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Results in Mathematics

On the Gleason-Kahane-Żelazko Theorem for Associative Algebras

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Abstract. The classical Gleason-Kahane-Żelazko Theorem states that a linear functional on a complex Banach algebra not vanishing on units, and such that $\Lambda(\mathbf{1}) = 1$, is multiplicative, that is, $\Lambda(ab) = \Lambda(a)\Lambda(b)$ for all $a, b \in A$. We study the GKŻ property for associative unital algebras, especially for function algebras. In a GKŻ algebra A over a field of at least 3 elements, and having an ideal of codimension 1, every element is a finite sum of units. A real or complex algebra with just countably many maximal left (right) ideals, is a GKŻ algebra. If A is a commutative algebra, then the localization A_P is a GKŻ-algebra for every prime ideal P of A. Hence the GKŻ property is not a local-global property. The class of GKŻ algebras is closed under homomorphic images. If a function algebra $A \subseteq \mathbb{F}^X$ over a subfield $\mathbb{F}$ of $\mathbb{C}$, contains all the bounded functions in $\mathbb{F}^X$, then each element of A is a sum of two units. If A contains also a discrete function, then A is a GKŻ algebra. We prove that the algebra of periodic distributions, and the unitisation of the algebra of distributions with support in $(0, \infty)$ satisfy the GKŻ property, while the algebra of compactly supported distributions does not.

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1. Introduction

We study the Gleason-Kahane-Żelazko property (Definition 1.1 below) of associative unital algebras, especially of function algebras. We will also investigate the validity of the GKŻ theorem in some natural convolution algebras of distributions.

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1.1. Background

The classical Gleason-Kahane-Żelazko Theorem, proved independently by Gleason [4], and by Kahane and Żelazko [8], provides a characterization of the maximal ideals of a commutative complex Banach algebra A: a subspace of A is a maximal ideal if and only if it has codimension 1 and contains no units. Equivalently, a linear functional $\Lambda : A \to \mathbb{C}$ not vanishing on units, and such that $\Lambda(\mathbf{1}_A) = 1$, is multiplicative, that is, $\Lambda(ab) = \Lambda(a)\Lambda(b)$ for all $a, b \in A$. The formulation in terms of linear functionals was extended by Żelazko to any complex Banach algebra [25], thus providing a characterization of the ideals of codimension 1 (that are necessarily maximal) in a complex Banach algebra.

Definition 1.1. Let A be a unital algebra over a field $\mathbb{F}$. The algebra A is said to have the **GKŻ property** (or to be a **GKŻ algebra**) if every linear functional $\Lambda : A \to \mathbb{F}$ not vanishing on units, and such that $\Lambda(\mathbf{1}_A) = 1$, is multiplicative.

In Definition 1.1 we allow the case that $\mathbf{1}_A = \mathbf{0}_A$, that is the case that A is the zero algebra because, although we always start with a nonzero algebra, we may obtain sometimes zero algebras. By the above definition, a zero algebra is a GKŻ algebra.

The converse of the GKŻ property is obvious: if Λ is a multiplicative nonzero linear functional on A, then Λ does not vanish on units and $\Lambda(\mathbf{1}_A) = 1$.

Remark 1.2. Let A be an algebra over a field $\mathbb{F}$. Every proper subspace of A is contained in a subspace of codimension 1. Thus the maximal subspaces of A are the subspaces of codimension 1.

An algebra A satisfies the GKŻ-property if and only if each subspace of A of codimension 1 not containing units is multiplicatively closed, equivalently it is an ideal (necessarily maximal).

There is an extensive literature on extensions of the Gleason-Kahane-Żelazko Theorem. See, e.g., the surveys [6] and [13].

Definition 1.3. A function algebra over a field $\mathbb{F}$ on a non-empty set X, is a unital subalgebra of the $\mathbb{F}$-algebra $\mathbb{F}^X$ (the set of all functions $X \to \mathbb{F}$, with addition and multiplication defined componentwise) such that $\mathbf{1}_{\mathbb{F}^X} \in A$.

1.2. Summary

In Sect. 2 we deal with the GKŻ property for algebras, in Sect. 3 for function algebras, and in Sect. 4 for distribution algebras.

The main result of §2.1 (*The GKŻ property of algebras and generation by units*), is that a GKŻ algebra A over a field of at least 3 elements and having an ideal of codimension 1, is generated by units, that is, every element of A is a finite sum of units (Theorem 2.1 (1)), equivalently, $A = \operatorname{span}(\mathcal{U}(A))$. A GKŻ function algebra over a field $\mathbb{F}$ with at least 3 elements is unit generated.

In §2.2 (*Linear coverings*), we prove that if the field $\mathbb{F}$ is infinite, and $A \setminus \mathcal{U}(A)$ is contained in a union of less than $|\mathbb{F}|$ proper one-sided ideals, then the algebra A satisfies the GKŻ property. (Proposition 2.15). Thus a real or complex algebra with just countably many maximal left ideals is a GKŻ algebra. We also show that if A is a commutative algebra, then the localization A_P is a GKŻ-algebra for every prime ideal P of A. Hence the GKŻ property is not a local-global property.

In §2.3 (*The GKŻ property for homomorphic images*) we show that the class of GKŻ algebras is closed under homomorphic images (Proposition 2.18), and provide a sufficient condition for the GKŻ property of A assuming that A/I is a GKŻ algebra for an ideal I of A (Proposition 2.19). Naturally, A satisfies the GKŻ property if and only if $A/\operatorname{Jac}(A)$ does (Proposition 2.21), where $\operatorname{Jac}(A)$ is the Jacobson radical of A.

In §4.2 (*Unitisation*), we show that the unitisation of a radical algebra is a GKŻ algebra, and provide a topological sufficient condition for an algebra to be radical (Proposition 2.26).

In Sect. 3 (*On the GKŻ property of function algebras*), we prove that if a function algebra $A \subseteq \mathbb{F}^X$ over a subfield $\mathbb{F}$ of $\mathbb{C}$ contains all the bounded functions in $\mathbb{F}^X$, then each element of A is a sum of two units. If A contains also a discrete function, then A is a GKŻ algebra.

In Sect. 4 (*Algebras of distributions*), we prove that the algebra of periodic distributions, and the unitisation of the algebra of distributions with support in $(0, \infty)$ satisfy the GKŻ property, while the algebra of compactly supported distributions does not. Actually, we started our paper with the study of the GKŻ property for some distribution algebras, and this study led us to more general results.

1.3. Conventions and Notations

Unless otherwise stated, all the algebras in this paper are nonzero unital associative algebras, not necessarily commutative, over a field $\mathbb{F}$. We use the notation $\mathbf{1}_A$ for the unity of the algebra A, and $\mathcal{U}(A)$ for the group of units (invertible elements) of A. Using the $\mathbb{F}$-algebra monomorphism $\mathbb{F} \to A,\ f \mapsto f\mathbf{1}_A$, we may assume that $\mathbb{F} \subseteq A$. In the whole paper, we denote by $\mathbb{F}$ a field, by A a nonzero associative unital algebra over $\mathbb{F}$, and by X a nonempty set. Thus a function algebra is a subalgebra of $\mathbb{F}^X$ with the same unity. We let $\mathbb{F}^\bullet = \mathbb{F} \setminus \{0\}$. The cardinality of a set X is denoted by $|X|$. If Y is a subset of X, we denote by I_Y the indicator (characteristic function) of Y. If $A \subseteq \mathbb{F}^X$ is a function algebra, and $x \in X$, the projection p_x is defined as the function $p_x : A \to X$ $(f \mapsto f(x))$. The field of two elements is denoted by $\mathbb{F}_2$. A *proper* subspace of an algebra A is a subspace of A different from A, possibly zero. Similarly, we use *proper* ideals, etc. If A is an algebra and $a \in A$, then $\rho(a)$ (the *resolvent* of a) is the set of all scalars $\lambda \in \mathbb{F}$ such that $a - \lambda\mathbf{1}$ is invertible, and $\sigma(a)$ (the *spectrum*) of a is the set $\mathbb{F} \setminus \rho(a)$.

2. The GKŻ Property of General Algebras

2.1. The GKŻ Property of Algebras and Generation by Units

There are several equivalent ways to define the property of an algebra to be generated by units: as an algebra, as a ring, as a vector space (that is, $A = \operatorname{span}(\mathcal{U}(A))$, and as an additive group (which means that every element of A is a finite sum of units). For two surveys on rings generated by units, see [18] and [19, Section 1].

Theorem 2.1. *Let A be a GKŻ algebra over a field $\mathbb{F}$ of at least 3 elements. If A has an ideal of codimension 1, (equivalently, if there exists a linear functional on A not vanishing on units), then A is generated by units.*

Proof. Let Λ be a linear functional that does not vanish on units. Replacing Λ by $\frac{\Lambda}{\Lambda(\mathbf{1})}$, we may assume that $\Lambda(\mathbf{1}) = 1$. Assume that A is not generated by units, that is, $A \neq \operatorname{span}(\mathcal{U}(A))$. There exists a subspace V of A of codimension 1 containing $\mathcal{U}(A)$. Let $a \in A \setminus V$. Thus $A = Fa \oplus V$. Let $a^2 = \lambda a + w$, where $\lambda \in \mathbb{F}$, and $w \in V$. Since $\mathbb{F}$ contains at least three elements, we may choose $c \in \mathbb{F}$ such that

$$c^2 - \lambda c - \Lambda(w) \neq 0.$$

There exists a unique linear functional $\Psi : A \to \mathbb{F}$ such that $\Psi(v) = \Lambda(v)$ for all $v \in V$, and $\Psi(a) = c$. Hence

$$\Psi(a^2) = \Psi(\lambda a + w) = \lambda c + \Lambda(w) \neq c^2 = (\Psi(a))^2,$$

so Ψ is not multiplicative. On the other hand, $\Psi(u) = \Lambda(u) \neq 0$ for all $u \in \mathcal{U}(A)$, and $\Psi(\mathbf{1}) = \Lambda(\mathbf{1}) = 1$, contradicting the assumption that A has the GKŻ property. □

Corollary 2.2. *A GKŻ function algebra over a field $\mathbb{F}$ with at least 3 elements is unit generated.*

Proof. Let $A \subseteq F^X$ be a nonzero function algebra. For all $x \in X$, the projection p_x is a multiplicative functional, and $p_x(I_X) = 1$. By Theorem 2.1, A is generated by units. □

An algebra A is said to satisfy the GKŻ property *vacuously* if each linear functional on A vanishes on some unit, equivalently, if each maximal subspace of A contains a unit. Clearly, if A is a GKŻ algebra vacuously, then A is a GKŻ algebra.

Remark 2.3. In Theorem 2.1, the condition on the existence of an ideal of codimension 1 is essential, but not necessary.

Indeed, consider the $\mathbb{C}$-algebras $\mathbb{C}(x)$ and $\mathbb{C}(x)[y]$, where x and y are two independent indeterminates over $\mathbb{C}$. Both these algebras satisfy the GKŻ property vacuously by the next Lemma 2.4. However, $\operatorname{span}(\mathcal{U}(\mathbb{C}(x)) = \mathbb{C}(x)$, so the condition is not necessary, and $\operatorname{span}(\mathcal{U}(\mathbb{C}(x)[y])) = \mathbb{C}(x) \neq \mathbb{C}(x)[y]$, so the condition is essential. □

Lemma 2.4. *Let $\mathbb{F} \subsetneqq \mathbb{K}$ be two fields, and let A be an algebra over $\mathbb{K}$. Then A is a GKŻ algebra vacuously over $\mathbb{F}$.*

Proof. Let V be an $\mathbb{F}$-subspace of A of codimension 1. Since $\mathbb{F} \subsetneqq \mathbb{K}$, we have $[\mathbb{K} : \mathbb{F}] > 1$, so V contains a nonzero scalar $k \in \mathbb{K}$, and $k \in \mathcal{U}(A)$. □

Theorem 2.1 implies:

Corollary 2.5. *Let A be an algebra over a field of cardinality ≥ 3, that is not generated by units. Then A is a GKŻ algebra $\Leftrightarrow$ A is a GKŻ algebra vacuously.*

Remark 2.6. If A is an algebra, then $\operatorname{span}(\mathcal{U}(A))$ is a subalgebra of A with the same units.

Proposition 2.7. *Let $\mathbb{F}$ be a field of at least 3 elements.*

(1) The following three conditions are equivalent:
- *(a) A is a GKŻ algebra.*
- *(b) A is a GKŻ algebra vacuously.*
- *(c) $\operatorname{span}(\mathcal{U}(A))$ is a GKŻ algebra vacuously.*

(2) If $C \subsetneqq A$ are two algebras with the same units, then either both A and C are GKŻ algebras vacuously, or both of them are not GKŻ algebras.

Proof. (1) $(a) \Rightarrow (b)$ If A is not a GKŻ algebra vacuously, then by Theorem 2.1, $A = \operatorname{span}(\mathcal{U}(A))$, a contradiction. $(b) \Rightarrow (a)$ Clear. $(b) \Rightarrow (c)$ If condition (c) does not hold, then there exists a linear functional Λ on C not vanishing on units and such that $\Lambda(\mathbf{1}_C) = 1$. Thus Λ can be extended to a linear functional $\widetilde{\Lambda}$ on A. We see that $\widetilde{\Lambda}$ does not vanish on units, since $\mathcal{U}(A) = \mathcal{U}(C)$, and $\widetilde{\Lambda}(\mathbf{1}_A) = 1$, contradicting (b). $(c) \Rightarrow (b)$ If (b) does not hold, then there exists a linear functional on A not vanishing on units and preserving unity. Its restriction to C has the same properties, contradicting (c).

(2) Clearly, (1) implies (2).

□

Corollary 2.8. *If A is a GKŻ algebra, then all proper subalgebras of A with the same units, are GKŻ algebras vacuously.*

For group rings, see [14].

Remark 2.9. If A is an algebra, then $\operatorname{span}(\mathcal{U}(A))$ is a homomorphic image of the group ring $\mathbb{F}[\mathcal{U}(A)]$.

Corollary 2.10. *No proper algebra extension of a group ring $\mathbb{F}[G]$ with the same units satisfies the GKŻ property.*

Proof. This follows from Theorem 2.1, since the augmentation homomorphism of $\mathbb{F}[G]$ is a multiplicative linear functional. □

For an application of Corollary 2.10 to distribution algebras, see §4.3.

Proposition 2.11. *(1) $\mathbb{F}_2$ is the only field for which all algebras generated by units satisfy the GKŻ property.*
(2) $\mathbb{F}_2$ is the only field for which the only unit generated function algebra is the field itself (up to isomorphism).

Proof. (1) Let A be a unit generated algebra over $\mathbb{F}_2$. If $\Lambda : A \to \mathbb{F}_2$ is a linear functional not vanishing on units, then $\Lambda(u) = 1$ for every unit u in A. Since $A = \text{span}(\mathcal{U}(A))$, we obtain that Λ is multiplicative, implying that A is a GKŻ algebra. On the other hand, if $|\mathbb{F}| > 2$, let $c \in \mathbb{F} \setminus \{0, 1\}$. Consider the polynomial ring $\mathbb{F}[x]$. Let $\Lambda : \mathbb{F}[x] \to \mathbb{F}$ be the unique linear functional over $\mathbb{F}$, such that $\Lambda(x^n) = 1$ for all nonnegative integers $n \neq 2$, and $\Lambda(x^2) = c$. Clearly, Λ is not multiplicative, implying that $\mathbb{F}[x]$ is not a GKŻ algebra.

(2) Let $A \subseteq \mathbb{F}_2^X$ be a function algebra. The only unit of $\mathbb{F}_2^X$ is I_X. Hence $\text{span}(\mathcal{U}(A)) = \mathbb{F}_2 I_X$. It follows that $A = \text{span}(\mathcal{U}(A)) \Leftrightarrow A = \mathbb{F}_2 I_X$. On the other hand, if $|\mathbb{F}| > 2$, then every GKŻ algebra over $\mathbb{F}$ is unit generated by Theorem 2.1, and by (1), there are GKŻ algebras A that are not isomorphic to $\mathbb{F}$.

□

As shown by Vamos [21, page 418], each element of a real or complex Banach algebra is a sum of two units, as an immediate consequence of the spectral theorem. Analogously, we have:

Remark 2.12. Let A be an $\mathbb{F}$-algebra. Let $a \in A$ such that $\rho(a) \neq \emptyset$. Then a is a sum of at most 2 units. Hence, if every element of A has a non-empty resolvent, then each element of A is a sum of at most 2 units.

Indeed, let $\lambda \in \rho(a)$. If $\lambda = 0$, then a is invertible. If $\lambda \neq 0$, then $a = (a - \lambda\mathbf{1}) + \lambda\mathbf{1}$, is a sum of two units. □

2.2. Linear Coverings

Remark 2.13. A one-sided ideal V of codimension 1 of the algebra A is a maximal ideal.

Indeed, V is multiplicatively closed, so V is an ideal. □

Lemma 2.14. *Let V be a subspace A of codimension 1 that is not an ideal. Then there exist a finitely generated subspace V_0 of V such that $AV_0 = V_0 A = A$.*

Proof. By Remark 2.13, V is not a left ideal, so $V \subsetneqq AV$, implying that $AV = A$. Similarly, $VA = A$. Hence there exist finitely generated subspaces V_1 and V_2 of V such that $AV_1 = V_2 A = A$. Set $V_0 = V_1 \cup V_2$. □

Proposition 2.15. *Let A be an $\mathbb{F}$-algebra. Then A is a GKŻ algebra under each of the following conditions:*

(1) $A \backslash \mathcal{U}(A)$ is a contained in a union of at most two proper one-sided ideals.

(2) $\mathbb{F}$ is infinite, and $A \setminus \mathcal{U}(A)$ is contained in a union of less than $|\mathbb{F}|$ proper one-sided ideals.

(3) $\mathbb{F}$ is infinite, and $A \setminus \mathcal{U}(A)$ is contained in a union of finitely many proper one-sided ideals.

Proof. Let V be a subspace of A of codimension 1 that contains no units.

(1) We have $V \subseteq A \setminus \mathcal{U}(A) \subseteq I_1 \cup I_2$, where I_1, I_2 are two one-sided ideals of A, not necessarily distinct. Viewing A as an additive group, we obtain that $V \subseteq I_k$ for some $k = 1, 2$. Since $\operatorname{codim} V = 1$, we see that $V = I_k$, so V is an ideal by Remark 2.13. Hence A is a GKŻ algebra.

(2) By Lemma 2.14, there exists a finitely generated subspace V_0 of V such that $AV_0 = V_0 A = A$. Since V_0 is contained in a union of less than $|\mathbb{F}|$ proper one-sided ideals, it follows from [10, Theorem 1.2] or from [1, Main Theorem], that V_0 is contained in one of the one-sided ideals I in this union. Hence $V = I$, so V is an ideal by Remark 2.13, and A is a GKŻ algebra.

(3) is a particular case of (2).

□

Corollary 2.16. *Let $A \subseteq \mathbb{F}^X$ be a function algebra. Assume that $|X| < |\mathbb{F}|$, and that each $f \in A$ such that $f(x) \neq 0$ for all $x \in X$ is invertible in A. Then A is a GKŻ-algebra. In particular, if $|X| < |\mathbb{F}|$, then $\mathbb{F}^X$ is a GKŻ algebra. More particularly, if $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and X is countable, then A is a GKŻ-algebra.*

Proof. We have $A \setminus \mathcal{U}(A) \subseteq \bigcup_{x \in X} p_x^{-1}(0)$. Hence A is a GKŻ algebra by Proposition 2.15. □

Proposition 2.17. *Let A be an algebra over a field $\mathbb{F}$.*

(1) If A has at most two maximal left (or at most two maximal right) ideals, then A satisfies the GKŻ property.

(2) Let A be an algebra over an infinite field $\mathbb{F}$. If the cardinality of the set of left ideals, or of the right ideals, is $< |\mathbb{F}|$, then A is a GKŻ algebra.

(3) If the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and A has just countably many left ideals, then A is a GKŻ algebra.

(4) If A is commutative, then the localization A_P is a GKŻ-algebra for every prime ideal P of A. Hence the GKŻ property is not a local-global property.

(5) Assume that A is commutative and that the field $\mathbb{F}$ is infinite. Let $\mathcal{P}$ be a set of less than $|\mathbb{F}|$ prime ideals. Then the algebra $B = \bigcap_{P \in \mathcal{P}} A_P$ satisfies the GKŻ property.

Proof. Items (1), (2) and (3) immediately follow from Proposition 2.15.

(4) A_P has just one maximal ideal, so it a GKŻ-algebra by (1).

(5) This follows from Proposition 2.15 since the set of non-units of the algebra B is contained in $\bigcup_{P \in \mathcal{P}} P A_P$.

□

2.3. The GKŻ Property for Homomorphic Images

We show first that the class of GKŻ algebras is closed under homomorphic images (Proposition 2.18). Equivalently, if I is an ideal of a GKŻ algebra A, then also A/I is a GKŻ algebra. Then we deal with the converse, and show that under suitable assumptions, if A/I is a GKŻ algebra for a certain ideal I, then A is a GKŻ algebra.

Proposition 2.18. *A homomorphic image of a GKŻ algebra is a GKŻ algebra.*

Proof. We have to show that if I is an ideal of a GKŻ algebra A, then A/I is a GKŻ algebra. Let W be a subspace of A/I of codimension 1 that contains no units. Hence $W = V/I$, where V is a subspace of A of codimension 1, that contains I, but no units modulo I. We infer that V contains no units of A, implying that V is an ideal of A, since A is a GKŻ algebra. Hence $W = V/I$ is an ideal of A/I. It follows that A/I is a GKŻ algebra. □

For a partial converse of Proposition 2.18, see the next Proposition 2.19, Recall that if I is an ideal of A, an element $t \in A/I$ is *liftable* to a unit in A if t contains a unit in A, that is, $t + I = u + I$ for some $u \in \mathcal{U}(A)$.

Proposition 2.19. *Let A be an algebra over $\mathbb{F}$.*

(1) Let I be an ideal of A. Then A/I is a GKŻ algebra $\Leftrightarrow$ each maximal subspace of A containing I, but no units modulo I, is an ideal.

(2) If each unit of A/I is liftable to A, then A is a GKŻ algebra $\Leftrightarrow$ A/I is a GKŻ algebra.

Proof. (1) $\Rightarrow$ See Proposition 2.18. $\Leftarrow$ Let V be a maximal subspace of A containing I, but no units modulo I. Thus V/I is a maximal subspace of A/I that contains no units. Hence V/I is an ideal of A/I, implying that V is an ideal of A.

(2) follows from (1) since in this case a subspace of A containing I, contains no units in A if and only if it contains no units modulo I.

□

By Proposition 2.17 (1), if A is a unital algebra with a unique maximal left ideal J, then A satisfies the GKŻ property. For another generalization, see Proposition 2.21 (2) below. Indeed, in this case $J = \text{Jac}(A)$, the Jacobson radical of A. Recall that $\text{Jac}(A)$ is the intersection of all maximal left ideals of A, and that in this definition 'left ' can be changed to 'right', so $\text{Jac}(A)$ is a two-sided ideal. For the Jacobson radical theory, see e.g., [5, section 13B], [9, Part II, §1], and [11, Chapter 2.4].

Proposition 2.20. *Each maximal subspace V of A without units contains* $\text{Jac}(A)$*, so* $\text{Jac}(A)$ *is the intersection of all maximal subspaces of A without units.*

Proof. There exists a unique linear functional Λ on A such that $\ker \Lambda = V$ and $\Lambda(\mathbf{1}) = 1$. Let $c \in \text{Jac}(A)$. We have $\Lambda(c) \in \sigma(c) = \{0\}$. Hence $\Lambda(c) = 0$, so $c \in V$. □

Proposition 2.21. *If I is an ideal contained in* Jac(A)*, then A is a GKŻ algebra $\Leftrightarrow$ A/I is a GKŻ algebra.*

Proof. This follows from Proposition 2.19, since $I \subseteq \mathrm{Jac}(A)$, so an element of A is a unit in A if and only if it is a unit modulo I. □

Proposition 2.22. *Let I be an ideal of the algebra A. Assume that each maximal subspace of A that does not contain units and does not contain I, is an ideal in A. Equivalently, assume that if Λ is a linear functional such that $\Lambda(u) \neq 0$ for all $u \in \mathcal{U}(A)$, $\Lambda(\mathbf{1}_A) = 1$ and $\Lambda(I) \neq (0)$, then Λ is multiplicative. Then A is a GKŻ algebra $\Leftrightarrow$ A/I is a GKŻ algebra.*

Proof. In view of Proposition 2.18, we have to prove just the implication $\Leftarrow$. Let V be a maximal subspace of A that does not contain units and does not contain I. If V is not an ideal, then $I \subseteq V$ and V/I is not an ideal in A/I, a contradiction. Thus every maximal subspace of A is an ideal, so A is a GKŻ algebra. □

Remark 2.23. If A contains an element a with empty spectrum, then A is GKŻ algebra vacuously.

Indeed, if Λ is a linear functional that does vanish on units such that $\Lambda(\mathbf{1}) = 1$, then $\Lambda(a) \in \sigma(a) = \emptyset$, a contradiction. □

Proposition 2.24. *Let I be an ideal of a generated by units algebra A satisfying the following two conditions:*

(1) There exists an element $c \in A$ such that $c + \lambda\mathbf{1}$ is invertible modulo I for all $\lambda \in \mathbb{F}$.
(2) Each unit in A/I is liftable to a unit in A.

Then A is a GKŻ algebra.

Proof. By (1), the element $c + I$ has an empty spectrum in A/I. Hence, A/I is a GKŻ algebra by Remark 2.23. By Proposition 2.19, A is a GKŻ algebra. □

2.4. Unitisation

Recall that the *unitisation* of an $\mathbb{F}$-algebra A (not necessarily unital) is the algebra $B = \mathbb{F} \oplus A$ (a direct sum of additive groups) with multiplication defined by $(\lambda + a)(\mu + b) = \lambda\mu + \lambda b + \mu a + ab$ for $\lambda, \mu \in \mathbb{F}$, and $a, b \in A$. We identify $\mathbb{F}$ with the subfield $\mathbb{F}\mathbf{1}_B$ of B.

For the next proposition recall that that if I is an ideal of an algebra A, not necessarily unital, then $J \subseteq \mathrm{Jac}(A)$ if and only if each element $a \in I$ is left quasiregular, that is, $1+a$ is left invertible in the unitisation of A, equivalently, there exists an element $b \in A$ such that $a+b+ab = 0$ (see e.g., [22]). Moreover, an ideal I of A is contained in Jac(A) if and only if each element of I is left invertible. Here, 'left' can be replaced by 'right' or omitted ([9] and [5]). Hence, a non zero non-unital algebra l A is *radical*)(that is, $A = \mathrm{Jac}(A)$), if and only if each element of A is quasiregular.

Lemma 2.25. *Let A be a nonzero $\mathbb{F}$-algebra, not necessarily unital, and let B the unitisation of A. The following conditions are equivalent:*

(1) A is radical algebra.
(2) $A = \mathrm{Jac}(B)$.
(3) $1 + A \subseteq \mathcal{U}(B)$ (here $1 \in \mathbb{F} \subseteq B$, so $1 \notin A$).
(4) $\mathcal{U}(B) = \mathbb{F}^{\bullet}(1 + A)$ (a direct product of multiplicative groups).
(5) $\mathcal{U}(B) = B \setminus A$.

Proof. We have:
$A = \mathrm{Jac}(A) \Leftrightarrow$ each element of A is quasiregular $\Leftrightarrow A = \mathrm{Jac}(B) \Leftrightarrow (3)$.

Thus the first three conditions are equivalent. Since $B = F + A$, we see that the last three conditions are also equivalent. □

Proposition 2.26. *The unitisation of a a radical algebra is a GKŻ algebra. Moreover, A is a radical algebra if there exists a Hausdorff topology on A such that $\lim_{n\to\infty} a^n = 0$ in this topology for all $a \in A$.*

Proof. Assume that A is a radical algebra, and B is the unitisation of A. By Lemma 2.25, $A = \mathrm{Jac}(B)$. Since $B/\mathrm{Jac}(B) = B/A \cong \mathbb{F}$, it follows that B satisfies the GKŻ property by Proposition 2.21. Alternatively, it follows from Lemma 2.25 that $\mathcal{U}(B) = B \setminus A$, so A is the unique maximal one-sided ideal of A. By Proposition 2.15, B is a GKŻ algebra.

Assume that A has a Hausdorff topology as described. Let $a \in A$. Since $\lim_{n\to\infty} a^n = 0$, the geometric series $\sum_{n=0}^{\infty}$ converges in A, and its sum s satisfies in B: $s(1-a) = (1-a)s = 1$, so $1 - a$ is invertible in B. By Lemma 2.25, B is a GKŻ-algebra. □

For an application of Proposition 2.26 to distribution algebras see §4.2.

3. On the GKŻ Property of Function Algebras

In this section we investigate the GKŻ property for function algebras, using ideas related to connections between ideals of a commutative ring and ultrafilters, although we do not use ultrafilters explicitly. Instead of ideals, we use maximal subspaces with no units. For ideals and ultrafilters see the classical book [3], for later results see [12], and for the basics of ultrafilter theory see [23].

Let A be an algebra over a subfield $\mathbb{F}$ of $\mathbb{C}$. We prove that if $\mathbb{F}$ contains all the bounded functions in $\mathbb{F}^X$, then each element of A is a sum of two units (Proposition 3.4). If A contains also a discrete function, then A is a GKŻ algebra (Corollary 3.12). The condition that A contains all bounded functions is implied by the following condition: if $f \in A$, $g \in \mathbb{F}^X$, and $|g| \le |f|$ (that is, $|g(x)| \le |f(x)|$ for all $x \in X$), then $g \in A$.

For the next Lemma 3.1, see [24] and especially Eric Wofsey's answer. In this lemma, the implications (1) ⇒ (2) ⇒ (3) are easily proved. Nevertheless, we indicate all the relevant implications.

Lemma 3.1. *Let A be a commutative algebra, and let Λ be a linear functional on A such that $\Lambda(u) \neq 0$ for all $u \in \mathcal{U}(A)$ and $\Lambda(\mathbf{1}) = 1$. Let e be an idempotent in A. We have:*

(1) Let $u, v \in \mathcal{U}(A)$. Then
   (a) $ue + v(1-e) \in \mathcal{U}(A)$.
   (b) Exactly one of the two scalars $\Lambda(ue)$ and $\Lambda(v(1-e))$ vanishes.
(2) (a) $\Lambda(e) = 0, 1$.
   (b) For all $u \in \mathcal{U}(A)$, $\Lambda(ue) = 0$ if $\Lambda(e) = 0$, and $\Lambda(u) = \Lambda(ue) \neq 0$ if $\Lambda(e) = 1$.
   (c) $\Lambda(ue) = \Lambda(u)\Lambda(e)$ for all $u \in \mathcal{U}(A))$ and idempotents $e \in A$.
(3) If A is generated by units, then
   (a) $\Lambda(ae) = 0$ for all $a \in A$ and idempotents e in $\ker\Lambda$, so $\ker\Lambda$ contains the ideal generated by the idempotents it contains.
   (b) $\Lambda(ae) = \Lambda(a)\Lambda(e)$ for all $a \in A$ and idempotents $e \in A$.

Proof. (1) (a) $(ue+v(1-e)(u^{-1}e+v^{-1}(1-e)) = 1$, so $ue+v(1-e) \in \mathcal{U}(A)$.
   (b) Since $\Lambda(e + (1-e)) = 1$, the scalars $\Lambda(e)$ and $\Lambda(1-e)$ cannot both vanish. Suppose that both of these scalars are nonzero. Then

$$\Lambda\left(\Lambda(v(1-e))u(e) - \Lambda(ue)v(1-e)\right) = 0,$$

   although $\Lambda(v(1-e))u(e) - \Lambda(ue)v(1-e) \in \mathcal{U}(A)$ by (1) (a), a contradiction.
(2) (a) follows from (1)(b) applied to $u = v = 1$.
   (b) follows from (1)(b) applied to u and to $v = 1$.
   (c) follows from (2)(b).
(3) (a) follows from (2)(b).
   (b) follows from (2)(c).

□

Proposition 3.2. *Let A be a function algebra that is generated by units. Suppose that A contains the indicator functions of all subsets of X. Let Λ be a linear functional such that $\Lambda(u) \neq 0$ for all $u \in \mathcal{U}(A)$ and $\Lambda(\mathbf{1}) = 1$. If $\Lambda(I_{\{x_0\}}) \neq 0$ for some $x_0 \in X$, then $\Lambda = p_{x_0}$. Hence $\Lambda(I_x) = 0$ for all $x \neq x_0$ in X.*

Proof. Suppose that $\Lambda(I_{\{x_0\}}) \neq 0$ for some $x_0 \in X$. Since $\Lambda(I_{\{x_0\}}) = 1$ by Lemma 3.1(2)(a), we obtain by Lemma 3.1 (3) (b) for every $f \in A$: $\Lambda(f)\Lambda(I_{\{x_0\}}) = \Lambda(f\ I_{\{x_0\}}) = \Lambda(f(x_0)I_{\{x_0\}}) = f(x_0)\Lambda(I_{\{x_0\}}) = f(x_0) = p_{x_0}(f)$. □

Remark 3.3. If a function algebra A on a set X contains the indicators of all subsets of X, then we may use definition by cases to obtain a function in A.

More precisely, given a partition of X into n disjoint subsets $S_1, \ldots, S_n$, and functions $f_1, \cdots, f_n$ in A, then the function $f = \sum_{i=1}^{n} f_i I_{S_i}$ belongs to A.
□

Proposition 3.4. *Let A be a function algebra over a subfield $\mathbb{F}$ of $\mathbb{C}$. Assume that A contains all the bounded functions in $\mathbb{F}^X$. Then every element of A is a sum of two units.*

Proof. Let $f \in A$. We define two functions g and h in $\mathbb{F}^X$ as follows:

$$g(x) = \begin{cases} f(x) + 3 & \text{if } |f(x)| \le 2 \\ f(x) - 1 & \text{if } |f(x| > 2 \end{cases}$$

$$h(x) = \begin{cases} \frac{1}{f(x)+3} & \text{if } |f(x)| \le 2 \\ \frac{1}{f(x)-1} & \text{if } |f(x)| > 2 \end{cases}$$

We have $g \in A$ by Remark 3.3, and $h \in A$ since h is a bounded function. Also $gh = I_X$, so g is invertible. We have $f = (f - g) + g$, and $f - g$ is invertible since $\frac{1}{f-g}$ is bounded. Hence f is a sum of two units in A. □

Corollary 3.5. *Let A be a function algebra over a subfield $\mathbb{F}$ of $\mathbb{C}$. Assume that if $f \in A, g \in \mathbb{F}^X$, and $|g| \le |f|$, then $g \in A$. Then A contains all the bounded functions in $\mathbb{F}^X$, so every element of A is a sum of two units.*

Proof. Let g be a bounded function in $\mathbb{F}^X$. Let n be a positive integer such that $|g(x)| \le n$ for all $x \in X$. Thus $|g| \le nI_X$, implying that $g \in A$. By Proposition 3.4, every element of A is a sum of two units in A. □

Remark 3.6. In the setting of the next Proposition 3.7, the ideal L is the set generated by the indicators of all finite subsets of X. Thus L consists of all functions $t \in A$ such that $t(x) = 0$ for all $x \in X$, except finitely many elements, that is, the set $t^{-1}(0)$ is cofinite in X.

Proposition 3.7. *Let $A \subseteq \mathbb{F}^X$ be a function algebra that contains the indicators of all singletons of X, and let L be the ideal of A generated by these indicators. Then*

(1) A is a GKŻ algebra $\Leftrightarrow$ A/L is a GKŻ algebra (Here A/L is the zero algebra when $L = A$).
(2) If there exists in A an element f such that $f - \lambda I_X$ is invertible modulo L for every scalar $\lambda \in \mathbb{F}$, then the only linear functionals Λ on A such that $\Lambda(u) \neq 0$ for all $u \in \mathcal{U}(A)$ and $\Lambda(\mathbf{1}) = 1$, are the projections p_x for $x \in X$. Thus A is a GKŻ algebra.

Proof. (1) By Proposition 3.2, the only linear functionals Λ on A such that $\Lambda(u) \neq 0$ for all $u \in \mathcal{U}(A)$, $\Lambda(\mathbf{1}) = 1$, and $\Lambda(L) \neq (0)$ are the projections p_x for $x \in X$. Hence, A is a GKŻ algebra $\Leftrightarrow$ A/L is a GKŻ algebra.

(2) By assumption, the spectrum of $f + L$ in A/L ie empty, so A/L is a GKŻ algebra by Remark 2.23. By (1), A is a GKŻ algebra.

□

Remark 3.8. Here is an alternative proof of Proposition 3.7 using Proposition 2.24.

By Proposition 3.7, it is enough to show that if $f \in A$ and f is a unit modulo L, then there exists $f_0 \in \mathcal{U}(A)$ such that $f - f_0 \in L$. By assumption, there exists $g \in A$ such that $fg = 1 + t$ for some $t \in L$. Define for $x \in X$:

$$f_0(x) = \begin{cases} f(x) & \text{if } t(x) = 0 \text{ (that is, if } f(x)(g(x) = 1) \\ 1 & \text{if otherwise} \end{cases}$$

Define g_0 similarly. Clearly, $f_0 g_0 = I_X$, and if $t(x) = 0$ for some $x \in X$, then $(f - f_0)(x) = 0$, so by Remark 3.6, $f - f_0 \in L$, as required. □

Corollary 3.9. *If the set X is finite, then a subspace of $\mathbb{F}^X$ that contains the indicators of all singletons of X is equal to $\mathbb{F}^X$. Hence, by Proposition 3.7, $\mathbb{F}^n$ is a GKŻ algebra for all positive integers n.*

Let $\mathbb{F}$ be a subfield of $\mathbb{C}$. A function $f \in \mathbb{F}^X$ is called *discrete* if the set $f(X)$ is discrete in $\mathbb{C}$, and the sets $f^{-1}(\lambda)$ are finite for all $\lambda \in \mathbb{C}$.

Remark 3.10. If $\mathbb{F}^X$ contains a discrete function, then $|X| = \aleph_0$. On the other hand, if $X = \{x_n \ (n \in \mathbb{N})\}$, where the elements x_n are distinct, and $f \in \mathbb{F}^X$, then f is discrete if and only if $\lim_{n\to\infty} |f(x_n)| = \infty$.

Theorem 3.11. *Let $A \subseteq \mathbb{F}^X$ be a unit generated function algebra over a field $\mathbb{F}$ contained in $\mathbb{C}$. Assume that A contains the indicators of all subsets of X, and a discrete function f. Then the only linear functionals Λ on A such that $\Lambda(u) \neq 0$ for all $u \in \mathcal{U}(A)$ and $\Lambda(\mathbf{1}) = 1$, are the projections p_x for $x \in X$. Thus A is a GKŻ algebra.*

Proof. By Proposition 3.7 and in the same notation, it is enough to show that $f - \lambda\mathbf{1}$ is invertible modulo L for all $\lambda \in \mathbb{F}$. Define the function $f_\lambda : X \to \mathbb{C}$ as follows:

$$f_\lambda(x) = \begin{cases} 1 & \text{if } f(x) = \lambda I_X \\ f(x) - \lambda I_X & \text{if } f(x) \neq \lambda I_X \end{cases}$$

Since f is discrete, $f(x) = \lambda I_X$ just for finitely many x's. Hence

$$(f(x) - \lambda I_X) - f_\lambda \in L.$$

Since the function f_λ is discrete, there exists $m = |\min f_\lambda(X)|$, and $m > 0$, implying that $|\frac{1}{f_\lambda}(x)| \leq \frac{1}{m}$, so $\frac{1}{f_\lambda} \in A$, and f_λ is invertible in A. Since $f(x) - \lambda I_X \equiv f_\lambda \pmod{L}$, we infer that $f(x) - \lambda I_X$ is invertible modulo L as required. □

Corollary 3.12. *Let $A \subseteq \mathbb{F}^X$ be a function algebra over a subfield $\mathbb{F}$ of $\mathbb{C}$. Assume that A contains all the bounded functions in $\mathbb{F}^X$, and a discrete function f. Then every element of A is a sum of two units, the only linear functionals Λ on A such that $\Lambda(u) \neq 0$ for all $u \in \mathcal{U}(A)$ and $\Lambda(\mathbf{1}) = 1$, are projections. Thus A is a GKŻ algebra.*

Proof. A contains the indicators of all subsets of X since the indicators are bounded functions. By Proposition 3.4, every element of A is a sum of two units. We conclude the proof by Theorem 3.11. □

Corollary 3.13. *Let $A \subseteq \mathbb{F}^X$ be a function algebra over a subfield $\mathbb{F}$ of $\mathbb{C}$. Assume that if $f \in A, g \in \mathbb{F}^X$, and $|g| \le |f|$, then $f \in A$, and that A contains a discrete function. Then every element of A is a sum of two units, A contains all the bounded functions in $\mathbb{F}^X$, and the only linear functionals Λ on A such that $\Lambda(u) \neq 0$ for all $u \in \mathcal{U}(A)$ and $\Lambda(\mathbf{1}) = 1$, are the projections p_x for $x \in X$. Thus A is a GKŻ algebra.*

Proof. By Corollary 3.5, A contains all bounded functions. We conclude the proof by Corollary 3.12. □

Corollary 3.14. *Let $\mathcal{H}$ be a set of functions in $[2, \infty)^X$ satisfying the following conditions:*

(1) For each $h_1, h_2 \in \mathcal{H}$ there exists $h \in \mathcal{H}$ such that for all $x \in X$, $h_1(x) + h_2(x) \le h(x)$.

(2) For each $h_1, h_2 \in \mathcal{H}$ there exists $t \in \mathcal{H}$ such that for all $x \in X$, $h_1(x)h_2(x) \le t(x)$.

Let $A = \{f \in \mathbb{F}^X \mid \exists h \in \mathcal{H}$ such that $\forall x \in X,\ |f(x)| \le h(x)\}$. We have:

(a) A is a unital $\mathbb{F}$-subalgebra of $\mathbb{F}^X$ containing $\mathcal{H}$.

(b) Every element of A is a sum of two units.

(c) If A contains a discrete function, then the only linear functionals Λ on A such that $\Lambda(u) \neq 0$ for all $u \in \mathcal{U}(A)$ and $\Lambda(\mathbf{1}) = 1$, are projections. Thus A is a GKŻ algebra.

Proof. (a) By item (1), we obtain inductively that for every $f \in A$ and every integer $n \ge 1$, we have $|nf| \le h$ for some $h \in \mathcal{H}$, so $\lambda f \in A$ for all $\lambda \in \mathbb{F}$. It is now easy to complete the proof of (a). We conclude the proof using Corollary 3.5. □

For an application of Corollary 3.14 to distribution algebras see §4.1.

4. The GKŻ Property for Algebras of Distributions

As usual, $\mathcal{D}(\mathbb{R})$ is the space of test functions (compact supported complex-valued infinitely differentiable functions on $\mathbb{R}$), and $\mathcal{D}'(\mathbb{R})$ is the space of distributions on $\mathbb{R}$. We study three subspaces of $\mathcal{D}'(\mathbb{R})$ that are algebras with convolution as multiplication.

4.1. Algebra of Periodic Distributions

For background on periodic distributions and its Fourier series theory, we refer the reader to [20, p. 527-529]. For $v \in \mathbb{R}$, the *translation operator* $\mathbf{S}_v : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$, is given by

$$\langle \mathbf{S}_v t, \varphi \rangle = \langle t, \varphi(\cdot + v) \rangle \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}), \text{ and for all } t \in \mathcal{D}'(\mathbb{R}).$$

A distribution $t \in \mathcal{D}'(\mathbb{R})$ is called *periodic with period* $v \in \mathbb{R} \setminus \{\mathbf{0}\}$ if $\mathbf{S}_v t = t$. We define $\mathcal{D}'_v(\mathbb{R})$ to be the set of the periodic distributions with period v. As is well known, $\mathcal{D}'_v(\mathbb{R})$ is a complex algebra with convolution as multiplication. Moreover, using Fourier transforms, one obtains that $\mathcal{D}'_v(\mathbb{R})$ is isomorphic as a complex algebra to the algebra $\mathcal{S}'(\mathbb{Z})$ of all complex-valued maps on $\mathbb{Z}$ of at most polynomial growth, that is,

$$\mathcal{S}'(\mathbb{Z}) := \left\{ \mathbf{a} : \mathbb{Z} \to \mathbb{C} \;\middle|\; \begin{array}{l} \exists \text{ an integer } k \geq 1 \text{ such that} \\ \forall n \in \mathbb{Z},\ |\mathbf{a}(n)| \leq (2+|n|)^k \end{array} \right\}.$$

Since $\mathcal{S}'(\mathbb{Z})$ is a function algebra satisfying the assumptions of Theorem 3.14, by letting $\mathcal{H}$ to be the set of all functions $h : \mathbb{Z} \to [2, \infty)$ of the form $h(n) = (2 + |n|)^k$ for all integers n, where $k \geq 1$ is an integer, we obtain

Proposition 4.1. *All the linear functionals on the complex function algebra $\mathcal{S}'$ not vanishing on units and preserving unity, are projections. Hence the algebra $\mathcal{D}'_v(\mathbb{R})$ of periodic distributions with period $v \in \mathbb{R} \setminus \{0\}$ satisfies the GKŻ property.*

4.2. The Algebra $\mathbb{C}\boldsymbol{\delta_0} + \boldsymbol{\mathcal{D}'}_+(\mathbb{R})$

Let $\mathcal{D}'_+(\mathbb{R})$ denote the set of all distributions $t \in \mathcal{D}'(\mathbb{R})$ having their distributional support supp t contained in the half line $(0, \infty)$. Then $\mathcal{D}'_+(\mathbb{R})$ is an algebra without unity, with convolution as multiplication. Let δ_0 denote the Dirac distribution supported at $0 \in \mathbb{R}$. Clearly, $\mathcal{A} := \mathbb{C}\delta_0 + \mathcal{D}'_+(\mathbb{R})$ is a unital subalgebra of $D'(\mathbb{R})$, and $\mathcal{A}$ is isomorphic to the unitisation of $\mathcal{D}'_+(\mathbb{R})$.

Proposition 4.2. *The unique linear functional on $\mathcal{A}$ not vanishing on units and preserving unity, is the functional induced by the canonical homomorphism $\mathcal{A} \to \mathcal{A}/\mathcal{D}'_+(\mathbb{R}) = \mathbb{C}$. Hence the algebra $\mathbb{C}\delta_0 + \mathcal{D}'_+(\mathbb{R})$ satisfies the GKŻ property.*

Proof. Since $\mathcal{A}$ is a subalgebra of the topological algebra $\mathcal{D}'_+(\mathbb{R})$, by Proposition 2.26, it is enough to prove that $\lim_{n\to\infty} a^{*n} = 0$ for all $a \in \mathcal{A}$. This follows from that supp $(s^{*n}) \subseteq n\,$supp s for all positive integers n, and so, for any test function $\varphi \in \mathcal{D}(\mathbb{R})$, (supp $(s^{*n})) \cap$ (supp $\varphi) \neq \emptyset$ for only finitely many $n \in \mathbb{N}$. □

4.3. The Algebra $\boldsymbol{\mathcal{E}'}(\mathbb{R})$

Let $\mathcal{E}'(\mathbb{R})$ denote the space of all distributions $t \in \mathcal{D}'(\mathbb{R})$ that have compact support. Thus $\mathcal{E}'(\mathbb{R})$ is an algebra with convolution as multiplication.

The Dirac distribution with support equal to $\{a\}$, where $a \in \mathbb{R}$, will be denoted by δ_a. We let $G = \{\delta_a \mid a \in \mathbb{R}\}$. Thus G is a group isomorphic to $(\mathbb{R}, +)$ by the map $\mathbb{R} \to G$ $(a \to \delta_a)$.

For the sake of completeness, in the next Proposition 4.3, we reproduce the characterization of units in $\mathcal{E}'$ and its proof from [15]:

Proposition 4.3. *[15, Proposition 4.2] $\mathcal{U}(\mathcal{E}'(\mathbb{R})) = \mathbb{C}^\bullet H$ (a direct product of two groups).*

Proof. Clearly, $\mathbb{C}^\bullet H \subseteq \mathcal{U}(\mathcal{E}'(\mathbb{R}))$. For the converse inclusion, suppose that t is invertible in $\mathcal{E}'(\mathbb{R})$. Then there exists a distribution $s \in \mathcal{E}'(\mathbb{R})$ such that $t * s = \delta_0$. By the theorem on supports [7, Theorem 4.3.3, p.107], we have

$$\text{c.h.supp}(t * s) = \text{c.h.supp}(t) + \text{c.h.supp}(s),$$

where, for a distribution $a \in \mathcal{E}'(\mathbb{R})$, the notation c.h.supp$(a)$ is used for the closed convex hull of supp a, that is, the intersection of all closed convex sets containing supp a. So we obtain

$$\{0\} = \text{c.h.supp}(\delta_0) = \text{c.h.supp}(t * s) = \text{c.h.supp}(t) + \text{c.h.supp}(s),$$

from which it follows that c.h.supp$(t) = \{a\}$ and c.h.supp$(s) = \{-a\}$ for some $a \in \mathbb{R}$. But then also supp $t = \{a\}$ and supp $s = \{-a\}$. As distributions with support in a point p are linear combinations of δ_p and its derivatives $\delta_p^{(n)}$ [20, Theorem 24.6, p.266], t and s have the form

$$t = \sum_{n=0}^{N} t_n \delta_a^{(n)}, \text{ and } s = \sum_{m=0}^{M} s_m \delta_{-a}^{(m)},$$

for some integers $N, M \geq 0$ and $t_n, s_m \in \mathbb{C}$ $(0 \leq n \leq N,\ 0 \leq m \leq M)$. Then $t * s = \delta_0$ implies that $N = M = 0$ and $t_0 s_0 = 1$, since for each $a \in \mathbb{R}$, the elements $\delta_c^{(n)}$, where $c \in \{0, -a, a\}$ and $n \geq 1$ is an integer, are linearly independent over $\mathbb{C}$ (if $a = 0$, then $c = 0$). It follows that $t_0 \neq 0$. Thus we have that $t = t_0 \delta_a \in \{c\delta_p : p \in \mathbb{R},\ 0 \neq c \in \mathbb{C}\}$. This completes the proof. □

Proposition 4.4. *$\mathcal{E}'(\mathbb{R})$ is not a GKŻ algebra.*

Proof. Using Proposition 4.3, we see that the distributions in span$(\mathcal{U}(\mathcal{E}'(\mathbb{R})))$ have finite support, so span$(\mathcal{U}(\mathcal{E}'(\mathbb{R})))$ is properly contained in $\mathcal{E}'(\mathbb{R})$. Since the distributions δ_a $(a \in \mathbb{R})$ are linearly independent over $\mathbb{C}$, it follows that the algebra span$(\mathcal{U}(\mathcal{E}'(\mathbb{R})))$ is isomorphic to the group ring $\mathbb{C}[G]$. From Corollary 2.10 it follows that $\mathcal{E}'(\mathbb{R})$ is not a GKŻ algebra. □

We can prove directly that $\mathcal{E}'(\mathbb{R})$ is not a GKŻ algebra.

Let φ be a function in $C^\infty(\mathbb{R})$, and let $\Lambda = \Lambda_\varphi : \mathcal{E}'(\mathbb{R}) \to \mathbb{C}$ be the functional $\Lambda(t) = \langle t, \varphi \rangle$ for all $t \in \mathcal{E}'(\mathbb{R})$. We have for all $a \in \mathbb{R}$: $\Lambda(\delta_a) = \varphi(a)$. For all $a, b \in \mathbb{R}$: $\Lambda(\delta_a) \cdot \Lambda(\delta_b) = \Lambda(\delta_{a+b}) = a + b$, and $\Lambda(\delta_a)\Lambda(\delta(b)) = \varphi(ab)$. Hence we have:

(1) Λ preserves units if and only if $\varphi(a) \neq 0$ for all $a \in \mathbb{R}$.
(2) $\Lambda(\delta_0) = 1$ if and only if $\varphi(0) = 1$.
(3) $\Lambda(uv) = \Lambda(u)(v)$ for all $u, v \in \mathcal{U}(\mathcal{E}'(\mathbb{R}))$ if and only if $\varphi(a+b) = \varphi(a)\varphi(b)$ for all $a, b \in \mathbb{R}$.

The only continuous functions φ with range contained in $\mathbb{R}$ satisfying the conditions stated in (2) and (3) above are the functions c^x, where $c \in (0, \infty)$. Thus the function $\varphi(x) = 1 + x^2$ for $x \in \mathbb{R}$ does not satisfy this conditions, but it does satisfy the properties in (1) and (2). Hence the corresponding functional

Λ_φ preserves units and the unity, but it is not multiplicative. It follows that $\mathcal{E}'(\mathbb{R})$ is not a GKŻ algebra.

For every φ the linear functional on $\Lambda : \mathcal{E}'(\mathbb{R}) \to \mathbb{C}$ given by $\Lambda(t) = \langle t, \varphi \rangle$ for all $t \in \mathcal{E}'(\mathbb{R})$, is continuous, when $\mathcal{E}'(\mathbb{R})$ is equipped with the weak dual/weak-$*$ topology $\sigma(\mathcal{E}'(\mathbb{R}), \mathcal{E}(\mathbb{R}))$; see e.g. [20].

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