A Fixed Point Theorem for Measurable Selection Valued Correspondences Induced by Upper Caratheodory Correspondences

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Abstract

We show that any measurable selection valued correspondence induced by the composition of an m-tuple of real-valued Caratheodory functions with an upper Caratheodory (uC) correspondence has fixed points if the underlying uC correspondence in the composition contains a continuum valued uC sub-correspondence. As an application we show that all uncountable-compact discounted stochastic games ($\mathcal{DSG}s$) satisfying the usual assumptions have Nash payoff selection correspondences having fixed points provided of course that the uC Nash correspondence contains a continuum valued uC Nash sub-correspondence. Fu and Page (2022) have shown that all such $\mathcal{DSG}s$, in fact, have uC Nash correspondences containing continuum valued uC Nash sub-correspondences - implying, therefore, that all $\mathcal{DSG}s$ satisfying the usual assumptions have stationary Markov perfect equilibria.

Key Words: m-tuples of Caratheodory functions, upper Caratheodory correspondences, continuum valued upper Caratheodory sub-correspondences, weak star upper semicontinuous measurable selection valued correspondences, approximate Caratheodory selections, fixed points of nonconvex, measurable selection valued correspondences induced by the composition of an m-tuple of Caratheodory functions with a continuum valued upper Caratheodory sub-correspondence, discounted stochastic games, stationary Markov perfect equilibria

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1 Introduction

We show that any measurable selection valued correspondence induced by the composition of an m-tuple of real-valued Caratheodory functions with an upper Caratheodory (uC) correspondence (a uC composition correspondence) has fixed points if the underlying uC correspondence in the composition contains a continuum valued uC sub-correspondence.

Fixed point problems involving measurable selection valued correspondences induced by uC compositions arise often in economics and game theory. One of the most interesting examples, but by no means the only example, is provided by game theoretic models of the formation of trading networks. The basic idea is to model the problem of forming a short term trading network as a discounted stochastic game of network formation in which the actions available to each player are the pieces of the network controlled by the player. Players' stationary Markov perfect equilibrium network formation strategies then generate the network and induce the equilibrium Markov process of network formation. It is this equilibrium Markov process of network formation which is the key to understanding the true nature, causes and remedies for *endogenous systemic risks* - risks which were realized in the financial crisis of 2007-2008 - a crisis which almost brought down the world's banking and monetary system. In order for this approach to systemic risk to work, we must be able to show that players have stationary Markov perfect equilibria ($SMPE$) in network formation strategies (i.e., that $SMPE$ exist in such models of strategic network formation). Therein lies the importance of the fixed point result we prove in this paper.

The existence or nonexistence of stationary Markov equilibria for uncountable-compact discounted stochastic games has been an open question from the time of the 1976 paper by Himmelberg, Parthasarathy, Raghavan, and Van Vleck on p-equilibria in stationary strategies.[1] Thanks to the seminal work of Blackwell (1965) on dynamic programming, we know that a discounted stochastic game ($\mathcal{DSG}$) has stationary Markov perfect equilibria if and only if the parameterized, state-contingent collection of one-shot games underlying the discounted stochastic game has a *Nash payoff correspondence* (an example of a uC composition correspondence) that induces a selection correspondence having fixed points. But to date, no fixed point result exists which can be used to show that the Nash payoff selection correspondence has fixed points. This is not surprising because Nash payoff selection correspondences are, in general, neither convex valued nor closed valued in the appropriate topology (in this case the weak star topology). However, as we will show here (and as was already suggested in Page, 2015), the problem can be solved by approximate fixed point methods, provided the underlying upper Caratheodory (uC) *Nash payoff correspondence* contains a contractible-valued uC sub-correspondence (implying that the induced selection correspondence has fixed points). We show here that this will be the case if the underlying uC *Nash correspondence* (in the uC composition) contains a uC *sub-correspondence taking closed, connected values* in the set of Nash equilibria.[2] By resolving this fixed point problem, and therefore by resolving the $SMPE$ existence problem for the class of $\mathcal{DSG}s$ most relevant to the analysis of endogenous systemic risk in large complex systems (such as banking systems), we will open a new pathway to understanding and controlling systemic risk. We will consider in more detail below the application of our fixed point result to resolving the $SMPE$ existence problem in uncountable-compact discounted stochastic games.

[1] We will often refer to a finite-player, nonzero-sum discounted stochastic game in which players'strategy sets are compact metric spaces and the state space is uncountable as an uncountable-compact discounted stochastic game. Such games are common in financial trading models.

[2] Fu and Page (2022) have shown that for the collecion of parameterized, state-contingent one-shot games underlying any discounted stochastic game (satisfying the usual assumptions) the uC Nash correspondence belonging to this one-shot game, always contains a uC *sub-correspondence taking closed, connected values* in the set of Nash equilibria.

2 Primitives, Assumptions, and Preview

Let (Ω, B_Ω, μ) be a probability space where Ω is a complete, separable metric space with metric ρ_Ω, B_Ω the Borel σ-field generated by the ρ_Ω-open sets in Ω, and μ a regular Borel probability measure. Let $Y_d := [-M, M]$ for $d = 1, 2, \ldots, m$, and $M > 0$ and $Y := Y_1 \times \cdots \times Y_m = [-M, M]^m \subset R^m$. Also, let $X := X_1 \times \cdots \times X_m$ where for each $d = 1, 2, \ldots, m$, X_d is a convex, compact metrizable subset of a locally convex Hausdorff topological vector space E_d equipped with a metric ρ_{X_d} compatible with the locally convex topology inherited from E_d. Finally, equip Y with sum of absolute values metric, $\rho_Y(y, y') := \sum_d \rho_{Y_d}(y_d, y'_d) := \sum_d |y_d - y'_d|$ and equip X with the sum metric, $\rho_X := \sum_d \rho_{X_d}$, compatible the product topology inherited from $E = E_1 \times \cdots \times E_m$.

Next, let $\mathcal{L}^\infty_Y := \mathcal{L}^\infty_{Y_1} \times \cdots \times \mathcal{L}^\infty_{Y_m}$, where for each $d = 1, 2, \ldots, m$, $\mathcal{L}^\infty_{Y_d}$ is a convex, weak star compact metrizable subset of $\mathcal{L}^\infty_R$, the Banach space of μ-equivalence classes of μ-essentially bounded, measurable, real-valued functions, where $v \in \mathcal{L}^\infty_Y$ if and only if $v(\omega) := (v_1(\omega), \ldots, v_m(\omega)) \in Y$ a.e. $[\mu]$. Equip $\mathcal{L}^\infty_Y$ with the sum metric, $\rho_{w^*} := \sum_d \rho_{w^*_d}$, compatible the weak star product topology inherited from $\mathcal{L}^\infty_{R^m}$. Finally, let $P_f(X)$ be the hyperspace of nonempty ρ_X-closed subsets of X.

Consider an *upper Caratheodory* (uC) correspondence,

$$\mathcal{N}(\cdot,\cdot) : \Omega \times \mathcal{L}^\infty_Y \longrightarrow P_f(X), \tag{1}$$

jointly measurable in (ω, v) and upper semicontinuous in v for each ω. We call the collection of upper semicontinuous correspondences, $\{\mathcal{N}(\omega,\cdot) : \omega \in \Omega\}$ the USCO part (Hola and Holy, 2015), and $\{\mathcal{N}(\cdot, v) : v \in \mathcal{L}^\infty_Y\}$ the measurable part of the uC correspondence $\mathcal{N}$. Denote by $\mathcal{UC}_{\Omega\times\mathcal{L}^\infty_Y\text{-}P_f(X)}$ the collection of all such uC correspondences.

Next consider the Y-valued Caratheodory function,

$$(\omega, v, x) \longrightarrow u(\omega, v, x) := (u_1(\omega, v, x), \ldots, u_m(\omega, v, x)) \in Y, \tag{2}$$

measurable in ω and jointly continuous in (v, x), and let

$$\mathcal{P}(\cdot,\cdot) : \Omega \times \mathcal{L}^\infty_Y \longrightarrow P_f(Y), \tag{3}$$

denote the composition of uC correspondence $\mathcal{N}(\cdot,\cdot)$ with the m-tuple of Caratheodory functions, $(u_1(\cdot,\cdot,\cdot), \ldots, u_m(\cdot,\cdot,\cdot))$. For each $(\omega, v) \in \Omega \times \mathcal{L}^\infty_Y$ we have

$$\left.\begin{aligned} \mathcal{P}(\omega, v) &:= u(\omega, v, \mathcal{N}(\omega, v)) \\ &:= \{(u(\omega, v, x), \ldots, u(\omega, v, x)) \in Y : x \in \mathcal{N}(\omega, v)\} \end{aligned}\right\} \tag{4}$$

The correspondence, $\mathcal{P}(\cdot,\cdot)$, is also a uC correspondence. As noted in the introduction, we will call such a correspondence a uC composition correspondence.

Each uC composition correspondence, $\mathcal{P}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times\mathcal{L}^\infty_Y\text{-}P_f(Y)}$, induces a measurable selection valued correspondence,

$$\left.\begin{aligned} v \longrightarrow \mathcal{S}^\infty(\mathcal{P}(\cdot, v)) &:= \mathcal{S}^\infty(\mathcal{P}_v) = \mathcal{S}^\infty(u(\cdot, v, \mathcal{N}(\cdot, v))), \\ &:= \{u(\cdot) \in \mathcal{L}^\infty_Y : u(\omega) \in \mathcal{P}(\omega, v) \text{ a.e. } [\mu]\} \end{aligned}\right\} \tag{5}$$

where for each $v \in \mathcal{L}^\infty_Y$, $\mathcal{S}^\infty(\mathcal{P}_v)$ is the collection of μ-equivalence classes of a.e. measurable selections of $\mathcal{P}(\cdot, v)$, i.e., functions $u(\cdot)$ in $\mathcal{L}^\infty_Y$ such that $u(\omega) \in \mathcal{P}(\omega, v)$ a.e. $[\mu]$. We will show that for all such uC composition correspondences,

$$v \longrightarrow \mathcal{S}^\infty(\mathcal{P}(\cdot, v)) = \mathcal{S}^\infty(u(\cdot, v, \mathcal{N}(\cdot, v))), \tag{6}$$

if the underlying uC correspondence, $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times\mathcal{L}^{\infty}_{Y}\text{-}P_f(X)}$, contains a *continuum valued sub-correspondence*, $\eta(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times\mathcal{L}^{\infty}_{Y}\text{-}P_f(X)}$ (i.e., a uC correspondence $\eta(\cdot,\cdot)$ taking continuum values such that $Gr\eta(\omega,\cdot) \subset Gr\mathcal{N}(\omega,\cdot)$ for all ω) then its uC composition correspondence,

$$(\omega,v) \longrightarrow p(\omega,v) := u(\omega,v,\eta(\omega,v)) := \{(u_1(\omega,v_1,x),\ldots,u_m(\omega,v_m,x)) \in Y : x \in \eta(\omega,v)\},$$

induces a selection sub-correspondence,

$$\left.\begin{aligned} v \longrightarrow \mathcal{S}^{\infty}(p(\cdot,v)) &= \mathcal{S}^{\infty}(u(\cdot,v,\eta(\cdot,v))) \\ := \{u(\cdot) \in \mathcal{L}^{\infty}_{Y} &: u(\omega) \in p(\omega,v) \text{ a.e. } [\mu]\}, \end{aligned}\right\} \tag{7}$$

that is weak star upper semicontinuous and has fixed points. Thus while the original selection correspondence, $v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}_v)$, may fail to be weak star upper semicontinuous, its selection sub-correspondence,

$$v \longrightarrow \mathcal{S}^{\infty}(p(\cdot,v))$$

induced by a continuum valued uC sub-correspondence $\eta(\cdot,\cdot)$ will be weak star upper semicontinuous, and more importantly, will have fixed points.

We will refer to all the assumptions made above concerning spaces and correspondences as [A-1].

2.1 Comments

(1) Given the probability space, $(\Omega, B_{\Omega}, \mu)$, metric spaces, (Z,ρ_Z) compact and (X,ρ_X) separable, consider an arbitrary set-valued mapping or a correspondence, Γ, from $\Omega \times Z$ into X taking *nonempty* values in X, denoted

$$\Gamma : \Omega \times Z \longrightarrow P(X). \tag{8}$$

For any metric space (X,ρ_X), $P(X)$ will denote the collection of all nonempty subsets of X, and $P_f(X) := P_{\rho_X f}(X)$ will denote the collection of all nonempty and ρ_X-closed subsets of X (we will often leave off the subscript denoting the metric). Given ω and z, we have for any subset S of X the following definitions,

$$\left.\begin{array}{c} \Gamma^{-}_{\omega}(S) := \{z \in Z : \Gamma_{\omega}(z) \cap S \neq \varnothing\}, \\ \text{and} \\ \Gamma^{-}_{z}(S) := \{\omega \in \Omega : \Gamma_{z}(\omega) \cap S \neq \varnothing\}, \end{array}\right\} \tag{9}$$

where for fixed ω, $\Gamma_{\omega}(\cdot) := \Gamma(\omega,\cdot)$, and for fixed z, $\Gamma_z(\cdot) := \Gamma(\cdot,z)$. Finally, let

$$\Gamma^{-}(S) := \{(\omega,z) \in \Omega \times Z : \Gamma(\omega,z) \cap S \neq \varnothing\}. \tag{10}$$

Let B_Z and B_X be the Borel σ-fields in Z and X (respectively). We have the following definitions. Given correspondence, $\Gamma(\cdot,\cdot)$, we say that,

(a) $\Gamma_z(\cdot)$ is weakly measurable (or measurable) if for all S open in X, $\Gamma^{-}_{z}(S) \in B_{\Omega}$,

(b) $\Gamma_{\omega}(\cdot)$ is upper semicontinuous if for all S closed X, $\Gamma^{-}_{\omega}(S)$ is ρ_Z-closed,

(c) $\Gamma(\cdot,\cdot)$ is product measurable if for all S open in X, $\Gamma^{-}(S) \in B_{\Omega} \times B_Z$.

(d) $\Gamma(\cdot,\cdot)$ is upper Caratheodory if $\Gamma(\cdot,\cdot)$ is product measurable and for each ω, $\Gamma_{\omega}(\cdot)$ is upper semicontinuous.

For X a separable metric space, weak measurability of $\Gamma_z(\cdot)$ implies that for each z,

$$Gr\Gamma_z(\cdot) := \{(\omega,x) \in \Omega \times X : x \in \Gamma_z(\omega)\} \in B_{\Omega} \times B_X. \tag{11}$$

Finally, for X compact and $\Gamma(\cdot,\cdot)$ upper Caratheodory, we have by Lemma 3.1 in Kucia and Nowak (2000) that the mapping

$$\omega \longrightarrow Gr\Gamma_{\omega}(\cdot) \in P_f(Z \times X) \tag{12}$$

is measurable - i.e., for S an open subset of $Z \times X$, $(Gr\Gamma_{(\cdot)}(\cdot))^{-}(S) \in B_{\Omega}$, where

$$(Gr\Gamma_{(\cdot)}(\cdot))^{-}(S) := \{\omega \in \Omega : Gr\Gamma_{\omega}(\cdot) \cap S \neq \varnothing\}. \tag{13}$$

(2) Let (Z, ρ_Z) be any metric space. Consider the hyperspace of nonempty, ρ_Z-closed subsets of Z, $P_f(Z)$. The distance from a point $z \in Z$ to a set $C \in P_f(Z)$ is given by

$$dist(z, C) := \inf_{z' \in C} \rho_Z(z, z'). \tag{14}$$

Given two sets B and C in $P_f(Z)$, the excess of B over C is given by

$$e_{\rho_Z}(B, C) := \sup_{z \in B} dist_{\rho_Z}(z, C). \tag{15}$$

The given two sets B and C in $P_f(Z)$, the Hausdorff distance in $P_f(Z)$ between B and C is given by

$$h_{\rho_Z}(B, C) = \max\{e_{\rho_Z}(B, C), e_{\rho_Z}(C, B)\}. \tag{16}$$

If (Z, ρ_Z) is separable, then $(P_f(Z), h_{\rho_Z})$ is a separable metric space. If (Z, ρ_Z) is compact, then $(P_f(Z), h_{\rho_Z})$ is a compact metric space (see Aliprantis and Border, 2006). Often we will write h rather than h_{ρ_Z} - when the underlying metric is clear.

(3) Again let (Z, ρ_Z) be any metric space. Z is said to be connected if it cannot be written as the union of two nonempty, disjoint open subsets of Z. Equivalently, Z is connected if and only if the only subsets of Z that are open and closed in Z are the empty set and Z itself. If Z is compact and connected it is called a continuum.

2.2 w^*-Convergence and K-Convergnece in $\mathcal{L}_Y^{\infty}$

A sequence, $\{v^n\}_n \subset \mathcal{L}_Y^{\infty}$, converges weak star to $v^* = (v_1^*(\cdot), \ldots, v_m^*(\cdot)) \in \mathcal{L}_Y^{\infty}$, denoted by $v^n \underset{\rho_{w^*}}{\longrightarrow} v^*$, if and only if

$$\int_{\Omega} \langle v^n(\omega), l(\omega) \rangle_{R^m} \, d\mu(\omega) \longrightarrow \int_{\Omega} \langle v^*(\omega), l(\omega) \rangle_{R^m} \, d\mu(\omega) \tag{17}$$

for all $l(\cdot) \in \mathcal{L}_{R^m}^1$.

A sequence, $\{v^n\}_n \subset \mathcal{L}_Y^{\infty}$, K-convergences (i.e., Komlos convergence - Komlos, 1967) to $v^* \in \mathcal{L}_Y^{\infty}$, denoted by $v^n \underset{K}{\longrightarrow} v^*$, if and only if every subsequence, $\{v^{n_k}(\cdot)\}_k$, of $\{v^n(\cdot)\}_n$ has an arithmetic mean sequence, $\{\widehat{v}^{n_k}(\cdot)\}_k$, where

$$\widehat{v}^{n_k}(\cdot) := \frac{1}{k} \sum_{q=1}^{k} v^{n_q}(\cdot), \tag{18}$$

such that

$$\widehat{v}^{n_k}(\omega) \underset{R^m}{\longrightarrow} v^*(\omega) \;\; a.e. \;\; [\mu]. \tag{19}$$

The relationship between w^*-convergence and K-convergence is summarized via the following results which follow from Balder (2000): For every sequence of value functions,

$\{v^n\}_n \subset \mathcal{L}_Y^\infty$, and $v^* \in \mathcal{L}_Y^\infty$ the following statements are true:

$$\left.\begin{array}{c} \text{(i) If the sequence } \{v^n\}_n \text{ } K\text{-converges to } v^*, \text{ then } \{v^n\}_n \text{ } w^*\text{-converges to } v^*. \\ \\ \text{(ii) The sequence } \{v^n\}_n \text{ } w^*\text{-converges to } v^* \text{ if and only if} \\ \text{every subsequence } \{v^{n_k}\}_k \text{ of } \{v^n\}_n \text{ has a further subsequence, } \{v^{n_{k_r}}\}_r, \\ K\text{-converging to } v^*. \end{array}\right\} \quad (20)$$

For any sequence of value function profiles, $\{v^n\}_n$, in $\mathcal{L}_Y^\infty$ it is automatic that

$$sup_n \int_\Omega \|v^n(\omega)\|_{R^m} \, d\mu(\omega) < +\infty. \quad (21)$$

Thus, by the classical Komlos Theorem (1967), any such sequence, $\{v^n\}_n$, has a subsequence, $\{v^{n_k}\}_k$ that K-converges to some K-limit, $v^* \in \mathcal{L}_Y^\infty$.

3 USCOs and Upper Caratheodory Correspondences

3.1 USCOs

We have compact metric spaces $(\mathcal{L}_Y^\infty, \rho_{w^*})$ and (X, ρ_X). Let $\mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)} := \mathcal{U}(\mathcal{L}_Y^\infty, P_f(X))$ denote the collection of all upper semicontinuous correspondences taking nonempty, ρ_X-closed (and hence ρ_X-compact) values in X. Following the literature, we will call such mappings, USCOs (see Crannell, Franz, and LeMasurier, 2005, Anguelov and Kalenda, 2009, and Hola and Holy, 2009). Given any $\mathcal{N} \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}$, denote by $\mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}[\mathcal{N}]$ the collection of all sub-USCOs belonging to N, that is, all USCOs $\phi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}$ whose graph,

$$Gr\phi := \{(v, x) \in \mathcal{L}_Y^\infty \times X : x \in \phi(v)\},$$

is contained in the graph of $\mathcal{N}$,

$$Gr\mathcal{N} := \{(v, x) \in \mathcal{L}_X^\infty \times X : x \in \mathcal{N}(v)\}.$$

We will call any sub-USCO, $\phi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}[\mathcal{N}]$ a minimal USCO belonging to $\mathcal{N}$, if for any other sub-USCO, $\psi \in \mathcal{U}_{\mathcal{L}_Y^\infty\text{-}P_f(X)}[\mathcal{N}]$, $Gr\psi \subseteq Gr\phi$ implies that $Gr\psi = Gr\phi$ (see Drewnowski and Labuda, 1990). We will use the special notation, $[\mathcal{N}]$, to denote the collection of all minimal USCOs belonging to $\mathcal{N}$.

3.2 Upper Caratheodory Sub-Correspondences

Consider the uC correspondence $(\omega, v) \longrightarrow \mathcal{N}(\omega, v)$, and let

$$\mathcal{UC}_{\Omega\times\mathcal{L}_Y^\infty\text{-}P_f(X)}[\mathcal{N}(\cdot,\cdot)] := \mathcal{UC}^{\mathcal{N}} \quad (22)$$

denote the collection of all upper Caratheodory mappings belonging to $\mathcal{N}(\cdot,\cdot)$. Thus, $\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$ if and only if $\eta(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times\mathcal{L}_Y^\infty\text{-}P_f(X)}$ and

$$Gr\eta(\omega,\cdot) \subset Gr\mathcal{N}(\omega,\cdot) \text{ for all } \omega.$$

We will refer to the uC correspondence $\eta(\cdot,\cdot)$ as a uC sub-correspondence belonging to $\mathcal{N}(\cdot,\cdot)$.

3.3 Connectedness and Caratheodory Approximability

Consider the uC composition correspondence,

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := u(\omega, v, \mathcal{N}(\omega, v)), \tag{23}$$

where $\mathcal{N}(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times\mathcal{L}_Y^\infty\text{-}P_f(X)}$ and the function,

$$(\omega, v, x) \longrightarrow u(\omega, v, x) := (u_1(\omega, v_1, x), \ldots, u_m(\omega, v_m, x)) \in Y, \tag{24}$$

is Caratheodory, measurable in ω and jointly continuous in (v, x). For all uC sub-correspondences, $\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$ the induced sub-correspondence,

$$\left.\begin{aligned} (\omega, v) \longrightarrow p(\omega, v) := u(\omega, v, \eta(\omega, v)) \\ := \{(u_1(\omega, v_1, x), \ldots, u_m(\omega, v_m, x)) \in Y : x \in \eta(\omega, v)\}, \end{aligned}\right\} \tag{25}$$

is a uC sub-correspondence belonging to $\mathcal{P}(\cdot,\cdot)$. Thus, $p(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{P}}$. Each uC sub-correspondence in $\mathcal{UC}^{\mathcal{P}}$ induces a selection sub-correspondence, $v \longrightarrow \mathcal{S}^\infty(p(\cdot, v))$, and we will show that if the underlying uC sub-correspondence, $\eta(\cdot,\cdot) \in \mathcal{UC}^{\mathcal{N}}$, is continuum valued then this selection sub-correspondence has fixed points. Thus, we will show that there exists $v^* \in \mathcal{L}_Y^\infty$, such that

$$v^* \in \mathcal{S}^\infty(p(\cdot, v^*)) \subset \mathcal{S}^\infty(\mathcal{P}(\cdot, v^*)) \subset \mathcal{L}_Y^\infty. \tag{26}$$

For a particular $d = 1, 2, \ldots, m$, consider the uC sub-correspondence,

$$(\omega, v) \longrightarrow p_d(\omega, v) := u_d(\omega, v_d, \eta(\omega, v)) \in P_f(Y_d). \tag{27}$$

Definitions 1 *(Caratheodory Approximable uC Correspondences)*
We say that $p_d(\cdot,\cdot) \in \mathcal{UC}_{\Omega\times\mathcal{L}_Y^\infty\text{-}P_f(Y_d)}$ is Caratheodory approximable if for each $\varepsilon > 0$ there is a Caratheodory function, $g_d^\varepsilon(\cdot,\cdot) : \Omega \times \mathcal{L}_Y^\infty \longrightarrow Y_d$, having the property that for each $(\omega, v) \in \Omega \times \mathcal{L}_Y^\infty$ and each $(v, g_d^\varepsilon(\omega, v)) \in \mathcal{L}_Y^\infty \times Y_d$ there exists $(\overline{v}^d, \overline{u}_d) \in Grp_d(\omega,\cdot)$ such that

$$\rho_{w^*}(v, \overline{v}^d) + \rho_{Y_d}(g_d^\varepsilon(\omega, v), \overline{u}_d) < \varepsilon. \tag{28}$$

We call this Caratheodory function, $g^\varepsilon(\cdot,\cdot)$, an ε-Caratheodory selection of $p_d(\cdot,\cdot)$ - or equivalently, a Caratheodory function, $g_d^\varepsilon : \Omega \times \mathcal{L}_Y^\infty \longrightarrow Y_d$, such that for each ω

$$Grg_d^\varepsilon(\omega,\cdot) \subset B_{\rho_{w^*\times Y_d}}(\varepsilon, Grp_d(\omega,\cdot)). \tag{29}$$

By Corollary 4.3 in Kucia and Nowak (2000), a sufficient condition for $p_d(\cdot,\cdot)$ to be Caratheodory approximable, and therefore, for $p_d(\cdot,\cdot)$ to have for each $\varepsilon > 0$ an ε-Caratheodory selection, is for the uC sub-correspondence, $p_d(\cdot,\cdot)$, to have closed, interval values. If $(\omega, v) \longrightarrow \eta(\omega, v)$ is continuum valued, then because the image of a continuum under a continuous real valued is a closed bounded interval, for each $d = 1, 2, \ldots, m$, the uC sub-correspondence,

$$(\omega, v) \longrightarrow p_d(\omega, v) := u_d(\omega, v_d, \eta(\omega, v)), \tag{30}$$

will be interval valued, and therefore Caratheodory approximable. As a consequence, we will be able to show that there exists $v^* = (v_1^*, \ldots, v_m^*) \in \mathcal{L}_Y^\infty$ such that for each $d = 1, \ldots, m$

$$v_d^*(\omega) \in u_d(\omega, v_d^*, \eta(\omega, v^*)), \tag{31}$$

or equivalently,

$$v^*(\omega) \in u_1(\omega, v_1^*, \eta(\omega, v^*)) \times \cdots \times u_m(\omega, v_m^*, \eta(\omega, v^*)). \tag{32}$$

By implicit measurable selection (Theorem 7.1, Himmelberg, 1975), there exists a X-valued, measurable function, $x^*(\cdot)$, such that $x^*(\omega) \in \eta(\omega, v^*)$ a.e. $[\mu]$ and

$$v^*(\omega) = (u_1(\omega, v_1^*, x^*(\omega)), \ldots, u_m(\omega, v_m^*, x^*(\omega))) \in p(\omega, v^*). \tag{33}$$

Thus, we will be able to conclude from our fixed point result that there exists (v^*, x^*) such that for ω a.e. $[\mu]$,

$$\left.\begin{array}{c} v^*(\omega) \in p(\omega, v^*) \subset \mathcal{P}(\omega, v^*) \\ \text{and} \\ x^*(\omega) \in \eta(\omega, v^*) \subset \mathcal{N}(\omega, v^*). \end{array}\right\} \tag{34}$$

To show this, all that is required is that we show that for each d, $v_d^*(\omega) \in u_d(\omega, v_d^*, \eta(\omega, v^*))$ a.e. $[\mu]$.

4 A Fixed Point Theorem for Measurable Selection Valued Correspondences Induced by uC Composition Correspondences

We will show here, under assumptions [A-1], that for any uC composition correspondence,

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := u(\omega, v, \mathcal{N}(\omega, v)), \tag{35}$$

if there exists a uC sub-correspondence, $\eta(\cdot, \cdot) \in \mathcal{UC}^{\mathcal{N}}$, taking *continuum values* in X (*closed and connected values* in X), then for each $d = 1, 2, \ldots, m$, the uC composition sub-correspondence,

$$(\omega, v) \longrightarrow p_d(\omega, v) := u_d(\omega, v_d, \eta(\omega, v)), \tag{36}$$

takes closed, interval values in Y_d, and therefore, by Corollary 4.3 in Kucia and Nowak (2000), $p_d(\cdot, \cdot)$ is Caratheodory approximable. As a consequence, we will be able to show that there exists a function $v^* \in \mathcal{L}_Y^\infty$ such that

$$v^*(\omega) \in \mathcal{P}(\omega, v^*) \text{ a.e. } [\mu],$$

or equivalently,

$$v^* \in \mathcal{S}^\infty(\mathcal{P}_{v^*}).$$

Here is our main result.

Theorem *(A selection correspondence induced by a uC composition correspondence has fixed points if the underlying uC correspondence contains a continuum valued uC sub-correspondence)*
Suppose assumptions [A-1] hold. Let

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := u(\omega, v, \mathcal{N}(\omega, v))$$

be a uC composition correspondence where $\mathcal{N}(\cdot, \cdot) \in \mathcal{UC}_{\Omega \times \mathcal{L}_Y^\infty\text{-}P_f(X)}$ *and* $(\omega, v, x) \longrightarrow u(\omega, v, x) \in Y$ *is Caratheodory. If the uC correspondence,* $\mathcal{N}(\cdot, \cdot)$*, contains a uC sub-correspondence,* $\eta(\cdot, \cdot)$*, taking closed connected values in* X*, then there exists* $v^* \in \mathcal{L}_Y^\infty$ *such that*

$$v^*(\omega) \in \mathcal{P}(\omega, v^*) \text{ a.e. } [\mu].$$

Proof: As noted above, because $\eta(\cdot,\cdot)$ takes closed and connected values, the induced uC composition sub-correspondence,

$$\left.\begin{array}{c}(\omega,v)\longrightarrow p(\omega,v):=\{(u_1(\omega,v_1,x),\ldots,u_m(\omega,v_m,x)):x\in\eta(\omega,v)\}\\ \subset u_1(\omega,v_1,\eta(\omega,v))\times\cdots\times u_m(\omega,v_m,\eta(\omega,v))\\ :=p_1(\omega,v)\times\cdots\times p_m(\omega,v),\end{array}\right\} \tag{37}$$

is such that for each $d=1,2,\ldots,m$, the uC sub-correspondence,

$$(\omega,v)\longrightarrow p_d(\omega,v):=\{u_d(\omega,v_d,x)\in Y_d : x\in\eta(\omega,v)\},$$

takes closed interval values in Y_d, implying via Corollary 4.3 in Kucia and Nowak (2000) that $p_d(\cdot,\cdot)$ is Caratheodory approximable. Thus, there is a sequence of m-tuples of Caratheodory functions,

$$\{g^n(\cdot,\cdot)\}_n:=\{(g_1^n(\cdot,\cdot),\ldots,g_m^n(\cdot,\cdot))\}_n, \tag{38}$$

such that for each n and for each $(\omega,v)\in\Omega\times\mathcal{L}_Y^\infty$ there exists for each d, $(\overline{v}^{nd},\overline{u}_d^n)\in Grp_d(\omega,\cdot)$ such that,

$$\rho_{w^*}(v,\overline{v}^{nd})+\rho_{Y_d}(g_d^n(\omega,v),\overline{u}_d^n)<\tfrac{1}{m\cdot n}. \tag{39}$$

Next, consider the mapping from $\mathcal{L}_Y^\infty$ to $\mathcal{L}_Y^\infty$ given by

$$v\longrightarrow T^n(v):=g^n(\cdot,v):=(g_1^n(\cdot,v),\ldots,g_m^n(\cdot,v))\in\mathcal{L}_Y^\infty. \tag{40}$$

Observe that for each n, $T^n(\cdot)$ is continuous (i.e., $v^k\underset{\rho_{w^*}}{\longrightarrow}v^*$ implies that $T^n(v^k)\underset{\rho_{w^*}}{\longrightarrow}T^n(v^*)$). This is true because for each n, $v^k\underset{\rho_{w^*}}{\longrightarrow}v^*$ implies that for each $\omega\in\Omega$, as $k\longrightarrow\infty$, $g^n(\omega,v^k)\underset{\rho_Y}{\longrightarrow}g^n(\omega,v^*)\in Y$. Therefore, for $l\in\mathcal{L}_{R^m}^1$ chosen arbitrarily, $\langle g^n(\omega,v^k),l(\omega)\rangle\underset{R}{\longrightarrow}\langle g^n(\omega,v^*),l(\omega)\rangle$ a.e. $[\mu]$, implying, by the Dominated Convergence Theorem (Ash, 1972) that as $k\longrightarrow\infty$,

$$\int_\Omega\langle g^n(\omega,v^k),l(\omega)\rangle\,d\mu(\omega)\longrightarrow\int_\Omega\langle g^n(\omega,v^*),l(\omega)\rangle\,d\mu(\omega).$$

Since the choice of $l\in\mathcal{L}_{R^m}^1$ was arbitrary, we can conclude that if $v^k\underset{\rho_{w^*}}{\longrightarrow}v^*$, then $g^n(\cdot,v^k)\underset{\rho_{w^*}}{\longrightarrow}g^n(\cdot,v^*)\in\mathcal{L}_Y^\infty$. By the Brouwer-Schauder-Tychonoff Fixed Point Theorem (e.g., see Aliprantis-Border, 17.56, 2006), for each n, there exists $v^n\in\mathcal{L}_Y^\infty$ such that

$$v^n=T^n(v^n):=g^n(\cdot,v^n). \tag{41}$$

Thus, we have for each n a set, N^n, of μ-measure zero such that

$$v^n(\omega)=g^n(\omega,v^n)\text{ for all }\omega\in\Omega\backslash N^n. \tag{42}$$

Letting $N^\infty:=\cup_n N^n$ - so that, $\mu(N^\infty)=0$ - we have for each $n=1,2,\ldots$and for each $d=1,2,\ldots,m$, that

$$v_d^n(\omega)=g_d^n(\omega,v^n)\text{ for all }\omega\in\Omega\backslash N^\infty,\ \mu(N^\infty)=0. \tag{43}$$

Call the equation (43), one for each n, the Caratheodory equation and call the sequence, $\{v^n\}_n$, in $\mathcal{L}_Y^\infty$ the *Caratheodory fixed point sequence*.

For each pair of m-tuples of Caratheodory approximating functions and fixed points, $(g^n(\cdot,\cdot), v^n)$, consider the measurable function,

$$\omega \longrightarrow \min_{(v,u_d)\in Grp_d(\omega,\cdot)}[\rho_{w^*}(v^n, v) + \rho_{Y_d}(g_d^n(\omega, v^n), u_d)], \tag{44}$$

By Lemma 3.1 in Kucia and Nowak (2000) the graph correspondence, $\omega \longrightarrow Grp_d(\omega,\cdot)$, is measurable, and therefore, by the continuity of the function

$$(v, u_d) \longrightarrow [\rho_{w^*}(v^n, v) + \rho_{Y_d}(g_d^n(\omega, v^n), u_d)]$$

on $\mathcal{L}_Y^\infty \times Y_d$, there exists for each n, a measurable (everywhere) selection of $Grp_d(\omega,\cdot)$,

$$\omega \longrightarrow (\overline{v}_\omega^{nd}, \overline{u}_{\omega d}^n) \in \mathcal{L}_Y^\infty \times Y_d \tag{45}$$

solving the minimization problem (44) state-by-state (see Himmelberg, Parthasarathy, and VanVleck, 1976). Moreover, we have by the Caratheodory approximability of uC sub-correspondences,

$$\{p_1(\cdot,\cdot), \ldots, p_m(\cdot,\cdot)\},$$

and (39) above that for the sequences of optimal selections, $\{(\overline{v}_{(\cdot)}^{nd}, \overline{u}_{(\cdot)d}^n)\}_n$, $d = 1, 2, \ldots, m$, where for each n and for each ω, $\overline{v}_\omega^{nd} \in \mathcal{L}_Y^\infty$ and $\overline{u}_{\omega d}^n \in Y_d$, we have for each n and for each ω,

$$\underbrace{\rho_{w^*}(v^n, \overline{v}_\omega^{nd})}_{A} + \underbrace{\rho_{Y_d}(g_d^n(\omega, v^n), \overline{u}_{\omega d}^n)}_{B} < \tfrac{1}{m\cdot n}. \tag{46}$$

Given (42) and (46), we have for the sequences,

$$\{g^n(\cdot,\cdot), v^n\}_n \text{ and } \{\overline{v}_{(\cdot)}^{nd}, \overline{u}_{(\cdot)d}^n\}_n, d = 1, 2, \ldots, m, \tag{47}$$

that for all $\omega \in \Omega \backslash N^\infty$ and for all n,

$$\rho_{w^*}(v^n, \overline{v}_\omega^{nd}) + \underbrace{\rho_{Y_d}(v_d^n(\omega), \overline{u}_{\omega d}^n)}_{C} < \tfrac{1}{m\cdot n}, \tag{48}$$

for each d, where for each d and for each n, $\omega \longrightarrow \overline{v}_\omega^{nd}$ is $\mathcal{L}_Y^\infty$-valued, while $\omega \longrightarrow \overline{u}_{\omega d}^n$ is Y_d-valued, and

$$\overline{u}_\omega^n := (\overline{u}_{\omega 1}^n, \ldots, \overline{u}_{\omega m}^n) \in p_1(\omega, \overline{v}_\omega^{n1}) \times \cdots \times p_m(\omega, \overline{v}_\omega^{nm}) \text{ for all } \omega \in \Omega. \tag{49}$$

Next, because $(\mathcal{L}_Y^\infty, \rho_{w^*})$ is a compact metric space we can assume without loss of generality that the sequence of fixed points in $\mathcal{L}_Y^\infty$, $\{v^n\}_n$, K-converges to some $v^* \in \mathcal{L}_Y^\infty$, implying that $v^n \underset{\rho_{w^*}}{\longrightarrow} v^*$, and therefore implying, via (46)A, that $\overline{v}_\omega^{nd} \underset{\rho_{w^*}}{\longrightarrow} v^*$ uniformly in d and ω. Moreover, by (48)C, we have that

$$\widehat{\overline{u}}_{\omega d}^n = \frac{1}{n}\sum_{k=1}^{n} \overline{u}_{\omega d}^k \underset{\rho_{Y_d}}{\longrightarrow} v_d^*(\omega) \text{ a.e. } [\mu], \tag{50}$$

where for each n, $\overline{u}_{\omega d}^n \in p_d(\omega, \overline{v}_\omega^{nd})$ for all ω. By the properties of K-convergence, for each $n = 1, 2, 3, \ldots$, there is a set, $\widehat{N}^n$, of μ-measure zero such that for all d and for all

$\omega \in \Omega \backslash \widehat{N}^n$ as $q \longrightarrow \infty$

$$\left.\begin{array}{c} \widehat{\overline{u}}^{n+q}_{\omega d} = \frac{1}{q}\sum_{r=1}^{q} \overline{u}^{n+r}_{\omega d} \underset{\rho_{Y_d}}{\longrightarrow} v^*_d(\omega), \\ \text{and} \\ \widehat{v}^{n+q}_d(\omega) = \frac{1}{q}\sum_{r=1}^{q} v^{n+r}_d(\omega) \underset{\rho_{Y_d}}{\longrightarrow} v^*_d(\omega). \end{array}\right\} \tag{51}$$

Letting $\widehat{N}^\infty := \cup_{n=1}^\infty \widehat{N}^n$ we have for $n = 1, 2, 3, \ldots$, that for each player the truncated sequences, $\{\overline{u}^{n+q}_{(\cdot)d}\}_{q=1}^\infty$ and $\{v^{n+q}_d(\cdot)\}_{q=1}^\infty$, have arithmetic mean sequences, $\{\widehat{\overline{u}}^{n+q}_{(\cdot)d}\}_{q=1}^\infty$ and $\{\widehat{v}^{n+q}_d(\cdot)\}_{q=1}^\infty$, converging pointwise to $v^*_d(\cdot)$ off the set $\widehat{N}^\infty$ of μ-measure zero where the exceptional set $\widehat{N}^\infty$ is independent of n.

Because $p_d(\omega, \cdot)$ is ρ_{w^*}-ρ_{Y_d}-upper semicontinuous and because for each d, $\overline{v}^{nd}_\omega \underset{\rho_{w^*}}{\longrightarrow} v^*$ uniformly in d and ω, we have for each d and ω and for any sequence of $k_\omega = 1, 2, \ldots$, increasing to ∞, that there is a sequence $\{n_{k_\omega}\}_{k_\omega}$ increasing to ∞, such that for all $n \geq n_{k_\omega}$ the ρ_{Y_d}-open ball, $B_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, v^*))$, about $p_d(\omega, v^*)$ of radius $\frac{1}{k_\omega}$ with *closure* given by the closed, convex ball, $\overline{B}_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, v^*))$, is such that for each $\omega \in \Omega$, $n \geq n_{k_\omega}$ and $q = 1, 2, \ldots$.

$$p_d(\omega, \overline{v}^{(n+q)d}_\omega) \subset B_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, v^*)) \subset \overline{B}_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, v^*)). \tag{52}$$

Moreover, for all $\omega \in \Omega \backslash (N^\infty \cup \widehat{N}^\infty)$, $n \geq n_{k_\omega}$, and $q = 1, 2, \ldots$, we have for each d

$$\overline{u}^{n+q}_{\omega d} \in p_d(\omega, \overline{v}^{(n+q)d}_\omega) \subset \overline{B}_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, v^*)), \tag{53}$$

where recall for each d, ω and $n+q$, $(\overline{v}^{(n+q)d}_\omega, \overline{u}^{n+q}_{\omega d}) \in Grp_d(\omega, \cdot)$ solves,

$$\min_{(v, u_d) \in Grp_d(\omega, \cdot)} [\rho_{w^*}(v^{n+q}, v) + \rho_{Y_d}(g^{n+q}_d(\omega, v^{n+q}), u_d)].$$

Because for each k_ω, $\overline{B}_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, v^*))$ is closed and convex, we have for each k_ω, each d, and each $\omega \in \Omega \backslash (N^\infty \cup \widehat{N}^\infty)$ that

$$\widehat{\overline{u}}^{n+q}_{\omega d} \in \overline{B}_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, v^*)) \text{ for all } n \geq n_{k_\omega} \text{ and } q = 1, 2, \ldots, \tag{54}$$

By K-convergence we have for each $\omega \in \Omega \backslash (N^\infty \cup \widehat{N}^\infty)$, each d, and each $n \geq n_{k_\omega}$, $\widehat{\overline{u}}^{n+q}_{\omega d} \underset{\rho_{Y_d}}{\longrightarrow} v^*_d(\omega)$ as $q = 1, 2, \ldots$, goes to ∞, implying that for each d and $\omega \in \Omega \backslash (N^\infty \cup \widehat{N}^\infty)$,

$$v^*_d(\omega) \in \overline{B}_{\rho_{Y_d}}(\frac{1}{k_\omega}, p_d(\omega, v^*)) \text{ for all } k_\omega. \tag{55}$$

Thus, we have for each d and for each $\omega \in \Omega \backslash (N^\infty \cup \widehat{N}^\infty)$

$$v^*_d(\omega) \in p_d(\omega, v^*) = u_d(\omega, v^*_d, \eta(\omega, v^*)),$$

implying via implicit measurable selection (Theorem 7.1, Himmelberg, 1975) that there exists

$$x^*(\omega) = (x^*_1(\omega), \ldots, x^*_m(\omega)) \in \eta(\omega, v^*) \text{ a.e. } [\mu]$$

such that

$$\left.\begin{aligned} v^*(\omega) &= (u_1(\omega, v_1^*, x^*(\omega)), \ldots, u_m(\omega, v_m^*, x^*(\omega))) \in p(\omega, v^*) \\ &:= \{(u_1(\omega, v_1^*, x), \ldots, u_m(\omega, v_m^*, x)) : x \in \eta(\omega, v^*)\} \\ &\subset \mathcal{P}(\omega, v^*) \text{ a.e. } [\mu]. \end{aligned}\right\} \tag{56}$$

Q.E.D.

5 One-Shot Games and Stationary Markov Perfect Equilibria for Discounted Stochastic Games

An m-player, non-zero sum, discounted stochastic game, $\mathcal{DSG}$, is given by the following primitives:

$$\mathcal{DSG} := \left\{ \underbrace{(\Omega, B_\Omega, \mu)}_{\text{probability space}}, \underbrace{\left\{(X_d, \Phi_d(\omega), \beta_d, u_d(\omega, v_d, \cdot))_{d \in D}\right\}_{(\omega, v)}}_{\text{the one-shot game}}, \underbrace{q(\cdot|\omega, \cdot)}_{\text{law of motion}} \right\}, \tag{57}$$

where Ω is the state space, B_Ω is the σ-field of events, and μ is a probability measure. For each player d, X_d is the set of all possible actions available to player d, while $\Phi_d(\omega)$ is the feasible set of actions available to player d in state ω. Finally, $\beta_d \in (0,1)$ is player $d's$ discount rate and $u_d(\omega, v_d, \cdot)$ is player $d's$ payoff function in state ω given valuations (or prices) $v_d \in \mathcal{L}^\infty_{Y_d}$ (see assumptions [A-1]), and $q(\cdot|\omega, \cdot)$ is the law of motion in state ω. If players holding value function profile $v = (v_1 \ldots, v_m) \in \mathcal{L}^\infty_Y$ choose feasible action profile,

$$x = (x_1, \ldots, x_m) \in \Phi_1(\omega) \times \cdots \times \Phi_m(\omega) = \Phi(\omega),$$

in state ω then the next state ω' is chosen in accordance with probability measure $q(\cdot|\omega, x) \in \Delta(\Omega)$ and player $d's$ expected payoff is given by

$$u_d(\omega, v_d, x) := (1 - \beta_d) r_d(\omega, x) + \beta_d \int_\Omega v_d(\omega') q(\omega'|\omega, x). \tag{58}$$

We will denote by, $\mathcal{G}_{(\omega,v)}$, the m-player one-shot game in state ω underlying the $\mathcal{DSG}$ when players hold valuations $v := (v_1, \ldots, v_m)$.

Formally, the $\mathcal{DSG}$s we will consider here satisfy the following list of assumptions (a list we think of as the usual assumptions), labeled $[G]$(1)-(11):[3]

(1) $D =$ the set of players, consisting of m players indexed by $d = 1, 2, \ldots, m$ and each having discount rate given by $\beta_d \in (0,1)$.
(2) (Ω, B_Ω, μ), the state space where Ω is a complete separable metric spaces with metric ρ_Ω, equipped with the Borel σ-field, B_Ω, upon which is defined a probability measure, μ.
(3) $Y := Y_1 \times \cdots \times Y_m$, the space of players' payoff profiles, $u := (u_1, \ldots, u_m)$, such that for each player d, $Y_d := [-M, M]$, $M > 0$, and is equipped with the absolute value metric, $\rho_{Y_d}(u_d, u'_d) := |u_d - u'_d|$ and Y is equipped with the sum metric, $\rho_Y := \sum_d \rho_{Y_d}$.
*(4) $X := X_1 \times \cdots \times X_m := \prod_d X_d \subset E := \prod_d E_d$, the space of player action profiles, $x := (x_1, \ldots, x_m)$, such that for each player d, X_d is a **convex**, compact metrizable*

[3] We note that assumptions $[G]$(1)-(11) include assumptions $[A$-1].

subset of a locally convex Hausdorff topological vector space E_d and is equipped with a metric, ρ_{X_d}, compatible with the locally convex topology inherited from E_d, and X is equipped with the sum metric, $\rho_X := \sum_d \rho_{X_d}$.
*(5) $\omega \longrightarrow \Phi_d(\omega)$, is player d's measurable action constraint correspondence, defined on Ω taking nonempty, **convex**, ρ_{X_d}-closed (and hence compact) values in X_d.*
(6) $\omega \longrightarrow \Phi(\omega) := \Phi_1(\omega) \times \cdots \times \Phi_m(\omega)$, players' measurable action profile constraint correspondence, defined on Ω taking nonempty, convex, and ρ_X-closed (hence compact) values in X.
(7) $\mathcal{L}^{\infty}_{Y_d}$, the Banach space of all μ-equivalence classes of measurable (value) functions, $v_d(\cdot)$, defined on Ω with values in Y_d a.e. $[\mu]$, equipped with metric $\rho_{w_d^*}$ compatible with the weak star topology inherited from $\mathcal{L}^{\infty}_R$.
(8) $\mathcal{L}^{\infty}_Y := \mathcal{L}^{\infty}_{Y_1} \times \cdots \times \mathcal{L}^{\infty}_{Y_m} \subset \mathcal{L}^{\infty}_{R^m}$, the Banach space of all μ-equivalence classes of measurable (value) function profiles, $v(\cdot) := (v_1(\cdot), \ldots, v_m(\cdot))$, defined on Ω with values in Y a.e. $[\mu]$, equipped with the sum metric $\rho_{w^*} := \sum_d \rho_{w_d^*}$ compatible with the weak star product topology inherited from $\mathcal{L}^{\infty}_{R^m}$.
(9) $\mathcal{S}^{\infty}(\Phi_d(\cdot))$, the set of all μ-equivalence classes of (B_Ω, B_{X_d})-measurable functions (selections), $x_d(\cdot)$, defined on Ω such that in $x_d(\omega) \in \Phi_d(\omega)$ a.e. $[\mu]$, and

$$\mathcal{S}^{\infty}(\Phi(\cdot)) = \mathcal{S}^{\infty}(\Phi_1(\cdot)) \times \cdots \times \mathcal{S}^{\infty}(\Phi_m(\cdot)) \tag{59}$$

the set of all μ-equivalence classes of measurable profiles (selection profiles), $x(\cdot) = (x_1(\cdot), \ldots, x_m(\cdot))$, defined on Ω such that

$$x(\omega) \in \Phi(\omega) := \Phi_1(\omega) \times \cdots \times \Phi_m(\omega) \text{ a.e. } [\mu].$$

(10) $r_d(\cdot,\cdot) : \Omega \times X \longrightarrow Y_d$ is player d's affine, Caratheodory stage payoff function (i.e., for each ω, $r_d(\omega,\cdot)$ is ρ_X-continuous on X, for each x, $r_d(\cdot,x)$ is (B_Ω, B_{Y_d})-measurable on Ω, and for each (ω, x_{-d}) and each x_d^0 and x_d^1 in X_d,

$$r_d(\omega, \gamma x_d^0 + (1-\gamma)x_d^1, x_{-d}) = \gamma r_d(\omega, x_d^0, x_{-d}) + (1-\gamma) r_d(\omega, x_d^1, x_{-d}), \tag{60}$$

for all $\gamma \in [0,1]$.
(11) $q(\cdot|\cdot,\cdot) : \Omega \times X \longrightarrow \Delta(\Omega)$ is the law of motion defined on $\Omega \times X$ taking values in the space of probability measures on Ω, having the following properties: (i) each probability measure, $q(\cdot|\omega,x)$, in the collection

$$Q(\Omega \times X) := \{q(\cdot|\omega,x) : (\omega,x) \in \Omega \times X\} \tag{61}$$

is absolutely continuous with respect to μ (denoted $Q(\Omega \times X) << \mu$), (ii) for each $E \in B_\Omega$, $q(E|\cdot,\cdot)$ is measurable on $\Omega \times X$, and (iii) the collection of probability density functions,

$$H_\mu := \{h(\cdot|\omega,x) : (\omega,x) \in \Omega \times X\}, \tag{62}$$

of $q(\cdot|\omega,x)$ with respect to μ is such that for each state ω, the function

$$(x_d, x_{-d}) \longrightarrow h(\omega'|\omega, x_d, x_{-d}) \tag{63}$$

is continuous in x and affine in x_d a.e. $[\mu]$ in ω'.

A one-shot game then is a collection of strategic form games,

$$\mathcal{G}(\Omega \times \mathcal{L}^{\infty}_Y) := \left\{\mathcal{G}_{(\omega,v)} : (\omega,v) \in \Omega \times \mathcal{L}^{\infty}_Y\right\}, \tag{64}$$

where each (ω, v)-game in the collection is given by

$$\mathcal{G}_{(\omega,v)} := \left\{ \underbrace{\Phi_d(\omega)}_{\text{feasible actions}} , \underbrace{u_d(\omega, v_d, (\cdot,\cdot))}_{\text{payoff function}} \right\}_{d\in D}, \tag{65}$$

Under assumptions [G], in a (ω, v)-game player $d's$ payoff function, given by

$$x \longrightarrow u_d(\omega, v_d, x) := (1-\beta_d) r_d(\omega, x) + \beta_d \int_\Omega v_d(\omega') h(\omega'|\omega, x) d\mu(\omega'), \tag{66}$$

is jointly continuous in action profiles, $x = (x_1, \dots, x_m)$, and for any sequence of value function-action profiles pairs, $\{(v^n, x^n)\}_n$, if $v^n \underset{\rho_{w^*}}{\longrightarrow} v^*$ and $x^n \underset{\rho_X}{\longrightarrow} x^*$ then for each ω, $u(\omega, v^n, x^n) \underset{\rho_Y}{\longrightarrow} u(\omega, v^*, x^*)$ (i.e., $u(\omega, \cdot, \cdot)$ is jointly continuous in (v, x)). Thus, the Y-valued players' payoff function, $u(\cdot,\cdot,\cdot)$, is a Caratheodory function: $\rho_{w^*\times X}$-continuous in (v, x) for each ω, and (B_Ω, B_Y)-measurable in ω on Ω for each (v, x).[4]

A stationary Markov strategy for player d, is a (B_Ω, B_{X_d})-measurable function, $x_d(\cdot) : \Omega \longrightarrow X_d$, such that $x_d(\omega) \in \Phi_d(\omega)$ a.e. $[\mu]$. Because the strategy, $x_d(\cdot)$, is not time dependent, the strategy, $x_d(\cdot)$, is stationary, and because the action in X_d chosen by the strategy, $x_d(\omega) \in X_d$, depends only on the current state, ω, for states off a set of μ-measure zero, the strategy, $x_d(\cdot)$, is Markov. Thus, the collection of all player d stationary Markov strategies is given by $\mathcal{S}^\infty(\Phi_d(\cdot))$, the collection of all μ-equivalence classes of a.e. measurable selections of $\Phi_d(\cdot)$.[5] A Markov strategy profile is given by,

$$(x_1(\cdot), \dots, x_m(\cdot)) \in \mathcal{S}^\infty(\Phi_1(\cdot)) \times \cdots \times \mathcal{S}^\infty(\Phi_m(\cdot)),$$

where

$$\mathcal{S}^\infty(\Phi(\cdot)) = \mathcal{S}^\infty(\Phi_1(\cdot)) \times \cdots \times \mathcal{S}^\infty(\Phi_m(\cdot))$$

is the collection of all such profiles.

Let

$$r_d^n(x(\cdot))(\omega) := \begin{cases} r_d(\omega, x(\omega)) & \text{for } n = 1 \\ \int_\Omega r_d(\omega', x(\omega')) q^{n-1}(\omega'|\omega, x(\omega)) & \text{for } n \geq 2, \end{cases} \tag{67}$$

denote the n^{th} period *expected* immediate payoff to player d under Markov strategy profile $x(\cdot)$ starting at state ω given law of motion $q(\cdot|\cdot,\cdot)$. Here, for $n \geq 2$, $q^n(\cdot|\omega, x(\omega))$ is defined recursively by

$$\left.\begin{aligned} & q^n(E|\omega, x(\omega)) \\ & = \int_\Omega q(E|\omega', x(\omega')) q^{n-1}(\omega'|\omega, x(\omega)). \end{aligned}\right\} \tag{68}$$

The discounted expected payoff to player d, with discount rate $\beta_d \in (0,1)$, over an infinite time horizon under stationary Markov strategy profile $x(\cdot)$ starting at state ω is given by

$$Er_d^\infty(x(\cdot))(\omega) := \textstyle\sum_{n=1}^\infty \beta_d^{n-1} Er_d^n(x(\cdot))(\omega). \tag{69}$$

[4] Abusing the notation a bit, $\rho_{w^*\times X}$ denotes the summ metric, $\rho_{w^*} + \rho_X$. Therefore,

$$\rho_{w^*\times X} := \rho_{w^*} + \rho_X.$$

[5] Such a strategy is stationary because it does not depend on time (the same strategy applies at all time points). Such a strategy is Markov because the action choice specified by the strategy is a function of the current state - and nothing else.

A stationary Markov strategy profile $x^*(\cdot) \in \mathcal{S}^\infty(\Phi(\cdot))$ is a *stationary Markov perfect equilibrium* if for all players d,

$$Er_d^\infty(x_d^*(\cdot), x_{-d}^*(\cdot))(\omega) \geq Er_d^\infty(x_d'(\cdot), x_{-d}^*(\cdot))(\omega), \text{ a.e. } [\mu] \text{ in } \omega, \tag{70}$$

for all other strategies, $x_d'(\cdot) \in \mathcal{S}^\infty(\Phi_d(\cdot))$. In particular, a $SMPE$, $x^*(\cdot) \in \mathcal{S}^\infty(\Phi(\cdot))$, is a profile of strategies such that for each player and for almost all initial states, ω, the discounted sum of player $d's$ expected payoffs is maximal under strategy $x_d^*(\cdot)$ given the strategies, $x_{-d}^*(\cdot)$ being used by the other players. We know from Blackwell (1965) (see also Himmelberg, Parthasarathy, and VanVleck, 1976), and their results for dynamic programming, that a discounted stochastic game satisfying assumptions $[G]$ will have a stationary Markov perfect equilibrium, $x^*(\cdot) \in \mathcal{S}^\infty(\Phi(\cdot))$ if and only if there exists a profile of value functions, $v^* \in \mathcal{L}_Y^\infty$ such that together, $(v^*, x^*(\cdot))$, are such that for each player d and a.e. $[\mu]$ in ω

$$\left.\begin{array}{c} (1)\ v_d^*(\omega) = u_d(\omega, v_d^*, x^*(\omega)) \\ := (1-\beta_d) r_d(\omega, x^*(\omega)) + \beta_d \int_\Omega v_d^*(\omega') h(\omega'|\omega, x^*(\omega)) d\mu(\omega'), \\ \text{and} \\ (2)\ u_d(\omega, v_d^*, x_d^*(\omega), x_{-d}^*(\omega)) = \max_{x_d \in \Phi_d(\omega)} u_d(\omega, v_d^*, x_d, x_{-d}^*(\omega)). \end{array}\right\} \tag{71}$$

The first condition is the Bellman condition. The second condition is the Nash condition.

The Bellman condition can be rewritten as

$$\left.\begin{array}{c} \text{for each player } d \\ v_d^*(\omega) = u_d(\omega, v_d^*, x^*(\omega)) \in u_d(\omega, v_d^*, \mathcal{N}(\omega, v^*)) \text{ a.e. } [\mu], \\ \text{or equivalently} \\ v^*(\omega) := (v_1^*(\omega), \ldots, v_m^*(\omega)) \in \mathcal{P}(\omega, v^*) \text{ a.e. } [\mu]. \end{array}\right\} \tag{72}$$

We note that if the latter inclusions holds, then the profile of value functions, $v^* \in \mathcal{L}_Y^\infty$, is a fixed point of the Nash payoff selection correspondence, $v \longrightarrow \mathcal{S}^\infty(\mathcal{P}_v)$, and in particular, we have, $v^* \in \mathcal{S}^\infty(\mathcal{P}_{v^*})$, and we can deduce via implicit measurable selection methods that there exists $x^* \in \mathcal{S}^\infty(\mathcal{N}_{v^*})$. In particular, by our fixed point result above we know that if the uC Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, has uC Nash sub-correspondence, $\eta(\cdot,\cdot)$, taking closed, connected values, then for the induced uC Nash payoff sub-correspondence, $p(\cdot,\cdot)$, given by

$$p(\omega, v) = u(\omega, v, \eta(\omega, v)), \tag{73}$$

there exists $v^* \in \mathcal{L}_Y^\infty$ such that

$$v^*(\omega) \in p(\omega, v^*) \subset \mathcal{P}(\omega, v^*) \text{ a.e. } [\mu]. \tag{74}$$

Then by implicit measurable selection (e.g., see Theorem 7.1 in Himmelberg, 1975), there exists a profile, $x^*(\cdot) = (x_1^*(\cdot), \ldots, x_m^*(\cdot))$, of a.e. measurable selections of $\omega \longrightarrow \eta(\omega, v^*)$, such that for each player $d = 1, 2, \ldots, m$,

$$v_d^*(\omega) = u_d(\omega, v_d^*, x^*(\omega)) \in u_d(\omega, v_d^*, \eta(\omega, v^*)) := p_d(\omega, v^*) \text{ a.e. } [\mu], \tag{75}$$

We have then for each player d that the state-contingent prices given by value function, $v_d^*(\cdot) \in \mathcal{L}_{Y_d}^\infty$, incentivizes the continued choice by each player d, of action strategy, $x_d^*(\cdot)$, and for the value function-strategy profile pair, $(v^*, x^*(\cdot)) \in \mathcal{S}^\infty(p_{v^*}) \times \mathcal{S}^\infty(\eta_{v^*})$, we have that

$$v^*(\omega) = u(\omega, v^*, x^*(\omega)) \in p(\omega, v^*) \text{ and } x^*(\omega) \in \eta(\omega, v^*) \text{ a.e. } [\mu], \tag{76}$$

implying that

$$v^*(\omega) \in \mathcal{P}(\omega, v^*) \text{ and } x^*(\omega) \in \mathcal{N}(\omega, v^*) \text{ a.e. } [\mu]. \tag{77}$$

Thus, for value function-strategy profile pair, $(v^*, x^*(\cdot))$, we have for each player $d = 1, 2, \ldots, m$ and for ω a.e. $[\mu]$, that $(v^*, x^*(\cdot))$ satisfies the Bellman condition (1 above) as well as satisfies the Nash condition (2 above). Thus, $x^*(\cdot) \in \mathcal{S}^\infty(\mathcal{N}_{v^*})$ is a stationary Markov perfect equilibrium of a $\mathcal{DSG}$ satisfying assumptions $[G]$, incentivized by state-contingent prices, $v^* \in \mathcal{S}^\infty(\mathcal{P}_{v^*})$.

6 Comments

(1) Note that, due to the fact that Komlos convergence implies weak star convergence, the arguments given in the latter part of the proof above of our Theorem (see expressions (50)-(55) above) establish that the uC Nash payoff sub-correspondence induces a weak star upper semicontinuous selection sub-correspondence, $v \longrightarrow \mathcal{S}^\infty(p_v)$.

(2) Fu and Page (2022) establish that all $\mathcal{DSG}s$ satisfying assumptions $[G]$ above have uC Nash correspondences given by a bundle of minimal uC Nash correspondences each of which takes minimally essential, closed connected Nash values. Given that all $\mathcal{DSG}s$ satisfying the usual assumptions have one-shot games satisfying assumptions $[G]$, all such $\mathcal{DSG}s$ have Nash payoff selection correspondences having fixed points - implying that all such $\mathcal{DSG}s$ have stationary Markov perfect equilibria ($SMPE$).

References

[1] Aliprantis, C. D. and Border, K. C. (2006) *Infinite Dimensional Analysis: A Hitchhiker's Guide*, 3rd Edition, Springer-Verlag, Berlin-Heidelberg.

[2] Anguelov, R. and Kalenda, O. F. K. (2009) "The Convergence Space of Minimal USCO Mappings," *Czechoslovak Mathematical Journal* 134, 101-128.

[3] Ash R. (1972) *Probability and Real Analysis*, John Wiley & Sons, New York.

[4] Balder, E. J. (2000) "Lectures on Young Measure Theory and Its Application in Economics," *Rend. Instit. Mat. Univ. Trieste* XXXI Suppl. 1, 1-69.

[5] Blackwell, D. (1965) "Discounted Dynamic Programming," *Annals of Mathematical Statistics* 36, 226-235.

[6] Crannell, A., Franz, M., and LeMasurier, M. (2005) "Closed Relations and Equivalence Classes of Quasicontinuous Functions," *Real Analysis Exchange* 31, 409-424.

[7] Drewnowski, L. and Labuda, I. (1990) "On Minimal Upper Semicontinuous Compact-Valued Maps," *Rocky Mountain Journal of Mathematics* 20, 737-752.

[8] Fu, J. and Page, F. (2022) "Parameterized State-Contingent Games, 3M Minimal Nash Correspondences, and Connectedness," Discussion Paper 116, Systemic Risk Centre, London School of Economics.

[9] Himmelberg, C. J. (1975) "Measurable Relations," *Fundamenta Mathematicae* 87, 53-72.

[10] Himmelberg, C. J., Parthasarathy, T., and VanVleck, F. S. (1976) "Optimal Plans for Dynamic Programming Problems," *Mathematical of Operations Research* 1, 390-394.

[11] Himmelberg, C. J., Parthasarathy, T., Raghavan, T. E. S., and VanVleck, F. S. (1976) "Existence of p-equilibrium and Optimal Stationary Strategies in Stochastic Games," Proceedings of the American Mathematical Society 60, 245-251.

[12] Hola, L. and Holy, D. (2009) “Minimal USCO Maps, Densely Continuous Forms, and Upper Semicontinuous Functions,” *Rocky Mountain Journal of Mathematics* 39, 545-562.

[13] Hola, L. and Holy, D. (2015) “Minimal USCO and Minimal CUSCO Maps,” *Khayyam Journal of Mathematics* 1, 125-150.

[14] Komlos, J. (1967) “A Generalization of a Problem of Steinhaus,” *Acta Mathematica Academiae Scientiarum Hungaricae* 18, 217-229.

[15] Kucia, A. and Nowak, A. (2000) “Approximate Caratheodory Selections,” *Bull. Pol. Acad. Sci. Math.* 48, 81-87.