Layered Networks, Equilibrium Dynamics, and Stable Coalitions

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Abstract
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Keywords: club networks, stable coalition structures, networks as partial functions, Harris recurrent sets, basins of attraction, discounted stochastic games, stationary Markov perfect equilibria. equilibrium
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Abstract

An important aspect of network dynamics that has been missing from our understanding of network dynamics in various applied settings is the influence of strategic behavior in determining equilibrium network dynamics. Our main objective hear to say what we can regarding the emergence of stable club networks - and therefore, stable coalition structures - based on the stability properties of strategically determined equilibrium network formation dynamics. Because club networks are layered networks, our work here can be thought of as a first work on the dynamics of layered networks. In addition to constructing a discounted stochastic game model (i.e., a *DSG* model) of club network formation, we show that (1) our *DSG* of network formation possesses a stationary Markov perfect equilibrium in players' membership-action strategies and (2) we identify the assumptions on primitives which ensure that the induced equilibrium Markov process of layered club network formation satisfies the Tweedie Stability Conditions (2001) and that (3) as a consequence, the equilibrium Markov network formation processes generates a unique decomposition of the set of state-network pairs into a transient set together with *finitely many* basins of attraction. Moreover, we show that if there is a basin containing a *vio set (a visited infinitely often set)* of club networks sufficiently close together, then the coalition structures across club networks in the vio set will be the same (i.e., closeness across networks in a vio set leads to invariance in coalition structure across networks in a vio set).

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1 Introduction

A coalition is a group of players who, through their own actions, can realize some set of outcomes for its own members (Wooders and Page, 2008). Here we will be interested in the equilibrium dynamics governing the formation and evolution of coalitions as well as the strategic forces which give rise to these dynamics. We will think of a coalition as a group of players belonging to the same club, and we will represent the prevailing club membership structure as a labeled, directed bipartite network. Because we will allow each player to be a *member of multiple clubs*, each player can be a member of multiple coalitions (see Page and Wooders, 2010).[1] Each club network consists of three primitives: a set of players, a set of clubs, and a set of arc labels. In our network model, a player's club membership is represented by a labeled directed arc from the node representing the player to the node representing the club. The arc label, which must be feasible for that player in that club, indicates the action chosen by the player to be taken in the chosen club. Thus, a player establishes a directed connection by choosing a club and a feasible club action. The set of all such player-specific directed club connections is *the player's club network* and together the union of these player club networks constitute the club network. At each of infinitely many time points players, in light of the prevailing state and club network, are free to noncooperatively alter their club memberships as well as their corresponding club action profiles in accordance with the rules of network formation. We will assume that after players have altered their own club networks, each player receives a stage payoff, a function of the prevailing state-network pair, then given the prevailing state and the new club network chosen by the players, a new state is generated in accordance with the law of motion. We will assume that players, in making their membership-action choices through discrete time, seek to maximize the discounted sum of their expected future payoffs. In particular, we will assume that players in forming their club networks, play a discounted stochastic game of club network formation in which they seek to choose stationary Markov perfect membership-action strategies that maximize the discounted sum of their expected payoffs. Taken together, the players noncooperative network formation strategies (i.e., membership and club specific action strategies) determine a network formation process. We say that this process is an *equilibrium network formation process* if in the underlying discounted stochastic game (DSG) of network formation players noncooperatively choose *stationary Markov perfect membership and club action strategies forming a Nash equilibrium.*

As an example, we consider the emergence of equilibrium market structure dynamics (i.e., coalition structure dynamics).[2] In our example, m firms (players) compete via the nonlinear prices and product lines (i.e., the catalogs - here arcs) each firm

[1] The paper by Arnold and Wooders (2015), "Dynamic Club Formation with Coordination," is closely related our paper - but differs in two important respects: (i) in Arnold and Wooders players are allowed to join only one club, whereas here we allow players to have multiple club memberships. and (ii) in Arnold and Wooders players are myopic. whereas here players are farsighted in the sense of discounted stochastic games.

[2] Here a coalition is a set of firms active in the same market (club). More on this below.

offers (or abstains from offering) in each of n markets (clubs). Each firms knows the consumer types populating a particular market only up to a conditional probability measure.

Our main contributions, in addition to constructing a DSG model of layered club network formation, are (1) to show that our DSG of network formation is *approximable*, and that as a consequence, it possesses a stationary Markov perfect equilibrium in players' membership-action strategies; (2) to identify the assumptions on primitives which ensure that the induced equilibrium Markov state and club network formation process satisfies the Tweedie Conditions (2001) and; (3) to show that as a consequence of satisfying the Tweedie Conditions, the equilibrium Markov perfect network formation processes generates a unique decomposition of the set of state-network pairs into a transient set together with *finitely many* basins of attraction. We then show that each basin, upon which resides a unique ergodic probability measure, has the property that if the Markov state-network process enters the basin, then the process will remain there for all future periods, visiting some unique, *further* subset of state-network pairs infinitely often. From a macroscopic perspective, it is these basin-specific club networks, visited infinitely often by the process (i.e., vio sets), that form the set of viable candidates for stable equilibrium club network. We show that if there is a basin having a *vio set of states* generating club networks each of whose induced coalition structure is invariant across the networks generated by the vio states, then this vio set of states generates club networks having a stable coalition structure.[3] This is what we mean by coalitional stability: persistent club network structures in which the underlying coalition structure is invariant across club networks generated by the set of vio states. We show that whether or not the vio states contained in a basin of attraction generate an invariant coalition structure depends on the distance between the club networks generated by the vio states (i.e., depends on the distance between the vio networks). In particular, we show that closeness (across networks) leads to invariance in coalition structure across club networks. Thus, if each of finitely many basins has network vio sets containing club networks that are sufficiently close together (i.e., are densely packed), then the equilibrium Markov process of club network formation will, in finite time with probability one, generate a stable coalition structure. While the coalition structures generated by the vio states in each basin are the same across the basin's densely packed club networks, these signature coalition structures can differ across basins. In the long run, whether or not the equilibrium Markov process of network formation generates a stable coalition structure depends on whether or not each basins network vio set is densely packed. If all basin-specific network vio sets are sufficiently dense, then with probability 1 in finite time, the equilibrium network formation process will arrive at a basin specific stable coalition structure. However, if some basins have network vio sets in which the club networks are not sufficiently close, while other basins possess densely packed network vio sets, then in equilibrium, there is a positive probability that a stable coalition structure will never be reached. What we conclude here is that, under mild conditions on primitives, the conditions which guarantee the emergence of basins of

[3] vio stands for "visited infinitely often."

attraction (i.e., the Tweedie conditions, 2001), also guarantee that each basin will contain a densely packed club network vio set. Thus, in finite time with probability 1, there will emerge from the equilibrium process of club network formation, a basin specific stable coalition structure.

Part I

Layered Club Networks

We begin with a formal definition of club networks and a discussion of their properties. The discussion here is based, in part, on prior work by the second author with M. Wooders (see Page and Wooders 2010, 2009, 2007, Wooders and Page 2008, and Page, Wooders, and Kamat 2005). *Multiple membership club networks, as defined in Page and Wooders (2010), are examples of layered networks in which connections between layers is brought about by overlapping club memberships.* In a club network where each player is the member of one and only one club, the induced club membership structure partitions the set of active players, making each club layer isolated - having no connections, via overlapping memberships, to other layers in the network.

In the club network model we construct here, the feasible action sets available to the players who are active in a particular club layer are subsets of a compact metric space of actions specific to that club layer - and these club specific action spaces can differ across layers. In Page and Wooders (2010), the underlying metric space of player actions - whose subsets form the various player feasible sets - is the same across club layers. Here the heterogeneity of player action sets across club layers makes defining a metric to measure the distance between club networks a much more delicate task - but we do accomplish this, thereby providing us with a compact metric hyperspace of club networks in which to carry out our game theoretic analysis of the emergence of equilibrium layered club network dynamics.

We begin by defining connection, layers and networks.

2 Connections, Club Layers, and Partial Function Spaces

2.1 Connections

Assume the following:

(1) D is a finite set of players, consisting of n players, equipped with the discrete metric η_D, having typical element d.[4]
(2) C is a finite set of clubs, consisting of m clubs, equipped with discrete metric η_C having typical element c.

[4] Under the discrete metric the distance between two nodes d and d' in D is given by

$$\eta_D(d,d') := \begin{cases} 1 & \text{if } d \neq d' \\ 0 & \text{if } d = d'. \end{cases}$$

(3) A_c is the feasible set of arcs representing actions player d can take in club c, where A_c is a weak star compact, metrizable, convex subset of the separable norm dual, $(E_c^*, \|\cdot\|_c^*)$, of a separable Banach space, $(E_c, \|\cdot\|_c)$, equipped with a metric ρ_c^* compatible with the weak star topology inherited from E_c^*.

Two examples of weak star compact, metrizable, convex arc sets are (1) $A_c := \Delta(X_c)$ the set of all probability measures defined on X_c a compact metric space of pure actions that can be taken in club c, and (2) $A_c := \mathcal{L}_{Y_c}^{\infty}$ the set of all μ-equivalence classes of state-contingent contracts, $f : \Omega \longrightarrow Y_c$, defined on a probability space of states, (Ω, B_Ω, μ), with contract payoffs in a convex, closed bounded set $Y_c \subset R^l$. Here Ω is a Polish space of states, B_Ω is its Borel σ-field, and μ is a probability measure.

In a club network a *connection* is represented by an ordered pair, $(a, (d, c))$, consisting of an arc label a representing the player's action choice and another ordered pair (d, c) - a player-club pair - indicating the club c player d has chosen to join. Thus, in a club network a connection is given by

$$(a, (d, c)) \in A \times (D \times C). \tag{1}$$

We will often call (d, c) a *pre-connection*, and we will often call the connection, $(a, (d, c))$, a *c-connection*. Finally, we will call the connection, $(a, (d, c))$, a feasible connection provided the action, a, chosen by player d is feasible for the club c player d has chosen join. This will be the case if $a \in A_{dc}$, where the correspondence, $(d, c) \longrightarrow A_{dc}$, is the *feasible arc correspondence*. We will assume that the feasible arc correspondence, $A_{(\cdot,\cdot)}$ takes nonempty values for all pre-connections $(d, c) \in D \times C$ and that for each club c, the correspondence $d \longrightarrow A_{dc}$ takes nonempty, ρ_c^*-closed (and hence ρ_c^*-compact) values in A_c.

2.2 Club Layers (c-Layers)

Club networks are networks layered by clubs. To understand precisely what this means consider the set of c-connections given by

$$K_c := A_c \times (D \times \{c\}). \tag{2}$$

A c-layer, G_c, is a closed subset of c-connections, K_c. A club network is given by an m-tuple of c-layers,

$$G := (G_1, \ldots, G_c, \ldots, G_m) \in 2^{A_1 \times (D \times \{1\})} \times \cdots \times 2^{A_c \times (D \times \{c\})} \times \cdots \times 2^{A_m \times (D \times \{m\})}.$$

Here, $2^{A_c \times (D \times \{c\})}$ is the collection (or hyperspace) of all ρ_c^*-closed subsets of $A_c \times (D \times \{c\})$ - including the empty set (allowing for club c to have no members). Thus, in a club network, layer G_c is either empty or is a nonempty, ρ_c^*-closed subset of $K_c := A_c \times (D \times \{c\})$ such that $(a, (d, c)) \in G_c$ if and only if $a \in A_{dc} \subset A_c$.

2.3 Player Layers (d-Layers)

Letting $K_{dc} := A_{dc} \times (\{d\} \times \{c\})$, we can further decompose a club network's c-layer into the pieces of the c-layer belonging to specific players. In particular, we have for each c that the c-layer, $G_c \subset 2^{A_c \times (D \times \{c\})}$, is given by an n-tuple

$$G_c := (G_{1c}, \ldots, G_{dc}, \ldots, G_{nc}),$$

where $G_{dc} \subset A_{dc} \times (\{d\} \times \{c\})$ is the piece of the c-layer belonging to player d. Letting $|G_{dc}|$ denote the cardinality of the set G_{dc} (with the convention that $|G_{dc}| = 0$ if and only if $G_{dc} = \varnothing$), we will assume that $|G_{dc}| \leq 1$ with $|G_{dc}| = 1$ for some $c = 1, 2, \ldots, m$. Thus, if $|G_{dc}| = 1$ then player d has one and only one connection to club c - so that, player $d's$ piece of club layer c is given by $G_{dc} = \{(a, (d, c))\}$ for some $a \in A_{dc}$.

Formally, we have the following definition of a feasible club network, $G \in \mathbb{K}$, as an $n \times m$ array of feasible player-club connections.

Definition 1 *(Feasible Club Networks, c-Layers, d-Layers, and Connection Arrays) A feasible club network $G \in \mathbb{K}$ is an $n \times m$ array G of feasible player-club connections given by*

$$G := \begin{pmatrix} G_{11} & \cdots & G_{1c} & \cdots & G_{1m} \\ \vdots & & \vdots & & \vdots \\ G_{d1} & \cdots & G_{dc} & \cdots & G_{dm} \\ \vdots & & \vdots & & \vdots \\ G_{n1} & \cdots & G_{nc} & \cdots & G_{nm} \end{pmatrix}_{n \times m}$$

where for each player-club pair, (d, c), $G_{dc} \in \mathbb{K}_{dc} \subset 2^{A_{dc} \times (\{d\} \times \{c\}}$ is player $d's$ part of c-layer, G_c, in club network G, and where the hyperspace $\mathbb{K}_{dc}$ of all such (d, c)-connections is such that $G_{dc} \in \mathbb{K}_{dc}$ if and only if

$$G_{dc} = \begin{cases} \{(a_{dc}, (d, c))\} & \text{if } d \text{ is a member of club } c \quad \text{i.e., if } |G_{dc}| = 1 \\ \varnothing & \text{if } d \text{ is not a member of club } c. \quad \text{i.e., if } |G_{dc}| = 0. \end{cases}$$

We will denote by $\mathbb{K} = (\mathbb{K}_{dc})_{dc}$ the hyperspace of feasible club network arrays where $\mathbb{K} = (\mathbb{K}_{dc})_{dc}$ is an array of feasible sets given by

$$\left.\begin{aligned} &\mathbb{K} := (\mathbb{K}_{dc})_{dc} \\ &:= \begin{pmatrix} \mathbb{K}_{11} & \cdots & \mathbb{K}_{1c} & \cdots & \mathbb{K}_{1m} \\ \vdots & & \vdots & & \vdots \\ \mathbb{K}_{d1} & \cdots & \mathbb{K}_{dc} & \cdots & \mathbb{K}_{dm} \\ \vdots & & \vdots & & \vdots \\ \mathbb{K}_{n1} & \cdots & \mathbb{K}_{nc} & \cdots & \mathbb{K}_{nm} \end{pmatrix}_{n\times m} := \underbrace{\begin{pmatrix} \cdots & \mathbb{K}^1 & \cdots \\ & \vdots & \\ \cdots & \mathbb{K}^d & \cdots \\ & \vdots & \\ \cdots & \mathbb{K}^n & \cdots \end{pmatrix}}_{\text{rows represent feasible d-layers}} \\ &:= \underbrace{\begin{pmatrix} \vdots & \vdots & \vdots \\ \mathbb{K}_1 & \mathbb{K}_c & \mathbb{K}_m \\ \vdots & \vdots & \vdots \end{pmatrix}}_{\text{columns represent feasible c-layers}} . \end{aligned}\right\} \tag{3}$$

Note that in addition to the full array representation of $G \in \mathbb{K}$, there are two other representations: the row representation where each row represents a d-layer and the column representations where each column represents a c-layer. For $G \in \mathbb{K}$, we have

$$\left.\begin{aligned} G &:= \begin{pmatrix} G_{11} & \cdots & G_{1c} & \cdots & G_{1m} \\ \vdots & & \vdots & & \vdots \\ G_{d1} & \cdots & G_{dc} & \cdots & G_{dm} \\ \vdots & & \vdots & & \vdots \\ G_{n1} & \cdots & G_{nc} & \cdots & G_{nm} \end{pmatrix}_{n\times m} := \underbrace{\begin{pmatrix} \cdots & G^1 & \cdots \\ & \vdots & \\ \cdots & G^d & \cdots \\ & \vdots & \\ \cdots & G^n & \cdots \end{pmatrix}}_{\text{rows represent d-layers}} \\ &:= \underbrace{\begin{pmatrix} \vdots & \vdots & \vdots \\ G_1 & G_c & G_m \\ \vdots & \vdots & \vdots \end{pmatrix}}_{\text{columns represent c-layers}} . \end{aligned}\right\} \tag{4}$$

If we agree to the *notational convention* that $G_{dc} := a_{dc}$ so that now,

$$a_{dc} = \begin{cases} \{(a_{dc}, (d, c))\} & \text{if } d \text{ is a member of club } c \\ \varnothing & \text{if } d \text{ is not a member of club } c. \end{cases}$$

then club networks in $\mathbb{K} = (\mathbb{K}_{dc})_{dc}$ can be given a reduced form array representation

- as an array of arc labels (without loss of information) - as follows:

$$\left.\begin{array}{l} G := \begin{pmatrix} a_{11} & \cdots & a_{1c} & \cdots & a_{1m} \\ \vdots & & \vdots & & \vdots \\ a_{d1} & \cdots & a_{dc} & \cdots & a_{dm} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nc} & \cdots & a_{nm} \end{pmatrix}_{n\times m} = \underbrace{\begin{pmatrix} \cdots & a^1 & \cdots \\ & \vdots & \\ \cdots & a^d & \cdots \\ & \vdots & \\ \cdots & a^n & \cdots \end{pmatrix}}_{\text{reduced form rows represent d-layers}} \\ \qquad = \underbrace{\begin{pmatrix} \vdots & \vdots & \vdots \\ a_1 & a_c & a_m \\ \vdots & \vdots & \vdots \end{pmatrix}}_{\text{reduced form columns represent c-layers}} . \end{array}\right\} \tag{5}$$

The c-layer in a club network G immediately above, given in reduced form by $a_c = (a_{1c}, \ldots, a_{nc})$ implicitly defines a coalition, $S_{cG} \subset D$, consisting of the members of club c in network G. Thus, player d is a member of coalition S_{cG} if and only if $(a, (d, c)) \in G_c$ and $a \in A_{dc}$. The coalition structure induced by club network G is given by

$$S_G := (S_{1G}, \ldots, S_{mG}) \in 2^D \times \cdots \times 2^D,$$

where 2^D is the collection of all subsets of D, including the empty set. For each feasible club network, $G \in \mathbb{K}$, we have $S_{cG} \neq \varnothing$ for at least one $c = 1, 2, \ldots, m$.

In a club network, *connections between layers are made through overlapping club memberships*. Without this, each club layer is isolated. For example, in layered club network, $G := (G_c)_{c \in C} \in \mathbb{K}$, if $S_{cG} \cap S_{c'G} \neq \varnothing$, then each player $d \in S_{cG} \cap S_{c'G}$ is a member of club c as well as a member of club c'. In this way, layers G_c and $G_{c'}$ are connected in network $G := (G_c)_{c \in C}$. Note that if in club network, G, players are members of one and only one club then if the network has multiple nonempty layers, then there are no connections between these layers - each layer is isolated precisely because there are no overlapping club memberships. In this case, the coalition structure induced by club network G given by

$$S_G := (S_{1G}, \ldots, S_{mG}) \in 2^D \times \cdots \times 2^D, \tag{6}$$

is a partition of the active club members - and players are siloed by their club memberships.

2.4 Partial Function Spaces

2.4.1 The Domain of a c-Layer

One way to think about the set of all possible networks making up each layer in a club network is as a set of functions in which the domains of the functions differ across the functions in the set. A set of functions with domains that differ across the functions in the set is called a partial function space - unlike a regular function space, where the domain of each function in the set is the same across functions in the set.[5] In particular, it turns out to be very useful to view the layers in a club network as being *the graphs of functions from a partial function space*. The usefulness of taking this point of view becomes clear when we start decomposing networks into sections (in the Cartesian product sense). Consider a *nonempty* c-layer,

$$G_c \in \mathbb{K}_c \subset 2^{K_c} = 2^{A_c \times (D \times \{c\})},$$

in a feasible club network $G \in \mathbb{K}$. Because G_c is a subset of the Cartesian product, $A_c \times (D \times \{c\})$, we can think of G_c as being the graph of a correspondence,

$$(d, c) \longrightarrow G_c(dc),$$

where for $(d, c) \in \mathcal{D}(G_c)$, $G_c(dc)$ is a *nonempty* ρ_c^*-closed subset of A_{dc}, given by,

$$G_c(dc) := \{a \in A_{dc} : (a, (d, c)) \in G_c\}.$$

But more importantly, under the assumptions we have made here, for $(d, c) \in \mathcal{D}(G_c)$, $G_c(dc) = \{a_{dc}\}$, for some $a_{dc} \in A_{dc}$. The set $G_c(dc)$ is called the section of layer G_c at pre-connection $(d, c) \in D \times \{c\}$. If (d, c) is in the domain of the correspondence (i.e., if $(d, c) \in \mathcal{D}(G_c)$), then $G_c(dc)$ is a *nonempty* ρ_c^*-closed singleton subset of A_{dc}, otherwise, $G_c(dc)$ is empty - indicating that in any network in which $G_c \in 2^{K_c}$ is a layer, player d is not a member of club c. We will call the correspondence (set-valued function) induced by the layer, G_c, the *arc correspondence* - and in this case, a correspondence taking singleton values. Formally the domain, $\mathcal{D}(G_c)$, and the range, $\mathcal{R}(G_c)$, of the arc correspondence, $G_c(\cdot, \cdot)$, induced by the nonempty c-layer, $G_c \in 2^{K_c}$, are given by

$$\left.\begin{array}{c} \mathcal{D}(G_c) := \{(d, c) \in D \times C : G_c(dc) \neq \emptyset\} \subset D \times \{c\} \\ \text{and} \\ \mathcal{R}(G_c) := \{E \in P_{*f}(A_c) : G_c(dc) = E \text{ for some } (d, c) \in D \times \{c\}\,. \end{array}\right\} \tag{7}$$

Conversely, any set-valued mapping, $(d, c) \longrightarrow G_c(dc)$, defined on a nonempty subset of pre-connections, $g \subseteq D \times \{c\}$, taking values in $P_{*f}(A_c)$, the hyperspace of *nonempty* ρ_c^*-closed subsets of A_c, uniquely identifies a nonempty c-layer via the graph of the $G_c(\cdot, \cdot)$ given by

$$GrG_c := \{(a, (d, c)) \in K_c : a \in G_c(dc)\} \tag{8}$$

Denote by

$$\mathcal{U}(g, P_{*f}(A_c)) \tag{9}$$

the collection of all arc correspondences, $G_c(\cdot)$, with domain $g \subseteq D \times \{c\}$ (a pre-network) and range contained in $P_{*f}(A_c)$. Thus, we have

[5] Because a collection of matrices of the same dimension can be used to represent the networks from a space of networks having the same domain (i.e., the same set of pre–connections) this is not the case for a partial function space of networks.

(i) for nonempty club layer $G_c \in 2^{K_c}$, the induced arc correspondence, $(d,c) \longrightarrow G_c(dc)$, is contained in $\mathcal{U}(g, P_{*f}(A_c))$, where $g := \mathcal{D}(G_c)$, and
(ii) for arc correspondence, $(d,c) \longrightarrow G_c(dc)$, contained in $\mathcal{U}(g, P_{*f}(A_c))$, GrG_c is nonempty and contained in 2^{K_c}.

2.4.2 Domain Equivalences Classes

We say that another club network $G' \in \mathbb{K}$ is in the same *domain equivalence class* with G if the layers in G and G' have the same domains - i.e., $\mathcal{D}(G'_c) = \mathcal{D}(G_c)$ for all $c \in C$ and we write $\mathcal{D}(G') = \mathcal{D}(G)$ Formally, the equivalence class of club networks determined by feasible network $G \in \mathbb{K}$ is given by

$$\mathbb{K}_{\mathcal{D}(G)} := \{G' \in \mathbb{K} : \mathcal{D}(G') = \mathcal{D}(G)\}. \tag{10}$$

We note that $(d,c) \in \mathcal{D}(G)$ if and only if $(d,c) \in \mathcal{D}(G_c)$.

2.5 Club Memberships and Coalition Structure

Recall from the discussion above that each feasible club network $G \in \mathbb{K}$ induces a particular membership or coalition structure in the set of players. In particular, given club network G, the club membership of club c in network G is given by

$$S_{cG} := \{d \in D : (d,c) \in \mathcal{D}(G_c)\},$$

and the coalition structure induced by club network $G \in \mathbb{K}$ is given by a profile,

$$S_G := (S_{c_1 G}, \ldots, S_{c_m G}) \in 2^D \times \cdots \times 2^D.$$

Formally, we have the following definition.

Definition 2: (Coalition Structure Implied by a Club Network)
Given feasible club network $G := (G_1, \ldots, G_m) \in \mathbb{K}$, with induced arc correspondence, $(d,c) \longrightarrow G_c(dc) \in 2^{A_{dc}}$, the implied coalition structure is

$$S_G := (S_{1G}, \ldots, S_{mG}) \in 2^D \times \cdots \times 2^D.$$

where for $c = 1, 2, \ldots, m$, S_{cG} is nonempty (i.e., club c has members in network G) if and only if $(d,c) \in \mathcal{D}(G_c)$.

3 Measuring the Distance Between Club Networks

In order to analyze the co-evolution of strategic behavior, club network structure and equilibrium dynamics, we require a topology for the space of club networks that is simultaneously *coarse* enough to guarantee compactness of the set of networks and *fine* enough to discriminate between differences across networks that are due to differences in the ways nodes are connected (via differing arc types) *versus differences across networks that are due to the complete absence of connections.* We resolve this topological dilemma by equipping the space club networks, $\mathbb{K}$, with the Hausdorff metric h_K - making the space of feasible club network connection arrays, a compact metric space.

It is easy to show that if the Hausdorff distance between any pair club networks G and G' is less than $\varepsilon \in (0,1)$, then the networks can differ only in the ways a given set of player-club pairs are connected - and not in the set of player-club pairs that are connected. In particular, if for networks G and G' , $h_K(G, G') < \varepsilon < 1$, then

$$(a,(d,c)) \in G \text{ if and only if } (a',(d,c)) \in G'$$

for arcs a and a' with $\rho^*_A(a,a') < \varepsilon$. Thus, if two club networks are at h_K-distance $< \varepsilon < 1$, then both club networks G and G' have the same coalition structures, i.e.,

$$S_G := (S_{1G}, \ldots, S_{mG}) = (S_{1G'}, \ldots, S_{mG'}) := S_{G'}. \tag{11}$$

Club networks which are close together - as measured by the Hausdorff metric - have identical coalition structures. Such closeness will often occur and can only persist in network vio sets belonging to basins of attraction generated by the equilibrium dynamics governing the movements of club networks.

To begin, equip the set of c-connections, $K_c := A_c \times (D \times \{c\})$, with the sum metric,

$$\rho_{K_c} := \rho^*_c + \eta_D + \eta_C. \tag{12}$$

Thus, the distance between c-connections, $(a,(d,c))$ and $(a',(d',c))$, is

$$\rho_{K_c}((a,(d,c)),(a',(d',c))) := \rho^*_c(a,a') + \eta_D(d,d') + \eta_C(c,c) = \rho^*_c(a,a') + \eta_D(d,d').$$

We will equip each hyperspace of c-layers, $2^{K_c} := 2^{A_c \times (D \times \{c\})}$, with the Hausdorff metric induced by the metric, ρ_{K_c}, on the set of c-connections. In defining the Hausdorff metric h_{K_c} on 2^{K_c}, we must allow for empty c-layers. For nonempty c-layer $G_c \in 2^{K_c}$ and connection $(a,(d,c)) \in K_c$, we define the distance from $(a,(d,c))$ to nonempty c-layer G'_c to be

$$dist((a,(d,c)), G'_c) := \min_{(a',(d',c)) \in G'_c} \rho_{K_c}((a,(d,c)),(a',(d',c))); \tag{13}$$

and for c-layers $G_c \neq \varnothing$, $G'_c \neq \varnothing$, we define the excess of G_c over G'_c to be

$$e(G_c, G'_c) := \max_{(a,(b,c)) \in G_c} dist((a,(d,c)), G'_c). \tag{14}$$

The Hausdorff distance between nonempty c-layers, G_c and G'_c is given by

$$h_{K_c}(G_c, G'_c) = \max\left\{e(G_c, G'_c), e(G'_c, G_c)\right\} \tag{15}$$

while

$$\left.\begin{array}{c} h_{K_c}(G_c, \varnothing) := h_{K_c}(\varnothing, G'_c) = diam(K_c) \\ \text{and} \\ h_{K_c}(\varnothing, \varnothing) = 0. \end{array}\right\} \tag{16}$$

The diameter, $diam(K_c)$, of the set of c-connection K_c,is given by

$$\left. \begin{array}{c} diam(K_c) \\ := \max_{(a',(d',c)) \text{ and } (a'',(d'',c)) \text{ in } K_c} \rho_{K_c}((a',(d',c)),(a'',(d'',c))). \end{array} \right\} \tag{17}$$

Thus, the Hausdorff metric on the hyperspace of c-layers, 2^{K_c}, is given by

$$h_{K_c}(G_c, G'_c) = \begin{cases} \max\left\{e(G_c, G'_c), e(G'_c, G_c)\right\} & \text{if } G_c \neq \varnothing,\ G'_c \neq \varnothing. \\ diam(K_c) & \text{if } G_c \neq \varnothing, G'_c = \varnothing \text{ or } G_c = \varnothing, G'_c \neq \varnothing. \\ 0 & \text{if } G_c = \varnothing,\ G'_c = \varnothing. \end{cases} \tag{18}$$

Given that the basic building block of a club network array is the hyperspace $\mathbb{K}_{dc}$ of feasible c-connections belonging to player d, with an underlying set of connections given by $K_{dc} := A_{dc} \times (\{d\} \times \{c\})$, and given that each player can take, at most one action in each club, we see that the Hausdorff metric $h_{K_{dc}}$ on $\mathbb{K}_{dc}$ reduces to

$$h_{K_{dc}}(G_{dc}, G'_{dc}) = \begin{cases} \rho^*_c(a_{dc}, a'_{dc}) & \text{if } G_{dc} \neq \varnothing \text{ and } G'_{dc} \neq \varnothing. \\ diam(A_{dc}) & \text{if } G_{dc} \neq \varnothing, G'_{dc} = \varnothing \text{ or } G_{dc} = \varnothing, G'_{dc} \neq \varnothing. \\ 0 & \text{if } G_c = \varnothing,\ G'_c = \varnothing. \end{cases} \tag{19}$$

Recall that $G_{dc} \in \mathbb{K}_{dc}$ if and only if

$$G_{dc} = \begin{cases} \{(a_{dc}, (d, c))\} & \text{if } d \text{ is a member of club } c \quad \text{i.e., if } |G_{dc}| = 1 \\ \varnothing & \text{if } d \text{ is not a member of club } c. \quad \text{i.e., if } |G_{dc}| = 0. \end{cases} \tag{20}$$

The Hausdorff metric on the hyperspace of feasible club networks, $\mathbb{K} = (\mathbb{K}_{dc})_{dc}$, is given by

$$h_K(G, G') := \sum_{d=1}^{n} \sum_{c=1}^{m} h_{K_{dc}}(G_{dc}, G'_{dc}). \tag{21}$$

For $G := (G_c)_{c \in C}$ and $G' := (G'_c)_{c \in C}$ in $\prod_{c \in C} 2^{K_c}$. Because (K_c, ρ_{K_c}) is a compact metric space, we have by Proposition C.2 in Bertsekas and Shreve (1976) that $(2^{K_c}, h_{K_c})$ is a compact metric space of c-layers, and because $\mathbb{K}_{dc}$ is an $h_{K_{dc}}$-closed subset of the $h_{K_{dc}}$-compact subset of $2^{K_{dc}}$, $\mathbb{K}_{dc}$ is $h_{K_{dc}}$-compact - implying that $(\mathbb{K}, h_K)$, given by

$$(\mathbb{K}, h_K) = ((\mathbb{K}_{dc})_{dc}, \sum_{d=1}^{n} \sum_{c=1}^{m} h_{K_{dc}}), \tag{22}$$

is a compact metric space.

4 An Example of a Club Network - Marketing Networks and Strategic Competition in Product Lines and Nonlinear Prices

Consider m firms engaged in strategic competition in n markets. Each firm, f_i, at various points in time, makes two decisions: (i) which markets, m_j, to enter, stay in, or exit, and (ii) what catalog of products and prices, C_{ij}, to offer in each new market the firm decides to enter, as well as what changes, if any, to make in the catalogs already being offered by the firm in the markets the firm has entered. Each firm is characterized by a technology and hence a cost function associated with bringing various catalog profiles to market and each market is characterized by a probability measure over the preferences and income types of the consumers who populate each market. Given the profile of market probability measures and the profile of cost functions, profiles known to all firms, each firm can compute its expected payoff generated by each profile of catalogs offered by firms. Referring to Figure 1, the connections array for the marketing network depicted in Figure 1 is given by

$$\begin{pmatrix} (C_{11},(f_1,m_1)) & (C_{12},(f_1,m_2)) \\ (C_{21},(f_2,m_1)) & \varnothing \\ (C_{31},(f_3,m_1)) & \varnothing \\ (C_{41},(f_4,m_1)) & (C_{42},(f_4,m_2)) \\ \varnothing & (C_{52},(f_5,m_2)) \end{pmatrix}_{5\times 2}.$$

The f_4-layer is given by the fourth row, $((C_{41},(f_4,m_1)),(C_{42},(f_4,m_2)))$, of the marketing connections matrix, while the m_2-layer is given by by the second column, $((C_{12},(f_1,m_2)),\varnothing,\varnothing,(C_{42},(f_4,m_2)),(C_{52},(f_5,m_2))$, of the marketing connections matrix.

Is there a profile of firm-specific catalog strategies that will emerge and persist under strategic catalog competition? This problem is a great example of a network formation game over club networks (with c-layers and d-layers) - and in particular, over a type of club network called a marketing network. In a marketing network, each node representing a particular firm can initiate, alter, or eliminate a catalog arc, labeled by the catalog C_{ij}, from the node representing the firm, f_i, to any node representing a market, m_j, and each firm can initiate several such market-specific catalog arcs, each running from the firm's node to a specific market. Figure 1 depicts just such a marketing network populated by 5 firms competing in 2 markets.

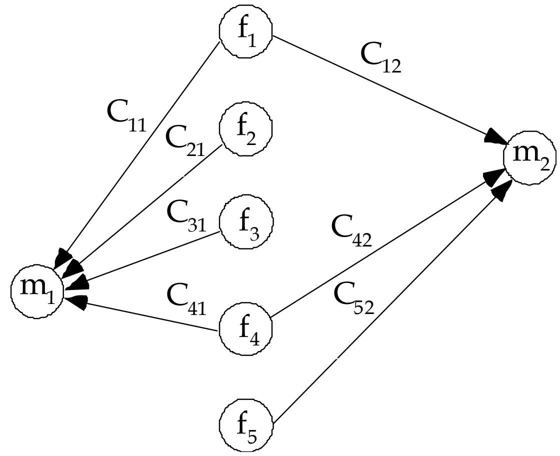

Figure 1: Marketing Network G

In marketing network G depicted in Figure 1, firms f_1 and f_4 are in both markets, m_1 and m_2. Firm f_1 offers catalog C_{11} in market m_1 and catalog C_{12} in market m_2, while firm f_4 offers catalog C_{41} in market m_1 and catalog C_{42} in market m_2. In a marketing network, firms are players in the game of network formation, while the market nodes (the club nodes) are passive, each characterized by a conditional probability measure over the consumer types who populate that market. The questions posed above concerning which marketing networks and which marketing strategies will emerge and persist can best be addressed by formulating the problem as a discounted stochastic game of network formation over marketing networks, and then by analyzing the stability properties of the resulting equilibrium Markov process of network formation generated by players' stationary Markov perfect equilibrium catalog strategies in forming marketing networks. In this dynamic game of marketing network formation, the m nodes representing firms are the players and the n nodes representing the markets are passive. Marketing networks are examples of club networks. The coalition structure implied by marketing network G in Figure 1 is given by $\{S_{m_1}, S_{m_2}\}$ where

$$S_{m_1} := \underbrace{\{f_1, f_2, f_3, f_4\}}_{\mathcal{D}(G_{m_1}) = \text{ domain of } m_1\text{-layer}} \quad \text{and } S_{m_2} := \underbrace{\{f_1, f_4, f_5\}}_{\mathcal{D}(G_{m_2}) = \text{ domain of } m_2\text{-layer}}.$$

Thus, the firms in S_{m_1} compete in market m_1 while the firms in S_{m_2} compete in market m_2. The c-layer decomposition of marketing network G, allows us to see the form this strategic competition takes by detailing for us the catalogs (product lines and prices) the competing firms in S_{m_1} and S_{m_2} offer in the markets in which they compete.

The m_1-Layer
$$\{(C_{11},(f_1,m_1)),(C_{21},(f_2,m_1),(C_{31},(f_3,m_1)),(C_{41},(f_4,m_1))\}.$$

The m_2-Layer
$$\{(C_{12},(f_1,m_2)),(C_{42},(f_4,m_2)),(C_{52},(f_5,m_2))\}.$$
c-Layer Decomposition of Marketing Network G

The d-decomposition of marketing network G is more granular and allows us to see what catalog strategies individual firms use to compete in each of the markets where they are present.

The f_1-Layer: $\{(C_{11},(f_1,m_1)),(C_{12},(f_1,m_2))\}$.
The f_2-Layer: $\{(C_{21},(f_2,m_1))\}$.
The f_3-Layer: $\{(C_{31},(f_3,m_1))\}$.
The f_4-Layer: $\{(C_{41},(f_4,m_1)),(C_{42},(f_4,m_2))\}$.
The f_5-Layer: $\{(C_{52},(f_5,m_2))\}$.
d-Decomposition of Marketing Network G

Is there a stable profile catalog strategies which firms can implement in their strategic competition in these two markets? This question will be answered if we can show that equilibrium marketing network dynamics which emerge from the strategic competition of these firms, modeled here as a discounted stochastic game of marketing network formation, generate a marketing network or set of marketing networks which emerge and persist through time.

Part II
Discounted Stochastic Games of Club Network Formation

In order to address the questions of whether or not and under what conditions the strategic formation of club networks will lead to the emergence of dynamically stable coalition structures, we must show that our discounted stochastic game (DSG) of club network formation possesses Nash equilibria in stationary Markov perfect behavioral club network formation strategies. It is the players' equilibrium behavioral network formation strategies which determine the equilibrium dynamics of club network formation. By identifying conditions under which stationary Markov perfect equilibria (SMPE) exist in such behavioral strategies and by showing that the resulting equilibrium club network dynamics are stable, we will be able to establish the conditions under which stable coalition structures will emerge and persist in the form of stable club networks. In this section we will construct a DSG model of club network formation and show that our model possess SMPE in behavioral club network formation strategies. The SMPE existence problem in the setting considered here (with uncountable states and compact metric action spaces) is quite difficult and its solution - and counterexamples - are of independent interest (for details see Levy 2013, Levy and McLennan 2015, Fu and Page 2022, and Page 2015, 2016).

An m-player, non-zero sum, discounted stochastic game, DSG, over the convex, weak star compact, metric product space of probability measures over player club networks (i.e., the convex, compact metric space of behavioral actions),

$$(\Delta(\mathbb{K}), \rho_{w^*_{ca}}) := (\prod_d \Delta(\mathbb{K}^d), \sum_d \rho_{w^{*d}_{ca}}), \tag{23}$$

is given by the following primitives:

$$DSG := \left\{ \underbrace{(\Omega, B_\Omega, \mu)}_{\text{state space}}, \underbrace{\left\{ \left(\Delta(\mathbb{K}^d), \Delta(\Phi_d(\omega)), \beta_d, U_d(\omega, v_d, \cdot) \right)_{d \in D} \right\}_{(\omega,v)}}_{\text{collection of one-shot games}}, \underbrace{q(\cdot|\omega, \cdot)}_{\text{law of motion}} \right\}, \tag{24}$$

where Ω is the state space, B_Ω is the Borel σ-field of events, and μ is a probability measure. For each player d, $\mathbb{K}^d$ is the set of all possible player club networks available to player d, while $\Delta(\Phi_d(\omega))$ is the convex, compact feasible set of behavioral action available to player d in state ω. A *feasible* behavioral action available to player d in state ω is a probability measure $\sigma_d \in \Delta(\mathbb{K}^d)$ with support contained in $\Phi_d(\omega)$ (i.e., $\sigma_d \in \Delta(\Phi_d(\omega))$ if and only if $\sigma_d(\Phi_d(\omega)) = 1$). Finally, $\beta_d \in (0,1)$ is player $d's$ discount rate and $U_d(\omega, v_d, \cdot)$ is player $d's$ payoff function in state ω given valuations (or prices) v_d, and $q(\cdot|\omega, \cdot)$ is the law of motion in state ω. If players holding value function profile $v = (v_1 \ldots, v_m)$ choose feasible profile of behavioral actions,

$$\sigma = (\sigma_1, \ldots, \sigma_m) \in \Delta(\Phi_1(\omega)) \times \cdots \times \Delta(\Phi_m(\omega)) = \Delta(\Phi(\omega)), \tag{25}$$

in state ω, then the next state ω' is chosen in accordance with probability measure $q(\cdot|\omega, G) \in \Delta(\Omega)$ and player $d's$ stage payoff is given by

$$\left. \begin{aligned} U_d(\omega, v_d, \sigma) &:= \textstyle\int_{\mathbb{K}} \left[(1-\beta_d) r_d(\omega, G) + \beta_d \int_\Omega v_d(\omega') q(\omega'|\omega, G) \right] \pi(\sigma(dG)) \\ &:= \textstyle\int_{\mathbb{K}} u_d(\omega, v_d, G) \pi\sigma(dG). \end{aligned} \right\} \tag{26}$$

Here $\pi\sigma(dG) := \pi(\sigma_1(dG^1), \ldots, \sigma_m(dG^m)) := \otimes_{d=1}^m \sigma_d(dG^d)$ is the *product probability measure* representing the random club network determined by the n-tuple of random player club networks, $(\sigma_1(dG^d), \ldots, \sigma_m(dG^m))$, chosen by the players.

We will denote by, $\mathcal{G}_{(\omega,v)} := (\Delta(\Phi_d(\omega)), U_d(\omega, v_d, \cdot))_{d \in D}$, the m-player (ω, v)-game in state ω underlying the DSG when players hold valuations (or state-contingent prices), $v := (v_1, \ldots, v_m)$.

5 Primitives and Assumptions

We will maintain the following assumptions throughout. Label these (i.e., assumptions (1)-(18) below) as [A-1]:

(1) $D =$ the set of players, consisting of m players indexed by $d = 1, 2, \ldots, m$ and each having discount rate given by $\beta_d \in (0, 1)$.
*(2) (Ω, B_Ω, μ), the state space where Ω is a complete separable metric spaces with metric ρ_Ω, equipped with the Borel σ-field, B_Ω, upon which is defined a **nonatomic** probability measure, μ.*[6]
(3) $Y := Y_1 \times \cdots \times Y_m$, the space of players' payoff profiles, $U := (U_1, \ldots, U_m)$, such that for each player d, $Y_d := [-M, M]$ and is equipped with the absolute value metric, $\rho_{Y_d}(U_d, U'_d) := |U_d - U'_d|$ and Y is equipped with the sum metric, $\rho_Y := \sum_d \rho_{Y_d}$.
(4) $\mathbb{K} := \mathbb{K}^1 \times \cdots \times \mathbb{K}^m$, the set of player pure action profiles, $G := (G^1, \ldots, G^m)$, where for each player d, G^d is player d's club network and where $\mathbb{K}^d$ is a compact metric space player club networks with typical element, G^d, equipped with metric, $h^*_{\mathbb{K}^d}$, and $\mathbb{K}$ is equipped with the sum metric, $h^*_{\mathbb{K}} := \sum_d h^*_{\mathbb{K}^d}$.
(5) $\Delta(\mathbb{K}^d)$ is the space of all probability measures, σ_d, with supports contained in player d's set of club networks, $\mathbb{K}^d$, equipped with the compact metrizable weak star topology (a topology denoted by w^{*d}_{ca}) inherited from $ca(\mathbb{K}^d)$, the Banach space of finite signed Borel measures on $\mathbb{K}^d$ with the total variation norm.[7] *We will equip $\Delta(\mathbb{K}^d)$ with a metric, $\rho_{w^{*d}_{ca}}$, compatible with the relative w^{*d}_{ca}-topology on $\Delta(\mathbb{K}^d)$ inherited from $ca(\mathbb{K}^d)$ and we will refer to σ_d as player d's random player club network.*
(6) $\Delta(\mathbb{K}) := \Delta(\mathbb{K}^1) \times \cdots \times \Delta(\mathbb{K}^m)$, the space of player behavioral action profiles, $\sigma := (\sigma_1, \ldots, \sigma_m)$, equipped with the sum metric, $\rho_{w^*_{ca}} := \sum_d \rho_{w^{*d}_{ca}}$, a metric compatible with the relative w^*_{ca}- product topology on $\Delta(\mathbb{K})$ inherited from $ca(\mathbb{K})$.
(7) $\omega \longrightarrow \Phi_d(\omega)$, is player d's measurable action constraint correspondence, defined on Ω taking nonempty $h^*_{\mathbb{K}^d}$-closed (and hence $h^*_{\mathbb{K}^d}$-compact) network values in $\mathbb{K}^d$.
(8) $\omega \longrightarrow \Phi(\omega) := \Phi_1(\omega) \times \cdots \times \Phi_m(\omega)$, players' measurable action profile constraint correspondence, defined on Ω taking nonempty $h^*_{\mathbb{K}}$-closed (and hence compact) network values in $\mathbb{K}$.
(9) $\omega \longrightarrow \Delta(\Phi_d(\omega))$, is player d's measurable behavioral action constraint correspondence, defined on Ω taking nonempty w^{*d}_{ca}-closed (and hence w^{*d}_{ca}-compact), convex random network values in $\Delta(\mathbb{K}^d)$, containing all probability measures, $\sigma_d(\omega)$, with supports contained in player d's feasible set of actions, $\Phi_d(\omega) \subset \mathbb{G}_d$, in state ω.
(10) $\omega \longrightarrow \Delta(\Phi(\omega)) := \Delta(\Phi_1(\omega)) \times \cdots \times \Delta(\Phi_m(\omega))$, players' measurable behavioral

[6] Note that the σ-field, B_Ω is countably generated. All the results we present here remain valid if instead we assume that Ω is an abstract set, but one equipped with a countably generated σ-field. We say that $E \subset \Omega$ is an atom of Ω relative to $\mu(\cdot)$ if the following implication holds: if $\mu(E) > 0$, then $H \subset E$ implies that $\mu(H) = 0$ or $\mu(E - H) = 0$. If Ω contains no atoms relative to $\mu(\cdot)$, Ω is said to be atomless or *nonatomic*. Because Ω, is a complete, separable metric space $\mu(\cdot)$ is atomless (or nonatomic) if and only if $\mu(\{\omega\}) = 0$ for all $\omega \in \Omega$ (see Hildenbrand, 1974, pp 44-45).

[7] Recall that the support of (a regular Borel) probability measure, $\sigma_d \in \Delta(\mathbb{K}^d)$, is the unique closed subset, $supp\sigma_d$, of $\mathbb{K}^d$ such that $\sigma_d(\mathrm{K}^d \backslash supp\sigma_d) = 0$, with the property that for any open set $\mathbb{E}^d \subset \mathbb{K}^d$ such that

$$\mathbb{E}^d \cap supp\sigma_d \neq \varnothing$$

$\sigma_d(\mathrm{E}^d \cap supp\sigma_d) > 0$. Also, note that $ca(\mathbb{K}^d)$ is a locally convex Hausdorff topological vector space with $\Delta(\mathbb{K}^d)$ a convex, $\rho_{w^{*d}_{ca}}$-compact subset of $ca(\mathbb{K}^d)$.

action profile constraint correspondence, defined on Ω *taking nonempty* w^*_{ca}*-closed (and hence* w^*_{ca}*-compact), convex values in* $\Delta(\mathbb{K})$*.*

(11) $\mathcal{L}^{\infty}_{Y_d}$*, the Banach space of all* μ*-equivalence classes of measurable (value) functions,* $v_d(\cdot)$*, defined on* Ω *with values in* Y_d *a.e.* $[\mu]$*, equipped with metric* $\rho_{w^*_d}$ *compatible with the weak star topology inherited from* $\mathcal{L}^{\infty}_R$*.*[8]

(12) $\mathcal{L}^{\infty}_Y := \mathcal{L}^{\infty}_{Y_1} \times \cdots \times \mathcal{L}^{\infty}_{Y_m} \subset \mathcal{L}^{\infty}_{R^m}$*, the Banach space of all* μ*-equivalence classes of measurable (value) function profiles,* $v(\cdot) := (v_1(\cdot), \ldots, v_m(\cdot))$*, defined on* Ω *with values in* Y *a.e.* $[\mu]$*, equipped with the sum metric* $\rho_{w^*} := \sum_d \rho_{w^*_d}$ *compatible with the weak star product topology inherited from* $\mathcal{L}^{\infty}_{R^m}$*.*

(13) $\mathcal{S}^{\infty}(\Delta(\Phi_d(\cdot)))$*, the set of all* μ*-equivalence classes of measurable functions (selections),* $\sigma_d(\cdot) \in \mathcal{L}^{\infty}_{\Delta(\mathbb{K}^d)}$*, defined on* Ω *such that in* $\sigma_d(\omega) \in \Delta(\Phi_d(\omega))$ *a.e.* $[\mu]$*, and*

$$\mathcal{S}^{\infty}(\Delta(\Phi(\cdot))) = \mathcal{S}^{\infty}(\Delta(\Phi_1(\cdot))) \times \cdots \times \mathcal{S}^{\infty}(\Delta(\Phi_m(\cdot))) \quad (27)$$

the set of all μ*-equivalence classes of measurable profiles (selection profiles),* $\sigma(\cdot) = (\sigma_1(\cdot), \ldots, \sigma_m(\cdot)) \in \mathcal{L}^{\infty}_{\Delta(\mathbb{K})}$*, defined on* Ω *such that in* $\sigma(\omega) \in \Delta(\Phi(\omega)) := \Delta(\Phi_1(\omega)) \times \cdots \times \Delta(\Phi_m(\omega))$ *a.e.* $[\mu]$*.*

(14) $r_d(\cdot, \cdot) : \Omega \times \mathbb{K} \longrightarrow Y_d$ *is player* d'*s Caratheodory stage payoff function (i.e., for all* $(\omega, G) \in \Omega \times \mathbb{K}$*,* $r_d(\omega, \cdot)$ *is* $h^*_{\mathbb{K}}$*-continuous on* $\mathbb{K}$ *and* $r_d(\cdot, G)$ *is* (B_Ω, B_{Y_d})*-measurable on* Ω*).*

(15) $q(\cdot|\cdot, \cdot) : \Omega \times \mathbb{K} \longrightarrow \Delta(\Omega)$ *is the law of motion defined on* $\Omega \times \mathbb{K}$ *taking values in the space of probability measures on* Ω*, having the following properties: (i) each probability measure,* $q(\cdot|\omega, G)$*, in the collection*

$$Q(\Omega \times \mathbb{K}) := \{q(\cdot|\omega, G) : (\omega, G) \in \Omega \times \mathbb{K}\}$$

is absolutely continuous with respect to μ *(denoted* $Q(\Omega \times \mathbb{K}) << \mu$*), (ii) for each* $E \in B_\Omega$*,* $q(E|\cdot, \cdot)$ *is measurable on* $\Omega \times \mathbb{K}$*, (iii) the collection of probability density functions,*

$$H_\mu := \{h(\cdot|\omega, G) : (\omega, G) \in \Omega \times \mathbb{K}\}, \quad (28)$$

of $q(\cdot|\omega, a)$ *with respect to* μ *is such that for each state* ω*, the function*

$$G \longrightarrow h(\omega'|\omega, G) \quad (29)$$

is $h^*_{\mathbb{K}}$*-continuous in* G *a.e.* $[\mu]$ *in* ω'*.*

(16) $\mathcal{L}^{\infty}_{ca(\mathbb{K}^d)}$ *is the Banach space of* μ*-equivalence classes of* $ca(\mathbb{K}^d)$*-valued, Bochner integrable functions equipped with the weak star topology, denoted* W^{*d}_{ca} *topology.*

(17) $\mathcal{L}^{\infty}_{\Delta(\mathbb{K}^d)}$ *is the nonempty, convex, weak star compact and metrizable subset of* μ*-equivalence classes of* $\Delta(\mathbb{K}^d)$*-valued, Bochner integrable functions* $\sigma_d(\cdot)$ *with*

[8] $\mathcal{L}^{\infty}_R$ is the Banach space of μ-equivalence classes of μ-essentially bounded functions, $v : \Omega \longrightarrow R$ with norm

$$\|v\|_\infty := esssup[v] := \inf\{y \in R : \mu\{\omega : |v(\omega)| > y\} = 0\}.$$

The space of μ-equivalence classes of functions $\mathcal{L}^{\infty}_R$ is the separable norm dual of the space of μ-equivalence classes of μ-integrable functions, $\mathcal{L}^1_R$. Because the Borel σ-field B_Ω is countably generated, $\mathcal{L}^1_R$, is separable. As a consequence, the subset of value function μ-equivalence classes, $\mathcal{L}^{\infty}_{Y_d}$, is a compact, convex, and metrizable subset of $\mathcal{L}^{\infty}_R$ for the weak star topology.

$\sigma_d(\omega) \in \Delta(\mathbb{K}^d)$ a.e. $[\mu]$, We will equip $\mathcal{L}^{\infty}_{\Delta(\mathbb{K}^d)}$ with a metric, $\rho_{W^{*d}_{ca}}$, compatible with the W^{*d}_{ca} topology on $\mathcal{L}^{\infty}_{\Delta(\mathbb{K}^d)}$ inherited from $\mathcal{L}^{\infty}_{ca(\mathbb{K}^d)}$.[9]

(18) $\mathcal{L}^{\infty}_{\Delta(\mathbb{K})} := \mathcal{L}^{\infty}_{\Delta(\mathbb{K}^1)} \times \cdots \times \mathcal{L}^{\infty}_{\Delta(\mathbb{K}^m)}$ is the set of all μ-equivalence classes of strategy profiles, equipped with the sum metric,

$$\rho_{W^*_{ca}} := \rho_{W^{*1}_{ca}} \times \cdots \times \rho_{W^{*m}_{ca}}.$$

6 Comments on Primitives and Assumptions

6.1 Weak Star Convergence of Random Player Club Networks

Under the metric, $\rho_{w^{*d}_{ca}}$, compatible with the relative w^{*d}_{ca}-topology on $\Delta(\mathbb{K}^d)$, we have

$$\sigma^n_d \underset{\rho_{w^{*d}_{ca}}}{\longrightarrow} \sigma^*_d \text{ if and only if } \int_{\mathbb{K}^d} c(G^d) d\sigma^n_d(G^d) \longrightarrow \int_{\mathbb{K}^d} c(G^d) d\sigma^*_d(G^d) \text{ for all } c(\cdot) \in \mathcal{C}(\mathbb{K}^d), \tag{30}$$

where $\mathcal{C}(\mathbb{K}^d)$ is the Banach space of continuous functions defined on the compact metric space, $\mathbb{K}^d$, with the sup norm. In fact, there exists a countable subcollection of continuous functions, $\mathcal{C}^0(\mathbb{G}_d)$, such that

$$\sigma^n_d \underset{\rho_{w^{*d}_{ca}}}{\longrightarrow} \sigma^*_d \text{ if and only if } \int_{\mathbb{K}^d} c_i(G_d) d\sigma^n_d(G_d) \longrightarrow \int_{\mathbb{K}^d} c_i(G^d) d\sigma^*_d(G^d) \tag{31}$$

for all $c_i(\cdot) \in \mathcal{C}^0(\mathbb{K}^d)$ (e.g., see Aliprantis and Border, 2006, Chapter 15).

By Theorem 3.2 in Billingsley (1968), we know that

$$\pi\sigma^n = \sigma^n_1 \otimes \cdots \otimes \sigma^n_m \underset{w^*_{ca}}{\longrightarrow} \sigma^*_1 \otimes \cdots \otimes \sigma^*_m = \pi\sigma^* \text{ if and only if for each player } \sigma^n_d \underset{w^{*d}_{ca}}{\longrightarrow} \sigma^*_d. \tag{32}$$

Thus a sequence of behavioral action profiles w^*_{ca}-converges to a particular behavioral action profile if and only if each player's sequence of behavioral actions w^{*d}_{ca}-converges to a particular behavioral action such that the w^*_{ca}-limit of behavioral action profiles is equal to the product of the w^{*d}_{ca}-limits of players' behavioral action sequences. Unfortunately, the mapping $(\sigma_d(\cdot), \sigma_{-d}(\cdot)) \longrightarrow \pi((\sigma_d(\cdot), \sigma_{-d}(\cdot)))$ is not jointly $\rho_{W^{*d}_{ca}}$-continuous. A good example of the failure of joint $\rho_{W^{*d}_{ca}}$-continuity can be found in Elliott, Kalton, and Markus, 1973, Example 3.16.

6.2 Weak Star and K-Convergence of Value Functions

A sequence, $\{v^n\}_n \subset \mathcal{L}^{\infty}_Y$, converges weak star to $v^* = (v^*_1(\cdot), \ldots, v^*_m(\cdot)) \in \mathcal{L}^{\infty}_Y$, denoted by $v^n \underset{\rho_{w^*}}{\longrightarrow} v^*$, if and only if

$$\int_{\Omega} \langle v^n(\omega), l(\omega) \rangle_{R^m} \, d\mu(\omega) \longrightarrow \int_{\Omega} \langle v^*(\omega), l(\omega) \rangle_{R^m} \, d\mu(\omega)$$

[9] *We will denote by $L^{\infty}_{\Delta(\mathbb{K}^d)}$ the prequotient of $\mathcal{L}^{\infty}_{\Delta(\mathbb{K}^d)}$ (i.e., the set of all $\Delta(\mathbb{K}^d)$-valued, Bochner integrable functions $\sigma_d(\cdot)$ with $\sigma_d(\omega) \in \Delta(\mathbb{K}^d)$ a.e. $[\mu]$).*

for all $l(\cdot) \in \mathcal{L}^1_{R^m}$.

A sequence, $\{v^n\}_n \subset \mathcal{L}^\infty_Y$, K-convergences (i.e., Komlos convergence - Komlos, 1967) to $\widehat{v} \in \mathcal{L}^\infty_Y$, denoted by $v^n \underset{K}{\longrightarrow} \widehat{v}$, if and only if every subsequence, $\{v^{n_k}(\cdot)\}_k$, of $\{v^n(\cdot)\}_n$ has an arithmetic mean sequence, $\{\widehat{v}^{n_k}(\cdot)\}_k$, where

$$\widehat{v}^{n_k}(\cdot) := \frac{1}{k}\sum_{q=1}^{k} v^{n_q}(\cdot),$$

such that

$$\widehat{v}^{n_k}(\omega) \underset{R^m}{\longrightarrow} \widehat{v}(\omega) \ a.e. \ [\mu].$$

The relationship between w^*-convergence and K-convergence is summarized via the following results from Balder (2000): For every sequence of value functions, $\{v^n\}_n \subset \mathcal{L}^\infty_Y$, and $\widehat{v} \in \mathcal{L}^\infty_Y$ the following are statements are true:

$$\left.\begin{array}{c} \text{(i) If the sequence } \{v^n\}_n \ K\text{-converges to } \widehat{v}, \text{ then } \{v^n\}_n \ w^*\text{-converges to } \widehat{v}. \\ \\ \text{(ii) The sequence } \{v^n\}_n \ w^*\text{-converges to } \widehat{v} \text{ if and only if} \\ \text{every subsequence } \{v^{n_k}\}_k \text{ of } \{v^n\}_n \text{ has a further subsequence, } \{v^{n_{k_r}}\}_r, \\ K\text{-converging to } \widehat{v}. \end{array}\right\} \tag{33}$$

For any sequence of value function profiles, $\{v^n\}_n$, in $\mathcal{L}^\infty_Y$ it is automatic that

$$sup_n \int_\Omega \|v^n(\omega)\|_{R^m} \, d\mu(\omega) < +\infty.$$

Thus, by the classical Komlos Theorem (1967), any such sequence, $\{v^n\}_n$, has a subsequence, $\{v^{n_k}\}_k$ that K-converges to some K-limit, $\widehat{v} \in \mathcal{L}^\infty_Y$.

6.3 Strong Stochastic Continuity of the Law of Motion

Under the stochastic continuity assumptions made above, [A-1](14), we have by Scheffee's Theorem (see Billingsley, 1986, Theorem 16.11) that for each $\omega \in \Omega$,

$$\left.\begin{array}{l} \sup_{E \in \mathcal{B}(\Omega)} |q(E|\omega, G^n) - q(E|\omega, G^*)|_R \\ \\ \leq \int_\Omega |h(\omega'|\omega, G^n) - h(\omega'|\omega, G^*)|_R \, d\mu(\omega') \longrightarrow 0, \end{array}\right\} \tag{34}$$

for any sequence of networks $\{G^n\}_n$ in $\Phi(\omega)$ converging to network $G^* \in \Phi(\omega)$ (i.e., for each $\omega \in \Omega$ the conditional density mapping, $G \longrightarrow h(\cdot|\omega, G)$, is continuous in L_1 norm with respect to G). Thus, by Scheffee's Theorem the L_1 norm continuity of $G \longrightarrow h(\cdot|\omega, G)$ with respect to networks G in each state ω is equivalent to the continuity of $G \longrightarrow q(E|\omega, G)$ in each state ω with respect to networks G uniformly in $E \in B_\Omega$ (i.e., for each $\omega \in \Omega$, $q(E|\omega, \cdot)$ is continuous in G, uniformly with respect to $E \in B_\Omega$).

6.4 Convergence and Continuity

Under assumptions [A-1], for each (ω, v) each player's payoff function,

$$G \longrightarrow u_d(\omega, v_d, G) := (1 - \beta_d) r_d(\omega, G) + \beta_d \int_\Omega v_d(\omega') q(\omega'|\omega, G), \tag{35}$$

is jointly continuous in $G = (G_1, \ldots, G_m)$, and for any sequence of value function-network pairs, $\{(v^n, G^n)\}_n$, if $v^n \underset{\rho_{w^*}}{\longrightarrow} v^*$ and $G^n \underset{h_K}{\longrightarrow} G^*$ then for each ω,

$$u(\omega, v^n, G^n) \underset{\rho_Y}{\longrightarrow} u(\omega, v^*, G^*),$$

(i.e., $u(\omega, \cdot, \cdot)$ is jointly continuous in (v, G)). Moreover, for each ω, the v-parameterized collection of m-tuples of integrands given by

$$\{u(\omega, v, \cdot) : v \in \mathcal{L}_Y^\infty\}, \tag{36}$$

where (see expression 35 above),

$$G \longrightarrow u(\omega, v, G) := (u_1(\omega, v_1, G), \ldots, u_m(\omega, v_m, G)) \tag{37}$$

is uniformly equicontinuous (see Solan, 1998, Lemma 3.6).[10] Thus, the Y-valued players' payoff function, $u(\cdot, \cdot, \cdot)$, is a Caratheodory function: $\rho_{w^* \times h_K}$-continuous in (v, G) for each ω, and (B_Ω, B_Y)-measurable in ω on Ω for each (v, G).

7 Stationary Markov Perfect Equilibria in Club Network Formation Strategies

Let DSG be a discounted stochastic game of club network formation satisfying assumptions [A-1] above, with one-shot game,

$$\mathcal{G}_{\Omega \times \mathcal{L}_Y^\infty} := \{\Big(\Delta(\mathbb{K}^d), \Delta(\Phi_d(\omega)), U_d(\omega, v_d, \cdot)\Big)_d\}_{(\omega, v) \in \Omega \times \mathcal{L}_Y^\infty}. \tag{38}$$

Definition 3 *(Nash Equilibria in Behavioral Strategies):*
A feasible profile of probability measures over player club networks, $\sigma^* := (\sigma_1^*, \ldots, \sigma_m^*) \in \Delta(\Phi(\omega))$ *is said to be a Nash equilibrium of the one-shot network formation game,*$(\Delta(\Phi_d(\omega)), U_d(\omega, v_d, \cdot))_d$*, provided that for each player d,*

$$U_d(\omega, v_d, \sigma_d^*, \sigma_{-d}^*) = \max_{\sigma_d \in \Delta(\Phi_d(\omega))} U_d(\omega, v_d, \sigma_d, \sigma_{-d}^*). \tag{39}$$

[10] The collection,

$$\{u(\omega, v, \cdot) : v \in \mathcal{L}_Y^\infty\},$$

is uniformly equicontinuous if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any G and G' in $\Phi(\omega)$ with $h_K(G, G') < \delta$,

$$\rho_Y(u(\omega, v, G), u(\omega, v, G')) < \varepsilon,$$

for all $v \in \mathcal{L}_Y^\infty$.

Denote by $\mathcal{N}(\omega, v)$ the set of all Nash equilibria belonging to $(\Delta(\Phi_d(\omega)), U_d(\omega, v_d, \cdot))_d$, and by $\mathcal{P}(\omega, v)$ the set of all Nash equilibria payoffs belonging to $(\Delta(\Phi_d(\omega)), U_d(\omega, v_d, \cdot))_d$. Thus

$$\left.\begin{aligned} \mathcal{P}(\omega, v) &:= \{U \in Y : U = U(\omega, v, \sigma) \text{ for some } \sigma \in \mathcal{N}(\omega, v)\} \\ &:= U(\omega, v, \mathcal{N}(\omega, v)). \end{aligned}\right\} \tag{40}$$

Under assumptions, [A-1], we know that $\mathcal{N}(\omega, v)$ is nonempty and $\rho_{w^*_{ca}}$-compact and therefore we know that $\mathcal{P}(\omega, v)$ is nonempty and ρ_Y-compact. Moreover, applying optimal measurable selection results (e.g., Himmelberg, Parthasarathy, and vanVleck, 1976) and Berge's Maximum Theorem (e.g., see 17.31 in Aliprantis and Border, 2006), we can show that the Nash correspondences, $\mathcal{N}(\cdot, \cdot)$ and $\mathcal{P}(\cdot, \cdot)$, are upper Caratheodory (also, see Proposition 4.2 in Page, 1992). In particular, the Nash correspondence, $\mathcal{N}(\cdot, \cdot)$, is jointly measurable in (ω, v) and $\mathcal{N}(\omega, \cdot)$, is upper semicontinuous in v for each ω, and the Nash payoff correspondence, $\mathcal{P}(\cdot, \cdot)$, is jointly measurable in (ω, v) and $\mathcal{P}(\omega, \cdot)$, is upper semicontinuous in v for each ω.

Let $\mathcal{UC}_{\Omega \times w^*\text{-}Y}$ denote the collection of all upper Caratheodory (uC) correspondences defined on $\Omega \times \mathcal{L}^\infty_Y$ taking nonempty closed values in Y. Also, let $\mathcal{U}_{w^*\text{-}Y}$ denote the collection of all upper semicontinuous correspondences defined on $\mathcal{L}^\infty_Y$ taking nonempty closed (and hence compact) values in Y. Following the literature, correspondences contained in $\mathcal{U}_{w^*\text{-}Y}$ are often called USCOs (see Hola and Holy, 2015).

Also, let $\mathcal{U}_{w^*-\Delta(\mathbb{K})}$ denote the collection of all USCOs taking nonempty, $\rho_{w^*_{ca}}$-closed (and hence $\rho_{w^*_{ca}}$-compact) values in $\Delta(\mathbb{K})$. Given any $\Psi \in \mathcal{U}_{w^*\text{-}\Delta(\mathbb{K})}$, denote by $\mathcal{U}_{w^*\text{-}\Delta(\mathbb{K})}[\Psi]$ the collection of all sub-USCOs belonging to Ψ, that is, all USCOs $\phi \in \mathcal{U}_{w^*-\Delta(\mathbb{K})}$ whose graph,

$$Gr\phi := \{(v, \sigma) \in \mathcal{L}^\infty_Y \times \Delta(\mathbb{K}) : \sigma \in \phi(v)\},$$

is contained in the graph of Ψ,

$$Gr\Psi := \{(v, \sigma) \in \mathcal{L}^\infty_Y \times \Delta(\mathbb{K}) : \sigma \in \Psi(v)\}.$$

We will call any sub-USCO, $\phi \in \mathcal{U}_{w^*\text{-}\Delta(\mathbb{K})}[\Psi]$ a minimal USCO belonging to Ψ, if for any other sub-USCO, $\psi \in \mathcal{U}_{w^*\text{-}\Delta(\mathbb{K})}[\Psi]$, $Gr\psi \subseteq Gr\phi$ implies that $Gr\psi = Gr\phi$. We will use the special notation, $[\Psi]$, to denote the collection of all minimal USCOs belonging to Ψ. We know that for any USCO Ψ, $[\Psi] \neq \varnothing$ (see Drewnowski and Labuda, 1990). In general, we say that ψ is a minimal USCO, if for any other USCO $\phi \in \mathcal{U}_{w^*\text{-}\Delta(\mathbb{K})}$, $Gr\phi \subseteq Gr\psi$ implies that $Gr\phi = Gr\psi$. Let $\mathcal{M}_{w^*\text{-}\Delta(\mathbb{K})}$ denote the collection of all minimal USCOs.

8 Connected-Valued Minimal uC Nash Correspondences and Equilibrium Value Functions

Let $\mathcal{UC}_{\Omega \times w^*\text{-}\Delta(\mathbb{K})}$ denote the collection of all upper Caratheodory mapping defined on $\Omega \times \mathcal{L}^\infty_Y$ taking nonempty $\rho_{w^*_{ca}}$-closed (and hence $\rho_{w^*_{ca}}$-compact) values in $\Delta(\mathbb{K})$.

For $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times w^*\text{-}\Delta(\mathbb{K})}$, let

$$\mathcal{UC}^{\mathcal{N}} := \mathcal{UC}_{\Omega\times w^*\text{-}\Delta(\mathbb{K})}[\mathcal{N}(\cdot,\cdot)] \tag{41}$$

denote the collection of all upper Caratheodory mappings, $\eta(\cdot,\cdot)$, belonging to $\mathcal{N}(\cdot,\cdot)$. Thus, $\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$ if and only if $\eta(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times w^*\text{-}\Delta(\mathbb{K})}$ and

$$Gr\eta(\omega,\cdot) \subset Gr\mathcal{N}(\omega,\cdot) \text{ for all } \omega.$$

We will be interested in sub-uC-mappings, $\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$, with the property that for each ω, $\eta(\omega,\cdot)$ is a *minimal USCO belonging to* $\mathcal{N}(\omega,\cdot)$. Already we know that for each ω, $[\mathcal{N}(\omega,\cdot)]$ is nonempty (e.g., see Drewnowski and Labuda, 1990). By Theorem 1 in Fu and Page (2022) we know that the uC mapping, $\mathcal{N}(\cdot,\cdot)$, contains a sub-uC-mapping, $\eta(\cdot,\cdot)$, such that for each ω, $\eta(\omega,\cdot)$ is a minimal USCO belonging to $\mathcal{N}(\omega,\cdot)$. We call any such sub-uC correspondence a minimal uC, and we denote by,

$$\mathcal{MUC}^{\mathcal{N}} := \left\{\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}} : \eta(\omega,\cdot) \in [\mathcal{N}(\omega,\cdot)] \text{ for all } \omega\right\}, \tag{42}$$

the collection of minimal uCs belonging to $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times w^*\text{-}\Delta(\mathbb{K})}$.

Let $C(\Delta(\mathbb{K}))$ denote the hyperspace of nonempty, $\rho_{w^*_{ca}}$-closed (and hence $\rho_{w^*_{ca}}$-compact) and *connected* subsets of $\Delta(A)$, and denote by $\mathcal{MUC}^{\mathcal{N}}_{C(\Delta(\mathbb{K}))}$ the collection of all minimal uC Nash correspondences belonging to $\mathcal{N}(\cdot,\cdot)$ talking continuum (i.e., nonempty, $\rho_{w^*_{ca}}$-compact and *connected*) values in $\Delta(\mathbb{K})$. For any minimal uC Nash correspondence, $\eta(\cdot,\cdot)$, taking continuum values in $\Delta(\mathbb{K})$ (i.e., for any $\eta(\cdot,\cdot)$ in $\mathcal{MUC}^{\mathcal{N}}_{C(\Delta(\mathbb{K}))}$), each player d has an induced Nash payoff subcorrespondence,

$$(\omega,v) \longrightarrow p_d(\omega,v) := U_d(\omega,v_d,\eta(\omega,v)), \tag{43}$$

taking closed bounded interval values in Y_d, and therefore, contractible values, in Y_d (the image of a connected set under a continuous function is connected - and connected sets in the interval $Y_d = [-M,M]$ are intervals and in this case closed intervals in Y_d). By Theorems 5.6 and 5.12 in Gorniewicz, Granas, and Kryszewski (1991), for each ω, the USCO part of each player's Nash payoff subcorrespondence, $v \longrightarrow p_d(\omega,v)$, is approximable, and therefore, by Theorem 4.2 in Kucia and Nowak (2000), for each $\varepsilon > 0$ each player's Nash payoff subcorrespondence, $p_d(\cdot,\cdot)$, has an ε-approximate Caratheodory Selection, $(\omega,v) \longrightarrow g^\varepsilon_d(\omega,v)$. Thus, for each $\varepsilon > 0$ and for each $(v,u_{(\cdot)d\cdot}) \in \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_{Y_d}$ with $u_{\omega d} = g^\varepsilon_d(\omega,v)$ for all ω there exists $(\overline{v}^\varepsilon,\overline{u}^\varepsilon_{(\cdot)d}) \in \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_{Y_d}$ with $\overline{u}^\varepsilon_{\omega d} \in p_d(\omega,\overline{v}^\varepsilon)$ for all ω such that for all ω

$$\rho_{w^*}(v,\overline{v}^\varepsilon) + \rho_{Y_d}(u_{\omega d},\overline{u}^\varepsilon_{\omega d}) < \varepsilon. \tag{44}$$

Thus, for the game's Nash payoff subcorrespondence there is a sequence of approximate Caratheodory Selections, $\{g^n(\cdot,\cdot)\}_n$ where for each n, the Caratheodory function, $(\omega,v) \longrightarrow g^n(\omega,v) := (g^n_1(\omega,v),\ldots,g^n_m(\omega,v))$, is such that for each $n = 1,2,3,\ldots$ and for each $(v,u_{(\cdot)}) \in \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_Y$ with $u_\omega = g^n(\omega,v)$ for all ω there exists $(\overline{v}^\varepsilon,\overline{u}^\varepsilon_{(\cdot)}) \in \mathcal{L}^\infty_Y \times \mathcal{L}^\infty_Y$ with $\overline{u}^\varepsilon_\omega \in p(\omega,\overline{v}^\varepsilon)$ for all ω such that for all ω

$$\rho_{w^*}(v,\overline{v}^\varepsilon) + \rho_Y(u_\omega,\overline{u}^\varepsilon_\omega) < \frac{1}{n}. \tag{45}$$

9 Existence of Stationary Markov Perfect Equilibria

The question of whether or not our discounted stochastic game of club network formation has a Nash equilibrium in stationary Markov perfect network formation strategies is a very difficult question - but one which we will answer here in the affirmative. We know from Blackwell's seminal 1965 paper on dynamic programming that our club network formation DSG will have stationary Markov perfect equilibria ($SMPE$) if and only if the Nash payoff selection correspondence, $v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}_v)$, belonging to the $DSG's$ underlying one-short game,

$$\mathcal{G}(\Omega \times \mathcal{L}^{\infty}_{Y}) := \left\{ \left(\Delta(\mathbb{K}^d), \Delta(\Phi_d(\omega)), \beta_d, U_d(\omega, v_d, \cdot) \right)_{d \in D} \right\}_{(\omega, v) \in \Omega \times \mathcal{L}^{\infty}_{Y}},$$

has fixed points. Because $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ is not convex valued nor closed valued nor upper semicontinuous in the appropriate topology (in this case the weak star topology in $\mathcal{L}^{\infty}_{Y}$) this fixed point question (or equivalently, the SMPE existence question) has been an open question since the 1976 paper by Himmelberg, Parthasarathy, Raghavan, and VanVleck on p-equilibria in uncountable-compact $DSGs$.[11] Our approach to resolving this $SMPE$ existence question is new. Fu and Page (2022) have show, under the usual assumptions specifying a DSG (see for example, Nowak and Raghavan, 1992) that because the upper semicontinuous (USCO) part, $\mathcal{N}(\omega, \cdot)$, of the uC Nash correspondence, $\mathcal{N}(\cdot, \cdot)$, is made up of strands of minimally essential Nash equilibria given by a minimal upper semicontinuous correspondence, $\eta(\omega, \cdot)$, taking continuum values, the $DSG's$ uC Nash payoff correspondence, $\mathcal{P}(\omega, \cdot)$, is made up of Nash payoff strands given by upper semicontinuous sub-correspondences, $p(\omega, \cdot)$, taking contractible values for each player. As a consequence, the $DSG's$ uC Nash payoff subcorrespondence, $p(\cdot, \cdot)$, is Caratheodory approximable (i.e., for any $\varepsilon > 0$ has an ε-approximate Caratheodory selection). We will show here that if our club network formation DSG has a Caratheodory approximable Nash payoff subcorrespondence (a fact established in Fu and Page, 2022), then our DSG has a Nash payoff selection correspondence with fixed points - further implying that our club network formation DSG has a Nash equilibrium in stationary Markov perfect network formation strategies. Thus here we will confirm that, under the usual assumptions specifying a discounted stochastic game (in this case a club network formation DSG), while the $DSG's$ Nash payoff selection may be badly behaved, nonetheless, it naturally possesses (without additional assumptions) subcorrespondences which are sufficiently well behaved (i.e., approximable) so as to have fixed points. He and Sun (2017), by making an *additional* assumption (that the DSG is $\mathcal{G}$-nonatomic or has *a coarser transition kernel*) guarantee that that $DSG's$ Nash payoff selection correspondence has a convex valued subcorrespondence - and therefore an approximable subcorrespondence.[12] Moreover, He and Sun (2017) show that Duggan (2012) accomplishes

[11] An uncountable-compact DSG is a game in which the state space is uncountable and the players' action choice sets are compact metric spaces. An uncountable-finite DSG is a game in which the state space is uncountable and the players' action choice sets are finite - a special case of an uncountable-compact DSG.

[12] In our club network formation model, we have assumed that the state space is a Polish space, Ω, equipped with the Borel σ-field, B_{Ω}, and a probability measure, μ, defined on B_{Ω}. Also, recall that

the same thing by assuming that the DSG has a noisy state. In the negative direction, Levy (2013) and Levy and McLennan (2015) construct counterexamples showing that not all uncountable-finite $DSGs$ have stationary Markov perfect equilibria. They accomplish this by constructing counterexamples in which the Nash correspondences are *not approximable* - which follows from the fact that in their counterexamples, there is an absence of fixed points. Because the club network formation DSG we analyze here is approximable, we avoid the Levy-McLennan counterexamples.

We state our main existence result here and refer the reader to Fu and Page (2022) for details and proofs.

Theorem 1 *(All nonatomic DSGs satisfying [A-1] have SMPE in behavioral strategies):*
Let

$$DSG := \left\{ (\Omega, B_\Omega, \mu), \left\{ \left(\Delta(\mathbb{K}^d), \Delta(\Phi_d(\omega)), \beta_d, U_d(\omega, v_d, \cdot) \right)_{d \in D} \right\}_{(\omega, v) \in \Omega \times \mathcal{L}_Y^\infty}, q(\cdot|\omega, \cdot) \right\}$$

be a discounted stochastic game of club network formation satisfying assumptions [A-1], with uC Nash correspondences, $\mathcal{N}(\cdot,\cdot)$ *and* $\mathcal{P}(\cdot,\cdot)$*.*
If μ *is nonatomic, then there exists a pair,* $(v^*, \sigma^*(\cdot)) \in \mathcal{L}_Y^\infty \times \mathcal{S}^\infty(\Delta(\Phi(\cdot)))$ *such that* $\sigma^*(\cdot) \in \mathcal{S}^\infty(\mathcal{N}_{v^*})$ *is a stationary Markov Perfect Equilibrium (SMPE) in* ***behavioral club network formation strategies*** *supported by Bellman prices* $v^* \in \mathcal{S}^\infty(\mathcal{P}_{v^*})$ *- i.e., there exists a pair,* $(v^*, \sigma^*(\cdot)) \in \mathcal{L}_Y^\infty \times \mathcal{S}^\infty(\Delta(\Phi(\cdot)))$ *such that a.e.* $[\mu]$

$$\sigma^*(\omega) \in \mathcal{N}(\omega, v^*) \text{ and } v^*(\omega) = U(\omega, v^*, \sigma^*(\omega)) \in \mathcal{P}(\omega, v^*).$$

Part III

Equilibrium Dynamics of Club Network Formation and Stable Coalitions

Under the profile of stationary Markov perfect equilibrium strategies,

$$\sigma^*(\cdot) := (\sigma_1^*(\cdot), \ldots, \sigma_m^*(\cdot)), \tag{46}$$

the equilibrium state and network formation process is given by,

$$\{(W_t^*, \sigma^*(W_t^*))\}_{t=0}^\infty, \tag{47}$$

when Ω is Polish, μ is nonatomic if and only if $\mu(\{\omega\}) = 0$ for all $\omega \in \Omega$ (see Hildenbrand, 1974). Suppose that $\mathcal{G}$ is a sub-σ-field of B_Ω. Denote by $\mu^{\mathcal{G}}(\cdot)$ a regular $\mathcal{G}$-conditional probability given sub-σ-field $\mathcal{G}$. Following Dynkin and Evstigneev (1976), $A \in B_\Omega$ is $\mathcal{G}$-atom if $\mu(A) > 0$ and for any $B \in B_\Omega$ such that $B \subset A$

$$\mu\left\{\omega \in \Omega : 0 < \mu^{\mathcal{G}}(B)(\omega) < \mu^{\mathcal{G}}(A)(\omega)\right\} = 0.$$

where the the discrete-time state process, $\{W_t^* : t = 0, 1, 2, \ldots\}$ consists of random objects, $\omega \longrightarrow W_t^*(\omega)$, having dynamics governed by the equilibrium Markov transition,

$$\left.\begin{aligned} p^*(E|\omega) &= q(E|\omega, \sigma^*(\omega)) \\ &= \int_{\mathbb{G}} q(E|\omega, G')d\sigma^*(G'|\omega). \end{aligned}\right\} \tag{48}$$

Thus, if the prevailing state at t is $W_t^* = \omega$, then, given equilibrium strategy profile $\sigma^*(\cdot) := (\sigma_1^*(\cdot), \ldots, \sigma_m^*(\cdot))$, the probability that the coming state, ω', is contained in $E \in B_\Omega$ is

$$p^*(E|\omega) = q(E|\omega, \sigma^*(\omega)) = \int_E q(d\omega'|\omega, \sigma^*(\omega)). \tag{49}$$

Moreover, if the current state-club network is $(\omega, G) = (\omega, G_1, \ldots, G_m)$, then the probability that the coming network lies in the feasible subset, $\mathbb{E} \in B_{\mathbb{K}}$, is

$$Q^*(\mathbb{E}|\omega, G) := \int_\Omega \sigma^*(\mathbb{E}|\omega')q(d\omega'|\omega, G) = \int_{\mathbb{E}}\int_\Omega \sigma^*(dG'|\omega')q(d\omega'|\omega, G). \tag{50}$$

Thus, $(\omega, G)) \longrightarrow Q^*(dG|\omega, G)$, is the state-contingent stationary $\rho_\Omega \times h_K$-continuous Markov club network transition

Under stationary Markov perfect strategy, $\sigma_d^*(\cdot)$, player d maximizes the discounted sum of $d's$ future payoffs. Moreover, as long as the other players continue to choose their player club networks in accordance with their behavioral strategies, $\sigma_{-d}^*(\cdot)$, player d has no incentive to defect to any other strategy - even a history dependent strategy (an implication of Blackwell's Theorem, 1965).

Whether or not a stable club network emerges depends on the stability properties of equilibrium state-strategy dynamics underlying club network formation. Our main objective is to say what we can regarding the emergence of stable club structures - and hence stable coalition structures - based on the stability properties of equilibrium network formation dynamics. *We would argue that this is one of the main aspects of network dynamics that has been missing from our understanding of network dynamics in various applied settings - the influence of strategic behavior on network dynamics.* Here we present a first attempt. We will proceed as follows: First we state the Tweedie (Stability) Conditions (Tweedie, 2001) and after discussing some of the elementary properties of Markov transitions, we will summarize some of the main implications of the Tweedie Conditions for the existence of basins of attraction and ergodic probabilities. We will then show that our Markov equilibrium (i.e., strategically informed) club network formation process satisfies the Tweedie Conditions. To begin we will consider an arbitrary (discrete time) Markov process satisfying the Tweedie conditions.

9.1 The Tweedie Conditions

We will assume that the Markov transition, $p(\cdot|\cdot)$, satisfies the *Tweedie conditions,* [T]:

(1) (Drift Condition) There exists (i) a nonnegative-valued measurable function, $V(\cdot):\Omega\longrightarrow[0,+\infty]$, with $V(\omega^0)<+\infty$ for some $\omega^0\in\Omega$; (ii) a subset $C\subset\Omega$, and (iii) a finite real number, $-\infty<b<+\infty$, such that for (i)-(iii) we have

$$\int_\Omega V(\omega')p(d\omega'|\omega)\leq V(\omega)-1+bI_C(\omega). \tag{51}$$

(2) (Uniform Countable Additivity) For any sequence of measurable sets, $\{B_n\}_n\subset B_\Omega$, with $B_n\downarrow\varnothing$,

$$\lim_{n\longrightarrow+\infty}\sup_{\omega\in C}p(B_n|\omega)=0. \tag{52}$$

10 Elementary Properties of Markov Transitions

Let $\{W_t\}$ be the Markov process governed by the transition kernel, $p(d\omega'|\omega)$, where for each $t=01,2,3,\ldots$, we have for each $\omega\in\Omega$, $W_t(\omega)=\omega_t$. Recall that

$$p(B|\omega):=prob\{W_{t+1}(\omega')\in B|W_t(\omega)=\omega\}. \tag{53}$$

10.1 Hitting and Return Times

The number of visits of the process, $W_t(\cdot)$, to the set of states B is given by $\eta_B:=\sum_{t=1}^{\infty}I_B(W_t(\omega))$. Let φ be a nontrivial, σ-finite measure on B_Ω.

We say that the process, $W_t(\cdot)$, is φ-irreducible if for all $B\in B_\Omega$ such that $\varphi(B)>0$,

$$U(\omega,B)=E_\omega(\eta_B)=\lim_{T\longrightarrow\infty}\sum_{t=1}^{T}p^t(B|\omega)=\sum_{t=1}^{\infty}p^t(B|\omega)>0. \tag{54}$$

If the process, $W_t(\cdot)$, is φ-irreducible, then we know that there exists a (maximal) dominating measure, ψ, such that $W_t(\cdot)$ is also ψ-irreducible and (see Proposition 4.2.2 in Meyn and Tweedie, 2009).

Let

$$\left.\begin{array}{c}\tau_B=\min\{t\geq 1:W_t(\omega)\in B\}=\text{first return time,}\\ \text{and}\\ \delta_B=\min\{t\geq 0:W_t(\omega)\in B\}=\text{first hitting time.}\end{array}\right\} \tag{55}$$

Return time probabilities and recurrent time probabilities are given by

$$\left.\begin{array}{c}L(\omega,B):=P_\omega(\tau_B<\infty)=P_\omega(W_t(\cdot)\text{ ever enters }B)\\ \text{and}\\ Q(\omega,B):=P_\omega(\eta_B=\infty)=P_\omega(W_t(\cdot)\text{ enters }B\text{ infinitely often})\end{array}\right\} \tag{56}$$

The set $B\in B_\Omega$ is *Harris recurrent* if $P_\omega(\eta_B=\infty)=1$ for all $\omega\in B$. The Markov process, $W_t(\cdot)$, is Harris recurrent if it is ψ-irreducible and every $B\in B_\Omega$ such that $\psi(B)>0$ is Harris recurrent. Thus a set B is Harris recurrent if when the Markov process $W_t(\cdot)$ starts at $\omega\in B$, it returns to B infinitely many times, except when the process starts at any state contained in a set of initial states having probability zero - thus, $P_\omega(\cdot)$-almost surely. In fact, for any Markov process $W_t(\cdot)$ that is Harris recurrent, $P_\omega(\eta_B = \infty) = 1$ for all $\omega \in \Omega$ and for all $B \in B_\Omega$ such that $\psi(B) > 0$.

Summarizing for all $B \in B_\Omega$ such that $\psi(B) > 0$

$$\left.\begin{array}{r} U(\omega, B) := E_\omega(\eta_B) > 0 \Longleftrightarrow W_t(\cdot) \text{ is } \psi\text{-irreducible,} \\ U(\omega, B) := E_\omega(\eta_B) = \infty \Longleftrightarrow W_t(\cdot) \text{ is recurrent,} \\ Q(\omega, B) := P_\omega(\eta_B = \infty) = 1 \Longleftrightarrow W_t(\cdot) \text{ is Harris recurrent.} \end{array}\right\} \tag{57}$$

10.2 Occupation Times

Given Markov process, $W_t(\cdot)$, with Markov transition, $p(\cdot|\cdot)$, the n-step occupation measure is given by

$$p^{(T)}(B|\omega) := \frac{1}{T}\sum_{t=0}^{T-1} p^t(B|\omega) \text{ for all } B \in B_\Omega,\ t = 1, 2, 3, \ldots \tag{58}$$

the pathwise occupation measure is given by

$$\pi^{(T)}(B) := \frac{1}{T}\sum_{t=0}^{T-1} I_B(W_t(\omega)). \tag{59}$$

Thus,

$$p^{(T)}(B|\omega) = E(\pi^{(T)}(B))|W_0(\omega) = \omega).$$

11 Main Implications of The Tweedie Conditions

Let $\{W_t\}$ be the Markov process governed by the transition kernel, $p(d\omega'|\omega)$ satisfying the Tweedie conditions [T]. We have the following results:

Theorem 2
Let $\{W_t\}$ be the Markov process governed by the transition kernel, $p(d\omega'|\omega)$ satisfying [T]. Then there exists a finite positive number of orthogonal invariant probability measures, $\pi_i(\cdot)$, $i = 1, 2, \ldots, I$ such that for each ω with $V(\omega) < +\infty$ and for every $B \in B_\Omega$,

$$\frac{1}{T}\sum_{t=1}^{T} p^t(B|\omega) \longrightarrow \sum_{i=1}^{I} \alpha_i(\omega)\pi_i(B)$$

for constants $\alpha_i(\omega) \geq 0$ such that $\sum_{i=1}^{I} \alpha_i(\omega) = 1$.

Theorem 3
Let $\{W_t\}$ be the Markov process governed by the transition kernel, $p(d\omega'|\omega)$ satisfying [T]. Then Ω contains at most a finite number of disjoint absorbing sets.

Recall that a set of states $B \in B_\Omega$ is a an invariant set or an absorbing set with respect to the transition kernel, $p(\cdot|\cdot)$, if $p(B|\omega) = 1$ for all $\omega \in B$.

Theorem 4
Let $\{W_t\}$ be the Markov process governed by the transition kernel, $p(d\omega'|\omega)$ satisfying [T]. Then there is a decomposition of the state space

$$\Omega = \left[\cup_{i=1}^{I} H_i\right] \cup E \tag{60}$$

into a finite nonzero number of maximal Harris sets, H_i, and a transient set, E, such that for each $\omega \in E$, $L(\omega, \cup_{i=1}^{I} H_i) = 1$.

Each Harris set H_i is a largest absorbing set. If we restrict the process to the maximal Harris set, H_i, giving us a sub-process, $W_t^{H_i}(\cdot) := W_t^i(\cdot)$, governed by the kernel, $p_{H_i}(\cdot|\cdot) := p_i(\cdot|\cdot)$, then the process is $\pi_i(\cdot)$-irreducible (i.e., for all $B \subset H_i$ with $B \in B_{H_i}$ and $\pi_i(B) > 0$, $L(\omega, B) > 0$ for all $\omega \in H_i$). By Theorem 2 above, it follows from Theorem 2.18 and Corollary 2.19 in Costa and Dufour (2005) that $W_t^i(\cdot)$, governed by the kernel, $p_i(\cdot|\cdot)$, is a $\pi_i(\cdot)$-irreducible, T-process (see p. 124 in Meyn and Tweedie, 2009, and Definitions 2.1-2.4 in Costa and Dufour, 2005). By Theorem 9.3.6, in Meyn and Tweedie, 2009, each Harris set H_i in expression (60) is positive Harris recurrent and can be further decomposed as,

$$H_i = R_i \cup E_i \tag{61}$$

where R_i is the set of topological Harris recurrent states, ω^{i*}, where $P_{\omega^{i*}}(\eta_{O_{\omega^{*i}}} = \infty) = 1$ (if and only if $L(\omega^{i*}, O_{\omega^{*i}}) = 1$ for all neighborhoods $O_{\omega^{*i}}$ of ω^{i*}) and where E_i is topologically transient (i.e., $E_{\omega^i}(\eta_{O_{\omega^i}}) < \infty$).[13] Essentially, as soon as the process, $W_t(\cdot)$, enters the maximal Harris set H_i it stays in H_i and becomes a $\pi_i(\cdot)$-irreducible, T-process, visiting the topological Harris recurrent states , $\omega^{i*} \in R_i$, infinitely often - passing through states in E_i on its way to states in R_i. But once the process enters H_i, the process stays in H_i. Thus, a refinement of the decomposition in Theorem 3 is given by

$$\Omega = \left[\cup_{i=1}^{I} (R_i \cup E_i)\right] \cup E. \tag{62}$$

The process will always leave E in finite time and enter into one of the basins, $H_i = R_i \cup E_i$, where it leaves E_i in finite time and travels for all future time in R_i visiting all states in R_i infinitely often.

Next, we strengthen our assumptions [A-1](15) by adding the assumption that

[A-1](15) The state space, Ω, is a compact metric space and for any sequence of state-club network pairs, $\{(\omega^n, G^n)\}_n$, converging to (ω^*, G^*) under the sum metric $\rho_{\Omega \times \mathbb{G}} := \rho_\Omega + h_K$ on the product of the state space Ω and the club network space $\mathbb{K}$,*

$$q(F|\omega^n, G^n) \longrightarrow q(F|\omega^*, G^*) \tag{63}$$

for all nonempty ρ_Ω-closed subsets F of Ω. (i.e., $F \in P_f(\Omega)$).

[13] Recall that $\eta_{O_{\omega^{*i}}} := \sum_{t=1}^{\infty} I_{O_{\omega^{*i}}}(W_t(\omega))$.

Let [A-1]* denote the list of assumptions [A-1] augmented by the assumption [A-1](15)* above. We have shown that under assumptions [A-1] our *DSG* of club network formation has stationary Markov perfect equilibria. Now we will show that if we strengthen assumptions [A-1] by adding [A-1](15)* then our equilibrium club network dynamics will satisfy the Tweedie Conditions. In order to accomplish this, it suffices to show that the equilibrium Markov transition kernel is globally uniform countably additive.

Theorem 5 *(Global Uniform Countable Additivity)*
Suppose assumptions [A-1] hold. Then the equilibrium Markov transition,* $p^*(\cdot|\cdot) := q(\cdot|\cdot, \sigma^*(\cdot))$ *is globally uniformly countably additive.*

Proof: Let

$$\Delta_{\Omega\times\mathbb{G}}(\Omega) := \{q(\cdot|\omega, G) : (\omega, G) \in \Omega \times \mathbb{G}\}.$$

We will show that $\Delta_{\Omega\times\mathbb{G}}(\Omega)$ is sequentially compact in the $\sigma(ca(\Omega), \mathcal{L}_R^\infty)$ topology.[14]

By the compactness of $\Omega \times \mathbb{G}$, for any sequence $\{q(\cdot|\omega^n, G^n)\}_n \subset \Delta_{\Omega\times\mathbb{G}}(\Omega)$, there is a subsequence, $\{q(\cdot|\omega^{n_k}, G^{n_k})\}_k$ such that $(\omega^{n_k}, G^{n_k}) \underset{\rho_{\Omega\times\mathbb{G}}}{\longrightarrow} (\omega^*, G^*)$ implying by assumption [A-1](15)* that for all nonempty, ρ_Ω-closed subsets E of Ω,

$$q(E|\omega^{n_k}, G^{n_k}) \longrightarrow q(E|\omega^*, G^*) \in \Delta_{\Omega\times\mathbb{G}}(\Omega).$$

Thus, by Delbaen's Lemma (1974) for each $f \in \mathcal{L}_R^\infty$, we have

$$\int_\Omega f(\omega')q(\omega'|\omega^{n_k}, G^{n_k}) \longrightarrow \int_\Omega f(\omega')q(\omega'|\omega^*, G^*).$$

Thus, $\Delta_{\Omega\times\mathbb{G}}(\Omega)$ is sequentially compact in the $\sigma(ca(\Omega), \mathcal{L}_R^\infty)$ topology. By Corollary 2.2 in Lassere (1998), $p^*(\cdot|\cdot)$ is globally uniformly countably additive. In particular, letting $\{B_k\}_k \subset B_\Omega$ be any decreasing sequence(i.e., $B_k \downarrow \emptyset$) and $\{f_k(\cdot)\}_k$ be the sequence of functions in $\mathcal{L}_R^\infty$ where for each k,

$$f_k(\omega) := I_{B_k}(\omega) \in \mathcal{L}_R^\infty,$$

we have by Corollary 2.2 in Lassere (1998) that the sequential compactness of $\Delta_{\Omega\times\mathbb{G}}(\Omega)$ implies that

$$\lim_{k\longrightarrow\infty} \sup_{(\omega,G)\in\Omega\times\mathbb{G}} \int_\Omega f_k(\omega')q(\omega'|\omega, G) = \lim_{k\longrightarrow\infty} \sup_{(\omega,G)\in\Omega\times\mathbb{G}} q(B_k|\omega, G) = 0.$$

Thus, because

$$\sup_{(\omega,G)\in\Omega\times\mathbb{G}} q(B_k|\omega, G) \geq \sup_{\omega\in\Omega} q(B_k|\omega, \sigma^*(\omega)) \geq 0,$$

we have

$$\lim_{k\longrightarrow\infty} \sup_{\omega\in\Omega} q(B_k|\omega, \sigma^*(\omega)) = \lim_{k\longrightarrow\infty} \sup_{\omega\in\Omega} p^*(B_k|\omega) = 0.$$

[14] Recall that $ca(\Omega)$ is the Banach space of finite signed Borel measures on (Ω, B_Ω) and $\mathcal{L}_R^\infty$ is the Banach space of μ-equivalence classes of real-valued, essentially bounded measurable functions on Ω.

Thus, the equilibrium Markov transition $p^*(\cdot|\cdot)$ governing the process of club network formation is globally uniformly countably additive. Letting $C = \Omega$, $V(\omega) = 1$ for all $\omega \in \Omega$, and $b = 2$, the drift condition is also satisfied. Thus, under assumptions [A-1]* the equilibrium Markov perfect transition, $p^*(\cdot|\cdot)$, satisfies the Tweedie conditions globally (i.e., with $C = \Omega$). **Q.E.D.**

12 Strategically Stable Coalition Structures

Under assumptions [A-1], our discounted stochastic game of club network formation will have a stationary Markov perfect equilibrium, $(v^*, \sigma^*(\cdot)) \in \mathcal{L}^\infty_Y \times \mathcal{S}^\infty(\Delta(\Phi(\cdot))$, where the profile of state-contingent prices, $v^* \in \mathcal{L}^\infty_Y$, incentivizes players, behaving farsightedly and optimally, to follow strategies, $\sigma^*(\cdot)) \in \mathcal{S}^\infty(\Delta(\Phi(\cdot))$, for all future time periods. Together, the price-strategy profile pair, $(v^*, \sigma^*(\cdot))$, and the law of motion, $q(\cdot|\cdot,\cdot)$, determine the equilibrium Markov state transition, $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma^*(\cdot))$, which generates the stationary equilibrium Markov state process, $\{W^*_t(\cdot)\}$.

If we strengthen the assumptions specifying our discounted stochastic game model of club network formation from [A-1] to [A-1]*, then the induced stationary Markov state process generated by the equilibrium Markov state transition, $p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma^*(\cdot))$ will satisfy the Tweedie Conditions. As a consequence, the equilibrium Markov state transition will generate a unique finite decomposition of the state space given by,

$$\Omega = \left[\cup_{i=1}^{I}(R_i \cup E_i)\right] \cup E, \tag{64}$$

with basins of attraction $\{R_1 \cup E_1, \ldots, R_I \cup E_I\}$ and transient set E, and there will emerge a finite set, $\mathcal{E}^* = \{\gamma_i(\cdot)\}_{i=1}^N$, of ergodic probability measures with $\gamma_i(R_i \cup E_i) = 1$ for all i. Once the equilibrium state process, $\{W^*_t(\cdot)\}$, enters the set of states $R_i \cup E_i$ it never leaves passing through states in E_i on its way to states in R_i, where it stays for all future periods, visiting the topological Harris recurrent states , $\omega^{i*} \in R_i$, infinitely often. Finally, the stationary Markov perfect equilibrium strategies, $\sigma^*(\cdot)$, of the players together with the law of motion determine a state-contingent stationary $\rho_\Omega \times h_K$-continuous Markov club network transition which in each state $\omega \in R_i$ is given by

$$Q^*(\mathbb{R}_i|\omega, G) := \int_\Omega \sigma^*(\mathbb{R}_i|\omega')q(d\omega'|\omega, G) = \int_{\mathbb{R}_i}\int_\Omega \sigma^*(dG'|\omega')q(d\omega'|\omega, G). \tag{65}$$

Under the equilibrium Markov dynamics determined by strategic behavior of the players in forming club networks in order for a set of state-club network pairs to be stable, not only must the state-club network pairs contained in the set be favored by the players involved and therefore chosen by their behavioral strategies, but they must also be favored by nature's law of motion (i.e., stated loosely, in order for a set of state-club network pairs to be stable, the state-club network pairs contained in the set must not only be chosen but they must be reachable - via the law of motion). Recall that given a *club network* $G \in \mathbb{K}$*, the implied coalition structure is given by* $\{S_{cG} : c \in C\}$ *where*

$$S_{cG} := S_{cG} := \{d \in D : (d, c) \in \mathcal{D}(G_c)\} \text{ and } \mathcal{D}(G_c) := \{(d, c) \in D \times \{c\} : G_c(dc) \neq \emptyset\}.$$

Thus, S_{cG} is the coalition of players who are members of club c in club network G.

Definition 4 (Strategically Stable Coalitions)
Let $\sigma^*(\cdot)$ be a stationary Markov perfect equilibrium strategy profile of the dynamic club network formation game with equilibrium Markov transition,
$p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma^*(\cdot))$ *and state space decomposition given by,*

$$\Omega = \left[\cup_{i=1}^{I}(R_i \cup E_i)\right] \cup E. \tag{66}$$

We say that a coalition structure, $\{S_{(c,G)} : (c, G) \in C \times \mathbb{K}\}$, is stable if for each club $c \in C$ the membership of club c is the same across the club networks in the vio set, $\mathbb{R}_i$, in each of finitely many basins, i.e., for each club $c \in C$ and for some fixed $S_c \subset D$,

$$S_{cG} = S_c \text{ for all } G \in \mathbb{R}_i \tag{67}$$

where for all $(\omega, G) \in R_i \times \mathbb{R}_i$,

$$p^*(R_i|\omega) = q(R_i|\omega, \sigma^*(\omega)) = 1 \tag{68}$$

and

$$Q^*(\mathbb{R}_i|\omega, G) := \int_{\Omega} \sigma^*(\mathbb{R}_i|\omega')q(d\omega'|\omega, G) = \int_{\mathbb{R}_i}\int_{\Omega} \sigma^*(dG'|\omega')q(d\omega'|\omega, G) = 1. \tag{69}$$

We saw in Section 3 above that if two club networks G and G' are at h_K-distance,

$$h_K(G, G') := \sum_{d=1}^{n}\sum_{c=1}^{m} h_{K_{dc}}(G_{dc}, G'_{dc}) < \varepsilon < 1, \tag{70}$$

then both club networks G and G' have the same coalition structures, i.e.,

$$S_G := (S_{c_1G}, \ldots, S_{c_mG}) = (S_{c_1G'}, \ldots, S_{c_mG'}) := S_{G'}. \tag{71}$$

We note that if two club networks, G and G', are sufficiently close, as measured by the h_K-distance, then they are in the same domain equivalence class. In particular, if $h_K(G, G') < \varepsilon < 1$ for all pairs of networks, G and G', in $\mathbb{R}_i$, then given the properties of the metric h_K all the networks in $\mathbb{R}_i$ have the same pre-network - implying that the set of clubs with members (active clubs) as well as club memberships are the same across networks in $\mathbb{R}_i$. Thus, the coalition structures, $\{S_{(c,G)} : (c, G) \in C \times \mathbb{R}_i\}$, underlying the club networks in $\mathbb{R}_i$ - networks chosen by the stationary Markov perfect equilibrium behavioral strategy profile, $\sigma^*(dG'|\omega')$, when the state process is in state vio set R_i - are the same across the networks in $\mathbb{R}_i$. If networks are close together, then their differences are due entirely to differences in the actions club members take in their respective clubs rather than differences in club memberships. Thus, if the equilibrium state dynamics generate vio sets such that the equilibrium behavioral network formation strategies generates club networks which are h_K-close together, then we can conclude that the equilibrium dynamics will lead to coalitional homogeneity within each network vio set $\mathbb{R}_i$ as represented by some pre-network, $g_i \in$

$P(D \times C)$ which is common to all the club networks in $\mathbb{R}_i$. This will be the case if the club networks in $\mathbb{R}_i$ are topologically Harris recurrent - meaning that for all h_K-open neighborhoods, $O_{G'}$, of $G' \in \mathbb{R}_i$ (for example, for all h_K-open balls about G' of any radius $\delta > 0$), it is true that players' club network formation strategies are such that the induced equilibrium process, $\omega \longrightarrow \sigma^*(W_t(\omega))$, visits any neighborhood $O_{G'}$ of $G' \in \mathbb{R}_i$ infinitely often. Stated formally, a condition sufficient to guarantee coalitional stability (or more to the point, coalitional homogeneity) of the club networks in $\mathbb{R}_i$ is that

$$Q^*(\eta_{O_{G'}} = \infty | \omega, G') = 1 \text{ for all } (\omega, G') \ \in R_i \times \mathbb{R}_i.$$

We summarize all of this in our last result.

Theorem 6 *(***Topological Harris Recurrence and Coalitional Homogeneity***)*
Let $\sigma^*(\cdot)$ be a stationary Markov perfect equilibrium strategy profile of the dynamic club network formation game with equilibrium Markov transition,
$p^*(\cdot|\cdot) = q(\cdot|\cdot, \sigma^*(\cdot))$ and state space decomposition given by,

$$\Omega = \left[\cup_{i=1}^{I}(R_i \cup E_i)\right] \cup E. \tag{72}$$

If each club network vio set, $\mathbb{R}_i$, $i = 1, 2, \ldots, I$, is topologically Harris recurrent - that is, if for any $(\omega, G') \in R_i \times \mathbb{R}_i$ and for any *neighborhood $O_{G'}$ of $G' \in \mathbb{R}_i$*

$$Q^*(\eta_{O_{G'}} = \infty | \omega, G') = 1, \tag{73}$$

then for each club network vio set, $\mathbb{R}_i$, $i = 1, 2, \ldots, I$, and any pair of club networks, G and G' in $\mathbb{R}_i$,

$$h_K(G, G') < \varepsilon < 1, \tag{74}$$

and therefore,

$$S_{(c,G)} = S_{(c,G')} \text{ for each club } c = 1, 2, \ldots, m. \tag{75}$$

If the club network vio set, $\mathbb{R}_i$, $i = 1, 2, \ldots, I$, is *topologically Harris recurrent*, then no matter what the starting state-network pair $(\omega, G') \in R_i \times \mathbb{R}_i$ is, the equilibrium network formation process will visit any neighborhood, $O_{G'}$, of $G' \in \mathbb{R}_i$ infinitely often. Using (65) and using argument similar to those used in the proof of Theorem 5 (especially, Corollary 2.2 in Lassere, 1998), it can be shown that the club network vio sets, $\mathbb{R}_i$, $i = 1, 2, \ldots, I$, are topologically Harris recurrent. Thus, if we strengthen the assumptions specifying our discounted stochastic game model of club network formation from [A-1] to [A-1]*, then the equilibrium club network formation process will, in the long run, generate stable coalition structures - which while they may differ across basins of attraction, will be homogeneous within each basin.

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