ISSN 0956-8549-806

Asset Pricing with Index Investing

By

Georgy Chabakauri

Oleg Rytchkov

DISCUSSION PAPER NO 806

PAUL WOOLLEY CENTRE WORKING PAPER No 68

June 2020

Asset Pricing with Index Investing*

Georgy Chabakauri† Oleg Rytchkov‡

Abstract

We theoretically analyze how index investing affects financial markets using a dynamic exchange economy with heterogeneous investors and two Lucas trees. We identify two effects of indexing: lockstep trading of stocks increases market volatility and stock return correlations but reduction in risk sharing decreases them. Overall, indexing decreases market volatility but has an ambiguous effect on the correlations. Also, index investing decreases an investor's welfare, but indexing by other investors partially offsets the loss. When the introduction of index trading opens financial markets for new investors, the improved risk sharing makes market returns more volatile and stock returns more correlated.

*We thank an anonymous referee, Suleyman Basak, Harjoat Bhamra, Michael Gallmeyer, Itay Goldstein, Vincent Grégoire, Burton Hollifield, Ron Kaniel, Serhiy Kozak, Dan Luo, Hanno Lustig, Dmitry Makarov, Igor Makarov, Gonzalo Morales, Anna Pavlova, Elizaveta Shevyakhova, Dimitri Vayanos, Jiang Wang, Hongjun Yan, seminar participants at Temple University, as well as participants of the 2015 Finance Down Under Conference, 2015 Financial Intermediation Research Society Conference, 11th World Congress of the Econometric Society, 2015 Financial Management Association Meeting, and 2016 SFS Finance Cavalcade for helpful comments and suggestions. Chabakauri is grateful to Paul Woolley Centre at LSE for financial support. All errors are our own.

†London School of Economics, Houghton Street, London WC2A 2AE, UK. Phone: +44 (0) 20 7107 5374. E-mail: G.Chabakauri@lse.ac.uk.

‡Fox School of Business, Temple University, 1801 Liacouras Walk, 423 Alter Hall (006-01), Philadelphia, PA 19122, USA. Phone: +1 (215) 204-4146. E-mail: rytchkov@temple.edu.

1. Introduction

Starting from the 1970s, passive index investing has been consistently gaining popularity among institutional and individual investors. According to the 2019 Investment Company Fact Book (http://www.icifactbook.org), 36 percent of households that invested in mutual funds in 2018 owned at least one equity index mutual fund. The proportion of index funds in all equity mutual fund assets increased from 8.7 percent in 1998 to 29 percent in 2018. Moreover, the funds benchmarked to the S&P 500 index managed almost 29 percent of all assets invested in index mutual funds. Index investing was initially promoted by proponents of the efficient market hypothesis (e.g., Malkiel, 1973; Samuelson, 1974) and has an increasing number of supporters due to the inability of money management industry as a whole to outperform the market (e.g., Malkiel, 1995; Fama and French, 2010; Lewellen, 2011) and high costs of active investment for society (e.g, French, 2008). It is blessed even by successful investors like Warren Buffett, who in his 2013 letter to Berkshire Hathaway shareholders argued that "the goal of the non-professional should not be to pick winners – neither he nor his 'helpers' can do that – but should rather be to own a cross-section of businesses that in aggregate are bound to do well. A low-cost S&P 500 index fund will achieve this goal."

Despite the growing popularity of index investing, its broad economic impact is not well understood. While many academics and practitioners tout indexing as the best investment strategy for ordinary investors, others raise concerns that the proliferation of index trading can increase volatility of stock returns, make the returns more correlated, and thereby hurt market participants (e.g., Wurgler, 2011; Sullivan and Xiong, 2012). The objective of our study is to assess those concerns from a theoretical perspective and provide a rigorous analysis of how index investing affects statistical properties of returns and investor welfare. We build a dynamic general equilibrium model of an exchange economy with two Lucas trees and two groups of investors dubbed type P investors (professional investors) and type I investors (index investors). We interpret the type P investors as professional market participants such as hedge funds, actively managed mutual funds, proprietary traders, etc., who can implement complex trading strategies that involve individual assets. The type I investors are unsophisticated market participants like individuals who manage their savings and retirement accounts and can trade only the market portfolio of Lucas trees (index). Consistent with our interpretation of the investors, we also assume that the type I and type P investors have different risk aversion.

The emergence of index mutual funds in the 1970s and the proliferation of exchange traded funds (ETFs) in the 2000s reduced the cost of owning well-diversified portfolios for ordinary investors and had two implications. First, some investors who held individual stocks switched to passive indexing to minimize transaction and asset management costs. Second, cheap index funds opened the equity market for many households that did not own stocks before.[1] We separately investigate those two effects by comparing our main economy with two benchmark economies. In the first benchmark, which we call the unconstrained economy, all investors can trade all assets individually. In the second benchmark, the type I investors are excluded from the market, and in each period they consume a fixed fraction of the total dividend. The type P investors can trade all assets. Such a setup provides a stylized description of financial markets before the proliferation of indexing, and it is dubbed the pre-indexing economy.

Our analysis delivers several results. Most importantly, we find that switching from trading individual stocks to index trading by a fraction of investors produces two effects: lockstep trading of the stocks, which occurs when investors trade the index, and the reduction in risk sharing produced by a smaller number of financial assets traded by some investors. The first effect increases market volatility and the correlation between stock returns, whereas the second effect decreases them. Overall, indexing decreases market volatility but can either increase or decrease the correlation depending on the state of the economy. In contrast, the possibility to trade indexes in the economy with investors who were previously excluded from financial markets increases the volatility of stock returns and the correlation between them. Those effects mainly result from improved risk sharing rather than from lockstep trading of stocks caused by indexing. We also analytically relate the differences between various equilibrium character-

[1] Index investing likely contributed to the increase in equity market participation from around 30% in the early 80s to almost 50% nowadays (e.g., Li, 2014).

istics in the economy with indexing and in the unconstrained economy to portfolio distortions produced by indexing. Finally, we conduct the welfare analysis and find that the inability to trade individual stocks decreases an investor's welfare, but indexing by other investors partially offsets the loss through improving investment opportunities.

To see the economic intuition behind those effects, consider an unconstrained economy in which all investors can trade all assets. Because the investors have different risk aversion, they trade to share risks, and risk sharing affects the statistical properties of stock returns. Assume, for example, that a positive cash flow shock hits one of the stocks. Because in equilibrium less risk-averse investors hold a larger share of their wealth in stocks than those who are more risk averse, the shock disproportionately increases wealth of the less risk averse investors and decreases the aggregate risk aversion in the economy. Thus, risk sharing produces a variation in the aggregate risk aversion, which in turn produces a common variation in the stock discount rates. As a result, the volatilities of returns become higher than the volatilities of cash flows, and the returns become correlated even when cash flows are independent.[2]

Compared to the unconstrained economy, indexing introduces two new effects. First, investors exert price pressure on all stocks simultaneously by buying and selling the market portfolio as a whole thereby increasing the volatility and correlation of returns.[3] This lockstep trading effect is responsible for the perception of indexing as a source of a positive correlation between stock returns, which is shared by practitioners (e.g., Sullivan and Xiong, 2012) and appeared in popular press.[4] Second, the risk sharing between investors, which is another source of the volatility and correlation, is lower in the economy with indexing than in the unconstrained economy. Overall, indexing decreases market volatility but has an ambiguous effect on the correlation of stock returns, which is determined by the relative strength of the two effects that work in opposite directions. To disentangle the lockstep trading and reduced risk sharing effects, we consider a modified specification of our economy dubbed a lockstep-trading econ-

[2]Xiong (2001), Kyle and Xiong (2001), Cochrane et al. (2008), Bhamra and Uppal (2009), Longstaff and Wang (2012), and Ehling and Heyerdahl-Larsen (2016) discuss in detail how risk sharing among investors affects the dynamics of stock returns.

[3]A similar effect arises in Barberis and Shleifer (2003), Basak and Pavlova (2013, 2016), and Grégoire (2020).

[4]"Simple Index Funds May Be Complicating the Markets", *The Wall Street Journal*, February 18, 2012.

omy. In this economy, the type I investors exogenously put in the index the same fraction of their wealth as they put in both stocks in the unconstrained economy and optimize only their consumption policy. Because the portfolio optimization is switched off, the type I investors do not change their allocation of wealth to stocks in response to reduced risk sharing opportunities. Thus, the differences in the equilibrium characteristics between the unconstrained and lockstep-trading economies are naturally interpreted as solely produced by lockstep trading. Overall, our analysis implies that the ongoing debates on indexing should consider not only the lockstep trading of stocks but also the general equilibrium effect of reduced risk sharing to portray a more complete picture of how indexing changes stock returns.

It is also instructive to compare the economy with indexing and the pre-indexing economy. When indexes become accessible to those investors who were initially prohibited from participating in financial markets, it opens up risk sharing which produces the effects described above. In particular, index trading inflates market volatility and makes stock returns correlated even when the fundamentals of the stocks move independently. Thus, our analysis provides a potential theoretical explanation for empirical findings that indexing can make stocks more volatile and more correlated (e.g., Vijh, 1994; Greenwood and Sosner, 2007; Boyer, 2011; Sullivan and Xiong, 2012; Leippold et al., 2016; Ben-David et al., 2018; Da and Shive, 2018; Grégoire, 2020). Because we assume that all investors are rational, our explanation substantially differs from that in Barberis and Shleifer (2003) and Barberis et al. (2005), who relate the comovement of stock returns to behavioral biases. We also find that allowing index trading in the pre-indexing economy affects the equilibrium characteristics and investors' welfare much stronger than imposing indexing constraints on some investors in the unconstrained economy. Thus, the economy with indexing is much closer to the unconstrained economy than to the pre-indexing economy, and index trading makes it possible to share risks at a highly efficient level.

Finally, we explore welfare implications of index investing. To quantify them, we use the certainty equivalent loss (CEL) of an investor from the unconstrained economy and the certainty equivalent gain (CEG) of an investor from the pre-indexing economy who become type I investors in the economy with indexing. The CEL contains two components: one is produced by the inability of the investor to arbitrarily adjust portfolio weights, and the other arises because indexing by other investors distorts investment opportunities. We find that the first component is positive (the inability to trade individual stocks decreases welfare), whereas the second component is typically negative (the change in investment opportunities increases welfare). However, the first component is larger, and the total CEL is positive, albeit small: an investor from the unconstrained economy would give up less than 0.004% of his wealth for not becoming a type I investor in the economy with indexing. In contrast, compared to the pre-indexing economy, investors are much better off in the economy with a tradable index: allowing index trading can be equivalent to increasing cash flows in the pre-indexing economy by tens of percentage points. Overall, despite making stock returns more volatile, index trading makes previously constrained investors much better off by facilitating risk sharing, but those unconstrained investors who switch to indexing lose only a small fraction of their welfare. Thus, indexing is not detrimental as feared in the literature (e.g., Wurgler, 2011; Sullivan and Xiong, 2012).

Our paper belongs to the large literature that uses the dynamic exchange economy framework with heterogeneous investors to study equilibrium effects of various economic frictions.[5] Such frictions include restricted stock market participation (e.g., Basak and Cuoco, 1998; Guvenen, 2009; Chien et al., 2011), intermittent trading (e.g., Chien et al., 2012), short-sale and borrowing constraints (e.g., Detemple and Murthy, 1997; Basak and Croitoru, 2000; Kogan et al., 2007; Gallmeyer and Hollifield, 2008; Gomes and Michaelides, 2008; Chabakauri, 2015a), portfolio concentration constraints (e.g., Pavlova and Rigobon, 2008), margin constraints (e.g., Gromb and Vayanos, 2002; Gârleanu and Pedersen, 2011; Chabakauri, 2013; Rytchkov, 2014; Brumm et al., 2015), and transaction costs (e.g., Buss et al., 2015; Buss and Dumas, 2019). Gromb and Vayanos (2010) survey the literature on the frictions that produce the limits to arbitrage. Dumas and Lyasoff (2012) develop a general approach to solving incomplete-market models with one Lucas tree.

[5]Dynamic exchange economies with one Lucas tree, heterogeneous investors, and complete markets are studied by Wang (1996), Chan and Kogan (2002), Weinbaum (2009, 2010), Xiouros and Zapatero (2010), Longstaff and Wang (2012), Cvitanić et al. (2012), and Bhamra and Uppal (2014), among others.

In particular, our paper is related to Chabakauri (2013) that analyzes a model with two stocks, two groups of investors with different risk aversion, and margin constraint. However, there are substantial differences between the papers. First, the indexing constraint in our paper is intrinsically different from the margin constraint: the former changes over time in line with the weights of the market portfolio, whereas the latter stays the same. As a result, Chabakauri (2013) and our paper have different economic implications: indexing amplifies stock return correlations in some states of the economy, whereas the margin constraint unambiguously decreases them. Second, the equilibrium processes in Chabakauri (2013) are stated in terms of the Lagrange multiplier of the portfolio constraint, whereas we explicitly solve the utility maximization problem of constrained investors and directly relate the equilibrium characteristics to portfolio distortions produced by indexing. Finally, our equilibrium is described by a system of two differential equations, in contrast to three equations in Chabakauri (2013).

Another related paper is Shapiro (2002). It considers a general equilibrium model in which a fraction of logarithmic investors can implement only the trading strategies consistent with the investor recognition hypothesis (IRH), and indexing is one of them. In contrast to our paper, Shapiro (2002) does not solve the model for the equilibrium characteristics and mostly focuses on qualitative implications of portfolio constraints for interest rates and risk premiums. Thus, our results are novel even for a model in which constrained investors have logarithmic preferences.

Dynamic models with indexing are also considered by He and Shi (2017) and Grégoire (2020). He and Shi (2017) consider an economy with two groups of unconstrained investors who have heterogeneous beliefs and study the effect of indexing on the welfare of an infinitesimal index investor. In contrast, all investors in our model have the same beliefs, and index investors have a nonnegligible weight. Grégoire (2020) uses the perturbation analysis to approximate the solution to the model and demonstrates that indexing increases comovement of stock returns. However, all investors in Grégoire (2020) have identical preferences. As a result, there is no risk sharing among them, which is one of the channels through which indexing affects the market equilibrium in our model.

Our result that indexing decreases the correlation of stock returns echoes the conclusion of Malamud (2016), who argues that the introduction of new ETFs can produce a similar effect. However, the mechanism of the effect in Malamud (2016) is different from ours. Malamud (2016) assumes that ETF investors are subject to exogenous demand shocks, and the ETF prices may deviate from the ETFs' net asset values because of the limits to arbitrage. The entry of new ETFs creates a demand substitution effect whereby a part of the investors' demand shifts to new ETFs. The demand substitution effectively produces two imperfectly correlated shocks to asset prices, which reduces the volatility and comovement of asset returns.

Our paper is also related to the studies of equilibrium effects produced by benchmarking of asset managers' compensation to a particular index (e.g., Gómez and Zapatero, 2003; Cuoco and Kaniel, 2011; Basak and Pavlova, 2013; Buffa et al., 2015; Basak and Pavlova, 2016; Buffa and Hodor, 2018). One of the insights of this research is endogenous arising of partial indexing in the presence of index-based incentives. Also, several papers examine the effects of benchmarking and indexing on informational efficiency and welfare in economies with heterogeneously informed investors (Bond and Garcia, 2018; Breugem and Buss, 2019). In contrast, we abstract from the origin of indexing and, therefore, identify the implications of indexing that are produced by indexing itself and that are uncontaminated by other frictions in the economy including the informational frictions.

2. Model

2.1. Assets

There are three assets in the economy: a risk-free short-term bond in zero net supply and two risky stocks. The supply of each stock is normalized to one share, which is a claim on a stream of dividends produced by a Lucas tree. The dividends D_{1t} and D_{2t} follow geometric Brownian motions

$$\frac{dD_{it}}{D_{it}} = \mu_{Di}dt + \Sigma_{Di}dB_t, \quad i = 1, 2, \tag{1}$$

where μ_{Di} are constant expected dividend growth rates, Σ_{Di} are constant 1×2 matrices of diffusions, and B_t is a 2×1 vector of independent Brownian motions. The rate of return on the bond r_t and the stock prices S_{1t} and S_{2t} are determined in the equilibrium. The excess return on each stock i is defined as

$$dQ_{it} = \frac{dS_{it} + D_{it}dt}{S_{it}} - r_t dt, \tag{2}$$

and the vector $Q_t = [Q_{1t} \;\; Q_{2t}]'$ follows a diffusion process

$$dQ_t = \mu_{Qt}dt + \Sigma_{Qt}dB_t, \tag{3}$$

where the matrix of the risk premiums $\mu_{Qt} = [\mu_{Q1t} \;\; \mu_{Q2t}]'$ and the matrix of the diffusions $\Sigma_{Qt} = [\Sigma'_{Q1t} \;\; \Sigma'_{Q2t}]'$ are also determined in the equilibrium. In those notations, the volatility of stock i is $\sigma_{it} = \sqrt{\Sigma_{Qit}\Sigma'_{Qit}}$, and the correlation between stock returns is $\rho_t = \Sigma_{Q1t}\Sigma'_{Q2t}/(\sigma_{1t}\sigma_{2t})$.

Taken together, the stocks constitute a market portfolio (index), which pays the aggregate dividend $D_t = D_{1t} + D_{2t}$ and has the price $S_t = S_{1t} + S_{2t}$. Using Itô's lemma and Eq. (1), the dynamics of the dividend D_t can be written as

$$\frac{dD_t}{D_t} = \mu_{Dt}dt + \Sigma_{Dt}dB_t, \tag{4}$$

where $\mu_{Dt} = u_t\mu_{D1} + (1 - u_t)\mu_{D2}$, $\Sigma_{Dt} = u_t\Sigma_{D1} + (1 - u_t)\Sigma_{D2}$, and $u_t = D_{1t}/D_t$. The excess return on the index is defined as

$$dQ_{It} = \frac{dS_t + D_t dt}{S_t} - r_t dt, \tag{5}$$

and using Eq. (3), its dynamics can be described as

$$dQ_{It} = \mu_{It}dt + \Sigma_{It}dB_t, \tag{6}$$

where $\mu_{It} = (\mu_{Q1t}S_{1t} + \mu_{Q2t}S_{2t})/S_t$ and $\Sigma_{It} = (\Sigma_{Q1t}S_{1t} + \Sigma_{Q2t}S_{2t})/S_t$. By construction, the index is value-weighted, and its expected return and diffusions are value-weighted averages of the expected returns and diffusions of the individual stocks. The volatility of the index is $\sigma_{It} = \sqrt{\Sigma_{It}\Sigma_{It}'}$.

2.2. Agents

The economy is populated by two groups of competitive agents dubbed type P investors (professional investors) and type I investors (index investors). Each group consists of a unit mass of identical investors who have CRRA preferences. The type P and type I investors differ in two respects. First, they have different coefficients of risk aversion, which are γ_P and γ_I, respectively. Second, the trading strategies that the investors can implement depend on their type: the type P investors can trade all assets individually, whereas the type I investors are constrained to trade only the risk-free bond and market portfolio. Specifically, the type P investors form a portfolio of the stocks $\omega_{Pt} = [\omega_{P1t} \ \ \omega_{P2t}]'$, where ω_{P1t} and ω_{P2t} are the fractions of their wealth W_{Pt} allocated to stocks 1 and 2, respectively, and invest the rest of their wealth $\alpha_{Pt} = 1 - \omega_{P1t} - \omega_{P2t}$ in the bond. In contrast, the type I investors allocate their wealth W_{It} between the index and the bond with the weights $\hat{\omega}_{It}$ and $\alpha_{It} = 1 - \hat{\omega}_{It}$, respectively. Thus, the weights of the individual stocks in the portfolio of the type I investors are $\omega_{It} = \hat{\omega}_{It}[S_{1t}/S_t \ \ S_{2t}/S_t]'$.

The types of investors admit a natural interpretation. The type P investors can be thought of as professional traders such as hedge funds, actively managed mutual funds, proprietary traders, etc., who are relatively risk tolerant and can implement sophisticated trading strategies that involve individual assets. The type I investors are unsophisticated market participants such as individual investors who manage their savings and retirement accounts. They are more risk averse than professional investors and trade only the index, not individual stocks. Lower risk aversion of the type P investors is also consistent with endogenous occupation choice models, which predict that risk tolerant individuals self-select themselves into entrepreneurial activities such as creating and managing hedge funds (Kihlstrom and Laffont, 1979). That prediction is consistent with the empirical evidence (Hvide and Panos, 2014).

We do not specify the reason why the type I investors can trade only the index, which allows us to study the implications of indexing without contaminating the analysis by other economic frictions. This approach follows many studies in economics and finance that investigate equilibrium implications of various impediments to risk sharing without endogenizing them.[6] In practice, indexing can be an optimal response to various factors such as information processing costs, organizational and management costs, transaction costs, etc. For example, investors with limited attention may allocate their learning capacity to macroeconomic factors rather than to firm-specific information (e.g., Peng and Xiong, 2006) and trade only the market portfolio. Investors may prefer to categorize assets in various classes and invest in indexes because this simplifies asset choice (e.g., Barberis and Shleifer, 2003). Even mutual fund and pension fund managers, whose compensation is related to index performance directly or indirectly through the response of fund flows to fund performance, may find it optimal to partially allocate assets under management to index portfolios (e.g., Basak and Pavlova, 2013). In Internet Appendix E, we present a modification of our model that demonstrates how the equilibrium with index investing endogenously arises when the type I investors are unconstrained in their portfolio choice but derive disutility from managing a complex portfolio of individual risky assets.

The optimization problem of the investors in our model has the standard form: each investor $j = P, I$ chooses a consumption stream C_{jt} and portfolio weights ω_{jt} that maximize the CRRA utility

$$U_t = \mathbb{E}_t \left[\int_t^\infty e^{-\beta \tilde{t}} \frac{C_{j\tilde{t}}^{1-\gamma_j}}{1-\gamma_j} d\tilde{t} \right] \tag{7}$$

subject to a budget constraint, which is

$$dW_{Pt} = (r_t W_{Pt} - C_{Pt})dt + W_{Pt}\omega'_{Pt}(\mu_{Qt}dt + \Sigma_{Qt}dB_t) \tag{8}$$

for the type P investors and

$$dW_{It} = (r_t W_{It} - C_{It})dt + W_{It}\hat{\omega}_{It}(\mu_{It}dt + \Sigma_{It}dB_t) \tag{9}$$

[6] An incomplete list includes Detemple and Murthy (1997), Basak and Cuoco (1998), Basak and Croitoru (2000), Gromb and Vayanos (2002), Kogan et al. (2007), Gallmeyer and Hollifield (2008), Gomes and Michaelides (2008), Pavlova and Rigobon (2008), Guvenen (2009), Gârleanu and Pedersen (2011), Chabakauri (2013), Rytchkov (2014), Chabakauri (2015a), and Buss et al. (2016).

for the type I investors.

2.3. State variables

The model has two Lucas trees and two types of investors. Therefore, it is natural to assume that the state of the economy is described by two variables. The first variable is the consumption share of the type I investors $s_t = C_{It}/D_t$, which is bounded between 0 and 1.[7] In general, s_t follows a diffusion process

$$ds_t = \mu_{st}dt + \Sigma_{st}dB_t, \tag{10}$$

where the scalar μ_{st} and the 1×2 matrix Σ_{st} are determined by equilibrium conditions. The second state variable $u_t = D_{1t}/D_t$ measures the relative share of the dividend on the first stock in the aggregate dividend.[8] The process for u_t is obtained by applying Itô's lemma to the definition of u_t and using Eqs. (1) and (4):

$$du_t = \mu_{ut}dt + \Sigma_{ut}dB_t, \tag{11}$$

where μ_{ut} and Σ_{ut} are determined by exogenous model parameters:

$$\mu_{ut} = u_t(1-u_t)(\mu_{D1} - \mu_{D2} - (\Sigma_{D1} - \Sigma_{D2})(u_t\Sigma_{D1} + (1-u_t)\Sigma_{D2})'), \tag{12}$$

$$\Sigma_{ut} = u_t(1-u_t)(\Sigma_{D1} - \Sigma_{D2}). \tag{13}$$

By construction, u_t belongs to the interval from 0 to 1. When $0.5 < u_t < 1$ ($0 < u_t < 0.5$), the first (second) stock has a larger dividend, and we refer to it as the larger stock.

[7] The consumption share of one of the agents is often used as a state variable in economies with heterogeneous agents (e.g., Bhamra and Uppal, 2009, 2014; Longstaff and Wang, 2012; Chabakauri, 2013; Rytchkov, 2014).

[8] This state variable is standard in the models with multiple Lucas trees (e.g., Menzly et al., 2004; Cochrane et al., 2008; Martin, 2013).

2.4. Equilibrium

We define the equilibrium in the model as a set of stochastic processes for the risk-free rate r_t, expected excess returns μ_{Qt}, diffusions of returns Σ_{Qt}, consumption streams C_{jt}, $j = P, I$, and portfolio strategies ω_{jt}, $j = P, I$, such that

1. C_{jt} and ω_{jt} solve the utility maximization problem of investor j;

2. the aggregate consumption is equal to the aggregate dividend: $C_{It} + C_{Pt} = D_t$;

3. the markets for the stocks and bond clear: $\omega_{Pit}W_{Pt} + \omega_{Iit}W_{It} = S_{it}$, $\alpha_{Pt}W_{Pt} + \alpha_{It}W_{It} = 0$.

Assuming that the state of the economy is fully described by s_t and u_t, we look for the equilibrium processes r_t, μ_{Qt}, Σ_{Qt}, μ_{It}, and Σ_{It} as functions of those variables: $r_t = r(s_t, u_t)$, $\mu_{Qt} = \mu_Q(s_t, u_t)$, $\Sigma_{Qt} = \Sigma_Q(s_t, u_t)$, $\mu_{It} = \mu_I(s_t, u_t)$, and $\Sigma_{It} = \Sigma_I(s_t, u_t)$. The same representation should exist for the drift and diffusion of s_t: $\mu_{st} = \mu_s(s_t, u_t)$, $\Sigma_{st} = \Sigma_s(s_t, u_t)$. It is also convenient to introduce i) the price-dividend ratios of the index and individual stocks $S_t/D_t = f(s_t, u_t)$ and $S_{it}/D_{it} = f_i(s_t, u_t)$, $i = 1, 2$, and ii) the wealth-consumption ratios of the type I and type P investors $W_{It}/C_{It} = h(s_t, u_t)$ and $W_{Pt}/C_{Pt} = h_P(s_t, u_t)$. The derivatives of those functions with respect to s_t and u_t will be denoted by the corresponding subscripts. Finally, we denote the risk aversion of a representative investor as

$$\Gamma_t = \left(\frac{s_t}{\gamma_I} + \frac{1 - s_t}{\gamma_P}\right)^{-1}. \tag{14}$$

The following proposition characterizes the equilibrium and describes how to compute various equilibrium characteristics. To simplify notation, in the rest of the paper we omit the subscript t of all variables as well as the arguments s and u of all functions.

Proposition 1 *The equilibrium in the model is characterized by the functions r, μ_s, Σ_s, Σ_I, f, and h that solve a system of algebraic and partial differential equations (A1) – (A6) from the Appendix. The market price of risk η and the expected excess returns on the index μ_I are given by Eq. (A7). The price-dividend ratio f_i of stock $i = 1, 2$ solves Eq. (A8). The expected* *excess returns on individual stocks μ_{Qi}, $i = 1, 2$, and return diffusions Σ_{Qi}, $i = 1, 2$, are given by Eq. (A9).*

Proof. See the Appendix.

The formulas from Proposition 1 reveal a technical trick that helps us simplify the analysis. In general, the description of an equilibrium in an economy with two trees and two types of investors involves three differential equations: two of them are for the stock price-dividend ratios and the third is for the wealth-consumption ratio of one of the investor types (e.g., Chabakauri, 2013). However, in the economy with index investors, the equilibrium can be found by sequentially solving two sets of equations: the first is a pair of quasilinear differential equations for the price-dividend ratio of the index and the wealth-consumption ratio of the type I investors; the second is a pair of linear differential equations for the price-dividend ratios of the individual stocks. Intuitively, the simplification occurs because only the index is traded in the equilibrium and only its price, not the prices of individual stocks, is needed to describe the dynamics of the economy. In Section 3, we numerically solve the system of equations (A1) – (A6) and demonstrate various equilibrium properties.

As we show in Internet Appendix D, our model can be generalized by assuming that all investors have recursive preferences of Duffie and Epstein (1992).[9] However, in this case the characterization of the equilibrium is more complicated because the stochastic discount factor directly depends on the investors' wealth-consumption ratios and their derivatives. Nevertheless, even in the case of recursive preferences, we managed to describe the equilibrium in terms of a system of differential and algebraic equations that can be solved by the same numerical techniques as in the CRRA case. Those equations are presented in Proposition ID1, which generalizes Proposition 1 to the model with recursive preferences.

[9] Models with investors who have heterogeneous recursive preferences have been considered by Isaenko (2008), Chabakauri (2015b), Gârleanu and Panageas (2015), Drechsler et al. (2018), and Borovička (2020), among others.

2.5. Analysis of the equilibrium

In this section, we analytically describe several properties of the equilibrium in our model. To identify the effects of indexing, we compare the equilibrium characteristics with their counterparts in the unconstrained economy. The latter differs from our main setting by the assumption that all investors can trade all assets individually. The equilibrium in the unconstrained economy with heterogeneous CRRA investors and two Lucas trees is described by Proposition 2 in Chabakauri (2013). Specifically, in the unconstrained economy the diffusion Σ_s is

$$\Sigma_s^{unc} = \frac{\gamma_P - \gamma_I}{\gamma_P \gamma_I} s(1-s)\Gamma\Sigma_D. \tag{15}$$

The equations for μ_s, r, and η can also be analytically solved, and the solutions are

$$\mu_s^{unc} = \frac{s(1-s)\Gamma}{\gamma_P\gamma_I}\left((\gamma_P - \gamma_I)(\mu_D - \Sigma_D\Sigma_D') + \frac{(\gamma_P - \gamma_I)\Gamma^2}{2\gamma_P\gamma_I}\Sigma_D\Sigma_D'\right), \tag{16}$$

$$r^{unc} = \beta + \Gamma\mu_D - \frac{\Gamma^3}{2}\left((1-s)\frac{1+\gamma_P}{\gamma_P^2} + s\frac{1+\gamma_I}{\gamma_I^2}\right)\Sigma_D\Sigma_D', \qquad \eta^{unc} = \Gamma\Sigma_D. \tag{17}$$

Eqs. (15) – (17) show that in the unconstrained economy the evolution of the endogenous state variable s (which is determined by μ_s^{unc} and Σ_s^{unc}) and the discount factor (which is determined by r^{unc} and η^{unc}) do not depend on the price-dividend ratio f or wealth-consumption ratio h. Therefore, as demonstrated in Chabakauri (2013), the differential equations for those ratios are linear, decoupled, and easy to solve.

Index investing changes the equilibrium compared to the unconstrained economy because it distorts the portfolio of the type I investors. To characterize the distortions, it is convenient to introduce a portfolio with the following weights:

$$\omega_0 = (\Sigma_Q')^{-1}\left(\frac{\Sigma_Q^{-1}\mu_Q}{\gamma_I} + \frac{h_s}{h}\Sigma_s' + \frac{h_u}{h}\Sigma_u'\right). \tag{18}$$

The portfolio ω_0 solves the utility maximization problem of an investor who is identical to a type I investor in all respects, including the wealth-consumption ratio, but who can trade individual stocks. The following proposition reveals a connection between the portfolio ω_0 and the optimal portfolio ω_I.

Proposition 2 *The optimal portfolio ω_I can be represented as*

$$\omega_I = \arg\min_{\omega \in \Omega} \, (\omega - \omega_0)' \Sigma_Q \Sigma_Q' (\omega - \omega_0), \tag{19}$$

where Ω is the set of all index portfolios, and ω_0 is defined by Eq. (18). The optimization implies that the vector of portfolio diffusions $\Sigma_Q' \omega_I$ is a projection of the portfolio diffusions $\Sigma_Q' \omega_0$ on the index returns, that is,

$$\Sigma_Q' \omega_I = \Pi_I \Sigma_Q' \omega_0, \tag{20}$$

where the projection operator is $\Pi_I = (\Sigma_I' \Sigma_I)/(\Sigma_I \Sigma_I')$.

Proof. See the Appendix.

Intuitively, Proposition 2 states that the portfolio ω_0 is an unfeasible target of a constrained investor, who chooses the index portfolio to be as close as possible to ω_0. As a result, the optimal index portfolio turns out to be a projection of the target portfolio ω_0 on the set of all feasible index portfolios. The next proposition shows explicitly how indexing changes the model equilibrium compared to its unconstrained counterpart.

Proposition 3 *In the equilibrium described by Proposition 1, the drift μ_s and diffusion Σ_s can be represented as*

$$\mu_s = \mu_s^{unc} + a \Sigma_D \Sigma_Q' (\omega_I - \omega_0) + b (\omega_I - \omega_0)' \Sigma_Q \Sigma_Q' (\omega_I - \omega_0), \tag{21}$$

$$\Sigma_s = \Sigma_s^{unc} + \frac{s(1-s)}{\gamma_P} \Gamma (\omega_I - \omega_0)' \Sigma_Q, \tag{22}$$

where μ_s^{unc} and Σ_s^{unc} are given by Eqs. (16) and (15), respectively, and the coefficients a and b

are

$$a = \frac{s(1-s)\Gamma}{\gamma_P}\left(\frac{\Gamma^2}{\gamma_I\gamma_P} + \Gamma - 1\right), \qquad b = \frac{s(1-s)\Gamma^3}{2\gamma_P^2}\left(\frac{(\gamma_I+1)(1-s)^2}{\gamma_P} - \frac{(\gamma_P+1)s^2}{\gamma_I}\right). \quad (23)$$

The risk-free rate r and the price of risk η can be represented as

$$r = \beta + \Gamma\mu_D - \frac{1}{2}\Gamma\left((1+\gamma_I)s\Sigma_{CI}\Sigma_{CI}' + (1+\gamma_P)(1-s)\Sigma_{CP}\Sigma_{CP}'\right), \quad (24)$$

$$\eta = \gamma_P\Sigma_{CP}, \quad (25)$$

where Σ_{CP} and Σ_{CI} are the consumption diffusions of the type P and type I investors:

$$\Sigma_{CP} = \frac{\Gamma\Sigma_D}{\gamma_P} - \frac{s\Gamma}{\gamma_P}(\omega_I - \omega_0)'\Sigma_Q, \qquad \Sigma_{CI} = \frac{\Gamma\Sigma_D}{\gamma_I} + \frac{(1-s)\Gamma}{\gamma_P}(\omega_I - \omega_0)'\Sigma_Q. \quad (26)$$

The partial differential equation for the wealth-consumption ratio h can be rewritten as

$$\tfrac{1}{2}h_{ss}\Sigma_s\Sigma_s' + \tfrac{1}{2}h_{uu}\Sigma_u\Sigma_u' + h_{su}\Sigma_s\Sigma_u' + h_s\left(\mu_s + \tfrac{1-\gamma}{\gamma}\Sigma_s\Sigma_Q^{-1}\mu_Q\right) + h_u\left(\mu_u + \tfrac{1-\gamma_I}{\gamma_I}\Sigma_u\Sigma_Q^{-1}\mu_Q\right)$$
$$+\tfrac{1}{\gamma_I}h\Big((1-\gamma_I)\Big(r - \tfrac{\gamma_I}{2}(\omega_I-\omega_0)'\Sigma_Q\Sigma_Q'(\omega_I-\omega_0)\Big) + \tfrac{1-\gamma_I}{2\gamma_I}\mu_Q'(\Sigma_Q\Sigma_Q')^{-1}\mu_Q - \beta\Big) + 1 = 0. \quad (27)$$

Proof. See the Appendix.

Proposition 3 highlights the channels through which indexing affects the equilibrium. Eqs. (21) – (26) imply that the equilibrium processes can be represented as the processes in the unconstrained economy adjusted by linear or quadratic functions of $\Sigma_Q'(\omega_I - \omega_0)$. This fact explicitly shows that the adjustments appear because index investors are constrained and cannot form the portfolio ω_0. The term $\Sigma_Q'(\omega_I - \omega_0)$ can be viewed as a measure of portfolio distortions produced by indexing. Using Eq. (20), it can be written as $-(I_2 - \Pi_I)\Sigma_Q'\omega_0$, where I_2 is a 2×2 identity matrix. Note that $\Sigma_Q'\omega_0$ is the transposed diffusion of returns on the unconstrained portfolio ω_0, so $(I_2 - \Pi_I)\Sigma_Q'\omega_0$ represents the variation in the returns on the portfolio ω_0 that is orthogonal to the index and cannot be hedged by index investors. Thus, the additional terms in Eqs. (21) – (26) can be viewed as a result of restrictions on risk sharing among investors imposed by indexing.

The constraints on risk sharing change the evolution of the state variable s, which determines the dynamics of all other equilibrium characteristics. Eq. (15) implies that in the unconstrained economy, the variation in s is produced solely by the variation in the total dividend D, which is the source of the aggregate risk in the economy. How the individual dividends D_1 and D_2 contribute to D is irrelevant because all investors trade all risky assets and perfectly hedge tree-specific shocks.[10] In contrast, Eq. (22) shows that in the presence of index investors, the diffusion Σ_s has an additional term that is not collinear to D. Therefore, the shocks to D_1 and D_2 individually affect s.

Proposition 3 also reveals how indexing changes the risk-free rate r and market price of risk η, and how its impact can be traced to reduced risk sharing and portfolio distortions. Eq. (26) demonstrates that portfolio distortions change the consumption diffusions Σ_{CP} and Σ_{CI} thereby affecting the volatilities of the investors' consumption processes. Indeed, reduced risk sharing is likely to make the consumption process of more risk averse type I investors more volatile than in the unconstrained economy, and the effect is opposite for the less risk averse type P investors. Eq. (24), which holds both in the constrained and unconstrained economies, shows that consumption volatilities determine the risk-free rate through the third term, which is associated with precautionary savings of investors (e.g., Cochrane, 2005). Indexing has a substantial impact on the equilibrium when the type I investors constitute a large fraction of all investors, that is, when s is close to 1. In this case, precautionary savings in the economy are mostly determined by the type I investors. Because the volatility of their consumption is amplified by indexing, the type I investors have stronger incentives to save than in the unconstrained economy, and this decreases the equilibrium risk-free rate.[11]

The relation between the market price of risk η and portfolio distortions is even more straightforward. Eq. (25) shows that η is proportional to the diffusion of the type P investors'

[10]The dividend share u is still a state variable because it affects the expected growth rate and volatility of the total dividend.

[11]We numerically demonstrate this effect in Section 3.2.2.

consumption diffusion Σ_{CP}, whose relation to portfolio distortion is described by Eq. (26).

The differential equation (27) for the wealth-consumption ratio h of index investors provides additional insights. Compared to its counterpart for unconstrained investors, Eq. (27) contains an additional term $\frac{1}{2}\gamma_I(\omega_I - \omega_0)'\Sigma_Q\Sigma_Q'(\omega_I - \omega_0)$, which is produced by the portfolio constraints faced by the type I investors. This term can be combined with the risk-free rate r, and this fact yields the following corollary.

Corollary 1. *Consider an unconstrained investor with the coefficient of risk aversion γ_I who becomes an index investor in the same economy. This transition changes the investor's wealth-consumption ratio exactly as a decrease of the risk-free rate by $\frac{1}{2}\gamma_I(\omega_I - \omega_0)'\Sigma_Q\Sigma_Q'(\omega_I - \omega_0)$.*

Corollary 1 states that the wealth-consumption ratio of the index investor is the same as that of an unconstrained investor in the economy with a lower risk-free rate $r - \frac{1}{2}\gamma_I(\omega_I - \omega_0)'\Sigma_Q\Sigma_Q'(\omega_I - \omega_0)$ but with the same other equilibrium processes. An unconstrained economy with a modified risk-free rate is a version of the fictitious unconstrained economy of Cvitanic and Karatzas (1992), which is often used to solve utility maximization problems of constrained investors (e.g., Basak and Cuoco, 1998; Chabakauri, 2013, 2015b). However, the Cvitanic and Karatzas (1992) fictitious economies are obtained by adjusting both the market price of risk and the risk-free rate, whereas according to Corollary 1 the indexing constraint is equivalent to an adjustment of the risk-free rate only.

The differential equation (27) also shows that the investor's wealth-consumption ratio is affected by indexing directly and indirectly. The direct effect is represented by the term $\frac{1}{2}\gamma_I(\omega_I - \omega_0)'\Sigma_Q\Sigma_Q'(\omega_I - \omega_0)$, which is quadratic in the portfolio distortions $\Sigma_Q'(\omega_I - \omega_0)$. The indirect effect is produced by changes in the equilibrium processes that determine the coefficients of Eq. (27). As immediately follows from Eqs. (22) – (25), those changes have linear terms in $\Sigma_Q'(\omega_I - \omega_0)$, so indexing has the first-order indirect effect on the investor's wealth-consumption ratio. We further investigate the direct and indirect effects of indexing in Section 3.4, in which we relate the wealth-consumption ratio to the investor's welfare and separate the welfare implications of portfolio constraints and distorted investment opportunities.

3. Numerical analysis

Even though we can characterize the impact of indexing on various equilibrium characteristics analytically, the differential equations that describe the equilibrium do not have closed-form solutions. Moreover, the most interesting characteristics such as the volatilities of stock returns and the correlation between them are nonlinear functions of Σ_s, f, f_u, and f_s. As a result, the impact of indexing on volatilities and correlations can be found only numerically. In this section, we compute the equilibrium from Proposition 1 using the finite-difference approximation described in Internet Appendix A, compare the equilibrium to its analogs in the two benchmark economies, decompose the impact of indexing into the lockstep trading effect and the reduced risk sharing effect, and examine how indexing affects investors' welfare.

3.1. Model parameters

We calibrate the model parameters so that the growth rates and volatilities of the two dividend processes are identical: $\mu_{D1} = \mu_{D2} = 0.018$, $\Sigma_{D1} = [0.045 \;\; 0]$, and $\Sigma_{D2} = [0 \;\; 0.045]$. We follow previous studies (e.g., Basak and Cuoco, 1998; Dumas and Lyasoff, 2012; Chabakauri, 2013) and identify the aggregate dividend with the aggregate consumption, and the chosen parameter values are in the ballpark of the estimated mean and volatility of the consumption growth rate in the United States. The dividends of the stocks are assumed to be uncorrelated.

Because we interpret the type P investors as financial professionals and the type I investors as individual investors, we set $\gamma_P = 1$ and $\gamma_I = 5$, and this choice is consistent with individual investors being more risk averse than professionals. In contrast to the vast majority of the papers that study equilibria in incomplete markets, we do not assume that constrained investors have logarithmic preferences. On the one hand, this complicates the analysis because the hedging demand of such investors affects the properties of the equilibrium and should be taken into account. On the other hand, the assumption $\gamma_I > 1$ makes the analysis more realistic. The time preference parameter β is 0.03 for all investors.

In Internet Appendix C, we explore the robustness of our results to changes in the model parameters and consider two alternative specifications. In the first of them, the Lucas trees have different dividend processes with the parameters $\mu_{D1} = 0.01$, $\mu_{D2} = 0.03$, $\Sigma_{D1} = [0.01\ \ 0]$, and $\Sigma_{D2} = [0\ \ 0.08]$. In the second specification, the index investors are less risk averse than the unconstrained investors, and we set $\gamma_I = 1$ and $\gamma_P = 5$.

3.2. Economy with indexing vs. unconstrained economy

We start with comparing the equilibrium variables in the economy with indexing and the unconstrained economy discussed in Section 2.5.[12] For variables that can take only positive values, we consider percentage changes produced by indexing. Because correlations can switch their signs, we present simple differences in them.

3.2.1. Market volatility and correlation

Consider first the impact of indexing on market volatility and the correlation between stock returns. The left panels of Fig. 1 plot the changes in σ_I and ρ as functions of the state variables. The graphs show that indexing unambiguously decreases market volatility but can increase or decrease the correlation depending on the state of the economy. In particular, the correlation increases when the stocks have comparable sizes (when u is close to 0.5) but decreases when the sizes are substantially different.

FIGURE 1 IS HERE

Those effects can be understood through examining how indexing distorts the investors' trading strategies. In the unconstrained economy, the stock volatilities and correlation are determined by the dividend processes and risk sharing among investors. Because less risk-averse type P investors hold a larger share of their wealth in stocks than more risk averse type

[12]The equilibrium characteristics of the unconstrained economy with the parameters from Section 3.1 are presented in Internet Appendix B.

I investors, a positive dividend shock to one of the stocks disproportionately increases their wealth and decreases the aggregate risk aversion in the economy. The induced variation in the aggregate risk aversion produces a common variation in the stock discount rates. As a result, the volatilities of stocks are higher than the volatilities of the dividends, and the returns are correlated even when the dividends are independent.

Indexing produces two new effects that work in opposite directions. On the one hand, it further increases market volatility and the return correlation because indexers hold an equal number of the shares of each stock and trade both stocks in lockstep. On the other hand, indexing reduces the volatility and correlation because it hampers risk sharing among investors. To isolate the lockstep trading effect from the reduced risk sharing effect, we consider a modified specification of the economy dubbed a lockstep-trading economy. In this economy, the type I investors exogenously put in the index the same fraction of their wealth as they put in both stocks in the unconstrained economy and optimize only their consumption policy. All other components of the model are the same as before. Because the portfolio optimization is switched off, the type I investors do not change their allocation of wealth to index in response to reduced risk-sharing opportunities. Thus, the differences in the equilibrium characteristics between the unconstrained and lockstep-trading economies are naturally interpreted as solely produced by lockstep trading.

More specifically, in the lockstep-trading economy, we exogenously set $\hat{\omega}_I = \omega_{I1}^{unc} + \omega_{I2}^{unc}$, where ω_{I1}^{unc} and ω_{I2}^{unc} are the optimal portfolio weights of the type I investors in the unconstrained economy, and assume that each type I investor maximizes utility (7) with respect to the consumption C_I subject to the budget constraint (9). The equilibrium in the lockstep-trading economy is described by Proposition 4.

Proposition 4 *The equilibrium in the lockstep-trading economy is characterized by the functions r, μ_s, Σ_s, Σ_I, f, and h that solve a system of algebraic and partial differential equations (A60) – (A65) from the Appendix.*

Proof. See the Appendix.

As in the case of the economy with indexing, the equilibrium in the lockstep-trading economy cannot be found analytically, and we compute its characteristics using the numerical techniques described in Internet Appendix A.

Fig. 1 presents the decomposition of the changes in market volatility and the correlation produced by indexing into the components associated with lockstep trading and reduced risk sharing. The middle column shows the differences in the characteristics between the unconstrained and lockstep-trading economies, which are solely produced by the lockstep trading effect. The right column presents the differences in the same variables between the lockstep-trading economy and the economy with indexing; those differences measure the reduced risk sharing effect.

The obtained decomposition delivers several insights. Consider first the graphs for the correlation, which is increased by indexing when the stocks have comparable sizes but decreased when one of the stocks is substantially larger than the other. Fig. 1 sheds new light on this pattern by demonstrating that lockstep trading increases the correlation in almost all states of the economy, whereas reduced risk sharing decreases it in all states. The reduced risk sharing effect is stronger and more than offsets the lockstep trading effect when the stocks have different sizes, so the correlation becomes lower than in the unconstrained economy. When the stocks have comparable sizes, risk sharing is almost unaffected by indexing, and the lockstep trading effect dominates. As a result, the correlation between stock returns is higher than without indexing. Overall, the interplay between the two effects explains the shape of the correlation surface.

Lockstep trading and reduced risk sharing also have opposite effects on market volatility. Fig. 1 demonstrates that lockstep trading unambiguously increases it, which is a consequence of a higher correlation between stock returns and reduced diversification when investors simultaneously buy or sell multiple assets. In contrast, market volatility, which is higher than the volatility of the aggregate dividend because of risk sharing, decreases when risk sharing is reduced. Both the lockstep trading and reduced risk sharing effects disappear when stocks have equal sizes, but the reduced risk sharing effect is more pronounced in the other states of the economy. As a result, the overall market volatility is lower in the economy with indexing than in the unconstrained economy.

In Internet Appendix D, we numerically explore how indexing changes market volatility and the stock correlation when investors have recursive preferences. We find that the main effects are qualitatively similar to those in the economy with the CRRA preferences: indexing typically decreases market volatility, but its effect on the correlation is ambiguous. However, indexing has a quantitatively stronger impact on market volatility in the economy with recursive preferences. Thus, in a model with more realistic preferences the consequences of indexing can be more pronounced and practically important.

3.2.2. Risk-free rate, market prices of risk, and individual stock volatilities

Next, we examine how indexing changes the risk-free rate, market prices of risk, and individual stock volatilities. Because the dividend processes of the stocks have identical parameters, we plot only the component η_1 and volatility σ_1; η_2 and σ_2 are easily obtainable by flipping the graphs around $u = 1/2$. To interpret the results, we also consider the first component of $\Sigma'_Q(\omega_I - \omega_0)$. As discussed in Section 2.5, the vector $\Sigma'_Q(\omega_I - \omega_0)$ represents the variation in returns on the unconstrained portfolio ω_0 that is orthogonal to the index, and it can be viewed as a measure of portfolio distortions produced by indexing.

FIGURE 2 IS HERE

The results are plotted in Figs. 2 and 3. The graph for portfolio distortions $[\Sigma'_Q(\omega_I - \omega_0)]_1$ demonstrates that they naturally disappear when either there is only one stock in the economy ($u = 0$ or $u = 1$) or the sizes of the stocks are identical ($u = 1/2$) because in those cases the indexing constraint is not binding. Furthermore, $[\Sigma'_Q\omega_I]_1 > [\Sigma'_Q\omega_0]_1$ ($[\Sigma'_Q\omega_I]_1 < [\Sigma'_Q\omega_0]_1$) when the first stock is small (large), so indexing makes the type I investors' portfolio returns more (less) sensitive to dividend shocks of the smaller (larger) stock. Also, the magnitude of the effect is larger when $u < 1/2$ because by trading only the index it is harder for the type I investors to share risks associated with the smaller stock.

The graphs for the changes in the risk-free rate and market price of risk are directly related to the graph for portfolio distortions. According to Eqs. (25) and (26), the changes in η produced by indexing are proportional to $\Sigma_Q'(\omega_I - \omega_0)$ with a negative coefficient, which is evident in the graph for $\Delta\eta_1/\eta_1^{unc}$. As discussed in Section 2.5, indexing amplifies the consumption volatility of the type I investors, which increases their precautionary savings and drives the risk-free rate down. This pattern is illustrated by the graph for $\Delta r/r^{unc}$, and the effect is stronger when the type I investors dominate the economy. The two troughs in the graph are explained by larger portfolio distortions produced by indexing in the states around $u = 0.25$ and $u = 0.75$.

The effects of indexing on the risk-free rate r and the market price of risk η_i explain why indexing increases the price-dividend ratios f_i of the smaller stock. Indeed, the approximate Gordon formula $f_i \approx 1/(r + \eta\Sigma_{Di}' - \mu_{Di})$ shows that the price-dividend ratio increases when both the risk-free rate and the market price of risk become lower but may decrease when a lower risk-free rate is offset by a higher market price of risk.[13] As follows from the discussion above, the former happens for the smaller stock and the latter may happen for the larger stock.

The impact of indexing on the individual stock volatilities σ_i can also be traced to the portfolio distortions $\Sigma_Q'(\omega_I - \omega_0)$. First, note that because $\sigma_i = \sqrt{\Sigma_{Qi}\Sigma_{Qi}'}$, the percentage change in the volatility produced by indexing in the first-order approximation can be written as $\Delta\sigma_i/\sigma_i^{unc} \approx \Sigma_{Qi}^{unc}\Delta\Sigma_{Qi}'/(\sigma_i^{unc})^2$. Second, using that $\Sigma_{Qi} = \Sigma_{Di} + (f_{is}/f_i)\Sigma_s + (f_{iu}/f_i)\Sigma_u$ (this is Eq. (A9)), the change in Σ_{Qi} in the first-order approximation can be decomposed as

$$\begin{aligned}
\Delta\Sigma_{Qi} &= \Delta\left(\frac{f_{is}}{f_i}\Sigma_s\right) + \Delta\left(\frac{f_{iu}}{f_i}\Sigma_u\right) \approx \Delta\left(\frac{f_{is}}{f_i}\right)\Sigma_s^{unc} + \frac{f_{is}^{unc}}{f_i^{unc}}\Delta\Sigma_s + \Delta\left(\frac{f_{iu}}{f_i}\right)\Sigma_u \\
&= \underbrace{\frac{f_{is}^{unc}}{f_i^{unc}}\frac{s(1-s)}{\gamma_P}\Gamma(\omega_I - \omega_0)'\Sigma_Q}_{\text{portfolio distortions}} + \underbrace{\Delta\left(\frac{f_{is}}{f_i}\right)\Sigma_s^{unc} + \Delta\left(\frac{f_{iu}}{f_i}\right)\Sigma_u}_{\text{changes in the price elasticities}}, \qquad (28)
\end{aligned}$$

where the last equality follows from Eq. (22). Thus, there are direct and indirect channels through which indexing changes the return diffusions. The first term in Eq. (28) is proportional to $\Sigma_Q'(\omega_I - \omega_0)$, and it represents the direct effect of portfolio distortions on Σ_{Qi}. The last two

[13] Chabakauri (2013) discusses the approximate Gordon formula in more detail.

terms are produced by the changes in the price elasticities with respect to the state variables s and u; this is the indirect effect of indexing. Fig. 3 plots the described decompositions of $\Delta\sigma_i/\sigma_i^{unc}$ and $\Delta\Sigma_{Qi}$. Because the stocks have identical dividend processes, we present the results only for the first of them; the graphs for the other stock can be obtained by flipping all graphs around $u = 1/2$.

FIGURE 3 IS HERE

The top row of panels in Fig. 3 demonstrates the decomposition of $\Delta\sigma_1/\sigma_1^{unc}$ into two terms associated with the changes in the diffusion components $\Sigma_{Q1,1}$ and $\Sigma_{Q1,2}$. In general, both shocks dB_1 and dB_2 contribute to the volatility of the first stock, but the change in $\Sigma_{Q1,1}$ has a much stronger effect than the change in $\Sigma_{Q1,2}$: the graphs for $\Delta\sigma_1/\sigma_1^{unc}$ and its first component are almost indistinguishable. Intuitively, the stock returns load more strongly on their own dividend shock (dB_1 in the case of the first stock), so the diffusion of that shock largely determines the stock volatility and the change in it. Thus, to understand the impact of indexing, we can focus on how it changes $\Sigma_{Q1,1}$.

The bottom row of panels in Fig. 3 decomposes $\Delta\Sigma_{Q1,1}$ according to Eq. (28), where the middle and right panels plot the contributions of portfolio distortions and changes in the price elasticities, respectively. The graphs unambiguously show that the former is an order of magnitude larger than the latter, and the shapes of the surfaces in the bottom left and middle panels are almost identical. Thus, the impact of indexing on the stock volatility is mostly determined by portfolio distortions that it creates.

The comparison of the top left panel of Fig. 2 and the bottom middle panel of Fig. 3 suggests that the contribution of portfolio distortions $[\Sigma_Q'(\omega_I - \omega_0)]_1$ into the volatility is shaped by two additional factors. First, when the share of the index investors' consumption decreases (s approaches 0), the impact of the portfolio distortions on diffusions and volatilities becomes weaker, even though the portfolio distortions themselves do not disappear. This result is intuitive since in the limit $s \to 0$ the influence of index investors on the equilibrium becomes negligible, although their portfolios still nontrivially deviate from the unconstrained portfolio. Second, Eq. (28) demonstrates that $[\Sigma'_Q(\omega_I - \omega_0)]_1$ is negatively related to the change in the volatility because it is premultiplied by the derivative f_{1s}^{unc}. The latter is negative because the price-dividend ratio decreases with s: for higher s the proportion of the more risk-averse investors in the economy and the required risk premium are higher and the prices are lower.

3.3. Economy with indexing vs. pre-indexing economy

So far we used the unconstrained economy as the benchmark for gauging the impact of indexing. In this section, we consider an alternative benchmark that differs from our main model by the assumption that the type I investors, who are still interpreted as unsophisticated investors, cannot trade financial assets at all. Such a setup provides a stylized description of financial markets before the advent of indexing and is dubbed pre-indexing economy. It helps us understand the degree to which the possibility to trade indexes, which simplify the access to financial markets for unsophisticated investors, alleviates trading constraints and completes the markets.

Specifically, we assume that in the pre-indexing economy the type I investors are endowed with the cash flow $s_0 D$, where s_0 is an exogenous parameter bounded between 0 and 1. Without access to financial markets, the investors must consume all cash flows in each period. The type P investors still can trade all assets individually, and their behavior determines asset prices. Because all type P investors have the same preferences, the dynamics of such an economy coincide with the dynamics of an economy with homogeneous investors and two trees that produce the dividends $(1 - s_0)D_1$ and $(1 - s_0)D_2$, respectively. Such economies are studied by Cochrane et al. (2008) and Martin (2013), and their equilibrium characteristics are readily available. Note that all characteristics are functions of only the first dividend share u. Moreover, because only the type P investors can trade with each other and the consumption share s_0 is an exogenous parameter, the equilibrium processes in the pre-indexing economy are the same as in the limit $s \to 0$ of our main economy. The equilibrium characteristics of the pre-indexing economy with the parameters from Section 3.1 are presented in Internet Appendix B.

FIGURE 4 IS HERE

Fig. 4 shows how the pre-indexing economy characteristics change as a result of allowing the type I investors to trade bonds and the market index. Most importantly, indexing makes the individual stock returns and the market returns more volatile, and this effect has an intuitive explanation. The ability of both types of investors to trade the index makes the market more complete and facilitates risk sharing among investors. As a result, the equilibrium characteristics become closer to their counterparts in the unconstrained economy. In particular, risk sharing inflates volatilities and makes the returns much more correlated than they are in the pre-indexing economy.[14] Thus, our results provide a potential theoretical explanation for empirical findings that indexing can make stocks more volatile and more correlated (e.g., Sullivan and Xiong, 2012; Leippold et al., 2016; Ben-David et al., 2018; Da and Shive, 2018; Grégoire, 2020). Note that the investors in our model are perfectly rational, and this fact distinguishes our explanation from those in Barberis and Shleifer (2003) and Barberis et al. (2005), who explain the comovement of stock returns by behavioral biases.

Fig. 4 also shows that index investing increases the risk-free rate and market prices of risk but decreases the price-dividend ratios. Indeed, the less risk-averse type P investors prefer to hold more risky assets than the type I investors. To finance their positions, they borrow from the type I investors, and this pushes the risk-free rate up. Also, the ability to trade the index allows the type I investors to partially unload their exposure to dividend shocks to the type P investors. As a result, the market price of risk increases but the stock prices decrease to make large positions in the risky assets sufficiently attractive to the type P investors.

3.4. Welfare analysis

Finally, we explore how indexing affects the investors' welfare. To quantify the welfare implications of indexing, we use the certainty equivalent loss (CEL). The CEL is defined as the

[14] A similar mechanism is responsible for excessive volatility produced by introducing a nonredundant derivative in Bhamra and Uppal (2009).

maximal fraction of wealth that an investor in the state (s,u) of the unconstrained economy would give up for not being an index investor in the same state of the economy with indexing. More formally, we consider the function $CEL(s,u)$ that solves the following equation:

$$J_{unc}((1-CEL(s,u))W,s,u,t)=J(W,s,u,t), \tag{29}$$

where $J_{unc}(W,s,u,t)$ and $J(W,s,u,t)$ are the investor's indirect utility functions in the unconstrained economy and in the economy with indexing, respectively. Using Eq. (A27), the CEL of a type I investor can be written as

$$CEL(s,u)=1-\left(\frac{h(s,u)}{h_{unc}(s,u)}\right)^{\frac{\gamma_I}{1-\gamma_I}}, \tag{30}$$

where $h(s,u)$ and $h_{unc}(s,u)$ are the wealth-consumption ratios of the type I investors in the constrained and unconstrained economies.

There are two channels through which indexing affects the investor. First, it restricts the portfolio choice and makes the investment policy suboptimal. Second, it distorts prices and investment opportunities because in the economy with indexing many investors are constrained. To disentangle the contributions of those factors to the CEL, we compute two additional certainty equivalents: the first one dubbed CEL_1 measures the loss in utility produced by constraints, and the second one dubbed CEL_2 measures the welfare effect of distorted investment opportunities. We define CEL_1 and CEL_2 as solutions to the following equations:

$$\tilde{J}((1-CEL_1(s,u))W,s,u,t)=J(W,s,u,t), \quad J_{unc}((1-CEL_2(s,u))W,s,u,t)=\tilde{J}(W,s,u,t), \tag{31}$$

where $\tilde{J}(W,s,u,t)$ is the indirect utility function of a hypothetical unconstrained investor with the coefficient of risk aversion γ_I who lives in the economy with index investors. Again, using

Eq. (A27), we find that

$$CEL_1(s,u) = 1 - \left(\frac{h(s,u)}{\tilde{h}(s,u)}\right)^{\frac{\gamma_I}{1-\gamma_I}}, \qquad CEL_2(s,u) = 1 - \left(\frac{\tilde{h}(s,u)}{h_{unc}(s,u)}\right)^{\frac{\gamma_I}{1-\gamma_I}}, \tag{32}$$

where $\tilde{h}(s,u)$ is the unconstrained investor's wealth-consumption ratio. It solves Eq. (27) with $\omega_I = \omega_0$. Note that $1 - CEL(s,u) = (1 - CEL_1(s,u))(1 - CEL_2(s,u))$, so when the utility losses are small, the CEL can be decomposed as $CEL(s,u) \approx CEL_1(s,u) + CEL_2(s,u)$.

FIGURE 5 IS HERE

Panel A of Fig. 5 presents CEL, CEL_1, and CEL_2 of a type I investor for our standard calibration of the parameters.[15] It shows that the CEL is positive in all states, so the investor unambiguously prefers to be in the unconstrained economy. The decomposition of CEL into CEL_1 and CEL_2 reveals that CEL_1 is always positive but CEL_2 is mostly negative. Thus, the welfare loss is largely produced by the inability of the investor to freely adjust portfolio weights in the economy with indexing (the component CEL_1), not by price distortions (the component CEL_2). Intuitively, $CEL_1 > 0$ because keeping the investment opportunities fixed, any constraint that makes the originally optimal portfolio infeasible reduces the investor's welfare. In general, the constraint distorts both the myopic and hedging demand of the investor: the former distortion worsens the risk-return tradeoff that the investor faces, the latter distortion hampers the investor's ability to hedge the time variation in investment opportunities. The sign of CEL_2 is more ambiguous. By definition, CEL_2 compares the welfare of an unconstrained investor in the economies with and without index investors, so it is determined by the difference in investment opportunities in those economies. Typically, CEL_2 is negative (an unconstrained investor benefits from constraints imposed on other investors) because a lower volatility of the market returns in the economy with indexing makes the investment opportunities more attractive.

[15] Because the type P investors have logarithmic preferences, their wealth-consumption ratios are equal to $1/\beta$ in both constrained and unconstrained economies, and their welfare is unaffected when the indexing constraints are imposed on other investors.

Fig. 5 also shows that the welfare implications of indexing are particularly strong when the economy is dominated by the type I investors (when s is close to 1). Indeed, in those states, the equilibrium distortions produced by indexing are the largest because the majority of the investors are impacted by the constraint. The effects are also stronger when one of the dividends constitutes approximately 25% of the total consumption in the economy. Indeed, as demonstrated in Section 2.5, the effect of indexing on the wealth-consumption ratio is determined by portfolio distortions through the term $\frac{1}{2}\gamma_I(\omega_I - \omega_0)'\Sigma_Q\Sigma_Q'(\omega_I - \omega_0)$ in Eq. (27). When $u = 0$ or $u = 1$, the index contains only one stock, and the economy is identical to an economy with one tree. Therefore, there are no distortions produced by indexing, and the welfare is unchanged. When $u = 0.5$, the stocks have identical sizes, so even in the unconstrained economy investors would put equal fractions of their wealth in each stock, that is, hold the index portfolio. As a result, the indexing constraint is effectively nonbinding, and the welfare distortion is low. Consequently, the effect of indexing is the strongest in the intermediate states around $u = 0.25$ or $u = 0.75$.

To quantify the welfare implications of opening index markets for all investors in the pre-indexing economy, we use the certainty equivalent gain (CEG) of a type I investor. In the pre-indexing economy, such an investor consumes the cash flow $s_0 D_t$ and has the indirect utility function $J_0(s_0 D_t, u, t)$. In the economy with indexing, an asset with the cash flow $s_0 D_t$ has the price $(S_t/D_t)s_0 D_t = f(s, u)s_0 D_t$, and it is equal to the investors' wealth. The CEG is defined as the minimal additional fraction of the aggregate dividend that the type I investor in the pre-indexing economy should receive to make him indifferent between being in the pre-indexing economy and in the economy in which he can trade the risk-free asset and the index. Formally, the CEG solves the following equation:

$$J_0((1 + CEG(s, u))s_0 D_t, u, t) = J(f(s, u)s_0 D_t, s, u, t), \qquad (33)$$

where $J(W, s, u, t)$ is the type I investor's indirect utility function in the economy with indexing.

To find the CEG, note that by definition

$$J_0(s_0 D_t, u, t) = \mathbb{E}_t \left[\int_t^\infty e^{-\beta \tilde{t}} \frac{(s_0 D_{\tilde{t}})^{1-\gamma_I}}{1-\gamma_I} d\tilde{t} \right], \tag{34}$$

where the dynamics of D_t are described by Eq. (4). This indirect utility function can be found in a closed form, and the result is stated in Proposition 5.

Proposition 5 *The indirect utility function of a type I investor who lives in the pre-indexing economy is*

$$J_0(s_0 D_t, u, t) = \frac{1}{1-\gamma_I} (s_0 D_t)^{1-\gamma_I} h_0(u) \exp(-\beta t), \tag{35}$$

where the function $h_0(u)$ is given by Eq. (A69) in the Appendix.

Proof. See the Appendix.

Using the indirect utility functions from Eqs. (35) and (A27), Eq. (33) can be solved for the CEG:

$$CEG(s, u) = f(s, u) \left(\frac{h(s, u)^{\gamma_I}}{h_0(u)} \right)^{\frac{1}{1-\gamma_I}} - 1. \tag{36}$$

Note that the CEG does not depend on the share of the total dividend s_0 consumed by the type I investors in the pre-indexing economy.

For our standard calibration of the parameters, the graph for the CEG is presented in Panel B of Fig. 5. The CEG is positive, so risk sharing facilitated by index trading is beneficial to the investors despite producing excessive volatility and comovement of asset returns. The comparison of Panels A and B of Fig. 5 shows that the gain in utility produced by introducing of index trading is several orders of magnitude larger than the loss produced by investors' switching from trading individual assets to trading the index: the former can reach 20%, whereas the latter does not exceed 0.004%. Although our model cannot provide accurate magnitudes of the effects, so large differences are unlikely to disappear in a more realistic model.

Fig. 5 also shows that the CEG decreases with s. Indeed, there is almost no benefit from trading the index when the type I investors dominate in the economy with indexing (when s is close to 1) because in this case the risk-absorbing capacity of the type P investors is small, and the type I investors are exposed to almost the same fluctuations in the aggregate dividend as in the pre-indexing economy. In contrast, when the share of the type I investors is small (when s is close to 0), a large part of the aggregate risk can be unloaded by the type I investors to the type P investors through trading the index and risk-free asset. This makes the type I investors substantially better off than in the pre-indexing economy.

4. Conclusion

In this paper, we investigate the impact of index investing on the market equilibrium. Our analysis reveals that switching of some investors from trading individual stocks to trading the index decreases the volatility of market returns but has an ambiguous effect on the return correlation. Although it is widely recognized that indexing changes the dynamics of stock returns because it induces lockstep trading of many individual securities, our analysis shows that indexing also affects risk-sharing opportunities, which determine the incentives to trade, and this by itself has an impact on volatilities and correlations of stock returns. When the introduction of index trading opens financial markets for new investors, improved risk sharing makes market returns more volatile and stock returns more correlated.

Our analysis can be extended in several directions. In particular, our model can accommodate alternative types of indexes such as fundamental indexes, which were proposed in the literature and implemented in practice (e.g., Arnott et al., 2005). Also, it could be interesting to consider a setting with multiple trees in which only a subset of the trees is included in the index. Such a model could help investigate how the choice of assets for the index affects the equilibrium properties and provide a fully-fledged general equilibrium analysis of the correlations between the assets included and excluded from the index. This extension is likely to be harder to analyze than our model because of a larger number of state variables. Finally, it may be interesting to endogenize the dividend processes using a production economy framework and examine the impact of indexing on firm behavior. The analysis of how portfolio constraints affect corporate policies could be a particularly fruitful direction for future research.

Appendix. Proofs.

Proof of Proposition 1. The equilibrium functions r, μ_s, Σ_s, Σ_I, f, and h solve the following system of equations:

$$r = \beta + \Gamma\left(\mu_D - \frac{1}{2}(\gamma_I + 1)s\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)' \right.$$
$$\left. -\frac{1}{2}(\gamma_P + 1)(1-s)\left(\Sigma_D - \frac{1}{1-s}\Sigma_s\right)\left(\Sigma_D - \frac{1}{1-s}\Sigma_s\right)'\right), \quad \text{(A1)}$$

$$\mu_s = -\Sigma_s\Sigma_D' + \frac{s(1-s)}{\gamma_I\gamma_P}\Gamma\left(\mu_D(\gamma_P - \gamma_I) + \frac{\gamma_I(\gamma_I+1)}{2}\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)' \right.$$
$$\left. -\frac{\gamma_P(\gamma_P+1)}{2}\left(\Sigma_D - \frac{1}{1-s}\Sigma_s\right)\left(\Sigma_D - \frac{1}{1-s}\Sigma_s\right)'\right), \quad \text{(A2)}$$

$$\Sigma_s = \frac{\gamma_P - \gamma_I}{\gamma_I\gamma_P}s(1-s)\Gamma\Sigma_D\Pi_I - \frac{s}{h + sh_s}(h\Sigma_D + h_u\Sigma_u)(I_2 - \Pi_I), \quad \text{(A3)}$$

$$\Sigma_I = \frac{\left(f\Sigma_D + f_u\Sigma_u + f_s\frac{\gamma_P-\gamma_I}{\gamma_I\gamma_P}s(1-s)\Gamma\Sigma_D\right)\left(f\Sigma_D + f_u\Sigma_u - \frac{sf_s}{h+sh_s}(h\Sigma_D + h_u\Sigma_u)\right)'}{\left(f\Sigma_D + f_u\Sigma_u - \frac{sf_s}{h+sh_s}(h\Sigma_D + h_u\Sigma_u)\right)\left(f\Sigma_D + f_u\Sigma_u - \frac{sf_s}{h+sh_s}(h\Sigma_D + h_u\Sigma_u)\right)'} \times$$
$$\times\left(\Sigma_D + \frac{f_u}{f}\Sigma_u - \frac{f_s}{f}\frac{s}{h+sh_s}(h\Sigma_D + h_u\Sigma_u)\right), \quad \text{(A4)}$$

$$\frac{1}{2}f_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}f_{uu}\Sigma_u\Sigma_u' + f_{su}\Sigma_s\Sigma_u' + f_s(\mu_s + (1-\Gamma)\Sigma_D\Sigma_s')$$
$$+ f_u(\mu_u + (1-\Gamma)\Sigma_D\Sigma_u') + (\mu_D - r - \Gamma\Sigma_D\Sigma_D')f + 1 = 0, \quad \text{(A5)}$$

$$\frac{1}{2}h_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}h_{uu}\Sigma_u\Sigma_u' + h_{us}\Sigma_u\Sigma_s'$$
$$+ h_s\left(\mu_s - (\gamma_I - 1)\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)\Sigma_s'\right) + h_u\left(\mu_u - (\gamma_I - 1)\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)\Sigma_u'\right)$$
$$-\left(\frac{\gamma_I - 1}{2}\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)' + \frac{(\gamma_I - 1)r + \beta}{\gamma_I}\right)h + 1 = 0, \quad \text{(A6)}$$

where $\Pi_I = (\Sigma_I' \Sigma_I)/(\Sigma_I \Sigma_I')$. The market price of risk η and the expected excess return on the index μ_I are

$$\eta = \gamma_P \left(\Sigma_D - \frac{1}{1-s} \Sigma_s \right), \qquad \mu_I = \Gamma \Sigma_I \Sigma_D'. \tag{A7}$$

The price-dividend ratio f_i solves the following differential equation:

$$\frac{1}{2} f_{iss} \Sigma_s \Sigma_s' + \frac{1}{2} f_{iuu} \Sigma_u \Sigma_u' + f_{isu} \Sigma_s \Sigma_u' + f_{is}(\mu_s + (\Sigma_{Di} - \eta)\Sigma_s') \\ + f_{iu}(\mu_u + (\Sigma_{Di} - \eta)\Sigma_u') + (\mu_{Di} - r - \eta \Sigma_{Di}') f_i + 1 = 0. \tag{A8}$$

The expected excess returns on individual stocks μ_{Qi} and return diffusions Σ_{Qi} are

$$\mu_{Qi} = \gamma_P \Sigma_{Qi} \left(\Sigma_D - \frac{1}{1-s} \Sigma_s \right)', \qquad \Sigma_{Qi} = \Sigma_{Di} + \frac{f_{is}}{f_i} \Sigma_s + \frac{f_{iu}}{f_i} \Sigma_u. \tag{A9}$$

We derive Eqs. (A1) – (A9) in several steps.

A. Price-dividend ratios

First, we derive equations for the price-dividend ratios f_1, f_2, and f. By definition, $S_i = D_i f_i$. Applying Itô's lemma to this equation, we get

$$\frac{dS_i}{S_i} = \frac{dD_i}{D_i} + \frac{df_i}{f_i} + \frac{dD_i}{D_i} \frac{df_i}{f_i}, \tag{A10}$$

where

$$df_i = f_{is}(\mu_s dt + \Sigma_s dB) + f_{iu}(\mu_u dt + \Sigma_u dB) + \frac{1}{2} f_{iss} \Sigma_s \Sigma_s' dt + \frac{1}{2} f_{iuu} \Sigma_u \Sigma_u' dt + f_{ius} \Sigma_u \Sigma_s' dt. \tag{A11}$$

Using Eq. (1),

$$\frac{dS_i + D_i dt}{S_i} - r dt = \Bigg(\mu_{Di} - r + \frac{1}{2} \frac{f_{iss}}{f_i} \Sigma_s \Sigma_s' + \frac{1}{2} \frac{f_{iuu}}{f_i} \Sigma_u \Sigma_u' + \frac{f_{ius}}{f_i} \Sigma_u \Sigma_s' \\ + (\mu_s + \Sigma_{Di} \Sigma_s') \frac{f_{is}}{f_i} + (\mu_u + \Sigma_{Di} \Sigma_u') \frac{f_{iu}}{f_i} + \frac{1}{f_i} \Bigg) dt + \left(\Sigma_{Di} + \frac{f_{is}}{f_i} \Sigma_s + \frac{f_{iu}}{f_i} \Sigma_u \right) dB. \tag{A12}$$

This process should coincide with the process for excess returns from Eq. (3), so

$$\mu_{Qi} = \mu_{Di} - r + \frac{1}{2} \frac{f_{iss}}{f_i} \Sigma_s \Sigma_s' + \frac{1}{2} \frac{f_{iuu}}{f_i} \Sigma_u \Sigma_u' + \frac{f_{ius}}{f_i} \Sigma_u \Sigma_s' \\ + (\mu_s + \Sigma_{Di} \Sigma_s') \frac{f_{is}}{f_i} + (\mu_u + \Sigma_{Di} \Sigma_u) \frac{f_{iu}}{f_i} + \frac{1}{f_i}, \tag{A13}$$

$$\Sigma_{Qi} = \Sigma_{Di} + \frac{f_{is}}{f_i}\Sigma_s + \frac{f_{iu}}{f_i}\Sigma_u. \tag{A14}$$

Eq. (A13) is a differential equation for f_i:

$$\frac{1}{2}f_{iss}\Sigma_s\Sigma_s' + \frac{1}{2}f_{iuu}\Sigma_u\Sigma_u' + f_{isu}\Sigma_s\Sigma_u' + f_{is}(\mu_s + \Sigma_{Di}\Sigma_s') \\ + f_{iu}(\mu_u + \Sigma_{Di}\Sigma_u') + (\mu_{Di} - r - \mu_{Qi})f_i + 1 = 0. \tag{A15}$$

By definition of the market price of risk η, $\mu_{Qi} = \Sigma_{Qi}\eta'$. Plugging this representation for μ_{Qi} in Eq. (A15) and using Eq. (A14), we arrive at Eq. (A8). The same steps applied to the index yield the differential equation for the index price-dividend ratio f:

$$\frac{1}{2}f_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}f_{uu}\Sigma_u\Sigma_u' + f_{su}\Sigma_s\Sigma_u' + f_s(\mu_s + \Sigma_D\Sigma_s') \\ + f_u(\mu_u + \Sigma_D\Sigma_u') + (\mu_D - r - \mu_I)f + 1 = 0. \tag{A16}$$

The index diffusion is related to the diffusions of the state variables as

$$\Sigma_I = \Sigma_D + \frac{f_s}{f}\Sigma_s + \frac{f_u}{f}\Sigma_u, \tag{A17}$$

and this equation is similar to Eq. (A14).

B. Utility maximization problem of the type P investors

The type P investors can trade individual stocks, so from their perspective the market is complete. The first-order conditions of their utility maximization problem can be interpreted as pricing equations that relate the risk-free rate r and the expected excess returns μ_Q to their discount factor Λ (e.g., Cochrane, 2005):

$$r = -\frac{1}{dt}E\left(\frac{d\Lambda}{\Lambda}\right), \qquad \mu_{Qi} = -\frac{1}{dt}E\left(\frac{d\Lambda}{\Lambda}\frac{dS_i}{S_i}\right), \quad i = 1, 2. \tag{A18}$$

Since the investors have the CRRA preferences, their discount factor is $\Lambda = \exp(-\beta t)(C_P)^{-\gamma_P}$. Hence,

$$\frac{d\Lambda}{\Lambda} = -\beta dt - \gamma_P\frac{dC_P}{C_P} + \frac{\gamma_P(\gamma_P+1)}{2}\left(\frac{dC_P}{C_P}\right)^2. \tag{A19}$$

Itô's lemma applied to $C_P = (1-s)D$ together with Eqs. (4) and (10) yields

$$\frac{dC_P}{C_P} = \mu_{CP}dt + \Sigma_{CP}dB, \tag{A20}$$

where

$$\mu_{CP} = \mu_D - \frac{\mu_s + \Sigma_D \Sigma_s'}{1-s}, \qquad \Sigma_{CP} = \Sigma_D - \frac{1}{1-s}\Sigma_s. \tag{A21}$$

Therefore,

$$\frac{d\Lambda}{\Lambda} = -\beta dt - \gamma_P \left(\mu_D - \frac{\mu_s + \Sigma_D \Sigma_s'}{1-s} - \frac{\gamma_P + 1}{2} \left(\Sigma_D - \frac{1}{1-s}\Sigma_s \right) \left(\Sigma_D - \frac{1}{1-s}\Sigma_s \right)' \right) dt \\ - \gamma_P \left(\Sigma_D - \frac{1}{1-s}\Sigma_s \right) dB. \tag{A22}$$

Using Eq. (A18), we find the risk-free rate r and the expected excess returns μ_Q and μ_I:

$$r = \beta + \gamma_P \mu_D - \frac{\gamma_P}{1-s}\left(\mu_s + \Sigma_s \Sigma_D'\right) - \frac{\gamma_P(\gamma_P + 1)}{2} \left(\Sigma_D - \frac{1}{1-s}\Sigma_s \right) \left(\Sigma_D - \frac{1}{1-s}\Sigma_s \right)', \tag{A23}$$

$$\mu_Q = \gamma_P \Sigma_Q \left(\Sigma_D - \frac{1}{1-s}\Sigma_s \right)', \qquad \mu_I = \gamma_P \Sigma_I \left(\Sigma_D - \frac{1}{1-s}\Sigma_s \right)'. \tag{A24}$$

C. Utility maximization problem of the type I investors

The type I investors maximize the CRRA utility from Eq. (7) subject to the budget constraint from Eq. (9). Because they can trade only the index and the risk-free bond, from their perspective the market is incomplete, and the utility maximization problem should be solved directly. In particular, their indirect utility function J satisfies the standard Hamilton-Jacobi-Bellman (HJB) equation

$$\max_{\{C_I, \hat{\omega}_I\}} \left[e^{-\beta t} \frac{C_I^{1-\gamma_I}}{1-\gamma_I} + \mathcal{D}J \right] = 0, \tag{A25}$$

where $\mathcal{D}J = \mathbb{E}[dJ]/dt$ is given by

$$\mathcal{D}J = J_W(rW_I - C_I + \hat{\omega}_I W_I \mu_I) + \frac{1}{2} J_{WW} W_I^2 \hat{\omega}_I^2 \Sigma_I \Sigma_I' + J_{Ws} \hat{\omega}_I W_I \Sigma_I \Sigma_s' + J_{Wu} \hat{\omega}_I W_I \Sigma_I \Sigma_u' \\ + J_s \mu_s + J_u \mu_u + \frac{1}{2} J_{ss} \Sigma_s \Sigma_s' + \frac{1}{2} J_{uu} \Sigma_u \Sigma_u' + J_{us} \Sigma_s \Sigma_u' + J_t, \tag{A26}$$

and the subscripts of J denote derivatives with respect to the corresponding variables. When investors have the CRRA preferences, it is standard to look for the indirect utility in the following form:

$$J = \frac{1}{1-\gamma_I} W_I^{1-\gamma_I} h^{\gamma_I} \exp(-\beta t), \tag{A27}$$

where the function h depends on the state variables s and u. The maximization in (A25) with respect to C_I together with Eq. (A27) yields the optimal consumption:

$$C_I = W_I h^{-1}, \tag{A28}$$

so h is the optimal wealth-consumption ratio. Similarly, the maximization in (A25) with respect to $\hat{\omega}_I$ gives the optimal weight of the index:

$$\hat{\omega}_I = \frac{1}{\Sigma_I \Sigma_I'} \left(\frac{\mu_I}{\gamma_I} + \frac{h_s}{h} \Sigma_I \Sigma_s' + \frac{h_u}{h} \Sigma_I \Sigma_u' \right). \tag{A29}$$

The substitution of Eqs. (A28) and (A29) back into (A25) yields a differential equation for h:

$$\begin{aligned} &\frac{1}{2} h_{ss} \Sigma_s \Sigma_s' + \frac{1}{2} h_{uu} \Sigma_u \Sigma_u' + h_{us} \Sigma_u \Sigma_s' + h_s \mu_s + h_u \mu_u \\ &\quad + \frac{\gamma_I - 1}{2} \left(\left(\frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right) \left(\frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right)' - \frac{1}{\Sigma_I \Sigma_I'} \left(\frac{\mu_I}{\gamma_I} + \frac{h_s}{h} \Sigma_I \Sigma_s' + \frac{h_u}{h} \Sigma_I \Sigma_u' \right)^2 \right) h \\ &\quad + \frac{1}{\gamma_I} ((1 - \gamma_I) r - \beta) h + 1 = 0. \end{aligned} \tag{A30}$$

D. Dynamics of the state variable s and returns on the index

Next, we find the expressions for μ_s, Σ_s, μ_I, and Σ_I. Itô's lemma applied to $C_I = sD$ yields

$$\frac{dC_I}{C_I} = \mu_{CI} dt + \Sigma_{CI} dB, \qquad \frac{dC_I^{-\gamma_I}}{C_I^{-\gamma_I}} = \left(-\gamma_I \mu_{CI} + \frac{1}{2} \gamma_I (\gamma_I + 1) \Sigma_{CI} \Sigma_{CI}' \right) dt - \gamma_I \Sigma_{CI} dB, \tag{A31}$$

where

$$\mu_{CI} = \mu_D + \frac{\mu_s + \Sigma_s \Sigma_D'}{s}, \qquad \Sigma_{CI} = \Sigma_D + \frac{1}{s} \Sigma_s. \tag{A32}$$

Using $C_I = W_I h^{-1}$, the indirect utility function from Eq. (A27) can be rewritten as

$$J = \frac{1}{1 - \gamma_I} C_I^{-\gamma_I} W_I \exp(-\beta t). \tag{A33}$$

Applying Itô's lemma to this equation and taking into account Eqs. (9) and (A31), we get

$$\frac{dJ}{J} = \left(-\beta - \gamma_I \mu_{CI} + \frac{1}{2} \gamma_I (\gamma_I + 1) \Sigma_{CI} \Sigma_{CI}' + r - h^{-1} + \hat{\omega}_I (\mu_I - \gamma_I \Sigma_I \Sigma_{CI}') \right) dt + (\hat{\omega}_I \Sigma_I - \gamma_I \Sigma_{CI}) dB. \tag{A34}$$

Alternatively, Itô's lemma applied to Eq. (A27) yields

$$\frac{dJ}{J} = \frac{\mathcal{D} J}{J} dt + \left((1 - \gamma_I) \hat{\omega}_I \Sigma_I + \gamma_I \frac{h_s}{h} \Sigma_s + \gamma_I \frac{h_u}{h} \Sigma_u \right) dB. \tag{A35}$$

Noting that Eqs. (A27) and (A28) imply that

$$e^{-\beta t}\frac{C_I^{1-\gamma_I}}{1-\gamma_I} = Jh^{-1} \tag{A36}$$

and using the HJB equation (A25), we get $\mathcal{D}J = -Jh^{-1}$ and rewrite Eq. (A35) as

$$\frac{dJ}{J} = -h^{-1}dt + \left((1-\gamma_I)\hat{\omega}_I\Sigma_I + \gamma_I\frac{h_s}{h}\Sigma_s + \gamma_I\frac{h_u}{h}\Sigma_u\right)dB. \tag{A37}$$

Matching the drifts and diffusions in Eqs. (A34) and (A37) and using μ_{CI} and Σ_{CI} from Eq. (A32), we get

$$\frac{1+\gamma_I}{2}\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)' + \frac{r-\beta}{\gamma_I} + \hat{\omega}_I\left(\frac{\mu_I}{\gamma_I} - \left(\Sigma_D + \frac{1}{s}\Sigma_s\right)\Sigma_I'\right)$$
$$= \mu_D + \frac{1}{s}(\mu_s + \Sigma_s\Sigma_D'), \tag{A38}$$

$$\hat{\omega}_I\Sigma_I - \frac{h_s}{h}\Sigma_s - \frac{h_u}{h}\Sigma_u = \Sigma_D + \frac{1}{s}\Sigma_s. \tag{A39}$$

Eq. (A39) helps to derive a system of equations for Σ_I and Σ_s. Plugging the optimal portfolio weight $\hat{\omega}_I$ from Eq. (A29) into Eq. (A39) yields

$$\frac{\mu_I\Sigma_I}{\gamma_I(\Sigma_I\Sigma_I')} - \left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\left(I_2 - \frac{\Sigma_I'\Sigma_I}{\Sigma_I\Sigma_I'}\right) = \Sigma_D + \frac{1}{s}\Sigma_s, \tag{A40}$$

where I_2 is a 2×2 identity matrix. Multiplying this equation by Σ_I', we get

$$\mu_I = \gamma_I\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)\Sigma_I', \tag{A41}$$

which together with the expression for μ_I from (A24) gives

$$\Sigma_s\Sigma_I' = \left(\frac{\gamma_I}{s} + \frac{\gamma_P}{1-s}\right)^{-1}(\gamma_P - \gamma_I)\Sigma_D\Sigma_I'. \tag{A42}$$

The substitution of this equation in Eq. (A41) yields $\mu_I = \Gamma\Sigma_D\Sigma_I'$, where Γ is defined in Eq. (14). This expression for μ_I is a part of Eq. (A7). Plugging it into Eq. (A40), introducing the matrix $\Pi_I = (\Sigma_I'\Sigma_I)/(\Sigma_I\Sigma_I')$, which is a projector operator on the vector Σ_I, and rearranging the terms, we get

$$\Sigma_s = (\gamma_P - \gamma_I)\left(\frac{\gamma_I}{s} + \frac{\gamma_P}{1-s}\right)^{-1}\Sigma_D - s\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u + \frac{1}{\gamma_I}\Gamma\Sigma_D\right)(I_2 - \Pi_I). \tag{A43}$$

The solution of this equation for Σ_s is

$$\Sigma_s = (\gamma_P - \gamma_I)\left(\frac{\gamma_I}{s} + \frac{\gamma_P}{1-s}\right)^{-1}\Sigma_D\Pi_I - \frac{s}{h+sh_s}(h\Sigma_D + h_u\Sigma_u)(I_2 - \Pi_I). \tag{A44}$$

Using the definition of Γ, we obtain Eq. (A3).

Eqs. (A17) and (A44) jointly determine Σ_s and Σ_I. The substitution of Σ_s from (A17) in (A44) yields an equation for Σ_I:

$$\Sigma_I = \frac{f_s}{f}\left[(\gamma_P - \gamma_I)\left(\frac{\gamma_I}{s} + \frac{\gamma_P}{1-s}\right)^{-1}\Sigma_D + \frac{s}{h+sh_s}(h\Sigma_D + h_u\Sigma_u)\right]\Pi_I + \left[\Sigma_D + \frac{f_u}{f}\Sigma_u - \frac{f_s}{f}\frac{s}{h+sh_s}(h\Sigma_D + h_u\Sigma_u)\right]. \tag{A45}$$

To solve this equation, we use the following lemma.

Lemma 1. Consider a linear space with a scalar product $(\cdot,\cdot)$ *and denote by* Π_x *the orthogonal projection on the vector* x. *Also, let* a *and* b *be two vectors and assume that* $(b,b) > 0$. *Then, the equation for* x

$$x = \Pi_x a + b \tag{A46}$$

has the unique solution

$$x = \frac{(a+b,b)}{(b,b)}b. \tag{A47}$$

Proof of Lemma 1. The application of the operator Π_x to both sides of Eq. (A46) gives $x = \Pi_x a + \Pi_x b$, which together with the initial Eq. (A46) implies that $\Pi_x b = b$. Hence, the vector b belongs to the subspace spanned by the vector x, so $x = \lambda b$, $\lambda \in \mathbb{R}$. The substitution of this expression in Eq. (A46) yields $\lambda b = \Pi_b a + b$, which implies $\lambda = (\Pi_b a + b, b)/(b,b)$. Finally, $(\Pi_b a, b) = (a - (I - \Pi_b)a, b) = (a,b)$, where I is the identity operator. Q.E.D.

Eq. (A45) has exactly the form of Eq. (A46) with Σ_I corresponding to x. Hence,

$$\Sigma_I = \frac{\left(f\Sigma_D + f_u\Sigma_u + f_s(\gamma_P - \gamma_I)\left(\frac{\gamma_I}{s} + \frac{\gamma_P}{1-s}\right)^{-1}\Sigma_D\right)\left(f\Sigma_D + f_u\Sigma_u - \frac{sf_s}{h+sh_s}(h\Sigma_D + h_u\Sigma_u)\right)'}{\left(f\Sigma_D + f_u\Sigma_u - \frac{sf_s}{h+sh_s}(h\Sigma_D + h_u\Sigma_u)\right)\left(f\Sigma_D + f_u\Sigma_u - \frac{sf_s}{h+sh_s}(h\Sigma_D + h_u\Sigma_u)\right)'} \times \left(\Sigma_D + \frac{f_u}{f}\Sigma_u - \frac{f_s}{f}\frac{s}{h+sh_s}(h\Sigma_D + h_u\Sigma_u)\right). \tag{A48}$$

This is Eq. (A4). To derive the expression for μ_s, we use Eq. (A38), which together with Eq.

(A41) yields

$$\frac{1+\gamma_I}{2}\left(\Sigma_D+\frac{1}{s}\Sigma_s\right)\left(\Sigma_D+\frac{1}{s}\Sigma_s\right)'+\frac{r-\beta}{\gamma_I}=\mu_D+\frac{1}{s}(\mu_s+\Sigma_s\Sigma_D'). \tag{A49}$$

Eqs. (A23) and (A49) can be viewed as a system of linear equations for r and μ_s. Its solution is given by Eqs. (A1) and (A2).

E. Differential equations for f and h

Eq. (A5) for the price-dividend ratio f follows from (A16) after noting that $\mu_I=\Gamma\Sigma_D\Sigma_I'$ and Σ_I is given by Eq. (A17). Eq. (A6) for the wealth-consumption ratio h is derived from Eq. (A30). Using the expression for μ_I from (A41) and noting that (A43) implies that

$$\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u+\Sigma_D+\frac{1}{s}\Sigma_s\right)(I_2-\Pi_I)=0, \tag{A50}$$

we get

$$\begin{aligned}
&\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)'-\frac{1}{\Sigma_I\Sigma_I'}\left(\frac{\mu_I}{\gamma_I}+\frac{h_s}{h}\Sigma_I\Sigma_s'+\frac{h_u}{h}\Sigma_I\Sigma_u'\right)^2\\
&\qquad=\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)'\\
&\qquad-\left(\Sigma_D+\frac{1}{s}\Sigma_s+\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)\Pi_I\left(\Sigma_D+\frac{1}{s}\Sigma_s+\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)'\\
&\qquad=\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)'\\
&\qquad-\left(\Sigma_D+\frac{1}{s}\Sigma_s+\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)\left(\Sigma_D+\frac{1}{s}\Sigma_s+\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)'\\
&\qquad=-2\left(\Sigma_D+\frac{1}{s}\Sigma_s\right)\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)'-\left(\Sigma_D+\frac{1}{s}\Sigma_s\right)\left(\Sigma_D+\frac{1}{s}\Sigma_s\right)'.
\end{aligned} \tag{A51}$$

This transformation allows us to eliminate the quadratic terms with h_s and h_u from Eq. (A30) and get Eq. (A6). Q.E.D.

Proof of Proposition 2. Any portfolio $\omega\in\Omega$ has the form $\omega=\theta\omega_M$, where θ is a scalar and $\omega_M=[S_1/S\ S_2/S]'$ is the market portfolio. Therefore, the optimization problem in (19) is one-dimensional and reduces to the minimization of $(\theta\omega_M-\omega_0)'\Sigma_Q\Sigma_Q'(\theta\omega_M-\omega_0)$ with respect to θ. The first-order condition is $\omega_M'\Sigma_Q\Sigma_Q'(\theta\omega_M - \omega_0) = 0$, and its solution for θ is

$$\theta = \frac{\omega_M'\Sigma_Q\Sigma_Q'\omega_0}{\omega_M'\Sigma_Q\Sigma_Q'\omega_M} = \frac{\Sigma_I(\Sigma_Q'\omega_0)}{\Sigma_I\Sigma_I'}. \tag{A52}$$

Using the definition of ω_0 from Eq. (18) and noting that $\Sigma_I\Sigma_Q^{-1}\mu_Q = \omega_M'\Sigma_Q\Sigma_Q^{-1}\mu_Q = \mu_I$, we get

$$\theta = \frac{\Sigma_I}{\Sigma_I\Sigma_I'}\left(\frac{\Sigma_Q^{-1}\mu_Q}{\gamma_I} + \frac{h_s}{h}\Sigma_s' + \frac{h_u}{h}\Sigma_u'\right) = \frac{1}{\Sigma_I\Sigma_I'}\left(\frac{\mu_I}{\gamma_I} + \frac{h_s}{h}\Sigma_I\Sigma_s' + \frac{h_u}{h}\Sigma_I\Sigma_u'\right) = \hat{\omega}_I. \tag{A53}$$

To obtain Eq. (20), note that

$$\Sigma_Q'\omega_I = \Sigma_Q'\omega_M\frac{\Sigma_I(\Sigma_Q'\omega_0)}{\Sigma_I\Sigma_I'} = \frac{\Sigma_I'\Sigma_I(\Sigma_Q'\omega_0)}{\Sigma_I\Sigma_I'} = \Pi_I\Sigma_Q'\omega_0. \tag{A54}$$

Q.E.D.

Proof of Proposition 3. First, we derive Eq. (22). Note that

$$\begin{aligned}
(\omega_I - \omega_0)'\Sigma_Q &\overset{\text{(i)}}{=} \omega_0'\Sigma_Q(\Pi_I - I_2) \overset{\text{(ii)}}{=} \left(\frac{\mu_Q'(\Sigma_Q')^{-1}}{\gamma_I} + \frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)(\Pi_I - I_2) \\
&\overset{\text{(iii)}}{=} \left(\frac{\gamma_P}{\gamma_I}\left(\Sigma_D - \frac{1}{1-s}\Sigma_s\right) + \frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)(\Pi_I - I_2) \\
&\overset{\text{(iv)}}{=} \left(\frac{\gamma_P}{\gamma_I}\Sigma_D + \frac{s\gamma_P}{\gamma_I(1-s)}\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u + \Sigma_D\right) + \frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)(\Pi_I - I_2) \\
&\overset{\text{(v)}}{=} \frac{\gamma_P\Gamma^{-1}}{1-s}\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u + \frac{1}{\gamma_I}\Gamma\Sigma_D\right)(\Pi_I - I_2) \overset{\text{(vi)}}{=} \frac{\gamma_P\Gamma^{-1}}{s(1-s)}\left(\Sigma_s - \Sigma_s^{unc}\right),
\end{aligned} \tag{A55}$$

where I_2 is the 2×2 identity matrix. Equality (i) in (A55) uses Eq. (20), equality (ii) exploits the definition of ω_0 from Eq. (18), equality (iii) is the result of substituting the market price of risk from Eq. (A7), equality (iv) applies Eq. (A50), equality (v) uses the definition of Γ from Eq. (14), and equality (vi) immediately follows from Eq. (A43). Resolving the equality between the first and last terms in (A55) for Σ_s, we get (22).

Eqs. (21) and (26) are obtained by plugging the representation of Σ_s from (22) into (21), (A32), and (A21) and using μ_s^{unc} from (16). Eq. (24) immediately follows from combining Eq. (A1) with (A21) and (A32). Eq. (25) is a combination of (A7) and (A21).

To derive Eq. (27), note that the indirect utility function of a type I investor J solves the following HJB equation:

$$0 = \max_{\{C,\omega\in\Omega\}} \left[e^{-\beta t}\frac{C^{1-\gamma_I}}{1-\gamma_I} + J_W(W_I(r+\omega'\mu_Q) - C_I) + \frac{1}{2}J_{WW}W_I^2\omega'\Sigma_Q\Sigma_Q'\omega + J_{Ws}W_I\omega'\Sigma_Q\Sigma_s' \right.$$
$$\left. + J_{Wu}W_I\omega'\Sigma_Q\Sigma_u' + J_s\mu_s + J_u\mu_u + \frac{1}{2}J_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}J_{uu}\Sigma_u\Sigma_u' + J_{us}\Sigma_s\Sigma_u' + J_t \right], \quad \text{(A56)}$$

where Ω is the set of all index portfolios. As in the proof of Proposition 1, we conjecture that the value function is given by Eq. (A27) and obtain the optimal consumption (A28). The substitution of (A27) and (A28) into (A56) yields

$$\frac{1}{2}h_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}h_{uu}\Sigma_u\Sigma_u' + h_{su}\Sigma_s\Sigma_u' + h_s\mu_s + h_u\mu_u + \frac{h}{\gamma_I}\left((1-\gamma_I)r - \beta\right) + 1$$
$$- \frac{(1-\gamma_I)h}{2}\min_{\omega\in\Omega}\left[\omega'\Sigma_Q\Sigma_Q'\omega - 2\omega'\left(\frac{\mu_Q}{\gamma_I} + \frac{h_s}{h}\Sigma_Q\Sigma_s + \frac{h_u}{h}\Sigma_Q\Sigma_u\right)\right.$$
$$\left. + \left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)'\right] = 0. \quad \text{(A57)}$$

Note that now the equation involves minimization, not maximization, because the optimized function has been multiplied by a negative factor. Using the definition of ω_0 from Eq. (18), the minimized expression from (A57) can be written as

$$\omega'\Sigma_Q\Sigma_Q'\omega - 2\omega'\left(\frac{\mu_Q}{\gamma_I} + \frac{h_s}{h}\Sigma_Q\Sigma_s + \frac{h_u}{h}\Sigma_Q\Sigma_u\right) + \left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)'$$
$$= \omega'\Sigma_Q\Sigma_Q'\omega - 2\omega'\Sigma_Q\Sigma_Q'\omega_0 + \left(\Sigma_Q'\omega_0 - \frac{\Sigma_Q^{-1}\mu_Q}{\gamma_I}\right)'\left(\Sigma_Q'\omega_0 - \frac{\Sigma_Q^{-1}\mu_Q}{\gamma_I}\right)$$
$$= (\omega-\omega_0)'\Sigma_Q\Sigma_Q'(\omega-\omega_0) - \frac{2}{\gamma_I}\omega_0'\mu_Q + \frac{1}{\gamma_I^2}\mu_Q'(\Sigma_Q\Sigma_Q')^{-1}\mu_Q$$
$$= (\omega-\omega_0)'\Sigma_Q\Sigma_Q'(\omega-\omega_0) - \frac{2h_s}{\gamma_I h}\Sigma_s\Sigma_Q^{-1}\mu_Q - \frac{2h_u}{\gamma_I h}\Sigma_u\Sigma_Q^{-1}\mu_Q - \frac{1}{\gamma_I^2}\mu_Q'(\Sigma_Q\Sigma_Q')^{-1}\mu_Q. \quad \text{(A58)}$$

Collecting all terms, we get

$$\frac{1}{2}h_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}h_{uu}\Sigma_u\Sigma_u' + h_{su}\Sigma_s\Sigma_u' + h_s\left(\mu_s - \frac{1-\gamma_I}{\gamma_I}\Sigma_s\Sigma_Q^{-1}\mu_Q\right) + h_u\left(\mu_u - \frac{1-\gamma_I}{\gamma_I}\Sigma_u\Sigma_Q^{-1}\mu_Q\right)$$
$$+ \frac{h}{\gamma_I}\left((1-\gamma_I)\left(r - \frac{\gamma_I}{2}\min_{\omega\in\Omega}(\omega-\omega_0)'\Sigma_Q\Sigma_Q'(\omega-\omega_0)\right) + \frac{1-\gamma_I}{2\gamma_I}\mu_Q'(\Sigma_Q\Sigma_Q')^{-1}\mu_Q - \beta\right) + 1 = 0.$$
$$\text{(A59)}$$

Using Proposition 2 and Eq. (19), we arrive at Eq. (27). Q.E.D.

Proof of Proposition 4. The equilibrium functions r, μ_s, Σ_s, Σ_I, f, and h solve the following system of equations:

$$\begin{aligned} r = \beta + \Gamma \Bigg(\mu_D - \frac{1}{2}(\gamma_I+1)s\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)' - \frac{1}{2}(\gamma_P+1)(1-s) \\ \times \left(\Sigma_D - \frac{1}{1-s}\Sigma_s\right)\left(\Sigma_D - \frac{1}{1-s}\Sigma_s\right)'\Bigg) - \frac{\gamma_P\hat{\omega}_I}{1-s}\left(\frac{\gamma_P-\gamma_I}{\gamma_I\gamma_P}s(1-s)\Gamma\Sigma_D - \Sigma_s\right)\Sigma_I', \end{aligned} \tag{A60}$$

$$\begin{aligned} \mu_s = -\Sigma_s\Sigma_D' + \frac{s(1-s)}{\gamma_I\gamma_P}\Gamma\Bigg(\mu_D(\gamma_P-\gamma_I) + \frac{\gamma_I(\gamma_I+1)}{2}\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)' \\ -\frac{\gamma_P(\gamma_P+1)}{2}\left(\Sigma_D - \frac{1}{1-s}\Sigma_s\right)\left(\Sigma_D - \frac{1}{1-s}\Sigma_s\right)'\Bigg) \\ + \hat{\omega}_I\left(\frac{\gamma_P-\gamma_I}{\gamma_I\gamma_P}s(1-s)\Gamma\Sigma_D - \Sigma_s\right)\Sigma_I', \end{aligned} \tag{A61}$$

$$\Sigma_s = -\frac{1}{\hat{\omega}_I\frac{f_s}{f} - \frac{h_s}{h} - \frac{1}{s}}\left(\Sigma_D(\hat{\omega}_I - 1) + \Sigma_u\left(\hat{\omega}_I\frac{f_u}{f} - \frac{h_u}{h}\right)\right), \tag{A62}$$

$$\Sigma_I = \frac{1}{\hat{\omega}_I\frac{f_s}{f} - \frac{h_s}{h} - \frac{1}{s}}\left(\Sigma_D\left(\frac{f_s}{f} - \frac{h_s}{h} - \frac{1}{s}\right) - \Sigma_u\left(\frac{f_u}{sf} + \frac{f_u}{f}\frac{h_s}{h} - \frac{f_s}{f}\frac{h_u}{h}\right)\right), \tag{A63}$$

$$\begin{aligned} \frac{1}{2}f_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}f_{uu}\Sigma_u\Sigma_u' + f_{su}\Sigma_s\Sigma_u' + f_s(\mu_s + \Sigma_D\Sigma_s') \\ + f_u(\mu_u + \Sigma_D\Sigma_u') + \left(\mu_D - r - \gamma_P\Sigma_I\left(\Sigma_D - \frac{1}{1-s}\Sigma_s\right)'\right)f + 1 = 0, \end{aligned} \tag{A64}$$

$$\begin{aligned} \frac{1}{2}h_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}h_{uu}\Sigma_u\Sigma_u' + h_{us}\Sigma_u\Sigma_s' + h_s\mu_s + h_u\mu_u \\ - \Bigg(\hat{\omega}_I\frac{\gamma_P(\gamma_I-1)}{\gamma_I}\left(\Sigma_D - \frac{1}{1-s}\Sigma_s\right)\Sigma_I' - \frac{\gamma_I-1}{2}\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)' \\ + \frac{(\gamma_I-1)r+\beta}{\gamma_I}\Bigg)h + 1 = 0. \end{aligned} \tag{A65}$$

The price-dividend ratio f_i solves Eq. (A8). The expected excess returns on individual stocks μ_{Qi} and return diffusions Σ_{Qi} are given by (A9).

The proof of the proposition closely follows the proof of Proposition 1, so to save space we present it with less details. In particular, the analysis of the price-dividend ratio presented in part A of the proof of Proposition 1 does not change, and Eqs. (A15) – (A17) still hold. Moreover, Eqs. (A14) and (A15) together with the definition $\mu_{Qi} = \Sigma_{Qi}\eta'$ again give (A8). Similarly, the utility maximization problem of the type P investors from part B is the same and yields Eqs. (A23) and (A24). Combining Eqs. (A16) and (A24), we get (A64). The utility maximization problem of the type I investors is now different and yields a different HJB equation:

$$\max_{\{C_I\}} \left[e^{-\beta t} \frac{C_I^{1-\gamma_I}}{1-\gamma_I} + \mathcal{D}J \right] = 0, \tag{A66}$$

where

$$\begin{aligned} \mathcal{D}J = J_W(rW_I - C_I + \hat{\omega}_I W_I \mu_I) + \frac{1}{2} J_{WW} W_I^2 \hat{\omega}_I^2 \Sigma_I \Sigma_I' + J_{Ws} \hat{\omega}_I W_I \Sigma_I \Sigma_s' + J_{Wu} \hat{\omega}_I W_I \Sigma_I \Sigma_u' \\ + J_s \mu_s + J_u \mu_u + \frac{1}{2} J_{ss} \Sigma_s \Sigma_s' + \frac{1}{2} J_{uu} \Sigma_u \Sigma_u' + J_{us} \Sigma_s \Sigma_u' + J_t \end{aligned} \tag{A67}$$

and $\hat{\omega}_I$ is fixed as $\hat{\omega}_I = \omega_{I1}^{unc} + \omega_{I2}^{unc}$. The indirect utility function and optimal consumption are still given by (A27) and (A28) but the equation for h becomes

$$\begin{aligned} &\frac{1}{2} h_{ss} \Sigma_s \Sigma_s' + \frac{1}{2} h_{uu} \Sigma_u \Sigma_u' + h_{us} \Sigma_u \Sigma_s' + h_s \mu_s + h_u \mu_u \\ &+ \frac{\gamma_I - 1}{2} \left(\left(\frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right) \left(\frac{h_s}{h} \Sigma_s + \frac{h_u}{h} \Sigma_u \right)' + \hat{\omega}_I^2 \Sigma_I \Sigma_I' - 2\hat{\omega}_I \left(\frac{\mu_I}{\gamma_I} + \frac{h_s}{h} \Sigma_I \Sigma_s' + \frac{h_u}{h} \Sigma_I \Sigma_u' \right) \right) h \\ &+ \frac{1}{\gamma_I} ((1-\gamma_I) r - \beta) h + 1 = 0. \end{aligned} \tag{A68}$$

The dynamics of the state variable s and returns on the index derived in part D also change, although the discussion that leads to Eqs. (A38) and (A39) is still valid. Eqs. (A17) and (A39) represent a system of linear equations for Σ_s and Σ_I, which can be easily resolved, and the solutions are given by (A62) and (A63). Also, using (A39), Eq. (A68) can be written as Eq. (A65). Finally, Eqs. (A23) and (A38) represent a system of equations for r and μ_s, and its solution is given by (A60) and (A61). Q.E.D.

Proof of Proposition 5. The expectation (34) can be found using the results of Proposition 4 from Martin (2013). In particular, the function $h_0(u)$ coincides with the stock price-dividend ratio in the Martin (2013) economy and can be written in terms of the hypergeometric functions

${}_2F_1(a,b;c;x)$ as

$$h_0(u) = \frac{4}{X^2(\lambda_1-\lambda_2)}\left(\frac{1}{(\gamma_I-1+2\lambda_1)u^{\gamma_I-1}}\,{}_2F_1\left(\gamma_I-1, \frac{\gamma_I-1+2\lambda_1}{2}; \frac{\gamma_I+1+2\lambda_1}{2}; \frac{u-1}{u}\right)\right.$$
$$\left.+\frac{1}{(\gamma_I-1-2\lambda_2)(1-u)^{\gamma_I-1}}\,{}_2F_1\left(\gamma_I-1, \frac{\gamma_I-1-2\lambda_2}{2}; \frac{\gamma_I+1-2\lambda_2}{2}; \frac{u}{u-1}\right)\right), \quad \text{(A69)}$$

where X^2, λ_1, λ_2, Z^2, and Y are constants given by

$$X^2 = (\Sigma_{D1}-\Sigma_{D2})(\Sigma_{D1}-\Sigma_{D2})', \qquad Y = \mu_{D1}-\mu_{D2}-\frac{\gamma_I+1}{2}(\Sigma_{D1}\Sigma_{D1}'-\Sigma_{D2}\Sigma_{D2}'), \quad \text{(A70)}$$

$$Z^2 = 2\beta + (\gamma_I-1)\left(\mu_{D1}+\mu_{D2}-\frac{\gamma_I+1}{4}(\Sigma_{D1}\Sigma_{D1}'+\Sigma_{D2}\Sigma_{D2}')-\frac{\gamma_I-1}{2}\Sigma_{D1}\Sigma_{D2}'\right), \quad \text{(A71)}$$

$$\lambda_1 = \frac{\sqrt{Y^2+X^2Z^2}-Y}{X^2}, \qquad \lambda_2 = -\frac{\sqrt{Y^2+X^2Z^2}+Y}{X^2}. \ \text{Q.E.D.} \quad \text{(A72)}$$

References

Arnott, R. D., Hsu, J., Moore, P., 2005. Fundamental indexation. Financial Analysts Journal 61, 83–99.

Barberis, N., Shleifer, A., 2003. Style investing. Journal of Financial Economics 68, 161–199.

Barberis, N., Shleifer, A., Wurgler, J., 2005. Comovement. Journal of Financial Economics 75, 283–317.

Basak, S., Croitoru, B., 2000. Equilibrium mispricing in a capital market with portfolio constraints. Review of Financial Studies 13, 715–748.

Basak, S., Cuoco, D., 1998. An equilibrium model with restricted stock market participation. Review of Financial Studies 11, 309–341.

Basak, S., Pavlova, A., 2013. Asset prices and institutional investors. American Economic Review 103, 1728–1758.

Basak, S., Pavlova, A., 2016. A model of financialization of commodities. Journal of Finance 71, 1511–1556.

Ben-David, I., Franzoni, F. A., Moussawi, R., 2018. Do ETFs increase volatility? Journal of Finance 73, 2471–2535.

Bhamra, H. S., Uppal, R., 2009. The effect of introducing a non-redundant derivative on the volatility of stock-market returns when agents differ in risk aversion. Review of Financial Studies 22, 2303–2330.

Bhamra, H. S., Uppal, R., 2014. Asset prices with heterogeneity in preferences and beliefs. Review of Financial Studies 27, 519–580.

Bond, P., Garcia, D., 2018. The equilibrium consequences of indexing. Unpublished working paper. University of Washington.

Borovička, J., 2020. Survival and long-run dynamics with heterogeneous beliefs under recursive preferences. Journal of Political Economy 128, 206–251.

Boyer, B. H., 2011. Style-related comovement: Fundamentals or labels? Journal of Finance 66, 307–332.

Breugem, M., Buss, A., 2019. Institutional investors and information acquisition: implications for asset prices and information efficiency. Review of Financial Studies 32, 2260–2301.

Brumm, J., Grill, M., Kubler, F., Schmedders, K., 2015. Collateral requirements and asset prices. International Economic Review 56, 1–25.

Buffa, A. M., Hodor, I., 2018. Institutional investors, heterogeneous benchmarks and the co-movement of asset prices. Unpublished working paper. Boston University.

Buffa, A. M., Vayanos, D., Woolley, P., 2015. Asset management contracts and equilibrium prices. Unpublished working paper. LSE.

Buss, A., Dumas, B., 2019. The dynamic properties of financial-market equilibrium with trading fees. Journal of Finance 74, 795–844.

Buss, A., Dumas, B., Uppal, R., Vilkov, G., 2016. The intended and unintended consequences of financial-market regulations: A general-equilibrium analysis. Journal of Monetary Economics 81, 25–43.

Buss, A., Uppal, R., Vilkov, G., 2015. Asset prices in general equilibrium with recursive utility and illiquidity induced by transactions costs. Unpublished working paper. INSEAD.

Chabakauri, G., 2013. Dynamic equilibrium with two stocks, heterogeneous investors, and portfolio constraints. Review of Financial Studies 26, 3104–3141.

Chabakauri, G., 2015a. Asset pricing with heterogeneous preferences, beliefs, and portfolio constraints. Journal of Monetary Economics 75, 21–34.

Chabakauri, G., 2015b. Dynamic equilibrium with rare events and heterogeneous Epstein-Zin investors. Unpublished working paper. LSE.

Chan, Y. L., Kogan, L., 2002. Catching up with the Joneses: Heterogeneous preferences and the dynamics of asset prices. Journal of Political Economy 110, 1255–1285.

Chien, Y., Cole, H., Lustig, H., 2011. A multiplier approach to understanding the macro implications of household finance. Review of Economic Studies 78, 199–234.

Chien, Y., Cole, H., Lustig, H., 2012. Is the volatility of the market price of risk due to intermittent portfolio rebalancing? American Economic Review 102, 2859–2896.

Cochrane, J. H., 2005. Asset pricing. Princeton University Press, Princeton, NJ.

Cochrane, J. H., Longstaff, F. A., Santa-Clara, P., 2008. Two trees. Review of Financial Studies 21, 347–385.

Cuoco, D., Kaniel, R., 2011. Equilibrium prices in the presence of delegated portfolio management. Journal of Financial Economics 101, 264–296.

Cvitanić, J., Jouini, E., Malamud, S., Napp, C., 2012. Financial markets equilibrium with heterogeneous agents. Review of Finance 16, 285–321.

Cvitanic, J., Karatzas, I., 1992. Convex duality in constrained portfolio optimization. Annals of Applied Probability 2, 767–818.

Da, Z., Shive, S., 2018. Exchange Traded Funds and asset return correlations. European Financial Management 24, 136–168.

Detemple, J., Murthy, S., 1997. Equilibrium asset prices and no-arbitrage with portfolio constraints. Review of Financial Studies 10, 1133–1174.

Drechsler, I., Savov, A., Schnabl, P., 2018. A model of monetary policy and risk premia. Journal of Finance 73, 317–373.

Duffie, D., Epstein, L. G., 1992. Stochastic differential utility. Econometrica 60, 353–394.

Dumas, B., Lyasoff, A., 2012. Incomplete-market equilibria solved recursively on an event tree. Journal of Finance 67, 1897–1941.

Ehling, P., Heyerdahl-Larsen, C., 2016. Correlations. Management Science 63, 1919–1937.

Fama, E. F., French, K. R., 2010. Luck versus skill in the cross-section of mutual fund returns. Journal of Finance 65, 1915–1947.

French, K. R., 2008. Presidential address: The cost of active investing. Journal of Finance 63, 1537–1573.

Gallmeyer, M., Hollifield, B., 2008. An examination of heterogeneous beliefs with a short-sale constraint in a dynamic economy. Review of Finance 12, 323–364.

Gârleanu, N., Panageas, S., 2015. Young, old, conservative, and bold: The implications of heterogeneity and finite lives for asset pricing. Journal of Political Economy 123, 670–685.

Gârleanu, N., Pedersen, L. H., 2011. Margin-based asset pricing and deviations from the law of one price. Review of Financial Studies 24, 1980–2022.

Gomes, F., Michaelides, A., 2008. Asset pricing with limited risk sharing and heterogeneous agents. Review of Financial Studies 21, 415–448.

Gómez, J.-P., Zapatero, F., 2003. Asset pricing implications of benchmarking: a two-factor CAPM. European Journal of Finance 9, 343–357.

Greenwood, R. M., Sosner, N., 2007. Trading patterns and excess comovement of stock returns. Financial Analysts Journal 63, 69–81.

Grégoire, V., 2020. The rise of passive indexing and index-linked comovement. North American Journal of Economics and Finance (forthcoming).

Gromb, D., Vayanos, D., 2002. Equilibrium and welfare in markets with financially constrained arbitrageurs. Journal of Financial Economics 66, 361–407.

Gromb, D., Vayanos, D., 2010. Limits of arbitrage. Annual Review of Financial Economics 2, 251–275.

Guvenen, F., 2009. A parsimonious macroeconomic model for asset pricing. Econometrica 77, 1711–1750.

He, X.-Z., Shi, L., 2017. Index portfolio and welfare analysis under heterogeneous beliefs. Journal of Banking & Finance 75, 64–79.

Hvide, H. K., Panos, G. A., 2014. Risk tolerance and entrepreneurship. Journal of Financial Economics 111, 200–223.

Isaenko, S., 2008. The term structure of interest rates in a pure exchange economy where investors have heterogeneous recursive preferences. Quarterly Review of Economics and Finance 48, 457–481.

Kihlstrom, R. E., Laffont, J.-J., 1979. A general equilibrium entrepreneurial theory of firm formation based on risk aversion. Journal of Political Economy 87, 719–748.

Kogan, L., Makarov, I., Uppal, R., 2007. The equity risk premium and the riskfree rate in an economy with borrowing constraints. Mathematics and Financial Economics 1, 1–19.

Kyle, A. S., Xiong, W., 2001. Contagion as a wealth effect. Journal of Finance 56, 1401–1440.

Leippold, M., Su, L., Ziegler, A., 2016. How index futures and ETFs affect stock return correlations. Unpublished working paper. University of Zurich.

Lewellen, J., 2011. Institutional investors and the limits of arbitrage. Journal of Financial Economics 102, 62–80.

Li, G., 2014. Information sharing and stock market participation: Evidence from extended families. Review of Economics and Statistics 96, 151–160.

Longstaff, F., Wang, J., 2012. Asset pricing and the credit market. Review of Financial Studies 25, 3169–3215.

Malamud, S., 2016. A dynamic equilibrium model of ETFs. Unpublished working paper. EPFL.

Malkiel, B. G., 1973. A random walk down Wall Street. W. W. Norton & Company, Inc., New York, NY.

Malkiel, B. G., 1995. Returns from investing in equity mutual funds 1971 to 1991. Journal of Finance 50, 549–572.

Martin, I., 2013. The Lucas orchard. Econometrica 81, 55–111.

Menzly, L., Santos, T., Veronesi, P., 2004. Understanding predictability. Journal of Political Economy 112, 1–47.

Pavlova, A., Rigobon, R., 2008. The role of portfolio constraints in the international propagation of shocks. Review of Economic Studies 75, 1215–1256.

Peng, L., Xiong, W., 2006. Investor attention, overconfidence, and category learning. Journal of Financial Economics 80, 563–602.

Rytchkov, O., 2014. Asset pricing with dynamic margin constraints. Journal of Finance 69, 405–452.

Samuelson, P. A., 1974. Challenge to judgement. Journal of Portfolio Management 1, 17–19.

Shapiro, A., 2002. The investor recognition hypothesis in a dynamic general equilibrium: Theory and evidence. Review of Financial Studies 15, 97–141.

Sullivan, R. N., Xiong, J. X., 2012. How index trading increases market vulnerability. Financial Analysts Journal 68, 70–84.

Vijh, A. M., 1994. S&P 500 trading strategies and stock betas. Review of Financial Studies 7, 215–251.

Wang, J., 1996. The term structure of interest rates in a pure exchange economy with heterogeneous investors. Journal of Financial Economics 41, 75–110.

Weinbaum, D., 2009. Investor heterogeneity, asset pricing and volatility dynamics. Journal of Economic Dynamics and Control 33, 1379–1397.

Weinbaum, D., 2010. Preference heterogeneity and asset prices: An exact solution. Journal of Banking and Finance 34, 2238–2246.

Wurgler, J., 2011. On the economic consequences of index-linked investing. In: Rosenfeld, G., Lorsch, J. W., Khurana, R. (Eds.), Challenges to Business in the Twenty-First Century. American Academy of Arts and Sciences.

Xiong, W., 2001. Convergence trading with wealth effects: An amplification mechanism in financial markets. Journal of Financial Economics 62, 247–292.

Xiouros, C., Zapatero, F., 2010. The representative agent of an economy with external habit formation and heterogeneous risk aversion. Review of Financial Studies 23, 3017–3047.

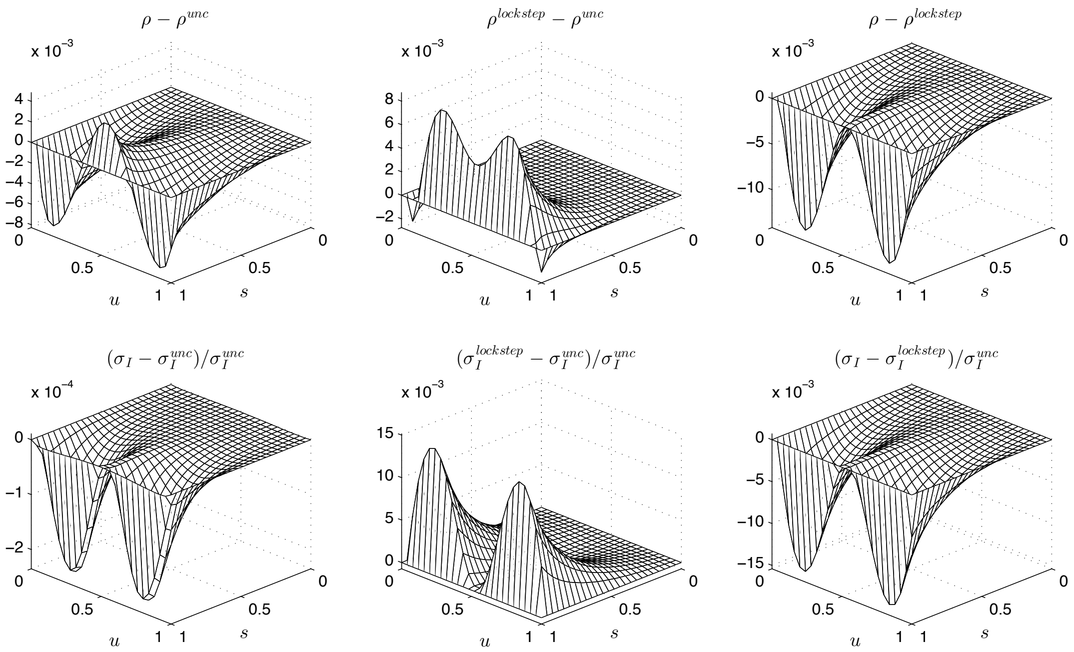

Fig. 1. Correlation of stock returns and market volatility. The left panels of the figure demonstrate how indexing changes the correlation of stock returns ρ and the market volatility σ_I compared to the unconstrained economy. The middle and right panels show the lockstep trading effect and the reduced risk sharing effect, respectively. The superscript *lockstep* indicates the variables from the lockstep-trading equilibrium described in Section 3.2.1; the superscript *unc* indicates the variables from the unconstrained equilibrium. All variables are functions of the consumption share s of the type I investors and the share u of the first dividend D_1 in the aggregate dividend D.

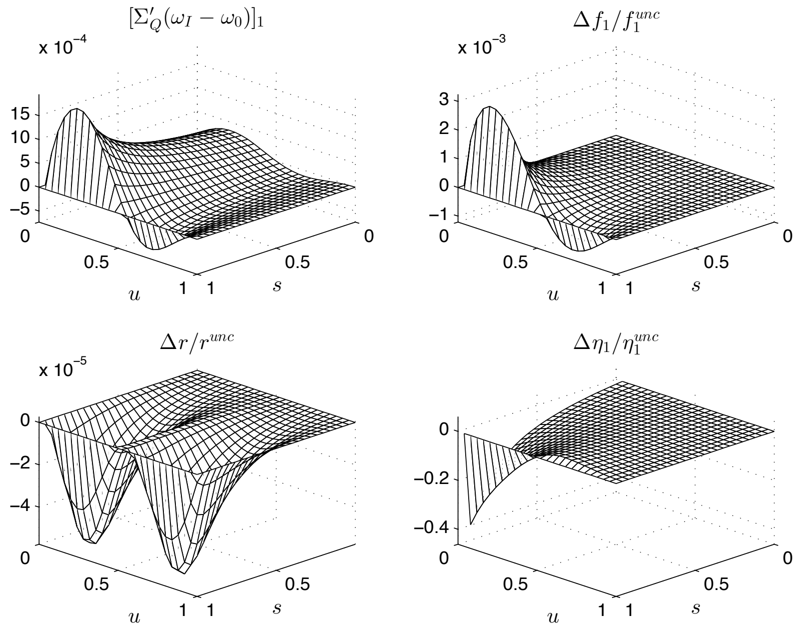

Fig. 2. Other equilibrium characteristics. This figure plots the measure of portfolio distortions $[\Sigma_Q'(\omega_I - \omega_0)]_1$ and shows how the price-dividend ratio f_1, the risk-free rate r, and the market price of risk η_1 change due to indexing relative to the unconstrained economy. The superscript *unc* indicates the variables from the unconstrained equilibrium. All variables are functions of the consumption share s of the type I investors and the share u of the first dividend D_1 in the aggregate dividend D.

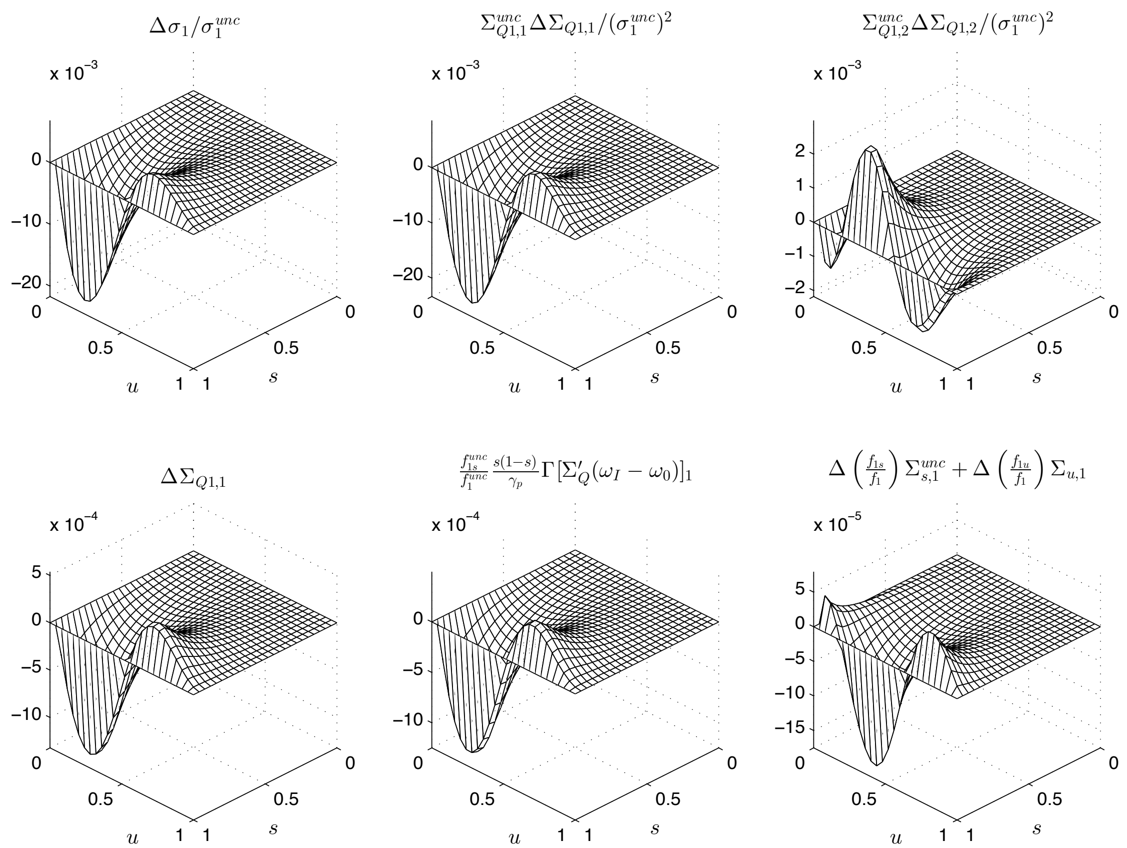

Fig. 3. Decomposition of the changes in σ_1 and $\Sigma_{Q1,1}$ produced by indexing. This figure shows how indexing affects the volatility of the first stock σ_1 and the diffusion component $\Sigma_{Q1,1}$. The first row of the panels relates the changes in σ_1 to the changes in the diffusion components $\Sigma_{Q1,1}$ and $\Sigma_{Q1,2}$. The second row of the panels decomposes $\Delta\Sigma_{Q1,1}$ into components produced by portfolio distortions and changes in the price elasticities. All variables are functions of the consumption share s of the type I investors and the share u of the first dividend D_1 in the aggregate dividend D.

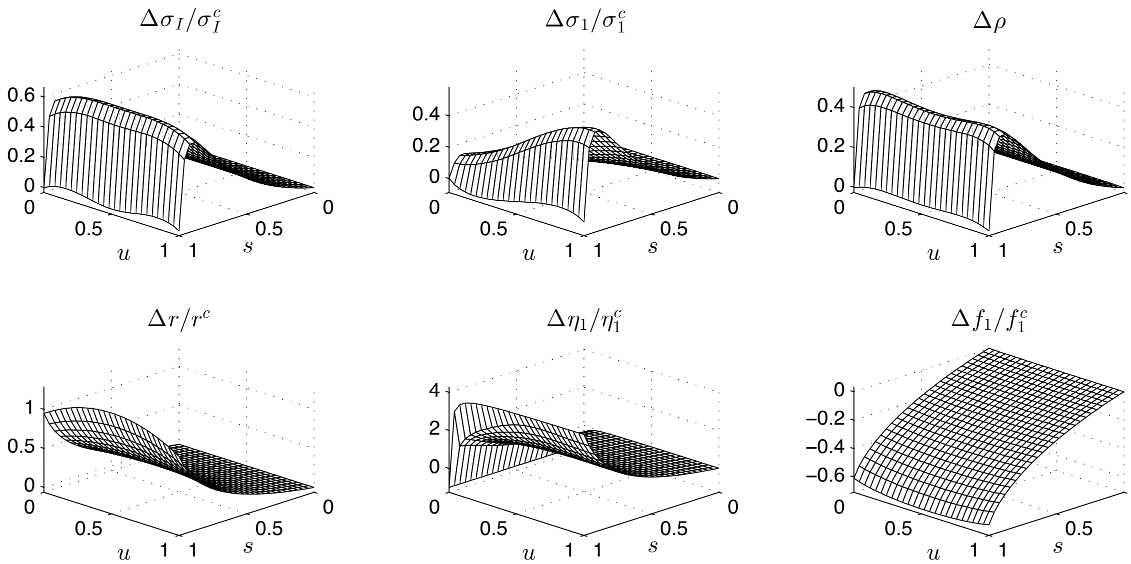

Fig. 4. Comparison with the pre-indexing economy. This figure shows how the market volatility σ_I, the volatility of the first stock σ_1, the correlation of returns ρ, the risk-free rate r, the market price of risk η_1, and the price-dividend ratio of the first stock f_1 change when indexing is introduced in the pre-indexing economy. The superscript c indicates the variables from the pre-indexing equilibrium. All variables are functions of the consumption share s of the type I investors and the share u of the first dividend D_1 in the aggregate dividend D.

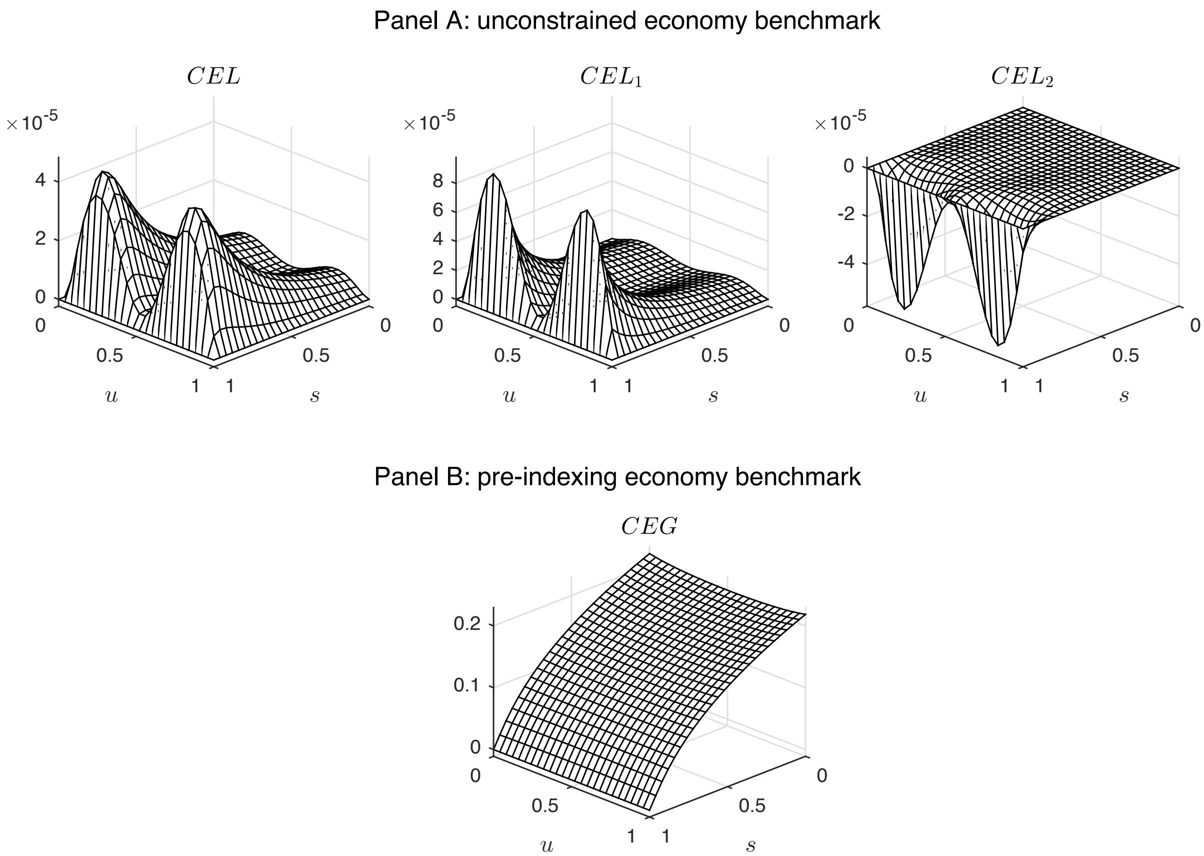

Fig. 5. Welfare analysis. Panel A presents the certainty equivalent losses CEL, CEL_1, and CEL_2 in the economy with indexing compared to the unconstrained economy. Panel B presents the certainty equivalent gain CEG in the economy with indexing compared to the pre-indexing economy. All variables are functions of the consumption share s of the type I investors and the share u of the first dividend D_1 in the aggregate dividend D.

Internet Appendices for
"Asset Pricing with Index Investing"

Internet Appendix A. Numerical methods

As follows from Proposition 1, all equilibrium processes in our model can be expressed in terms of the price-dividend ratio f and the wealth-consumption ratio h, which satisfy the system of quasilinear differential equations (A5) and (A6). Those equations do not admit an analytical solution, so we solve them numerically. We use the standard finite-difference approach, which prescribes to approximate an infinite-horizon economy by an economy with a large finite horizon T, discretize the time interval $[0, T]$ and domains of state variables, and solve the discretized equations backward as a sequence of systems of linear algebraic equations (e.g., Lapidus and Pinder, 1999).

Specifically, we introduce a vector of functions $F = [f \;\; h]'$, denote the first and second partial derivatives of F with respect to the state variables s and u as F_s, F_u, F_{ss}, F_{uu}, and F_{us}, and write the system of equations (A5) and (A6) adjusted for a finite horizon economy as

$$A_{ss}(F, F_s, F_u, s, u)F_{ss} + A_{uu}(F, F_s, F_u, s, u)F_{uu} + A_{us}(F, F_s, F_u, s, u)F_{us} + A_s(F, F_s, F_u, s, u)F_s$$
$$+ A_u(F, F_s, F_u, s, u)F_u + A(F, F_s, F_u, s, u)F + 1 + \frac{\partial F}{\partial t} = 0, \quad \text{(IA1)}$$

where A_{ss}, A_{uu}, A_{us}, A_s, A_u, and A are diagonal 2×2 matrices with elements that correspond to the coefficients of differential equations (A5) and (A6). Note that Eq. (IA1) includes the time derivative $\partial F/\partial t$, which appears as an additional term in Itô's lemma applied to the time-dependent price-dividend ratio and indirect utility function in the derivation of Eqs. (A5) and (A6) presented in the Appendix.

Next, we set $T = 500$ and using a backward recursion solve Eq. (IA1) at the discrete moments $t = T,\; T - \Delta t, \ldots, \Delta t,\; 0$ and in the discrete states $s = 0,\; \Delta s,\; 2\Delta s, \ldots, 1$, and $u = 0,\; \Delta u,\; 2\Delta u, \ldots, 1$, where $\Delta t = 0.1$, $\Delta s = 0.01$, and $\Delta u = 0.01$. In particular, the time

t solution $F_{(t)}$ is found by solving discretized equation (IA1) in which all derivatives of $F_{(t)}$ are replaced with their finite-difference approximations and the equation coefficients are computed using the solution $F_{(t+\Delta t)}$ at time $t + \Delta t$ obtained in the previous step. Thus, the coefficients of the discretized equation do not depend on the time t solution, and $F_{(t)}$ solves a system of linear algebraic equations. Because the time horizon T is large, the sequence $F_{(t)}$, $t = T,\ T - \Delta t, \ldots, \Delta t,\ 0$, converges to a time-independent solution F, which describes an equilibrium in the infinite-horizon economy. We verify the convergence by observing that the discrete approximation of the derivative $\partial F/\partial t$ has the order of magnitude 10^{-7} at $t = 0$.

The iteration procedure starts from the terminal solution $F_{(T)} = [\Delta t \quad \Delta t]'$. Indeed, the index price and the type I investors' wealth at the terminal date are equal to $S_T = D_T \Delta t$ and $W_{IT} = C_{IT}\Delta t$, respectively, so the price-dividend ratio and wealth-consumption ratio at time T are $f_{(T)} = \Delta t$ and $h_{(T)} = \Delta t$. The spacial boundary conditions for the discretized version of Eq. (IA1) are obtained by taking the limits $s \to 0$, $u \to 0$, $s \to 1$, and $u \to 1$ in Eq. (IA1). The computation of the boundary conditions is incorporated directly into the numerical algorithm. Appendix B in Chabakauri (2013) provides further details.

Having solved Eq. (IA1) and obtained f and h, we find r, μ_s, Σ_s, and Σ_I as functions of the state variables using Eqs. (A1) – (A4). Also, we compute η and μ_I from Eq. (A7). To find the price-dividend ratios f_i, we solve differential equations (A8). Note that those equations are linear because their coefficients are known functions of the state variables. Therefore, they are solved using the finite-difference approximation that no longer requires a backward recursion. The remaining equilibrium variables are obtained from Eq. (A9).

To find the equilibrium in the unconstrained benchmark economy, we also use the finite-difference approximation. However, in this case the differential equations for the price-dividend ratios and wealth-consumption ratios are linear and decoupled, so each of them is solved individually without a backward recursion. Those computations closely follow Chabakauri (2013).

Internet Appendix B. Benchmark economies

In this Internet Appendix, we present the equilibrium characteristics of the unconstrained economy and pre-indexing economy, which are the benchmarks in the analysis conducted in the main part of the paper.

Unconstrained economy

Consider first an unconstrained economy in which all investors can trade all assets individually. The equilibrium in this economy is characterized by Proposition 2 in Chabakauri (2013), and the corresponding Σ_s, μ_s, and r are given by Eqs. (15), (16), and (17). For the parameters from Section 3.1, the equilibrium characteristics are presented in Fig. IB.1. Because the fundamentals of the stocks are symmetric, we plot the characteristics of the first stock only.

Fig. IB.1 demonstrates that the volatilities of returns on the market and individual stocks tend to be higher than the volatilities of the corresponding dividends, and this is a consequence of dynamic risk sharing among investors with different risk preferences (e.g., Bhamra and Uppal, 2009; Longstaff and Wang, 2012). Fig. IB.1 also shows that this effect is stronger for the larger stock (the first stock when $u > 1/2$) since this stock is traded more actively when the type P investors rebalance their portfolios. The stock returns are positively correlated even though the correlation between dividends is zero because the prices of both stocks are affected by time varying aggregate risk aversion (e.g., Cochrane et al., 2008; Ehling and Heyerdahl-Larsen, 2016). Also, dividend shocks of the larger stock have a higher price of risk. This is not surprising because the larger stock is a better proxy for the whole market, and the risk associated with it has a stronger effect on the investors' consumption.

Fig. IB.1 also demonstrates that the interest rate r and the market price of risk η_1 are increasing functions of the type I investors' consumption share s. As discussed in Wang (1996), the interest rate is determined by both the investors' equilibrium expected consumption growth and consumption volatility. On the one hand, the investors who are more risk averse have a lower elasticity of intertemporal substitution (EIS) and prefer to borrow more to smooth their consumption over time. This drives the equilibrium interest rate up. On the other hand, the higher

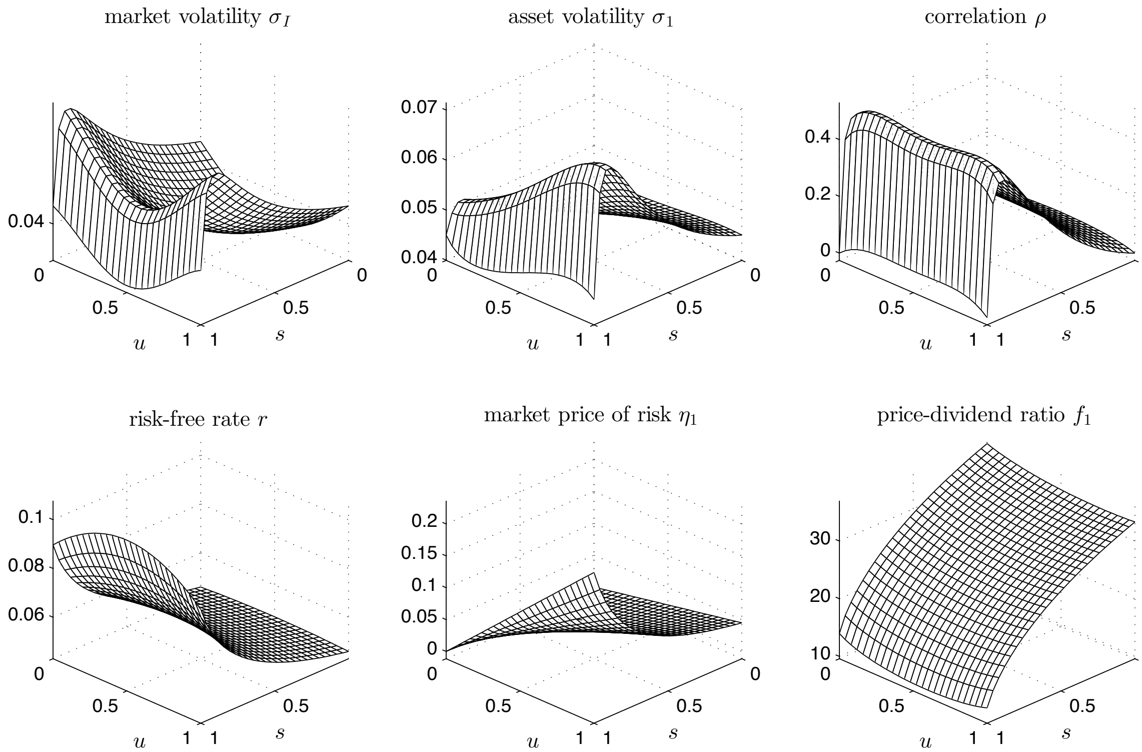

Figure IB.1: This figure presents the equilibrium characteristics of the unconstrained economy as functions of the consumption share s of the type I investors and the share u of the first dividend D_1 in the aggregate dividend D. The model parameters are as follows: $\mu_{D1} = \mu_{D2} = 0.018$, $\Sigma_{D1} = [0.045 \;\; 0]$, $\Sigma_{D2} = [0 \;\; 0.045]$, $\beta = 0.03$, $\gamma_I = 5$, and $\gamma_P = 1$.

risk aversion of the investors makes them less tolerant to consumption volatility and increases their precautionary savings. This drives the equilibrium interest rate down. In our case, the first effect is stronger than the second one.[1] As a result, the interest rate is higher when more risk averse type I investors dominate the market (when s is close to 1) and lower when the type P investors dominate (when s is close to 0).

The increasing relation between η_1 and s is also intuitive: the more risk averse type I investors, whose impact increases with s, require a higher compensation for holding risk than the type P investors. Also, η_1 is an increasing function of the size of the first tree u because investors require a higher compensation for holding the risk associated with a shock to a larger tree. Finally, because both the interest rate and market prices of risk are increasing functions of s, future dividends are more heavily discounted when s is high, and the price-dividend ratio f_1 decreases with s.

[1] As demonstrated by Wang (1996), the relation between the risk aversion and interest rate can be nonmonotonic.

Pre-indexing economy

As another benchmark, we consider the pre-indexing economy, in which the type I investors cannot trade at all, and the prices are determined by optimal behavior of the type P investors. More details on this economy are provided in Section 3.3. Because the type P investors have logarithmic preferences, this benchmark economy has exactly the same properties as the economy in Cochrane et al. (2008).

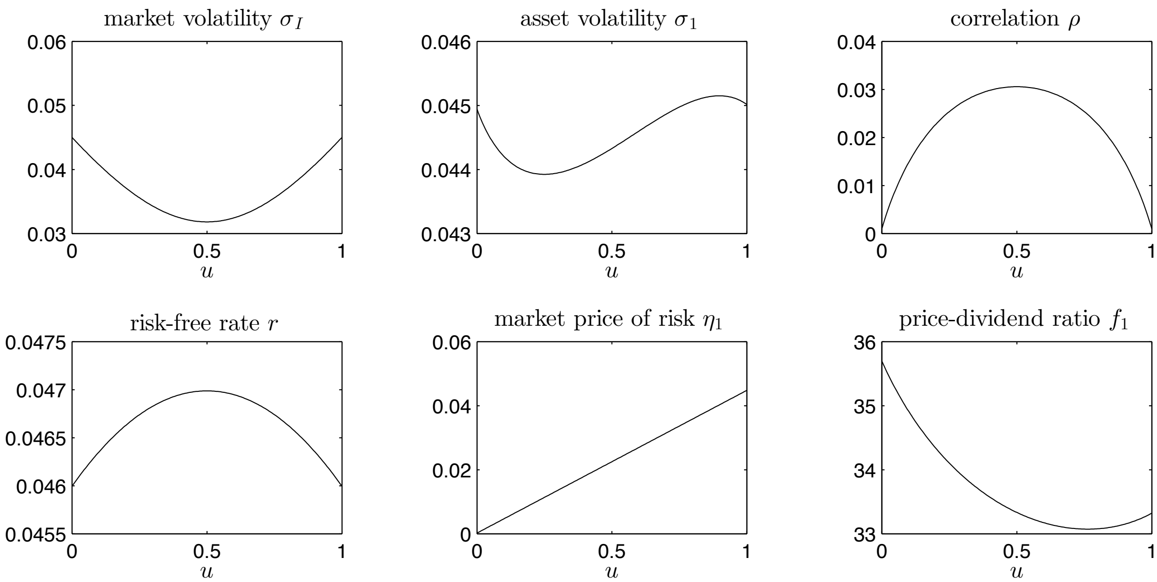

Figure IB.2: This figure presents the equilibrium characteristics of the pre-indexing economy as functions of the share u of the first dividend D_1 in the aggregate dividend D. The preference parameters of market participants are $\beta = 0.03$ and $\gamma_P = 1$. The other parameters are $\mu_{D1} = \mu_{D2} = 0.018$, $\Sigma_{D1} = [0.045 \;\; 0]$, and $\Sigma_{D2} = [0 \;\; 0.045]$.

The main equilibrium characteristics for the parameters from Section 3.1 are presented in Fig. IB.2. As before, we plot only the characteristics of the first stock because those of the other stock can be obtained by flipping the graphs around $u = 1/2$. For a better comparison with the economy with indexing, the price-dividend ratio in Fig. IB.2 uses the tradable fraction of the dividend as the denominator.

The graphs highlight several properties of the equilibrium resulting from market clearing. As in Cochrane et al. (2008), the volatilities of returns differ from fundamental volatilities, returns appear to be correlated even though the dividend shocks of the trees are independent, and the market price of risk rises with the size of a tree to compensate investors for a higher risk associated

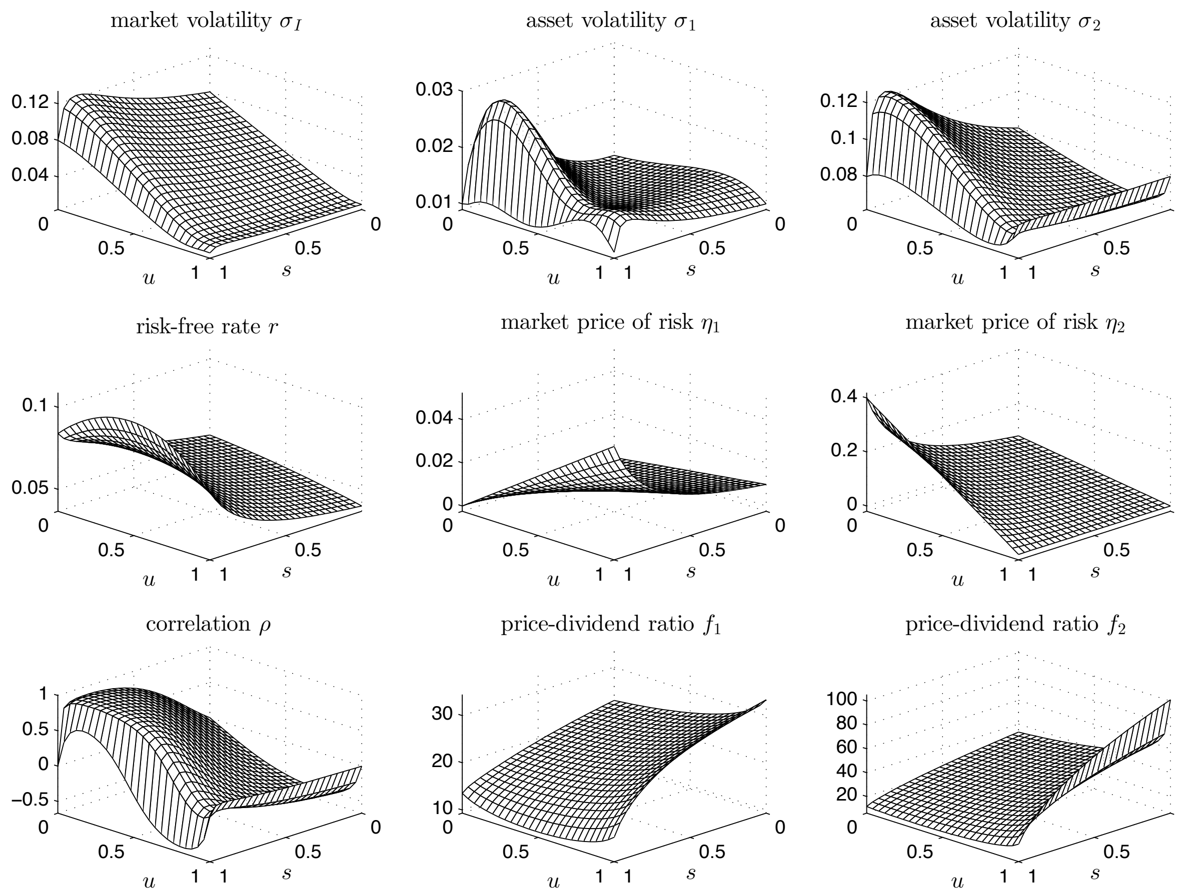

Figure IC.1: This figure presents equilibrium variables in the unconstrained economy as functions of the consumption share s of the type I investors and the share u of the first dividend D_1 in the aggregate dividend D. The model parameters are as follows: $\mu_{D1} = 0.01$, $\mu_{D2} = 0.03$, $\Sigma_{D1} = [0.01\ \ 0]$, $\Sigma_{D2} = [0\ \ 0.08]$, $\beta = 0.03$, $\gamma_I = 5$, and $\gamma_P = 1$.

with a larger tree.

Internet Appendix C. Alternative parameters of the model

In this Internet Appendix, we explore the robustness of our results to various changes in the model parameters. In particular, we consider the Lucas trees that have different dividend processes and investors with alternative coefficients of risk aversion.

Heterogeneous trees

In our main analysis, we assume that the dividend growth rates and diffusions of the dividend processes are identical (the trees are homogeneous), so all heterogeneity across the trees is produced by the endogenous difference in the tree sizes. As a result, the distortions of the unconstrained economy caused by index investing are relatively small, and all effects are relatively weak. However, in reality there is a substantial heterogeneity in the dividend processes, which may amplify the magnitude of the considered effects. To entertain this possibility, we consider a specification with heterogeneous Lucas trees in which the expected dividend growth rates and diffusions of dividends are different and calibrated as $\mu_{D1} = 0.01$, $\mu_{D2} = 0.03$, $\Sigma_{D1} = [0.01 \quad 0]$, and $\Sigma_{D2} = [0 \quad 0.08]$. Thus, the first tree can be interpreted as a mature firm with a relatively low expected dividend growth rate but stable cash flows, and the second tree can be viewed as a young firm with a high dividend growth rate but relatively volatile cash flows. All other parameters are the same as in our main specification described in Section 3.1.

Fig. IC.1 presents the equilibrium in the unconstrained economy with heterogeneous trees. Because the stocks are not symmetric any more, we plot the characteristics for both of them. The graphs confirm many observations made in the case of homogeneous trees and reveal new patterns. In particular, the volatilities of both individual stocks and the market tend to be higher when the economy is dominated by the more volatile second tree. This happens because both the fundamental volatility and the volatility produced by risk sharing are higher.

Fig. IC.1 also shows that the correlation between stock returns is high and positive in some states but negative in the others. To explain the sign of the correlation, we follow Cochrane et al. (2008) and decompose the covariance between stock returns as

$$\operatorname{cov}(dQ_1, dQ_2) = \operatorname{cov}\left(\frac{dD_1}{D_1}, \frac{dD_2}{D_2}\right) + \operatorname{cov}\left(\frac{df_1}{f_1}, \frac{df_2}{f_2}\right) + \operatorname{cov}\left(\frac{dD_1}{D_1}, \frac{df_2}{f_2}\right) + \operatorname{cov}\left(\frac{dD_2}{D_2}, \frac{df_1}{f_1}\right). \quad \text{(IC1)}$$

The first term in Eq. (IC1) is zero because the dividends are uncorrelated. The second term is small, so, as in Cochrane et al. (2008), the covariance between stock returns is mainly determined by the last two terms. The third term is positive because f_2 is an increasing function of u, and the shock dD_1 is positively correlated with changes in $u = D_1/(D_1 + D_2)$. The fourth term is negative when $u \in (0.5, 1)$ because in that range f_1 is an increasing function of u, and the change in u is negatively correlated with the shock dD_2. Moreover, the fourth term is quantitatively large in absolute terms because the dividend D_2 has a larger volatility than the dividend D_1. As a result, the correlation of stock returns in positive for $u \in (0, 0.5)$, but can become negative for $u \in (0.5, 1)$.

It remains to explain why the price-dividend ratio f_1 increases with u when $u \in (0.5, 1)$. Fig. IC.1 shows that in this region the interest rate r is a decreasing function of u. As a result, the cash flows are discounted at a lower rate in the vicinity of $u = 1$ (where the first tree dominates the economy) than in the vicinity of $u = 1/2$, and the price-dividend ratios of both stocks are higher. Thus, the ratios f_i are increasing functions of u, and this fact gives rise to the negative correlation between stock returns. Note that the described effect crucially relies on the heterogeneity of both drifts and diffusions of the dividend processes: when either of them is homogeneous, the correlation is positive in all states of the economy.

Fig. IC.2 compares the equilibrium characteristics in the economy with indexing and in the unconstrained economy in the case of heterogeneous trees. It demonstrates that many effects of indexing are qualitatively similar to those documented for the economy from the main part of the paper and illustrated in Figs. 1 and 2. In particular, we find that indexing reduces the volatility of market returns σ_I and the risk-free rate r. It also again has an ambiguous effect on the correlation between stock returns. As before, the presence of index investors tends to increase (decrease) the volatility of the larger (smaller) stock. Finally, the sizes of all effects tend to be large when the risk-averse type I investors consume a substantial fraction of the total dividend. Thus, our main conclusions are robust to the heterogeneity in the stock dividend process.

Nevertheless, the heterogeneity in the fundamental processes quantitatively modifies the impact of indexing on the equilibrium characteristics. In particular, the graphs in Fig. IC.2 show

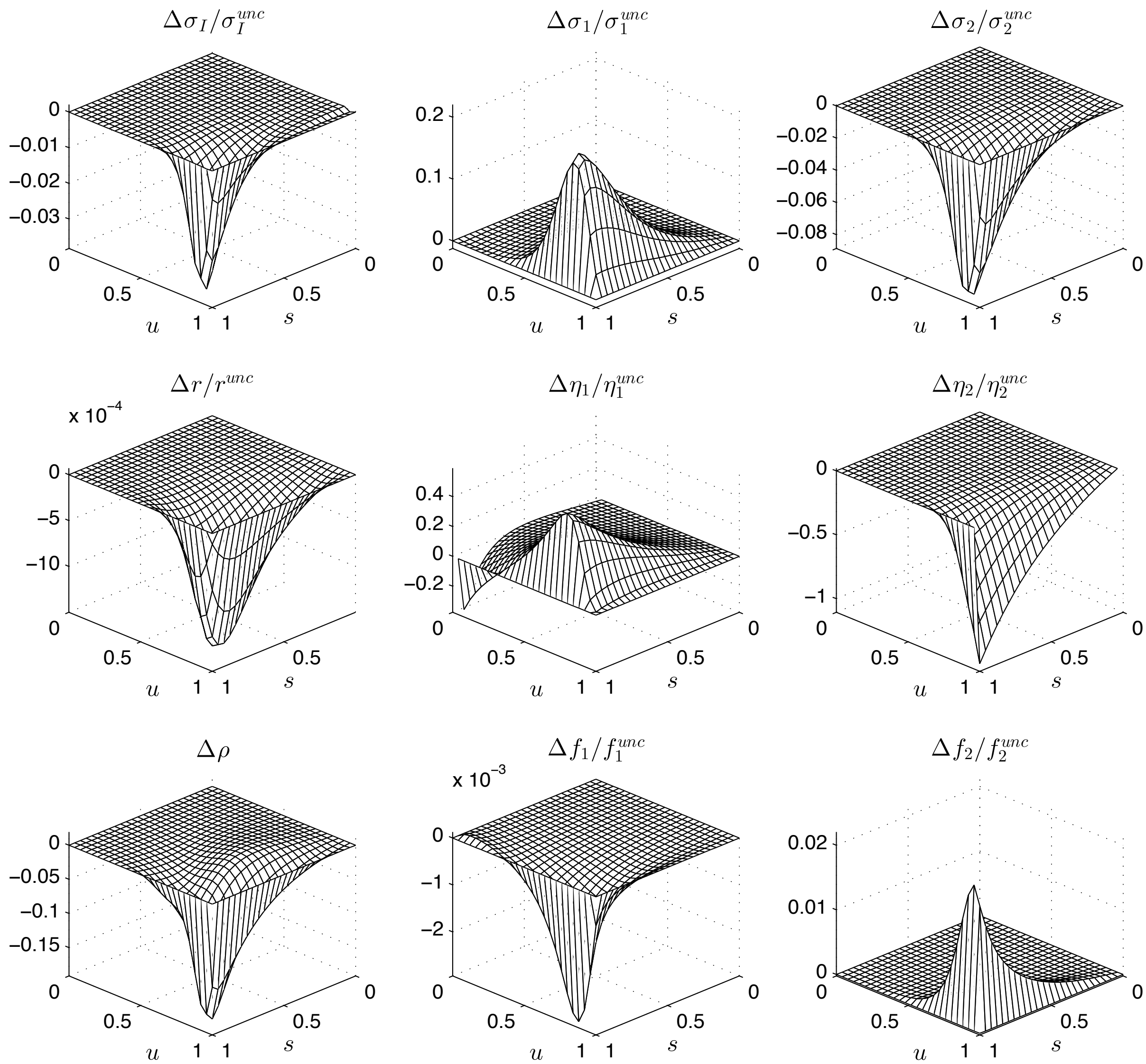

Figure IC.2: This figure shows how the equilibrium in the unconstrained economy with heterogeneous trees changes due to indexing. All variables are functions of the consumption share s of the type I investors and the share u of the first dividend D_1 in the aggregate dividend D. The model parameters are as follows: $\mu_{D1} = 0.01$, $\mu_{D2} = 0.03$, $\Sigma_{D1} = [0.01 \;\; 0]$, $\Sigma_{D2} = [0 \;\; 0.08]$, $\beta = 0.03$, $\gamma_I = 5$, and $\gamma_P = 1$.

that it makes many effects much stronger than they are in the economy with homogeneous trees. For example, the correlation between stock returns in the economy with indexing can be lower by almost 0.15 than in the unconstrained economy, whereas this difference does not exceed 0.01 when the trees are homogeneous. Similarly, the differences in the volatilities and risk-free rates reported in Fig. IC.2 differ by an order of magnitude from their counterparts in Figs. 1 and 2.

Alternative coefficients of risk aversion

Next, we investigate the sensitivity of our conclusions to the assumption $\gamma_P = 1$ and $\gamma_I = 5$, which implies that the index investors are more risk averse than the unconstrained investors. Specifically, we find the equilibrium in exactly the same model as in the main part of the paper but set $\gamma_P = 5$ and $\gamma_I = 1$. Because now the type I investors have logarithmic preferences, they have a constant wealth-consumption ratio $h = 1/\beta$. As a result, the system of equations that describes the equilibrium simplifies, and instead of two differential equations (for the wealth-consumption ratio h and index price-dividend ratio f) it contains only one of them (for the index price-dividend ratio f). Nevertheless, the equation does not have an analytical solution, and, as in Section 3, we solve it numerically.

Because in the unconstrained case the switch in the coefficients of risk aversion is equivalent to relabeling the agents, the graphs of all equilibrium variables in the unconstrained economy can be obtained from the graphs in Figs. IB.1 and IB.2 by flipping them around the plane $s = 1/2$. Therefore, we reproduce only the graphs for the changes in the equilibrium variables produced by indexing. The results are presented in Fig. IC.3.

As a result, the impact of indexing on volatilities is also the same: it typically increases (decreases) the volatilities of larger (smaller) stocks.

One of the most interesting results reported in Section 3.2.2 is an ambiguous effect of indexing on the correlation between stock returns: the latter can increase in some states and decrease in the others. The graph for $\Delta\rho$ in Fig. IC.3 shows that the same observation holds when the less risk-averse investors are indexers. Recall that the impact of indexing on the correlation results from the reduction in risk sharing, which decreases the correlation, and the impossibility for the

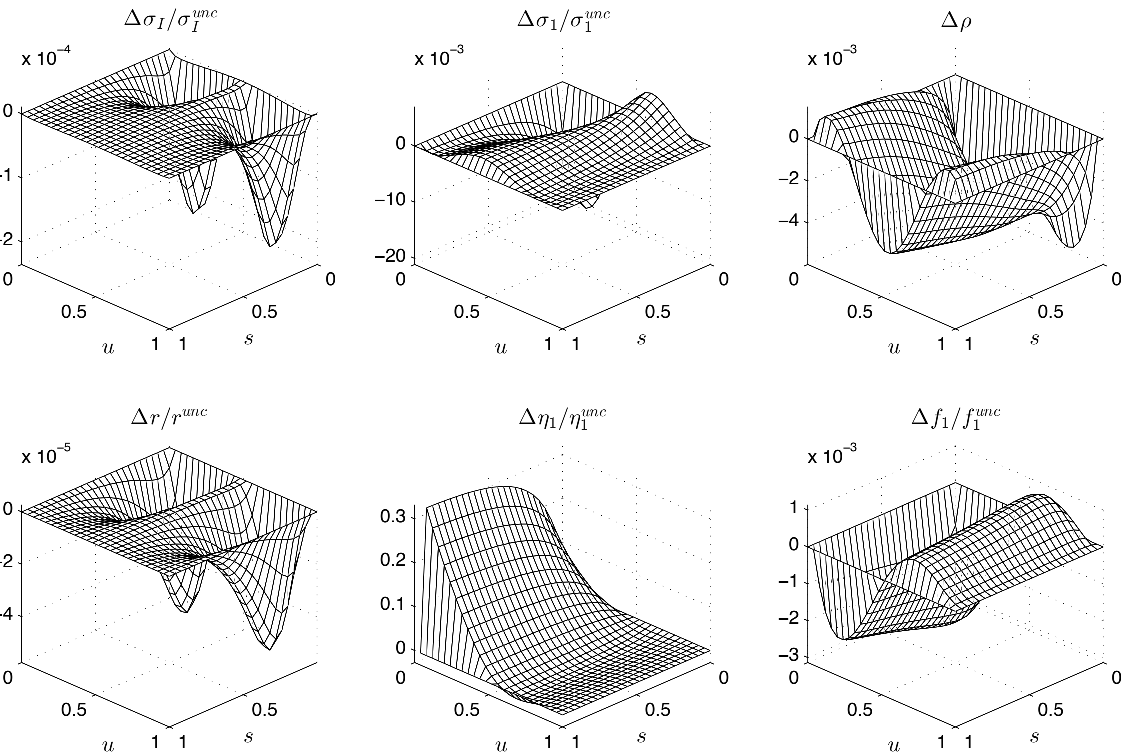

Figure IC.3: This figure shows how the unconstrained equilibrium changes due to indexing. All variables are functions of the consumption share s of the type I investors and the share u of the first dividend D_1 in the aggregate dividend D. The model parameters are as follows: $\mu_{D1} = \mu_{D2} = 0.018$, $\Sigma_{D1} = [0.045 \;\; 0]$, $\Sigma_{D2} = [0 \;\; 0.045]$, $\beta = 0.03$, $\gamma_I = 1$, and $\gamma_P = 5$.

constrained investors to trade the stocks individually, which increases the correlation. Fig. IC.3 implies that the first effect dominates when the trees have comparable sizes, but the second effect dominates when the trees have different sizes.

Finally, Fig. IC.3 shows that all effects of indexing are more pronounced when s is close to 0, that is, when the more risk-averse type P investors dominate the economy. This result replicates the findings from Section 3.2.2, in which all effects are stronger when $s > 0.5$, that is, when the more risk-averse type I investors dominate the economy.

Overall, the comparison of the constrained economies with $\gamma_I = 1$, $\gamma_P = 5$ and $\gamma_I = 5$, $\gamma_P = 1$ shows that our conclusions about the impact of indexing on the volatilities of returns, on the risk-free rate, and on the correlation between returns are insensitive to whether the more risk-averse or less risk-averse investors are assumed to be constrained.

Internet Appendix D. Alternative preferences

In this Internet Appendix, we consider the same model as in the main part of the paper but assume that all investors have recursive preferences in the form of Duffie and Epstein (1992). In particular, each investor solves the following optimization problem:

$$J_t = \max_{\{C_t, \omega_t \in \Omega\}} \mathbb{E}_t \left[\int_t^{\infty} f(C_\tau, J_\tau) d\tau \right], \tag{ID1}$$

where

$$f(C,V) = \begin{cases} \dfrac{\beta(1-\gamma)V}{1-1/\psi} \left[\left(\dfrac{C}{((1-\gamma)V)^{\frac{1}{1-\gamma}}} \right)^{1-1/\psi} - 1 \right], & \psi \neq 1, \\ \beta(1-\gamma)V \left[\log(C) - \dfrac{\log((1-\gamma)V)}{1-\gamma} \right], & \psi = 1, \end{cases} \tag{ID2}$$

subject to the budget constraint

$$dW_t = (r_t W_t - C_t)dt + W_t \omega_t' (\mu_{Qt} dt + \Sigma_{Qt} dB_t). \tag{ID3}$$

The risk aversion parameters γ and elasticities of intertemporal substitution ψ are $\gamma = \gamma_P$, $\psi = \psi_P$ for professional investors and $\gamma = \gamma_I$, $\psi = \psi_I$ for index investors. As in the main part of the paper, the type P investors are unconstrained ($\Omega_P = \mathbb{R}^2$), but the type I investors can invest only in bonds and index (Ω_I consists of index potfolios). The following proposition, which is an analog of Proposition 1, describes the equilibrium in the model.

Proposition ID1. *The equilibrium in the model is characterized by the functions r, η, μ_s, Σ_s, Σ_I, h, h_P, and f that solve a system of algebraic and differential equations. The functions h and*

h_P satisfy the following partial differential equations:

$$\begin{aligned}
&\frac{1}{2}h_{Pss}\Sigma_s\Sigma_s' + \frac{1}{2}h_{Puu}\Sigma_u\Sigma_u' + h_{Psu}\Sigma_s\Sigma_u' \\
&+ h_{Ps}\left(\mu_s + \frac{1}{2}\Sigma_s\left(\frac{1-2\gamma_P}{\gamma_P}\eta + \Sigma_D - \frac{\Sigma_s}{1-s}\right)'\right) + h_{Pu}\left(\mu_u + \frac{1}{2}\Sigma_u\left(\frac{1-2\gamma_P}{\gamma_P}\eta + \Sigma_D - \frac{\Sigma_s}{1-s}\right)'\right) \\
&+ h_P\left((\psi_P - 1)r + \frac{\psi_P - 1}{2\gamma_P}\eta'\eta - \beta\psi_P\right) + 1 = 0, \qquad \text{(ID4)}
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}h_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}h_{uu}\Sigma_u\Sigma_u' + h_{su}\Sigma_s\Sigma_u' \\
&+ h_s\left(\mu_s + \frac{1}{2}\Sigma_s\Pi_I\left(\frac{1-2\gamma_I}{\gamma_I}\eta + \Sigma_D + \frac{\Sigma_s}{s}\right)'\right) + h_u\left(\mu_u + \frac{1}{2}\Sigma_u\Pi_I\left(\frac{1-2\gamma_I}{\gamma_I}\eta + \Sigma_D + \frac{\Sigma_s}{s}\right)'\right) \\
&+h\left((\psi_I - 1)r + \frac{\psi_I - 1}{2\gamma_I}\eta'\Pi_I\eta - \beta\psi_I + \frac{2-\gamma_I-\psi_I}{2(\psi_I - 1)}\left(\Sigma_D + \frac{\Sigma_s}{s}\right)(I - \Pi_I)\left(\Sigma_D + \frac{\Sigma_s}{s}\right)'\right) + 1 = 0, \qquad \text{(ID5)}
\end{aligned}$$

where the market price of risk η is given by

$$\eta = \gamma_P\left(\Sigma_D - \frac{\Sigma_s}{1-s}\right) + \frac{\gamma_P\psi_P - 1}{\psi_P - 1}\left(\frac{h_{Ps}}{h_P}\Sigma_s + \frac{h_{Pu}}{h_P}\Sigma_u\right) \qquad \text{(ID6)}$$

and $\Pi_I = (\Sigma_I'\Sigma_I)/(\Sigma_I\Sigma_I')$. The rest of the equations are algebraic:

$$\begin{aligned}
r = \beta + \frac{1}{s\psi_I + (1-s)\psi_P}\Bigg[&\mu_D + \frac{(\psi_I - \gamma_I)}{2(1-\psi_I)}s\left(\Sigma_D + \frac{\Sigma_s}{s}\right)\left(\Sigma_D + \frac{\Sigma_s}{s}\right)' \\
&- \frac{\psi_I(\gamma_I\psi_I - 1)}{2(\psi_I - 1)}s\left(\Sigma_D + \frac{\Sigma_s}{s} + \frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\left(\Sigma_D + \frac{\Sigma_s}{s} + \frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)' \\
&+ \frac{(\gamma_P - \psi_P)}{2(\psi_P - 1)}(1-s)\left(\Sigma_D - \frac{\Sigma_s}{1-s}\right)\left(\Sigma_D - \frac{\Sigma_s}{1-s}\right)' - \frac{\psi_P(\gamma_P\psi_P - 1)}{2(\psi_P - 1)}(1-s)\times \\
&\times\left(\Sigma_D - \frac{\Sigma_s}{1-s} + \frac{h_{Ps}}{h_P}\Sigma_s + \frac{h_{Pu}}{h_P}\Sigma_u\right)\left(\Sigma_D - \frac{\Sigma_s}{1-s} + \frac{h_{Ps}}{h_P}\Sigma_s + \frac{h_{Pu}}{h_P}\Sigma_u\right)'\Bigg], \qquad \text{(ID7)}
\end{aligned}$$

$$\mu_s = -\Sigma_s\Sigma_D' + \frac{s(1-s)}{s\psi_I + (1-s)\psi_P}\left[(\psi_I - \psi_P)\mu_D + \frac{(\psi_I - \gamma_I)\psi_P}{2(\psi_I - 1)}\left(\Sigma_D + \frac{\Sigma_s}{s}\right)\left(\Sigma_D + \frac{\Sigma_s}{s}\right)'\right.$$

$$+ \frac{\psi_I\psi_P(\gamma_I\psi_I - 1)}{2(\psi_I - 1)}\left(\Sigma_D + \frac{\Sigma_s}{s} + \frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\left(\Sigma_D + \frac{\Sigma_s}{s} + \frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)'$$

$$+ \frac{(\gamma_P - \psi_P)\psi_I}{2(\psi_P - 1)}\left(\Sigma_D - \frac{\Sigma_s}{1-s}\right)\left(\Sigma_D - \frac{\Sigma_s}{1-s}\right)' + \frac{\psi_I\psi_P(1 - \gamma_P\psi_P)}{2(\psi_P - 1)}\times$$

$$\left.\times\left(\Sigma_D - \frac{\Sigma_s}{1-s} + \frac{h_{Ps}}{h_P}\Sigma_s + \frac{h_{Pu}}{h_P}\Sigma_u\right)\left(\Sigma_D - \frac{\Sigma_s}{1-s} + \frac{h_{Ps}}{h_P}\Sigma_s + \frac{h_{Pu}}{h_P}\Sigma_u\right)'\right], \quad \text{(ID8)}$$

$$f = (1-s)h_P + sh, \quad \text{(ID9)}$$

$$\Sigma_I = \frac{(f\Sigma_D + f_u\Sigma_u + f_s\Psi_1)(f\Sigma_D + f_u\Sigma_u + f_s\Psi_2)'}{(f\Sigma_D + f_u\Sigma_u + f_s\Psi_2)(f\Sigma_D + f_u\Sigma_u + f_s\Psi_2)'}\left(\Sigma_D + \frac{f_u}{f}\Sigma_u + \frac{f_s}{f}\Psi_2\right), \quad \text{(ID10)}$$

$$\Sigma_s = \Psi_2 + (\Psi_1 - \Psi_2)\Pi_I, \quad \text{(ID11)}$$

where Ψ_1 *and* Ψ_2 *are given by*

$$\Psi_1 = \frac{(\gamma_P - \gamma_I)\Sigma_D + \left(\frac{\psi_P\gamma_P - 1}{\psi_P - 1}\frac{h_{Pu}}{h_P} - \frac{\psi_I\gamma_I - 1}{\psi_I - 1}\frac{h_u}{h}\right)\Sigma_u}{\frac{\gamma_P}{1-s} + \frac{\gamma_I}{s} - \left(\frac{\psi_P\gamma_P - 1}{\psi_P - 1}\frac{h_{Ps}}{h_P} - \frac{\psi_I\gamma_I - 1}{\psi_I - 1}\frac{h_s}{h}\right)}, \quad \text{(ID12)}$$

$$\Psi_2 = -\frac{s}{h + sh_s}\left(h\Sigma_D + h_u\Sigma_u\right). \quad \text{(ID13)}$$

Proof. Because many steps of the proof are identical or very similar to those of the proof of Proposition 1 from the main part of the paper, we present them with less details and focus on the modifications produced by recursive preferences. In the derivations, we assume that $\psi \neq 1$.

A. Utility maximization problem

The investors of type P solve the optimization problem (ID1) – (ID3) with $\gamma = \gamma_P$ and $\psi = \psi_P$, whereas the investors of type I solve the same problem but with $\gamma = \gamma_I$ and $\psi = \psi_I$ as well as with an additional portfolio constraint $\omega_t \in \Omega_I$, where Ω_I is a set of index portfolios. Below we solve the optimization problem (ID1) – (ID3) with arbitrary γ, ψ, and Ω and omit the subscripts P and I. To simplify notations, we also omit the subscript t.

The value function J solves the following HJB equation:

$$\max_{\{C,\omega\in\Omega\}} \Big\{ f(C,J) + J_W(W(r+\omega'\mu_Q) - C) + \frac{1}{2}J_{WW}W^2\omega'\Sigma_Q\Sigma_Q'\omega + J_{Ws}W\omega'\Sigma_Q\Sigma_s' \\ + J_{Wu}W\omega'\Sigma_Q\Sigma_u' + J_s\mu_s + J_u\mu_u + \frac{1}{2}J_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}J_{uu}\Sigma_u\Sigma_u' + J_{us}\Sigma_s\Sigma_u' \Big\} = 0. \quad \text{(ID14)}$$

The first-order condition with respect to C gives

$$\frac{\beta(1-\gamma)JC^{-1/\psi}}{((1-\gamma)J)^{\frac{1-1/\psi}{1-\gamma}}} = J_W. \quad \text{(ID15)}$$

Assuming that the value function is

$$J = (\beta^{\psi}h)^{\frac{1-\gamma}{\psi-1}}\frac{W^{1-\gamma}}{1-\gamma}, \quad \text{(ID16)}$$

where the function h depends on the state variables s and u, and substituting (ID16) into (ID15), we find that $C = Wh^{-1}$. Thus, the function h coincides with the investor's wealth-consumption ratio. Note that

$$f(C,J) - CJ_W = \frac{\beta(1-\gamma)J}{\psi-1}\left(\frac{C}{((1-\gamma)J)^{\frac{1}{1-\gamma}}}\right)^{1-1/\psi} - \frac{\beta(1-\gamma)J}{1-1/\psi} = \left(\frac{1}{h} - \beta\psi\right)\frac{(1-\gamma)J}{\psi-1}.$$

Using this fact and substituting the value function from (ID16) into Eq. (ID14), we obtain the following equation for h:

$$\begin{aligned}\frac{1}{2}h_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}h_{uu}\Sigma_u\Sigma_u' + h_{su}\Sigma_s\Sigma_u' + h_s\mu_s + h_u\mu_u + h((\psi-1)r-\beta\psi)+1 \\ + \frac{(2-\gamma-\psi)h}{2(\psi-1)}\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)' \\ + (\psi-1)h\max_{\{\omega\in\Omega\}}\left\{\omega'\mu_Q + \frac{1-\gamma}{\psi-1}\omega'\Sigma_Q\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)' - \frac{\gamma}{2}\omega'\Sigma_Q\Sigma_Q'\omega\right\} = 0. \quad \text{(ID17)}\end{aligned}$$

In the case of the type P investors, $\Omega_P = \mathbb{R}^2$ and the optimization with respect to ω yields

$$\omega = \frac{1}{\gamma_P}\left(\Sigma_Q'\right)^{-1}\left(\eta + \frac{1-\gamma_P}{\psi_P-1}\left(\frac{h_{Ps}}{h_P}\Sigma_s + \frac{h_{Pu}}{h_P}\Sigma_u\right)\right)', \quad \text{(ID18)}$$

where $\eta' = \Sigma_Q^{-1}\mu_Q$. Putting this solution back into Eq. (ID17) and rearranging the terms, we get

$$\begin{aligned}&\tfrac{1}{2}h_{Pss}\Sigma_s\Sigma_s' + \tfrac{1}{2}h_{Puu}\Sigma_u\Sigma_u' + h_{Psu}\Sigma_s\Sigma_u' + h_{Ps}\left(\mu_s + \tfrac{1-\gamma_P}{\gamma_P}\Sigma_s\eta'\right) + h_{Pu}\left(\mu_u + \tfrac{1-\gamma_P}{\gamma_P}\Sigma_u\eta'\right) \\ &+h_P\left((\psi_P-1)r + \tfrac{\psi_P-1}{2\gamma_P}\eta\eta' - \beta\psi_P + \tfrac{1-\gamma_P\psi_P}{2\gamma_P(\psi_P-1)}\left(\tfrac{h_{Ps}}{h_P}\Sigma_s + \tfrac{h_{Pu}}{h_P}\Sigma_u\right)\left(\tfrac{h_{Ps}}{h_P}\Sigma_s + \tfrac{h_{Pu}}{h_P}\Sigma_u\right)'\right) + 1 = 0.\end{aligned} \quad \text{(ID19)}$$

In the case of the type I investors, $\Omega = \{\omega : \omega = \hat{\omega}[S_1/S \;\; S_2/S]'\}$, where $\hat{\omega}$ is a scalar weight of the index in the investor's portfolio. Therefore, Eq. (ID17) becomes

$$\begin{aligned}\frac{1}{2}h_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}h_{uu}\Sigma_u\Sigma_u' + h_{su}\Sigma_s\Sigma_u' + h_s\mu_s + h_u\mu_u + h((\psi_I-1)r-\beta\psi_I)+1 \\ + \frac{(2-\gamma_I-\psi_I)h}{2(\psi_I-1)}\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)' \\ + (\psi_I-1)h\max_{\hat{\omega}}\left\{\hat{\omega}\mu_I + \frac{1-\gamma_I}{\psi_I-1}\hat{\omega}\Sigma_I\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)' - \frac{\gamma_I}{2}\hat{\omega}^2\Sigma_I\Sigma_I'\right\} = 0. \quad \text{(ID20)}\end{aligned}$$

The optimal portfolio of the type I investors is given by

$$\hat{\omega}_I = \frac{1}{\gamma_I\Sigma_I\Sigma_I'}\left(\mu_I + \frac{1-\gamma_I}{\psi_I-1}\Sigma_I\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)'\right), \quad \text{(ID21)}$$

and Eq. (ID20) reduces to

$$\frac{1}{2}h_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}h_{uu}\Sigma_u\Sigma_u' + h_{su}\Sigma_s\Sigma_u' + h_s\mu_s + h_u\mu_u + h((\psi_I - 1)r - \beta\psi_I) + 1 \\ + \frac{(2-\gamma_I-\psi_I)h}{2(\psi_I - 1)}\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)' \\ + \frac{(\psi_I - 1)h}{2\gamma_I\Sigma_I\Sigma_I'}\left(\mu_I + \frac{1-\gamma_I}{\psi_I - 1}\Sigma_I\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)'\right)^2 = 0. \quad \text{(ID22)}$$

B. Dynamics of the state variable s and returns on the index

The definition of the consumption share s implies that $C_P = (1-s)D$ and $C_I = sD$. Using Itô's lemma, we obtain the consumption processes of the type P and type I investors:

$$\frac{dC_P}{C_P} = \mu_{CP}dt + \Sigma_{CP}dB, \qquad \frac{dC_I}{C_I} = \mu_{CI}dt + \Sigma_{CI}dB, \quad \text{(ID23)}$$

where

$$\mu_{CP} = \mu_D - \frac{\mu_s + \Sigma_D\Sigma_s'}{1-s}, \qquad \Sigma_{CP} = \Sigma_D - \frac{\Sigma_s}{1-s}, \quad \text{(ID24)}$$

and

$$\mu_{CI} = \mu_D + \frac{\mu_s + \Sigma_D\Sigma_s'}{s}, \qquad \Sigma_{CI} = \Sigma_D + \frac{\Sigma_s}{s}. \quad \text{(ID25)}$$

Next, we represent dJ/J of both types of investors in two different ways and match the drifts and diffusions of the obtained processes. As in the discussion of the optimization problem, we omit the subscripts P and I. On the one hand, using $h = W/C$, we rewrite Eq. (ID16) as

$$J = \frac{1}{1-\gamma}(\beta W)^{\frac{\psi(1-\gamma)}{\psi-1}}C^{\frac{1-\gamma}{1-\psi}}.$$

Applying Itô's lemma to this equation and taking into account Eqs. (ID3) and (ID23), we get

$$\frac{dJ}{J} = \frac{\psi(1-\gamma)}{\psi-1}\Big(r - h^{-1} - \frac{1}{\psi}\mu_C + \omega'\left(\mu_Q - \frac{1-\gamma}{\psi-1}\Sigma_Q\Sigma_C'\right) - \frac{\psi-\gamma}{2\psi(1-\psi)}\Sigma_C\Sigma_C' \\ + \frac{1-\psi\gamma}{2(\psi-1)}\omega'\Sigma_Q\Sigma_Q'\omega\Big)dt + \frac{\psi(1-\gamma)}{\psi-1}\Big(\omega'\Sigma_Q - \frac{1}{\psi}\Sigma_C\Big)dB. \quad \text{(ID26)}$$

On the other hand, Itô's lemma applied to Eq. (ID16) yields

$$\frac{dJ}{J} = \frac{\mathcal{D}J}{J}dt + (1-\gamma)\left(\omega'\Sigma_Q + \frac{1}{\psi-1}\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\right)dB. \tag{ID27}$$

The HJB equation (ID14) implies that $f(C,J) + \mathcal{D}J = 0$, so

$$\frac{\mathcal{D}J}{J} = -\frac{f(C,J)}{J} = \frac{\psi(1-\gamma)}{\psi-1}\left(\beta - \frac{1}{h}\right),$$

where we use $f(C,J)$ from (ID2) and J from (ID16). Matching the drifts and diffusions in Eqs. (ID26) and (ID27), we get

$$r - \beta - \frac{1}{\psi}\mu_C + \omega'\left(\mu_Q - \frac{1-\gamma}{\psi-1}\Sigma_Q\Sigma_C'\right) - \frac{\psi-\gamma}{2\psi(1-\psi)}\Sigma_C\Sigma_C' + \frac{1-\psi\gamma}{2(\psi-1)}\omega'\Sigma_Q\Sigma_Q'\omega = 0, \tag{ID28}$$

$$\omega'\Sigma_Q = \Sigma_C + \frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u. \tag{ID29}$$

First, we consider this system of equations for the type P investors. Plugging the optimal portfolio weight ω from (ID18) and Σ_{CP} from (ID24) into (ID29), solving for η, and restoring the subscript P, we get Eq. (ID6). Using that $\mu_Q = \Sigma_Q\eta'$, $\omega'\Sigma_Q$ from (ID29), Σ_{CP} from (ID24), and η from (ID6), we transform Eq. (ID28) into

$$\begin{aligned} r - \beta - \frac{1}{\psi_P}\left(\mu_D - \frac{\mu_s + \Sigma_D\Sigma_s'}{1-s}\right) - \frac{\gamma_P - \psi_P}{2\psi_P(\psi_P-1)}\left(\Sigma_D - \frac{\Sigma_s}{1-s}\right)\left(\Sigma_D - \frac{\Sigma_s}{1-s}\right)' \\ - \frac{1-\gamma_P\psi_P}{2(\psi_P-1)}\left(\Sigma_D - \frac{\Sigma_s}{1-s} + \frac{h_{Ps}}{h_P}\Sigma_s + \frac{h_{Pu}}{h_P}\Sigma_u\right)\left(\Sigma_D - \frac{\Sigma_s}{1-s} + \frac{h_{Ps}}{h_P}\Sigma_s + \frac{h_{Pu}}{h_P}\Sigma_u\right)' = 0. \end{aligned} \tag{ID30}$$

Eqs. (ID28) and (ID29) for the type I investors yield two additional relations for the equilibrium functions. Using that for the type I investors $\omega'\Sigma_Q = \hat{\omega}\Sigma_I$, where $\hat{\omega}$ is given by Eq. (ID21), noting that Σ_C is given by Eq. (ID25), and multiplying both sides of Eq. (ID29) by Σ_I', we get

$$\mu_I = \gamma_I\left(\Sigma_D + \frac{\Sigma_s}{s}\right)\Sigma_I' + \frac{\psi_I\gamma_I - 1}{\psi_I - 1}\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\Sigma_I'. \tag{ID31}$$

Plugging this representation of μ_I back into Eq. (ID21), we find that

$$\hat{\omega} = \frac{\Sigma_I}{\Sigma_I \Sigma_I'} \left(\Sigma_D + \frac{\Sigma_s}{s} + \frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u \right)'. \tag{ID32}$$

Using Eq. (ID29) again, we get

$$\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u + \Sigma_D + \frac{1}{s}\Sigma_s \right) (I_2 - \Pi_I) = 0. \tag{ID33}$$

Note that Eq. (ID33) coincides with Eq. (A50) from the proof of Proposition 1. To simplify Eq. (ID28) for the type I investors, note that $\omega' \mu_Q = \hat{\omega}\mu_I$ and $\omega' \Sigma_Q \Sigma_Q' \omega = \hat{\omega}^2 \Sigma_I \Sigma_I'$, where μ_I and $\hat{\omega}$ are given by (ID31) and (ID32), respectively. Using μ_C and Σ_C from (ID25), Eq. (ID28) for the type I investors becomes

$$\begin{aligned} r - \beta - \frac{1}{\psi_I}\left(\mu_D + \frac{\mu_s + \Sigma_D \Sigma_s'}{s} \right) - \frac{\gamma_I - \psi_I}{2\psi_I(\psi_I - 1)} \left(\Sigma_D + \frac{\Sigma_s}{s} \right) \left(\Sigma_D + \frac{\Sigma_s}{s} \right)' \\ - \frac{1 - \gamma_I \psi_I}{2(\psi_I - 1)} \left(\Sigma_D + \frac{\Sigma_s}{s} + \frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u \right) \Pi_I \left(\Sigma_D + \frac{\Sigma_s}{s} + \frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u \right)' = 0, \end{aligned}$$

which together with (ID33) yields

$$\begin{aligned} r - \beta - \frac{1}{\psi_I}\left(\mu_D + \frac{\mu_s + \Sigma_D \Sigma_s'}{s} \right) - \frac{\gamma_I - \psi_I}{2\psi_I(\psi_I - 1)} \left(\Sigma_D + \frac{\Sigma_s}{s} \right) \left(\Sigma_D + \frac{\Sigma_s}{s} \right)' \\ - \frac{1 - \gamma_I \psi_I}{2(\psi_I - 1)} \left(\Sigma_D + \frac{\Sigma_s}{s} + \frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u \right) \left(\Sigma_D + \frac{\Sigma_s}{s} + \frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u \right)' = 0. \end{aligned} \tag{ID34}$$

Eqs. (ID30) and (ID34) can be viewed as a system of linear equations for r and μ_s. Its solution is given by Eqs. (ID7) and (ID8).

To find equations for Σ_s and Σ_I, we use that $\mu_I = \eta \Sigma_I'$, where μ_I is from Eq. (ID31) and η is from Eq. (ID6). After collecting the terms, this equation yields $\Sigma_s \Sigma_I' = \Psi_1 \Sigma_I'$ or $\Sigma_s \Pi_I = \Psi_1 \Pi_I$, where Ψ_1 is defined by (ID12). Using this fact, Eq. (ID33) can be resolved for Σ_s as

$$\Sigma_s = \Psi_1 \Pi_I + \Psi_2 (I_2 - \Pi_I), \tag{ID35}$$

where Ψ_2 is defined by (ID13). This is Eq. (ID11). As in the proof of Proposition 1 in the main part of the paper, the second equation in the system for Σ_s and Σ_I is given by the diffusion of index returns represented in terms of the index price-dividend ratio f:

$$\Sigma_I = \Sigma_D + \frac{f_s}{f}\Sigma_s + \frac{f_u}{f}\Sigma_u. \tag{ID36}$$

This is Eq. (A17). Substituting Σ_s from (ID35) into (ID36) and rearranging the terms, we obtain that

$$\Sigma_I = \frac{f_s}{f}(\Psi_1 - \Psi_2)\Pi_I + \left(\Sigma_D + \frac{f_u}{f}\Sigma_u + \frac{f_s}{f}\Psi_2\right). \tag{ID37}$$

This is an analog of Eq. (A45) from the proof of Proposition 1. As there, we solve it by applying Lemma 1, which yields (ID10).

C. Equations for h_P, h, and f

Finally, we obtain quasilinear differential equations for h_P and h. Note that Eq. (ID6) implies that

$$\frac{1-\gamma_P\psi_P}{\psi_P - 1}\left(\frac{h_{Ps}}{h_P}\Sigma_s + \frac{h_{Pu}}{h_P}\Sigma_u\right) = \gamma_P\left(\Sigma_D - \frac{\Sigma_s}{1-s}\right) - \eta. \tag{ID38}$$

Therefore, Eq. (ID19) can be rewritten as (ID4). To transform Eq. (ID22) into quasilinear differential equation, note first that Eq. (ID33) can be rewritten as

$$\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u = -\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)(I_2 - \Pi_I) + \left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\Pi_I$$

and, therefore,

$$\begin{aligned}\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)' &= \left(\Sigma_D + \frac{1}{s}\Sigma_s\right)(I_2 - \Pi_I)\left(\Sigma_D + \frac{1}{s}\Sigma_s\right)' \\ &\quad + \left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\Pi_I\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)'.\end{aligned} \tag{ID39}$$

Using (ID39), the last two terms of Eq. (ID22) become

$$\begin{aligned}
&\frac{(2-\gamma_I-\psi_I)h}{2(\psi_I-1)}\left(\left(\Sigma_D+\frac{1}{s}\Sigma_s\right)(I_2-\Pi_I)\left(\Sigma_D+\frac{1}{s}\Sigma_s\right)'+\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)\Pi_I\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)'\right)\\
&\quad+\frac{(\psi_I-1)h}{2\gamma_I}\left(\frac{\mu_I^2}{\Sigma_I\Sigma_I'}+\frac{2(1-\gamma_I)\mu_I}{(\psi_I-1)\Sigma_I\Sigma_I'}\Sigma_I\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)'+\right.\\
&\quad\left.\frac{(1-\gamma_I)^2}{(\psi_I-1)^2}\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)\Pi_I\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)'\right)\\
&=\frac{(2-\gamma_I-\psi_I)h}{2(\psi_I-1)}\left(\Sigma_D+\frac{1}{s}\Sigma_s\right)(I_2-\Pi_I)\left(\Sigma_D+\frac{1}{s}\Sigma_s\right)'+\frac{(\psi_I-1)h}{2\gamma_I}\eta\Pi_I\eta'\\
&+\frac{(1-\gamma_I)h}{\gamma_I}\eta\Pi_I\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)'+\frac{(1-\psi_I\gamma_I)h}{2\gamma_I(\psi_I-1)}\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)\Pi_I\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)',
\end{aligned} \tag{ID40}$$

where it has been used that $\mu_I = \Sigma_I\eta$. Multiplying Eq. (ID31) by $\Sigma_I\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)'/(\Sigma_I\Sigma_I')$ and using again that $\mu_I = \Sigma_I\eta$, we get

$$\left(\gamma_I\left(\Sigma_D+\frac{1}{s}\Sigma_s\right)-\eta\right)\Pi_I\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)'=\frac{1-\psi_I\gamma_I}{\psi_I-1}\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)\Pi_I\left(\frac{h_s}{h}\Sigma_s+\frac{h_u}{h}\Sigma_u\right)'.$$

Combining this equation with (ID40) and collecting all terms, we arrive at Eq. (ID5).

Finally, we derive the equation for f. The market clearing conditions imply that $W_P+W_I = S$. Noting that $W_I = hC_I$, $W_P = h_PC_P$, and $S = fD$, we get that $f = (1-s)h_P + sh$, and this is Eq. (ID9). Q.E.D.

As in the main part of the paper, we demonstrate the impact of indexing by comparing several equilibrium variables in the economy with indexing and in the unconstrained economy. The equilibrium in the unconstrained economy is described by Proposition ID2.

Proposition ID2. *The equilibrium in the economy without indexing is characterized by the functions r, η, μ_s, Σ_s, h, and h_P, that solve a system of algebraic and differential equations. The* *functions h and h_P satisfy the following partial differential equations:*

$$
\begin{aligned}
&\frac{1}{2}h_{Pss}\Sigma_s\Sigma_s' + \frac{1}{2}h_{Puu}\Sigma_u\Sigma_u' + h_{Psu}\Sigma_s\Sigma_u' \\
&+ h_{Ps}\left(\mu_s + \frac{1}{2}\Sigma_s\left(\frac{1-2\gamma_P}{\gamma_P}\eta + \Sigma_D - \frac{\Sigma_s}{1-s}\right)'\right) + h_{Pu}\left(\mu_u + \frac{1}{2}\Sigma_u\left(\frac{1-2\gamma_P}{\gamma_P}\eta + \Sigma_D - \frac{\Sigma_s}{1-s}\right)'\right) \\
&\qquad + h_P\left((\psi_P-1)r + \frac{\psi_P-1}{2\gamma_P}\eta'\eta - \beta\psi_P\right) + 1 = 0, \quad \text{(ID41)}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{2}h_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}h_{uu}\Sigma_u\Sigma_u' + h_{su}\Sigma_s\Sigma_u' \\
&+ h_s\left(\mu_s + \frac{1}{2}\Sigma_s\left(\frac{1-2\gamma_I}{\gamma_I}\eta + \Sigma_D + \frac{\Sigma_s}{s}\right)'\right) + h_u\left(\mu_u + \frac{1}{2}\Sigma_u\left(\frac{1-2\gamma_I}{\gamma_I}\eta + \Sigma_D + \frac{\Sigma_s}{s}\right)'\right) \\
&\qquad + h\left((\psi_I-1)r + \frac{\psi_I-1}{2\gamma_I}\eta'\eta - \beta\psi_I\right) + 1 = 0, \quad \text{(ID42)}
\end{aligned}
$$

where the market price of risk η is given by (ID6) and

$$
\Sigma_s = \frac{(\gamma_P-\gamma_I)\Sigma_D + \left(\frac{\psi_P\gamma_P-1}{\psi_P-1}\frac{h_{Pu}}{h_P} - \frac{\psi_I\gamma_I-1}{\psi_I-1}\frac{h_u}{h}\right)\Sigma_u}{\frac{\gamma_P}{1-s} + \frac{\gamma_I}{s} - \left(\frac{\psi_P\gamma_P-1}{\psi_P-1}\frac{h_{Ps}}{h_P} - \frac{\psi_I\gamma_I-1}{\psi_I-1}\frac{h_s}{h}\right)}. \quad \text{(ID43)}
$$

The other equations coincide with Eqs. (ID7) and (ID8).

Proof. The proof of Proposition ID2 closely follows the proof of Proposition ID1. In particular, the utility maximization problem of the type P investors yields the same equation (ID19) for h_P. Because the type I investors are also unconstrained, the equation for h is identical to (ID19) in which ψ_P is replaced with ψ_I and γ_P is replaced with γ_I:

$$
\begin{aligned}
&\tfrac{1}{2}h_{ss}\Sigma_s\Sigma_s' + \tfrac{1}{2}h_{uu}\Sigma_u\Sigma_u' + h_{su}\Sigma_s\Sigma_u' + h_s\left(\mu_s + \tfrac{1-\gamma_I}{\gamma_I}\Sigma_s\eta'\right) + h_u\left(\mu_u + \tfrac{1-\gamma_I}{\gamma_I}\Sigma_u\eta'\right) \\
&+h\left((\psi_I-1)r + \tfrac{\psi_I-1}{2\gamma_I}\eta\eta' - \beta\psi_I + \tfrac{1-\gamma_I\psi_I}{2\gamma_I(\psi_I-1)}\left(\tfrac{h_s}{h}\Sigma_s + \tfrac{h_u}{h}\Sigma_u\right)\left(\tfrac{h_s}{h}\Sigma_s + \tfrac{h_u}{h}\Sigma_u\right)'\right) + 1 = 0.
\end{aligned} \quad \text{(ID44)}
$$

Also, removing of the indexing constraint does not change Eqs. (ID23) – (ID30), and the equation for the market price of risk η is again given by (ID6). This equation can be rewritten as (ID38), and a simple algebraic manipulation yields Eq. (ID41).

Because the type I investors are unconstrained, Eqs. (ID28) and (ID29) that follow from their optimization problem produce equations that are similar to those in the case of type P investors. Plugging the optimal portfolio weight

$$\omega = \frac{1}{\gamma_I} \left(\Sigma'_Q\right)^{-1} \left(\eta + \frac{1-\gamma_I}{\psi_I - 1} \left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\right)' \tag{ID45}$$

and Σ_{CI} from (ID25) into (ID29), solving for η, and restoring the subscript I, we get

$$\eta = \gamma_I \left(\Sigma_D + \frac{\Sigma_s}{s}\right) + \frac{\gamma_I \psi_I - 1}{\psi_I - 1} \left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right). \tag{ID46}$$

Combining Eqs. (ID46) and (ID6) and solving for Σ_s, we get Eq. (ID43). Also, rewriting (ID46) as

$$\frac{1-\gamma_I \psi_I}{\psi_I - 1} \left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right) = \gamma_I \left(\Sigma_D + \frac{\Sigma_s}{s}\right) - \eta,$$

we transform Eq. (ID44) into (ID42).

Finally, using that $\mu_Q = \Sigma_Q \eta'$, $\omega' \Sigma_Q$ from (ID29), Σ_{CI} from (ID25), and η from (ID46), we transform Eq. (ID28) into (ID34). Solving Eqs. (ID30) and (ID34) for r and μ_s yields Eqs. (ID7) and (ID8). Q.E.D.

To compare the equilibria in the constrained and unconstrained economies, we solve the systems of equations from Propositions ID1 and ID2 numerically using the same techniques as in the main part of the paper. We calibrate the preference parameters following Gârleanu and Panageas (2015) and set $\gamma_I = 10$, $\gamma_P = 1.5$, $\psi_I = 0.05$, $\psi_P = 0.7$, and $\beta = 0.02$. The dividends follow the same processes as in the main part of the paper: $\mu_{D1} = \mu_{D2} = 0.018$, $\Sigma_{D1} = [0.045 \;\; 0]$, and $\Sigma_{D2} = [0 \;\; 0.045]$. Fig. ID.1, which is an analog of Fig. 2 from the main part of the paper, reports relative changes in various statistics produced by indexing (for those variables that can be equal or close to zero we present absolute changes).

The comparison of Fig. ID.1 and Fig. 2 reveals that the graphs of the changes in equilibrium characteristics are qualitatively similar in the economies with the CRRA and recursive utility functions. Thus, our main conclusions are robust to the choice of the preferences. In particular, we again observe that indexing typically decreases the market volatility and interest rates, but its

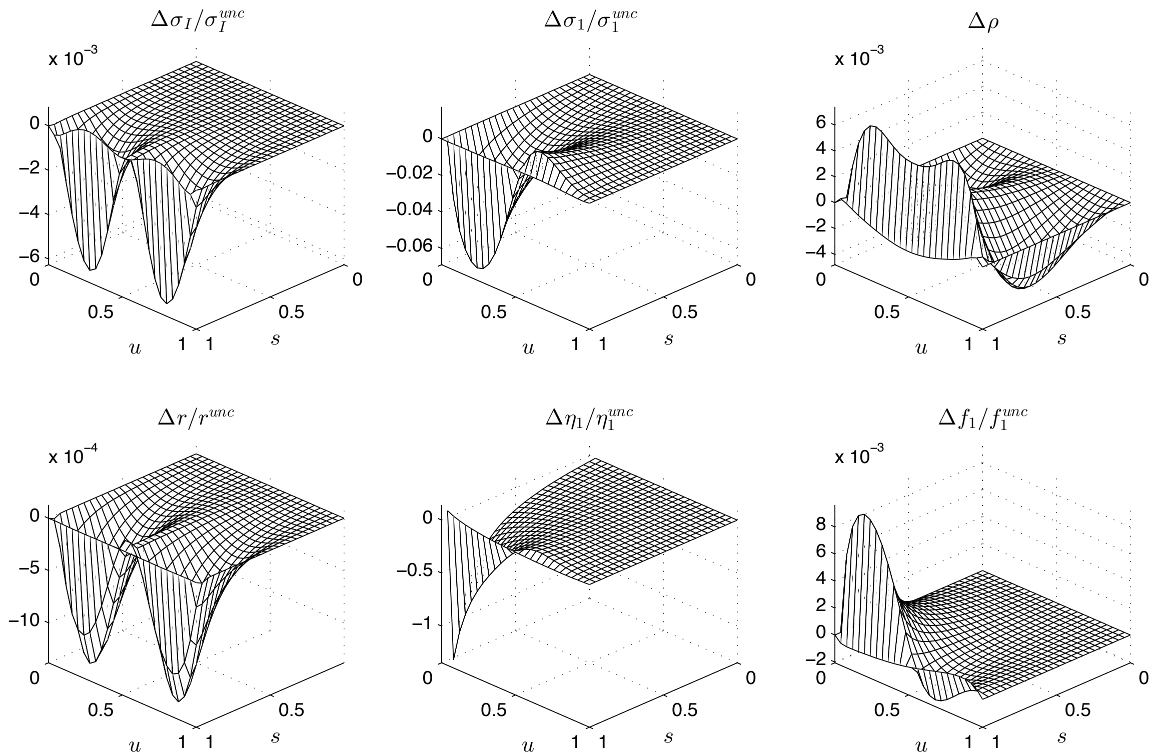

Figure ID.1: This figure shows how the unconstrained equilibrium changes due to indexing in the model with recursive preferences. All variables are functions of the consumption share s of the type I investors and the share u of the first dividend D_1 in the aggregate dividend D. The model parameters are as follows: $\mu_{D1} = \mu_{D2} = 0.018$, $\Sigma_{D1} = [0.045 \quad 0]$, $\Sigma_{D2} = [0 \quad 0.045]$, $\gamma_I = 10$, $\gamma_P = 1.5$, $\psi_I = 0.05$, $\psi_P = 0.7$, and $\beta = 0.02$.

effect on the correlations between stocks and their volatilities is ambiguous. For example, indexing tends to increase market volatility when the assets have comparable sizes but decrease when one of the assets is substantially larger than the other. Finally, note that quantitatively many effects are substantially stronger in the economy with recursive preferences than their analogs in the economy with the CRRA preferences. Thus, a more realistic specification of preferences makes the impact of indexing more pronounced and practically relevant.

Internet Appendix E. Index investing as an outcome of complexity aversion

In the main part of the paper, we assume that the type I investors trade only the index because of unspecified exogenous reasons. In this Internet Appendix, we present a modification of the model that demonstrates how the equilibrium with index investing endogenously arises when the type I investors are unconstrained in their portfolio choice but have complexity aversion, that is, derive disutility from managing a complex portfolio of individual risky assets.[2]

The main components of the modified model are the same as in the main model of the paper. We again consider a pure exchange economy with two Lucas trees that follow geometric Brownian motions and with two types of investors. The type P investors have the standard CRRA preferences and can trade all assets. The difference between the models is in the preferences of the type I investors. Specifically, we assume that the type I investors have the following utility function:

$$U_t = \mathbb{E}_t \left[\int_t^{\infty} e^{-\beta \tilde{t}} \frac{C_{I\tilde{t}}^{1-\gamma_I}}{1-\gamma_I} g(A_{I\tilde{t}}) d\tilde{t} \right], \tag{IE1}$$

where $\gamma_I > 1$. Thus, the type I investors' utility is determined not only by the consumption C_{It} but also by A_{It}, which is a set of risky assets in their portfolio at time t and a subset of $A = \{\text{stock 1}, \text{ stock 2}, \text{ index}\}$. Eq. (IE1) represents the CRRA preferences with the new factor $g(A_{It})$ that describes complexity aversion: it decreases utility when the investor's portfolio includes assets that are hard to analyze and understand. Without losing generality, we set $g(\emptyset) = 1$. Also, we assume that and $g(\{\text{stock 1}\}) = g(\{\text{stock 2}\}) = g(\{\text{stock 1}, \text{ stock 2}\}) = g(\{\text{stock 1}, \text{ index}\}) = g(\{\text{stock 2}, \text{ index}\}) = \bar{g} > 1$. The assumption implies that i) individual stocks are "complex," so holding them reduces the investor's utility, ii) the disutility of a portfolio with individual stocks is the same irrespectively of how many individual assets the portfolio contains. To capture the idea

[2] The idea that economic agents have bounded rationality and prefer to reduce the complexity of decision making goes back to Simon (1955, 1956). Rubinstein (1998) provides a textbook exposition of various approaches to modeling bounded rationality. Recent theoretical studies of preferences that involve complexity include Ortoleva (2013), who provides an axiomatic treatment of "thinking aversion," and Fudenberg and Strzalecki (2015), who introduce "choice aversion" into a dynamic choice problem. Experimental evidence of complexity aversion is presented by Huck and Weizsäcker (1999), Sonsino et al. (2002), and Moffatt et al. (2015), among others.

that indexes simplify the portfolio choice and reduce the disutility from complexity of financial markets, we set $g(\{\text{index}\}) = 1$. Because the index can be replicated by a portfolio of individual stocks and the investors cannot benefit from combining individual stocks with the index, the portfolios {stock 1, index} and {stock 2, index} are never optimal.

The utility function (IE1) is maximized subject to the standard budget constraint

$$dW_{It} = (r_t W_{It} - C_{It})dt + W_{It}\omega'_{It}(\mu_{Qt}dt + \Sigma_{Qt}dB_t), \tag{IE2}$$

where $\omega'_{It} = [\omega_{I(1)t} \;\; \omega_{I(2)t}]$ is a vector of portfolio weights consistent with the composition of the portfolio A_{It}.[3] This condition implies that

$$\omega'_{It} = \begin{cases} [0 \;\; 0], & \text{if } A_{It} = \{\emptyset\}, \\ [\omega_{I(1)t} \;\; 0], & \text{if } A_{It} = \{\text{stock 1}\}, \\ [0 \;\; \omega_{I(2)t}], & \text{if } A_{It} = \{\text{stock 2}\}, \\ [\omega_{I(1)t} \;\; \omega_{I(2)t}], & \text{if } A_{It} = \{\text{stock 1, stock 2}\}, \\ \omega_{I(ind)t}\left[\frac{uf_1}{uf_1+(1-u)f_2} \;\; \frac{(1-u)f_2}{uf_1+(1-u)f_2}\right], & \text{if } A_{It} = \{\text{index}\}. \end{cases} \tag{IE3}$$

Note that in contrast to the main part of the paper, the type I investors optimally choose both the composition of the portfolio A_{It} and the portfolio weights ω_{It}.

The main result of Internet Appendix E is stated in the following proposition.

Proposition IE1. *There exists $\bar{g}_{min} > 1$ such that for any $\bar{g} > \bar{g}_{min}$ the equilibrium in the modified economy coincides with the equilibrium described in Proposition 1. In particular, in any state of the economy, the type I investors hold only the risk-free asset and the index, and the functions r, μ_s, Σ_s, Σ_I, f, and h solve the system of algebraic and differential Eqs. (A1) – (A6). The market price of risk η and the expected excess returns on the index μ_I are given by Eq. (A7). The price-dividend ratio f_i of stock $i = 1, 2$ solves Eq. (A8). The expected excess returns on individual stocks μ_{Qi}, $i = 1, 2$, and return diffusions Σ_{Qi}, $i = 1, 2$, are given by Eq. (A9).*

[3] For convenience, we have slightly changed the notations compared to the main part of the paper. In particular, the portfolio weights of the type I investors are now denoted as $\omega_{I(1)t}$ and $\omega_{I(2)t}$ instead of ω_{I1t} and ω_{I2t}, and the scalar weight of the index is denoted as $\omega_{I(ind)t}$ instead of $\hat{\omega}_{It}$.

Proof. The proof closely follows the proof of Proposition 1, which is presented in the Appendix. We do not reproduce the equations that are the same in both proofs.

A. Price-dividend ratios

This part of the proof is the same as in the proof of Proposition 1, and the equations for the price-dividend ratios f_i, expected returns μ_{Qi}, and diffusions Σ_{Qi} coincide with Eqs. (A15), (A13), and (A14), respectively.

B. Utility maximization problem of the type P investors

This part is also the same as in the proof of Proposition 1 and yields equations for the risk-free rate and expected returns that are identical to Eqs. (A23) and (A24).

C. Utility maximization problem of the type I investors

Because of the modified utility function and the lack of exogenous constraints on the portfolio weights, the utility maximization problem of the type I investors is different from its analog in the main part of the paper. In particular, the indirect utility function of the type I investors $J(s, u)$ solves a different HJB equation:

$$\max_{\{C_I, A_I, \omega_I\}} \left[e^{-\beta t} \frac{C_I^{1-\gamma_I}}{1-\gamma_I} g(A_I) + \mathcal{D}J \right] = 0, \tag{IE4}$$

where $\mathcal{D}J = \mathbb{E}[dJ]/dt$ is given by

$$\mathcal{D}J = J_W(rW_I - C_I) + J_s\mu_s + J_u\mu_u + \frac{1}{2}J_{ss}\Sigma_s\Sigma_s' + \frac{1}{2}J_{uu}\Sigma_u\Sigma_u' + J_{us}\Sigma_s\Sigma_u' + J_t + W_I \times$$

$$\times \begin{cases} 0, & A_I = \{\emptyset\}, \\ J_W\omega_{I(1)}\mu_{Q1} + \frac{1}{2}J_{WW}W_I\omega_{I(1)}^2\Sigma_{Q1}\Sigma_{Q1}' + J_{Ws}\omega_{I(1)}\Sigma_{Q1}\Sigma_s' + J_{Wu}\omega_{I(1)}\Sigma_{Q1}\Sigma_u', & A_I = \{\text{stock 1}\}, \\ J_W\omega_{I(2)}\mu_{Q2} + \frac{1}{2}J_{WW}W_I\omega_{I(2)}^2\Sigma_{Q2}\Sigma_{Q2}' + J_{Ws}\omega_{I(2)}\Sigma_{Q2}\Sigma_s' + J_{Wu}\omega_{I(2)}\Sigma_{Q2}\Sigma_u', & A_I = \{\text{stock 2}\}, \\ J_W\omega_I'\mu_Q + \frac{1}{2}J_{WW}W_I\omega_I'\Sigma_Q\Sigma_Q'\omega_I + J_{Ws}\omega_I'\Sigma_Q\Sigma_s' + J_{Wu}\omega_I'\Sigma_Q\Sigma_u', & A_I = \{\text{stocks 1, 2}\}, \\ J_W\omega_{I(ind)}\mu_I + \frac{1}{2}J_{WW}W_I\omega_{I(ind)}^2\Sigma_I\Sigma_I' + J_{Ws}\omega_{I(ind)}\Sigma_I\Sigma_s' + J_{Wu}\omega_{I(ind)}\Sigma_I\Sigma_u', & A_I = \{\text{index}\}. \end{cases}$$

The subscripts of J denote derivatives, and μ_I and Σ_I are the drift and diffusion of index returns.

As in the proof of Proposition 1, we look for the indirect utility function in the following form:

$$J = \frac{1}{1-\gamma_I} W_I^{1-\gamma_I} h(s,u)^{\gamma_I} \exp(-\beta t). \tag{IE5}$$

The maximization in Eq. (IE4) with respect to C_I together with Eq. (IE5) yields the optimal consumption:

$$C_I = W_I h^{-1} g(A_I)^{\frac{1}{\gamma_I}}. \tag{IE6}$$

The optimal portfolio weights depend on the choice of A_I:

$$\omega'_{It} = \begin{cases} [0 \ \ 0], & \text{if } A_{It} = \{\emptyset\}, \\ \left[\frac{1}{\Sigma_{Q1}\Sigma'_{Q1}}\left(\frac{\mu_{Q1}}{\gamma_I} + \frac{h_s}{h}\Sigma_{Q1}\Sigma'_s + \frac{h_u}{h}\Sigma_{Q1}\Sigma'_u\right) \ \ 0\right], & \text{if } A_{It} = \{\text{stock 1}\}, \\ \left[0 \ \ \frac{1}{\Sigma_{Q2}\Sigma'_{Q2}}\left(\frac{\mu_{Q2}}{\gamma_I} + \frac{h_s}{h}\Sigma_{Q2}\Sigma'_s + \frac{h_u}{h}\Sigma_{Q2}\Sigma'_u\right)\right], & \text{if } A_{It} = \{\text{stock 2}\}, \\ \left(\frac{\mu_Q}{\gamma_I} + \frac{h_s}{h}\Sigma_Q\Sigma'_s + \frac{h_u}{h}\Sigma_Q\Sigma'_u\right)'(\Sigma_Q\Sigma'_Q)^{-1}, & \text{if } A_{It} = \{\text{stocks 1, 2}\}, \\ \frac{1}{\Sigma_I\Sigma'_I}\left(\frac{\mu_I}{\gamma_I} + \frac{h_s}{h}\Sigma_I\Sigma'_s + \frac{h_u}{h}\Sigma_I\Sigma'_u\right)\left[\frac{uf_1}{uf_1+(1-u)f_2} \ \ \frac{(1-u)f_2}{uf_1+(1-u)f_2}\right], & \text{if } A_{It} = \{\text{index}\}. \end{cases} \tag{IE7}$$

Plugging Eqs. (IE6) and (IE7) into Eq. (IE4), we get a differential equation for h:

$$\begin{aligned} &\frac{1}{2}h_{ss}\Sigma_s\Sigma'_s + \frac{1}{2}h_{uu}\Sigma_u\Sigma'_u + h_{us}\Sigma_u\Sigma'_s + h_s\mu_s + h_u\mu_u \\ &\quad + \frac{\gamma_I - 1}{2}\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)\left(\frac{h_s}{h}\Sigma_s + \frac{h_u}{h}\Sigma_u\right)' h + \frac{1}{\gamma_I}((1-\gamma_I)r - \beta)h + \min_{A_I}\left[\frac{(1-\gamma_I)h}{2}\times\right. \\ &\times \begin{cases} 0, & A_I = \{\emptyset\}, \\ \frac{1}{\Sigma_{Q1}\Sigma'_{Q1}}\left(\frac{\mu_{Q1}}{\gamma_I} + \frac{h_s}{h}\Sigma_{Q1}\Sigma'_s + \frac{h_u}{h}\Sigma_{Q1}\Sigma'_u\right)^2, & A_I = \{\text{stock 1}\}, \\ \frac{1}{\Sigma_{Q2}\Sigma'_{Q2}}\left(\frac{\mu_{Q2}}{\gamma_I} + \frac{h_s}{h}\Sigma_{Q2}\Sigma'_s + \frac{h_u}{h}\Sigma_{Q2}\Sigma'_u\right)^2, & A_I = \{\text{stock 2}\}, \\ \left(\frac{\mu_Q}{\gamma_I} + \frac{h_s}{h}\Sigma_Q\Sigma'_s + \frac{h_u}{h}\Sigma_Q\Sigma'_u\right)'\left(\Sigma_Q\Sigma'_Q\right)^{-1}\left(\frac{\mu_Q}{\gamma_I} + \frac{h_s}{h}\Sigma_Q\Sigma'_s + \frac{h_u}{h}\Sigma_Q\Sigma'_u\right), & A_I = \{\text{stocks 1, 2}\}, \\ \frac{1}{\Sigma_I\Sigma'_I}\left(\frac{\mu_I}{\gamma_I} + \frac{h_s}{h}\Sigma_I\Sigma'_s + \frac{h_u}{h}\Sigma_I\Sigma'_u\right)^2, & A_I = \{\text{index}\}, \end{cases} \\ &\hspace{20em} \left. + g(A_I)^{\frac{1}{\gamma_I}}\right] = 0. \end{aligned} \tag{IE8}$$

Note that when $A_I = \{\text{index}\}$, equation (IE8) coincides with Eq. (A30).

D. Dynamics of the state variable s

The definition of the consumption share s implies that $C_I = sD$, so using Itô's lemma, we get

$$\frac{dC_I}{C_I} = \mu_C dt + \Sigma_C dB, \qquad \frac{dC_I^{-\gamma_I}}{C_I^{-\gamma_I}} = \left(-\gamma_I \mu_C + \frac{1}{2}\gamma_I(\gamma_I+1)\Sigma_C\Sigma_C'\right) dt - \gamma_I \Sigma_C dB, \tag{IE9}$$

where

$$\mu_C = \mu_D + \frac{\mu_s + \Sigma_s \Sigma_D'}{s}, \qquad \Sigma_C = \Sigma_D + \frac{1}{s}\Sigma_s. \tag{IE10}$$

Taking into account Eq. (IE6), the indirect utility function from Eq. (IE5) can be rewritten as

$$J = \frac{1}{1-\gamma_I} C_I^{-\gamma_I} W_I g(A_I) \exp(-\beta t).$$

Applying Itô's lemma to this equation and using Eqs. (IE2) and (IE9), we get

$$\frac{dJ}{J} = \left(-\beta - \gamma_I \mu_C + \frac{1}{2}\gamma_I(\gamma_I+1)\Sigma_C\Sigma_C' + r - h^{-1} g(A_I)^{\frac{1}{\gamma_I}} + \omega_I'(\mu_Q - \gamma_I \Sigma_Q \Sigma_C')\right) dt + (\omega_I'\Sigma_Q - \gamma_I \Sigma_C) dB. \tag{IE11}$$

Alternatively, Itô's lemma applied to Eq. (IE5) yields

$$\frac{dJ}{J} = \frac{\mathcal{D}J}{J} dt + \left((1-\gamma_I)\omega_I'\Sigma_Q + \gamma_I \frac{h_s}{h}\Sigma_s + \gamma_I \frac{h_u}{h}\Sigma_u\right) dB. \tag{IE12}$$

Noting that Eqs. (IE5) and (IE6) imply that

$$e^{-\beta t} \frac{C_I^{1-\gamma_I}}{1-\gamma_I} = J h^{-1} g(A_I)^{\frac{1-\gamma_I}{\gamma_I}}$$

and using the HJB equation (IE4), we get $\mathcal{D}J = -Jh^{-1}g(A_I)^{\frac{1}{\gamma_I}}$ and rewrite Eq. (IE12) as

$$\frac{dJ}{J} = -h^{-1} g(A_I)^{\frac{1}{\gamma_I}} dt + \left((1-\gamma_I)\omega_I'\Sigma_Q + \gamma_I \frac{h_s}{h}\Sigma_s + \gamma_I \frac{h_u}{h}\Sigma_u\right) dB. \tag{IE13}$$

Matching the drifts and diffusions in Eqs. (IE11) and (IE13) and using μ_C and Σ_C from Eq.

(IE10), we get

$$\frac{1+\gamma_I}{2}\left(\Sigma_D+\frac{1}{s}\Sigma_s\right)\left(\Sigma_D+\frac{1}{s}\Sigma_s\right)'+\frac{r-\beta}{\gamma_I}+\omega_I'\left(\frac{\mu_Q}{\gamma_I}-\Sigma_Q\left(\Sigma_D+\frac{1}{s}\Sigma_s\right)'\right)$$
$$=\mu_D+\frac{1}{s}(\mu_s+\Sigma_s\Sigma_D'), \quad \text{(IE14)}$$

$$\omega_I\Sigma_Q-\frac{h_s}{h}\Sigma_s-\frac{h_u}{h}\Sigma_u=\Sigma_D+\frac{1}{s}\Sigma_s. \quad \text{(IE15)}$$

Note that Eqs. (IE14) and (IE15) hold for any set of assets A_I. Moreover, when $A_I = \{\text{index}\}$, they coincide with Eqs. (A38) and (A39), so in this case Eqs. (A40) – (A49) also apply. As a result, the dynamics of the state variable s are identical to its dynamics in the economy with indexing constraints and described by Eqs. (A2) – (A4).

E. Optimal choice of A_I

The arguments presented above demonstrate that if the type I investors choose $A_I = \{\text{index}\}$ in all states of the economy, the equilibrium coincides with the equilibrium from Proposition 1. Hence, to prove the statement of Proposition IE1, it is sufficient to show that there exists $\bar{g}_{min} > 1$ such that for any $\bar{g} > \bar{g}_{min}$ the type I investors do not deviate from $A_I = \{\text{index}\}$ taking the investment opportunities as given.

Note that the value of an objective function in the optimum produced by unconstrained maximization cannot be lower than that produced by maximization with constraints. Applying this observation to the maximization in the HJB equation with respect to the portfolio weights, we get that in all states

$$\frac{1}{\Sigma_{Q1}\Sigma_{Q1}'}\left(\frac{\mu_{Q1}}{\gamma_I}+\frac{h_s}{h}\Sigma_{Q1}\Sigma_s'+\frac{h_u}{h}\Sigma_{Q1}\Sigma_u'\right)^2$$
$$\le\left(\frac{\mu_Q}{\gamma_I}+\frac{h_s}{h}\Sigma_Q\Sigma_s'+\frac{h_u}{h}\Sigma_Q\Sigma_u'\right)'\left(\Sigma_Q\Sigma_Q'\right)^{-1}\left(\frac{\mu_Q}{\gamma_I}+\frac{h_s}{h}\Sigma_Q\Sigma_s'+\frac{h_u}{h}\Sigma_Q\Sigma_u'\right),$$

$$\frac{1}{\Sigma_{Q2}\Sigma_{Q2}'}\left(\frac{\mu_{Q2}}{\gamma_I}+\frac{h_s}{h}\Sigma_{Q2}\Sigma_s'+\frac{h_u}{h}\Sigma_{Q2}\Sigma_u'\right)^2 \leq \left(\frac{\mu_Q}{\gamma_I}+\frac{h_s}{h}\Sigma_Q\Sigma_s'+\frac{h_u}{h}\Sigma_Q\Sigma_u'\right)'\left(\Sigma_Q\Sigma_Q'\right)^{-1}\left(\frac{\mu_Q}{\gamma_I}+\frac{h_s}{h}\Sigma_Q\Sigma_s'+\frac{h_u}{h}\Sigma_Q\Sigma_u'\right),$$

and

$$\frac{1}{\Sigma_I\Sigma_I'}\left(\frac{\mu_I}{\gamma_I}+\frac{h_s}{h}\Sigma_I\Sigma_s'+\frac{h_u}{h}\Sigma_I\Sigma_u'\right)^2 \leq \left(\frac{\mu_Q}{\gamma_I}+\frac{h_s}{h}\Sigma_Q\Sigma_s'+\frac{h_u}{h}\Sigma_Q\Sigma_u'\right)'\left(\Sigma_Q\Sigma_Q'\right)^{-1}\left(\frac{\mu_Q}{\gamma_I}+\frac{h_s}{h}\Sigma_Q\Sigma_s'+\frac{h_u}{h}\Sigma_Q\Sigma_u'\right).$$

In particular, these inequalities hold for the equilibrium with indexing. Also note that by definition, $h > 0$, so $(1-\gamma_I)h < 0$. Hence, the optimization with respect to A_I in Eq. (IE8) yields $A_I = \{\text{index}\}$ when in all states of the economy

$$\frac{(1-\gamma_I)h}{2}\frac{1}{\Sigma_I\Sigma_I'}\left(\frac{\mu_I}{\gamma_I}+\frac{h_s}{h}\Sigma_I\Sigma_s'+\frac{h_u}{h}\Sigma_I\Sigma_u'\right)^2+1 < \frac{(1-\gamma_I)h}{2}\left(\frac{\mu_Q}{\gamma_I}+\frac{h_s}{h}\Sigma_Q\Sigma_s'+\frac{h_u}{h}\Sigma_Q\Sigma_u'\right)'\left(\Sigma_Q\Sigma_Q'\right)^{-1}\left(\frac{\mu_Q}{\gamma_I}+\frac{h_s}{h}\Sigma_Q\Sigma_s'+\frac{h_u}{h}\Sigma_Q\Sigma_u'\right)+\bar{g}^{\frac{1}{\gamma_I}},$$

where all functions are the same as in the equilibrium with indexing. This condition holds if

$$\bar{g} > \bar{g}_{min} = \max_{s\in[0,1],\,u\in[0,1]} \bar{g}(s,u)^{\gamma_I},$$

where

$$\bar{g}(s,u) = \frac{(\gamma_I-1)h}{2}\left[\left(\frac{\mu_Q}{\gamma_I}+\frac{h_s}{h}\Sigma_Q\Sigma_s'+\frac{h_u}{h}\Sigma_Q\Sigma_u'\right)'\left(\Sigma_Q\Sigma_Q'\right)^{-1}\left(\frac{\mu_Q}{\gamma_I}+\frac{h_s}{h}\Sigma_Q\Sigma_s'+\frac{h_u}{h}\Sigma_Q\Sigma_u'\right) - \frac{1}{\Sigma_I\Sigma_I'}\left(\frac{\mu_I}{\gamma_I}+\frac{h_s}{h}\Sigma_I\Sigma_s'+\frac{h_u}{h}\Sigma_I\Sigma_u'\right)^2\right]+1. \quad \text{(IE16)}$$

This completes the proof. Q.E.D.

To demonstrate how the condition on $\bar{g}$ from Proposition IE1 applies in the numerical examples presented in the paper, we compute $\bar{g}(s,u)$ and plot $\bar{g}(s,u)-1$ in Fig. IE.1.

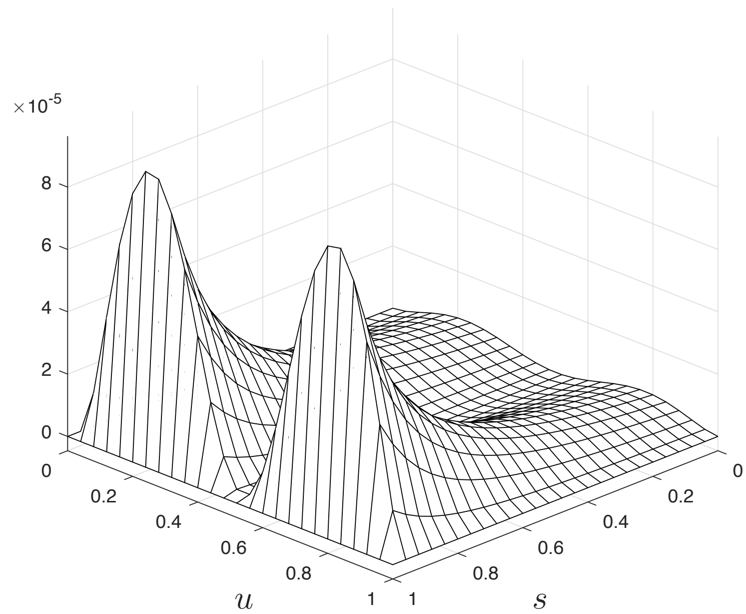

Figure IE.1: This figure plots the function $\bar{g}(s,u) - 1$ with $\bar{g}(s,u)$ from equation (IE16). The model parameters are as follows: $\mu_{D1} = \mu_{D2} = 0.018$, $\Sigma_{D1} = [0.045 \;\; 0]$, $\Sigma_{D2} = [0 \;\; 0.045]$, $\beta = 0.03$, $\gamma_I = 5$, and $\gamma_I = 5$.

The figure demonstrates that, as expected, $\bar{g}(s,u) \geq 1$ in all states, so investors hold the index instead of individual stocks only when they receive disutility from the latter choice. However, the magnitude of disutility that justifies indexing is small: investors shun individual assets in all states if $\bar{g} > 1.0001$. This observation is consistent with a relatively small welfare loss associated with switching from trading two stocks to trading the index that we document in Section 3.4 of the main part of the paper. It also demonstrates that the equilibrium with indexing constructed in the main part of the paper endogenously arises for a quantitatively realistic cost of complexity.

References

Bhamra, H. S., Uppal, R., 2009. The effect of introducing a non-redundant derivative on the volatility of stock-market returns when agents differ in risk aversion. Review of Financial Studies 22, 2303–2330.

Chabakauri, G., 2013. Dynamic equilibrium with two stocks, heterogeneous investors, and portfolio constraints. Review of Financial Studies 26, 3104–3141.

Cochrane, J. H., Longstaff, F. A., Santa-Clara, P., 2008. Two trees. Review of Financial Studies 21, 347–385.

Duffie, D., Epstein, L. G., 1992. Stochastic differential utility. Econometrica 60, 353–394.

Ehling, P., Heyerdahl-Larsen, C., 2016. Correlations. Management Science 63, 1919–1937.

Fudenberg, D., Strzalecki, T., 2015. Dynamic logit with choice aversion. Econometrica 83, 651–691.

Gârleanu, N., Panageas, S., 2015. Young, old, conservative, and bold: The implications of heterogeneity and finite lives for asset pricing. Journal of Political Economy 123, 670–685.

Huck, S., Weizsäcker, G., 1999. Risk, complexity, and deviations from expected-value maximization: Results of a lottery choice experiment. Journal of Economic Psychology 20, 699–715.

Lapidus, L., Pinder, G. F., 1999. Numerical solution of partial differential equations in science and engineering. Wiley-Interscience.

Longstaff, F., Wang, J., 2012. Asset pricing and the credit market. Review of Financial Studies 25, 3169–3215.

Moffatt, P. G., Sitzia, S., Zizzo, D. J., 2015. Heterogeneity in preferences towards complexity. Journal of Risk and Uncertainty 51, 147–170.

Ortoleva, P., 2013. The price of flexibility: Towards a theory of thinking aversion. Journal of Economic Theory 148, 903–934.

Rubinstein, A., 1998. Modeling Bounded Rationality. The MIT Press, Cambridge, MA.

Simon, H. A., 1955. A behavioral model of rational choice. Quarterly Journal of Economics 69, 99–118.

Simon, H. A., 1956. Rational choice and the structure of the environment. Psychological Review 63, 129–138.

Sonsino, D., Benzion, U., Mador, G., 2002. The complexity effects on choice with uncertainty – Experimental evidence. Economic Journal 112, 936–965.

Wang, J., 1996. The term structure of interest rates in a pure exchange economy with heterogeneous investors. Journal of Financial Economics 41, 75–110.