Locally Minimizing the Credit Risk

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NON TECHNICAL SUMMARY

The aim of this paper is the valuation and hedging of defaultable bonds and options on defaultable bonds. The Heath/Jarrow/Morton-framework is used to model the interest rate risk, and the time of default is determined by the first jump time of a point process.

In the first part, we consider valuation and hedging of a defaultable bond. The firm value process is modelled explicitly, and is used to determine the default intensity or the payout ratio after default. This means that default intensity or payout ratio are not exogenously given, but determined implicitly by the specification of the firm value process. Incompleteness of markets arises naturally, and therefore we apply the local risk-minimizing methodology introduced by Föllmer, Schweizer and Sondermann to determine a martingale measure and to calculate hedging strategies. In incomplete markets, the total risk of a contingent claim can be divided into traded risk and totally non-tradeable (intrinsic) risk. Therefore, a contingent claim cannot be hedged perfectly. We can only reduce the risk to the intrinsic component. In our model, we can hedge partly against the risk of default because we assume that the firm value is a traded asset.

In the second part, we consider the valuation and hedging of options on defaultable bonds. Again, we are in an incomplete market. In addition to the traded assets, we introduce a virtual asset to the market which represents non-hedgeable risk. We derive the partial differential equation which is satisfied by the value process of the option and show how the risk-minimizing strategy can be computed.

Contents

1 Introduction

Each model which aims at pricing contingent claims on defaultable bonds has to specify three characteristic points:

- Which interest rate model is used?
- When does default occur?
- What is the payoff after default?

Known models for the valuation of defaultable bonds can be subdivided into two groups. The first one, so-called "classical group", explicitly models the evolution of firm value, and default takes place when the firm value falls below an exogenously specified boundary. The second one, so-called "intensity group", models default as the first jump of a point process with deterministic or stochastic intensity.

The classical approach was started by Black and Scholes [1973] and Merton [1974]. Newer papers of this group include Longstaff and Schwartz [1995] and Zhou [1997]. Here, default occurs when firm value falls below a certain threshold level, which is exogenously given. The default time τ can then be expressed formally as

$$\tau = \inf\{t \geq 0 | V(t) \leq K\};$$

that is the first passage time for $V(t)$ to cross the lower bound K. The firm value is mostly modelled as a diffusion process (with the exception of Zhou [1997]), and this has several implications:

- Firms never default unexpectedly.
- The firm's probability of defaulting on very short-term debt is zero and therefore its short-term debt should have zero credit spread.
- The firm has a constant value upon default.

All of these implications of the diffusion approach are strongly rejected. A generalization to firm value processes with jumps is difficult, because explicit solutions for passage times, except in the case of some very special diffusion processes, are not known. The last step in this direction is Zhou [1997]. He models the firm value with jumps and obtains an exact formula for the value of defaultable bonds in a simplified model with a predetermined date of possible default. In his general model, he gives an approximation for the value of the defaultable bond.

The intensity approach models the time of default as the first jump-time of a point process, which is totally unpredictable. This approach was adopted by

Duffie and Singleton [1994], Jarrow and Turnbull [1995], Madan and Unal [1994], and others. It has the attractive property of tractability, while its main draw-back is the missing link between firm value and corporate default. In most models of this type the intensity of the point process as well as the payout ratio are imposed exogenously, and are not linked explicitly to the firm value.

In this paper, we will combine the classical and the intensity approach. We will model the time of default as the first jump-time of a point process, but we will allow the firm value process to influence either the time of default through the intensity of the point process or the payoff after default. This paper extends Jarrow and Turnbull [1995] by introducing the firm value process and making default intensity or payout ratio dependent on the firm value, and also by relaxing the assumption of independence between the default process and default-free interest rates. By introducing the firm value, we are able to endogenize the default intensity and payout ratio. Assuming that the firm value is a traded asset, we are able to hedge partly against the loss from default. Because of the incompleteness of the markets under consideration, we will introduce the local risk-minimizing approach of Föllmer, Schweizer and Sondermann (Föllmer/Sondermann [1986], Schweizer [1991]) into the context of markets for defaultable bonds. In incomplete markets the martingale measure is no longer unique, and contingent claims cannot be perfectly replicated. However, a hedging strategy which minimizes risk in a certain sense can be computed, and the initial investment required is equal to the expectation of the contingent claim under the local risk-minimizing martingale measure.

The basic framework of the bond market is similar to Jarrow and Madan [1995], which allows bonds to depend on point processes as well as the usual, well known diffusion processes. Most of the results can easily be generalized to include marked point processes, using setup and results from Björk et al. [1996]. However, we refrain from including this to keep everything clear and simple, and to concentrate on the key results.

The structure of the paper is as follows. In section 2, we give an exposition of the basic bond market framework and review some useful results from Jarrow/Madan [1995]. At the end of this section, we introduce the reader into local risk-minimizing hedging and valuation in incomplete markets. In section 3, we compute the value of defaultable bonds when a non-defaultable bond and the firm value are traded in the market. We consider two different specifications of default intensity and payoff after default:

- The intensity of the point process depends on the firm value, and the payoff after default is constant.
- The intensity of the point process is deterministic, and the payoff after default depends on the firm value.

In section 4, we value options on defaultable bonds when a non-defaultable bond and a defaultable bond are traded in the market. For this purpose, again we use the local risk-minimizing approach, and the option pricing problem is formulated as a partial differential equation. Simultaneously we obtain expressions for the local risk-minimizing hedging strategy.

2 The Bond Market

In the present section, we introduce the basic setting of the bond market, which we will build upon in the following chapters to value and hedge bonds subject to credit risk and options on risky bonds.

We begin by presenting basic definitions and results concerning point processes. Subsequently, we define forward rates and bond prices. Our setting is similar to that of Jarrow/Madan [1995] and Jarrow/Turnbull [1995], and we note some of their results which we will use later on. Finally, we mention some results of Björk et al. [1995], which deal with the existence and uniqueness of martingale measures in a more general setup.

2.1 Mathematical Setup

We consider a continuous trading economy with trading interval $[0, T^{\infty}]$ for a fixed $T^{\infty} > 0$. In the present model, random shocks driving the market are generated by two distinct processes: A point process as well as the usual n-dimensional Brownian Motion. The uncertainty in our model is specified by a probability space $(\Omega, \mathcal{A}, P)$, and a complete, right-continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. Adapted to this filtration are the following processes:

- A point process $N(t) = 1_{[\tau, T^{\infty}[}(t)$, where τ is a $\mathbb{F}$-stopping time, with $\mathbb{F}$-predictable intensity, and
- an n-dimensional Brownian Motion $\{W(t) = (W_1(t), \ldots, W_n(t)) : t \in [0, T^{\infty}]\}$ starting in 0.

The key characteristic of a point process is its intensity, which can be defined as follows (Brémaud [1981], p. 27):

Definition 1. Let $N(t)$ be a point process adapted to $\mathbb{F}$ and let $\lambda(t)$ be a nonnegative $\mathbb{F}$-predictable process such that for all $t \geq 0$

$$\int_0^t \lambda(s)\, ds < \infty \quad P\text{-f.s.}$$

If for all nonnegative $\mathbb{F}$-predictable processes $C(t)$ the equality

$$E\left[\int_0^{T^1} C(s)\,dN(s)\right] = E\left[\int_0^{T^1} C(s)\,\lambda(s)\,ds\right]$$

is verified, then we say: $N(t)$ admits the intensity $\lambda(t)$.

The compensated point process, defined by

$$\tilde{N}(t) := N(t) - \int_0^t \lambda(s)\,ds$$

is a martingale, and we have the following formula for the probability of no jump up to time t:

$$P[N(t) = 0] = E\left[\exp\left\{-\int_0^t \lambda(s)\,ds\right\}\right]$$

The next lemma contains some useful results on the (conditional) quadratic variation of point processes:

Lemma 1. The previously defined processes satisfy the following equations:

$$[N,N](t) = [\tilde{N},\tilde{N}](t) = N(t)$$
$$\langle N,N\rangle(t) = \langle \tilde{N},\tilde{N}\rangle(t) = \int_0^t \lambda(u)\,du$$
$$[N,W_i](t) = [\tilde{N},W_i](t) = 0$$

Proof. See Protter [1990], pp. 62ff.

Finally, we note the Itô-formula in a version for point processes, also taken from Protter [1990]:

Lemma 2. Let $X = (X_1,\ldots,X_m)$ be an m-tuple of semimartingales, and let $f : \mathbb{R}^m \to \mathbb{R}$ have continuous second order partial derivatives. Then $f(X)$ is a semimartingale and the following formula holds:

$$\begin{aligned} f(X(t)) - f(X(0)) = {} & \int_{0+}^t \nabla f(X(s-))\,dX^c(s) \\ & + \frac{1}{2}\sum_{1\le i,j\le m}\int_{0+}^t \frac{\partial^2 f}{\partial x_i \partial x_j}(X(s-))\,d[X_i,X_j]^c(s) \\ & + \sum_{0<s\le t}\{f(X(s)) - f(X(s-))\} \end{aligned}$$

2.2 Setup of the Bond Market

On the stochastic basis of the previous section we will now build the economic model of a credit market with default risk. We will first introduce non-defaultable bonds, using the approach of Heath et al. [1992]. Afterwards, we consider defaultable bonds by adding the influence of the point process $N(t)$.

Assumption 1. The dynamics of non-defaultable forward rates are given by the following stochastic process:

$$df_0(t,T) = \alpha_0(t,T)\,dt + \sigma_0(t,T)\,dW(t);$$

where σ_0 is deterministic and satisfies certain technical integrability conditions.

Remark. The first jump of the point process indicates default. Therefore, forward rates belonging to non-defaultable bonds do not depend on the point process and are defined exactly as in Heath et al. [1992] with deterministic volatilities.

Proposition 3. Under Assumption 1, the non-defaultable short rate satisfies

$$dr_0(t) = [\alpha_0(t,t) + \frac{\partial f_0}{\partial T}(t,t)]\,dt + \sigma_0(t,t)\,dW(t)$$

and the non-defaultable bond prices are given by

$$\begin{aligned} dp(t,T) &= p(t,T)\,[r_0(t) + A_0(t,T)]\,dt + p(t,T)\,S_0(t,T)\,dW(t) \\ \Leftrightarrow\; p(t,T) &= p(0,T)\exp\left\{\int_0^t [r_0(u) + A_0(u,T)]\,du\right\}\mathcal{E}\left\{\int_0^t S_0(u,T)\,dW(u)\right\} \end{aligned}$$

where

$$\begin{aligned} S_0(t,T) &:= -\int_t^T \sigma_0(t,u)\,du \\ A_0(t,T) &:= -\int_t^T \alpha_0(t,u)\,du + \frac{1}{2}\|S_0(t,T)\|^2 \end{aligned}$$

Proof. See Heath et al. [1992].

Remark. Again, we want to mention that we reserve the point process for defaultable bonds. Non-defaultable bond prices are only influenced by the Brownian Motions.

We now turn our attention to defaultable bonds. As mentioned above, the time of default is the first jump time of the point process and at that time, defaultable forward rates have a jump. Following Jarrow/Turnbull [1995], we can then introduce

Assumption 2. The dynamics of defaultable forward rates are given by the following stochastic process:

$$df_1(t;T) = \alpha_1(t;T)\,dt + \sigma_1(t;T)\,dW(t) + \vartheta(t;T)\,dN(t);$$

where σ_1 and ϑ are deterministic and satisfy certain technical integrability conditions.

Furthermore, we denote the (random) payoff after default with Δ. We want to allow the intensity of $N(t)$ or the payoff after default to depend on the firm value, and therefore we introduce

Assumption 3. The dynamics of the firm value are given by the following stochastic process:

$$dV(t) = V(t)\alpha_2(t)\,dt + V(t)\sigma_2(t)\,dW(t)$$
$$\Leftrightarrow \quad V(t) = V(0)\exp\left(\int_0^t \alpha_2(u)\,du - \int_0^t \sigma_2(u)\,dW(u)\right)$$

Following Jarrow/Turnbull [1995], the defaultable short rate and bond prices can be calculated. Both have a jump at the same time as the forward rates:

Proposition 4. Under Assumption 2, the defaultable short rate satisfies

$$dr_1(t) = [\alpha_1(t;t) + \frac{\partial f_1}{\partial T}(t;t)]\,dt + \sigma_1(t;t)\,dW(t) + \vartheta(t;t)\,dN(t)$$

and the defaultable bond prices are given by

$$\begin{aligned}
dq(t;T) = \;& q(t-;T)\,[r_1(t) + A_1(t;T)]\,dt \\
&+ q(t-;T)\,D(t;T)\,\lambda(t)\,dt \\
&+ q(t-;T)\,S_1(t;T)\,dW(t) \\
&+ q(t-;T)\,D(t;T)\,d\tilde{N}(t) \\
\Leftrightarrow \quad q(t;T) =\;& q(0;T)\exp\left\{\int_0^t [r_1(u) + A_1(u;T)]\,du\right\} \\
& \exp\left\{\int_0^t D(u;T)\,\lambda(u)\,du\right\} \\
& -\left\{\int_0^t S_1(u;T)\,dW(u)\right\} - \left\{\int_0^t D(u;T)\,d\tilde{N}(u)\right\}
\end{aligned}$$

where

$$S_1(t;T) := -\int_t^T \sigma_1(t;u)\,du$$

$$A_1(t;T) := -\int_t^T \alpha_1(t;u)\,du + \frac{1}{2}\|S_1(t;T)\|^2$$

$$\Theta(t;T) := -\int_t^T \vartheta(t;u)\,du$$

$$D(t;T) := \Delta e^{\Theta(t;T)} - 1$$

Proof. See Jarrow/Turnbull [1995].

Remark. To obtain this result, Jarrow and Turnbull employ a very interesting foreign currency analogy. Please note that we have not said anything yet about the connection between non-defaultable and defaultable rates and prices. We will do that in section 3 where we value a defaultable bond when a non-defaultable bond and the firm value are traded in the market.

The next two sections will contain some results on absence of arbitrage and completeness of our market. For these two sections, we make the following assumption:

Assumption 4. A continuum of non-defaultable and defaultable bonds as well as the firm value process are traded in the market.

2.3 Absence of arbitrage and existence of martingale measures

The following proposition is a well known result (see, for example, Björk et al., Proposition 3.9):

Proposition 5. If there exists an equivalent martingale measure, then the model is arbitrage-free.

In order to characterize the set of equivalent martingale measures, the following lemma is fundamental:

Lemma 6. Let $F = (F_t)_{t\in[0;T^*]}$ denote the internal filtration generated by the Brownian Motion and the point process, satisfying the usual conditions. Then every square integrable, F-martingale $M(t)$ has a representation

$$M(t) = M(0) + \int_0^t \sigma^M(u)\,dW(u) + \int_0^t \vartheta^M(u)\,d\tilde{N}(u); \qquad (1)$$

where the integrands σ^M and ϑ^M satisfy

- σ^M is measurable, $\mathcal{F}$-predictable and fulfills for $0 \le t \le T^*$

$$\int_0^t \|\sigma^M(u)\|^2\,du < \infty \quad P\text{-f.s.} \tag{2}$$

- ϑ^M is measurable, $\mathcal{F}$-predictable and fulfills for $0 \le t \le T^*$

$$\int_0^t |\vartheta^M(u)|^2 \lambda(u)\,du < \infty \quad P\text{-f.s.} \tag{3}$$

Proof. See Björk et al. [1996], Remark 3.2.

We can now proceed to characterize the set of all equivalent measures by a suitable version of Girsanov's theorem (see Björk et al. [1996]):

Theorem 7. Let $\tilde{P}$ be a measure equivalent to P and let G be the density process of $\tilde{P}$ given by

$$G(t) = E\left[\frac{d\tilde{P}}{dP}\,\middle|\,\mathcal{F}_t\right]; \quad 0 \le t \le T^* \tag{4}$$

Then there exist $\mathcal{F}$-predictable processes $\{\gamma(t); \mu(t)\}$ such that

1)

$$\int_0^{T^*} \|\gamma(u)\|^2\,du < \infty \quad P\text{-f.s.}$$

2) $\mu(t) \ge 0$ and

$$\int_0^{T^*} \mu(u)\lambda(u)\,du < \infty \quad P\text{-f.s.}$$

3) The P-martingale G is given by

$$G(t) = \exp\Big\{\int_0^t \gamma(u)\,dW(u) - \frac{1}{2}\int_0^t \|\gamma(u)\|^2\,du + \int_0^t \log\mu(u)\,dN(u) - \int_0^t [\mu(u) - 1]\lambda(u)\,du\Big\}$$

4) Under $\tilde{P}$, the processes

$$d\tilde{W}(t) = dW(t) - \gamma(t)\,dt$$
$$d\tilde{N}(t) = dN(t) - \lambda(t)\mu(t)\,dt$$

are martingales.

Conversely, every P-Martingale of the type given in 3) is the density of a measure equivalent to P.

The next theorem gives conditions under which the bond price processes become martingales under an equivalent measure:

Theorem 8. Under assumptions 1, 2, 3 and 4, a martingale measure exists if and only if the following conditions hold:

- There exist predictable processes $\{\gamma(t), \mu(t)\}$ such that for all $T \le T^{\infty}$, on $[0, T]$ we have

$$A_0(t,T) + S_0(t,T)\gamma(t) = 0; \tag{5}$$
$$r_1(t) - r_0(t) + A_1(t,T) + D(t,T)\lambda(t)\mu(t) + S_1(t,T)\gamma(t) = 0 \tag{6}$$

  and

$$\alpha_2(t) - r_0(t) + \sigma_2(t)\gamma(t) = 0 \tag{7}$$

- The predictable processes $\{\gamma(t), \mu(t)\}$ satisfy the integrability conditions of theorem 7 and are such that $E^P[G(t)] = 1$.

Proof. See Jarrow/Madan [1995].

For the model to possess a martingale measure, the forward rate drift can not be chosen freely, but it is determined by the volatilities:

Proposition 9. The existence of an equivalent martingale measure implies

$$\alpha_0(t,T) = -\sigma_0(t,T)S_0(t,T) - \sigma_0(t,T)\gamma(t) \tag{8}$$
$$\alpha_1(t,T) = -\sigma_1(t,T)S_1(t,T) - \delta e^{\Gamma(t,T)}\vartheta(t,T)\lambda(t)\mu(t) - \sigma_1(t,T)\gamma(t) \tag{9}$$
$$r_1(t) - r_0(t) = (1 - \delta)\lambda(t)\mu(t) \tag{10}$$

Proof. For the first two equations see Björk et al. [1996]. The last relationship follows from equation (6) by setting $T = t$.

Remark. Equation (10) shows that the difference between the non-defaultable and defaultable short rate is equal to the expected loss-rate. Duffie/Singleton [1994] model directly non-defaultable and defaultable short-rates, and this is the central equation connecting both rates. It is interesting to see that the same relationship can be obtained in a Heath/Jarrow/Morton-type of setup, where instead of the short rates the forward rates are modelled. As Schönbucher [1997] shows, conditions (8) to (10) are necessary and sufficient for the existence of a martingale measure when only bonds are present. Here, the firm value is also a traded asset and so additionally condition (7) has to be fulfilled.

2.4 Completeness and uniqueness of martingale measures

In our context we know that the existence of a unique martingale measure is sufficient for completeness of the market. The set of equivalent martingale measures is uniquely defined by the set of possible processes $\{\gamma, \mu\}$ satisfying the conditions of theorem 8. For each ω and t, these $n + 1$ variables, the so-called market prices of risk, are the solution of the following system of equations with $d = n + 1$ equations (see Jarrow/Madan [1995]):

$$\begin{pmatrix} S_{0;1}(t;T_1) & \cdots & S_{0;n}(t;T_1) & 0 \\ \vdots & & \vdots & \vdots \\ S_{0;1}(t;T_{d_0}) & \cdots & S_{0;n}(t;T_{d_0}) & 0 \\ S_{1;1}(t;T_{d_0+1}) & \cdots & S_{1;n}(t;T_{d_0+1}) & D(t;T_{d_0+1}) \\ \vdots & & \vdots & \vdots \\ S_{1;1}(t;T_d) & \cdots & S_{1;n}(t;T_d) & D(t;T_d) \\ S_{2;1}(t) & \cdots & S_{2;n}(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma_1(t) \\ \vdots \\ \gamma_n(t) \\ \lambda(t)\mu(t) \end{pmatrix} = - \begin{pmatrix} A_0(t;T_1) \\ \vdots \\ A_0(t;T_{d_0}) \\ r_1(t) - r_0(t) + A_1(t;T_{d_0+1}) \\ \vdots \\ r_1(t) - r_0(t) + A_1(t;T_d) \\ \alpha_2(t) - r_0(t) \end{pmatrix} \qquad (11)$$

Here we have taken d_0 non-defaultable bonds, $n - d_0$ defaultable bonds and the firm value to calculate the market prices of risk. The equivalent martingale measure exists and is uniquely determined if and only if this system of equations possesses a unique solution, which is independent of the choice of bonds (Jarrow/Madan [1995]). This is sufficient to ensure completeness of the market.

2.5 The presence of marked point processes

Most previous results can be easily generalized to the case when marked point processes are present, and only minor changes have to be made for the bond price processes and the conditions for the existence of a martingale measure. However, concerning the completeness of the market, marked point processes with infinitely many marks add some new features into the theory. Here, the processes $\{\gamma, \mu\}$ are no longer solutions of a finite-dimensional system of equations. Rather, we have to define the martingale operator (see Björk et al. [1996])

$$K_t(\omega): \mathbb{R}^n \times L^2(E, \mathcal{E}, \lambda(t)\Phi(t;dx)) \to C[0, \infty[$$
$$(\gamma(t), \mu(t;x)) \mapsto \int_X (\cdot\, e^{\Theta(t;x;T)} - 1)\mu(t;x)\lambda(t)\Phi(t;dx) + a(t;T)\gamma(t)$$

which operates on the market prices of risk and takes the place of the matrix multiplication on the left-hand side of equation 11 above. Now, uniqueness of the martingale measure is equivalent to

$$\ker K_t = 0 \quad dP\, dt\text{-a.e.}$$

It is no longer true that uniqueness of the martingale measure is equivalent to completeness of the market. Instead, it can be shown that the martingale measure is unique if and only if the market is approximately complete. For more on the theory of bond markets in the presence of marked point processes see Björk et al. [1996].

2.6 Incompleteness and local risk-minimization

The conditions for uniqueness of the martingale measure and thus completeness of the market are not always satisfied. This is especially true for the market of defaultable bonds. The government issues bonds in regular intervals, so that at each time there are many bonds of different maturities traded in the market. Firms, however, issue bonds only infrequently, and so the number of assets which is traded on the market is smaller than the number of stochastic processes driving the market. As a result from this, the equation system (11) has less equations than variables and therefore, many possible martingale measures exist. The approximate completeness, which arises when using marked point processes, is another argument to consider incomplete markets.

If markets are incomplete, the martingale measure is no longer uniquely defined and riskless hedging of arbitrary derivatives is no longer possible. On the contrary, the typical claim has an intrinsic risk, and all one can do is reduce the actual risk to the intrinsic part. This can be done by local risk-minimizing hedging and the local risk-minimizing martingale measure.

Let us denote the (discounted) price processes of traded assets with $X = (X_1, \ldots, X_d)$. In a complete market every contingent claim H with maturity T is attainable. Under the martingale measure $\tilde{P}$ it can be written as

$$H = V(T) = V(0) + \int_0^T \xi^H(s)\, dX(s);$$

where ξ^H is the invested part of a self-financing trading strategy with value process V. In this way, the total risk from the contingent claim can be eliminated. In an incomplete market, however, we have to incur additional costs. The additional costs at time T are given by

$$C(T) = H - V(0) - \int_0^T \xi^H(s)\,dX(s)$$

Föllmer/Sondermann [1986] suggested to minimize riskiness, which they defined by

$$R(t) := E[(C(T) - C(t))^2 | \mathcal{F}_t]$$

Under a martingale measure, it turns out that the risk-minimizing strategy is given by the Kunita-Watanabe decomposition of the claim, namely

$$H = E[H] + \int_0^T \xi^H(s)\,dX(s) + L(T).$$

Here, L is a martingale orthogonal to X and stands for the additional costs, while ξ^H is the invested part of the risk-minimizing strategy (see Föllmer/Schweizer [1990]). Please note that, as the process of additional costs is a martingale, a risk-minimizing strategy is mean-self-financing. This implies that, in the average, the additional costs are zero or

$$E[C(T) - C(t) | \mathcal{F}_t] = 0$$

Let us now turn to the general case, where asset prices $X = (X_1, \ldots, X_d)$ are semimartingales with a Doob-Meyer-decomposition

$$X(t) = X(0) + A(t) + M(t)$$

In this case, Schweizer [1991] introduced the criterion of local risk-minimization and showed that a replicating strategy is locally risk-minimizing if it is mean-self-financing and its cost process $C(t)$ follows a martingale strongly orthogonal to $M(t)$. This strategy corresponds to the Föllmer-Schweizer-decomposition, which is defined as follows:

Definition 2. A random variable $H \in L^2(\Omega, \mathcal{F}, P)$ admits a Föllmer-Schweizer decomposition if it can be written as

$$H = H_0 + \int_0^T \xi^H(s)\,dX(s) + L(T), \quad P\text{-f.s.},$$

where H_0 is an $\mathcal{F}_0$-measurable random variable, X is a semimartingale with a decomposition $X = X_0 + M + A$, $\xi^H \in L^2(M)$ and $L = (L(t))_{0 \le t \le T}$ is a martingale in $\mathcal{M}_0^2$, strongly orthogonal to $\int \theta\,dM$ for all $\theta \in L^2(M)$.

Again, ξ^H is the invested part of the risk-minimizing strategy, and $L(T)$ coincides with the additional cost. However, this time X is not a martingale, but a semimartingale. Under very general conditions Monat and Stricker [1995] show the existence, uniqueness and continuity of the Föllmer-Schweitzer-decomposition.

The Föllmer-Schweizer-decomposition can be calculated in the case of continuous processes by first going over to the so-called local risk-minimizing martingale measure, and then using the Kunita-Watanabe projection. The expectation under the local risk-minimizing martingale measure is the initial value of the local risk-minimizing hedging strategy.

The local risk-minimizing martingale measure P^* is characterized by the fact that all P-martingales which are orthogonal to M under P stay martingales under P^*. In the following, we will construct the local risk-minimizing martingale measure, as done in Schweizer [1991].

From the Doob-Meyer decomposition of traded assets above, we define the following processes:

$$\Sigma_{ij}(t) := \frac{d\langle M_i; M_j\rangle(t)}{dt}$$
$$a_i(t) := \frac{dA_i(t)}{dt}$$

The density process of the risk minimizing martingale measure P^* is given by

$$G^*(t) := \mathcal{E}\left\{ -\sum_{j=1} \int_0^t \psi_j(u)\, dM_j(u) \right\}$$

for certain ψ_j. For the X_i to be martingales under the minimal martingale measure, it is necessary that

$$A_i(t) = \sum_{i=1} \int_0^t \psi_j(u)\, d\langle M_i; M_j\rangle(u):$$

Therefore, the ψ_j are given by the solutions of the following system of linear equations:

$$\sum_{j=1} \Sigma_{ij}(t)\psi_j(t) = a_i(t)$$

In the next section, we will show how this technique can be used to value defaultable bonds in a local risk-minimizing way.

3 Valuation of a defaultable bond

This section concentrates on the valuation of defaultable bonds in incomplete markets. We first introduce the traded assets. Subsequently we compute the minimal martingale measure, which is determined by the traded assets. We introduce the general formula for a defaultable bond and go over to the forward measure to simplify calculation. Finally, we present two alternatives of modelling the defaultable bond and in each case give an approximation of the local risk-minimizing value of the defaultable bond. Because the firm value is a traded asset in our model, we are able to hedge against the loss from default. However, because the market is not complete, the hedge is not perfect and there still remains some unhedgeable risk.

3.1 Traded assets

We assume that the following assets are traded in the market and can be used for hedging purposes:

Assumption 5.

- A bank account with the interest rate of non-defaultable bonds

$$B(t) = e^{\int_0^t r_0(s)\,ds}$$

- One non-defaultable bond with maturity T, given by

$$\begin{aligned} dp(t;T) &= p(t;T)[r_0(t) + A_0(t;T)]\,dt + p(t;T)S_0(t;T)\,dW(t) \\ \Leftrightarrow\ p(t;T) &= p(0;T)\exp\left\{\int_0^t [r_0(u) + A_0(u;T)]\,du\right\}\mathcal{E}\left\{\int_0^t S_0(u;T)\,dW(u)\right\} \end{aligned}$$

- Additionally, we introduce the firm value, given by the process

$$\begin{aligned} dV(t) &:= V(t)\alpha_2(t)\,dt + V(t)\sigma_2(t)\,dW(t) \\ \Leftrightarrow\ V(t) &= \exp\left\{\int_0^t \alpha_2(s)\,ds\right\}\mathcal{E}\left\{\int_0^t \sigma_2(u)\,dW(u)\right\} \end{aligned}$$

Please recall that we have assumed from the beginning that all volatilities are deterministic.

In all of the above, the Brownian Motion is at least two-dimensional.

Remark.

- The whole analysis that follows can be done in exactly the same way if marked point processes are included in the bond price and firm value processes. However, formulas become more complicated, and that is the reason why we refrain from using them here.

- It seems reasonable to assume that the Brownian Motion is at least two-dimensional. This allows the non-defaultable bond and the firm value to be correlated only partially. Because the number of random sources driving the market is three (Brownian Motion plus point process, which governs default), but the number of traded assets is only two, the market is incomplete.

In the next subsection we will show how to compute the local risk-minimizing martingale measure.

3.2 The minimal martingale measure

We denote the local risk-minimizing martingale measure by $\hat{P}$. It is completely determined by the assets which are traded in the market. For the existence of the local risk-minimizing martingale measure, we need the following assumption:

Assumption 6. Suppose that the following linear system of equations has a unique solution $(\psi^p; \psi^V)$:

$$\begin{pmatrix} \frac{dA^p}{dt} \\ \frac{dA^V}{dt} \end{pmatrix} = \begin{pmatrix} \frac{d\langle M^p\rangle}{dt} & \frac{d\langle M^p; M^V\rangle}{dt} \\ \frac{d\langle M^V; M^p\rangle}{dt} & \frac{d\langle M^V\rangle}{dt} \end{pmatrix} \begin{pmatrix} \psi^p \\ \psi^V \end{pmatrix}$$

Here $A^p; A^V; M^p; M^V$ are the parts of finite variation resp. martingale parts of the discounted processes $p(t;T)/B(t)$ resp. $V(t)/B(t)$.

Remark. There are two cases in which the assumption above is not satisfied:

- Either one of the martingale parts vanishes or
- both assets are perfectly correlated.

This seems reasonable: In both cases, we can no longer use both assets independently to hedge against changes.

The next theorem provides an explicit formula for the minimal martingale measure in our setup:

Theorem 10. Under assumptions 5, 6 define

$$\hat{G}(t) := \mathcal{E}\left\{ -\int_0^t \left[\frac{\|\sigma_2(u)\|^2 A_0(u;T) - S_0(u;T)\sigma_2(u)(\alpha_2(u) - r_0(u))}{\|S_0(u;T)\|^2\|\sigma_2(u)\|^2 - (S_0(u;T)\sigma_2(u))^2} S_0(u;T) + \frac{\|S_0(u;T)\|^2(\alpha_2(u) - r_0(u)) - S_0(u;T)\sigma_2(u)A_0(u;T)}{\|S_0(u;T)\|^2\|\sigma_2(u)\|^2 - (S_0(u;T)\sigma_2(u))^2}\sigma_2(u) \right] dW(u) \right\}$$

Suppose further that $E[\hat{G}] = 1$. Then $\hat{G}$ is the density of the minimal martingale measure. The Brownian Motion under the new measure is given by

$$\hat{W}(t) = W(t) + \int_0^t \Big[\frac{\|\sigma_2(u)\|^2 A_0(u,T) - S_0(u,T)\sigma_2(u)(\alpha_2(u) - r_0(u))}{\|S_0(u,T)\|^2\|\sigma_2(u)\|^2 - (S_0(u,T)\sigma_2(u))^2} S_0(u,T) + \frac{\|S_0(u,T)\|^2(\alpha_2(u) - r_0(u)) - S_0(u,T)\sigma_2(u)A_0(u,T)}{\|S_0(u,T)\|^2\|\sigma_2(u)\|^2 - (S_0(u,T)\sigma_2(u))^2}\sigma_2(u) \Big] du$$

Proof. See Appendix.

We know that after the change of measure, the price processes of the riskless bond and the firm value can be written as

$$p(t,T) = p(0,T)\exp\left\{\int_0^t r_0(u)\,du \quad E \quad \int_0^t S_0(u,T)\,d\hat{W}(u)\right\}$$

$$V(t) = V(0)\exp\left\{\int_0^t r_0(u)\,du \quad E \quad \int_0^t \sigma_2(u,T)\,d\hat{W}(u)\right\}$$

where $\hat{W}$ is a Brownian Motion under $\hat{P}$.

3.3 Modelling a defaultable bond

Having changed to the minimal martingale measure, we are now in a position to determine the risk-minimizing price of a defaultable bond. Here by risk-minimizing price we mean the initial investment in a trading strategy which allows us to hedge the defaultable bond in a risk-minimizing way. To model a defaultable bond with maturity T we have to specify two characteristics:

- The time of default τ and
- the payoff after default δ as percentage of the bonds face value.

As we noted in the introduction, there are basically two approaches to the modelling of credit risk: The so-called classical approach, where default occurs when the firm value falls below a prespecified boundary, and the intensity approach.

We want to employ the intensity approach, where the time of default τ is the first jump time of a point process. In this case, the modelling of the default intensity is of particular interest. In the following, we will consider in detail two possibilities of modelling the intensity of the default-governing point process:

- The intensity is stochastic. While the point process itself is independent of the other processes driving the market, its intensity will depend on the firm value.

• The intensity is deterministic. This seems like a step backwards, but while restricting the intensity to be deterministic, we can allow the payoff after default to be stochastic and depend on the firm value.

In general, the value of a defaultable bond before the time of default can be written as

$$q(t;T) = B(t)\hat{E}\left[\frac{1}{B(T)}1_{\{\tau > T\}} + \frac{\delta(\tau)}{B(T)}1_{\{\tau \le T\}}\Big|\mathcal{F}_t\right] \qquad (12)$$

In order to get rid of $B(T)$ inside the expectation, we change the numeraire and go over to the forward measure $\hat{P}^T$.

3.4 The forward measure

The change of numeraire from "money today" to "money at time T" corresponds to a change of measure from the standard martingale measure to the so-called T-forward-measure. For an exposition of the usage of the T-foward-measure in the Heath-Jarrow-Morton model see Rutkowski [1996].

Definition 3. A probability measure $\hat{P}^T$ equivalent to $\hat{P}$ with the Radon-Nikodym density given by the formula

$$\frac{d\hat{P}^T}{d\hat{P}} = \frac{B(T)^{-1}}{\hat{E}[B(T)^{-1}]} = \frac{1}{B(T)p(0;T)} =: G^T$$

is called a forward probability measure for the settlement date T.

In our setting, an explicit representation for the density process G^T is available.

Theorem 11. The density process of the forward measure is given by

$$\hat{G}^T(t) = \frac{p(t;T)}{B(t)p(0;T)} = \mathcal{E}\left\{\int_0^t S_0(u;T)\, d\hat{W}(u)\right\}$$

and so the Brownian Motion under the forward measure is given by

$$\hat{W}^T(t) = \hat{W}(t) - \int_0^t S_0(u;T)\, du$$

Proof. We have

$$\frac{1}{B(T)p(0;T)} = \frac{B(t)}{B(T)}\frac{1}{B(t)p(0;T)} = \frac{p(t;T)}{B(t)p(0;T)}$$

The bond price process and the firm value process can be calculated under $\hat{P}^T$ to be

$$p(t;T) = p(0;T)B(t)\exp\left\{\int_0^t S_0(u;T)^2\,du - \int_0^t S_0(u;T)\,d\hat{W}^T(u)\right\}$$
$$V(t) = V(0)B(t)\exp\left\{\int_0^t \sigma_2(u)S_0(u;T)\,du - \int_0^t \sigma_2(u)\,d\hat{W}^T(u)\right\}$$

Under the forward measure, expression (12) becomes

$$\begin{aligned} q(t;T) &= p(t;T)\hat{E}^T\left[1_{\{\tau>T\}} + \delta(\tau)1_{\{\tau\le T\}}|F_t\right] \\ &= p(t;T)\left(1 - \hat{E}^T\left[(1-\delta(\tau))1_{\{\tau\le T\}}|F_t\right]\right) \end{aligned} \tag{13}$$

Because of

$$\hat{P}^T[\tau \le T|F_t] = 1 - \hat{E}^T[e^{-\int_t^T \lambda_\tau(u)\,du}|F_t];$$

the defaultable bond can be written as

$$\begin{aligned} q(t;T) = \; & p(t;T)\left(1 - \hat{E}^T[1 - e^{-\int_t^T \lambda_\tau(u)\,du}|F_t]\hat{E}^T[1-\delta(\tau)|F_t]\right) \\ & + \widehat{Cov}^T(e^{-\int_t^T \lambda_\tau(u)\,du}; \delta(\tau)|F_t) \end{aligned}$$

However, as the covariance between payoff after default and intensity of default-time is usually not known, this expression is difficult to evaluate. The simplest way around this problem is to take δ constant or a random variable independent of the time of default and λ_τ deterministic. Under these very restrictive assumptions (see Jarrow/Turnbull [1995]), the value of the defaultable bond before default is equal to

$$q(t;T) = p(t;T)(1 + \hat{E}^T[\delta - 1](1 - e^{-\int_t^T \lambda_\tau(u)\,du}))$$

In the following, we will derive explicit formulae for the value of a defaultable bond in a more general setup. To calculate the expectation, we have to impose the following assumptions on our model:

1) We can either assume that the payoff after default is independent of the time of default and that the intensity of the point process is stochastic, or

2) we can assume that the payoff after default is stochastic, but the intensity of the point process is deterministic.

In the following, we will treat both cases.

3.5 Constant payoff, stochastic intensity

In this subsection, we work under the following additional assumptions:

Assumption 7. 1) Let the payoff after default be constant,

$$\Delta(\tau) \equiv \Delta = \text{const}$$

2) Let the intensity of the default-governing jump process depend on the discounted firm value and be given by

$$\lambda^{\tau}(t) = \left(K - C \log \frac{V(t)}{B(t)}\right)^{+} = l\left(\frac{V(t)}{B(t)}\right);$$

where K and C are positive constants which have to be chosen and can be used to adjust the model to market data.

Remark. The functional form 2) has the following characteristics: For discounted firm values greater than a threshold, default risk vanishes,

$$\lambda^{\tau}(t) = 0 \quad \Leftrightarrow \quad \frac{V(t)}{B(t)} \geq e^{\frac{K}{C}};$$

while for very small discounted firm values, default risk is very high,

$$\lambda^{\tau}(t) \to +\infty \quad \Leftrightarrow \quad \frac{V(t)}{B(t)} \to 0:$$

It can be seen that the constants K and C can be chosen in such a way that firm values where default is possible, but not certain comprise a specific interval $]0; \exp(\frac{K}{C})[\subset \mathbb{R}$. Moreover, the functional form exhibits the property that a doubling of the firm value induces a constant decrease in default risk:

$$\begin{aligned} l\left(2\frac{V(t)}{B(t)}\right) &= K - C \log 2\frac{V(t)}{B(t)} = K - C \log \frac{V(t)}{B(t)} - C \log 2 \\ &= l\left(\frac{V(t)}{B(t)}\right) - C \log 2 \end{aligned} \qquad (14)$$

as long as

$$\frac{V(t)}{B(t)} \leq \frac{1}{2} \exp\left(\frac{K}{C}\right)$$

Madan/Unal [1994] use a similar function as intensity:

$$\phi(x) = \theta + \frac{c}{(\log(x/\delta))^2}$$

Here x stands for the discounted firm value. This function, however, has not all the convenient characteristics of our specification.

Because of assumption 7.1), we can simplify expression (13) to

$$q(t,T) = p(t,T)\left(\delta + (1-\delta)\hat{E}^T\left[\exp\left\{-\int_t^T \lambda^\tau(u)\,du\right\}\Big|\mathcal{F}_t\right]\right) \tag{15}$$

$$= p(t,T)\left(\delta + (1-\delta)\hat{E}^T\left[\exp\left\{-\int_t^T\left(K - C\log\frac{V(u)}{B(u)}\right)1_{\{\log\frac{V(u)}{B(u)}\le\frac{K}{C}\}}\,du\right\}\Big|\mathcal{F}_t\right]\right);$$

and it remains to calculate the conditional expectation on the right side. This can be done numerically, for example, by Monte-Carlo simulations: Paths of the firm value are generated, the associated default intensity λ^τ is calculated, and after checking the boundary the integral and expectation is computed.

In the following, we will give an analytic formula for the value of the defaultable bond. For this purpose, we have to simplify the problem. We will now argue that we can leave aside the indicator function inside the expectation, and will introduce our final assumptions:

Assumption 8. The discounted firm value is almost always smaller than $\frac{K}{C}$, or formally

$$\hat{P}^T\left[\log\left(\frac{V(t)}{B(t)}\right) > \frac{K}{C}\right] \approx 0 \quad \forall t \in [0,T^1] \tag{16}$$

Remark. Assumption 8 is not as restrictive as it looks at first sight. In the dependence of default intensity on firm value, we have the two constants C and K, and while C can be chosen to fulfill equation (14), we still have K available to satisfy (16). Because under this assumption negative default possibilities are taken into account, the defaultable bond will tend to be overvalued. However, numerical testing and comparison with computations of expression (15) will have to show if this is a realistic assumption.

Under these assumptions, we can formulate the main result of this subsection:

Theorem 12. Under the assumptions 5, 6, 7 and 8, the value of a defaultable bond before the time of default can be approximated by

$$q(t,T) \approx p(t,T)\Bigg(\delta + (1-\delta)e^{-(T-t)\lambda^\tau(t)}$$

$$\exp\left\{C\int_t^T (T-v)S_0(v)\sigma_2(v)\,dv + \frac{1}{2}C\int_t^T \sigma_2(v)^2\left((T-v)^2-(T-v)\right)dv\right\}\Bigg)$$

Proof. See Appendix.

Remark 1. The theorem gives a very easy formula for the valuation of defaultable bonds. All that is needed is the price of a non-defaultable bond of the same maturity and the firm value, estimates of the volatilites of non-defaultable bond and firm value, and an estimate of the payoff after default.

It can be seen that the write-down on the defaultable bond which is due to the time of default consists of two parts: The first one covers default risk based on the current firm value, while the second one captures variations in the firm value until maturity of the bond.

The drift of the firm value, $\alpha_2(t)$, does not enter into the formula due to the change to the local risk-minimizing martingale measure.

The effect of the parameters on the defaultable bond price is as follows: An increase in the volatility of the risk-free bond decreases the value of the defaultable bond, as long as risk-free bond and firm value are positively correlated. The correlation itself also has a negative effect on the defaultable bond price. The effect of an increase in the firm value volatility is also negative, as far as the first term is concerned. Its effect in the second term depends on the time to maturity: It can have a negative effect on the value of the defaultable bond, if $(T - v)^2 < (T - v)$, which is the case for a short time to maturity. On the other hand, if the time to maturity is longer, and we have $(T - v)^2 > (T - v)$, an increased volatility of the firm value can in principle have a positive effect on the value of the defaultable bond. However, this is not likely to happen because the firm value volatility enters as a square into the second term, and therefore the overall impact of the second term is much smaller than that of the first term. All of these findings are consistent with the results of Merton [1974] and Shimko/Tejima/Deventer [1993], who considered classic firm value models. Therefore, the present model can be seen as a link between the classical and the intensity approach. It captures many important characteristics of firm value models and in addition allows for unpredictable default.

Remark 2. The same calculation can be done, of course, when the payoff after default is not constant, but stochastic and independent of the time of default. In this case, ¢ has to be replaced by $\hat{E}^T[¢]$.

Because the market under consideration is incomplete, there exists no perfect hedging strategy for the defaultable bond. However, we can use the traded assets p and V to hedge the risk partly. As discussed in subsection I.6, we use a hedging strategy which minimizes local risk. The local risk-minimizing hedging strategy consists of the three parts $(\eta; \xi^p; \xi^V)$, where ξ^p and ξ^V are the investments into the assets p and V and η is put onto the savings account. Due to the Kunita-Watanabe decomposition, they are given by the solution to the following linear system of equations:

$$\begin{pmatrix} d\langle q,p\rangle \\ d\langle q,V\rangle \end{pmatrix} = \begin{pmatrix} d\langle p,p\rangle & d\langle V,p\rangle \\ d\langle p,V\rangle & d\langle V,V\rangle \end{pmatrix} \begin{pmatrix} \xi^p \\ \xi^V \end{pmatrix}$$

$$\eta(t) = q(t) - \xi^p(t)p(t) - \xi^V(t)V(t);$$

3.6 Stochastic payoff, deterministic intensity

In this subsection, we change our assumptions:

Assumption 9. 1) Let the payoff after default depend on the firm value and be given by the following expression:

$$\delta(t) = \min\left\{ \left(K + C \log \frac{V(t)}{B(t)} \right)^+ ; 1 \right\} = \tilde{\delta}\left(\frac{V(t)}{B(t)} \right)$$

Again, K and C are constants which have to be chosen and can be used to adjust the model to market data.

2) Let the intensity of the default-governing point process λ^τ be deterministic and known.

Remark. Specifying δ like in 1) implies the following: For discounted firm values below $e^{-\frac{K}{C}}$, payoff after default is zero, while for discounted firm values over $e^{\frac{1-K}{C}}$, the full amount is paid back. Again, a doubling of the firm value means that payoff after default increases by $C \log 2$ as long as

$$e^{-\frac{K}{C}} \le \frac{V(t)}{B(t)} \le \frac{1}{2} e^{-\frac{K}{C} + \frac{1}{C}}$$

Because the volatility is deterministic, expression (13) simplifies to

$$\begin{aligned} q(t;T) &= p(t;T)\hat{E}^T\left[1_{\{\tau > T\}} + \delta(\tau) 1_{\{\tau \le T\}} | \mathcal{F}_t \right] \\ &= p(t;T) \left(e^{-\int_t^T \lambda^\tau(u)\,du} + \hat{E}^T\left[\delta(\tau) 1_{\{\tau \le T\}} | \mathcal{F}_t \right] \right) \end{aligned}$$

In order to compute the last expectation, we make use of the fact that we know the distribution of the default time and thus its density function:

$$\hat{P}^T[\tau \le T | \mathcal{F}_t] = 1 - e^{-\int_t^T \lambda^\tau(u)\,du}$$

$$f(t;s) = e^{-\int_t^s \lambda^\tau(u)\,du} \lambda^\tau(s)$$

With the density function, we can rewrite the expectation as

$$\hat{E}^T[\delta(\tau) 1_{\{\tau \le T\}} | \mathcal{F}_t] = \hat{E}^T\left[\int_t^T \delta(s) f(t;s)\,ds | \mathcal{F}_t \right]$$

In contrast to the previous subsection, here we can give an exact formula for the value of the defaultable bond. However, as we will see, it is not very descriptive:

Theorem 13. Under the assumptions 5, 6 and 9, the value of a defaultable bond is given by

$$q(t;T) = p(t;T)\left(e^{-\int_t^T \lambda(u)\,du} + \int_t^T \int_{-\frac{K}{C}}^{\frac{1-K}{C}} (K + Cy)\, g(t;s;y)\, dy\, f(t;s)\, ds \right.$$
$$\left. + \int_t^T \int_{\frac{1-K}{C}}^{\infty} g(t;s;y)\, dy\, f(t;s)\, ds \right)$$

where $g(t;s;y)$ is the density function of a normal distribution with mean

$$V(t) + \int_t^s S_0(v;T)\sigma_2(v)\, dv - \frac{1}{2}\int_t^s \|\sigma_2(v)\|^2\, dv$$

and variance

$$\int_t^s \|\sigma_2(v)\|^2\, dv$$

Proof. See Appendix.

However, we can obtain a result which is analogous to that of the previous subsection if we introduce the following assumption, which is an analogon to assumption 8.

Assumption 10. Let

$$\hat{P}^T\left[-\frac{K}{C} \leq \log\frac{V(t)}{B(t)} < \frac{1-K}{C}\right] \approx 1 \quad \forall t \in [0;T] \tag{17}$$

Under this assumption, we can forget about the indicator function inside the expectation, and we have the following result:

Theorem 14. Under the assumptions 5, 6, 9 and 10, the value of a defaultable bond can be approximated by

$$q(t;T) \approx p(t;T)\left(\Delta(t) + (1 - \Delta(t))e^{-\int_t^T \lambda(u)\,du} \right.$$
$$\left. + C\int_t^T F(t;v)S_0(v)\sigma_2(v)\, dv - \frac{1}{2}C\int_t^T F(t;v)\|\sigma_2(v)\|^2\, dv \right)$$

where

$$F(t;v) := \int_v^T f(t;u)\, du$$

Proof. See Appendix.

Remark. Comparing theorem 14 with theorem 12, we see that both formulas are very similar. Instead of the time to maturity $T - v$, in the present case we have $F(v)$, which measures the remaining default risk. This time, the write-down on the defaultable bond which is due to the payoff after default, consists of two parts: The first one is based on the current, time t firm value, while the second one captures changes in the firm value until maturity of the bond. These changes are weighted with the probability that a default occurs until T.

Again, the local risk-minimizing hedging strategy can be computed as at the end of the previous subsection.

4 Valuation and hedging of options on defaultable bonds

In this section, we will concentrate on the valuation and hedging of options on defaultable bonds, again in the setting of an incomplete market. We will make use of the Föllmer-Schweizer-decomposition to split the value process of an option into a traded part, the risk of which can be hedged, and a totally untraded part, which cannot be hedged. The totally untraded part corresponds to the additional cost mentioned at the end of section 2, while hedging the traded part can be seen as a local risk-minimizing hedging strategy for the option. In order to calculate the value and the hedging process for an option, we will make use of the partial differential equations approach.

The article by Colwell and Elliot [1993] has a technically similar setup. However, they concentrate on another derivation of the local risk-minimizing martingale measure, while in the present approach no change of measure is carried out.

There is a particular reason why we do not change our measure: In a market where only diffusion processes are involved, we know that the change to the local risk-minimizing martingale measure preserves orthogonality (Schweizer [1990]). This means that martingales which are strongly orthogonal under the original measure stay strongly orthogonal under the local risk-minimizing martingale measure. As a consequence of this the Kunita-Watanabe decomposition under the local risk-minimizing martingale measure is equivalent to the Föllmer-Schweizer decomposition under the original measure. As soon as point processes are involved, however, this useful property of the local risk-minimizing martingale measure is no longer valid. There are still cases where one can proceed as in the continuous situation, but we are not in such a context. A strategy, computed by the Kunita-Watanabe decomposition under the local risk-minimizing martingale measure, would no longer minimize the risk with respect to the original measure.

4.1 Traded assets

In this section, we want to keep the setup as simple as possible in order to minimize notation. Therefore, we assume that the market is driven only by one Brownian Motion and by a single point process with deterministic intensity. We assume that the following assets are traded in the market and can be used for hedging purposes:

Assumption 11.

- One non-defaultable bond $p(t;T^p)$ with maturity T^p, given by

$$\begin{aligned} dp(t;T^p) &= p(t;T^p)[r_0(t) + A_0(t;T^p)]\,dt \\ &\quad + p(t;T^p)S_0(t;T^p)\,dW(t) \\ \Leftrightarrow\ p(t;T^p) &= p(0;T^p)\exp\left\{\int_0^t [r_0(u) + A_0(u;T^p)]\,du\right\} \\ &\quad \mathcal{E}\left\{\int_0^t S_0(u;T^p)\,dW(u)\right\} \end{aligned}$$

- One defaultable bond $q(t;T^q)$ with maturity $T^q \geq T^p$, given by

$$\begin{aligned} dq(t;T^q) &= q(t-;T^q)[r_1(t) + A_1(t;T^q)]\,dt \\ &\quad + q(t-;T^q)D^q(t;T^q)\lambda(t)\,dt \\ &\quad + q(t-;T^q)S_1(t;T^q)\,dW(t) \\ &\quad + q(t-;T^q)D^q(t;T^q)\,d\tilde{N}(t) \\ \Leftrightarrow\ q(t;T^q) &= q(0;T^q)\exp\left\{\int_0^t [r_1(u) + A_1(u;T^q)]\,du\right\} \\ &\quad \exp\left\{\int_0^t D^q(u;x;T^q)\lambda(u)\,du\right\} \\ &\quad \mathcal{E}\left\{\int_0^t S_1(u;T^q)\,dW(u)\right\}\mathcal{E}\left\{\int_0^t D^q(u;T^q)\,d\tilde{N}(u)\right\} \end{aligned}$$

In the following, we will leave out the dates of maturity $T^p; T^q$ to simplify notation.

We assume that all volatilities and the intensity of the point process are deterministic.

Remark. Please note that we have not assumed the existence of a bank account. The reason for this is that the interest rate $r_0(t)$ is not necessarily a Markov process in our model. In the following, we will think of the non-defaultable bond as a numeraire. The significance of this will become apparent later.

Because we have only one traded asset (the non-defaultable bond plays the role of the numeraire), it is enough to assume that the Brownian Motion is one-dimensional to make the market incomplete.

In order to calculate the locally risk-minimizing hedging strategies for contingent claims in this incomplete market, we will make use of the Föllmer-Schweizer decomposition.

In our case, the market can be completed by the introduction of one other asset. However, our model is not fully in line with the Föllmer- Schweizer case because of the missing bank account.

We solve this problem by choosing p as numéraire, thus going over to the forward market. Let C(t) be the discounted price of an option with maturity T^p at time t, and Z(t) the price of the additional asset introduced to complete the market. Then in our model the value process of a trading strategy replicating the option price would be

$$\tilde{V}(t) = \phi_0(t)p(t) + \phi_1(t)q(t) + \phi_2(t)Z(t) = \tilde{C}(t).$$

Dividing this equation by p(t), we see

$$\frac{\tilde{V}(t)}{p(t)} = V(t) = \phi_0(t) + \phi_1(t)\frac{q(t)}{p(t)} + \phi_2(t)\frac{Z(t)}{p(t)} = \frac{\tilde{C}(t)}{p(t)} = C(t)$$

The interpretation is the following: Instead of the spot market, we use the forward market to hedge. Introducing new definitions, we set

$$X_1(t) := \frac{q(t)}{p(t)}, \quad X_2(t) := \frac{Z(t)}{p(t)}$$

and together with the self-financing condition for V(t) we can write

$$dV(t) = \phi_1(t)\,dX_1(t) + \phi_2(t)\,dX_2(t)$$

The next proposition states the exact formula for $X_1(t)$.

Proposition 15. The process $X_1(t)$ is given by

$$dX_1(t) = X_1(t-)\mu^{X_1}(t)\,dt + X_1(t-)\Delta S(t)\,dW(t) + X_1(t-)D^q(t)\,dN(t)$$

where

$$\mu^{X_1}(t) := r_1(t) + A_1(t) - r_0(t) - A_0(t) + S_0(t)^2 - S_0(t)S_1(t)$$

and

$$\Delta S(t) := S_1(t) - S_0(t)$$

subject to the initial condition

$$X_1(0) = \frac{q(0)}{p(0)}$$

Proof. An application of Ito's formula to $X_1(t) = q(t)/p(t)$ yields the result.

□

To complete the market subject to the conditions of the Föllmer-Schweizer theorem, we introduce an asset which is strongly orthogonal to the martingale part of $X_1(t)$. This asset is given by the next proposition.

Proposition 16. The asset with the following differential equation is strongly orthogonal to the martingale part of $X_1(t)$:

$$dX_2(t) = -X_2(t-)D^q(t)\lambda(t)\,dW(t) + X_2(t-)\,\Delta S(t)\,d\tilde{N}(t)$$

We choose $X_2(0) = 1$.

Proof. The process X_2 has the form

$$dX_2(t) = X_2(t-)\sigma^{X_2}(t)\,dW(t) + X_2(t-)D^{X_2}(t)\,d\tilde{N}(t)$$

The conditional quadratic covariation of X_1 and X_2 is given by

$$d\langle X_1, X_2\rangle(t) = X_1(t-)X_2(t-)\,\Delta S(t)\sigma^{X_2}(t) + X_1(t-)X_2(t-)D^q(t)D^{X_2}(t)\lambda(t)\,dt$$

This should equal zero, and so we choose

$$\sigma^{X_2}(t) = -D^q(t)\lambda(t)$$
$$D^{X_2}(t) = \Delta S(t)$$

□

After the introduction of this asset, we can continue to derive the partial differential equation which is satisfied by the price process of a derivative security like in a complete market. The result of this will be a replicating strategy $\phi = (\phi_0, \phi_1, \phi_2)$ in terms of the three assets $(1, X_1, X_2)$. However, because X_2 is orthogonal to the traded asset X_1, and because of the Föllmer-Schweizer decomposition, the strategy (ϕ_0, ϕ_1) just using the assets $(1, X_1)$ will be the risk-minimizing strategy in the incomplete market. Here, risk is measured with respect to the forward measure.

4.2 Partial differential equations

A good reference for the technique used in this subsection is Rutkowski [1996]. We will consider a European pathwise independent claim C associated with the defaultable bond q and with expiry date $T^C = T^p$. The discounted price process of this claim is denoted by $C(t) = \tilde{C}(t)/p(t)$. We want to express the discounted price of this claim $C(t)$ as a function of the price of the discounted defaultable bond $X_1(t)$, the discounted virtual asset $X_2(t)$ and of time t. Therefore, we assume that the discounted value of the claim admits the representation

$$C(t) = \hat{C}(X_1(t), X_2(t), t) \text{ for all } t \in [0, T^C]$$

and satisfies a certain terminal condition, which, in the case of a call option on the defaultable bond, would be

$$\hat{C}(x, y, T^C) = (x - K)^+ \text{ for all } y \in \mathbb{R}^+.$$

A replicating trading strategy φ of the option has the form

$$\begin{aligned}\varphi(t) &= (\varphi_0(t), \varphi_1(t), \varphi_2(t)) \\ &= (g_0(X_1(t), X_2(t), t), g_1(X_1(t), X_2(t), t), g_2(X_1(t), X_2(t), t))\end{aligned}$$

where g_0, g_1, g_2 are functions not yet known. Because the trading strategy replicates the payoff of the option, the value process of the strategy satisfies

$$V(t) = \varphi_0(t) + \varphi_1(t)X_1(t) + \varphi_2(t)X_2(t) = \hat{C}(X_1(t), X_2(t), t) = C(t) \tag{18}$$

Because the trading strategy is self-financing, its value process satisfies

$$dV(t) = \varphi_1(t-)dX_1(t) + \varphi_2(t-)dX_2(t)$$

Substituting the dynamics of the price processes, we get

$$\begin{aligned}dV(t) = &\left(\varphi_1(t-)X_1(t-)\mu^{X_1}(t) - \varphi_2(t-)X_2(t-)\Delta S(t)\lambda(t)\right) dt \\ &+ \left(\varphi_1(t-)X_1(t-)\Delta S(t) - \varphi_2(t-)X_2(t-)D^q(t)\lambda(t)\right) dW(t) \\ &+ \left(\varphi_1(t-)X_1(t-)D^q(t) + \varphi_2(t-)X_2(t-)\Delta S(t)\right) dN(t).\end{aligned} \tag{19}$$

In the next step, we assume that $\hat{C} = \hat{C}(x, y, t)$ satisfies the necessary differentiability conditions to apply Ito's formula:

$$\begin{aligned}dC(t) = &\frac{\partial \hat{C}}{\partial t}dt + \frac{\partial \hat{C}}{\partial x}dX_1^c(t) + \frac{\partial \hat{C}}{\partial y}dX_2^c(t) \\ &+ \frac{1}{2}\frac{\partial^2 \hat{C}}{\partial x^2}d\langle X_1^c\rangle(t) + \frac{1}{2}\frac{\partial^2 \hat{C}}{\partial y^2}d\langle X_2^c\rangle(t) \\ &+ \frac{\partial^2 \hat{C}}{\partial x \partial y}d\langle X_1^c, X_2^c\rangle(t) + \Delta C(t)\, dN(t)\end{aligned}$$

Here the arguments $(X_1(t-); X_2(t-); t-)$ of $\tilde{C}$ have been omitted. $\Delta C(t)$ denotes the jump height of C in case a jump happens at time t and can be expressed as

$$\Delta C(t) = \tilde{C}(X_1(t-) + \Delta X_1(t); X_2(t-) + \Delta X_2(t); t) - \tilde{C}(X_1(t-); X_2(t-); t-);$$

where

$$\begin{aligned} \Delta X_1(t) &= X_1(t-)(D^{q}(t) - 1) \\ \Delta X_2(t) &= X_2(t-)(\Delta S(t) - 1) \end{aligned}$$

Substitution of the dynamics of price processes yields

$$\begin{aligned} dC(t) = {} & \frac{\partial \tilde{C}}{\partial t} dt \\ & + X_1(t-)\mu^{X_1}(t)\frac{\partial \tilde{C}}{\partial x} dt + X_1(t-)\Delta S(t)\frac{\partial \tilde{C}}{\partial x} dW(t) \\ & - X_2(t-)\Delta S(t)\lambda(t)\frac{\partial \tilde{C}}{\partial y} dt - X_2(t-)D^{q}(t)\lambda(t)\frac{\partial \tilde{C}}{\partial y} dW(t) \\ & + \frac{1}{2}X_1(t-)^2\Delta S(t)^2\frac{\partial^2 \tilde{C}}{\partial x^2} dt \\ & + \frac{1}{2}X_2(t-)^2 D^{q}(t)^2\lambda(t)^2\frac{\partial^2 \tilde{C}}{\partial y^2} dt \\ & - X_1(t-)X_2(t-)\Delta S(t)D^{q}(t)\lambda(t)\frac{\partial^2 \tilde{C}}{\partial x \partial y} dt \\ & + \Delta C(t)\, dN(t) \end{aligned}$$

Comparing the last equation with equation (19), we can derive the following two relationships:

$$\text{I} \quad \Delta C(t) = \varphi_1(t-)X_1(t-)D^{q}(t) + \varphi_2(t-)X_2(t-)\Delta S(t) \text{ for every } t \in [0; T^C]$$

$$\begin{aligned} \text{II} \quad & X_1(t-)\Delta S(t)\frac{\partial \tilde{C}}{\partial x} - X_2(t-)D^{q}(t)\lambda(t)\frac{\partial \tilde{C}}{\partial y} \\ & = \varphi_1(t-)X_1(t-)\Delta S(t) - \varphi_2(t-)X_2(t-)D^{q}(t)\lambda(t) \text{ for every } t \in [0; T^C] \end{aligned}$$

These two equations can be solved for φ_1 and φ_2, respectively:

$$\text{I'} \quad \varphi_1(t-) = \frac{1}{X_1(t-)D^{q}(t)}\left(\Delta C(t) - \varphi_2(t-)X_2(t-)\Delta S(t)\right) \text{ for all } t \in [0; T^C]$$

$$\begin{aligned} \text{II'} \quad \varphi_2(t-) = {} & \frac{1}{X_2(t-)D^{q}(t)\lambda(t)}\left(\varphi_1(t-)X_1(t-)\Delta S(t) - X_1(t-)\Delta S(t)\frac{\partial \tilde{C}}{\partial x}\right. \\ & \left. + X_2(t-)D^{q}(t)\lambda(t)\frac{\partial \tilde{C}}{\partial y}\right) \text{ for all } t \in [0; T^C] \end{aligned}$$

Substitution of $\phi_2(t^-)$ in I' from II' yields

$$\phi_1(t^-) = \frac{1}{X_1(t^-)}\frac{D\surd(t)\lambda(t)}{D\surd(t)^2\lambda(t) + \Delta S(t)^2}\left(\Delta C(t) + X_1(t^-)\frac{\Delta S(t)^2}{D\surd(t)\lambda(t)}\frac{\partial \tilde{C}}{\partial x} - X_2(t^-)\Delta S(t)\frac{\partial \tilde{C}}{\partial y}\right);$$

and inserting this into II', we see

$$\begin{aligned}\phi_2(t^-) = {} & \frac{1}{X_2(t^-)}\frac{\Delta S(t)}{D\surd(t)^2\lambda(t) + \Delta S(t)^2}\Delta C(t) \\ & + \frac{1}{X_2(t^-)D\surd(t)\lambda(t)}\left(\frac{\Delta S(t)^2}{D\surd(t)^2\lambda(t) + \Delta S(t)^2} - 1\right) \\ & \left[X_1(t^-)\Delta S(t)\frac{\partial \tilde{C}}{\partial x} - X_2(t^-)D\surd(t)\lambda(t)\frac{\partial \tilde{C}}{\partial y}\right]\end{aligned}$$

Substituting for ϕ_1 and ϕ_2 in the dt-part of dV and setting this equal to the dt-part of dC gives us the following equation:

$$\begin{aligned} & \frac{\mu_{X_1}(t)D\surd(t)\lambda(t) + \Delta S(t)^2\lambda(t)}{D\surd(t)^2\lambda(t) + \Delta S(t)^2}\Delta C(t) \\ & + \frac{\mu_{X_1}(t)\Delta S(t) - D\surd(t)\lambda(t)\Delta S(t)}{D\surd(t)^2\lambda(t) + \Delta S(t)^2}\left[X_1(t^-)\Delta S(t)\frac{\partial \tilde{C}}{\partial x} - X_2(t^-)D\surd(t)\lambda(t)\frac{\partial \tilde{C}}{\partial y}\right] \\ = {} & \frac{\partial \tilde{C}}{\partial t} + X_1(t^-)\mu_{X_1}(t)\frac{\partial \tilde{C}}{\partial x} \\ & - X_2(t^-)\Delta S(t)\lambda(t)\frac{\partial \tilde{C}}{\partial y} \\ & + \frac{1}{2}X_1(t^-)^2\Delta S(t)^2\frac{\partial^2 \tilde{C}}{\partial x^2} \\ & + \frac{1}{2}X_2(t^-)^2D\surd(t)^2\lambda(t)^2\frac{\partial^2 \tilde{C}}{\partial y^2} \\ & - X_1(t^-)X_2(t^-)\Delta S(t)D\surd(t)\lambda(t)\frac{\partial^2 \tilde{C}}{\partial x\partial y}\end{aligned}$$

Sorting this equation according to derivatives, we arrive at the final partial differential equation which is satisfied by any contingent claim in our market:

$$
\begin{aligned}
&\frac{\partial \tilde{C}}{\partial t} - \frac{\mu_{X_1}(t)D^q(t)\lambda(t) + \cdot S(t)^2\lambda(t)}{D^q(t)^2\lambda(t) + \cdot S(t)^2} \cdot C(t) \\
&+ \frac{\mu_{X_1}(t)D^q(t)^2\lambda(t) + D^q(t)\lambda(t)\cdot S(t)^2}{D^q(t)^2\lambda(t) + \cdot S(t)^2} X_1(t-)\frac{\partial \tilde{C}}{\partial x} \\
&+ \left(\frac{\mu_{X_1}(t)D^q(t)\lambda(t)\cdot S(t) - D^q(t)^2\lambda(t)^2\cdot S(t)}{D^q(t)^2\lambda(t) + \cdot S(t)^2} - \cdot S(t)\lambda(t)\right) X_2(t-)\frac{\partial \tilde{C}}{\partial y} \\
&+ \frac{1}{2}X_1(t-)^2\cdot S(t)^2\frac{\partial^2 \tilde{C}}{\partial x^2} \\
&+ \frac{1}{2}X_2(t-)^2 D^q(t)^2\lambda(t)^2\frac{\partial^2 \tilde{C}}{\partial y^2} \\
&- X_1(t-)X_2(t-)\cdot S(t)D^q(t)\lambda(t)\frac{\partial^2 \tilde{C}}{\partial x \partial y} \\
&= 0
\end{aligned}
$$

This partial differential equation, subject to some terminal condition according to the option's payoff at maturity, can be solved numerically with well-known methods such as finite elements for the function $\tilde{C}$. From this, the risk-minimizing hedging strategy can be computed as follows: φ_1 and φ_2 are given by equations I' and II' above. From these, φ_0 can be calculated from equation (18), and $(\varphi_0; \varphi_1)$ constitute the risk-minimizing trading strategy in the forward market with value process $\varphi_0(t) + \varphi_1(t)X_1(t)$.

This model can easily be generalized to multiple Brownian Motions and multiple point processes. However, in this case more than one virtual asset has to be constructed to complete the market, and the PDE gets more complicated.

The same technique can be used to compute the risk-minimizing value and hedging portfolio of a defaultable bond if only a non-defaultable bond and the firm value are tradeable. This would be an alternative to our approach taken in section 3.

5 Conclusion

Pricing formulae for defaultable bonds in two different specifications of our model have been derived. They combine the advantages of the classical approach of modelling credit risky bonds, based on the firm value, and of the intensity-based approach. The formulae allow for partial hedging of default risk through trading in the firm value as well as in a non-defaultable bond. We consider in detail the possibility that the firm value enters into the intensity, and therefore time of default, or that the ¯rm va

same way to all random shocks. Under this assumption we can solve the system of equations for $(\tilde{A}^p; \tilde{A}^V)$:

$$\begin{pmatrix} \tilde{A}^p \\ \tilde{A}^V \end{pmatrix} = \frac{1}{\det} \begin{pmatrix} \frac{d\langle M^V \rangle}{dt} & -\frac{d\langle M^p; M^V \rangle}{dt} \\ -\frac{d\langle M^V; M^p \rangle}{dt} & d\langle M^p \rangle \end{pmatrix}$$

and changing the order of integration yields

$$= \int_t^T \int_v^T S_0(v;T)\sigma_2(v)\,du\,dv + \int_t^T \int_v^T \sigma_2(v)\,du\,d\hat{W}^T(v) - \frac{1}{2}\int_t^T \int_v^T \|\sigma_2(v)\|^2\,du\,dv$$

$$= \int_t^T (T-v)S_0(v;T)\sigma_2(v)\,dv + \int_t^T (T-v)\sigma_2(v)\,d\hat{W}^T(v) - \frac{1}{2}\int_t^T (T-v)\|\sigma_2(v)\|^2\,dv$$

Because we have to take the exponential of these integrals, and because we want some of the exponentials to be stochastic exponentials, we add and subtract some terms:

$$= \int_t^T (T-v)S_0(v;T)\sigma_2(v)\,dv + \frac{1}{2}\int_t^T \|\sigma_2(v)\|^2 \left((T-v)^2 - (T-v)\right) dv$$
$$+ \int_t^T (T-v)\sigma_2(v)\,d\hat{W}^T(v) - \frac{1}{2}\int_t^T (T-v)^2\|\sigma_2(v)\|^2\,dv$$

Finally, taking the exponential,

$$\exp\left\{ C\int_t^T \left[\frac{V(u)}{B(u)} - \frac{V(t)}{B(t)}\right] du \right\}$$
$$= \exp\left(C\int_t^T (T-v)S_0(v;T)\sigma_2(v)\,dv + \frac{1}{2}C\int_t^T \|\sigma_2(v)\|^2\left((T-v)^2 - (T-v)\right) dv \right)$$
$$\mathcal{E}\left\{ +C\int_t^T (T-v)\sigma_2(v)\,d\hat{W}^T(v) \right\}$$

Now, we are in a position to calculate the expectation. We have

$$\hat{E}^T[e^{-\int_t^T \lambda^{\tau}(u)\,du} | F_t]$$
$$= e^{-(T-t)\lambda^{\tau}(t)}$$
$$\exp\left\{ C\int_t^T (T-v)S_0(v;T)\sigma_2(v)\,dv + \frac{1}{2}C\int_t^T \|\sigma_2(v)\|^2\left((T-v)^2 - (T-v)\right) dv \right\}$$

where the last stochastic exponential vanishes because of its martingale property.

$\square$

Proof of Theorem 13. We can write

$$\hat{E}^T[\Delta(\tau)1_{\{\tau \le T\}} | F_t]$$
$$= \hat{E}^T\Big[\int_t^T 0 \cdot 1_{\{\log \frac{V(s)}{B(s)} < -\frac{K}{C}\}} f(t;s)\,ds$$
$$+ \int_t^T \Delta(s) 1_{\{-\frac{K}{C} \le \log \frac{V(s)}{B(s)} < \frac{1-K}{C}\}} f(t;s)\,ds$$
$$+ \int_t^T 1_{\{\frac{1-K}{C} \le \log \frac{V(s)}{B(s)}\}} f(t;s)\,ds \,|\, F_t \Big]$$

and interchanging expectation and integration, we get

$$
\begin{aligned}
&= p(t;T)\int_t^T \hat{E}^T[\Delta(s) 1_{\{-\frac{K}{C} \le \log\frac{V(s)}{B(s)} < \frac{1-K}{C}\}} | \mathcal{F}_t] f(ts)\, ds \\
&+ \int_t^T \hat{E}^T[1_{\{\frac{1-K}{C} \le \log\frac{V(s)}{B(s)}\}} | \mathcal{F}_t] f(t;s)\, ds
\end{aligned}
$$

But we know the distribution of $\frac{V(t)}{B(t)}$ under the measure $\hat{P}^T[\cdot|\mathcal{F}_t]$:

$$
\begin{aligned}
\frac{V(t)}{B(t)} &= V(0)\exp\{\int_0^t S_0(u;T)\sigma_2(u)\, du\} \\
&\quad E\{\int_0^t \sigma_2(u)\, d\hat{W}^T(u)\} \\
\Rightarrow \log\frac{V(u)}{B(u)} &\sim N\Big(\frac{V(t)}{B(t)} + \int_t^u S_0(v;T)\sigma_2(v)\, dv - \frac{1}{2}\int_t^u \|\sigma_2(v)\|^2\, dv; \\
&\quad \int_t^u \|\sigma_2(v)\|^2\, dv
\end{aligned}
$$

Denoting the density of this normal distribution by $g(t;u;y)$, we get

$$
\begin{aligned}
&\hat{E}^T[\Delta(\tau)1_{\{\tau \le T\}}|\mathcal{F}_t] \\
&= \int_t^T \int_{-\frac{K}{C}}^{\frac{1-K}{C}} (K + Cy)\, g(t;s;y)\, dy\, f(s)\, ds \\
&\quad + \int_t^T \int_{\frac{1-K}{C}}^{1} g(t;s;y)\, dy\, f(s)\, ds\Bigg)
\end{aligned}
$$

$\Box$

Proof of Theorem 14. Under the last assumption, the expectation becomes

$$
\begin{aligned}
&\hat{E}^T\left[\int_t^T (K + C\log\frac{V(u)}{B(u)}) f(t;u)\, du \,\Big|\, \mathcal{F}_t\right] \\
&= K\left(1 - e^{-\int_t^T \lambda_\tau(u)\, du}\right) + C\hat{E}^T\left[\int_t^T \log\frac{V(u)}{B(u)} f(t;u)\, du \,\Big|\, \mathcal{F}_t\right] \\
&= K\left(1 - e^{-\int_t^T \lambda_\tau(u)\, du}\right) + C\log\frac{V(t)}{B(t)}\left(1 - e^{-\int_t^T \lambda_\tau(u)\, du}\right) \\
&\quad + C\hat{E}^T\left[\int_t^T \left(\log\frac{V(u)}{B(u)} - \log\frac{V(t)}{B(t)}\right) f(t;u)\, du \,\Big|\, \mathcal{F}_t\right] \\
&= \Delta(t)\left(1 - e^{-\int_t^T \lambda_\tau(u)\, du}\right) + C\hat{E}^T\left[\int_t^T \left(\log\frac{V(u)}{B(u)} - \log\frac{V(t)}{B(t)}\right) f(t;u)\, du \,\Big|\, \mathcal{F}_t\right]
\end{aligned}
$$

Examining only the last expectation and substituting for the firm value, we see

$$
\begin{aligned}
&\int_t^T \left(\log\frac{V(u)}{B(u)} - \log\frac{V(t)}{B(t)}\right) f(t;u)\,du \\
&= \int_t^T \left\{ \int_t^u S_0(v)\sigma_2(v)\,dv + \int_t^u \sigma_2(v)\,d\hat{W}^T(v) - \frac{1}{2}\int_t^u \|\sigma_2(v)\|^2\,dv \right\} f(t;u)\,du \\
&= \int_t^T \int_t^u S_0(v)\sigma_2(v)\,dv\, f(t;u)\,du + \int_t^T \int_t^u \sigma_2(v)\,d\hat{W}^T(v)\, f(t;u)\,du \\
&\quad - \frac{1}{2}\int_t^T \int_t^u \|\sigma_2(v)\|^2\,dv\, f(t;u)\,du
\end{aligned}
$$

Changing the order of integration yields

$$
\begin{aligned}
&= \int_t^T \int_v^T S_0(v)\sigma_2(v) f(t;u)\,du\,dv + \int_t^T \int_v^T \sigma_2(v) f(t;u)\,du\,d\hat{W}^T(v) \\
&\quad - \frac{1}{2}\int_t^T \int_v^T \|\sigma_2(v)\|^2 f(t;u)\,du\,dv
\end{aligned}
$$

We define

$$
\begin{aligned}
F(t;v) :&= \int_v^T f(t;u)\,du \\
&= -e^{-\int_t^T \lambda^\tau(s)\,ds} + e^{-\int_t^v \lambda^\tau(s)\,ds} \\
&= P[\tau > v|\mathcal{F}_t] - P[\tau > T|\mathcal{F}_t] \\
&= P[\tau \in]v;T]\,|\mathcal{F}_t];
\end{aligned}
$$

so that $F(t;v)$ can be interpreted as the default probability between v and maturity T. Then

$$
\begin{aligned}
&\hat{E}^T\left[\int_t^T \left(\log\frac{V(u)}{B(u)} - \log\frac{V(t)}{B(t)}\right) f(t;u)\,du \,\middle|\, \mathcal{F}_t\right] \\
&= \hat{E}^T\left[\int_t^T F(t;v)S_0(v)\sigma_2(v)\,dv + \int_t^T F(t;v)\sigma_2(v)\,d\hat{W}^T(v) - \frac{1}{2}\int_t^T F(t;v)\|\sigma_2(v)\|^2\,dv \,\middle|\, \mathcal{F}_t\right]
\end{aligned}
$$

Because of the conditional expectation, the martingale parts drop out:

$$
= \int_t^T F(t;v)S_0(v)\sigma_2(v)\,dv - \frac{1}{2}\int_t^T F(t;v)\|\sigma_2(v)\|^2\,dv
$$

Here, again we have used the assumption that volatilities are deterministic.

$\square$

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