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Testing idealness in the filter oracle model

Ahmad Abdi [a,*], Gérard Cornuéjols [b], Bertrand Guenin [c], Levent Tunçel [c]

[a] London School of Economics and Political Science, Houghton Street, London, WC2A 2AE, UK
[b] Carnegie Melon University, United States of America
[c] University of Waterloo, Canada

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ABSTRACT

A *filter oracle* for a clutter consists of a finite set V and an oracle which, given any set $X \subseteq V$, decides in unit time whether X contains a member of the clutter. Let $\mathfrak{A}_{2n}$ be an algorithm that, given any clutter $\mathcal{C}$ over $2n$ elements via a filter oracle, decides whether $\mathcal{C}$ is ideal. We prove that in the worst case, $\mathfrak{A}_{2n}$ makes at least 2^{n-1} calls to the filter oracle. Our proof uses the theory of cuboids.

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1. Background

Let V be a finite set, and $\mathcal{C}$ a family of subsets of V, called *members*. $\mathcal{C}$ is a *clutter* over *ground set* V if no member contains another one [9]. $\mathcal{C}$ is *ideal* if the set covering polyhedron $\left\{x \in \mathbb{R}^V : \sum_{u\in C} x_u \geq 1 \ \forall C \in \mathcal{C}; x \geq \mathbf{0}\right\}$ is integral. The terminology was coined in [6]. However, the notion goes back to a 1963 manuscript wherein Alfred Lehman extended the *width-length inequality* of Moore and Shannon [13] and Duffin [8] for two-terminal networks to arbitrary ideal clutters [11] (the manuscript was published years later in 1979). This manuscript reached Ray Fulkerson in 1965 which reportedly influenced his work in the area, prominently on *blocking theory of polyhedra* [10].

An important aspect of all ideal clutters, going all the way back to Lehman's first manuscript on the topic, has been the *structure* of such clutters. This aspect remains largely mysterious to this date, in part due to the fact that there are several structurally different examples of ideal clutters coming from undirected and directed graphs, binary matroids, and the unit hypercube (see [1]). Thinking about this problem from a computational complexity perspective leads to the following: What is the time complexity of detecting the property of idealness?

Using basic polyhedral theory, one can show easily that testing idealness belongs to co-NP. In fact, rather surprisingly Ding, Feng and Zang [7] showed that testing idealness is co-NP-complete (even for clutters where every element of the ground set belongs to at most two members), and so testing idealness is NP-hard.

Many examples of clutters from Combinatorial Optimization, such as arborescences, cuts, T-joins, and dijoins, have exponentially many members (in the size of the ground set). For this reason, for some problems, it may be more appropriate to work in a model where $\mathcal{C}$ is inputted via an oracle. More precisely, a *filter oracle* for a clutter $\mathcal{C}$ consists of V along with an oracle which, given any set $X \subseteq V$, decides in unit time whether or not X contains a member.

In the filter oracle model, it is no longer clear that testing idealness belongs to co-NP. Using a seminal theorem of Lehman on *minimally non-ideal* clutters [12], Seymour showed that testing idealness indeed belongs to co-NP [15]. Given Ding et al.'s co-NP-completeness result in the explicit model, one would not expect the classification to be any different in the filter oracle. In particular, one would not expect testing idealness in the filter oracle model to belong to NP.

In this brief note, we prove that in fact the situation in the filer oracle is determined independently of the "P versus NP" question. We prove that in the filter oracle model, testing idealness cannot be done in polynomial time, period. Our proof also proves that even the task of "finding a Δ_3 minor", a first test for detecting non-idealness, cannot be done in polynomial time.

* Corresponding author.
E-mail address: a.abdi1@lse.ac.uk (A. Abdi).

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2. Cuboids

Our main result is proved by using the concept of *cuboids*, initiated in [4] and developed in [2], which allows us to get an understanding of the "local geometry" of ideal clutters.

A *cuboid* is a clutter $\mathcal{C}$ whose ground set can be partitioned into pairs $\{u_i, v_i\}$, $i \in [n]$ such that $|\{u_i, v_i\} \cap C| = 1$ for all $i \in [n]$ and $C \in \mathcal{C}$. $\mathcal{C}$ can be represented as a subset of $\{0,1\}^n$. More precisely, for each $C \in \mathcal{C}$, let $p(C)$ be the point in $\{0,1\}^n$ such that $p(C)_i = 0$ if and only if $C \cap \{u_i, v_i\} = \{u_i\}$. Let $S := \{p(C) : C \in \mathcal{C}\}$. We call $\mathcal{C}$ the *cuboid of* S, denote by $\mathrm{cuboid}(S) := \mathcal{C}$ and by $C(p)$ the member of $\mathcal{C}$ corresponding to $p \in \{0,1\}^n$. Note that the operator cuboid$(\cdot)$ takes any subset of $\{0,1\}^n$ to a cuboid. S is *cube-ideal* if cuboid(S) is an ideal clutter. It is known that S is cube-ideal if, and only if, the convex hull of S can be described by $\mathbf{0} \le x \le \mathbf{1}$ and inequalities of the form $\sum_{i\in I} x_i + \sum_{j \in J}(1 - x_j) \ge 1$ for disjoint $I, J \subseteq [n]$ [4,2]. Thus, the set $\{0,1\}^n$ is cube-ideal. Moreover, if S is cube-ideal then so is every restriction of it, where a *restriction* is defined as any set obtained from S after fixing some coordinates to 0 or 1 and then dropping the coordinates.

Let $p \in \{0,1\}^n$. The set $S \triangle p$ is defined as $\{x \triangle p : x \in S\}$, where the second $\triangle$ denotes coordinate-wise sum mod 2; we call $S \triangle p$ the *twisting of* S *with respect to* p. It can be readily seen that twisting preserves cube-idealness. The *localization of* S *at* p, denoted by $\mathrm{loc}(S; p)$, is the clutter over ground set $[n]$ whose members are the inclusionwise minimal sets in $\{C \subseteq [n] : \chi_C \in S \triangle p\}$. In particular, if $p \in S$ then $\mathrm{loc}(S; p) = \{\emptyset\}$. In the original paper [2] where this notion was developed, "the localization of S at p" was referred to instead as "the induced clutter of S with respect to p". We feel this new terminology is more appropriate.

The localizations of S at points outside the set are very helpful in studying cube-idealness. A key insight for this note is that S is cube-ideal if, and only if, the localization of S at every point in $\{0,1\}^n - S$ is ideal [2]. Consequently, if for example S excludes a unique point p of $\{0,1\}^n$, then S is cube-ideal, because $\mathrm{loc}(S; p) = \{\{1\}, \{2\}, \dots, \{n\}\}$ is clearly an ideal clutter.

3. The result

We are almost ready to prove the main result of this note. Let $n \ge 1$ be an integer, and let G_n denote the skeleton graph of the unit hypercube $[0,1]^n$. Given $S \subseteq \{0,1\}^n$, if $G_n[\{0,1\}^n - S]$ has maximum degree at most 2, then S is cube-ideal. This result was first proved in [5], and further studied in [3]. It can also be readily shown using the characterization of cube-idealness in terms of the localizations. The result, however, does not extend from 2 to 3. Let $S_3 := \{e_1 + e_2, e_2 + e_3, e_1 + e_3, e_1 + e_2 + e_3\} \subseteq \{0,1\}^3$, where e_i denotes the i^{th} standard unit vector of appropriate dimension. Then S_3 is not cube-ideal because its convex hull has a facet-defining inequality of the form $x_1 + x_2 + x_3 \ge 2$. Moreover, in $G_3[\{0,1\}^3 - S_3]$, the vertex $\mathbf{0}$ has 3 neighbors e_1, e_2, e_3.

Theorem 1. *Let $\mathfrak{A}_{2n}$ be an algorithm that, given any clutter $\mathcal{C}$ over $2n$ elements via a filter oracle, decides whether or not $\mathcal{C}$ is ideal. Then in the worst case, $\mathfrak{A}_{2n}$ must make at least 2^{n-1} calls to the filter oracle.*

Proof. For all $p \in \{0,1\}^n$ and distinct $i, j, k \in [n]$, let $S_{(p:i,j,k)} := \{0,1\}^n - \{p, p\triangle e_i, p\triangle e_j, p\triangle e_k\}$. Then $S_{(p:i,j,k)}$ is not cube-ideal as it has an S_3 restriction, while every proper superset S' of $S_{(p:i,j,k)}$ is cube-ideal as $G_n[\{0,1\}^n - S']$ has degree at most 2. In particular, $\mathrm{cuboid}(S_{(p:i,j,k)})$ is a non-ideal clutter, while $\mathrm{cuboid}(S')$ is ideal for every $S' \supsetneq S_{(p:i,j,k)}$. Thus, $\mathfrak{A}_{2n}$ must distinguish between $\mathrm{cuboid}(S_{(p:i,j,k)})$ and $\mathrm{cuboid}(S')$ for every $S' \supsetneq S_{(p:i,j,k)}$. Consequently, for every point $q \in \{p, p\triangle e_i, p\triangle e_j, p\triangle e_k\}$, the algorithm must query the set $C(q)$ or a superset of it. In fact, we can say more.

Given neighbors r, r' of G_n, it can be readily seen that $C(r), C(r')$ differ in exactly one element. This observation implies that by adding a new element to $C(r)$ one obtained another set which contains $C(r')$ for some neighbor r' of r. Now, for each $q \in \{p, p\triangle e_i, p\triangle e_j, p\triangle e_k\} - \{p\}$, every neighbor of q in G_n except for p belongs to both $S_{(p:i,j,k)}$ and S', $S' \supsetneq S_{(p:i,j,k)}$, so in order to distinguish between the two sets the algorithm must query either $C(q)$ or $C(q) \cup C(p)$.

By applying the argument above to every $p \in \{0,1\}^n$ and distinct $i, j, k \in [n]$, we conclude the following: For every q and every neighbor of it p in G_n, $\mathfrak{A}_{2n}$ must query at least one of $C(q), C(q) \cup C(p)$. Let $S' := \{q \in \{0,1\}^n : \mathbf{1}^\top q \equiv 0 \pmod 2\}$. Then $\{C(q), C(q) \cup C(q \triangle e_1) : q \in S'\}$ consists of $2|S'| = 2^n$ distinct sets, and $\mathfrak{A}_{2n}$ queries at least one of $C(q), C(q) \cup C(q\triangle e_1)$ for each $q \in S'$. This implies that $\mathfrak{A}_{2n}$ queries at least 2^{n-1} sets, as required. □

Let $\mathcal{C}$ be a clutter over ground set V. Let I, J be disjoint subsets of V. The *minor of $\mathcal{C}$ obtained after deleting I and contracting J*, denoted $\mathcal{C} \setminus I / J$, is the clutter over ground set $V - (I \cup J)$ whose members are the inclusionwise minimal sets in $\{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}$. Given a filter oracle for $\mathcal{C}$, we also have one for every minor $\mathcal{C} \setminus I / J$ [15].

Being ideal is closed under taking minor operations [14]. Two clutters are *isomorphic* if one can be obtained from the other by relabeling its ground set. Denote by Δ_3 any clutter isomorphic to $\{\{1,2\}, \{2,3\}, \{3,1\}\}$. It can be readily checked that Δ_3 is the only non-ideal clutter over a ground set of size at most three. In particular, if a clutter has a Δ_3 minor, then it is non-ideal.

Let $S \subseteq \{0,1\}^n$. It can be readily seen that every localization of S is a (contraction) minor of cuboid(S). Thus, since $\mathrm{loc}(S_3; \mathbf{0}) = \{\{1,2\}, \{2,3\}, \{1,3\}\}$, $\mathrm{cuboid}(S_3)$ has a Δ_3 minor, proving once again that S_3 is not cube-ideal. It can also be readily seen that if R is a restriction of S, then cuboid(R) is a minor of cuboid(S). Consequently, in the proof of Theorem 1, it can be readily seen that $\mathrm{cuboid}(S_{(p:i,j,k)})$ has a Δ_3 minor, while $\mathrm{cuboid}(S')$ is ideal and therefore has no Δ_3 minor for every $S' \supsetneq S$. Thus, the proof also implies the following.

Theorem 2. *Let $\mathfrak{D}_{2n}$ be an algorithm that, given any clutter $\mathcal{C}$ over $2n$ elements via a filter oracle, decides whether or not $\mathcal{C}$ has a Δ_3 minor. Then in the worst case, $\mathfrak{D}_{2n}$ must make at least 2^{n-1} calls to the filter oracle.* □

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