big)^2. \tag{22}$$

If the function $w(t,\cdot,s)$ is convex for all $(t,s) \in [0,T] \times \mathbb{R}_+$, then the infimum in this PDE is achieved by

$$\pi^{\dagger}(t,x,s) = -\frac{\sigma(t,s)s w_{xs}(t,x,s) + \vartheta(t,s)w_x(t,x,s)}{\sigma(t,s)w_{xx}(t,x,s)} \tag{23}$$

and Equation (21) is equivalent to

$$w_t(t,x,s) - \frac{\big(\sigma(t,s)s w_{xs}(t,x,s) + \vartheta(t,s)w_x(t,x,s)\big)^2}{2w_{xx}(t,x,s)}$$

$$+\frac{1}{2}\sigma^2(t,s)s^2 w_{ss}(t,x,s) + r(t)x w_x(t,x,s)$$

$$+\mu(t,s)s w_s(t,x,s) - \big(2r(t) + \ell(t,s)\big)w(t,x,s) + \ell(t,s)\big(x - F_{\mathrm{E}}(t,s)\big)^2 = 0. \tag{24}$$

In view of the quadratic structure of the problem we consider, we look for a solution to this PDE of the form

$$w(t,x,s) = f(t,s)\big(x - g(t,s)\big)^2 + h(t,s), \tag{25}$$

for some functions f, g, and h. Substituting this expression for w in Equation (24), we can see that the functions f, g, and h should satisfy the PDEs

$$f_t(t,s) + \frac{1}{2}\sigma^2(t,s)s^2 f_{ss}(t,s) + \mu(t,s)s f_s(t,s) - \ell(t,s)f(t,s) + \ell(t,s)$$

$$-\frac{\big(\sigma(t,s)s f_s(t,s) + \vartheta(t,s)f(t,s)\big)^2}{f(t,s)} = 0, \tag{26}$$

$$g_t(t,s) + \frac{1}{2}\sigma^2(t,s)s^2 g_{ss}(t,s) + r(t)s g_s(t,s) - \left(r(t) + \frac{\ell(t,s)}{f(t,s)}\right)g(t,s)$$

$$+\frac{\ell(t,s)F_{\mathrm{E}}(t,s)}{f(t,s)} = 0, \tag{27}$$

$$h_t(t,s) + \frac{1}{2}\sigma^2(t,s)s^2 h_{ss}(t,s) + \mu(t,s)s h_s(t,s) - \big(2r(t) + \ell(t,s)\big)h(t,s)$$

$$+\ell(t,s)\big(F_{\mathrm{E}}(t,s) - g(t,s)\big)^2 = 0 \tag{28}$$

in $[0,T] \times]0,\infty[$, with boundary conditions

$$f(T,s) = 1, \quad g(T,s) = F_{\mathrm{T}}(s), \quad \text{and} \quad h(T,s) = 0. \tag{29}$$

Before addressing the solvability of these PDEs, we need to consider the following result.

Lemma 3.1. *Consider Lipschitz continuous functions* χ, σ *such that* $|\chi(t,s)| + \sigma^2(t,s) \le C$ *for all* $t \in [0,T]$ *and* $s > 0$, *where* $C > 0$ *is a constant. The SDE*

$$\mathrm{d}\hat{S}_t = \chi(t,\hat{S}_t)\hat{S}_t\,\mathrm{d}t + \sigma(t,\hat{S}_t)\hat{S}_t\,\mathrm{d}B_t, \quad \hat{S}_0 = s > 0, \tag{30}$$

which is driven by a standard one-dimensional Brownian motion B, *has a unique strong solution such that, given constants* $\xi_1 \in \mathbb{R}$ *and* $\xi_2 > 0$,

$$\mathbb{E}\left[\hat{S}_t^{\xi_1}\right] \le \hat{C}_1 s^{\xi_1} \quad \text{and} \quad \mathbb{E}\left[\sup_{0\le t\le T} \hat{S}_t^{2\xi_2}\right] < \hat{C}_2 s^{2\xi_2} \quad \text{for all } (t,s) \in [0,T]\times\,]0,\infty[, \tag{31}$$

where $\hat{C}_i = \hat{C}_i(C,\xi_i) > 0$, $i = 1,2$, *are constants.*

Proof. The SDE (30) has a unique strong solution because the functions χ, σ satisfy the required boundedness and Lipschitz continuity assumptions. Given any constant $\xi_1 \in \mathbb{R}$, the boundedness of σ^2 implies that the process $M^{(\xi_1)}$ defined by

$$M_t^{(\xi_1)} = \exp\left(-\frac{1}{2}\xi_1^2 \int_0^t \sigma^2(u,\hat{S}_u)\,\mathrm{d}u + \xi_1 \int_0^t \sigma(u,\hat{S}_u)\,\mathrm{d}B_u\right), \quad \text{for } t \ge 0,$$

is a martingale because Novikov's condition is satisfied.

In view of this observation, we can see that

$$\begin{aligned}\mathbb{E}\left[\hat{S}_t^{\xi_1}\right] &= s^{\xi_1}\,\mathbb{E}\left[\exp\left(\xi_1 \int_0^t \left(\chi(u,\hat{S}_u) + \frac{1}{2}(\xi_1 - 1)\sigma^2(u,\hat{S}_u)\right)\mathrm{d}u\right) M_t^{(\xi_1)}\right] \\ &\le \exp\left(|\xi_1|CT + \frac{1}{2}\left(\xi_1^2 + |\xi_1|\right)CT\right) s^{\xi_1} \quad \text{for all } (t,s) \in [0,T]\times\,]0,\infty[,\end{aligned}$$

and the first estimate in Equation (31) follows. On the other hand, given any $\xi_2 > 0$, we use Itô's formula, Jensen's inequality, the Burkholder–Davis–Gundy inequalities (see Karatzas & Shreve, 1988, Theorem 3.3.28) and Fubini's theorem to obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T} \hat{S}_t^{2\xi_2}\right] &= \mathbb{E}\left[\sup_{0\le t\le T}\left(s^{\xi_2} + \int_0^t \left(\xi_2\chi(u,\hat{S}_u) + \frac{1}{2}\xi_2(\xi_2 - 1)\sigma^2(u,\hat{S}_u)\right)\hat{S}_u^{\xi_2}\,\mathrm{d}u\right.\right. \\ &\qquad\left.\left. + \int_0^t \xi_2\sigma(u,\hat{S}_u)\hat{S}_u^{\xi_2}\,\mathrm{d}B_u\right)^2\right] \\ &\le 4s^{2\xi} + 4T\,\mathbb{E}\left[\int_0^T \left(\xi_2\chi(u,\hat{S}_u) + \tfrac{1}{2}\xi_2(\xi_2 - 1)\sigma^2(u,\hat{S}_u)\right)^2 \hat{S}_u^{2\xi_2}\,\mathrm{d}u\right] \\ &\quad + 4C_{\mathrm{BDG}}\,\mathbb{E}\left[\int_0^T \xi_2^2\sigma^2(u,\hat{S}_u)\hat{S}_u^{2\xi_2}\,\mathrm{d}u\right] \\ &\le 4s^{2\xi_2} + 4T(2\xi_2 + \xi_2^2)^2 C^2 \int_0^T \mathbb{E}\left[\hat{S}_u^{2\xi_2}\right]\mathrm{d}u + 4C_{\mathrm{BDG}}\xi_2^2 C \int_0^T \mathbb{E}\left[\hat{S}_u^{2\xi_2}\right]\mathrm{d}u,\end{aligned}$$

where $C_{\mathrm{BDG}} > 0$ is a constant. Combining these inequalities with the first estimate in Equation (31), we derive the second estimate in Equation (31).

The following is the main result of the section.

Theorem 3.2. *The following statements, which involve the constants $\varkappa \geq 1$ and $\xi \geq 1$ appearing in Assumptions 2.1 and 2.3, hold true:*

(I) The PDE (26) with the corresponding boundary condition in Equation (29) has a $C^{1,2}$ solution such that

$$\underline{K}_f \leq f(t,s) \leq 1 \quad \textit{and} \quad \left|f_s(t,s)\right| \leq \bar{K}_f s^{-1} \quad \textit{for all } t \in [0,T] \textit{ and } s > 0, \tag{32}$$

for some $0 < \underline{K}_f < \bar{K}_f$ that do not depend on (t,s). Furthermore, if $\vartheta^2 > 0$, then

$$f(t,s) < 1 \quad \textit{for all } t \in [0,T[\textit{ and } s > 0. \tag{33}$$

(II) The PDE (27) with the corresponding boundary condition in Equation (29) has a $C^{1,2}$ solution such that

$$0 \leq g(t,s) \leq K_g(1+s^{\xi}) \quad \textit{for all } t \in [0,T] \textit{ and } s > 0 \tag{34}$$

$$\textit{and} \quad \left|g_s(t,s)\right| \leq K_g(1+s^{\xi})s^{-1} \quad \textit{for all } t \in [0,T] \textit{ and } s > 0, \tag{35}$$

for some $K_g > 0$ that does not depend on (t,s).

(III) The PDE (28) with the corresponding boundary condition in Equation (29) has a $C^{1,2}$ solution such that

$$0 \leq h(t,s) \leq K_h(1+s^{2\xi}) \quad \textit{for all } t \in [0,T] \textit{ and } s > 0, \tag{36}$$

for some $K_h > 0$ that does not depend on (t,s).

Proof. At several places in the proof, we adopt the standard convention of using a generic constant $\mathfrak{C} > 0$, which may depend on the constant $\varkappa$ appearing in Assumptions 2.1, 2.3, and on the fixed time horizon $T > 0$, to indicate an upper bound of a function. For instance, we will write $\left|2e^z\vartheta(t,e^z)\vartheta_s(t,e^z) - e^{\varkappa t}e^z\ell_s(t,e^z)\right| \leq \mathfrak{C}$ for all $(t,z) \in [0,T] \times \mathbb{R}$ instead of the more accurate $\left|2e^z\vartheta(t,e^z)\vartheta_s(t,e^z) - e^{\varkappa t}e^z\ell_s(t,e^z)\right| \leq 2\varkappa^{3/2} + \varkappa e^{\varkappa T}$.

Proof of (I). If we define

$$f(t,s) = \frac{1}{\phi(t,s)}, \quad \text{for } t \in [0,T] \text{ and } s > 0, \tag{37}$$

then we can see that f satisfies the PDE (26) with the corresponding boundary condition in Equation (29) if and only if ϕ satisfies the PDE

$$\begin{aligned}\phi_t(t,s) + \frac{1}{2}\sigma^2(t,s)s^2\phi_{ss}(t,s) + \big(r(t) - \sigma(t,s)\vartheta(t,s)\big)s\phi_s(t,s) \\ -\big(\ell(t,s)\phi(t,s) - \ell(t,s) - \vartheta^2(t,s)\big)\phi(t,s) = 0\end{aligned} \tag{38}$$

in $[0, T] \times]0, \infty[$ with boundary condition

$$\phi(T, s) = 1, \quad \text{for } s > 0. \tag{39}$$

Furthermore, if we write

$$\phi(t, s) = e^{2\kappa(T-t)} \varphi(t, \ln s), \quad \text{for } t \in [0, T] \text{ and } s > 0, \tag{40}$$

for some function $(t, z) \mapsto \varphi(t, z)$, and we define

$$\zeta(t, s) = 2\kappa - \ell(t, s) - \vartheta^2(t, s) \geq 0,$$

then we can check that ϕ satisfies the PDE (38) with boundary condition (39) if and only if φ satisfies the PDE

$$\varphi_t(t, z) + \frac{1}{2}\sigma^2(t, e^z)\varphi_{zz}(t, z) + \left(r(t) - \sigma(t, e^z)\vartheta(t, e^z) - \frac{1}{2}\sigma^2(t, e^z)\right)\varphi_z(t, z)$$
$$- \left(e^{2\kappa(T-t)}\ell(t, e^z)\varphi(t, z) + \zeta(t, e^z)\right)\varphi(t, z) = 0 \tag{41}$$

in $[0, T] \times \mathbb{R}$ with boundary condition

$$\varphi(T, z) = 1, \quad \text{for } z \in \mathbb{R}. \tag{42}$$

To solve this nonlinear PDE, we consider the family of linear PDEs

$$\varphi_t^\psi(t, z) + \frac{1}{2}\sigma^2(t, e^z)\varphi_{zz}^\psi(t, z) + \chi_0(t, z)\varphi_z^\psi(t, z) - \delta^\psi(t, z)\varphi^\psi(t, z) = 0 \tag{43}$$

in $[0, T] \times \mathbb{R}$ with boundary condition

$$\varphi^\psi(T, z) = 1, \quad \text{for } z \in \mathbb{R}, \tag{44}$$

which is parametrized by smooth positive functions ψ, where

$$\chi_0(t, z) = r(t) - \sigma(t, e^z)\vartheta(t, e^z) - \frac{1}{2}\sigma^2(t, e^z) \tag{45}$$

$$\text{and} \quad \delta^\psi(t, z) = e^{2\kappa(T-t)}\ell(t, e^z)\psi(t, z) + \zeta(t, e^z) \geq 0. \tag{46}$$

In particular, we note that a solution φ to Equation (41) satisfies Equation (43) for $\psi = \varphi$. Conversely, a solution φ^ψ to Equation (43) such that $\psi = \varphi^\psi$ satisfies Equation (41).

Consider a $C^{1,2}$ function ψ satisfying

$$0 \leq \psi(t, z) \leq 1 \quad \text{and} \quad |\psi_z(t, z)| \leq C_1 \quad \text{for all } t \in [0, T] \text{ and } z \in \mathbb{R}, \tag{47}$$

for some constant C_1. The properties of such a function and Assumption 2.1 imply the uniform parabolicity, boundedness, and Lipschitz conditions required for the existence of a unique $C^{1,2}$ function φ^ψ of polynomial growth that solves the Cauchy problem (43)–(44) (see Friedman, 2006, Section 6.4 or Friedman, 2008, Section 1.7). In view of the Feynman–Kac formula (see Friedman, 2006, Section 6.5 or Karatzas & Shreve, 1988, Theorem 5.7.6), this function admits the probabilistic representation

$$\varphi^{\psi}(t,z) = \mathbb{E}\left[\exp\left(-\int_t^T \delta^{\psi}(u,Z_u)\,\mathrm{d}u\right)\,\middle|\, Z_t = z\right] \in \,]0,1], \tag{48}$$

for $t \in [0,T]$ and $z \in \mathbb{R}$, where Z is the strong solution to the SDE

$$\mathrm{d}Z_t = \chi_0(t,Z_t)\,\mathrm{d}t + \sigma(t,e^{Z_t})\,\mathrm{d}B_t, \quad Z_0 \in \mathbb{R}, \tag{49}$$

which is driven by a standard one-dimensional Brownian motion B. Note that this SDE indeed has a unique strong solution thanks to Assumption 2.1. Furthermore, Assumption 2.1 and the assumptions on ψ imply that φ^{ψ}_z is $C^{1,2}$ (see Friedman, 2008, Section 3.5). Differentiating Equation (43), we can see that φ^{ψ}_z satisfies

$$\varphi^{\psi}_{tz}(t,z) + \frac{1}{2}\sigma^2(t,e^z)\varphi^{\psi}_{zzz}(t,z) + \chi_1(t,z)\varphi^{\psi}_{zz}(t,z) - \chi_2(t,z)\varphi^{\psi}_z(t,z) + \mathcal{X}_1(t,z) = 0$$

in $[0,T]\times\mathbb{R}$, where

$$\chi_1(t,z) = r(t) - \sigma(t,e^z)\vartheta(t,e^z) - \frac{1}{2}\sigma^2(t,e^z) + e^z\sigma(t,e^z)\sigma_s(t,e^z),$$

$$\chi_2(t,z) = \delta^{\psi}(t,z) + e^z\vartheta(t,e^z)\sigma_s(t,e^z) + e^z\sigma(t,e^z)\vartheta_s(t,e^z) + e^z\sigma(t,e^z)\sigma_s(t,e^z)$$

$$\text{and}\quad \mathcal{X}_1(t,z) = \Big(e^z\ell_s(t,e^z) + 2e^z\vartheta(t,e^z)\vartheta_s(t,e^z) \\ - e^{2\kappa(T-t)}e^z\ell_s(t,e^z)\psi(t,z) - e^{2\kappa(T-t)}\ell(t,e^z)\psi_z(t,z)\Big)\varphi^{\psi}(t,z).$$

In particular, we note that Assumption 2.1 implies that the Cauchy problem given by this PDE with boundary condition

$$\varphi^{\psi}_z(T,z) = 0, \quad \text{for } z \in \mathbb{R},$$

has a unique $C^{1,2}$ solution of polynomial growth.

Using Assumption 2.1, the assumptions in Equation (47) and the fact that $|\varphi^{\psi}| \le 1$ (see Equation 48), we can see that

$$|\chi_1(t,z)| \le \mathfrak{C}, \quad |\chi_2(t,z)| \le \mathfrak{C} \quad \text{and} \quad |\mathcal{X}_1(t,z)| \le \mathfrak{C} \quad \text{for all } t \in [0,T] \text{ and } z \in \mathbb{R},$$

for some constant $\mathfrak{C} > 0$. Furthermore, Assumption 2.1 implies that the SDE

$$\mathrm{d}\widetilde{Z}_t = \chi_1(t,\widetilde{Z}_t)\,\mathrm{d}t + \sigma(t,e^{\widetilde{Z}_t})\,\mathrm{d}B_t, \quad \widetilde{Z}_0 \in \mathbb{R},$$

which is driven by a standard one-dimensional Brownian motion B, has a unique strong solution. In view of these observations, we can use the Feynman–Kac formula to obtain

$$|\varphi^{\psi}_z(t,z)| \le \mathbb{E}\left[\int_t^T \exp\left(-\int_t^u \chi_2(q,\widetilde{Z}_q)\,\mathrm{d}q\right)|\mathcal{X}_1(u,\widetilde{Z}_u)|\,\mathrm{d}u \,\middle|\, \widetilde{Z}_t = z\right] \\ \le \mathfrak{C} \quad \text{for all } t \in [0,T] \text{ and } z \in \mathbb{R},$$

where $\mathfrak{C} > 0$ is a constant. It follows that φ^ψ inherits all of the properties that we have assumed for ψ above, namely, it is $C^{1,2}$ and satisfies Equation (47). Furthermore, the regularity of φ^ψ implies that its restriction in $]\underline{t},\bar{t}[\times]\underline{z},\bar{z}[$ belongs to the Hölder space $C_{1+a,2+a}\big(]\underline{t},\bar{t}[\times]\underline{z},\bar{z}[\big)$ for all $0 < \underline{t} < \bar{t} < T$, $\underline{z} < \bar{z}$ and $a \in\,]0,1]$, where $C_{1+a,2+a}(\mathcal{D})$ is the Banach space with norm

$$\|\varphi\|^{\mathcal{D}}_{1+a,2+a} = \|\varphi\|^{\mathcal{D}}_a + \|\varphi_t\|^{\mathcal{D}}_a + \|\varphi_z\|^{\mathcal{D}}_a + \|\varphi_{zz}\|^{\mathcal{D}}_a, \tag{50}$$

in which definition, $\mathcal{D}$ is an open and bounded subset of $\mathbb{R}^2$ and

$$\|\varphi\|^{\mathcal{D}}_a = \sup_{(t,z)\in\mathcal{D}} |\varphi(t,z)| + \sup_{\substack{(t,z),(t',z')\in\mathcal{D}\\(t,z)\neq(t',z')}} \frac{|\varphi(t,z)-\varphi(t',z')|}{(|t-t'|+|z-z'|^2)^{a/2}}.$$

Given any $0 < \underline{t} < \bar{t} < T$, $\underline{z} < \bar{z}$ and $\varepsilon > 0$ such that $0 < \underline{t} - \varepsilon$ and $\bar{t} + \varepsilon < T$, Schauder's interior estimates for parabolic PDEs given by Theorem 3.5 in Friedman (2008) with

$$\mathcal{D} = \,]\underline{t}-\varepsilon, \bar{t}+\varepsilon[\,\times\,]\underline{z}-\varepsilon, \bar{z}+\varepsilon[$$

imply that

$$\|\varphi^\psi\|^{]\underline{t},\bar{t}[\times]\underline{z},\bar{z}[}_{1+a,2+a} \le C_2 \sup_{(t,z)\in]\underline{t},\bar{t}[\times]\underline{z},\bar{z}[} \big|\varphi^\psi(t,z)\big| \le C_2, \tag{51}$$

where C_2 depends only on a, $\underline{t}$, $\bar{t}$, $\underline{z}$, $\bar{z}$, ε and the constant $\varkappa$ in Assumptions 2.1 and 2.3. Here, we should note that the consideration of $\varepsilon > 0$ is needed to account for the difference of the norm defined by Equation (50) from the weighted norm defined by Equation (2.11) in Friedman (2008, Section 3.2).

To proceed further, we denote by $\varphi^{(0)}$ the solution to Equations (43) and (44) for $\psi \equiv 0$ and by $\varphi^{(j+1)}$ the solution to Equations (43) and (44) for $\psi = \varphi^{(j)}$ and $j \ge 0$. By appealing to a simple induction argument, we can see that each $\varphi^{(j)}$ has all of the properties that we have assumed for ψ, namely, it is $C^{1,2}$ and satisfies Equation (47). Therefore, the functions $\varphi^{(j)}$, $j \ge 0$, satisfy the estimate (51) for a constant C_2 that does not depend on j. We next argue as in the proof of Theorem 3 in Friedman (2008, Section 3.2), which considers the same norm as the one given by Equation (50) above. The inequalities

$$\sup_{(t,z)\in]\underline{t},\bar{t}[\times]\underline{z},\bar{z}[} |\varphi^{(j)}_{zz}(t,z)| + \sup_{\substack{(t,z),(t',z')\in]\underline{t},\bar{t}[\times]\underline{z},\bar{z}[\\(t,z)\neq(t',z')}} \frac{|\varphi^{(j)}_{zz}(t,z)-\varphi^{(j)}_{zz}(t',z')|}{(|t-t'|+|z-z'|^2)^{a/2}} \le C_2$$

imply that the restrictions of the functions $\varphi^{(j)}_{zz}$, $j \ge 1$, in $[\underline{t},\bar{t}] \times [\underline{z},\bar{z}]$ provide a uniformly bounded and equicontinuous family of functions. This observation and the Arzelà–Ascoli theorem imply that there exists a subsequence $(\varphi^{(j_n)})$ of $(\varphi^{(j)})$ such that $(\varphi^{(j_n)}_{zz})$ is uniformly convergent in $[\underline{t},\bar{t}] \times [\underline{z},\bar{z}]$. Repeating the same argument and passing to further subsequences if necessary, we obtain a $C^{1,2}$ function φ on $[\underline{t},\bar{t}] \times [\underline{z},\bar{z}]$ such that

$$\varphi^{(j_n)} \xrightarrow[n\to\infty]{} \varphi, \quad \varphi^{(j_n)}_t \xrightarrow[n\to\infty]{} \varphi_t, \quad \varphi^{(j_n)}_z \xrightarrow[n\to\infty]{} \varphi_z \quad \text{and} \quad \varphi^{(j_n)}_{zz} \xrightarrow[n\to\infty]{} \varphi_{zz}, \tag{52}$$

uniformly on $[\underline{t},\bar{t}] \times [\underline{z},\bar{z}]$. Furthermore, passing to further subsequences using the same arguments with each of the domains $]\underline{t}_n,\bar{t}_n[\times]\underline{z}_n,\bar{z}_n[$, where $(\underline{t}_n)$ and $(\underline{z}_n)$ (respectively, $(\bar{t}_n)$ and $(\bar{z}_n)$)

are strictly decreasing (respectively, strictly increasing) sequences such that

$$\underline{t}_0 < \overline{t}_0, \quad \underline{z}_0 < \overline{z}_0, \quad \lim_{n\uparrow\infty} \underline{t}_n = 0, \quad \lim_{n\uparrow\infty} \overline{t}_n = T, \quad \lim_{n\uparrow\infty} \underline{z}_n = -\infty, \quad \text{and} \quad \lim_{n\uparrow\infty} \overline{z}_n = \infty,$$

we obtain a $C^{1,2}$ function φ on $]0, T[\times\mathbb{R}$ such that Equation (52) hold true uniformly on compacts. This limiting function is a solution to Equation (43) for $\psi = \varphi$, namely, a solution to the nonlinear PDE (41).

To proceed further, we note that Equation (48) yields the representations

$$\varphi^{(0)}(t,z) = \mathbb{E}\left[\exp\left(-\int_t^T \delta^0(u, Z_u)\,\mathrm{d}u\right) \,\middle|\, Z_t = z\right]$$

$$\text{and} \quad \varphi^{(j+1)}(t,z) = \mathbb{E}\left[\exp\left(-\int_t^T \delta^{\varphi^{(j)}}(u, Z_u)\,\mathrm{d}u\right) \,\middle|\, Z_t = z\right], \tag{53}$$

for $t \in [0, T]$, $z \in \mathbb{R}$, and $j \geq 0$, where Z is the solution to the SDE (49). Combining these expressions with the definition (46) of the functions $\delta^{\varphi^{(j)}}$, we can see that $\varphi^{(0)} > \varphi^{(j+1)}$ for all $j \geq 0$ because

$$-\delta^0 = -\zeta(t, e^z) > -e^{2\kappa(T-t)}\ell(t, e^z)\varphi^{(j)}(t,z) - \zeta(t, e^z) = -\delta^{\varphi^{(j)}}.$$

Furthermore, we obtain the implications

$$\begin{aligned}
&\varphi^{(0)} > \varphi^{(1)} \quad \Rightarrow \quad -\delta^{\varphi^{(0)}} < -\delta^{\varphi^{(1)}} \quad \Rightarrow \quad \varphi^{(1)} < \varphi^{(2)},\\
&\varphi^{(1)} < \varphi^{(2)} \quad \Rightarrow \quad -\delta^{\varphi^{(1)}} > -\delta^{\varphi^{(2)}} \quad \Rightarrow \quad \varphi^{(2)} > \varphi^{(3)},\\
&\varphi^{(0)} > \varphi^{(2)} \quad \Rightarrow \quad -\delta^{\varphi^{(0)}} < -\delta^{\varphi^{(2)}} \quad \Rightarrow \quad \varphi^{(1)} < \varphi^{(3)},\\
\text{and} \quad &\varphi^{(1)} < \varphi^{(3)} \quad \Rightarrow \quad -\delta^{\varphi^{(1)}} > -\delta^{\varphi^{(3)}} \quad \Rightarrow \quad \varphi^{(2)} > \varphi^{(4)}.
\end{aligned}$$

Iterating these observations, we can see that the sequence of functions $(\varphi^{(2j)})$ is strictly decreasing, while the sequence of functions $(\varphi^{(2j+1)})$ is strictly increasing. It follows that

$$\varphi^{(1)} \leq \varphi \leq \varphi^{(0)} \leq 1. \tag{54}$$

The first two inequalities in Equation (54) imply immediately that φ satisfies the boundary condition (42). On the other hand, the last two inequalities in Equation (54) imply that

$$f(t,s) = \frac{1}{\phi(t,s)} = \frac{1}{e^{2\kappa(T-t)}\varphi(t, \ln s)} \geq e^{-2\kappa T}.$$

Furthermore, if we define

$$\lambda(t,z) = \varphi(t,z) - e^{-2\kappa(T-t)}, \quad \text{for } t \in [0,T] \text{ and } z \in \mathbb{R},$$

then we can use Equations (41) and (42) to see that λ satisfies the PDE

$$\lambda_t(t,z) + \frac{1}{2}\sigma^2(t, e^z)\lambda_{zz}(t,z) + \chi_0(t,z)\lambda_z(t,z) - \overline{\delta}(t,z)\lambda(t,z) + \vartheta^2(t, e^z)e^{-2\kappa(T-t)} = 0$$

in $[0,T]\times\mathbb{R}$ with boundary condition

$$\lambda(T,z)=0, \quad \text{for } z\in\mathbb{R},$$

where χ_0 is defined by Equation (45) and

$$\overline{\delta}(t,z)=e^{2\kappa(T-t)}\ell(t,e^z)\varphi(t,z)+2\kappa-\vartheta^2(t,e^z).$$

Using the Feynman–Kac formula, it follows that

$$\lambda(t,z)=\mathbb{E}\left[\int_t^T \exp\left(-\int_t^u \overline{\delta}(q,Z_q)\,\mathrm{d}q\right)e^{-2\kappa(T-u)}\vartheta^2(u,e^{Z_u})\,\mathrm{d}u \,\middle|\, Z_t=z\right] \tag{55}$$

for all $t\in[0,T]$ and $z\in\mathbb{R}$. This representation implies that

$$\varphi(t,z)\geq e^{-2\kappa(T-t)} \quad \Rightarrow \quad f(t,s)=\frac{1}{e^{2\kappa(T-t)}\varphi(t,\ln s)}\leq 1,$$

with the inequalities being strict if $\vartheta^2>0$. However, these arguments imply the estimates for f in Equation (32) as well as Equation (33).[3]

To derive the estimate for $|f_s|$ in Equation (32), we first note that the solution ϕ to the PDE (38) with boundary condition (39) is such that ϕ_s satisfies the PDE

$$\begin{aligned}\phi_{ts}(t,s)+\frac{1}{2}\sigma^2(t,s)s^2\phi_{sss}(t,s)+\chi_3(t,s)s\phi_{ss}(t,s)-\chi_4(t,s)\phi_s(t,s)&\\ +\mathcal{X}_2(t,s)\phi(t,s)=0&\end{aligned} \tag{56}$$

in $[0,T]\times\,]0,\infty[$, as well as the boundary condition

$$\phi_s(T,s)=0, \quad \text{for } s>0, \tag{57}$$

where

$$\begin{aligned}\chi_3(t,s)&=r(t)-\sigma(t,s)\vartheta(t,s)+\sigma^2(t,s)+s\sigma(t,s)\sigma_s(t,s),\\ \chi_4(t,s)&=2\ell(t,s)\phi(t,s)-r(t)-\ell(t,s)-\vartheta^2(t,s)+\sigma(t,s)\vartheta(t,s)\\ &\quad+s\vartheta(t,s)\sigma_s(t,s)+s\sigma(t,s)\vartheta_s(t,s)\\ \text{and}\quad \mathcal{X}_2(t,s)&=-\ell_s(t,s)\phi(t,s)+\ell_s(t,s)+2\vartheta(t,s)\vartheta_s(t,s).\end{aligned}$$

Using the Assumption 2.1 and the estimates for $f=1/\phi$ given by Equation (32), we can see that these functions are such that

$$\left|\chi_3(t,s)\right|\leq\mathfrak{C}, \quad \left|\chi_4(t,s)\right|\leq\mathfrak{C}$$
$$\text{and}\quad \left|\mathcal{X}_2(t,s)\right|\leq\Big(s\left|\ell_s(t,s)\right|\phi(t,s)+s\left|\ell_s(t,s)\right|+2s\left|\vartheta(t,s)\right|\left|\vartheta_s(t,s)\right|\Big)s^{-1}\leq\mathfrak{C}s^{-1},$$

for all $t\in[0,T]$ and $s>0$, where $\mathfrak{C}>0$ is a constant. Furthermore, Lemma 3.1 with $\xi_1=-1$ implies that the solution to the SDE

$$\mathrm{d}\bar{S}_u=\chi_3(u,\bar{S}_u)\bar{S}_u\,\mathrm{d}u+\sigma(u,\bar{S}_u)\bar{S}_u\,\mathrm{d}B_u,$$

which is driven by a standard one-dimensional Brownian motion B, is such that

$$\mathbb{E}\big[\bar{S}_u^{-1} \mid \bar{S}_t = s\big] \le \mathfrak{C} s^{-1} \quad \text{for all } 0 \le t \le u \le T \text{ and } s > 0,$$

for some constant $\mathfrak{C} > 0$ that does not depend on t. In view of these observations, the Feynman–Kac formula and Jensen's inequality, we can see that the solution to Equations (56)–(57) satisfies

$$\begin{aligned} |\phi_s(t,s)| &\le \mathbb{E}\left[\int_t^T \exp\left(-\int_t^u \chi_4(q,\bar{S}_q)\,\mathrm{d}q\right)|\mathcal{X}_2(u,\bar{S}_u)|\,\phi(u,\bar{S}_u)\,\mathrm{d}u \,\middle|\, \bar{S}_t = s\right] \\ &\le \int_0^T e^{\mathfrak{C}u}\mathfrak{C}^2\,\mathbb{E}\big[\bar{S}_u^{-1} \mid \bar{S}_t = s\big]\,\mathrm{d}u \le \mathfrak{C}s^{-1} \quad \text{for all } t \in [0,T] \text{ and } s > 0, \end{aligned} \tag{58}$$

where $\mathfrak{C} > 0$ stands for constants. Combining this estimate with the estimates for $f = 1/\phi$ given by Equation (32), we obtain

$$\big|f_s(t,s)\big| = \frac{|\phi_s(t,s)|}{\phi^2(t,s)} \le \mathfrak{C}s^{-1} \quad \text{for all } t \in [0,T] \text{ and } s > 0,$$

as claimed in Equation (32) follows.

Proof of (II). If we write

$$g(t,s) = \tilde{g}(t,\ln s), \qquad \text{for } t \in [0,T] \text{ and } s > 0,$$

for some function $(t,z) \mapsto \tilde{g}(t,z)$, then g satisfies the PDE (27) in $[0,T] \times]0,\infty[$ with the corresponding boundary condition in Equation (29) if and only if $\tilde{g}$ satisfies the PDE

$$\begin{aligned} &\tilde{g}_t(t,z) + \frac{1}{2}\sigma^2(t,e^z)\tilde{g}_{zz}(t,z) + \left(r(t) - \frac{1}{2}\sigma^2(t,e^z)\right)\tilde{g}_z(t,z) \\ &- \left(r(t) + e^{2\kappa(T-t)}\ell(t,e^z)\varphi(t,z)\right)\tilde{g}(t,z) + e^{2\kappa(T-t)}\ell(t,e^z)\varphi(t,z)F_{\mathrm{E}}(t,e^z) = 0, \end{aligned}$$

in $[0,T] \times \mathbb{R}$, where φ is introduced by Equation (40), with boundary condition

$$\tilde{g}(T,z) = F_{\mathrm{T}}(e^z), \qquad \text{for } z \in \mathbb{R}.$$

In view of the assumptions on ℓ, F_{E}, and F_{T}, and the smoothness and boundedness of φ that we have established above, there exists a unique $C^{1,2}$ function $\tilde{g}$ of polynomial growth that solves this Cauchy problem (see Friedman, 2008, Section 1.7).

Using the Feynman–Kac formula (Friedman, 2006, Section 6.5 or Karatzas & Shreve, 1988, Theorem 5.7.6), we can see that the solution to the PDE (27) with the corresponding boundary condition in Equation (29) admits the probabilistic expression

$$\begin{aligned} g(t,s) = \mathbb{E}\Bigg[&\int_t^T \exp\left(-\int_t^u \left(r(q) + \frac{\ell(q,\tilde{S}_q)}{f(q,\tilde{S}_q)}\right)\mathrm{d}q\right)\frac{\ell(u,\tilde{S}_u)F_{\mathrm{E}}(u,\tilde{S}_u)}{f(u,\tilde{S}_u)}\,\mathrm{d}u \\ &+ \exp\left(-\int_t^T \left(r(q) + \frac{\ell(q,\tilde{S}_q)}{f(q,\tilde{S}_q)}\right)\mathrm{d}q\right)F_{\mathrm{T}}(\tilde{S}_T) \,\Bigg|\, \tilde{S}_t = s\Bigg] \\ \ge 0 \quad &\text{for all } t \in [0,T] \text{ and } s > 0, \end{aligned} \tag{59}$$

where $\tilde{S}$ is the solution to the SDE

$$\mathrm{d}\tilde{S}_t = r(t)\tilde{S}_t\,\mathrm{d}t + \sigma(t,\tilde{S}_t)\tilde{S}_t\,\mathrm{d}B_t, \tag{60}$$

which is driven by a standard one-dimensional Brownian motion B. Given the constant $\xi \geq 1$ appearing in Assumption 2.3, Lemma 3.1, implies that

$$\mathbb{E}\big[\tilde{S}_u^{\xi} \mid \tilde{S}_t = s\big] \leq \mathfrak{C}s^{\xi} \quad \text{for all } 0 \leq t \leq u \leq T \text{ and } s > 0,$$

for some constant $\mathfrak{C} > 0$ that does not depend on t. Using this estimate and Assumptions 2.1, 2.3 and Equation (32), we can see that the identity in Equation (59) implies that

$$\begin{aligned} g(t,s) &\leq \mathfrak{C}\int_t^T \Big(1 + \mathbb{E}\big[\tilde{S}_u^{\xi} \mid \tilde{S}_t = s\big]\Big)\,\mathrm{d}u + \mathfrak{C}\Big(1 + \mathbb{E}\Big[\tilde{S}_T^{\xi} \mid \tilde{S}_t = s\Big]\Big) \\ &\leq \mathfrak{C}\int_0^T (1 + s^{\xi})\,\mathrm{d}u + \mathfrak{C}(1 + s^{\xi}) \quad \text{for all } t \in [0,T] \text{ and } s > 0. \end{aligned}$$

It follows that g admits an upper bound as in Equation (34).

Similarly to the proof of (I) above, we can verify that g_s satisfies the PDE

$$g_{ts}(t,s) + \frac{1}{2}\sigma^2(t,s)s^2 g_{sss}(t,s) + \chi_5(t,s)sg_{ss}(t,s) - \chi_6(t,s)g_s(t,s) + \mathcal{X}_3(t,s) = 0 \tag{61}$$

in $[0,T] \times]0,\infty[$ with boundary condition

$$g_s(T,s) = F_{\mathrm{T}}'(s), \quad \text{for } s > 0, \tag{62}$$

where

$$\begin{aligned} &\chi_5(t,s) = r(t) + \sigma^2(t,s) + s\sigma(t,s)\sigma_s(t,s), \\ &\chi_6(t,s) = \ell(t,s)\phi(t,s) \\ \text{and} \quad &\mathcal{X}_3(t,s) = -\big(\ell_s(t,s)\phi(t,s) + \ell(t,s)\phi_s(t,s)\big)\big(g(t,s) - F_{\mathrm{E}}(t,s)\big) \\ &\qquad + \ell(t,s)\phi(t,s)(F_{\mathrm{E}})_s(t,s), \end{aligned}$$

with ϕ being as in Equation (37).

Combining the bounds in Assumptions 2.1, 2.3 with Equation (34), and the estimates for $f = 1/\phi$ given by Equations (32) and (58), we can see that there exists a constant $\mathfrak{C} > 0$ such that

$$\big|\chi_5(t,s)\big| \leq \mathfrak{C} \quad \text{and} \quad 0 \leq \chi_6(t,s) \leq \mathfrak{C} \quad \text{for all } t \in [0,T] \text{ and } s > 0,$$

as well as

$$\begin{aligned} \big|\mathcal{X}_3(t,s)\big| &\leq \Big(s\big|\ell_s(t,s)\big|\phi(t,s) + \ell(t,s)s\big|\phi_s(t,s)\big|\Big)\big(g(t,s) + F_{\mathrm{E}}(t,s)\big)s^{-1} \\ &\quad + \ell(t,s)\phi(t,s)s\big|(F_{\mathrm{E}})_s(t,s)\big|s^{-1} \\ &\leq \mathfrak{C}(1 + s^{\xi})s^{-1} \quad \text{for all } t \in [0,T] \text{ and } s > 0. \end{aligned}$$

Furthermore, Lemma 3.1 implies that the solution to the SDE

$$\mathrm{d}\check{S}_t = \chi_5(t, \check{S}_t)\check{S}_t\,\mathrm{d}t + \sigma(t, \check{S}_t)\check{S}_t\,\mathrm{d}B_t,$$

which is driven by a standard one-dimensional Brownian motion B, is such that, given the constant $\xi \geq 1$ appearing in Assumption 2.3,

$$\mathbb{E}\big[\check{S}_u^{-1} \mid \check{S}_t = s\big] \leq \mathfrak{C} s^{-1} \text{ and } \mathbb{E}\big[\check{S}_u^{\xi-1} \mid \check{S}_t = s\big] \leq \mathfrak{C} s^{\xi-1} \text{ for all } 0 \leq t \leq u \leq T \text{ and } s > 0,$$

for some constant $\mathfrak{C} > 0$ that does not depend on t. In view of these results, we can see that Equations (61) and (62), the Feynman–Kac formula, and Jensen's inequality imply that

$$\begin{aligned}
\big|g_s(t,s)\big| &\leq \mathbb{E}\bigg[\int_t^T \exp\bigg(-\int_t^u \chi_6(q, \check{S}_q)\,\mathrm{d}q\bigg)\big|\mathcal{X}_3(u, \check{S}_u)\big|\,\mathrm{d}u \\
&\qquad + \exp\bigg(-\int_t^T \chi_6(q, \check{S}_q)\,\mathrm{d}q\bigg)\big|F_{\mathrm{T}}'(\check{S}_u)\big| \,\bigg|\, \check{S}_t = s\bigg] \\
&\leq \int_0^T \mathfrak{C}\Big(\mathbb{E}\big[\check{S}_u^{-1} \mid \check{S}_t = s\big] + \mathbb{E}\big[\check{S}_u^{\xi-1} \mid \check{S}_t = s\big]\Big)\,\mathrm{d}u \\
&\quad + \mathfrak{C}\Big(\mathbb{E}\big[\check{S}_T^{-1} \mid \check{S}_t = s\big] + \mathbb{E}\big[\check{S}_T^{\xi-1} \mid \check{S}_t = s\big]\Big) \\
&\leq \mathfrak{C}(1 + s^{\xi})s^{-1} \quad \text{for all } t \in [0,T] \text{ and } s > 0,
\end{aligned}$$

where $\mathfrak{C} > 0$ stands for constants. It follows that $|g_s|$ admits a bound as in Equation (35).

<u>*Proof of (III).*</u> We can show that the PDE (28) in $[0,T] \times \,]0,\infty[$ with the corresponding boundary condition in Equation (29) has a $C^{1,2}$ solution in the same way as in the proof of (II). Using the Feynman–Kac formula once again, we can see that this solution admits the probabilistic expression

$$\begin{aligned}
h(t,s) &= \mathbb{E}\bigg[\int_t^T \exp\bigg(-\int_t^u \big(2r(q) + \ell(q, S_q)\big)\,\mathrm{d}q\bigg)\ell(u, S_u)\big(F_{\mathrm{E}}(u, S_u) - g(u, S_u)\big)^2\,\mathrm{d}u \,\bigg|\, S_t = s\bigg] \\
&\geq 0,
\end{aligned} \tag{63}$$

where S is the solution to the SDE (2). In this case, Lemma 3.1 implies that, given the constant $\xi \geq 1$ appearing in Assumption 2.3,

$$\mathbb{E}\big[S_u^{2\xi} \mid S_t = s\big] \leq \mathfrak{C} s^{2\xi} \quad \text{for all } 0 \leq t \leq u \leq T \text{ and } s > 0,$$

for some constant $\mathfrak{C} > 0$ that does not depend on t. Combining this expression with Assumptions 2.1 and 2.3 as well as Equation (34), we can see that

$$\begin{aligned}
h(t,s) &\leq \mathbb{E}\bigg[\int_0^T 2\kappa e^{2\kappa u}\big(F_{\mathrm{E}}^2(u, S_u) + g^2(u, S_u)\big)\,\mathrm{d}u \,\bigg|\, S_t = s\bigg] \\
&\leq \mathfrak{C}\int_0^T \Big(1 + \mathbb{E}\big[S_u^{2\xi} \mid S_t = s\big]\Big)\,\mathrm{d}u \\
&\leq \mathfrak{C}\big(1 + s^{2\xi}\big) \quad \text{for all } t \in [0,T] \text{ and } s > 0,
\end{aligned}$$

as claimed in Equation (36).

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In view of Equations (23) and (25), we can see that a solution to the stochastic control problem we have considered in the last two sections is given by the portfolio strategy defined by

$$\pi_t^\star = \pi^\dagger(t, X_t^\star, S_t) = -\alpha(t, S_t) X_t^\star + \beta(t, S_t), \tag{64}$$

where $\pi^\dagger$ is the function defined by Equation (23),

$$\alpha(t, S_t) = \frac{\sigma(t, S_t) S_t f_s(t, S_t) + \vartheta(t, S_t) f(t, S_t)}{\sigma(t, S_t) f(t, S_t)}, \tag{65}$$

$$\beta(t, S_t) = \alpha(t, S_t) g(t, S_t) + S_t g_s(t, S_t), \tag{66}$$

and $X^\star$ is the associated solution to Equation (8).

The following result presents the solution to the control problem arising from the mean–variance hedging of a randomly liquidated American claim.

Theorem 4.1. *Consider the stochastic control problem defined by Equations (2), (8), (19), and (20), and suppose that the assumptions of Theorem 3.2 hold true. The problem's value function v identifies with the solution w to the HJB PDE (21)–(22) that is as in Equations (25)–(29), namely,*

$$v(T, x, s) = w(0, x, s) \equiv f(0, s)\big(x - g(0, s)\big)^2 + h(0, s) \quad \textit{for all } x \in \mathbb{R} \textit{ and } s > 0. \tag{67}$$

Furthermore, the portfolio strategy $\pi^\star$ defined by Equations (64)–(66) is optimal.

Proof. We fix any initial condition (x, s), as well as any admissible portfolio $\pi \in \mathcal{A}$, and let X be the associated solution to Equation (8) (throughout the proof, we drop the superscript "x, π" for notational simplicity).

Using Itô's formula, we calculate

$$\int_0^t e^{-\Lambda_u} \ell(u, S_u)\big(X_u - F_{\mathrm{E}}(u, S_u)\big)^2 \, \mathrm{d}u + e^{-\Lambda_t} w(t, X_t, S_t) = w(0, x, s) + A_t + M_t, \quad \text{for } t \in [0, T], \tag{68}$$

where

$$\begin{aligned} A_t = \int_0^t e^{-\Lambda_u} \bigg(& w_t(u, X_u, S_u) + \frac{1}{2}\sigma^2(u, S_u)\pi_u^2 w_{xx}(u, X_u, S_u) \\ & + \sigma^2(u, S_u) S_u \pi_u w_{xs}(u, X_u, S_u) + \frac{1}{2}\sigma^2(u, S_u) S_u^2 w_{ss}(u, X_u, S_u) \\ & + \big(r(u) X_u + \sigma(u, S_u)\vartheta(u, S_u)\pi_u\big) w_x(u, X_u, S_u) + \mu(u, S_u) S_u w_s(u, X_u, S_u) \\ & - \big(2r(u) + \ell(u, S_u)\big] w(u, X_u, S_u) + \ell(u, S_u)\big[X_u - F_{\mathrm{E}}(u, S_u)\big)^2 \bigg) \, \mathrm{d}u \end{aligned}$$

and

$$M_t = \int_0^{t\wedge T} e^{-\Lambda_u}\sigma(u,S_u)\big(\pi_u w_x(u,X_u,S_u) + S_u w_s(u,X_u,S_u)\big)\,\mathrm{d}W_u.$$

Since w satisfies the PDE (21) and the boundary condition (22), we can see that this identity implies that

$$\mathbb{E}^{\mathbb{P}}\Bigg[\int_0^{T\wedge T_n} e^{-\Lambda_u}\ell(u,S_u)\big(X_u - F_{\mathrm{E}}(u,S_u)\big)^2\,\mathrm{d}u \\ + e^{-\Lambda_T}\big(X_T - F_{\mathrm{T}}(S_T)\big)^2\mathbf{1}_{\{T\le T_n\}} + e^{-\Lambda_{T_n}}w(T_n,X_{T_n},S_{T_n})\mathbf{1}_{\{T_n<T\}}\Bigg] \ge w(0,x,s), \tag{69}$$

where (T_n) is any sequence of localizing stopping times for the local martingale M.

In view of the estimates in Equations (32), (34), and (36), we can see that

$$\begin{aligned}\big|w(t,x,s)\big| &\le 2f(t,s)\big(x^2 + g^2(t,s)\big) + h(t,s)\\ &\le K_w(1 + x^2 + s^{2\xi}) \quad \text{for all } t\in[0,T],\ x\in\mathbb{R} \text{ and } s>0,\end{aligned}$$

for some constant $K_w = K_w(T) > 0$, where $\xi \ge 1$ is as in Assumption 2.3. On the other hand, the admissibility condition (11), Fubini's theorem, Jensen's inequality, Itô's isometry, and Assumption 2.1 imply that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\big[X_t^2\big] &= \mathbb{E}^{\mathbb{P}}\left[B_t^2\left(x + \int_0^t B_u^{-1}\sigma(u,S_u)\vartheta(u,S_u)\pi_u\,\mathrm{d}u + \int_0^t B_u^{-1}\sigma(u,S_u)\pi_u\,\mathrm{d}W_u\right)^2\right]\\ &\le 9B_t^2\left(x^2 + \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^t B_u^{-1}\sigma(u,S_u)\vartheta(u,S_u)\pi_u\,\mathrm{d}u\right)^2\right] + \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^t B_u^{-1}\sigma(u,S_u)\pi_u\,\mathrm{d}W_u\right)^2\right]\right)\\ &\le 9B_t^2\left(x^2 + \varkappa(t+1)\,\mathbb{E}^{\mathbb{P}}\left[\int_0^t B_u^{-2}\pi_u^2\,\mathrm{d}u\right]\right)\\ &< \infty.\end{aligned}$$

These results and Lemma 3.1 imply that the random variable $\sup_{t\in[0,T]}\big|w(t,X_t,S_t)\big|$ is integrable. We can, therefore, pass to the limit as $n\to\infty$ in Equation (69) using the monotone and the dominated convergence theorems to obtain

$$\begin{aligned}J_{T,x,s}(\pi) \equiv \mathbb{E}^{\mathbb{P}}&\left[\int_0^T e^{-\Lambda_t}\ell(t,S_t)\big(X_t - F_{\mathrm{E}}(t,S_t)\big)^2\,\mathrm{d}t + e^{-\Lambda_T}\big(X_T - F_{\mathrm{T}}(S_T)\big)^2\right]\\ &\ge w(0,x,s).\end{aligned} \tag{70}$$

Since the initial condition (x,s) and the portfolio strategy $\pi\in\mathcal{A}$ have been arbitrary, it follows that

$$v(T,x,s) \ge w(0,x,s) \quad \text{for all } x\in\mathbb{R} \text{ and } s>0. \tag{71}$$

To prove the reverse inequality and establish (67) as well as the optimality of the portfolio strategy $\pi^\star$ defined by Equations (64)–(66), we first show that $\pi^\star$ is admissible, namely, $\pi^\star\in\mathcal{A}$. To this end, we first note that

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[X_t^{\star 2}\right] \le 25\Bigg(x^2 &+ \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^t \big(r(u) - \sigma(u,S_u)\vartheta(u,S_u)\alpha(u,S_u)\big)X_u^\star \,\mathrm{d}u\right)^2\right] \\ &+ \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^t \sigma(u,S_u)\vartheta(u,S_u)\beta(u,S_u)\,\mathrm{d}u\right)^2\right] \\ &+ \mathbb{E}^{\mathbb{P}}\left[\int_0^t \sigma^2(u,S_u)\alpha^2(u,S_u)X_u^{\star 2}\,\mathrm{d}u\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^t \sigma^2(u,S_u)\beta^2(u,S_u)\,\mathrm{d}u\right]\Bigg), \qquad (72)\end{aligned}$$

where we have also used Itô's isometry. Assumption 2.1 and the estimates (32) imply that there exists a constant $K_\alpha > 0$ such that

$$\left|\alpha(t,S_t)\right| \le \frac{\sigma(t,S_t)S_t\left|f_s(t,S_t)\right| + \vartheta(t,S_t)f(t,S_t)}{\sigma(t,S_t)f(t,S_t)} \le K_\alpha \quad \text{for all } t \in [0,T], \qquad (73)$$

while the estimates (34)–(35) imply that there exists a constant $K_\beta > 0$ such that

$$\beta^2(t,S_t) \le 2\alpha^2(t,S_t)g^2(t,S_t) + 2S_t^2 g_s^2(t,S_t) \le K_\beta(1 + S_t^{2\xi}) \quad \text{for all } t \in [0,T]. \qquad (74)$$

Using these inequalities, Assumption 2.1 and and Lemma 3.1, we can see that, for example,

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^t \sigma^2(u,S_u)\alpha^2(u,S_u)X_u^{\star 2}\,\mathrm{d}u\right] \le \kappa K_\alpha^2 \int_0^t \mathbb{E}^{\mathbb{P}}\left[X_u^{\star 2}\right]\mathrm{d}u \quad \text{for all } t \in [0,T]$$

and

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\left[\left(\int_0^t \sigma(u,S_u)\vartheta(u,S_u)\beta(u,S_u)\,\mathrm{d}u\right)^2\right] &\le T\,\mathbb{E}^{\mathbb{P}}\left[\int_0^t \sigma^2(u,S_u)\vartheta^2(u,S_u)\beta^2(u,S_u)\,\mathrm{d}u\right] \\ &\le \kappa^2 K_\beta T\left(t + \int_0^t \mathbb{E}^{\mathbb{P}}\left[S_u^{2\xi}\right]\mathrm{d}u\right) \\ &\le C_1(1 + s^{2\xi}) \quad \text{for all } t \in [0,T],\end{aligned}$$

where $C_1 > 0$ is a constant. In view of these inequalities and similar ones for the other terms, we can see that Equation (72) implies that there exists $C_2 = C_2(T,x,s) > 0$ such that

$$\mathbb{E}^{\mathbb{P}}\left[X_t^{\star 2}\right] \le C_2 + C_2\int_0^t \mathbb{E}^{\mathbb{P}}\left[X_u^{\star 2}\right]\mathrm{d}u.$$

It follows that

$$\mathbb{E}^{\mathbb{P}}\left[X_t^{\star 2}\right] \le C_2 e^{C_2 t} \quad \text{for all } t \in [0,T],$$

thanks to Grönwall's inequality. Combining this result with Equations (73) and (74), we obtain

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \pi_t^{\star 2}\,\mathrm{d}t\right] \le 2\,\mathbb{E}^{\mathbb{P}}\left[\int_0^T \left(\alpha^2(t,S_t)X_t^{\star 2} + \beta^2(t,S_t)\right)\mathrm{d}t\right]$$

$$\leq 2\int_0^T \left(K_\alpha^2 C_2 e^{C_2 t} + K_\beta\left(1 + \mathbb{E}^{\mathbb{P}}\left[S_t^{2\xi}\right]\right)\right) \mathrm{d}t < \infty,$$

and the admissibility of $\pi^\star$ follows.

Finally, it is straightforward to check that the portfolio strategy $\pi^\star$ defined by Equations (64)–(66) is such that Equation (69) as well as Equation (70) hold true with equality, which combined with the inequality (71), implies that $\pi^\star$ is optimal and that Equation (67) indeed holds true.

Remark 4.2. Back to the original optimization problem given by Equation (6), we can see that the previous theorem implies that the mean–variance hedging value of the claim we have considered, namely, the portfolio's initial endowment x^{mvh} that minimizes $v(T, x, s)$, is given by

$$x^{\mathrm{mvh}} = g(0, s).$$

We can express this value as the claim's expected with respect to a martingale measure discounted payoff. To this end, we consider the exponential martingale $(L_t,\ t \in [0, T])$ that solves the SDE

$$\mathrm{d}L_t = \big(1/f(t, S_t) - 1\big)L_{t-}\,\mathrm{d}M_t - \vartheta(t, S_t)L_t\,\mathrm{d}W_t,$$

where M is the $(\mathcal{G}_t)$-martingale defined by Equation (16), and is given by

$$L_t = \exp\bigg(- \mathbf{1}_{\{\eta \leq t\}} \ln f(\eta, S_\eta) - \int_0^{t\wedge\eta} \ell(u, S_u)\big(1/f(u, S_u) - 1\big)\,\mathrm{d}u$$
$$- \frac{1}{2}\int_0^t \vartheta^2(u, S_u)\,\mathrm{d}u - \int_0^t \vartheta(u, S_u)\,\mathrm{d}W_u\bigg), \quad \text{for } t \in [0, T].$$

If we denote by $\mathbb{Q}$ the probability measure on $(\Omega, \mathcal{G}_T)$ that has Radon–Nikodym derivative with respect to $\mathbb{P}$ given by $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{G}_T} = L_T$, then Girsanov's theorem implies that the process $\big(\tilde{W}_t,\ t \in [0, T]\big)$ is a standard Brownian motion under $\mathbb{Q}$, while the process $\big(\tilde{M}_t,\ t \in [0, T]\big)$ is a martingale under $\mathbb{Q}$, where

$$\tilde{W}_t = \int_0^t \vartheta(u, S_u)\,\mathrm{d}u + W_t \quad \text{and} \quad \tilde{M}_t = \mathbf{1}_{\{\eta \leq t\}} - \int_0^{t\wedge\eta} \frac{\ell(u, S_u)}{f(u, S_u)}\,\mathrm{d}u$$

(see also Remark 2.5). Furthermore, the price process of the risky asset satisfies the SDE

$$\mathrm{d}S_t = r(t)S_t\,\mathrm{d}t + \sigma(t, S_t)S_t\,\mathrm{d}\tilde{W}_t, \quad S_0 = s > 0, \tag{75}$$

in the time interval $[0, T]$, while the conditional distribution of η is given by

$$\mathbb{Q}(\eta > t \mid \mathcal{F}_t) = \exp\left(-\int_0^t \frac{\ell(u, S_u)}{f(u, S_u)}\,\mathrm{d}u\right), \quad \text{for } t \in [0, T].$$

In view of these observations and the Feynman–Kac formula (see also Equations 59–60), we obtain the expression

$$x^{\mathrm{mvh}} \equiv g(0, s) = \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \exp\left(-\int_0^t \left(r(u) + \frac{\ell(u, S_u)}{f(u, S_u)}\right)\mathrm{d}u\right)\frac{\ell(t, S_t)F_{\mathrm{E}}(t, S_t)}{f(t, S_t)}\,\mathrm{d}t\right.$$

$$+\exp\left(-\int_0^T\left(r(u)+\frac{\ell(u,S_u)}{f(u,S_u)}\right)\mathrm{d}u\right)F_{\mathrm{T}}(S_T)\Bigg]$$

$$=\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{\eta\wedge T}r(u)\,\mathrm{d}u}\Big(F_{\mathrm{E}}(\eta,S_\eta)\mathbf{1}_{\{\eta<T\}}+F_{\mathrm{T}}(S_T)\mathbf{1}_{\{\eta\geq T\}}\Big)\right], \tag{76}$$

as claimed at the beginning of the remark. We call $\mathbb{Q}$ “mean–variance hedging martingale measure” in what follows.

Remark 4.3. As we have discussed in Remark 2.5, the minimal martingale measure $\mathbb{Q}_1$, which identifies with the minimum entropy martingale measure as well as the variance-optimal martingale measure in the context that we consider here, has Radon–Nikodym derivative with respect to $\mathbb{P}$ that is given by

$$\left.\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_T}=L_T^1\equiv\exp\left(-\frac{1}{2}\int_0^T\vartheta^2(u,S_u)\,\mathrm{d}u-\int_0^T\vartheta(u,S_u)\,\mathrm{d}W_u\right). \tag{77}$$

For γ being as in Remark 2.5, the mean–variance hedging martingale measure $\mathbb{Q}$, which we identified in the previous remark, corresponds to the choice

$$\gamma_u=\frac{1}{f(u,S_u)},$$

while $\mathbb{Q}_1$ corresponds to the choice $\gamma=1$. If $\vartheta\equiv(\mu-r)/\sigma=0$, then both of these measures are the same as the physical probability measure $\mathbb{P}$ because the constant function $f=1$ satisfies the PDE (26). On the other hand, if $\vartheta^2>0$, then the measures $\mathbb{Q}$, $\mathbb{Q}_1$, and $\mathbb{P}$ are all different. In particular, the measures $\mathbb{Q}$ and $\mathbb{Q}_1$ are different because the mean–variance hedging is implemented in the random time interval $[0,\eta\wedge T]$ rather than in the deterministic time interval $[0,T]$.

To appreciate the last observation, recall the definition (12) of the random time η and consider a so-called “survival claim” that pays $Z\mathbf{1}_{\{\eta\geq T\}}$ at its maturity time T, where the “promised payoff” Z is any $\mathcal{F}_T$-measurable random variable satisfying suitable integrability conditions. Minimizing the performance index

$$\widetilde{J}_{T,x,s}(\pi)=\mathbb{E}^{\mathbb{P}}\left[\left(e^{-\int_0^T r(u)\,\mathrm{d}u}\big(X_T^{x,\pi}-Z\mathbf{1}_{\{\eta\geq T\}}\big)\right)^2\right] \tag{78}$$

over all admissible portfolio processes π and all portfolio initial endowments x gives rise to a mean–variance problem that can be solved as follows. In view of Equations (12) and (15), we can see that

$$\mathbb{E}^{\mathbb{P}}\big[X_T^{x,\pi}Z\mathbf{1}_{\{\eta\geq T\}}\big]=\mathbb{E}^{\mathbb{P}}\big[X_T^{x,\pi}ZD_T\big]\quad\text{and}\quad\mathbb{E}^{\mathbb{P}}\big[Z^2\mathbf{1}_{\{\eta\geq T\}}\big]=\mathbb{E}^{\mathbb{P}}\big[Z^2D_T\big],$$

where $D_T=\exp\big(-\int_0^T\ell(u,S_u)\,\mathrm{d}u\big)$. Therefore,

$$\widetilde{J}_{T,x,s}(\pi)=e^{-2\int_0^T r(u)\,\mathrm{d}u}\left(\mathbb{E}^{\mathbb{P}}\Big[\big(X_T^{x,\pi}-ZD_T\big)^2\Big]+\mathbb{E}^{\mathbb{P}}\Big[Z^2D_T(1-D_T)\Big]\right).$$

Now, there exists a self-financing portfolio strategy that starts with an initial investment

$$\tilde{x}^{\mathrm{mvh}} = \mathbb{E}^{\mathbb{Q}_1}\left[e^{-\int_0^T r(u)\,\mathrm{d}u} ZD_T\right]$$

and perfectly replicates ZD_T because this random variable is $\mathcal{F}_T$-measurable. It follows that the mean–variance hedging value of the survival claim that pays $Z\mathbf{1}_{\{\eta \geq T\}}$ at its maturity time $T > 0$ is equal to $\tilde{x}^{\mathrm{mvh}}$. Furthermore, in view of Equation (17) with $\gamma = 1$, we can see that

$$\begin{aligned}
\tilde{x}^{\mathrm{mvh}} &= \mathbb{E}^{\mathbb{Q}_1}\left[e^{-\int_0^T r(u)\,\mathrm{d}u} Z\mathbf{1}_{\{\eta \geq T\}}\right] \\
&= \mathbb{E}^{\mathbb{Q}_1}\left[\exp\left(-\int_0^T \big(r(u) + \ell(u, S_u)\big)\,\mathrm{d}u\right) Z\right],
\end{aligned} \tag{79}$$

which expresses the claim's mean–variance hedging value as its expected with respect to the minimal martingale measure discounted payoff.

The expression (79) for the mean–variance hedging value of a survival claim that is associated with the performance index (78) can be found in Bielecki et al. (2004, Proposition 29), who also consider claims with more general payoff structures. The same result has been established in a more general setting by Biagini and Cretarola (2007, Section 6). In this research direction, Choulli et al. (2020, 2021) study a general model of defaultable and life-insurance securities.

Minimizing the performance index

$$\mathbb{E}^{\mathbb{P}}\left[\left(e^{-\int_0^{\eta\wedge T} r(u)\,\mathrm{d}u}\big(X^{x,\pi}_{\eta\wedge T} - Z\mathbf{1}_{\{\eta \geq T\}}\big)\right)^2\right]$$

over all admissible portfolio processes π and all portfolio initial endowments x gives rise to a mean–variance hedging problem that is in line with the problem we study in this paper. If we focus on survival claims rather than on claims with more general payoff structures, then this performance index is plainly more suitable than the one in Equation (78), particularly, if the claim is long-dated. Indeed, on the event $\{\eta < T\}$, the claim pays nothing. Therefore, it is pointless to continue hedging beyond time η.

Intuition suggests that mean–variance hedging of a survival claim over the random time interval $[0, \eta \wedge T]$ rather than over the time interval $[0, T]$ should be associated with an optimal initial endowment x^{mvh} that is strictly less than $\tilde{x}^{\mathrm{mvh}}$. In the special case that arises when $Z = F_T(S_T)$, the results that we have obtained show that this is indeed the case when $\vartheta^2 > 0$. Recalling that the dynamics of the risky asset price process S are given by Equation (75) under the martingale measure $\mathbb{Q}$, which we introduced in the previous remark, as well as under the minimal martingale measure $\mathbb{Q}_1$, we can see that

$$\begin{aligned}
x^{\mathrm{mvh}} &= \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T \left(r(u) + \frac{\ell(u, S_u)}{f(u, S_u)}\right)\mathrm{d}u\right) F_{\mathrm{T}}(S_T)\right] \\
&< \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_0^T \big(r(u) + \ell(u, S_u)\big)\,\mathrm{d}u\right) F_{\mathrm{T}}(S_T)\right] \\
&= \mathbb{E}^{\mathbb{Q}_1}\left[\exp\left(-\int_0^T \big(r(u) + \ell(u, S_u)\big)\,\mathrm{d}u\right) F_{\mathrm{T}}(S_T)\right] = \tilde{x}^{\mathrm{mvh}},
\end{aligned}$$

thanks to the expressions (76) and (79), as well as to the strict upper bound of f in Equation (33).

Remark 4.4. Suppose that $\vartheta \equiv (\mu - r)/\sigma = 0$. As we have noted at the beginning of the previous remark, the constant function $f = 1$ satisfies the PDE (26) and the physical probability measure $\mathbb{P}$ identifies with the minimal martingale measure $\mathbb{Q}_1$ as well as the martingale measure $\mathbb{Q}$ introduced in Remark 4.3. It is worth noting that this special case is closely related with the family of models studied by Choulli et al. (2020, 2021).

Remark 4.5 (The infinite time horizon case). In many cases, randomly exercised American type claims, such as ESOs, are very long-dated. It is, therefore, of interest to consider the form that the solution to the problem we have studied takes as $T \to \infty$. For the purposes of this remark, suppose that r, μ, σ, and ℓ are constants. Also, suppose that F_{E} satisfies the polynomial growth condition (14) in Assumption 2.3 but does not depend explicitly on time (we do not need any differentiability properties of F_{E} here). For the problem to be well-posed, we assume further that

$$\ell + \vartheta^2 > \left(r + \frac{1}{2}\sigma^2\xi\right)(\xi - 1) \quad \text{and} \quad \ell > 2r(\xi - 1) + 2\sigma\vartheta\xi + \sigma^2\xi(2\xi - 1), \tag{80}$$

where ξ is as in Assumption 2.3. Note that, if $\xi = 1$, then these inequalities are equivalent to the simpler

$$\ell > 2\left(\mu - r + \frac{1}{2}\sigma^2\right). \tag{81}$$

In this context, the solution to the control problem becomes stationary, namely, it does not depend on time. In particular, the value function v_∞ identifies with the function w_∞ defined by

$$w_\infty(x, s) = f_\infty(s)\big(x - g_\infty(s)\big)^2 + h_\infty(s),$$

and the optimal portfolio strategy is given by

$$\pi_t^\star = -\frac{\sigma S_t f'_\infty(S_t) + \vartheta f_\infty(S_t)}{\sigma f_\infty(S_t)}\big(X_t^\star - g_\infty(S_t)\big) + S_t g'_\infty(S_t),$$

where $X^\star$ is the associated solution to Equation (8), and the functions f_∞, g_∞, and h_∞ are suitable solutions to the ODEs

$$\frac{1}{2}\sigma^2 s^2 f''_\infty(s) + \mu s f'_\infty(s) - \ell f_\infty(s) + \ell - \frac{\big(\sigma s f'_\infty(s) + \vartheta f_\infty(s)\big)^2}{f_\infty(s)} = 0, \tag{82}$$

$$\frac{1}{2}\sigma^2 s^2 g''_\infty(s) + r s g'_\infty(s) - \left(r + \frac{\ell}{f_\infty(s)}\right)g_\infty(s) + \frac{\ell F_{\mathrm{E}}(s)}{f_\infty(s)} = 0, \tag{83}$$

$$\frac{1}{2}\sigma^2 s^2 h''_\infty(s) + \mu s h'_\infty(s) - \big(2r + \ell\big)h_\infty(s) + \ell\big(F_{\mathrm{E}}(s) - g_\infty(s)\big)^2 = 0. \tag{84}$$

It is straightforward to verify that the solution to Equation (26) that satisfies the corresponding boundary condition in Equation (29) is given by

$$f(t, s) = \frac{\ell}{\ell + \vartheta^2} + \frac{\vartheta^2}{\ell + \vartheta^2}e^{-(\ell + \vartheta^2)(T - t)}. \tag{85}$$

In view of this observation, we can see that the constant function given by $f_\infty(s) = \frac{\ell}{\ell+\vartheta^2}$, for $s > 0$, trivially satisfies Equation (82). Furthermore, Equations (63) and (76) suggest that the functions given by

$$g_\infty(s) = (\ell + \vartheta^2)\,\mathbb{E}^{\mathbb{Q}}\left[\int_0^\infty e^{-(r+\ell+\vartheta^2)t} F_{\mathrm{E}}(S_t)\,\mathrm{d}t\right]$$

$$\text{and}\quad h_\infty(s) = \ell\,\mathbb{E}^{\mathbb{P}}\left[\int_0^\infty e^{-(2r+\ell)t}\big(F_{\mathrm{E}}(S_t) - g_\infty(S_t)\big)^2\,\mathrm{d}t\right]$$

should satisfy the ODEs (83) and (84). Note that the polynomial growth condition (14) in Assumption 2.3 that F_{E} satisfies and the conditions in (80) are sufficient for these functions to be real-valued because

$$\begin{aligned} g_\infty(s) &\leq (\ell + \vartheta^2)\kappa \int_0^\infty e^{-(r+\ell+\vartheta^2)t}\Big(1 + \mathbb{E}^{\mathbb{Q}}\big[S_t^\xi\big]\Big)\mathrm{d}t \\ &= (\ell + \vartheta^2)\kappa \int_0^\infty e^{-(r+\ell+\vartheta^2)t}\left(1 + s^\xi e^{\left(\frac{1}{2}\sigma^2\xi(\xi-1)+r\xi\right)t}\right)\mathrm{d}t \\ &\leq K_{g_\infty}(1 + s^\xi), \end{aligned}$$

where K_{g_∞} is a constant, and

$$\begin{aligned} h_\infty(s) &\leq 4\ell\left(\kappa^2 + K_{g_\infty}^2\right)\int_0^\infty e^{-(2r+\ell)t}\Big(1 + \mathbb{E}^{\mathbb{P}}\big[S_t^{2\xi}\big]\Big)\mathrm{d}t \\ &= 4\ell\left(\kappa^2 + K_{g_\infty}^2\right)\int_0^\infty e^{-(2r+\ell)t}\left(1 + s^{2\xi} e^{\left(\sigma^2\xi(2\xi-1)+2(\sigma\vartheta+r)\xi\right)t}\right)\mathrm{d}t \\ &< \infty. \end{aligned}$$

In view of standard analytic expressions of resolvents (e.g., see Knudsen et al., 1998, Proposition 4.1 or Lamberton & Zervos, 2013, Theorem 4.2), these functions admit the analytic expressions

$$g_\infty(s) = \frac{\ell + \vartheta^2}{\sigma^2(n_g - m_g)}\left(s^{m_g}\int_0^s u^{-m_g-1}F_{\mathrm{E}}(u)\,du + s^{n_g}\int_s^\infty u^{-n_g-1}F_{\mathrm{E}}(u)\,\mathrm{d}u\right)$$

$$\begin{aligned} \text{and}\quad h_\infty(s) = \frac{\ell}{\sigma^2(n_h - m_h)}\bigg(& s^{m_h}\int_0^s u^{-m_h-1}\big(F_{\mathrm{E}}(u) - g_\infty(u)\big)^2\,\mathrm{d}u \\ &+ s^{n_h}\int_s^\infty u^{-n_h-1}\big(F_{\mathrm{E}}(u) - g_\infty(u)\big)^2\,\mathrm{d}u\bigg), \end{aligned}$$

where the constants $m_g < 0 < n_g$ and $m_h < 0 < n_h$ are defined by

$$m_g, n_g = -\frac{r - \frac{1}{2}\sigma^2}{\sigma^2} \mp \sqrt{\left(\frac{r - \frac{1}{2}\sigma^2}{\sigma^2}\right)^2 + \frac{2}{\sigma^2}\left(r + \frac{\ell^2}{\ell + \vartheta^2}\right)},$$

$$m_h, n_h = -\frac{\mu - \frac{1}{2}\sigma^2}{\sigma^2} \mp \sqrt{\left(\frac{\mu - \frac{1}{2}\sigma^2}{\sigma^2}\right)^2 + \frac{2(2r + \ell)}{\sigma^2}}.$$

We can check that these functions indeed satisfy the ODEs (83) and (84) by direct substitution. Furthermore, we can use these expressions to calculate g_∞ and h_∞ in closed analytic form for a wide range of choices for F_{E}.

5 | NUMERICAL INVESTIGATION

We now present a numerical investigation of the theory we have developed focusing on ESOs because there is a rather large body of relevant literature, particularly, on the more applied side. ESOs are options granted by a firm to its employees as a form of benefit in addition to salary. They are typically long-dated options with maturities up to several years. Also, they typically have a vesting period of up to several years, during which, they cannot be exercised. After the expiry of their vesting period, they are of American type. Typical examples of ESO payoff functions include the one of a call option with strike K, in which case, $F(s) = (s - K)^+$, as well as the payoff of a capped call option that pays out no more than the double of its strike, in which case, $F(s) = \big(s \wedge (2K) - K\big)^+$.

If ESO holders have their jobs terminated (voluntarily or because of being fired), they forfeit their unvested ESOs, while they have a short time (typically, up to a few months) to exercise their vested ESOs. As a result, the possibility of job termination injects additional uncertainty into the structure of an ESO, which is referred to as the "job termination risk." On the other hand, ESOs are not allowed to be sold by their holders. Furthermore, ESO holders face restrictions in trading their employers' stocks. Therefore, they cannot hedge the initial values of their granted ESOs or use them as loss protections for speculation on their underlying stock price declines. These trading restrictions make ESO holders, who may be in a need for liquidity or want to diversify their portfolios, to exercise ESOs earlier than dictated by risk-neutrality. The early exercise behavior that is explained by these considerations has been documented in the empirical literature (e.g., see Huddart & Lang, 1996) and has been theoretically investigated by means of expected utility maximization techniques (e.g., see Leung & Sircar, 2009, or Carpenter et al., 2010).

To account for the early exercise behavior discussed in the previous paragraph, Jennergren and Näslund (1993) proposed the modeling of an ESO's exercise time $\eta \wedge T$ by letting η be the first jump of a Poisson process that is independent of the underlying stock price. This early research paper gave rise to the intensity-based framework for the modeling and valuation of ESOs, which has attracted most significant interest in the literature (e.g., see Carpenter, 1998; Carr & Linetsky, 2000; Sircar & Xiong, 2007; and Szimayer, 2004). In this framework, an ESO's payoff is given by Equations (3) and (4) in the introduction of this paper.

An intensity-based model for an ESO is associated with market incompleteness because the ESO's exercise time $\eta \wedge T$ is a random time rather than a stopping time. In this context, Jennergren and Näslund (1993) and Carr and Linetsky (2000) propose the valuation of such an ESO using the minimal martingale measure $\mathbb{Q}_1$ as in Equation (5) in the introduction by appealing to the idea that a large number of ESOs written by a firm "diversifies away" an individual ESO's early exercise risk.

In the rest of this section, we consider the numerical solution of the stochastic control problem associated with Equation (6) by solving its discrete time counterpart that arises if we approximate the geometric Brownian motion S by a binomial tree with 1000 time steps. To this end, we use the same parametrization as in Section 5 of Carr and Linetsky (2000). In particular, we consider an ESO granted at the money ($S_0 = K = 100$), with a ten year maturity ($T = 10$) and with payoff function $F(s) = \max(s - K, 0)$. Contrary to Carr and Linetsky (2000), who consider immediate vesting, we assume a vesting period of 3 years ($T_v = 3$). The intensity function ℓ is given by

$$\ell(t,s) = \ell_f + \ell_e(\ln s - \ln K)^+ \mathbf{1}_{\{T_v \le t\}}, \quad \text{for } t \in [0,T] \text{ and } s > 0,$$

and for $\ell_f = \ell_e = 10\%$. The constants ℓ_f and ℓ_e account for the ESO holder's job termination risk and the fact that the ESO holder's desire to exercise increases as the option's moneyness increases, respectively. We assume that the risk-free rate is 5% and the stock price volatility is 30%. Furthermore, we consider four values of the drift rate, specifically 15, −5, 25, and −15%. Note that we have selected two pairs of a positive and a negative drift rate. In each of the two pairs, the drift rates have the same distance from the risk-free rate, namely, 10 and 20%, respectively. In the next subsection, we observe that the distance of the drift rate from the risk-free rate may have a rather noticeable effect on the ESO's mean–variance hedging initial endowment.

5.1 | The mean–variance frontier

We have numerically solved the recursive equations arising from the discrete time approximation of the stochastic control problem we have considered in the previous sections. For each initial portfolio endowment x, we have, thus, computed the expected squared hedging error at the random time of the ESO's liquidation over all self-financing portfolio strategies with initial endowment x. The red curves in Figures 1 and 2, which we call "mean-variance frontiers," are plots of the square root of this error, to which we refer as the "root mean squared hedging error" (RMSHE), against the value of the initial endowment x. The ESO's mean–variance initial endowment x^{mvh} at time 0 corresponds to the apex of each curve.

We have also used backward induction to compute the risk-neutral initial endowment x^{rn} proposed by Carr and Linetsky (2000), as well as the super-replication endowment x^{sr}. In each of these two cases, we have computed the corresponding RMSHEs using Monte Carlo simulation along the lines described in the next subsection, and we have located the associated points in Figures 1 and 2.

As expected, the ESO's risk-neutral and super-replication values x^{rn} and x^{sr} do not depend on the drift rate. On the other hand, the ESO's mean–variance initial endowment x^{mvh} is sensitive to the value of the market price of risk. The absolute value of the difference between the drift rate and the risk-free rate is 10% in Figure 1, whereas it is 20% in Figure 2. This increase of the distance of the drift rate from the risk-free rate leads to a substantial decrease of the mean–variance initial endowment.

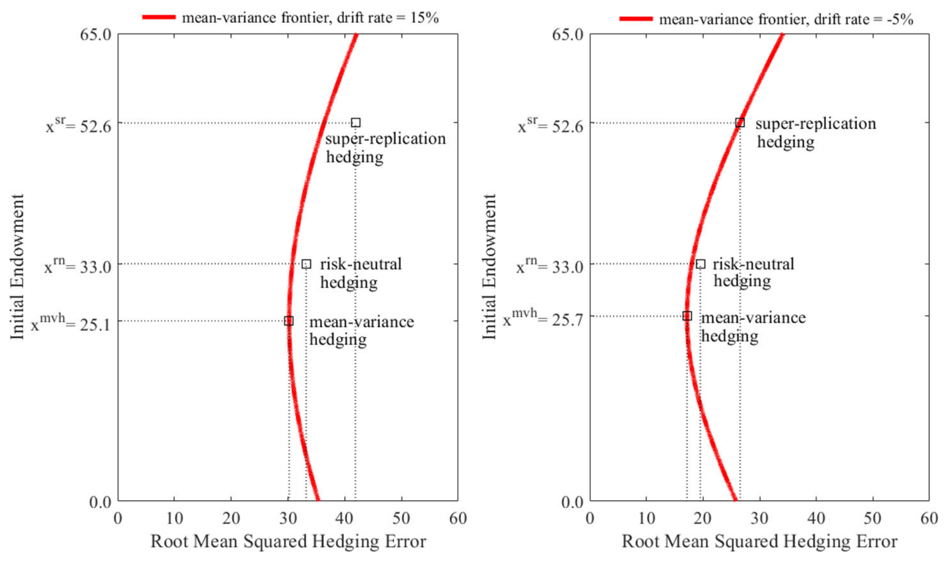

FIGURE 1 Mean–variance frontier, initial endowments, and root mean squared hedging errors for drift rates of 15 and −5%. [Color figure can be viewed at wileyonlinelibrary.com]

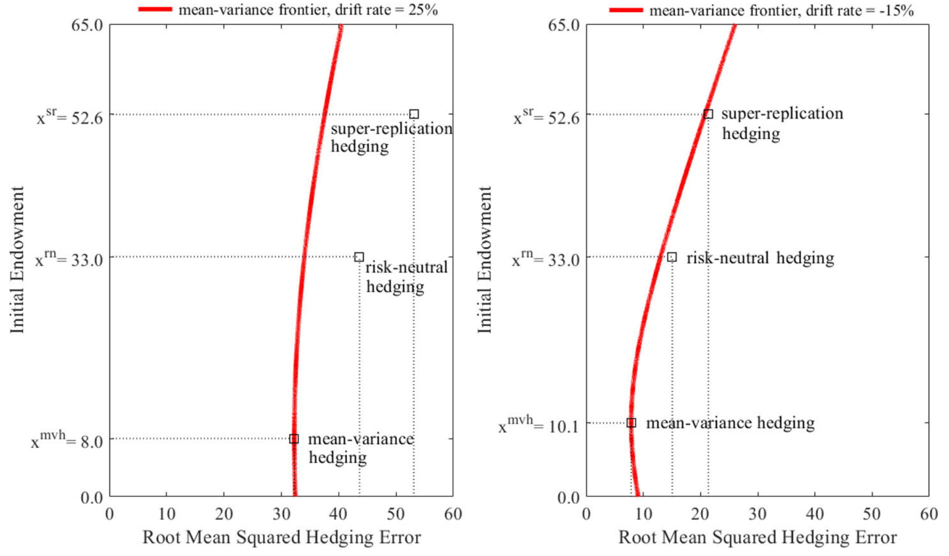

FIGURE 2 Mean–variance frontier, initial endowments, and root mean squared hedging errors for drift rates of 25 and −15%. [Color figure can be viewed at wileyonlinelibrary.com]

5.2 | Distribution of the hedging errors

In the case of mean–variance hedging, we have considered the mean–variance optimal portfolio strategy that starts with initial capital x^{mvh}. In the cases of risk-neutral and super-replication hedging, we have considered the standard Black and Scholes delta hedging strategy. The risk-neutral strategy starts with initial endowment x^{rn} and hedges the American option that yields the payoff $F(S_\eta)$ if exercised at a random time $\eta \in [T_{\mathrm{v}}, T]$. On the other hand, the super-replicating

TABLE 1 Mean hedging errors (MHE), root mean squared hedging errors (RMSHE), and 1, 5, 10, 50, 90, 95, 99% percentiles of the hedging errors.

Endowment	MHE	RMSHE	1%	5%	10%	50%	90%	95%	99%
Drift rate set to 15%									
$x^{mvh} = 25.1$	0.0	30.2	−63.4	−34.4	−24.6	−5.4	32.2	50.4	108.0
$x^{rn} = 33.0$	0.0	33.2	−47.1	−34.5	−29.0	−9.3	41.2	60.2	117.4
$x^{sr} = 52.6$	24.8	41.9	0.0	0.0	0.0	15.4	64.4	88.9	157.0
Drift rate set to −5%									
$x^{mvh} = 25.7$	0.0	17.2	−48.0	−24.1	−14.9	−1.6	21.2	29.1	50.6
$x^{rn} = 33.0$	0.0	19.5	−31.6	−23.0	−18.7	−4.9	27.7	37.4	63.5
$x^{sr} = 52.6$	14.6	26.6	0.0	0.0	0.0	2.3	45.9	59.3	93.1
Drift rate set to 25%									
$x^{mvh} = 8.0$	0.0	32.3	−79.7	−34.2	−20.0	−1.4	18.2	40.4	119.1
$x^{rn} = 33.0$	0.0	43.6	−58.2	−42.4	−35.4	−13.7	52.2	80.5	161.7
$x^{sr} = 52.6$	30.5	53.1	0.0	0.0	0.0	17.4	78.8	113.9	208.6
Drift rate set to −15%									
$x^{mvh} = 10.1$	0.0	7.9	−26.4	−4.6	−2.4	−0.1	5.7	9.5	16.7
$x^{rn} = 33.0$	0.0	15.1	−25.0	−16.9	−12.9	−4.1	22.2	31.9	49.6
$x^{sr} = 52.6$	10.6	21.4	0.0	0.0	0.0	0.1	38.3	51.6	75.5

strategy ignores the possibility of the option's early liquidation and hedges the option as if it were a European one.

In each of the three cases, we have used Monte Carlo simulation to compute the associated discounted to time 0 hedging errors, namely, the differences of the portfolios' values and the ESO's payoff at the ESO's random liquidation time. We have derived empirical distributions of these hedging errors by simulating 50 million sample paths of the stock price process (each sample path consisting of 1000 time steps) and 50 million realizations of the random time η.[4]

We plot the empirical distributions of the hedging errors for $\mu = 15\%$ and $\mu = -5\%$ in Figure 3. The corresponding plots for $\mu = 25\%$ and $\mu = -15\%$ are shown in Figure 4. Furthermore, we report the portfolios' initial endowments used, namely, the values of x^{mvh}, x^{rn}, and x^{sr}, as well as the corresponding mean hedging error (MHE), RMSHE, and selected percentiles of the hedging errors in Table 1.

We note that a negative value of the hedging error means that the portfolio's value has not covered the ESO's payoff. As expected, the super-replication strategy never leads to a negative hedging error. The "hump" that appears in the frequency of positive hedging errors is due to the fact that, if the ESO liquidation occurs during the vesting period, then the ESO forfeits without yielding a payoff. Indeed, if the vesting period is changed from 3 years to immediate vesting, then the bimodality of the hedging error distribution disappears.

5.3 | Convergence for long time horizons

To illustrate the convergence of the mean–variance valuation scheme as time to horizon becomes large, we have considered the ESO described at the beginning of the section but with a 20-year maturity ($T = 20$) and with immediate vesting. We have also assumed that $\ell_f = 20\%$, $\ell_e = 0$, $\mu =$

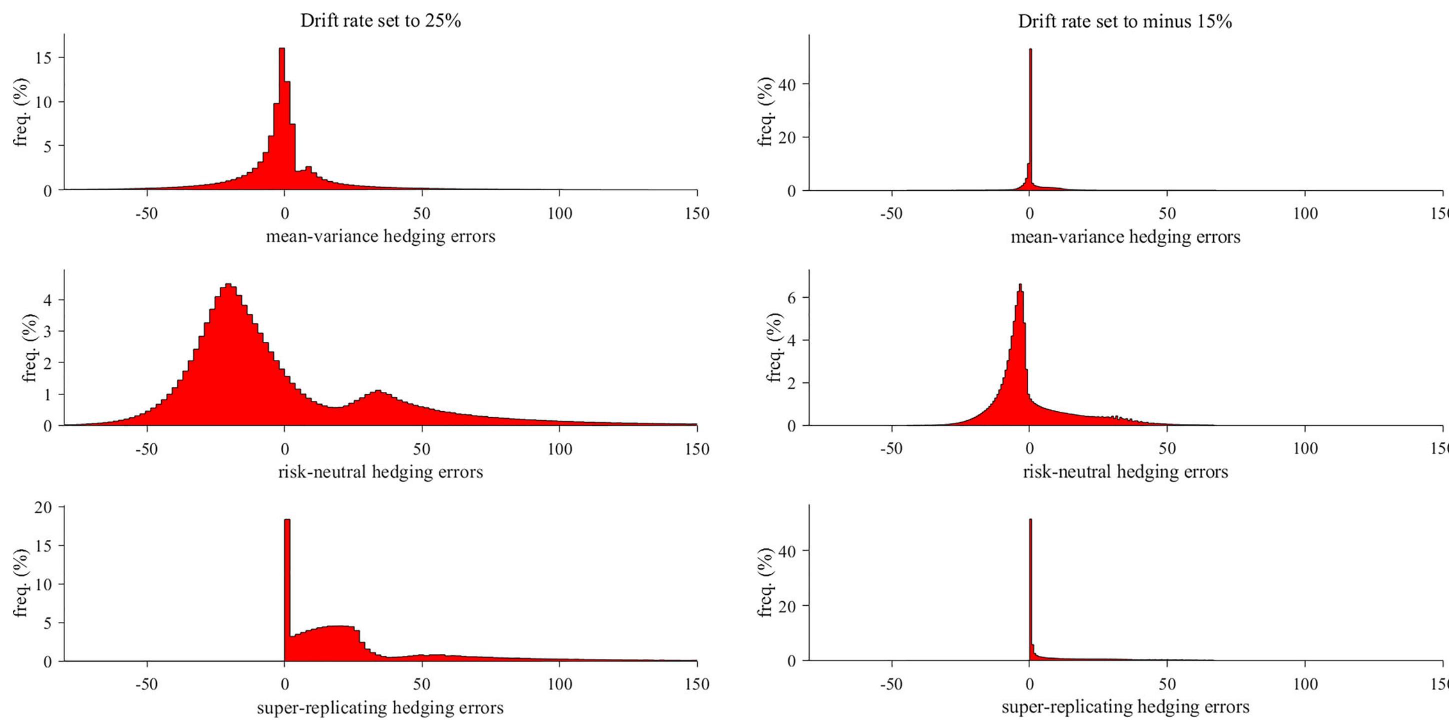

FIGURE 3 Histograms of hedging errors. The left-hand side captures the histograms for the drift rate set equal to 15% ($\mu = 0.15$). The right-hand side captures the histogram for the drift rate set equal to −5% ($\mu = -0.05$). [Color figure can be viewed at wileyonlinelibrary.com]

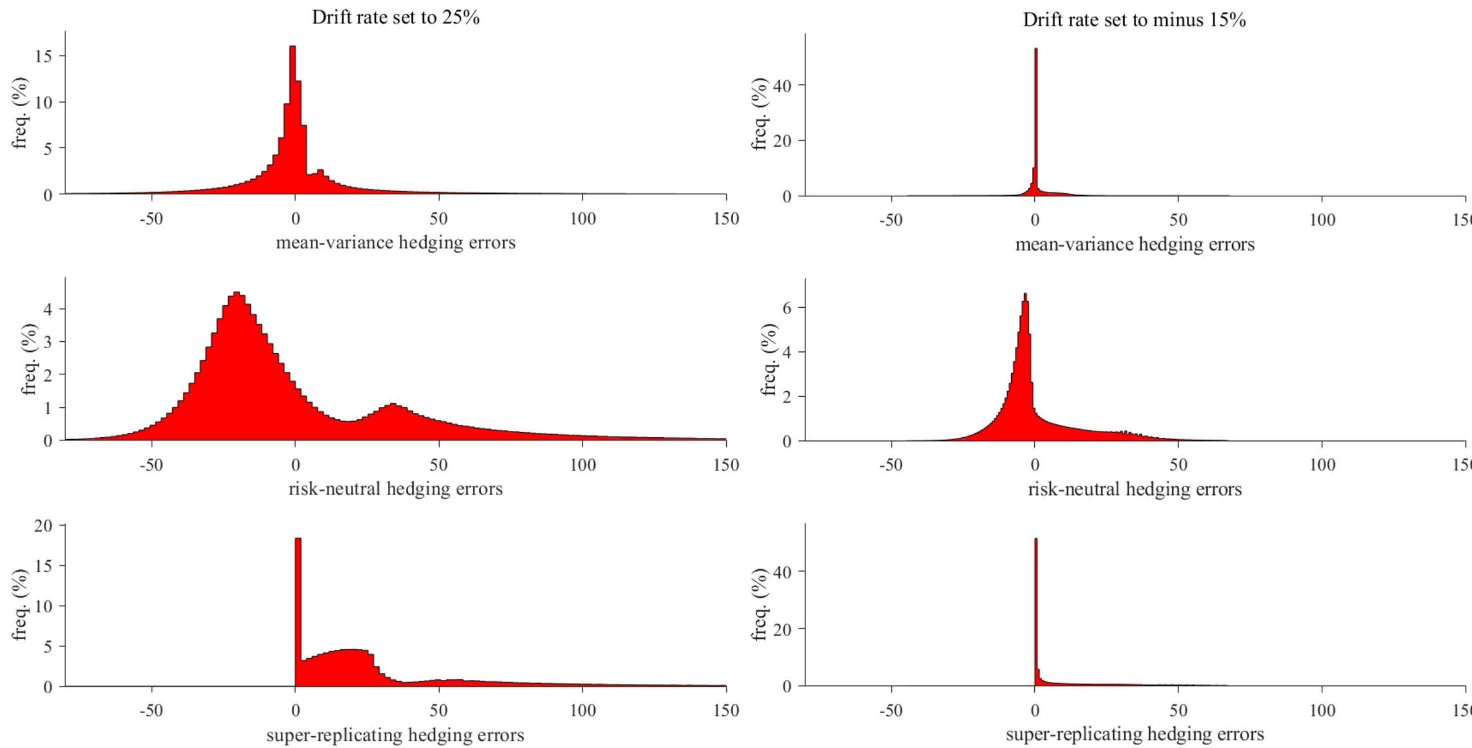

FIGURE 4 Histograms of hedging errors. The left-hand side captures the histograms for the drift rate set equal to 25% ($\mu = 0.25$). The right-hand side captures the histogram for the drift rate set equal to −15% ($\mu = -0.15$). [Color figure can be viewed at wileyonlinelibrary.com]

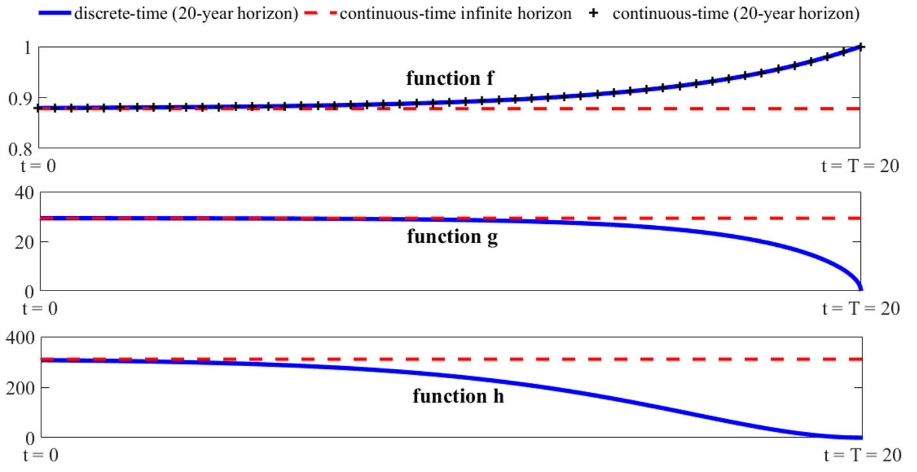

FIGURE 5 Illustration of convergence for long time horizons. [Color figure can be viewed at wileyonlinelibrary.com]

10%, $\sigma = 30\%$, and $r = 5\%$. Such choices put us in the context of Equation (81) in Remark 4.5. In Figure 5, we plot the functions f, g, and h as computed using the binomial tree model. In the first chart of the figure, we also plot the function f arising in the context of the continuous time model, which is given by Equation (85). We also plot the level given by the functions f_∞, g_∞, and h_∞ evaluated at the initial stock price $S_0 = 100$ using the closed form formulas derived in Remark 4.5.

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DATA AVAILABILITY STATEMENT

No data used.

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ENDNOTES

[1] Note that the definition of ϑ and the bounds in Equations (9) and (10) imply that the function $(t, s) \mapsto |\mu(t, s)| + s|\mu_s(t, s)|$ is also bounded by a constant.

[2] Without loss of generality, we use the same symbol κ here as well as in Assumption 2.1.

[3] The representation (55) implies that, if $\vartheta = 0$, then

$$\varphi(t, z) = e^{-2\kappa(T-t)} \quad \Rightarrow \quad f(t, s) = \frac{1}{e^{2\kappa(T-t)}\varphi(t, \ln s)} = 1,$$

which is a result that can be alternatively verified by a straightforward substitution in the PDE (26).

[4] With this number of samples, the simulated mean squared hedging error of the mean–variance hedging strategy matches its theoretically computed one up to the second decimal point.

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