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This paper offers an option pricing framework grounded in econometric microstructure modelling. We consider a model where stock price dynamics follow a pure jump process with constant jump size similar to a binomial setting with random time steps. Jump arrival times are described as an Autoregressive Conditional Duration (ACD) process while conditional probabilities of up-moves and down-moves are given by the logistic transformation of an autoregressive process. We derive no-arbitrage pricing formulae under the minimal martingale measure and illustrate the use of our Autoregressive Conditional Binomial (ACB) option pricing model on intraday IBM stock data.

Nh| zrugv= incomplete market, option pricing, minimal martingale measure, marked point process, microstructure, high frequency data, ACD model, volatility smile.

MHO Fodvvlilfdwlrq= C41, D52, G13.

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1 Introduction

On financial markets, option traders typically readjust their hedging portfolios when the underlying stock price has moved by a given percentage. This trading rule implies that rebalancing occurs at random times and that the Black-Scholes (1973) model which relies on the assumption of continuous rebalancing is no longer appropriate. Although this continuity assumption is clearly unrealistic for transaction cost reasons, because prices are quoted in ticks, or because of the mere impossibility of continuous trading, it has received comparatively less academic attention than other assumptions such as constant volatility.

As a discrete approximation to the continuous time model of Black and Scholes, the binomial tree of Cox, Ross and Rubinstein (1979) is widely employed by practitioners. Thanks to its elegant simplicity it is used as an introductory example to pricing theory in most finance textbooks. The simple structure of this model unfortunately yields its major drawback. Observed price variations do not follow i.i.d. binomial variables and the model cannot cope well with empirical data.

The purpose of this paper is to introduce an easily implementable pricing methodology which captures some of the salient features of observed market data and trading behaviour while enabling to derive option prices and deltas consistent with the trading rule described above. To this aim, we propose a model of discrete trading where market participants rebalance their position at random times triggered by variations in the underlying stock price by a given percentage a. In essence we introduce a binomial tree with random time spacings and probabilities in which the size of price changes is kept fixed. The structure of the traditional binomial pricing model is modified in order to improve its empirical fit and relax some of its unrealistic assumptions.

First we relax the stringent constraint of a fixed time interval between two successive price variations. The random intervals between two arrival times are called durations. Their conditional expectations depend on past durations in an autoregressive way. Such modelling for high frequency transaction data has been proposed by Engle and Russell (1997,1998), and has already proved successful in examining empirical predictions of microstructure theory (see O'Hara (1995) for a thorough exposition of microstrucure theory) on how the frequency of transactions (clustering) should carry information about the state of the market (see also Engle (1996)).

Second we allow for a similar past dependence in the probabilities of up-moves and down-moves. We therefore acknowledge the possibility that the direction of previous price jumps may influence forthcoming variations in the stock price. Such a relationship can occur if traders have a herding behaviour (positive relationship) or if a deviation from the fundamental value of the stock tends to be corrected by market participants (negative relationship). The autoregressive specification for up-move probabilities mainly follows the framework of Cox (1970, 1981) and its recent extension by Russell and Engle (1998). It consists of a pure time series version of the specification adopted by Hausman, Lo and MacKinlay (1992) for analysing transaction stock prices. This joint dynamic modelling of the price transition probabilities and the arrival times of the transactions is rich and flexible enough to capture the historical behaviour of price change data.

The structure of the paper is the following. We first review in section 2 some basic concepts about marked point processes (MPP) which embody our binomial tree specification. We outline our framework and introduce the main notations and mathematical tools used in the modelling. Dynamic specifications for the arrival times and the jumps (namely the ACD and ACB models) are discussed in some detail. These models will be implemented in the empirical application. Section 3 provides an introduction to the minimal martingale measure (MMM), before giving the option pricing formula based on it. The minimal martingale measure is the main building block of our pricing strategy. Derivative asset prices are obtained by taking discounted expectations of future payoffs with respect to this measure. The option pricing formulae rely on pricing tools derived in Prigent, Renault and Scaillet (1999). In Section 4, an empirical application is provided using IBM intraday transaction data on the NYSE (New York Stock Exchange). Model parameters are estimated and used as input to compute European call option prices. These prices are compared with Black-Scholes prices based on an historical volatility estimate from daily closing prices. The model is able to capture the shape of the volatility smile usually observed on stock option markets. Section 5 concludes.

2 Framework

We first start by reviewing some basic facts about marked point processes (MPP). The initials JS stand for the book of Jacod and Shiryaev (1987), which gathers major contributions to the theory of MPPs.

Let us consider an increasing sequence of non overlapping random times T , $j = 1, ...$ To each of them, we associate a random variable Z , called a mark and defined on the same probability space : $(\Omega, \mathcal{F}, P)$. Each (T , Z) is said to be a marked point and the sequence $((T , Z))$ of marked points is referred to as a marked point process (see Figure 1).

There are therefore three elements which caracterize a MPP. The law of arrival times of the jumps, the size (or amplitude) of the marks and their "direction" (either up or down). In our framework, we restrict the mark size to be constant and equal to a and denote the mark space by $E = \{a, -a\}$, therefore leaving two elements to be specified.

We assume that the logarithm of the stock price $S_|$ follows such a MPP so that :

$$S_| = S_0 e^{\mathsf{f}\, t}, \tag{1}$$

where the random variable :

$$X_| = \sum_{:\mathsf{A}_j \leq |} Z \ , \tag{2}$$

corresponds to the sum of jumps taken over the random times T of their arrivals. The process X is a purely discontinuous process with jumps $Z = \Delta X_{\mathsf{A}_j}$. The logarithmic variations of the stock price : $\Delta \log S_{\mathsf{A}_j} = \Delta X_{\mathsf{A}_j}$ thus either take the values a or $-a$.

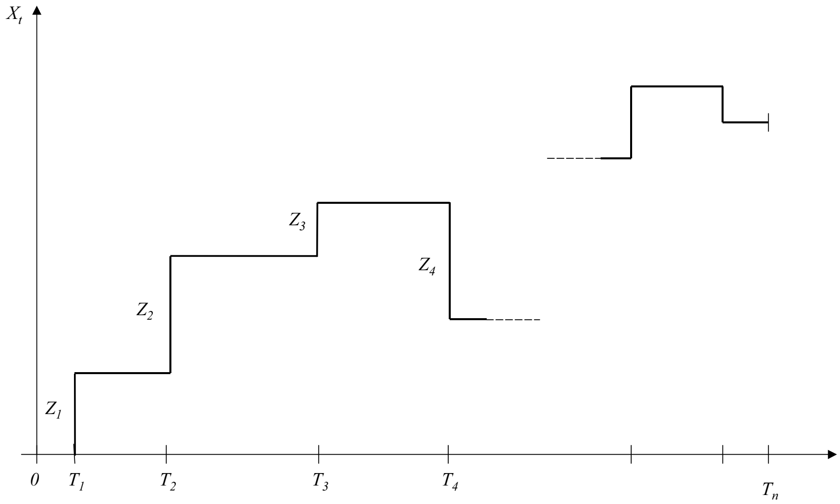

Figure 1: A typical realisation of a marked point process.

The process X can be written as the sum of jumps over time and over the mark space (see JS p. 69-72) :

$$X_{|} = \int_0^{|} \int_{.} x\mu(dt, dx)$$

The random measure $\mu(dt, dx)$ is the counting measure associated to the marked point process. This measure records the number of jumps occuring in the time interval dt and whose size falls in the interval dx (here $dx = a$ or $-a$). Introducing the predictable measure ν, called the compensator, with the property that $\mu - \nu$ is a local martingale measure, we can write :

$$X_{|} = \int_0^{|} \int_{.} x\nu(dt, dx) + \int_0^{|} \int_{.} x(\mu - \nu)(dt, dx). \tag{3}$$

Although this rewritting may look only technical, it enables us to see the crucial role of the compensator in our pricing model. We will be concerned with the expected returns on the stock price or equivalently with the expected value of X. By definition of the compensator, we know that the second term in (3) is a martingale, only leaving the first term to specify and estimate from the data.

Recall that two elements were left to fully specify our marked point process : the law of arrival times and the probability of an up-move (the jump size a being held constant). The compensator can be disintegrated (JS p. 67) so that these two elements are clearly identified :

$$\nu(dt, dx) = d\Lambda_{|} K(t, dx),$$

where Λ is a predictable integrable increasing process and K is a transition kernel.

The process Λ represents the intensity of the arrival times of jumps, and when $d\Lambda_{|} = \lambda_{|} dt$, the process λ is called the directing intensity and corresponds to a conditional hazard function. The transition kernel $K(t, dx)$ is given by : $\sum \mathbb{1}_{\{\mathsf{A}_j \square | \leq \mathsf{A}_{j+1}\}} P(Z_{\ +1} \in dx | \mathcal{F}_{\mathsf{A}_j}, T_{\ +1})$ i.e. the conditional probability of an up-jump (if $x = a$) or a down-jump (if $x = -a$) given the current information set $\mathcal{F}_{\mathsf{A}_j}$ (made of current and past realisations of the MPP) and the fact that there is a jump at time $T_{\ +1}$.

Note that the standard Poisson process corresponds to the constant directing intensity case : $d\Lambda_{|} = \lambda dt$. Its jumps are always 1, so all the probability is concentrated on the point $dx = 1$ which implies that the compensator can be written :

$$\nu(dt, dx) = \lambda dt \epsilon_1(dx),$$

where $\epsilon_1(dx)$ denotes the Dirac measure at point 1 with property $\epsilon_1(dx) = 1$ for $dx = 1$ and $\epsilon_1(dx) = 0$ otherwise.

Let us summarize what we have obtained so far. We have a model where the log of the stock price follows a marked point process with constant jump size a. We have shown the usefulness of the compensator which enables us to calculate expected price variations. This measure can be broken down into two parts, one of which is the intensity of jump arrival times and the second which determines the up-move probability. The model is fully specified up to the choice of these two terms which we will now discuss.

In Prigent, Renault and Scaillet (1999), we studied two specifications for the compensator. We considered the case where the true stock price is an unobservable geometric Brownian motion whose values were only known when its logarithm crossed boundaries spaced by a. Although intuitive from a continuous time finance point of view, this specification was not appropriate for empirical purposes because of the cumbersome and restrictive form of the kernel $K(t, dx)$. A marked Poisson model (i.e. a Poisson process whose jumps follow i.i.d. binomial variables) was then proposed. Both parameters (the directing intensity λ and the up-move probability p) were easily estimated on IBM transaction data and prices were derived for European call options on IBM stock for values of a between 3% and 5%. The marked Poisson specification was in the same spirit as the stochastic volatility model proposed by Bossaerts, Ghysels and Gourieroux (1996), and based on a time deformed binomial model.

However, we noticed that the hypothesis of exponentially distributed inter-trade durations was rejected for small values of a ($< 3\%$). This was due to the phenomenon of overdispersion of durations (the standard deviation of the durations exceeds their mean) in intraday data, a well documented fact in the microstructure literature. Besides, a constant up-move probability appears not to be a satisfactory assumption as already mentioned because of a possible herding behaviour of traders or, on the contrary, because of a tendancy to revert towards the fundamental value of the stock.

We will now tackle these two issues by turning to other possible specifications of the compensator which will be easily estimable and testable from market data. These specifications will also allow to derive expressions for option prices. Building on the econometrics of high frequency data (ENGLE (1996)), we have chosen to use autoregressive conditional specifications for the conditional distributions of both arrival times and marks.

2.1 Conditional Distribution of Durations

The conditional distribution of arrival times is specified according to an ACD(m,q) model proposed by ENGLE and RUSSELL (1997, 1998). The Autoregressive Conditional Duration (ACD) class of models consists in assuming that the durations $d_{\ +1} = T_{\ +1} - T_{\ }$ are such that :

$$d_{\ +1} = \psi_{\ +1}\xi_{\ +1},$$

where $\xi_{\ }$ are positive i.i.d. variables and the conditional expectation : $\psi_{\ } = E\left[d_{\ } | \mathcal{F}_{\mathsf{A}_{j-1}}\right]$ is :

$$\psi_{\ } = \omega + \sum_{\&=1}^{6} \alpha_{\&} d_{\ -\&} + \sum_{\&=1}^{\wedge} \beta_{\&} \psi_{\ -\&}.$$

These models are analogous to ARCH and GARCH models (ENGLE (1982), BOLLERSLEV (1986)) and share many of the same properties. For example, some constraints must be satisfied by the parameters, specifically :

$$\omega > 0, \alpha_{\&} \geq 0 \text{ and } \beta_{\&} \geq 0\,^{2}, \tag{4}$$

in order to ensure the positivity of the durations, and

$$\sum\nolimits_{\&=1}^{6} \alpha_{\&} + \sum\nolimits_{\&=1}^{\wedge} \beta_{\&} < 1, \tag{5}$$

to guarantee model stationarity and allow to use Maximum Likelihood (see ENGLE and RUSSELL (1998), CARRASCO and CHEN (1999)). The conditional hazard function of an ACD model for t in the random time interval $]]T_{\ }, T_{\ +1}]]$ is given by :

$$\lambda_{|} = \psi_{\ +1}^{-1} \lambda_0 \left(\frac{t - T_{\ }}{\psi_{\ +1}} \right),$$

where λ_0 is the baseline hazard of ξ (the ratio of the density and survival functions of ξ). Two choices are usually adopted for the distribution of ξ, either the exponential or the Weibull, which give respectively :

$$\lambda_{|} = \psi_{\ +1}^{-1},$$

or :

$$\lambda_{|} = \left(\psi_{\ +1}^{-1} \Gamma(1 + 1/\gamma) \right)^{\ } (t - T_{\ })^{\ -1} \gamma, \tag{6}$$

where Γ is the gamma function and γ is the second Weibull parameter. The first Weibull parameter must be equal to $\Gamma(1+1/\gamma)$ in order to ensure that ξ has mean 1. When $\gamma = 1$, the Weibull distribution coincides with the exponential distribution.

[2] The positivity of the parameters is a sufficient but not necessary condition to ensure that ψ_i be positive. These constraints can be weakened as shown by NELSON AND CAO (1992).

In trying to fit the above ACD model to IBM transaction data, we found that the constraints on the parameters ((4) and (5)) were not always satisfied. We thus propose to use the Log-ACD model proposed by Bauwens and Giot (1999) (see also Russell and Engle (1998)) which is the analogue of the Log-GARCH model of Geweke (1996) applied to durations. It is specified as :

$$d_{\ +1} = \exp\left(\psi_{\ +1}\right)\xi_{\ +1},$$

with a conditional expectation $\exp\left(\psi_{\ }\right)$ satisfying :

$$\psi_{\ } = \omega + \sum_{\&=1}^{6} \alpha_{\&} \ln\left(d_{\ -\&}\right) + \sum_{\&=1}^{\wedge} \beta_{\&}\psi_{\ -\&}.$$

and its conditional hazard function in the random time interval $]]T_{\ }, T_{\ +1}]]$ is given by :

$$\lambda_{|} = \exp\left(-\psi_{\ +1}\right),$$

for the exponential case and

$$\lambda_{|} = \left(\exp\left(-\psi_{\ +1}\right)\Gamma(1+1/\gamma)\right)^{\ } \left(t - T_{\ }\right)^{\ -1}\gamma,$$

for the Weibull case.

2.2 Conditional distribution of marks

Concerning the conditional distribution of marks we use the extension of the logistic linear model of Cox (1970,1981) given by Russell and Engle (1998). We define :

$$Y_{\ } = \begin{cases} 1 & \text{if} \quad Z_{\ } = +a, \\ 0 & \text{if} \quad Z_{\ } = -a. \end{cases}$$

The probability $\pi_{\ } = P[Y_{\ } = 1|\mathcal{F}_{\mathsf{A}_{j-1}}]$, resp. $1-\pi_{\ }$, gives the conditional probability of an up-move, resp. a down-move. In a logistic linear model, it satisfies :

$$l(\pi_{\ }) = \bar{\omega} + \sum_{\&=1}^{6} \bar{\alpha}_{\&} Y_{\ -\&}$$

with $l(\pi) = \log\left(\pi/(1-\pi)\right)$. The logistic transformation l ensures the interpretation of π as a probability. Russell and Engle (1998) propose to extend this specification by incorporating lagged values of the conditional probabilities themselves :

$$l(\pi_{\ }) = \bar{\omega} + \sum_{\&=1}^{6} \bar{\alpha}_{\&} Y_{\ -\&} + \sum_{\&=1}^{\mathsf{R}} \bar{\beta}_{\&}\pi_{\ -\&} + \sum_{\&=1}^{\mathsf{\breve{O}}} \bar{\kappa}_{\&} l(\pi_{\ -\&}).$$

This model has an Autoregressive Conditional Binomial (ACB) structure and is a binomial version of the Autoregressive Conditional Multinomial (ACM) model of Russell and

ENGLE (1998). We refer to it as an ACB($\bar{m}$,$\bar{p}$,$\bar{r}$) model. The transition kernel of the MPP is taken for $t \in]]T\ , T\ _{+1}]]$ equal to :

$$K(t, dx) = \begin{cases} \pi\ _{+1} & \text{for} \quad dx = +a, \\ 1 - \pi\ _{+1} & \text{for} \quad dx = -a. \end{cases}$$

Both autoregressive conditional models, ACD for the arrival times and ACB for the marks, are thus the building blocks of our specification for the stock price dynamics. This is summarized by the next assumption.

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The compensator $\nu(dt, dx)$ on $I\!R_+ \times \{a, -a\}$ satisfies :

$$\nu(dt, dx) = \lambda_{|} dt K(t, dx),$$

where for $t \in]]T\ , T\ _{+1}]]$, the directing intensity $\lambda_{|}$ is given by the conditional hazard function of an ACD(m,q) model and the transition kernel $K(t, dx)$ corresponds to an ACB($\bar{m}$,$\bar{q}$,$\bar{r}$) model.

Now that the setting is described we may turn to the next step : the choice of an equivalent martingale measure and the derivation of an option pricing formula.

3 Option pricing and the Minimal Martingale Measure

From HARRISON and KREPS (1979) and HARRISON and PLISKA (1981), we know that in order to preclude arbitrage in a market, there must exist an equivalent martingale measure (EMM) under which discounted asset prices are martingales. However, this measure needs not be unique unless markets are complete.

The presence of jumps in our framework implies that the market is incomplete. We therefore need a criterion by which to choose among all EMM, one measure under which to calculate option prices as expectations of discounted future payoffs. In this paper we have chosen the minimal martingale measure (MMM) initially proposed by FÖLLMER and SCHWEIZER (1991). Let us first recall some results about pricing in incomplete markets before motivating our choice of measure. We consider for simplicity the case of a market with one risky asset (the stock with price $S_{|}$) and one riskless asset whose growth rate is set to zero.

The standard approach to pricing by arbitrage consists of finding a portfolio (i.e. an investment policy of α in the riskless asset and β in the stock) which replicates the payoffs H of the option we want to price. Let $V_{|} = \alpha_{|} + \beta_{|} S_{|}$ denote the value of our hedging portfolio. The cost process of following a trading strategy from time 0 to time t is $C_{|} = V_{|} - V_0 - \int_0^{|} \beta\ dS$

Recall that a trading strategy is said to be self-financing if the cost is 0. When markets are incomplete, no self-financing strategy will provide a perfect hedge ($V_{\mathsf{A}} = H$ a.s.) for the option, or conversely, if we adopt a strategy which replicates the payoff of the contingent claim perfectly, it will in general not be self-financing. The natural way forward is to try to minimize some definition of the remaining risk or equivalently the cost associated with the replicating strategy.

One possibility is to choose a strategy (α^*, β^*) which minimizes the total risk under the historical measure P as defined by $R_t = E\left[(C_T - C_t)^2 \middle| \mathcal{F}_t\right]$. This corresponds to the choice of pricing under the variance optimal measure (for this measure see e.g. Föllmer and Sondermann (1986), Bouleau and Lamberton (1989), Duffie and Richardson (1991), Schweizer (1992,1994), Gouriéroux, Laurent and Pham (1998), Laurent and Scaillet (1998)). This strategy is mean self-financing, i.e., on average, its cost is zero. However the associated measure has two main drawbacks. First, an optimal (i.e. minimizing the total risk) strategy does not always exist. Second, the variance optimal measure does not have an analytic form in general and is therefore unpractical.

Another possibility is to adopt another (also self-financing) strategy which minimizes the local risk in the sense of Schweizer (1991). Instead of considering the total variation of the cost between dates t and T, it consists in minimizing all the variations of the costs over successive "small" periods between t and T (see Frey (1997)).

This policy corresponds to the choice of the minimal martingale measure which we will be using in this paper. This measure is characterized by the fact that it sets to zero all risk premia on sources of risk orthogonal to the martingale part of the underlying's price process. An example, borrowed from Hofmann, Platen and Schweizer (1992), will help understand this statement. One of the most famous models of option pricing with stochastic volatility has been proposed by Hull and White (1987). It is particularly convenient to model volatility smiles (see Renault and Touzi (1996)). The authors assume that the square of the volatility follows a geometric Brownian motion which is uncorrelated to the Brownian motion driving the price process. The market is incomplete because there are two sources of risk and only one risky asset to trade with. However Hull and White argue that if one assumes that the CAPM holds and that volatility risk is diversifiable, this source of uncertainty should not bear a risk premium. They then proceed to derive their option price. This price coincides with that given by the MMM which precisely assigns a zero value to the market price of risks orthogonal to the martingale part (here the Brownian motion) of the price.

The MMM has several appealing features which motivate its use in practical applications. First, there always exists an explicit form for the Radon-Nikodym derivative enabling to switch from the historical probability measure to the minimal measure. This makes it a computationally convenient tool as will become apparent later on. Then, recent work on the topic shows that it induces good convergence properties (Runggaldier and Schweizer (1995), Prigent (1995), Mercurio and Vorst (1996), Lesne, Prigent and Scaillet (1998)). Furthermore, in our framework (see Prigent, Renault and Scaillet (1999)), jump boundedness ensure that the MMM is a probability measure (i.e. is always positive) and therefore that the value of the trading strategy is an actual no-arbitrage price.

Finally this measure can also be linked to other possible choices of measures. For example Schweizer (1993) shows that in some cases the expectation of the final payoff under the minimal measure is equal to the value of the variance optimal hedging strategy (α^*, β^*) described above. When the mean-variance trade-off (i.e. the market price of risk) of the price process is deterministic, the MMM is the closest of all EMM to the historical measure P, as measured by the relative entropy criterion (FÖLLMER and SCHWEIZER (1991)). Concerning existence and uniqueness of the minimal measure, we refer to ANSEL and STRICKER (1992, 1993).

We now turn to the derivation of the option pricing formula under the minimal measure. We take as discount factor (or numéraire) a savings account whose growth rate r_{A_j} on the random time interval : $]]T_{-1}, T\,]]$ satisfies :

$$r_{\mathsf{A}_j} = e^{4(\mathsf{A}_j - \mathsf{A}_{j-1})} - 1, \tag{7}$$

with $\rho > 0$.

The discounted stock price is then equal to :

$$\tilde{S}_{|} = S_{|} / \prod_{:\mathsf{A}_j \leq |} (1 + r_{\mathsf{A}_j}).$$

Let us introduce the discounted excess return process δ such that for $t \in]]T\,, T_{+1}]]$:

$$\delta(t, x) = \delta(T\,, Z\,) = \frac{e^{\sim j} - \left(1 + r_{\mathsf{A}_j}\right)}{1 + r_{\mathsf{A}_j}} = e^{\sim j - 4(\mathsf{A}_j - \mathsf{A}_{j-1})} - 1. \tag{8}$$

The minimal measure is characterized by its Radon-Nicodym derivative w.r.t. P. It takes the following form in our framework (PRIGENT, RENAULT and SCAILLET (1999)).

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Under Assumption 1, the minimal martingale measure $\hat{P}$ is a probability measure characterized by its density process $\hat{\eta}$ relative to P :

$$\hat{\eta}_{|} = \prod_{:0\square\mathsf{A}_j \leq |} (1 + \hat{h}\,(Z\,)) \exp\left(-\int_0^{|} \int_{.} \hat{H}(s, x) K(s, dx) \lambda_{\mathsf{r}} ds\right). \tag{9}$$

where for $t \in]]T\,, T_{+1}]]$, $\hat{H}(t, x) = \hat{h}\,(x)$ with :

$$\hat{h}\,(x) = -\frac{\delta(T\,, a)\pi_{\,+1} + \delta(T\,, -a)(1 - \pi_{\,+1})}{\delta^2(T\,, a)\pi_{\,+1} + \delta^2(T\,, -a)(1 - \pi_{\,+1})} \delta(T\,, x).$$

$\hat{H}(t, x)$ takes the interpretation of a jump risk premium process. Once the Radon-Nicodym derivative $\hat{\eta}_{|}$ is computed, it is straightforward to derive the price $C_{|} = C(t, S_{|})$ of a contingent claim with final payoff $C(T, S_{\mathsf{A}})$. For a European call option with maturity T and strike price $\overline{K}$, the final payoff is $(S_{\mathsf{A}} - \overline{K})_+ = \max(0, S_{\mathsf{A}} - \overline{K})$. By taking its expectation under $\hat{P}$ after an adequate discounting, we get :

$$C(t, S_{|}) = E^{\hat{}} \left[(S_{\mathsf{A}} - \overline{K})_+ \prod_{:|\square\mathsf{A}_j \leq \mathsf{A}} (1 + r_{\mathsf{A}_j})^{-1} | \mathcal{F}_{|} \right], \tag{10}$$

which leads to :

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Under Assumption 1, the call price given by the minimal martingale measure is :

$$C(t, S_{|}) = E \quad \left[(S_{\mathsf{A}} - \overline{K})_{+} \frac{\hat{\eta}_{\mathsf{A}}}{\hat{\eta}_{|}} \prod_{:|\square \mathsf{A}_j \leq \mathsf{A}} (1 + r_{\mathsf{A}_j})^{-1} | \mathcal{F}_{|} \right]. \tag{11}$$

If the exponential or the Weibull distribution underlies the ACD model, we have :

$$\begin{aligned} \frac{\hat{\eta}_{\mathsf{A}}}{\hat{\eta}_{|}} \quad = \quad & \prod_{:|\square \mathsf{A}_j \leq \mathsf{A}} \left(1 - \frac{\delta(T \ , a)\pi \ _{+1} + \delta(T \ , -a)(1 - \pi \ _{+1})}{\delta^2(T \ , a)\pi \ _{+1} + \delta^2(T \ , -a)(1 - \pi \ _{+1})} \delta(T \ , Z \ ) \right) \\ & \exp \left(- \frac{(\delta(T \ , a)\pi \ _{+1} + \delta(T \ , -a)(1 - \pi \ _{+1}))^2}{\delta^2(T \ , a)\pi \ _{+1} + \delta^2(T \ , -a)(1 - \pi \ _{+1})} \log G(\min(T \ _{+1}, T) - T \ ) \right), \end{aligned} \tag{12}$$

where $G(u)$ is the survival function (i.e. the probability of not jumping over the period u). $-\log G(\min(T \ _{+1}, T) - T \ )$ is equal to $(\min(T \ _{+1}, T) - T \ ) / \psi \ _{+1}$ for the exponential ACD, to $(\min(T \ _{+1}, T) - T \ ) / \exp(\psi \ _{+1})$ for the exponential Log-ACD. For Weibull distributed noise $-\log G(\min(T \ _{+1}, T) - T \ )$ is equal to $(\Gamma(1 + 1/\gamma)(\min(T \ _{+1}, T) - T \ ) / \psi \ _{+1})$ for the ACD model and to $(\Gamma(1 + 1/\gamma)(\min(T \ _{+1}, T) - T \ ) / \exp(\psi \ _{+1}))$ for the Log-ACD model.

Expectation (11) can in principle be valuated by Monte-Carlo integration. Indeed S_{A}, $\hat{\eta}_{\mathsf{A}} / \hat{\eta}_{|}$, and (r_{A_j}) can be computed from simulated paths of the MPP $((T \ , Z \ ))$ once the parameters of the ACD and ACB models have been estimated. However, since $\hat{\eta}_{\mathsf{A}} / \hat{\eta}_{|}$ is made of a product of terms, it will not be accurately estimated through simulations. Therefore it is wiser to use expression (10) and work directly with the dynamics of $S_{|}$ under $\hat{P}$. The process of $S_{|}$ under $\hat{P}$ can be derived using relationships between the directing intensities and transition kernels under P and $\hat{P}$ (see appendix). These relationships come from a direct application of Girsanov theorem for jumps (JS p.157).

4 An empirical illustration on IBM trades

This section illustrates the empirical application of our marked point model to the pricing of European call options.[3] The parameters of the ACD and ACB models are estimated from intraday transaction data. The data were extracted from the Trades and Quotes Database (TAQ Database) released by the NYSE and span the period beginning on Thursday January 2th 1997 and ending on Wednesday September 30th 1997 (9 months). We report results obtained on trades of the IBM stock, which is one of the most liquid stocks in this market and is the support of actively traded options.

The observations are the trades recorded every second from market opening (9:30:00) to market closure (16:00:00). The trades dataset consists of 498,692 transactions. We removed all trades which took place outside market opening hours and were left with

[3] Gauss programs developed for this section are available on request.

486, 506 data points. We also adjusted our series for a 2:1 stock split which occurred on 28th May 1997 before market opening.

We now proceed to estimate the intensity and the kernel, namely the ACD and ACB models. Given Assumption 1, the likelihood function is separable as in Russell and Engle (1998) and we can estimate the parameters of the two models separately.

4.1 Estimation of the Log-ACD model

We first consider the dynamics of durations between trades. As mentioned above, we have chosen a Logarithmic Autoregressive Conditional Duration specification. Recall that in a Log-ACD model durations d are assumed to follow :

$$d_{\ +1} = \exp\left(\psi_{\ +1}\right) \xi_{\ +1},$$

where ξ are i.i.d. variables (typically exponential for the Log-EACD model and Weibull for the Log-WACD model) and $\psi = \omega + \sum_{\&=1}^{6} \alpha_{\&} \ln\left(d_{\ -\&}\right) + \sum_{\&=1}^{\wedge} \beta_{\&} \psi_{\ -\&}$.

Parameter estimates are obtained by maximum likelihood for $m = 1, 2$ and $q = 1, 2$ and both specifications of the distribution of ξ (exponential or Weibull).

The most successful specification is the Log-WACD(1,1) whose parameter estimates are given in Table 1. Parametrizations with more lags are rejected by a likelihood ratio test at the 5% level and the hypothesis of exponentially distributed ξ is strongly rejected (γ is ranging from 0.40 to 0.48 for different jump sizes and is statistically different from unity).

Wdeoh 4 = Sdudphwhu hvwlpdwhv ri wkh Orj0ZDFG+4/4, prgho

a	0.5%	1%	2%
ω	0.6111*	0.8607*	2.3173*
α	0.1844*	0.1701*	0.2293*
β	0.7740*	0.7705*	0.5912*
γ	0.4806*	0.4180*	0.3998*
$LB(40)$	21.0283**	13.4486**	9.5661**
$BI-1$	3.2917	2.8791	0.3300
$BI-2$	1.7112	2.3959	1.0351

* significant at 1% level, ** significant at 5% level.

These results are in line with what was previously documented in tick-by-tick transaction studies : durations exhibit clustering, i.e. a short time between two price variations tends to be followed by another short interval. This is suggested by Figure 2 and is confirmed by the positivity of $\widehat{\alpha}$ and $\widehat{\beta}$. Estimates of ω, α and β are all significantly different from zero at the 1% confidence level in the Log-WACD(1,1) model.

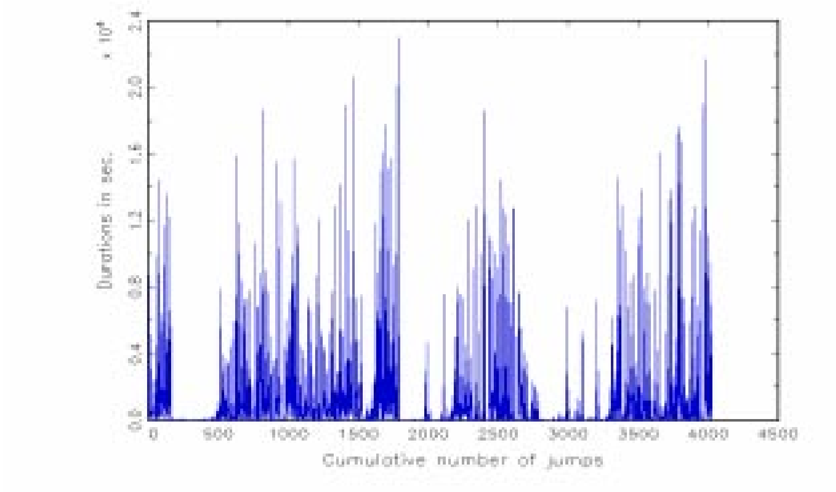

Figure 2: Durations between two 0.5% jumps.

Under our hypothesis, the residuals $\widehat{\xi}$ should be i.i.d. Weibull. We start by testing the absence of autocorrelation. $LB(40)$ reported in Table 1 denotes the Ljung-Box test with 40 lags. The hypothesis of zero autocorrelation in the residuals cannot be rejected at the 5% level for $a = 0.5\%$, $a = 1\%$ and $a = 2\%$.

We now want to test for Weibull-distributed residuals. Recall that if ξ is Weibull with parameter γ, then $(\xi\ )$ follows an exponential distribution. This is the latter hypothesis which we test using Bartlett identities tests for the exponential distribution introduced by Chesher, Dhaene, Gourieroux and Scaillet (1999).

The Bartlett Identity test of order 1 considers the equality between mean and standard deviation (overdispersion test on residuals), while the Bartlett Identity test of order 2 examines a restriction on the first three moments. Taking this second restriction into account helps gaining power against alternative specifications. Results of these tests (BI-1 and BI-2) are provided in the last two rows of Table 1. All values are below their critical $\chi^2(1)$ distribution at the 5% level $(\chi^2_{0\ 05}(1) = 3.841)$, so we cannot reject the hypothesis of Weibull distributed residuals.

4.2 Estimation of the ACB model

We now turn to the estimation of the process governing up-moves and down-moves from trades data. We have adopted the specification of Autoregressive Conditional Binomial jumps where the logistic transformation $l(.)$ of the up-move probability π is given by :

$$l(\pi\ ) = \bar{\omega} + \sum_{\&=1}^{6} \bar{\alpha}_{\&} Y_{-\&} + \sum_{\&=1}^{R} \bar{\beta}_{\&} \pi_{-\&} + \sum_{\&=1}^{\sigma} \bar{\kappa}_{\&} l(\pi_{-\&}),$$

and $Y = 1$ if the jump at the period j is an up-move and 0 otherwise.

Again, we find that a simple specification with one lag in all parameters was successful in capturing the dynamics of the kernel. Table 2 reports the parameter estimates obtained for this $ACB(1,1,1)$.

Wdeoh 5 = Sdudphwhu hvwlpdwhv ri wkh $ACB(1,1,1)$ prgho

a	0.5%	1%	2%
$\overline{\omega}$	0.3675*	0.5018*	0.3608*
$\overline{\alpha}$	−0.6420*	−0.8899*	−0.8043*
$\overline{\beta}$	0.0634*	0.1582*	0.4863**
$\bar{\kappa}$	−0.7790*	−0.6979*	−0.2827*
# jumps	3715	1065	222

* significant at 1% level, ** significant at 10% level.

All parameters but one are significant at the 1% level. The dynamics are similar for the various values of a, with the past jump and the logistic transform entering negatively in the recursion and the past probability entering positively.

We show in Figure 3 the relationship between the successive up-move probabilities π and π_{-1} setting alternatively $Y_{-1} = 0$ (down-move) or $Y_{-1} = 1$ (up-move), for $a = 0.5\%$. We can observe that π is a decreasing function of π_{-1} in both cases. It means that a high probability of an up-move will be followed by a lower probability at the next arrival time. This observation is thus not in favour of a herding behaviour of market participants. This decrease is less pronounced when a down-move ($Y_{-1} = 0$) has occurred, introducing an asymmetry in the response of the probability levels. Such an effect reinforces the mean reversion type of behaviour of the stock price since the up-move probability is comparatively higher when the stock has gone down ($Y_{-1} = 0$) than when it has gone up ($Y_{-1} = 1$). This remark also applies for other jump sizes.

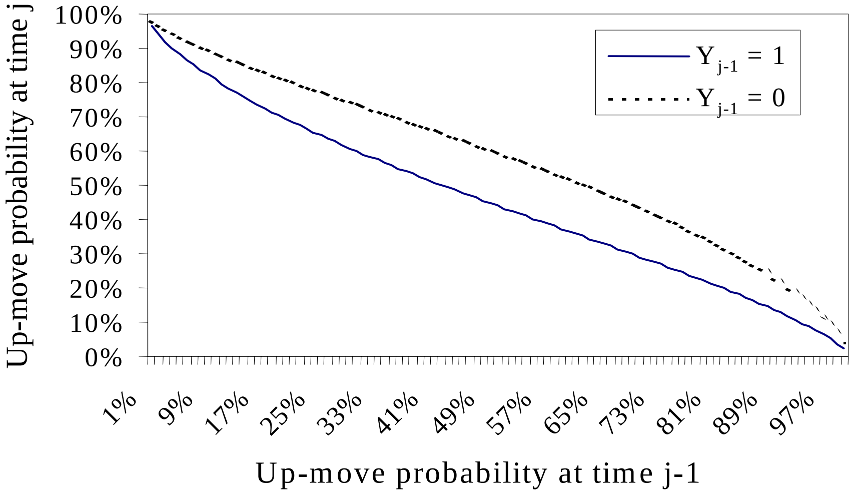

Figure 3 : Up-move probability π as a function of past probability π_{-1}

Figure 4 plots the jump risk premium $\widehat{h}\ (x)$ for $a = 0.5\%$ over the whole observation period. We can clearly see a clustering phenomenon in the risk premium. Large changes in the risk premium tend indeed to be followed by other large changes of either sign, and small changes tend to be followed by small changes. This type of clustering corresponds to the well known MANDELBROT (1963) observation on price changes (also valid for the IBM stock).

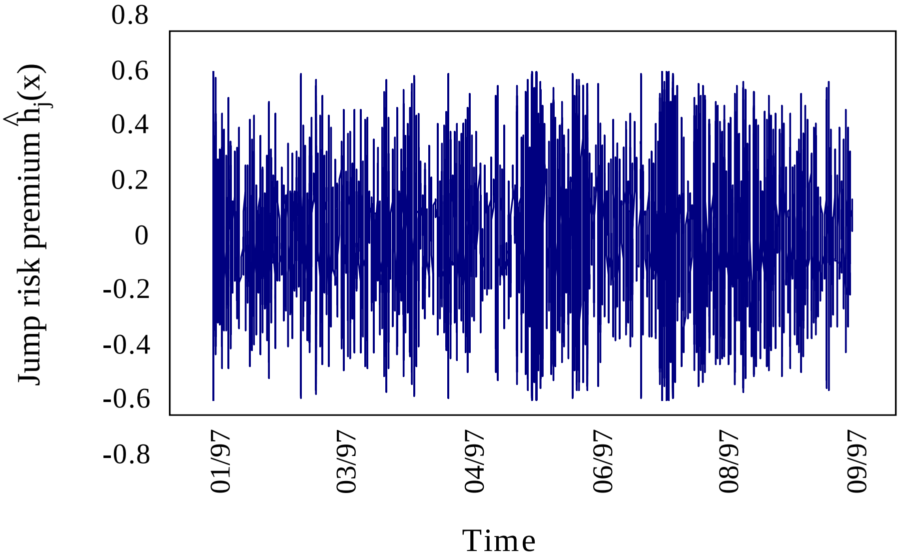

Figure 4 : Evolution of the risk premium $\widehat{h}\ (x)$

Trade prices used so far are appealing because they correspond to prices at which real transactions take place while quotes are only indicative values at which market markers are willing to trade. However, our database does not enable us to identify when a trade corresponds to a purchase of stocks from the market-maker or to a sale to the market-maker. Trade prices are thus potentially influenced by the bid-ask bounce (a trade at bid price followed by a trade at offer price leading to an observed price change although the mid price remained unchanged). This phenomenon has been reported to generate spurious autocorrelation (see ROLL (1984)) and could also lead to an over-estimation of the number of jumps for small a. Typical values for the bid-ask spread are one or two ticks ($0.125 or $0.25) as reported in the NYSE fact book which represents about 0.125% or 0.25% (the IBM stock price oscillated around $100 in our sample period). We do not believe that the bid-ask bounce should substantially affect the dynamics of arrival times of the jumps but it could bias the estimation of the ACB specification for $a = 0.5\%$ as considered in this paper. Especially the negative sign of $\overline{\alpha}$ may partially be attributable to it. However we have seen that the negative sign of $\overline{\alpha}$ persists even for large values of a where the bid-ask bounce surely is not at work. Furthermore the same methodology has been applied to mid-prices (i.e. the mean of bid and ask prices) which are free of any bid-ask bounce effect and has lead to similar results. We prefer to proceed further with trades rather than quotes because we believe it is more appropriate to work with actual transaction prices, especially for option pricing and hedging purposes.

4.3 Option pricing with the ACB model

We are now able to price options with the underlying Log-WACD and ACB models and formulae (11) and (12). We carry out Monte Carlo simulations with 100,000 replications and use S_{A} as control variate device, knowing that $S_{|} = E^{\hat{}}\left[S_{\mathsf{A}} \prod_{:|\square \mathsf{A}_j \leq \mathsf{A}} (1 + r_{\mathsf{A}_j})^{-1} | \mathcal{F}_{|}\right]$, to reduce the variance of Monte Carlo estimates.

Wde0h 6 = Rswlrq sulfhv = DFE yv E0dfn0Vfkrohv

$a \setminus \overline{K}$	90	95	100	105	110
0.5%	12.2633	8.5415	5.5638	3.4205	2.0076
1%	12.8271	9.3088	6.4706	4.3594	2.8766
2%	13.1964	9.8069	7.0430	4.9518	3.4201
BS	13.5285	10.2305	7.5030	5.3400	3.6926

Option prices based on the estimated Log-WACD(1,1) and ACB(1,1,1) models are compared with Black-Scholes prices in Table 3. Black-Scholes prices are computed using historical volatility of daily IBM stock returns taken over a 3 month period (from 2nd January to 27th March 1997). This 3 month length corresponds to market standards. Over this period the annualised volatility was 34.67%. For comparison, calculating the volatility over the whole sample (9 months) would yield an annualised volatility of 31.98%. All ACB prices are below their Black-Scholes equivalent. However this needs not be the case for all specifications. We have indeed tested the model on other data and other values of a and we have found that the Black-Scholes price cannot be seen as an upper bound on the values of the ACB model. Finally, prices generated by the ACB model can be inverted using the Black-Scholes formula in order to derive implied volatilities. Implied volatilities for various strike prices and values of a are displayed in Table 4. We find that the ACB model exhibits a volatility smile for all a which is asymetric with respect to the at-the-money level $\overline{K}$. The implied volatility ratio is here defined as the implied volatility for a given strike price $\overline{K}$ divided by the implied volatility for $\overline{K} = 100$, while moneyness is $\left(\overline{K} - S_0\right)/S_0$. The asymetry (smirk) with respect to $\overline{K}$ is usually observed on stock option markets (for recent evidence see Dumas, Fleming and Whaley (1998) or Peña, Rubio and Serna (1999)). Such a smile is present for all option maturities (ranging from one month $T = 1/12$ to one year $T = 1$) and is steeper for short dated options as shown on Figure 5. This result is in accordance with the observed fact reported in Das and Sundaram (1999), that the smile is deepest for short maturities in most markets and flattens out as the time to maturity increases.

Wdeoh 7 = Lpsolhg yrodwlolwlhv lq wkh DFE prgho

$a \setminus \overline{K}$	90	95	100	105	110
0.5%	25.44%	25.04%	24.82%	24.90%	25.16%
1%	29.77%	29.49%	29.43%	29.70%	30.18%
2%	32.40%	32.30%	32.33%	32.71%	33.19%

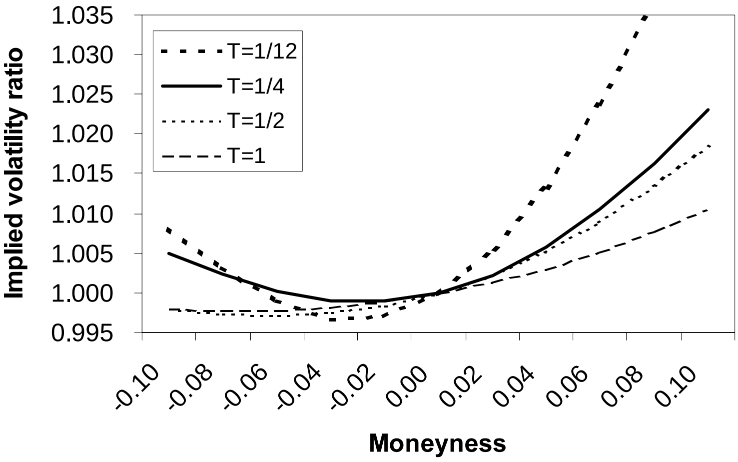

Figure 5 : Implied volatility smile for various maturities

5 Conclusion

In this paper, we have derived option-pricing formulae grounded in microstructure econometric modelling. Our results are very general and can be applied to various price dynamics which can be described as pure jump processes. This kind of process can in particular arise if traders rebalance their portfolio whenever the underlying price process has changed by a given percentage. We believe that this model is new to the literature not only because we propose new option pricing formulae for discontinuous processes but also because our approach to contingent claim pricing is essentially data-oriented (see Renault (1994) for related empirical option modelling issues). Most models since Black and Scholes (1973) have proceeded to derive their pricing formulae by first assuming a process and then pricing options based on this *a priori*. In this paper we first let the dynamics of the price process be estimated by econometric techniques and then derive option pricing formulae based on the estimated dynamics.

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6 Appendix

In this appendix, we derive the dynamics of the price process under $\widehat{P}$. This allows to use expression (10) directly. We know that the compensator under the MMM is linked to the compensator under the historical measure (see Girsanov theorem for jumps in JS p.157) by the following relation :

$$\widehat{\nu}\left(dt, dx\right) = \nu\left(dt, dx\right)\left(\widehat{h}\left(t, x\right) + 1\right),$$

where $\nu\left(dt, dx\right) = d\Lambda_{|} K\left(t, dx\right)$. Besides the compensator $\widehat{\nu}\left(dt, dx\right)$ under $\widehat{P}$ can also be disintegrated into a kernel part $\widehat{K}\left(t, dx\right)$ and an intensity component $d\widehat{\Lambda}_{|}$. Hence after identification, we deduce :

$$d\widehat{\Lambda}_{|} = \left(\int_{.} \left(\widehat{h}\left(t, x\right) + 1\right) K\left(t, dx\right)\right) d\Lambda_{|}, \qquad (13)$$

$$\widehat{K}\left(t, dx\right) = \frac{K\left(t, dx\right)\left(\widehat{h}\left(t, x\right) + 1\right)}{\int_{.} \left(\widehat{h}\left(t, x\right) + 1\right) K\left(t, dx\right)}. \qquad (14)$$

The normalisation factor $\int_{.} \left(\widehat{h}\left(t, x\right) + 1\right) K\left(t, dx\right)$ in (14) comes from the condition on the transition kernel to integrate to 1.

Thus, by using the directing intensity and transition kernel of the ACD and ACB models estimated under P, we can immediately deduce the dynamics of the stock price under $\widehat{P}$ thanks to (13) and (14).

In the specific case of Log-WACD distributed durations with ACB marks under P, we obtain under $\widehat{P}$:

$$\widehat{h}\left(T\ , x\right) = -\delta\left(T_{\ +1}, x\right) \frac{\pi_{\ +1}\delta\left(T_{\ +1}, a\right) + \left(1 - \pi_{\ +1}\right)\delta\left(T_{\ +1}, -a\right)}{\left(\pi_{\ +1}\delta^2\left(T_{\ +1}, a\right) + \left(1 - \pi_{\ +1}\right)\delta^2\left(T_{\ +1}, -a\right)\right)},$$

$$\widehat{K}\left(t, a\right) = \widehat{p}\ \left(a\right) = \frac{\pi_{\ +1}\left(\widehat{h}\left(T\ , a\right) + 1\right)}{I\left(T\ , a\right)},$$

with :

$$I\left(T\ , a\right) = 1 - \frac{\left(\pi_{\ +1}\delta\left(T_{\ +1}, a\right) + \left(1 - \pi_{\ +1}\right)\delta\left(T_{\ +1}, -a\right)\right)^2}{\left(\pi_{\ +1}\delta^2\left(T_{\ +1}, a\right) + \left(1 - \pi_{\ +1}\right)\delta^2\left(T_{\ +1}, -a\right)\right)}.$$

We know that the conditional intensity of the Log-WACD model under the historical measure is $\lambda_{|} = \left(\exp\left(-\psi_{\ +1}\right)\Gamma(1 + 1/\gamma)\right)\ \ \left(t - T\ \right)^{\ -1}\gamma$, corresponding to the process

$$d_{\ +1} = \exp\left(\psi_{\ +1}\right)\xi_{\ +1}.$$

Under the MMM, we have $\widehat{\lambda}_{|} = I\left(T\ ,a\right)\lambda_{|}$, thus corresponding to the modified process

$$d_{\ +1} = \frac{\exp\left(\psi_{\ +1}\right)}{I\left(T\ ,a\right)^{1^{*}}}\xi_{\ +1}.$$

This enables us to simulate durations directly under the MMM.