MARTINGALE CONDITIONS FOR OPTIMAL SAVING: DISCRETE TIME

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Abstract. Necessary and sufficient conditions are derived for optimal saving in a stochastic neo-classical one-good world with discrete time. The usual technique of dynamic programming is replaced by classical variational and concavity arguments, modified to take account of conditions of measurability which represent the planner's information structure. Familiar conditions of optimality are thus extended to admit production risks represented by quite general random processes - no i.i.d.r.v.s, stationarity or Markov dependence are assumed - while utility and length of life also may be taken as random. It is found that the 'Euler' conditions may be interpreted as martingale properties of shadow prices.

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I. INTRODUCTION

This Paper derives necessary and sufficient conditions for optimal saving in a one-commodity world where income, utility and the planning horizon are subject to risk. The usual technique of dynamic programming is replaced by an approach combining classical variational and concavity arguments with some concepts from the theory of discrete time martingales. The results generalise known conditions of optimality, notably by admitting a much wider class of risks.

The model considered is of standard neo-classical design, extended to allow a sufficiently general description of risk and change. Income is partly exogenous, partly produced under diminishing or constant returns by preceding inputs of capital; this allows interpretation of the model as a theory of either personal or social saving. The influence of risk and change - whether due to climate, technology, population, disease, fashion or any other exogenous source - is represented by treating income for each level of input, and utility for each level of consumption, as random processes, i.e. as functions of state and time. The only assumptions of substance made about these processes are (i) those which specify that the planner has complete, up-to-date knowledge of the past evolution of the processes, (ii) those implicit in the formulation of the problem of optimal saving, in particular the finite supremum condition mentioned below and the existence of an optimum, and (iii) those which preclude constrained solutions. Conditions of stationarity, period-by-period independence or Markov dependence, often imposed in models of saving, are not required. Welfare is defined as the sum over time of expected utilities of consumption, and the welfare functional is supposed to have a finite supremum on the set of feasible plans. In the main discussion the horizon is infinite, but variants are considered which allow for an exogenous random stopping time at which utility ceases to accrue and assets become worthless to the planner.

The main result for the infinite horizon model asserts that a plan for saving and consumption is optimal if, and only if, (a) the welfare functional evaluated at that plan is finite, (b) the shadow prices (marginal utilities) form a martingale when goods are denominated in 'reduced' units, i.e. net of random compound interest at rates defined by the marginal productivity of capital, separately for each state of nature, and (c) the mathematical expectation of the 'shadow value' of the capital stock (product of shadow price and reduced quantity) tends to zero as time tends to infinity. More precisely, these conditions are sufficient under the general terms of the model, and also necessary if further assumptions are imposed to rule out constrained solutions and ensure the convergence of certain sums. Modified results apply in the case of a random horizon; they depend on whether the stopping time varies with the state of nature and, if so, whether it can be predicted by observing antecedent events.

It is worthwhile to dwell briefly on the economic interpretation of condition (b). A discrete time martingale - see Neveu (1972) - is a finite random process whose past and current values are known at each time t and whose current value at t is equal to the conditional expectation, relative to information at t, of the prospective value at t+1. Thus (b) expresses an inter-temporal, stochastic version of the principle of equi-marginal utility: at each time and state the planner chooses between consumption and saving in such a way that the marginal utility of current consumption equals the current conditional expectation of the marginal utility of future consumption, adjusted for random compound interest. The necessity of this condition, although intuitively appealing and demonstrated previously in special cases, should not too readily be taken for granted. A general proof must pay attention to the problems of passing to the limit under expectation and summation signs and to the non-negativity constraints for capital and consumption, which are required to hold in each state of nature. These difficulties can be overcome in discrete time, but they cause the proof to break down when the present model is replaced by a corresponding continuous one.[1]

For brevity, discussion of previous literature will be limited to Levhari and Srinivasan (1969), Mirman and Zilcha (1976) and Zilcha (1976); reference may be made to Brock and Mirman (1972) for a survey and a full description of the model used by Mirman and Zilcha. The conditions of optimality stated above are, apart from the terminology of martingales[2] and reduced units, essentially those given in the Papers cited, but their the method of proof here is different and the scope of the conditions is extended in certain respects. The following points may be noted. Levhari and Srinivasan (1969) assume that returns are proportional to capital and prove the sufficiency of the conditions. Mirman and Zilcha (1976) introduce

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1 See Foldes (1978), where it is found that the reduced shadow prices defined by an optimal plan will in general form a martingale only if a random time change is first performed, whereby time in each state of nature is measured by the depletion of reduced capital instead of the clock. It is this difference which has mainly prompted the writing of the present paper as a companion to the above work. A systematic comparison between the theories would unduly complicate the discussion, but a few points may be noted. The model in Foldes (1978) assumes constant returns to capital, whereas diminishing returns and an exogenous income are permitted here because in the discrete case these extensions demand little extra effort; in the continuous case they are less straightforward, and will be considered separately elsewhere. Much of the present derivation of the conditions of optimality for the infinite horizon model can be translated into continuous time, but certain steps break down crucially: the 'integration by parts' in (17) below, and the use made of the resulting equation in proving the sub-martingale inequality - see (18) and part [4] of the necessity argument for Theorem 2. The theory of random stopping times is also richer in the continuous case. Finally, the existence of optima is not studied here, but the new methods developed in the continuous-time paper could be adapted to the present model.

2 The mere re-interpretation of known 'Euler' conditions of optimality as martingale properties of shadow prices, though trivial enough mathematically, is in itself not without interest. For example, it points to easy proofs of results about convergence and unboundedness of shadow price processes, and it suggests an approach based on capital theory to the study of martingale properties of market prices.

a concave production function and add a proof of necessity by dynamic programming. Both of these Papers assume that production risks in distinct periods are identically and independently distributed. Zilcha (1976) uses a more elaborate model, based on work by Radner and by Dana, in which there are several goods and the 'i.i.d.' assumption is replaced by one of stationarity; the proof of necessity again relies on dynamic programming. The assumptions made in the above Papers about the processes representing risks in production are a good deal more restrictive than those made here, as regards both inter-temporal dependence and boundedness of random variables; in addition we allow random utility and a random horizon. Certain regularity conditions for production and utility functions are also relaxed here, notably the 'no input, no output' condition and the assumption (made by Mirman and Zilcha and by Zilcha) that utility is non-negative. Finally, it is not difficult to extend the present approach to models with several goods or securities.[3]

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[3] It should be mentioned that, whereas the finite supremum condition is postulated directly here, it can be inferred from assumptions in the models of Brock and Mirman (1972), and Zilcha (1976); this condition is, of course, needed for the necessity proofs in all cases. The sufficiency proof in Levhari and Srinivasan (1969) is valid without a finite supremum if optimality is defined by the 'overtaking criterion', and an analogous extension of the present approach is given in Section IV below. Infinite supremum problems generally are beyond the scope of this Paper.

The rest of the Paper is arranged as follows. Section II states basic definitions and formulates the model for the case of an infinite horizon. A brief discussion of directional derivatives of concave functionals appears in Section III, where alternative expressions for derivatives are obtained and an optimum is characterised as a plan such that the derivative of the welfare functional in the direction of any other feasible plan is non-positive (Theorem 1). These results are used in Section IV to show that conditions (a), (b) and (c) are sufficient, and under slight restrictions necessary, for optimality (Theorem 2). The changes needed when the horizon is random are indicated in Section V.

II. THE MODEL

We begin with a formal description of the structure of events and information. Let $\underline{T} = \{0,1,\ldots\}$ represent discrete time and $(\Omega, \underline{A}, P)$ a probability space; Ω is the set of states of nature (possible histories of the environment for all time), $\underline{A}$ a σ-algebra of subsets called events, and P a probability on $\underline{A}$. We write 'a.s.' for 'almost surely' or 'except for an event of zero probability'. The information available at a fixed time t is defined by a sub σ-algebra $\underline{A}_t$ of events, called events at t, of which the planner knows at t whether or not they occur. The overall <u>information structure</u> is represented by the family $(\underline{A}_t; t\varepsilon\underline{T})$; it is assumed that this family is ascending, i.e. that $A_T \subseteq \underline{A}_t$ for $T \leq t$, to represent learning without forgetting. For simplicity, it is also assumed that $\underline{A}_o$ is the σ-algebra generated by all the P-null sets of $\underline{A}$.

All random variables considered below are finite or extended real-valued, $\underline{A}$-measurable functions on Ω defined up to P-null sets, i.e. two variables which are a.s. equal are treated as identical. The assumption that the value of a random variable $v = v(\omega)$ is known to the planner at t is expressed formally by the condition that v is $\underline{A}_t$-measurable, i.e. that for each numerical interval I the event $\{\omega: v(\omega) \varepsilon I\}$ belongs to $\underline{A}_t$. All random processes considered are families of random variables indexed by the whole of $\underline{T}$ or by $\{1,2,\ldots\}$. Processes are denoted by symbols like z, $(z_t; t\varepsilon\underline{T})$ or (z_t) when regarded as families of variables, by z or $z(.,.)$ when regarded as functions of the pair (ω,t), the two points of view being used interchangeably; particular variables belonging to a process z are denoted by z_t, $z_t(.)$ or $z(.,t)$, particular values by $z_t(\omega)$ or $z(\omega,t)$. A process (z_t) is called <u>adapted</u> - more precisely, adapted to the family $(\underline{A}_t)$ - if, for each t, the variable z_t is $\underline{A}_t$-measurable. Thus an adapted process is one whose previous history is known at each time. All processes considered are assumed, or can be shown to be, adapted, though for two distinct reasons: observed processes such as the income from a given level of capital because the saver is supposed to observe them immediately and precisely; controlled processes such as consumption or capital plans because at each time the current value can be chosen only on the basis of available information.

Turning now to the description of opportunities, we postulate a production function $F=F(K;\omega,t)$, defined for $K \in [0,\infty]$, $\omega\epsilon\Omega$ and $t > 0$, with the following properties. For each fixed (ω,t), $F(.;\omega,t)$ is finite for $K < \infty$, continuous and concave, and possesses a positive[4] derivative $f(.;\omega,t)$; moreover $F(0;\omega,t) \geq 0$, to allow for the possibility of a purely exogenous, random income. For each fixed K, the process $F(K;.,.)$ is assumed to be adapted, and it follows that $f(K;.,.)$ has the same property. The influence of exogenous changes in technology, size of labour force, etc. is implicit in the definition of the production function.

Let $W > 0$ denote the initial endowment. A (feasible) capital plan, or simply plan, is defined as a non-negative, adapted process $k = (k_t)$ such that a.s. the system

$$\begin{aligned} k(\omega,t) + c(\omega,t) &= F[k(\omega,t-1),\omega,t] \qquad t=1,2,\ldots, \\ k(\omega,0) + c(\omega,0) &= W \end{aligned} \tag{1}$$

has a solution $c(\omega,t)$ which is non-negative on $\underline{T}$; of course, the

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[4] Note that the terms positive, negative, increasing and decreasing are used throughout in their strict sense.

variables $k_o = k(\omega,0)$ and $c_o = c(\omega,0)$ are degenerate, and the process $c = (c_t)$ is the consumption plan corresponding to k. We denote by $\underline{K}$ the set of all capital plans, considered as a subset of the vector space of adapted processes. This set is not empty (because c = 0 is feasible) and it is convex by the concavity of F.

Next, the utility function $U = U(C;\omega,t)$ is supposed to be defined (with an arbitrary but fixed choice of scale and origin) for $C\varepsilon[0,\infty]$, $\omega\varepsilon\Omega$ and $t\varepsilon\underline{T}$ and to have the following properties. For each fixed (ω,t), $U(.;\omega,t)$ is finite on $(0,\infty)$, continuous and concave, and possesses a non-negative derivative $u(.;\omega,t)$. For each fixed C, $U(C;.,.)$ and $u(C;.,.)$ are adapted processes. The welfare functional is postulated in the form

$$\phi(k) = E\,\Sigma_0^\infty\, U[c(\omega,t);\omega,t] \qquad k\varepsilon\underline{K}\ , \tag{2}$$

where c is the consumption plan corresponding to k. It is assumed that for each $k\varepsilon\underline{K}$ the expected sum of the positive terms appearing in (2) is not $+\infty$; this ensures that the functional is well defined, and incidentally that $E\Sigma$ can be replaced by ΣE. We further assume the important condition that

$$\phi^* = \sup\{\phi(k): k\varepsilon\underline{K}\} \quad \text{is finite;} \tag{3}$$

this of course entails the existence of some $k\varepsilon\underline{K}$ for which $\phi(k) > -\infty$. The effect of (3) is to impose implicit restrictions on F and U jointly; various explicit conditions which imply (3) could be adopted, but for our purposes it is unnecessary to make a choice. As an example, it is enough if U can be represented in the form

$$U(C;\omega,t) = V(C;\omega,t)q_t \tag{4}$$

for all C, ω and t, where V is a bounded function and (q_t) is a summable sequence of positive numbers (discount factors). As usual, an element $k^* \epsilon \underline{K}$, and the corresponding c^*, are called optimal if $\phi(k^*) = \phi^*$, and the problem of optimal saving is to find such an element if one exists.

Consider a fixed capital plan, say k^*; together with the corresponding consumption plan c^* this induces a number of further processes, namely a production plan $F^* = (F_t^*)$, a marginal productivity plan f^*, a utility plan U^* and a marginal utility plan u^*. To define these, and at the same time establish an abridged notation, we write

$$F_t^* = F^*(\omega,t) = F_t(k_{t-1}^*) = F[k^*(\omega,t-1);\omega,t] \qquad t=1,2,... \qquad (5)$$

$$f_t^* = f^*(\omega,t) = f_t(k_{t-1}^*) = f[k^*(\omega,t-1);\omega,t] \qquad t=1,2,... \qquad (6)$$

$$U_t^* = U^*(\omega,t) = U(c_t^*) = U[c^*(\omega,t);\omega,t] \qquad t=0,1,... \qquad (7)$$

$$u_t^* = u^*(\omega,t) = u(c_t^*) = u[c^*(\omega,t);\omega,t] \qquad t=0,1,... \qquad (8)$$

Analogous notation will be used below for processes induced by other capital plans; for example, c^1, F^1 and U^1 correspond to k^1, while c^α, F^α and U^α correspond to k^α. It can be verified that all these processes are adapted.

The particular element denoted by k^* - usually the one whose optimality is in question - plays a special part. From it we define the (compound) interest process $r = (r_t;\ t \epsilon \underline{T})$ by writing

$$r_t = r(\omega,t) = f_1^* f_2^* \ldots f_t^* \qquad t=1,2,... \qquad (9)$$

and $r_o = 1$. Sometimes it will be convenient to reckon quantities of goods in reduced units, i.e. net of random compound interest, separately for for each state of nature.[5] The transformation from natural to reduced

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[5] The term 'discounted' is avoided for fear of confusion with either subjective or ordinary commercial discounting. Both of these operations apply the same discount factors to all states of nature.

units, written

$$\tilde{k}(\omega,t) = k(\omega,t)/r(\omega,t) \; ; \quad \tilde{c}(\omega,t) = c(\omega,t)/r(\omega,t), \qquad \omega\epsilon\Omega, \quad t\epsilon\underline{T}, \tag{10}$$

will always be defined by the fixed plan k* but applied to all k and c. Thus we have, for example,

$$\tilde{k}_t^* = k_t^*/r_t, \quad \tilde{c}_t^* = c_t^*/r_t \; ; \quad \tilde{k}_t^1 = k_t^1/r_t, \quad \tilde{c}_t^1 = c_t^1/r_t. \tag{11}$$

When units of goods are transformed by dividing by r_t, the corresponding marginal utilities or shadow prices must obviously be multiplied by the same number. Accordingly we define - for the plan k* only - the (reduced) <u>shadow price process</u> $y = (y_t; \; t\epsilon\underline{T})$ by setting

$$y(\omega,t) = r(\omega,t)u^*(\omega,t) \qquad \omega\epsilon\Omega, \quad t\epsilon\underline{T}. \tag{12}$$

This process is non-negative, finite and adapted. Speaking informally, one might imagine the values y(ω,t) implemented as prices at zero time in a perfect market for 'dated contingent goods at (ω,t) expressed in reduced units', one contract being a promise to deliver one reduced unit of goods at t iff history until then is consistent with ω.[6]

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[6] The assumption that f > 0 everywhere, which is designed to ensure that the transformation to reduced units is well defined, can be weakened to $f \geq 0$ if certain conventions are observed. Suppose that $f_T^* = 0$ on $A\epsilon\underline{A}_T$ for some T. Then $r_t = y_t = 0$ on A for $t \geq T$, so that $\tilde{k}_t$ and $\tilde{c}_t$ are either infinite or undefined; however, the products $\tilde{k}_t y_t = k_t u_t^*$ and $\tilde{c}_t y_t = c_t u_t^*$ may still be read as zero. On this basis the argument of Sections III-IV goes through, subject to minor changes in part (3) of the necessity proof for Theorem 2.

III. DIRECTIONAL DERIVATIVES

We begin with conditions of optimality expressed in terms of directional derivatives of the functional ϕ on the convex set $\underline{K}$.[7] Let k^* and $k^1 = k^* + \delta k$ be arbitrary elements of $\underline{K}$ with $\phi(k^*) > -\infty$; the process δk is called a (feasible) direction or variation at k^*, and the directional derivative of ϕ at k^* in the direction δk is defined by

$$D\phi = D\phi(k^*,\delta k) = \lim_{\alpha\downarrow 0}(1/\alpha)\left[\phi(k^*+\alpha\delta k) - \phi(k^*)\right] \tag{13}$$

This limit certainly exists (although it may take the values $\pm\infty$), since $g(\alpha) = \phi(k^*+\alpha\delta k) - \phi(k^*)$ is an ordinary concave function of α on $[0,1]$, and $g'(\alpha+) = D\phi$. Now write $k^\alpha = k^*+\alpha\delta k$, let c^α be the consumption plan corresponding to k^α - see (1) - and abridge the notation as in (5)-(8); then the definition (13) reads

$$D\phi = \lim_{\alpha\downarrow 0} (1/\alpha)\ E\ \Sigma_0^\infty (U_t^\alpha - U_t^*). \tag{14}$$

We shall show that, if $\phi(k) > -\infty$, the limit in (14) can be evaluated by differentiating under $E\Sigma$ to obtain

$$D\phi = E\{\Sigma_0^\infty \left[f_t^*\delta k_{t-1} - \delta k_t\right] u_t^* \} > -\infty \tag{15}$$

where $f_0^*\delta k_{-1}$ is to be read as zero. To see this, restrict α to $(0,1]$ and check that for each (ω,t) the concavity of $U(.;\omega,t)$ and $F(.;\omega,t)$ implies

$$U_t^1-U_t^* \leq (1/\alpha)(U_t^\alpha-U_t^*) \uparrow (dU_t^\alpha/d\alpha)\big|_{\alpha=0} = (f_t^*\ \delta k_{t-1}-\delta k_t)u_t^*, \quad \alpha\downarrow 0 \tag{16}$$

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[7] Discussions of the derivaties of functionals usually assume more analytic structure in the domain than is required here. It is therefore convenient to give a short self-contained statement.

The functions of (ω,t) defined by $(1/\alpha)(U^{\alpha}-U^*)$ are thus bounded below by the function U^1-U^*, which is summable (with respect to $E\Sigma$), and they ascend pointwise to the limit function; the result follows from the monotone convergence theorem.

It is useful for later reference to rewrite (15) in various forms, using the definitions (10-12) of reduced quantities and shadow prices:

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\begin{aligned}
D\phi &= E\{\Sigma_0^{\infty} (\delta\tilde{k}_{t-1} - \delta\tilde{k}_t)y_t\} \\
&= \lim_{T\to\infty} \Sigma_0^T E\{(\delta\tilde{k}_{t-1} - \delta\tilde{k}_t)y_t\} \\
&= \lim_{T\to\infty} E\{\Sigma_0^{T-1} \delta\tilde{k}_t(y_{t+1} - y_t) - \delta\tilde{k}_T y_T\} \qquad (17)
\end{aligned}
$$

According to elementary properties of conditional expectation we have $E = EE^t$ for each t, and $E^t[\delta\tilde{k}_t y_{t+1}] = \delta\tilde{k}_t E^t y_{t+1}$ a.s. since $\delta\tilde{k}_t$ is $\underline{A}_t$-measurable; consequently

$$D\phi = \lim_{T\to\infty} E\{ \Sigma_0^{T-1} \delta\tilde{k}_t(E^t y_{t+1} - y_t) - \delta\tilde{k}_T y_T\}. \qquad (18)$$

It is easily shown that an element $k^*\varepsilon\underline{K}$ is optimal iff $\phi(k^*) > -\infty$ and $D\phi(k^*,\delta k) \leq 0$ for all $k^1 = k^*+\delta k \ \varepsilon\ \underline{K}$. Indeed, if k^* is optimal, then $\phi(k^*)$ is finite by definition and $\phi(k^*) \geq \phi(k^*+\alpha\delta k)$ for any feasible δk and $\alpha\varepsilon(0,1]$; hence $[\phi(k^*+\alpha\delta k)-\phi(k^*)]/\alpha \leq 0$ and it only remains to go to the limit. Conversely, concavity implies $\phi(k^*+\delta k)-\phi(k^*) \leq D\phi$ and the assertion follows.

The preceding paragraph does not rely on (15); but for the necessity argument in Section IV we shall require this formula to hold when k^* is optimal for every δk, even if $\phi(k^*+\delta k) = -\infty$. To ensure this, we shall impose a further slight restriction on ϕ, to the effect that the value of the functional ϕ remains finite when an optimal consumption plan is scaled down in a small proportion. To be precise, let k^* with corresponding c^* be optimal, and for $\alpha\varepsilon(0,1)$ let $k^*_{1-\alpha}$ be the capital plan corresponding to $(1-\alpha)c^*$; such plans are clearly feasible. We state

ASSUMPTION (i). If k^* is optimal, there is an $a\varepsilon(0,1)$ such that $\phi(k^*_{1-\alpha}) > -\infty$ for $\alpha\varepsilon[0,a]$.

For natural choices of F and U this assumption is satisfied by all plans, not just optima; in particular, it holds if $U \geq 0$ everywhere.

The proof of (15) is like that given above, except that the inequality in (16) is replaced by

$$(1/a)\left[U(c^*_t-ac^*_t)-U(c^*_t)\right] \leq (1/\alpha)\left[U(c^*_t-\alpha c^*_t)-U(c^*_t)\right] \leq (1/\alpha)\left[U(c^\alpha_t)-U(c^*_t)\right]$$

$$= (1/\alpha)(U^\alpha_t-U^\alpha_t) \qquad (19)$$

$0 < \alpha \leq a$. Here the first inequality follows from the concavity of U, the second from $(1-\alpha)c^*_t \leq c^\alpha_t$, which in turn results from the concavity of F. The expected sum of the left-hand side of (19) is

$$(1/a)E\Sigma_0^\infty\left[U(c^*_t-ac^*_t) - U(c^*_t)\right] = (1/a)\left[\phi(k^*_{1-a}) - \phi(k^*)\right], \qquad (20)$$

which by A.(i) is finite. The function on the left of (19) thus serves as a summable lower bound for the functions in (16), and the result follows as before by monotone convergence. Reference to (14) further shows that (20) defines a finite lower bound for all values of $D\phi(k^*,\delta k)$ when k^* is optimal. To sum up, we have

<u>Theorem 1.</u> Let $k^*\varepsilon\underline{K}$ be such that $\phi(k^*) > -\infty$, and let y be defined from k^* as in (12). Then (a) k^* is optimal iff the directional derivative $D\phi(k^*,\delta k)$ is non-positive for every $k^1 = k^* + \delta k \,\varepsilon\, \underline{K}$, and for $\phi(k^1) > -\infty$ the value of $D\phi(k^*,\delta k)$ is given by (15), (17) or (18). Moreover (b) if k^* is optimal and A.(i) holds, the value of $D\phi(k^*,\delta k)$ is given by (15), (17) or (18) for every $k^1 \,\varepsilon\, K$, all these numbers are non-positive and are bounded below by the finite number (20).

IV. MARTINGALE CONDITIONS

An optimum will now be characterised by means of 'Euler' and 'transversality' conditions involving the shadow prices. Two further assumptions, designed to rule out constrained solutions, will be adopted in the necessity argument; these assumptions, and the first two parts of the necessity proof below, may be omitted if attention is confined to an optimum such that $c_t^* > 0$ and $k_t^* > 0$ a.s. for each $t \in \underline{T}$.

ASSUMPTION (ii). For each (ω,t), $u(0;\omega,t) = \infty$.

ASSUMPTION (iii). For each $t\in\underline{T}$ and each event $A \in \underline{A}_t$ with positive probability, either

(α) the event $A \cap \{\omega: F(0;\omega,t+1) = 0\}$ has positive probability; or

(β) $u(C;\omega,t) > 0$ for $C < \infty$ and $f(0;\omega,t) = \infty$, a.s. for $\omega \in A$.

Theorem 2. Let $k^* \in \underline{K}$ be a capital plan such that

(a) $\phi(k^*) > -\infty$;

(b) the corresponding shadow process $y = (y_t; t\in\underline{T})$, defined as in (12), is a martingale; and

(c) $\lim_{T\to\infty} E\{\tilde{k}_T^* y_T\} = \lim_{T\to\infty} E\{k_T^* u_T^*\} = 0$;

then k^* is optimal. Conversely, if Assumptions (i), (ii) and (iii) hold, an optimal plan k^* satisfies (a), (b) and (c).

Proof of sufficiency. Let $k^1 = k^* + \delta k \in \underline{K}$ and suppose that $\phi(k^1) > -\infty$; (otherwise there is nothing to prove). The assertion that y is a martingale means that this process is finite and adapted, and satisfies

$$y(\omega,t) = E^t y(\omega,t+1) \quad \text{a.s.} \tag{21}$$

for each $t\in\underline{T}$. When (21) is substituted into the formula (18) for

$D\phi(k^*,\delta k)$ the latter is reduced to

$$D\phi = \lim_{T\to\infty} E\{-\delta\tilde{k}^1_T y_T\} = \lim_{T\to\infty} E\{(\tilde{k}^*_T - \tilde{k}^1_T) y_T\}. \tag{22}$$

Now (c) implies $D\phi \leq 0$ since $\tilde{k}^1_T y_T \geq 0$, and the result follows from Theorem 1(a).

Proof of necessity. From now on, k^* with corresponding c^* is a fixed optimum and $k^1 = k^*+\delta k$ with corresponding c^1 another plan.

[1] We first check that $c^*_t > 0$ a.s. for each $t\varepsilon\underline{T}$.[8] If not, let T be the earliest time such that, for some $A\varepsilon\underline{A}_T$ with $PA > 0$, we have $c^*_T = 0$ on A. Then $u^*_T = \infty$ on A by A.(ii). Suppose first that, at the given time T, we can choose A such that $k^*_T > 0$ on A. Then clearly we can construct a variation δk such that $\delta k_T < 0$ on A, while $\delta k_t = 0$ on Ω for $t < T$ (if $T > 0$). For such a variation, the formula (15) for $D\phi$ contains the term

$$E\{(f^*_T\ \delta k_{T-1} - \delta k_T) u^*_T\ I(A)\}\ , \tag{23}$$

where I(A) denotes the indicator of the event A. The value of this term is $+\infty$, which by Theorem 1(b) contradicts optimality, since it implies that either $D\phi = \infty$ or (15) is undefined because it contains infinite terms of opposite sign. Alternatively, suppose that at T the only possible choice of A is such that $k^*_T = 0$ on A; this can happen only if $T > 0$, since we are assuming that $c^*_T = 0$ on A. Now $c^*_{T-1} > 0$ on Ω by the choice of T, so that we can construct a variation with $\delta k_{T-1} > 0$ on Ω and $\delta k_T = 0$ on A; once again the value of (23) is $+\infty$ and the result follows.

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[8] From now on, the qualification 'a.s.' is sometimes omitted.

[2] We next check that $k_t^* > 0$ a.s. for each $t \in \underline{T}$. Suppose, on the contrary, that $k_{T-1}^* = 0$ on $A \in \underline{A}_{T-1}$ with $T \geq 1$. If A.(iii)(α) applies, the event $A \cap \{c_T^* = 0\}$, which belongs to $\underline{A}_T$, has positive probability, contrary to [1] above. If A.(iii)(β) applies, we choose δk so that $\delta k_{T-1} > 0$ on A; then, since $f_T^* = \infty$ on A while δk_T is finite, the coefficient of u_T^* in (23) is ∞. Since $u_T^* > 0$, the value of (23) is ∞ also.

[3] Turning now to the proof proper, we show that $(y_t) = (r_t u_t^*)$ is a super-martingale. Since the process is certainly finite and adapted, we need only show that

$$y_T \geq E^T y_{T+1} \qquad \text{a.s.} \tag{24}$$

for an arbitrary $T \in \underline{T}$. For this purpose we construct a variation which increases reduced capital at T and consumes the additional output at T+1. To be precise, choose any $A \in \underline{A}_T$ and $\varepsilon > 0$, write $A_\varepsilon = A \cap \{\tilde{c}_T^* > \varepsilon\}$, and define δk by setting

$$\delta \tilde{k}_T = \varepsilon I(A_\varepsilon), \qquad \delta \tilde{k}_t = 0 \text{ for } t \neq T. \tag{25}$$

It is clear that this variation is feasible, and Theorem 1(b) with (15) yields

$$0 \geq D\phi = \varepsilon E\{(y_{T+1} - y_T) I(A_\varepsilon)\}. \tag{26}$$

Now cancel ε, rewrite as

$$E\{y_T I(A_\varepsilon)\} \geq E\{y_{T+1} I(A_\varepsilon)\} \tag{27}$$

and let $\varepsilon \downarrow 0$; then $I(A_\varepsilon) \uparrow I(A)$ since $\tilde{c}_T^* > 0$ a.s. by (1) above, and since $y \geq 0$ the monotone convergence theorem yields

$$E\{y_T I(A)\} \geq E\{y_{T+1} I(A)\}. \tag{28}$$

This implies (24) since A is an arbitrary event in $\underline{A}_T$.

[4] To show that y is a martingale, we consider any variation such that $\delta k_t < 0$ a.s. for each $t \geq 0$ - such variations clearly exist by virtue of [2] above - and refer to the formula (18) for $D\phi$. It follows from (24) that each of the variables $\delta\tilde{k}_t(E^t y_{t+1} - y_t)$ is a.s. non-negative, and the same is true of $-\delta\tilde{k}_T y_T$. Thus, if any of the numbers $E\{\delta\tilde{k}_t(E^t y_{t+1} - y_t)\}$ were positive, we should have $D\phi > 0$, contrary to Theorem 1(b). Consequently each of the variables $\delta\tilde{k}_t(E^t y_{t+1} - y_t)$ vanishes a.s. and (24) may be read as an equality.

[5] The transversality condition (c) follows immediately. Indeed, Theorem 1(b) and (18) as simplified by the preceding argument yield

$$0 \geq D\phi = \lim_{T\to\infty} E\{-\delta\tilde{k}_T y_T\} = \lim_{T\to\infty} E\{(\tilde{k}^*_T - \tilde{k}^1_T) y_T\} \tag{29}$$

for each $k^1 \in \underline{K}$, and it suffices to set $k^1 = 0$ and note that $\tilde{k}^*_T y_T \geq 0$ a.s. This completes the proof.

Extension. The sufficiency argument is easily extended to certain problems - otherwise beyond the scope of this Paper - in which the supremum ϕ^* is not finite. Suppose, for example, that U is bounded, so that for each plan and each $t\epsilon\underline{T}$ the expectation $EU_t = EU[c(\omega,t);\omega,t]$ is defined and finite. We say that k* overtakes $k^1 = k^*+\delta k$ if

$$\limsup_{T\to\infty} E\,\Sigma_0^T\,(U_t^1 - U_t^*) \leq 0\ , \tag{30}$$

and call k* optimal if it overtakes any other plan. Conditions (b) and (c) of Theorem 2 are now sufficient for k* to be optimal, and 'lim' may be replaced by 'limsup' in (c). Explicitly, concavity implies

$$E\,\Sigma_0^T(U_t^1 - U_t^*) \leq E\{\Sigma_0^T(f_t^*\delta k_{t-1} - \delta k_t)u_t^*\}\ ,\qquad T\epsilon\underline{T} \tag{31}$$

- cf. (16) - and the right-hand side may be rewritten

$$E\{ \Sigma_{o}^{T-1} \delta\tilde{k}_t (E^t y_{t+1} - y_t) - \delta\tilde{k}_T y_T \} \tag{32}$$

as in (18). If y is a martingale the terms under the summation sign drop out, and the result follows from

$$\limsup_{T\to\infty} E\{-\delta\tilde{k}_T^* y_T\} \leq \limsup_{T\to\infty} E\{\tilde{k}_T^* y_T\} = 0. \tag{33}$$

This incidentally shows that the sufficiency argument is easily stated without reference to directional derivatives.

V. RANDOM HORIZON

The planning horizon has been treated so far as infinite, but it is not difficult to modify the theory to allow for an exogenous random stopping time at which utility ceases to accrue and remaining assets are worthless to the planner. We shall not introduce the usual 'utility of bequest' function, but rather suppose that any vicarious satisfaction which today's planner derives from the prospect of consumption by his heirs is reflected in the formulation of the utility function. Thus, although we shall refer to the stopping time as the planner's 'time of death' to make the model more vivid, it may be interpreted in various ways, for example as the time of death or departure from office of the last successor whose welfare concerns today's planner.

Recall that a random variable $\Delta = \Delta(\omega)$ is called a (random) stopping time if it takes values in $\underline{T} \cup \{\infty\}$ and if, for each $t \in \underline{T}$, the event $\{\omega: \Delta(\omega) = t\}$ belongs to $\underline{A}_t$ - see Neveu (1972) p.19. We amend the model of saving by introducing the time of death Δ as a positive stopping time, set

$$U(.;\omega,t) = 0 \text{ identically for } t \geq \Delta(\omega), \quad \omega \in \Omega, \tag{34}$$

and modify Assumptions (ii) and (iii) accordingly. It is convenient still to regard the information structure $(\underline{A}_t)$ and all processes, including plans, as defined formally for all values of t, while identifying elements of $\underline{K}$ which agree a.s. for $0 \leq t \leq \Delta(\omega)$. Informally, we think of observation and action as stopping at $\Delta(\omega) - 1$. In particular, the actual time of death is excluded from the interval when action can affect utility; we therefore imagine 'death at t' as occurring before action can be taken at that time to consume the remaining capital. Note that in general Δ can take infinite as well as finite values, although $\Delta = 0$ is excluded; various special cases may be considered when Δ is finite, for example $E\Delta$ finite, Δ bounded, or Δ equal to a fixed time H.

The implications of a random time of death depend mainly on the accuracy with which it can be predicted; two extreme cases will be considered. We say that Δ is <u>predictable</u> if it is a.s. finite and for each $t > 0$ the event $\{\Delta = t\}$ is in $\underline{A}_{t-1}$, i.e. a sure augury of death is observed at least one period in advance. We say that Δ is <u>unpredictable</u> if, for each t and each $A \in \underline{A}_t$ with $PA > 0$ and $A \subseteq \{t < \Delta\}$, we have $P\{A \cap \{t+1 < \Delta\}\} > 0$; in other words, whatever event of positive probability is observed during life, the conditional probability of survival beyond the next period is positive.

A review of earlier Sections shows that the theory is almost unchanged in the unpredictable case. Theorems 1 and 2 remain true, the proofs being altered in trivial ways to allow for the arbitrary definition of plans after death. Note that the capital will never be entirely depleted during life, for fear of a terminal period without consumption; thus, if $\Delta(\omega) < \infty$ there will be a 'reluctant bequest' $k^*[\omega, \Delta(\omega)] > 0$. The martingale condition (b) is formally the same, although its content is slightly altered: if at t the planner is alive, he equates the current value of y_t with a value of $E^t y_{t+1}$ which takes into account the conditional probability of death at t+1; and if he is dead, the equality $y_t = y_{t+1} = \ldots = 0$ holds by definition.[9]

In the predictable case, it is found that T.1 and the sufficiency part of T.2 hold true but that the necessary conditions must be amended. Suppose for example that $u(C;\omega,t) > 0$ for all $C < \infty$ when $t < \Delta(\omega)$; then clearly $(y_t; t\varepsilon \underline{T})$ cannot be a martingale, because for $t = \Delta(\omega) - 1$ there can a.s. be no equality between the positive value of y_t and the zero value of $E^t y_{t+1}$. On the other hand the terminal condition can be strengthened, since if the planner knows at the finite time $\Delta-1$ that he will die at Δ he will consume everything. More precisely, it can now be shown that the following conditions are sufficient for an element of $k^*\varepsilon \underline{K}$ to be optimal, and also necessary under A.(i), (ii) and (iii) as modified:

(a) $\phi(k^*) > -\infty$;

(b) the process $(y_{t\wedge(\Delta-1)}; t\varepsilon\underline{T})$ is a martingale, where $\wedge$ means 'the lesser of' and $y_{t\wedge(\Delta-1)} = y[\omega; t\wedge[\Delta(\omega)-1]]$; and

(c) $k^*_{\Delta-1} = 0$ a.s.

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9 In the special case where the time of death is independent of all $\underline{A}_t$ (and is therefore unpredictable), the machinery of stopping times can be avoided and the model of earlier Sections used with no more than a change of interpretation. Briefly, it suffices to write $U = V_q$ as in (4) and to interpret $V(C;\omega,t)$ as the utility of consumption at t conditional upon survival beyond t, q_t as the probability of such survival. Here V has the properties previously ascribed to U and (q_t) is a non-increasing sequence of non-negative numbers with $q_o = 1$, the probability assigned to infinite life being $\lim q_t$; (the special assumptions previously made in connection with (4) do not necessarily apply here). The functional (2) now represents 'expected lifetime utility' in the ordinary sense rather than a sum of expected utilities. The optimal plans obtained are the same as in the case of an infinite horizon, except that they are now to be interpreted as plans whose execution is conditional on survival.

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