

# **Sustainability in a Risky World**

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**DISCUSSION PAPER NO 830**

**PAUL WOOLLEY CENTRE WORKING PAPER No 76**

**April 2021**

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# Sustainability in a Risky World

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## Abstract

This paper studies the restrictions on consumption, portfolio choice, and social discounting implied by a sustainability constraint, that utility should not be expected to decline over time, in an economy with risky investment opportunities. The sustainability constraint does not distort portfolio choice and implies a consumption-wealth ratio and social discount rate that can be considerably higher than the riskless interest rate.

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# 1 Introduction

Do ethical considerations restrict the rate at which society consumes, or its preference for the present over the future? Economists have answered this question in different ways.

One view is that preferences, social or individual, must be taken as given. If society discounts the future at a high rate, strongly preferring present consumption over future consumption, that preference must be respected; and if it leads to high consumption today, declining over time, that outcome must be accepted.

An alternative view, famously expressed by Ramsey (1928), is that at least for long-term discounting over the lifetimes of multiple generations, society should not discount the future at all because to do so is unethically to privilege the generation alive today over those yet unborn. Recently, this view has found powerful expression in the Stern Review (Stern 2006), which argued for aggressive action to combat climate change in large part on the basis of a social discount rate, or rate of time preference, close to zero.

A third view is that social choices over consumption and saving should be subjected to an external “sustainability” constraint. Sustainability was defined by the World Commission on Environment and Development (1987) as a consumption plan that “meets the needs of the present without compromising the ability of future generations to meet their own needs”. Arrow et al. (2004) have formalized this as a requirement that social value—the expected discounted value of utility from the present to the infinite future—should not decline over time.

In a deterministic economy with a single form of capital that has a constant riskless rate of return, the Arrow et al. constraint requires that the social rate of time preference does not exceed the exogenous riskless interest rate. When the constraint binds, the rate of time preference equals the riskless interest rate, implying that society consumes the riskless return generated by its wealth

and leaves the capital stock unchanged. Wealth, consumption, the utility and marginal utility of current consumption, and social value are then all constant over time. Sustainable consumption is only feasible when the riskless interest rate is positive, and then the rate of time preference should also be positive. The sustainability constraint therefore responds to the availability of an investment opportunity with a positive rate of return by allowing higher time preference and greater current consumption than would be required by Ramsey.<sup>2</sup>

In this paper we extend Arrow et al.'s concept of sustainability to allow for risk. In a risky economy, where the return on capital is uncertain, it is not possible to guarantee that social value remains constant over time. Instead, we impose a weaker sustainability constraint that social value follows a martingale with non-negative drift. Under this condition social value can decline because of negative shocks, but should not be expected to decline.

We study a model with two forms of capital, one safe and one risky, so that society faces a portfolio choice problem as well as a consumption-savings decision. We also consider a special case where the safe asset is in zero net supply, so the risk premium on the risky asset must adjust to ensure a risky portfolio share equal to one. We assume that both assets have iid returns, but we do not impose that the risky return is normally distributed. The model is set in continuous time and allows the risky return to be driven both by a Brownian motion and by a Poisson jump process. We assume that society has a standard time-separable power utility function.

Our main results are as follows. First, our sustainability constraint does not distort portfolio choice, which is always the same whether or not the constraint binds. In the absence of jumps, the portfolio rule is the classic one derived by Merton (1969, 1971). Second, we solve for the

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<sup>2</sup>By adjusting the rate of time preference to the available rate of return, the sustainability constraint responds to a famous critique of Ramsey made by Koopmans (1960, 1967). As Koopmans (1967) put it: "One cannot adopt ethical principles without regard to... the anticipated technological possibilities. Any proposed optimality criterion needs to be subjected to a mathematical screening, to determine whether it does indeed bear on the problem at hand, under the circumstances assumed. More specifically, too much weight given to generations far in the future turns out to be self-defeating. It does nobody any good. How much weight is too much has to be determined in each case."

consumption-wealth ratio when the constraint binds and show that it exceeds the riskless interest rate. In the absence of jumps, the sustainable consumption-wealth ratio is the riskless interest rate plus one-half the squared Sharpe ratio on the risky asset divided by risk aversion. In the presence of jumps, the solution is defined implicitly but is straightforward to calculate numerically. Third, we show that the constraint binds whenever the unconstrained social rate of time preference exceeds the sustainable consumption-wealth ratio. The constrained solution can be implemented by reducing the rate of time preference to equal the sustainable consumption-wealth ratio. Fourth, we show that in a special case where the riskless asset is in zero net supply, the sustainable consumption-wealth ratio—equivalently, the sustainable rate of time preference—lies between the riskless interest rate and the risky asset return. In the absence of jumps, it lies exactly at the midpoint between these two rates of return. In the presence of jumps, it can be as low as the riskless rate plus one-half risk aversion times the variance of the risky return, or as high as the risky return less one-half risk aversion times the variance of the risky return. We characterize the position of the sustainable consumption-wealth ratio within these bounds using methods derived from Martin (2013).

The risky model we consider is different from the previously analyzed riskless model in two fundamental ways. First, in the presence of risk a zero drift for social value does not imply zero drifts for consumption, wealth, or the marginal utility of consumption. We characterize this distinction and emphasize that our sustainability constraint is therefore different from the zero drift in consumption and wealth imposed by Campbell and Sigalov (2021). The Campbell and Sigalov constraint distorts portfolio choice, whereas ours does not.

Second, the sustainable rate of time preference is not the same as the sustainable discount rate that society should apply to riskless investment projects. That discount rate is given by the riskless interest rate in the sustainable equilibrium, which is lower than the sustainable rate of time preference when the economy is exposed to risk. As a salient example, investments to mitigate climate change can rationally be discounted at low rates if the investments are riskless and the sustainable equilibrium has a low riskless interest rate. They should be discounted at even lower

rates if climate investments pay off in bad states of the world—that is, if they are analogous to insurance policies—an important point emphasized by Weitzman (2009) and Gollier (2021).

The literature on discounting and sustainability is enormous, and we do not attempt a complete review here. Dasgupta (2008, 2021) and Zeckhauser and Viscusi (2008) provide recent surveys. Within the literature on climate change, there has been debate between those such as Cline (1992) and Stern (2006) who argue for very low social rates of time preference, and Nordhaus (1994) who uses a higher rate of time preference. Our analysis implies that low rates of time preference are not required by the ethical criterion of sustainability in a risky world.

The organization of the paper is as follows. Section 2 sets up our unconstrained continuous-time model with portfolio choice over a safe and a risky asset. Section 3 introduces the sustainability constraint and solves the constrained model. Section 4 imposes the equilibrium condition that the risky asset share equals one. Section 5 concludes. An appendix presents details of key derivations.

## 2 Unconstrained Consumption and Portfolio Choice

We consider an investor faced with two assets, one riskless and one risky. The investor chooses her consumption,  $C_t$ , and risky portfolio share,  $\alpha$ , to maximize the expected discounted value of a power utility function,

$$U_0 = \mathbb{E} \int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} dt. \quad (1)$$

We assume that the time discount rate  $\rho > 0$  and the coefficient of relative risk aversion  $\gamma > 1$ . In the appendix we show that all the results extend in the expected way to the log utility case,  $\gamma = 1$ . It would also be easy to handle the case  $0 < \gamma < 1$ , but as this case requires occasional sign flips in our logic below, we rule it out to streamline the exposition.

The riskless asset has gross return  $R_f$ . It will generally be convenient to think in terms of the

log riskless rate,  $r_f = \log R_f$ . The risky asset has expected excess return  $\mu = \log(ER/R_f) > 0$ , Brownian volatility  $\sigma$ , and is exposed to jumps arriving according to a Poisson counting process  $N_t$  with constant arrival rate  $\omega$ ; we assume that  $\mu$ ,  $\sigma$ , and  $\omega$  are each constant. We write  $W_t$  for wealth at time  $t$  and  $\theta = C_t/W_t$  for the consumption-wealth ratio. Under our assumptions,  $\theta$  is also constant. Thus

$$\frac{dC}{C} = \frac{dW}{W} = \left[ r_f + \alpha \underbrace{(\mu + \omega EL)}_{\hat{\mu}} - \theta \right] dt + \alpha \sigma dZ - \alpha L dN, \quad (2)$$

where we suppress time subscripts on random variables for simplicity.

Jumps are captured by the third term on the right-hand side of equation (2). When a jump occurs, an agent who is fully invested in the risky asset loses a fraction  $L$  of her capital. We assume that  $L$  is a random variable that is drawn in iid fashion each time a jump occurs. We also assume (with one eye on an equilibrium in which  $\alpha = 1$ ) that  $L$  is strictly less than 1, so that someone who invests fully in the risky asset is not bankrupted. We can allow  $L$  to take negative values; these represent good news for the risky asset. We write  $\hat{\mu} = \mu + \omega EL$  to denote the expected excess return in the absence of jumps.

It follows from (2) that

$$d \log C = \left( r_f + \alpha \hat{\mu} - \frac{1}{2} \alpha^2 \sigma^2 - \theta \right) dt + \alpha \sigma dZ + \log(1 - \alpha L) dN. \quad (3)$$

This equation is derived in the Appendix.

Integrating forwards, exponentiating, using  $C_0 = \theta W_0$ , and raising to the power  $1 - \gamma$ , we have

$$C_t^{1-\gamma} = W_0^{1-\gamma} \theta^{1-\gamma} \exp \left\{ (1 - \gamma) \left( r_f + \alpha \hat{\mu} - \frac{1}{2} \alpha^2 \sigma^2 - \theta \right) t + \alpha (1 - \gamma) \sigma Z_t \right\} \prod_{i=1}^{N_t} (1 - \alpha L_i)^{1-\gamma}. \quad (4)$$

Writing  $L$  for a representative of the iid  $L_i$ , the Appendix shows that

$$EC_t^{1-\gamma} = W_0^{1-\gamma} \theta^{1-\gamma} \exp \left\{ (1-\gamma) \left( r_f + \alpha \hat{\mu} - \frac{1}{2} \gamma \alpha^2 \sigma^2 - \theta \right) t + \omega E \left[ (1-\alpha L)^{1-\gamma} - 1 \right] t \right\}. \quad (5)$$

Hence the objective function can be evaluated explicitly, as

$$U_0 = \frac{W_0^{1-\gamma} \theta^{1-\gamma}}{1-\gamma \rho - (1-\gamma) \left( r_f + \alpha \hat{\mu} - \frac{1}{2} \gamma \alpha^2 \sigma^2 - \theta \right) - \omega E \left[ (1-\alpha L)^{1-\gamma} - 1 \right]}. \quad (6)$$

The optimal investment and consumption choices are identified by maximizing (6) with respect to  $\alpha$  and  $\theta$ . The optimal consumption-wealth ratio is

$$\theta_{\text{unc}} = \frac{\rho + (\gamma - 1) \left( r_f + \alpha \hat{\mu} - \frac{1}{2} \gamma \alpha^2 \sigma^2 \right) - \omega E \left[ (1-\alpha L)^{1-\gamma} - 1 \right]}{\gamma}. \quad (7)$$

We assume that  $\theta_{\text{unc}}$  is positive. This implies that the denominator of (6) is positive when  $\alpha$  and  $\theta$  are chosen optimally, and hence that the integral in the definition of expected utility converges.

The optimal risky portfolio share is defined implicitly by

$$\hat{\mu} - \alpha \gamma \sigma^2 = \omega E \left[ L (1-\alpha L)^{-\gamma} \right]. \quad (8)$$

In the absence of jumps, where the risky asset follows a pure Brownian motion, this simplifies to the classic Merton formula,

$$\alpha_{\text{BM}} = \frac{\mu}{\gamma \sigma^2}. \quad (9)$$

Using (8) to eliminate  $\hat{\mu}$  in (7), we can also write

$$\theta_{\text{unc}} = \frac{\rho + (\gamma - 1) \left( r_f + \frac{1}{2} \gamma \alpha^2 \sigma^2 \right) - \omega E \left[ (1-\alpha \gamma L) (1-\alpha L)^{-\gamma} - 1 \right]}{\gamma}. \quad (10)$$



### 3 A Sustainability Constraint

We now introduce a sustainability constraint that the individual might choose to impose on herself. The constraint requires that expected utility should not be allowed to decline, in expectation, over time. (Expected utility is itself a random variable, because it is a function of current wealth.) If the consumer is thought of as the currently living representative of an infinitely lived dynasty, then the constraint is appropriate if she does not want her descendants to expect a lower quality of life than she does.

Equation (6) shows that expected utility at time  $t$ ,  $U_t$ , is proportional to  $W_t^{1-\gamma}/(1-\gamma)$ . We multiply by  $(1-\gamma)$ , which is negative under our maintained assumption that  $\gamma > 1$ , and work with a rescaled variable  $X_t = W_t^{1-\gamma}$ . This follows the process

$$\frac{dX}{X} = (1-\gamma) \left( r_f + \alpha\hat{\mu} - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2 \right) dt + (1-\gamma)\alpha\sigma dZ + [(1-\alpha L)^{1-\gamma} - 1] dN. \quad (11)$$

The derivation of equation (11) is given in the Appendix.

The *sustainability constraint*—which requires that utility should not decline, in expectation, over time—dictates that the drift of  $X$  should be negative, and hence that

$$\theta \leq r_f + \alpha\hat{\mu} - \frac{1}{2}\gamma\alpha^2\sigma^2 + \omega \mathbb{E} \frac{[(1-\alpha L)^{1-\gamma} - 1]}{1-\gamma}, \quad (12)$$

recalling that  $\mathbb{E}dN = \omega dt$ .

Let us write  $\theta_{\text{con}}$  for the case in which the constraint binds,

$$\theta_{\text{con}} = r_f + \alpha\hat{\mu} - \frac{1}{2}\gamma\alpha^2\sigma^2 + \omega \frac{\mathbb{E} [(1-\alpha L)^{1-\gamma} - 1]}{1-\gamma}, \quad (13)$$

and  $\theta_{\text{unc}}$  for the case in which the constraint does not bind, so that the optimal choice is given by

equation (10). The optimal consumption-wealth ratio is independent of  $\rho$  if the constraint binds; this is not true of the unconstrained case.

If the constraint binds, then we can use it to eliminate  $\theta$  from the objective function (6), giving

$$U_{\text{con},0} = \frac{W_0^{1-\gamma} \left( r_f + \alpha \hat{\mu} - \frac{1}{2} \gamma \alpha^2 \sigma^2 + \omega \frac{\mathbb{E}[(1-\alpha L)^{1-\gamma} - 1]}{1-\gamma} \right)^{1-\gamma}}{1-\gamma} \frac{1}{\rho}. \quad (14)$$

The optimal investment choice is unaffected by the presence of the sustainability constraint. Maximizing equation (14) with respect to  $\alpha$ , we find the same first-order condition as before, equation (8). We can use this condition to eliminate  $\hat{\mu}$  from equation (13), giving

$$\theta_{\text{con}} = r_f + \frac{1}{2} \gamma \alpha^2 \sigma^2 + \omega \frac{\mathbb{E}[(1-\alpha \gamma L)(1-\alpha L)^{-\gamma} - 1]}{1-\gamma}. \quad (15)$$

When risk and the risky portfolio share are positive, the second two terms on the right hand side of equation (15) are positive. Hence (15) shows that the constrained consumption-wealth ratio  $\theta_{\text{con}}$  exceeds the riskfree interest rate  $r_f$  in a risky economy.

### 3.1 Comparing the constrained and unconstrained solutions

The optimal sustainable consumption-wealth ratio,  $\theta_{\text{sus}}$ , is given by whichever of  $\theta_{\text{con}}$  and  $\theta_{\text{unc}}$  is smaller. If the unconstrained case features a lower consumption-wealth ratio, then it certainly satisfies the constraint and delivers higher utility. If not, the unconstrained case does not satisfy the constraint, so that  $\theta_{\text{con}}$  is the best we can do. Thus

$$\theta_{\text{sus}} = \min \{ \theta_{\text{unc}}, \theta_{\text{con}} \}. \quad (16)$$

Equivalently,  $\theta_{\text{con}}$  is the highest possible sustainable consumption-wealth ratio.

It follows from equations (10) and (15) that

$$\theta_{\text{unc}} = \frac{1}{\gamma}\rho + \left(1 - \frac{1}{\gamma}\right)\theta_{\text{con}}. \quad (17)$$

Equations (16) and (17) have several interesting implications. First, the sustainability constraint binds if and only if  $\rho > \theta_{\text{con}}$ , or equivalently if and only if  $\rho > \theta_{\text{unc}}$ . Related to this, we can show that in the absence of a sustainability constraint,

$$E_0 X_t = X_0 e^{(\rho - \theta_{\text{unc}})t}. \quad (18)$$

If impatience is sufficiently high that  $\rho > \theta_{\text{unc}}$ , then  $X_t = W_t^{1-\gamma}$  is expected to grow without limit in an unconstrained equilibrium and expected utility is expected to decline without limit.<sup>3</sup> The sustainability constraint, which rules out declining expected utility, binds in this circumstance.

Second, equation (17) shows that the moderating influence of  $\rho$  makes  $\theta_{\text{unc}}$  less sensitive than  $\theta_{\text{con}}$  to changes in other parameters of the model, holding  $\rho$  fixed.

Third, equation (17) implies that the behavior of an extremely risk-averse individual is little affected by the presence or absence of a sustainability constraint, as  $\theta_{\text{unc}} \approx \theta_{\text{con}}$  if  $\gamma$  is large. This reflects the fact that highly concave utility leads an agent to choose a flat consumption path that is close to sustainable, regardless of the level of  $\rho$ .

Fourth, equations (16) and (17) show that  $\theta_{\text{sus}}$  and  $\theta_{\text{unc}}$  can easily be calculated from knowledge of  $\theta_{\text{con}}$ , so we can focus our analysis on the determinants of  $\theta_{\text{con}}$ .

Finally, we note that a planner could implement the sustainable optimum by using a modified

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<sup>3</sup>Expected wealth will also decline without limit if  $\rho$  is sufficiently large; but if  $\rho$  is sufficiently close to  $\theta_{\text{con}}$  then wealth has positive drift despite the negative drift in expected utility.

“social discount rate”  $\hat{\rho}$ , chosen to satisfy

$$\theta_{\text{unc}}(\hat{\rho}) = \theta_{\text{sus}} = \min\{\theta_{\text{unc}}(\rho), \theta_{\text{con}}\}. \quad (19)$$

(The first equality defines the social discount rate. The second is equation (16). The notation emphasizes the dependence of  $\theta_{\text{unc}}$  on the discount rate.) If the sustainability constraint is not binding,  $\theta_{\text{unc}}(\rho) \leq \theta_{\text{con}}$ , then we can set  $\hat{\rho} = \rho$ . If the constraint does bind, then we must ensure that  $\theta_{\text{unc}}(\hat{\rho}) = \theta_{\text{con}}$ . By equation (17), this requires that  $\hat{\rho} = \theta_{\text{con}}$ . As the constraint binds precisely when  $\rho > \theta_{\text{con}}$ , we can summarize all this by saying that we should set

$$\hat{\rho} = \min\{\rho, \theta_{\text{con}}\}. \quad (20)$$

### 3.2 Sustainable drifts in wealth and marginal utility

A binding sustainability constraint implies zero drift in expected utility, but this does not imply a zero drift in wealth. Instead wealth drifts upwards over time under our assumption that  $\gamma > 1$ , because

$$\frac{dW}{W} = \left\{ \frac{1}{2}\gamma\alpha^2\sigma^2 - \frac{\omega}{1-\gamma}\text{E}[(1-\alpha L)^{1-\gamma} - 1] \right\} dt + \alpha\sigma dZ - \alpha L dN, \quad (21)$$

so the drift is

$$\text{E}\frac{dW}{W} = \left( \frac{1}{2}\gamma\alpha^2\sigma^2 + \frac{\omega}{\gamma-1}\text{E}[(1-\alpha L)^{1-\gamma} - 1 + \alpha L(1-\gamma)] \right) dt, \quad (22)$$

and both terms in the brackets are positive.

We can also show the stronger result that log wealth drifts upwards over time under a binding

sustainability constraint. We have

$$\mathbb{E} d \log W = \left( \frac{1}{2}(\gamma - 1)\alpha^2\sigma^2 + \frac{\omega}{\gamma - 1}\mathbb{E} [(1 - \alpha L)^{1-\gamma} - 1 - (1 - \gamma)\log(1 - \alpha L)] \right) dt, \quad (23)$$

and again both terms in the brackets are positive.

These facts illustrate the distinction between our sustainability constraint and the arithmetic and geometric constraints considered by Campbell and Sigalov (2021), which impose zero drift in wealth or in log wealth, respectively. Campbell and Sigalov's constraints distort portfolio choice, unlike the sustainability constraint we consider.

A binding sustainability constraint also does not imply zero drift in marginal utility. The process for marginal utility,  $M = W^{-\gamma}$ , is

$$\frac{dM}{M} = \left\{ \frac{1}{2}\gamma\alpha^2\sigma^2 + \frac{\omega\gamma}{1 - \gamma}\mathbb{E} [(1 - \alpha L)^{1-\gamma} - 1] \right\} dt - \gamma\alpha\sigma dZ + [(1 - \alpha L)^{-\gamma} - 1] dN, \quad (24)$$

so the drift is

$$\mathbb{E} \frac{dM}{M} = \left( \frac{1}{2}\gamma\alpha^2\sigma^2 + \frac{\omega}{1 - \gamma} \left\{ \gamma\mathbb{E} [(1 - \alpha L)^{1-\gamma} - 1] + (1 - \gamma)\mathbb{E} [(1 - \alpha L)^{-\gamma} - 1] \right\} \right) dt, \quad (25)$$

which is positive when  $\gamma > 1$ . The drift in marginal utility equals the drift in wealth in the Brownian case where  $L = 0$ , but differs from it in the general case with jumps.

The positive drift in marginal utility in the constrained economy is another way to understand the result that the constrained consumption-wealth ratio exceeds the riskfree interest rate. The first-order condition for optimal investment in a riskless asset implies that discounted marginal utility drifts downward at the riskfree interest rate, and hence that the drift in undiscounted marginal utility is the constrained social rate of time preference (equivalently, the constrained consumption-

wealth ratio) less the riskfree interest rate. That is, we have

$$\mathbb{E} \frac{dM}{M} = \theta_{\text{con}} - r_f. \quad (26)$$

Our solutions have this property, as can be verified by comparing the right hand sides of equations (15) and (25).

With power utility, the driftless variable  $X = W^{1-\gamma}$  is the product of marginal utility and wealth:  $X = MW$ . At first sight it might seem surprising that  $X$  has no drift while both  $M$  and  $W$  have positive drift. But we must also take into account comovement in  $M$  and  $W$ , whose effect is visible in the product rule:

$$\frac{dX}{X} = \frac{dM}{M} + \frac{dW}{W} + \frac{dM}{M} \frac{dW}{W}. \quad (27)$$

The product is

$$\frac{dM}{M} \frac{dW}{W} = -\gamma \alpha^2 \sigma^2 dt - \alpha L [(1 - \alpha L)^{-\gamma} - 1] dN, \quad (28)$$

and the fact that this quantity is negative makes it possible for  $X = MW$  to be driftless even though  $M$  and  $W$  each have positive drift:

$$\mathbb{E} \frac{dX}{X} = \mathbb{E} \frac{dM}{M} + \mathbb{E} \frac{dW}{W} + \mathbb{E} \left( \frac{dM}{M} \frac{dW}{W} \right) = 0. \quad (29)$$

We conclude by noting one other implication of the upwards drift in log wealth shown in equation (23). Although the sustainability constraint requires that the average future value of expected utility  $U_t$  is the same as expected utility today, as one looks into the far-distant future expected utility is overwhelmingly likely to be higher than its current value. This counterintuitive fact, which echoes the result of Martin (2012), follows from the fact that  $\log W_t$  has positive drift, so that expected utility at time  $t$  approaches its upper bound of zero almost surely as  $t$  approaches infinity. The invariance of expected utility to the horizon is achieved by a vanishingly small number

of paths in which expected utility in the future is arbitrarily low.

### 3.3 Pure Brownian motion and pure jumps

There are some cases in which we can characterize the optimal investment choice explicitly. We can, for example, contrast the pure Brownian case,  $\omega = 0$ , with the pure jump case in which  $\sigma = 0$  and  $L \in (0, 1)$  is deterministic. In the pure Brownian case, the risky share satisfies the Merton formula (9). Substituting into equation (15) we have

$$\theta_{\text{con,BM}} = r_f + \frac{1}{2} \frac{\mu^2}{\gamma \sigma^2}. \quad (30)$$

In this case the constrained consumption-wealth ratio is the riskless interest rate plus one half the squared Sharpe ratio of the risky asset, divided by risk aversion. This can be much larger than the riskless interest rate: for example, if the riskless rate is 2%, the Sharpe ratio of the risky asset is 0.4, and risk aversion is 2, then the constrained consumption-wealth ratio is 6%. The constrained consumption-wealth ratio only approaches the riskless rate, its value in a deterministic model, as risk aversion becomes extremely high.

In the pure jump case, we can solve equation (8) to give

$$\alpha_{\text{jump}} = \frac{1}{L} \left[ 1 - \left( \frac{\omega L}{\mu + \omega L} \right)^{1/\gamma} \right]. \quad (31)$$

This solution has  $\alpha L$  invariant to  $L$  if  $\mu/L$  is invariant to  $L$ . In other words, the portfolio exposure to jumps does not alter with jump size provided that the risk premium per unit of jump risk is constant. This is the jump equivalent of the well known property under Brownian motion that  $\alpha \sigma$  is invariant to  $\sigma$  if the Sharpe ratio  $\mu/\sigma$  is invariant to  $\sigma$ .

The pure jump model has two free parameters,  $\omega$  and  $L$ , to compare with the single parameter

$\sigma$  in the Brownian case. To put them on the same footing, we choose  $\omega$  and  $L$  so that variance is the same in each case, i.e., we set  $\omega = \sigma^2/L^2$ . We can imagine fixing  $\sigma$  and then choosing the parameter  $L$  freely. High values of  $L$  correspond to rare extreme disasters— $\omega$  moves in the opposite direction to  $L$ —whereas values of  $L$  close to zero correspond to frequent small jumps. With this renormalization, equation (31) becomes

$$\alpha_{\text{jump}} = \frac{1}{L} \left[ 1 - \left( \frac{\sigma^2}{\mu L + \sigma^2} \right)^{1/\gamma} \right], \quad (32)$$

and the Appendix shows that

$$\alpha_{\text{jump}} < \alpha_{\text{BM}} \text{ if } \gamma \geq 1. \quad (33)$$

When jumps are larger and rarer—with mean and variance held constant—the Appendix shows that the investor trades less aggressively, that is,

$$\frac{\partial \alpha_{\text{jump}}}{\partial L} < 0. \quad (34)$$

Conversely, in the limit in which jumps are very small and very frequent, we recover the same allocation as in the Brownian motion case: by l'Hôpital's rule,  $\lim_{L \rightarrow 0} \alpha_{\text{jump}} = \frac{\mu}{\gamma \sigma^2}$ .

Substituting the expression for  $\alpha$  given in equation (32) into the expression for  $\theta_{\text{con}}$  given in equation (15), we have

$$\theta_{\text{con}} = r_f + \frac{\sigma^2}{L^2} \frac{(1 - \gamma)(K^{-\gamma} - 1) + \gamma(K^{1-\gamma} - 1)}{1 - \gamma} \quad \text{where} \quad K = \left( \frac{\sigma^2}{\mu L + \sigma^2} \right)^{1/\gamma}. \quad (35)$$

This tends to the corresponding expression in the pure Brownian case (30) as  $L \rightarrow 0$ .



### 3.4 Numerical examples

In this section we present some numerical examples to illustrate the properties we have discussed. Table 1 reports numerical results for a Brownian model without jumps, in which the riskless interest rate  $r_f$  equals 1%, the risk premium  $\mu$  equals 8%, and the standard deviation of the risky asset  $\sigma$  equals 20%, implying a Sharpe ratio of 0.4. The four rows of the table consider risk aversion coefficients  $\gamma$  of 1, 2, 5, and 10. The columns report the constrained consumption-wealth ratio  $\theta_{\text{con}}$ , the risky portfolio share  $\alpha$ , and the corresponding drifts in wealth, log wealth, and marginal utility.

Table 1: Numerical examples in the Brownian case  
 Baseline calibration sets  $r_f = 1\%$ ,  $\mu = 8\%$ ,  $\sigma = 20\%$ .

$\gamma$	$\theta_{\text{con}}$	$\alpha$	$E \frac{dW}{W}$	$E d \log W$	$E \frac{dW^{-\gamma}}{W^{-\gamma}}$
1	0.09	2	0.08	0	0.08
2	0.05	1	0.04	0.02	0.04
5	0.026	0.4	0.016	0.0128	0.016
10	0.018	0.2	0.008	0.0072	0.008

In a Brownian model, the risky portfolio share is inversely proportional to risk aversion. Given our assumed parameters an investor with log utility ( $\gamma = 1$ ) sets  $\alpha = 2$ , leveraging the risky asset two for one. The constrained consumption-wealth ratio is 9% and the corresponding expected growth rate of wealth is 8%. However, the expected growth rate of log wealth is zero since this is the growth rate of utility for a log investor so the sustainability constraint sets it to zero. The expected growth rate of marginal utility is the difference between the constrained consumption-wealth ratio and the riskfree rate, or 8%, and in the Brownian model it also equals the expected growth rate of wealth as we noted earlier.

As risk aversion increases, the constrained consumption-wealth ratio and the risky portfolio share both decline. For example, when  $\gamma = 2$  the constrained consumption-wealth ratio is 5% and

the risky portfolio share  $\alpha = 1$ . However, the constrained consumption-wealth ratio declines slowly and is still 1.8%, almost twice the riskless interest rate, when  $\gamma = 10$ . The expected growth rates of wealth and marginal utility also decline with risk aversion, but the expected growth rate of log wealth is hump-shaped in risk aversion, first increasing and then ultimately declining towards zero.

Table 2: Numerical examples with jumps

Baseline calibration sets  $r_f = 2\%$ ,  $\mu = 4\%$ ,  $\sigma = 10\%$ ,  $\omega = 4\%$ ,  $L = 0.4$ ,  $\gamma = 2$ .

	$\gamma$	$r_f$	$\mu$	$\sigma$	$\omega$	$L$	$\theta_{\text{con}}$	$\alpha$	$E \frac{dW}{W}$	$E d \log W$	$E \frac{dW^{-\gamma}}{W^{-\gamma}}$
Baseline	2	0.02	0.04	0.10	0.02	0.40	0.045	1.11	0.019	0.010	0.025
High $\gamma$	5						0.031	0.49	0.009	0.007	0.011
Low $\gamma$	1						0.065	1.83	0.028	0	0.045
High $r_f$		0.04					0.065	1.11	0.019	0.010	0.025
Low $r_f$		0					0.025	1.11	0.019	0.010	0.025
High $\mu$			0.08				0.10	1.56	0.045	0.026	0.080
Low $\mu$			0.02				0.027	0.66	0.006	0.003	0.007
High $\sigma$				0.15			0.035	0.72	0.014	0.007	0.015
Low $\sigma$				0.06			0.055	1.36	0.019	0.011	0.035
High $\omega$					0.04		0.040	0.88	0.016	0.008	0.020
Low $\omega$					0		0.060	2.00	0.040	0.020	0.040
High $L$						0.60	0.037	0.72	0.012	0.007	0.017
Low $L$						0.20	0.056	1.73	0.034	0.017	0.036
Negative $L$						-0.40	0.054	1.74	0.036	0.017	0.034

Table 2 reports numerical results for a more general model allowing for jumps. Here, in the benchmark case the riskless interest rate  $r_f$  equals 2%, the risk premium  $\mu$  equals 4%, and the Brownian standard deviation of the risky asset  $\sigma$  equals 10%. In addition a jump of size  $L = 40\%$  occurs with a probability  $\omega$  of 4% per period. The benchmark case sets  $\gamma = 2$ , in which case the optimal constrained consumption-wealth ratio is 4.5% and the risky portfolio share  $\alpha = 1.11$ . These values are fairly close to those in the Brownian case reported in Table 1, indicating that the calibration with jumps is broadly comparable in its overall level of risk.

The remaining rows of the table consider variations of the benchmark model with higher and lower risk aversion, then higher and lower riskfree rates, risk premium, Brownian volatility,

jump probability, and jump size. The results are intuitive and in all cases imply a sustainable consumption-wealth ratio well above the riskfree interest rate, or equivalently a substantial positive expected growth rate of marginal utility.

## 4 Sustainability Without a Riskless Asset

We might alternatively deal with  $\alpha$  by requiring that it should equal one in equilibrium. In this case we are implicitly normalizing the risky asset return to equal the return on society's invested wealth.

It will be convenient to write  $L = 1 - e^{-J}$  where  $J$  is a random variable that can take positive or negative values, and whose moment-generating function we write, following Martin (2013), as  $\mathbf{m}(x) = \text{E}e^{Jx}$ . (Large positive  $J$  is very bad news, large negative  $J$  is very good news.) Setting  $\alpha = 1$  in equation (8),

$$\mu = \gamma\sigma^2 + \omega \{ \mathbf{m}(\gamma) - \mathbf{m}(\gamma - 1) - [\mathbf{m}(0) - \mathbf{m}(-1)] \}. \quad (36)$$

Setting  $\alpha = 1$  in (15),

$$\theta_{\text{con}} = r_f + \frac{1}{2}\gamma\sigma^2 + \omega \left[ \mathbf{m}(\gamma) - \mathbf{m}(\gamma - 1) - \frac{\mathbf{m}(\gamma - 1) - \mathbf{m}(0)}{\gamma - 1} \right]. \quad (37)$$

The terms in square brackets on the right hand side of equation (37) are positive because  $\mathbf{m}(x)$  is convex. This provides a lower bound on the maximum sustainable consumption-wealth ratio,  $\theta_{\text{con}}$ , that is greater than the riskless rate.

To derive a bound in the opposite direction, we can use equation (36) to rewrite equation (37)

as

$$\theta_{\text{con}} = \mu + r_f - \frac{1}{2}\gamma\sigma^2 - \omega \left[ \frac{\mathbf{m}(\gamma - 1) - \mathbf{m}(0)}{\gamma - 1} - \frac{\mathbf{m}(0) - \mathbf{m}(-1)}{1} \right]. \quad (38)$$

The terms in square brackets on the right hand side of equation (38) are also positive (by the convexity of  $\mathbf{m}(x)$  once again), so equation (38) supplies an upper bound on the maximum sustainable consumption-wealth ratio that is lower than the return on the risky asset. To sum up, we have

$$r_f + \frac{1}{2}\gamma\sigma^2 \leq \theta_{\text{con}} \leq \mu + r_f - \frac{1}{2}\gamma\sigma^2. \quad (39)$$

Moreover, it is possible to choose the distribution of  $J$  so that the terms in square brackets in (38) are close to zero while the terms in square brackets in (37) are not; then the upper bound is tight. This happens if the MGF of  $J$  is roughly flat between  $-1$  and  $\gamma - 1$  and steeply upward-sloping from  $\gamma - 1$  to  $\gamma$ , as can happen in the frightening case with occasional *bad news* jumps.<sup>4</sup> Conversely, it is possible to arrange for the terms in square brackets in (37) to be close to zero, while those in (38) are not; then the lower bound binds. This happens if the MGF of  $J$  is steeply downward-sloping between  $-1$  and  $0$  and roughly flat on the range  $[0, \gamma]$ , as can happen if there are occasional *good news* jumps. Meanwhile in the pure Brownian case,  $\omega = 0$  or  $J = 0$ , the sustainable consumption-wealth ratio is the equally weighted average of  $r_f$  and  $r_f + \mu$ :

$$\theta_{\text{BM,con}} = r_f + \frac{1}{2}\gamma\sigma^2. \quad (40)$$

## 4.1 Numerical examples without a riskless asset

We illustrate these properties numerically in two different ways. Figure 1 shows the constrained consumption-wealth ratio and the upper and lower bounds given in equation (39) for a model with

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<sup>4</sup>For any given  $\gamma > 1$ , we can make  $\theta_{\text{con}}$  as close as we like to either of the bounds by appropriately choosing some very small  $\omega$  and a fixed jump size  $J$ . To make the lower bound tight, we will want to choose  $J < 0$ ; to make the upper bound tight, we will want to choose  $J > 0$ .

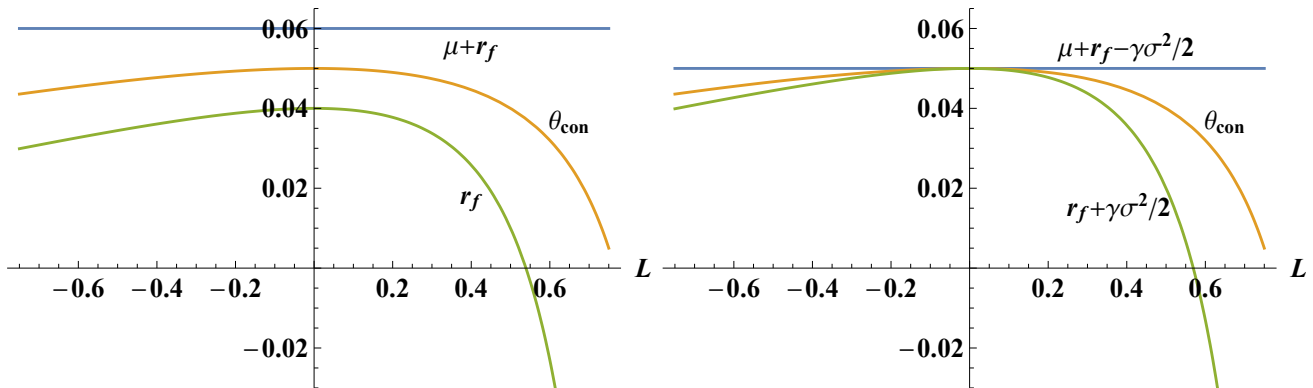


Figure 1:  $\theta_{\text{con}}$  and the upper and lower bounds for various deterministic jump sizes  $L$ , with  $\gamma = 2$ ,  $\sigma = 0.1$ ,  $\omega = 0.02$ ,  $\mu + r_f = 0.06$ . Jumps are bad news if  $L$  is positive and good news if  $L$  is negative.

risk aversion  $\gamma = 2$ , an expected return on the risky asset  $r_f + \mu = 6\%$ , Brownian volatility  $\sigma = 10\%$ , a jump probability  $\omega = 2\%$ , and jumps of deterministic size  $L$ . The horizontal axis shows different values for  $L$ , where positive values correspond to negative jumps (losses) in wealth, and negative values correspond to positive jumps in wealth. In the left panel, the constrained consumption-wealth ratio  $\theta_{\text{con}}$  is plotted along with the expected risky asset return  $r_f + \mu$  (constant at 6%) and the riskfree interest rate  $r_f$ . The constrained consumption-wealth ratio is halfway between the two returns in the Brownian case; it is closer to the risky asset return in the bad-jump region where  $L > 0$ , and closer to the riskfree interest rate in the good-jump region where  $L < 0$ . In the right panel, the constrained consumption-wealth ratio is plotted along with the upper and lower bounds from equation (39). The bounds are tight in the Brownian case, and widen out as the absolute jump size increases.

Table 3 reports numerical results for variations of this model. Here, in the benchmark case the jump size  $L = 0.4$ . The constrained consumption-wealth ratio equals 4.5% and the riskfree interest rate is only 2.6%. The remaining rows of the table consider variations of the benchmark model with higher and lower risk aversion, then higher and lower risky asset returns, Brownian volatility,

jump probability, and jump size. The results are intuitive and in all cases imply a sustainable consumption-wealth ratio well above the riskfree interest rate, or equivalently a substantial positive expected growth rate of marginal utility.

Table 3: Numerical examples without a riskless asset

Baseline calibration sets  $\mu + r_f = 6\%$ ,  $\sigma = 10\%$ ,  $\omega = 2\%$ ,  $L = 0.4$ ,  $\gamma = 2$  in equilibrium with  $\alpha = 1$ .

	$\gamma$	$\mu + r_f$	$\sigma$	$\omega$	$L$	$\theta_{\text{con}}$	$r_f$	$\mu$	$E \frac{dW}{W}$	$E d \log W$	$E \frac{dW^{-\gamma}}{W^{-\gamma}}$
Baseline	2	0.06	0.10	0.02	0.40	0.045	0.026	0.034	0.015	0.008	0.019
High $\gamma$	5					0.009	-0.085	0.145	0.051	0.043	0.094
Low $\gamma$	1					0.053	0.045	0.015	0.007	0	0.008
High $\mu + r_f$		0.10				0.085	0.066	0.034	0.015	0.008	0.019
Low $\mu + r_f$		0.02				0.005	-0.014	0.034	0.015	0.008	0.019
High $\sigma$			0.15			0.032	0.001	0.059	0.028	0.014	0.031
Low $\sigma$			0.06			0.051	0.039	0.021	0.009	0.005	0.012
High $\omega$				0.04		0.039	0.012	0.048	0.021	0.011	0.028
Low $\omega$				0		0.050	0.040	0.020	0.010	0.005	0.010
High $L$					0.50	0.040	0.010	0.050	0.020	0.011	0.030
Low $L$					0.30	0.047	0.034	0.026	0.013	0.006	0.014
Negative $L$					-0.40	0.048	0.036	0.024	0.012	0.006	0.012

## 5 Conclusion

In this paper we have argued, in the spirit of Koopmans (1960, 1967), that the implication of an ethical criterion—sustainability—for social discounting and consumption decisions depends on the production technology available to society. Specifically, in a risky world with a binding sustainability constraint, the sustainable social rate of time preference and consumption-wealth ratio, which equal one another, are not equal to either the riskless interest rate or the risky return on invested wealth, but lie in between these two. In the special case where invested wealth has only Brownian risk and no jump risk, the sustainable social rate of time preference is the equal-weighted average

of the riskless interest rate and the risky return.

We have made this point in the context of an extremely simple model with iid returns in which the parameters governing the distribution of returns are known. We have therefore ignored parameter uncertainty, a phenomenon emphasized by Weitzman (2001). We have also ignored the possibility that returns may not be iid, because expected returns or risks change over time. Models with non-iid returns in general imply time-varying consumption growth and a term structure of discount rates. When consumption growth is persistent, this term structure is generally downward-sloping for safe investments and upward-sloping for risky ones as in the long-run risk model of Bansal and Yaron (2004). Gollier (2002) emphasizes the potential importance of a downward-sloping term structure of discount rates for social discounting. Our iid model has discount rates that are invariant to the horizon of an investment.

Although we have emphasized the sustainable social rate of time preference in this paper, we conclude by noting that this is not the same as the appropriate social discount rate that should be applied to an investment project. That discount rate depends on the project's risk. For a riskless project, the appropriate discount rate is the riskless interest rate, which is lower than the sustainable social rate of time preference in a risky world; and for a project that has the same risk as society's invested wealth, the appropriate discount rate is the expected risky return, which is higher than the sustainable social rate of time preference. Some previous discussions of social discounting have obscured these distinctions by ignoring the risk that society faces. Our analysis is deliberately simple in order to achieve clarity about these issues.

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## Appendix

*Derivation of equation (3).*

Equation (3) follows from (2) and Itô's formula for semimartingales. See, for example, Proposition 8.19 of Cont and Tankov (2004). Heuristically, we can derive it by writing

$$d \log C = \frac{1}{C} dC - \frac{1}{2} \frac{1}{C^2} (dC)^2 + \frac{1}{6} \frac{2}{C^3} (dC)^3 - \frac{1}{24} \frac{6}{C^4} (dC)^4 + \dots$$

and using the relationships  $dt dN = dZ dN = 0$  and  $dN^k = dN$  for all  $k > 0$ , in addition to the standard properties of  $dZ$  and the fact that  $\log(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \dots$  if  $|x| < 1$ , which holds when  $x = -\alpha L$  because the agent will never risk bankruptcy.

*Derivation of equation (5).*

The calculation exploits the fact that  $N_t$ ,  $Z_t$ , and  $L_i$  are independent. In particular, note that by the law of iterated expectations, the fact that  $N_t$  is a Poisson random variable with parameter  $\omega t$ , the iid nature of the  $L_i$ , and the series definition of the exponential function,

$$\begin{aligned} \mathbb{E} \prod_{i=1}^{N_t} (1 - \alpha L_i)^{1-\gamma} &= \mathbb{E} \left[ \mathbb{E} \left( \prod_{i=1}^{N_t} (1 - \alpha L_i)^{1-\gamma} \mid N_t \right) \right] \\ &= \sum_{n=0}^{\infty} e^{-\omega t} \frac{(\omega t)^n}{n!} \mathbb{E} \prod_{i=1}^n (1 - \alpha L_i)^{1-\gamma} \\ &= \sum_{n=0}^{\infty} e^{-\omega t} \frac{(\omega t)^n}{n!} (\mathbb{E} [(1 - \alpha L)^{1-\gamma}])^n \\ &= \exp \{ \omega \mathbb{E} [(1 - \alpha L)^{1-\gamma} - 1] t \}. \end{aligned}$$

*Derivation of equation (11).*

We write

$$\begin{aligned}
\frac{dX}{X} &= (1-\gamma)\frac{dW}{W} + \frac{\gamma(\gamma-1)}{2}\left(\frac{dW}{W}\right)^2 - \frac{\gamma(\gamma-1)(\gamma+1)}{6}\left(\frac{dW}{W}\right)^3 + \dots \\
&= (1-\gamma)\left(r_f + \alpha\hat{\mu} - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2\right)dt + (1-\gamma)\alpha\sigma dZ + \\
&\quad + \left[(\gamma-1)\alpha L + \frac{\gamma(\gamma-1)}{2}\alpha^2 L^2 + \frac{\gamma(\gamma-1)(\gamma+1)}{6}\alpha^3 L^3 + \dots\right]dN \\
&= (1-\gamma)\left(r_f + \alpha\hat{\mu} - \theta - \frac{1}{2}\gamma\alpha^2\sigma^2\right)dt + (1-\gamma)\alpha\sigma dZ + [(1-\alpha L)^{1-\gamma} - 1]dN.
\end{aligned}$$

*Derivation of equation (33).*

This follows because

$$\left(\frac{\sigma^2}{\mu L + \sigma^2}\right)^{1/\gamma} = \left(1 - \frac{\mu L}{\mu L + \sigma^2}\right)^{1/\gamma} > \left(1 - \frac{\mu L}{\sigma^2}\right)^{1/\gamma} \geq 1 - \frac{\mu L}{\sigma^2} \quad \text{if } \gamma \geq 1.$$

*Derivation of equation (34).*

We differentiate equation (32) with respect to  $L$ . The result is negative if and only if

$$1 + \frac{\mu L}{\gamma(\mu L + \sigma^2)} < \left(1 - \frac{\mu L}{\mu L + \sigma^2}\right)^{-1/\gamma}.$$

But this holds, because

$$1 + \frac{\mu L}{\gamma(\mu L + \sigma^2)} < \left(1 + \frac{\mu L}{\mu L + \sigma^2}\right)^{1/\gamma} < \left(\frac{1}{1 - \frac{\mu L}{\mu L + \sigma^2}}\right)^{1/\gamma},$$

where the first inequality is Bernoulli's inequality.

*The log utility case.*

With log utility, the investor's objective function is

$$U = \mathbb{E} \int_0^\infty e^{-\rho t} \log C_t dt, \quad \text{where } \rho > 0.$$

It follows from equation (3) that

$$\log C_t = \log C_0 + \left( r_f + \alpha \hat{\mu} - \frac{1}{2} \alpha^2 \sigma^2 - \theta \right) dt + \alpha \sigma Z_t + \sum_{i=1}^{N_t} \log(1 - \alpha L_i),$$

and hence

$$\mathbb{E} \log C_t = \log C_0 + \left( r_f + \alpha \hat{\mu} - \frac{1}{2} \alpha^2 \sigma^2 - \theta \right) dt + \omega \mathbb{E} [\log(1 - \alpha L)] t.$$

Thus the objective function can be evaluated explicitly as

$$U = \frac{\log W_0 + \log \theta}{\rho} + \frac{r_f + \alpha \hat{\mu} - \frac{1}{2} \alpha^2 \sigma^2 - \theta + \omega \mathbb{E} [\log(1 - \alpha L)]}{\rho^2}.$$

Maximizing with respect to  $\theta$  and  $\alpha$  we find the first-order conditions for an unconstrained optimum,

$$\theta = \rho \quad \text{and} \quad \hat{\mu} - \alpha \sigma^2 = \omega \mathbb{E} [L(1 - \alpha L)^{-1}].$$

The objective function at time  $t$  is affine in  $\log W_t$ , so the sustainability condition requires that  $d \log W_t$ , or equivalently  $d \log C_t$ , is driftless, i.e. that

$$\theta \leq r_f + \alpha \hat{\mu} - \frac{1}{2} \alpha^2 \sigma^2 + \omega \mathbb{E} [\log(1 - \alpha L)].$$

We define the constrained solution as before, giving

$$\theta_{\text{con}} = r_f + \alpha \hat{\mu} - \frac{1}{2} \alpha^2 \sigma^2 + \omega \mathbb{E} [\log(1 - \alpha L)].$$

When the constraint binds, we have

$$U = \frac{\log W_0 + \log \theta}{\rho},$$

so  $\alpha$  is chosen to maximize the constrained consumption-wealth ratio. We end up with the same first-order condition as in the unconstrained case. Thus the optimal investment choice is the same in the constrained and unconstrained cases, as before. Equations (16) and (17) also hold as before. Thus, all the results stated previously for risk aversion  $\gamma > 1$  carry over to the log case where  $\gamma = 1$ .