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Triangles in randomly perturbed graphs[†]

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Abstract

We study the problem of finding pairwise vertex-disjoint triangles in the randomly perturbed graph model, which is the union of any n-vertex graph G satisfying a given minimum degree condition and the binomial random graph $G(n, p)$. We prove that asymptotically almost surely $G \cup G(n, p)$ contains at least $\min\{\delta(G), \lfloor n/3 \rfloor\}$ pairwise vertex-disjoint triangles, provided $p \geq C \log n/n$, where C is a large enough constant. This is a perturbed version of an old result of Dirac.

Our result is asymptotically optimal and answers a question of Han, Morris, and Treglown [RSA, 2021, no. 3, 480–516] in a strong form. We also prove a stability version of our result, which in the case of pairwise vertex-disjoint triangles extends a result of Han, Morris, and Treglown [RSA, 2021, no. 3, 480–516]. Together with a result of Balogh, Treglown, and Wagner [CPC, 2019, no. 2, 159–176], this fully resolves the existence of triangle factors in randomly perturbed graphs.

We believe that the methods introduced in this paper are useful for a variety of related problems: we discuss possible generalisations to clique factors, cycle factors, and 2-universality.

1. Introduction and main result

The problem of subgraph containment is one of the most studied questions in extremal and probabilistic combinatorics. In this paper, we are interested in conditions that guarantee the containment of pairwise vertex-disjoint triangles in different graph models. In particular, when a graph G on $v(G)$ vertices contains $\lfloor v(G)/3 \rfloor$ pairwise vertex-disjoint copies of triangles, we say that G contains a triangle factor. We will consider minimum degree conditions in dense graphs, lower bounds on the edge-probability in random graphs, and a combination of both.

From Mantel's theorem on the maximum number of edges in a triangle-free graph, it follows that any n-vertex graph with minimum degree larger than $n/2$ contains a triangle. The first result on the containment of a triangle factor is due to Corrádi and Hajnal [11], who proved that any n-vertex graph G with minimum degree $\delta(G) \geq 2n/3$ contains a triangle factor. From this, it is not hard to derive a more general result for graphs with smaller minimum degree, which was first proved by Dirac [12].

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Theorem 1.1 (Dirac [12]). *Any n-vertex graph G with $n/2 \leq \delta(G) \leq 2n/3$ contains at least $2\delta(G) - n$ pairwise vertex-disjoint triangles.*

Given an integer m with $n/2 \leq m \leq 2n/3$, the tripartite complete graph with parts of size $2m - n$, $n - m$, and $n - m$ shows the result is best possible. Moreover, a stability version of Theorem 1.1 was proved by Hladký, Hu, and Piguet [18].

Sparser graphs do not necessarily contain triangles, but most of them do if there are enough edges. To determine a cut-off point for the containment of a subgraph in the *binomial random graph* $G(n, p)$, we define a threshold function. Before stating its definition, we remark that we say that a property $\mathcal{A}$ holds asymptotically almost surely (a.a.s.) in the random graph $G(n, p)$ if $\lim_{n \to \infty} \mathbb{P}[G(n, p) \in \mathcal{A}] = 1$. Given a graph H, the function $\hat{p} : \mathbb{N} \to [0, 1]$ is called the *threshold* for the containment of H in $G(n, p)$ if a.a.s. $H \subseteq G(n, p)$ for $p = \omega(\hat{p})$ and a.a.s. $H \nsubseteq G(n, p)$ for $p = o(\hat{p})$. Already in one of the early papers on random graphs by Erdős and Rényi [14] from 1960, the threshold for a single triangle was determined as $1/n$. However, the problem for a triangle factor is much harder and was eventually solved by Johannson, Kahn, and Vu [20], in 2008, as part of a more general result, and the threshold was located at $n^{-2/3} \log^{1/3} n$, with an even sharper transition than in our definition of threshold. When one requires εn to $(1 - \varepsilon)n$ pairwise vertex-disjoint triangles, the problem is easier and the threshold is $n^{-2/3}$ as proved by Ruciński [29] in 1992. Note that the the spanning version requires an extra $\log^{1/3} n$, and this logarithmic term is essential to ensure that a.a.s. every vertex is contained in a triangle.

Bohman, Frieze, and Martin [7] combined the random graph model and the deterministic minimum-degree model by asking how many random edges one needs to add to a dense graph with small linear minimum degree, such that it contains a Hamilton cycle. More precisely, they introduced the model of *randomly perturbed graphs* as the union of an n-vertex graph G_α with minimum degree at least αn and the random graph $G(n, p)$. Given H and fixed α, we are then interested in lower bounds on the edge-probability p such that a.a.s. $G_\alpha \cup G(n, p)$ contains H for any G_α. A lower bound $\hat{p}$ is optimal when in addition there exists a G_α for which a.a.s. $G_\alpha \cup G(n, p)$ does not contain H if $p = o(\hat{p})$, in which case we call $\hat{p}$ the threshold for the containment of H in the randomly perturbed model. Note that this threshold $\hat{p}$ only depends on α and H. In recent years, there has been a lot of work on embeddings of spanning graphs in randomly perturbed graphs. Most results in this model focus on the extreme cases with small $\alpha > 0$ [4, 6, 8, 21, 23–26] or small p [2, 5, 13, 27]. More recently, Han, Morris, and Treglown [17] started a more thorough investigation of the intermediate regime. The goal is to determine the perturbed threshold for every α from $\alpha = 0$, where we can rely only on $G(n, p)$, to the α where the structure already exists in G_α alone and $p = 0$ is sufficient.

It is easy to see that with $0 < \alpha \leq 1/2$, $G_\alpha \cup G(n, p)$ a.a.s. contains a triangle when $p \geq C/n^2$ and C is a sufficiently large constant depending on α. This is asymptotically optimal, as $G(n, p)$ with $p = o(n^2)$ a.a.s. is empty. Together with the cases for $\alpha = 0$ and $\alpha > 1/2$ discussed above, this completes all the range of α for the containment of a single triangle. For a triangle factor, we already know the threshold when $\alpha = 0$ from the random graph model, and when $\alpha \geq 2/3$ we do not need random edges at all. For any $\alpha > 0$, Balogh, Treglown, and Wagner [4] showed that $p \geq Cn^{-2/3}$ is always sufficient (with C depending on α). This is asymptotically optimal in the case $0 < \alpha < 1/3$, as with G_α the complete bipartite graph with classes of size αn and $(1 - \alpha n)$, we need a linear number of triangles with all edges from the random graph, for which $n^{-2/3}$ is the threshold as discussed above. For $1/3 < \alpha < 2/3$, Han, Morris, and Treglown [17] proved that then already $p \geq C/n$ is enough for a triangle factor. Again this is asymptotically optimal as, when G_α is the complete tripartite graph with classes of size $\alpha n/2$, $\alpha n/2$ and $(1 - \alpha)n$, we need a linear number of edges from $G(n, p)$. Together, these results can be summarised as follows.

Theorem 1.2 (Balogh, Treglown, and Wagner [4] and Han, Morris, and Treglown [17]). *Given any $\alpha \in (0, 1/3) \cup (1/3, 2/3)$ there exists $C > 0$ such that the following holds. For any n-vertex graph*

Table 1. Triangle factor containment in $G_\alpha \cup G(n, p)$, where $\delta(G_\alpha) \geq \alpha n$

α	$\alpha = 0$	$0 < \alpha < 1/3$	$\alpha = 1/3$	$1/3 < \alpha < 2/3$	$2/3 \leq \alpha$
p	$n^{-2/3} \log^{1/3} n$	$n^{-2/3}$	$n^{-1} \log n$	n^{-1}	0

G with minimum degree $\delta(G) \geq \alpha n$, a.a.s. there is a triangle factor in $G \cup G(n, p)$ provided that $p \geq Cn^{-2/3}$ if $\alpha \in (0, 1/3)$ and $p \geq Cn^{-1}$ if $\alpha \in (1/3, 2/3)$.

In this paper, we close the remaining open case for the triangle factor, that is $\alpha = 1/3$, for which we show that $p \geq C \log n/n$ is sufficient.

Theorem 1.3. *There exists $C > 0$ such that for any n-vertex graph G with minimum degree $\delta(G) \geq n/3$, we can a.a.s. find a triangle factor in $G \cup G(n, p)$, provided that $p \geq C \log n/n$.*

To see that the bound on p in Theorem 1.3 is asymptotically optimal consider the complete bipartite graph $G = K_{n/3, 2n/3}$ and denote the partition classes by A and B with $|A| < |B|$. By Markov's inequality and with $p \leq \frac{1}{2} \log n/n$ a.a.s. there are $O(\log^4 n)$ triangles within B and a.a.s. there is a polynomial number of vertices in the class B without any neighbours in B [19, Theorem 6.36]. However, for a triangle factor to exist, for each triangle with at most one vertex in B, there must be at least one triangle fully contained in B. In conclusion, a.a.s. $G \cup G(n, p)$ does not contain a triangle factor, and the $\log n$-term is needed for local reasons similarly as discussed above for the triangle factor in $G(n, p)$.

This result closes the problem of determining, given $\alpha \in [0, 1]$, the threshold for a triangle factor in $G_\alpha \cup G(n, p)$ and we refer to Table 1 for a summary. Note that the threshold is within a constant factor in the intervals $(0, 1/3)$ and $(1/3, 2/3)$, while it jumps at $\alpha = 0, 1/3$, and $2/3$.

Theorem 1.3 is a special case of the following theorem, which is our main result. It answers the question which minimum degree condition is needed in the randomly perturbed graph model with $p = C \log n/n$ to enforce k vertex-disjoint triangles for any $1 \leq k \leq \lfloor n/3 \rfloor$.

Theorem 1.4 (Main result). *There exists $C > 0$ such that for any n-vertex graph G we can a.a.s. find at least $\min\{\delta(G), \lfloor n/3 \rfloor\}$ pairwise vertex-disjoint triangles in $G \cup G(n, p)$, provided that $p \geq C \log n/n$.*

This is a perturbed version of the result by Dirac on vertex-disjoint triangles in dense graphs (Theorem 1.1). We are not aware of other results in the randomly perturbed graph model that consider large but not spanning structures.

Theorem 1.4 is basically optimal in terms of the number of triangles, because given $1 \leq m < n/3$, then $G = K_{m, n-m}$ has minimum degree $\delta(G) = m$, and there can be at most m pairwise vertex-disjoint triangles using each at least one edge of G, and at most $O(\log^4 n)$ additional triangles solely coming from $G(n, p)$. The bound on p is asymptotically optimal as it is in Theorem 1.3, but we remark that when m is 'significantly smaller' than $n/3$, then already $p \geq C/n$ is sufficient to a.a.s. find m pairwise vertex-disjoint triangles in $K_{m, n-m} \cup G(n, p)$. We call $K_{m, n-m}$ the extremal graph. See the concluding remarks (Section 9) for more details.

In addition, we prove a stability version (Theorem 2.2) of our main result, which allows us to work with edge probability only $p = C/n$ for graphs G that are not 'close'[1] to the extremal graph.

We believe that the methods we introduce for proving our results are valuable for other questions concerning randomly perturbed graphs; we discuss some possible directions and open problems in Section 9. One important novel ingredient in our proofs is that we can find a triangle factor in a graph on three vertex sets U, V, W of the same size, where (V, U) and (V, W) are

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[1]Basically this is in edit distance, but in addition we require that no vertex can have a degree in G that is much smaller than the corresponding degree in the extremal graph. A precise condition is given in Definition 2.1.

super-regular pairs and between U and W we have random edges with probability $p \geq C \log n/n$ (see Lemma 4.1).

Organisation

The rest of this paper is organised as follows. In Section 2, we state our Stability Theorem (Theorem 2.2), one result (Theorem 2.3) that deals with the 'extremal' case of Theorem 1.4, and one result (Theorem 2.4) that deals with the case of small minimum degrees; we shall show that these three theorems together imply our main result (Theorem 1.4).

In Section 3, we then introduce some tools that we will use later. In Section 4, we outline the proofs of Theorems 2.2, 2.3, and 2.4 and we state the auxiliary lemmas we use in their proofs. In Section 5 we prove Theorem 2.3, in Section 6 we prove Theorem 2.2, and in Section 7 we prove Theorem 2.4. The auxiliary lemmas are proved in Section 8.

Finally, we give concluding remarks and pose some open questions in Section 9. A few supplementary proofs are moved to Appendix A.

Notation

For numbers a, b, c, we write $a = b \pm c$ for $b - c \leq a \leq b + c$. Moreover, for non-negative a,b, we write $0 < a \ll b$, when we require $a \leq f(b)$ for some function $f: \mathbb{R}_{>0} \mapsto \mathbb{R}_{>0}$. We will only use this to improve readability and in addition to the precise dependencies of the constants.

We use standard graph theory notation. For a graph G on vertex set V and two disjoint sets $A, B \subseteq V$, let $G[A]$ be the subgraph of G induced by A, $G[A, B]$ be the bipartite subgraph of G induced by sets A and B, $e(A)$ be the number of edges with both endpoints in A and $e(A, B)$ be the number of edges with one endpoint in A and the other one in B. We will also use standard Landau notation for $f, g : \mathbb{N} \to \mathbb{R}_{>0} : f = o(g)$ if and only if $\lim_{n \to \infty} f(n)/g(n) = 0$ and $f = \omega(g)$ if and only if $g = o(f)$.

2. Stability version and proof of the main result

We already discussed how the probability $p \geq C \log n/n$ cannot be significantly lowered in Theorem 1.4. However, we are able to show that when the minimum degree of G is linear in n, then with $m = \min\{\delta(G), n/3\}$, the complete bipartite graph $K_{m,n-m}$ is the unique extremal graph for Theorem 1.4, in the sense that if the graph G is not 'close' to $K_{m,n-m}$ then a.a.s. $G \cup G(n, p)$ contains m pairwise vertex-disjoint triangles already at probability $p \geq C/n$ and we can even assume a slightly smaller minimum degree on G. To formalise this, we introduce the following notion of stability for an n-vertex graph G.

Definition 2.1 ((α, β)-stable). For $0 < \beta < \alpha < 1/2$, we say that an n-vertex graph G is (α, β)-stable if there exists a partition of $V(G)$ into two sets A and B of size $|A| = (\alpha \pm \beta)n$ and $|B| = (1 - \alpha \pm \beta)n$ such that the minimum degree of the bipartite subgraph $G[A, B]$ of G induced by A and B is at least $\alpha n/4$, all but at most βn vertices from A have degree at least $|B| - \beta n$ into B, all but at most βn vertices from B have degree at least $|A| - \beta n$ into A, and $G[B]$ contains at most βn^2 edges.

The stability condition with $\alpha = 1/3$ says that the size of B is roughly double the size of A, there is a minimum degree condition between A and B, in each part all but at most a few vertices see most of the other part, and the set B is almost independent. Note that for $0 < \alpha \leq 1/3$ and $m = \alpha n$ an integer, the complete bipartite graph $K_{m,n-m}$ is (α, β)-stable with $\beta = 0$. We prove the following stability result in Section 6.

Theorem 2.2 (Stability Theorem). *For $0 < \beta < 1/12$ there exist $\gamma > 0$ and $C > 0$ such that for any α with $4\beta \leq \alpha \leq 1/3$ the following holds. Let G be an n-vertex graph with minimum degree $\delta(G) \geq$*

$(\alpha - \gamma) n$ *that is not (α, β)-stable. With $p \geq C/n$ a.a.s. the perturbed graph $G \cup G(n, p)$ contains at least $\min\{\alpha n, \lfloor n/3 \rfloor\}$ pairwise vertex-disjoint triangles.*

The result is best possible as G can be bipartite and have no triangles, in which case we need at least a linear number of edges from the random graph to find a linear number of pairwise vertex-disjoint triangles in $G \cup G(n, p)$. On the other hand, the logarithmic factor is needed for the extremal graph. When the graph G is (α, β)-stable for a small enough $\beta > 0$, then we prove the following in Section 5.

Theorem 2.3 (Extremal Theorem). *For $0 < \alpha_0 \leq 1/3$ there exist $\beta, \gamma > 0$ and $C > 0$ such that for any α with $\alpha_0 \leq \alpha \leq 1/3$ the following holds. Let G be an n-vertex graph with minimum degree $\delta(G) \geq (\alpha - \gamma) n$ that is (α, β)-stable. With $p \geq C \log n/n$ a.a.s. the perturbed graph $G \cup G(n, p)$ contains at least $\min\{\delta(G), \lfloor \alpha n \rfloor\}$ pairwise vertex-disjoint triangles.*

Indeed our argument will give slightly more. If G is (α, β)-stable and $|A| \geq \alpha n$, then we can a.a.s. find $\lceil \alpha n \rceil$ if $\alpha < 1/3$ and $\lfloor n/3 \rfloor$ if $\alpha = 1/3$ pairwise vertex-disjoint triangles in $G \cup G(n, p)$ (even when the minimum degree in G is smaller than αn). Also, although as discussed above the $\log n$-factor cannot be avoided in general, when $|A| - \alpha n$ is linear in n our proof does not need such a $\log n$-factor.

The proofs of Theorem 2.2 and 2.3 use regularity. When the minimum degree gets smaller, we can avoid the transition to sparse regularity and prove the following in Section 7 with a more elementary argument.

Theorem 2.4 (Sublinear Theorem). *There exists $C > 0$ such that the following holds for any $1 \leq m \leq n/256$ and any n-vertex graph G of minimum degree $\delta(G) \geq m$. With $p \geq C \log n/n$ a.a.s. the perturbed graph $G \cup G(n, p)$ contains at least m pairwise vertex-disjoint triangles.*

Theorem 1.4 easily follows from Theorems 2.2, 2.3, and 2.4.

Proof of Theorem 1.4. Let $\beta_{2.3}, \gamma_{2.3} > 0$ and $C_{2.3}$ be given by Theorem 2.3 on input $\alpha_0 = 1/256$. Then let $\gamma_{2.2} > 0$ and $C_{2.2}$ be given by Theorem 2.2 on input $\beta = \min\{\alpha_0/4, \beta_{2.3}\}$. Moreover, let $C_{2.4}$ be given by Theorem 2.4. Define $C = \max\{C_{2.3}, C_{2.4}\}$ and $\gamma = \min\{\gamma_{2.3}, \gamma_{2.2}\}$.

Let G be any n-vertex graph and $p \geq C \log n/n$, and define $m = \min\{\delta(G), \lfloor n/3 \rfloor\}$. If $m \leq n/256$, then we get from Theorem 2.4 that a.a.s. $G \cup G(n, p)$ contains at least m pairwise vertex-disjoint triangles, as $C \geq C_{2.4}$. Otherwise, $m > n/256$ and we can choose $\alpha \in (\alpha_0, 1/3]$ such that $(\alpha - \gamma)n \leq m \leq \alpha n$. If G is (α, β)-stable, then G is also $(\alpha, \beta_{2.3})$-stable and, by Theorem 2.3, there are a.a.s. at least $\min\{\delta(G), \lfloor \alpha n \rfloor\} \geq m$ pairwise vertex-disjoint triangles in $G \cup G(n, p)$, as $\alpha_0 < \alpha \leq 1/3$ and $C \geq C_{2.3}$. Otherwise, G is not (α, β)-stable and, by Theorem 2.2, a.a.s. $G \cup G(n, p)$ contains at least $\min\{\alpha n, \lfloor n/3 \rfloor\} \geq m$ pairwise vertex-disjoint triangles, as $p = \omega(1/n)$. □

3. Tools

We will repeatedly use the following concentration inequality due to Chernoff (see e.g. [19, Corollaries 2.3 and 2.4]).

Lemma 3.1 (Chernoff's inequality). *Let X be the sum of independent binomial random variables, then for any $\delta \in (0, 1)$ we have*

$$\mathbb{P}\left[|X - \mathbb{E}[X]| \geq \delta \, \mathbb{E}[X]\right] \leq 2 \exp\left(-\frac{\delta^2}{3} \mathbb{E}[X]\right).$$

Moreover, for any $k \geq 7 \, \mathbb{E}[X]$, we have $\mathbb{P}[X > k] \leq \exp(-k)$.

The following lemma allows us to find specific triangles in a dense graph with additional random edges. The proof is a standard application of Janson's inequality (see e.g. [19, Theorem 2.18]), and it is included in Appendix A.

Lemma 3.2. *For any $d > 0$ there exists $C > 0$ such that the following holds. Let U, V, W be three sets of vertices of size n, G be a bipartite graph on (U, W) with $e(G) \geq dn^2$, $p > C/n$ and $G(U \cup W, V, p)$ be the random bipartite graph. Then with probability at least $1 - 2^{-4n/d}$ there is a triangle in $G \cup G(U \cup W, V, p)$ with one vertex in each of U, V, W.*

We will use Szemerédi's Regularity Lemma [30] and some of its consequences. Before stating it, we introduce the relevant terminology. The *density* of a pair (A, B) of disjoint sets of vertices is defined by

$$d(A, B) = \frac{e(A, B)}{|A| \cdot |B|}$$

and the pair (A, B) is called ε-*regular*, if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have $|d(A, B) - d(X, Y)| \leq \varepsilon$. Without further mentioning it, we will repeatedly use that when $\varepsilon < 1/2$ any $A' \subseteq A$ with $|A'| \geq |A|/2$ and $B' \subseteq B$ with $|B'| \geq |B|/2$ give a 2ε-regular pair (A', B') with density at least $d(A, B) - \varepsilon$.

We will also use the following well-known result that follows from the definition.

Lemma 3.3 (Minimum Degree Lemma). *Let (A, B) be an ε-regular pair with $d(A, B) = d$. Then for every $Y \subseteq B$ with $|Y| \geq \varepsilon|B|$, the number of vertices from A with degree into Y less than $(d - \varepsilon)|Y|$ is at most $\varepsilon|A|$.*

With $d \in [0, 1)$ a pair (A, B) is called (ε, d)-*super-regular*, if for all sets $X \subseteq A$ and $Y \subseteq B$ with $|X| \geq \varepsilon|A|$ and $|Y| \geq \varepsilon|B|$ we have $d(X, Y) \geq d$ and $\deg(a) \geq d|B|$ for all $a \in A$ and $\deg(b) \geq d|A|$ for all $b \in B$. It is easy to prove with Hall's Theorem that a super-regular pair with parts of the same size contains a perfect matching.

Lemma 3.4. *For any $d > 0$ there exists $\varepsilon > 0$ such that any (ε, d)-super-regular pair (U, V) with $|U| = |V|$ contains a perfect matching.*

We will use the following well-known degree form of the regularity lemma that can be derived from the original version [30].

Lemma 3.5 ([22]). *For every $\varepsilon > 0$ and integer t_0 there exists an integer $T > t_0$ such that for any graph G on at least T vertices and $d \in [0, 1]$ there is a partition of $V(G)$ into $t_0 < t + 1 \leq T$ sets $V_0, \ldots, V_t$ and a subgraph G' of G such that*

(P1) *$|V_i| = |V_j|$ for all $1 \leq i, j \leq t$ and $|V_0| \leq \varepsilon|V(G)|$,*

(P2) *$\deg_{G'}(v) \geq \deg_G(v) - (d + \varepsilon)|V(G)|$ for all $v \in V(G)$,*

(P3) *the set V_i is independent in G' for all $1 \leq i \leq t$,*

(P4) *for all $1 \leq i < j \leq t$, the pair (V_i, V_j) is ε-regular in G' and has density either 0 or at least d.*

The sets $V_1, \ldots, V_t$ are also called clusters, and we refer to V_0 as the set of exceptional vertices. We call a partition $V_0, \ldots, V_t$, which satisfies (P1)–(P4), a (ε, d)-*regular partition* of G. Given this partition, we define the (ε, d)-*reduced graph* R for G, that is the graph on the vertex set $[t]$, where ij is an edge if and only if (V_i, V_j) is an ε-regular pair in G' and has density at least d.

We will also use the following result on perfect matchings in random subgraphs of bipartite graphs with large minimum degree. For any given graph G, we denote by G_p the random graph model, where we keep each edge of G with probability p, independently from all other choices.

Lemma 3.6. *For any $\varepsilon > 0$ there exists $C > 0$ such that the following holds for any bipartite graph G with partition classes $|U| = |W| = n$ and minimum degree $\delta(G) \geq (1/2 + \varepsilon)n$. With $p \geq C \log n/n$ there is a.a.s. a perfect matching in G_p.*

The proof is standard and closely follows the proof for the perfect matching threshold in the random bipartite graph $G(n, n, p)$ (see [19, Theorem 4.1]). For completeness, we include it in Appendix A.

4. Proof overview and main Lemmas

In this section, we sketch the ideas behind our proof of Theorem 2.2, 2.3, and 2.4, and we give the statements of the lemmas we use. For simplicity, when outlining the proof of Theorem 2.2 and 2.3, we assume $\alpha = 1/3$, n is a multiple of 3, and G is an n-vertex graph with minimum degree $\delta(G) \geq n/3$, in which case both theorems give a triangle factor in $G \cup G(n, p)$.

4.1 Extremal case

Assume that G is $(1/3, \beta)$-stable and let $p \geq C \log n/n$. The definition of stability (Definition 2.1) gives a partition of $V(G)$ into $A \cup B$ where the size of B is roughly the double of the size of A, there is a minimum degree condition between A and B, and in each part all but at most a few vertices see all but at most few a vertices of the other part. Our proof will follow three steps. Firstly, we find a collection of triangles $\mathcal{T}_1$, such that after removing the triangles of $\mathcal{T}_1$, we are left with two sets $A_1 = A \setminus V(\mathcal{T}_1)$ and $B_1 = B \setminus V(\mathcal{T}_1)$ with $|B_1| = 2|A_1|$. The way we find these triangles depend on the sizes of A and B, and we will use two different approaches when $|B| > 2n/3$ and $|B| \leq 2n/3$. In particular, when $|B| > 2n/3$, we need to find some triangles entirely within B, just using the minimum degree $n/3 - |A|$ and random edges. For that we will use Theorem 2.4.

Our second step is to cover the vertices in A_1 and B_1 that do not have a high degree to the other part; this will give two collections of triangles $\mathcal{T}_2$ and $\mathcal{T}_3$. Each such triangle has one vertex in A_1 and two vertices in B_1 so that we still have $|B_2| = 2|A_2|$, where $A_2 = A_1 \setminus V(\mathcal{T}_2 \cup \mathcal{T}_3)$ and $B_2 = B_1 \setminus V(\mathcal{T}_2 \cup \mathcal{T}_3)$. Moreover, at this point, each vertex sees all but at most a few vertices of the other part. We are now ready for the last step. We split B_2 arbitrarily into two subsets B_2' and B_2'' of equal size and we obtain that (B_2', A_2, B_2'') is a super-regular cherry, i.e. both (B_2', A_2) and (B_2'', A_2) are super-regular pairs. We want to find a triangle factor covering the cherry, with the help of random edges between B_2' and B_2''. The next lemma, which encapsulates the main idea of our paper, takes care of this and will be proved in Section 8.

Lemma 4.1. *For any $0 < d < 1$ there exist $\varepsilon > 0$ and $C > 0$ such that the following holds. Let U, V, W be sets of size n, let (V, U) and (V, W) be (ε, d)-super-regular pairs and let $G(U, W, p)$ be a random bipartite graph with $p \geq C \log n/n$. Then a.a.s. there exists a triangle factor.*

Thus, we are able to cover the cherry with a triangle factor $\mathcal{T}_4$, and we conclude observing that the collection of triangles $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4$ gives a triangle factor in $G \cup G(n, p)$.

4.2 Non-extremal case

Before giving an overview of Theorem 2.2, it is worth to make some comments about Lemma 4.1. We point out that the probability p cannot be significantly lowered. Indeed, a triangle factor in the setting of Lemma 4.1 gives a perfect matching in the random bipartite graph on vertex set $U \cup W$ and this is a.a.s. not possible with $p \leq \frac{1}{2} \log n/n$ [19, Theorem 4.1]. However, we would like to be able to find a triangle factor in a super-regular cherry with the help of the random edges also in the proof of Theorem 2.2, where we claimed that when the graph G is not $(1/3, \beta)$-stable, already $p \geq C/n$ is sufficient. For that we will use the following variation of Lemma 4.1, where

the improvement on the probability comes from the assumption that the super-regular cherry U, V, W is a bit unbalanced as the sizes of U and W are smaller than the size of V, and thus the random bipartite graph on vertex set $U \cup W$ will be used to build a large matching covering all but a small linear fraction of vertices, which is possible already with $p \geq C/n$. Note that here we need to use random edges within V.

Lemma 4.2. *For any $0 < \delta' \leq d < 1$ there exist $\delta_0, \delta, \varepsilon$ with $\delta' \geq \delta_0 > \delta > \varepsilon > 0$ and $C > 0$ such that the following holds. Let U, V, W be sets of size $|V| = n$ and $(1 - \delta_0)n \leq |U| = |W| \leq (1 - \delta)n$ where $|V| + |U| + |W| \equiv 0 \pmod 3$. Further, let (V, U) and (V, W) be (ε, d)-super-regular pairs and let $G(V, p)$, $G(U, W, p)$ be random graphs with $p \geq C/n$. Then a.a.s. there exists a triangle factor.*

From Lemma 4.2, we also derive the following result about the existence of a triangle factor in a super-regular pair edge, again with the help of the random edges. Lemmas 4.2, and 4.3 will be proved in Section 8.

Lemma 4.3. *For any $0 < d < 1$ there exist $\varepsilon > 0$ and $C > 0$ such the following holds for sets U, V of size $|V| = n$ and $3n/4 \leq |U| \leq n$ where $|V| + |U| \equiv 0 \pmod 3$. If (U, V) is an (ε, d)-super-regular pair and $G(U, p)$ and $G(V, p)$ are random graphs with $p \geq C/n$, then a.a.s. there exists a triangle factor.*

Now we turn to the overview of the proof for Theorem 2.2. Assume that G is not $(1/3, \beta)$-stable and let $p \geq C/n$. We apply the regularity lemma to G and obtain the reduced graph R. By adjusting an argument of the fourth author with Balogh and Mousset [3], we can prove the following stability result.

Lemma 4.4. *For any $0 < \beta < 1/12$ there exists $d > 0$ such that the following holds for any $0 < \varepsilon < d/4$, $4\beta \leq \alpha \leq 1/3$, and $t \geq 10/d$. Let G be an n-vertex graph with minimum degree $\delta(G) \geq (\alpha - d/2)n$ that is not (α, β)-stable and let R be the (ε, d)-reduced graph for some (ε, d)-regular partition $V_0, \ldots, V_t$ of G. Then R contains a matching M of size $(\alpha + 2d)t$.*

For completeness, we give the proof in Appendix A. It follows that we can cover the vertices of R with cherries $K_{1,2}$ and matching edges $K_{1,1}$, such that there are not too many cherries.[2] Before we can apply Lemma 4.2 to each cherry and Lemma 4.3 to each matching edge, some preliminary steps are needed. We remove some vertices from each cherry to make it unbalanced and ensure that both edges are super-regular. Then we cover all vertices that are not contained in any of the cherries or edges by finding a collection of triangles $\mathcal{T}_1$. We construct another collection of triangles $\mathcal{T}_2$ to ensure that in each cherry the relations between the three sets are as required by Lemma 4.2. For constructing $\mathcal{T}_1$ and $\mathcal{T}_2$, we will mainly rely on the minimum degree condition of G and the fact that in the probability $p \geq C/n$, the constant C can be chosen large enough so that a.a.s. the following holds: each linear-sized set contains a random edge and for any not too small part of a regular pair and a linear-sized set there is a triangle containing an edge form the pair and the third vertex from the set. Finally, we can use Lemma 4.2 and Lemma 4.3 to cover the remaining vertices with a collection of triangles $\mathcal{T}_3$. Together $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ gives a triangle factor in $G \cup G(n, p)$.

We mention already now that when α is sufficiently smaller than $1/3$ and the condition on the minimum degree of G reads as $\delta(G) \geq (\alpha - \gamma)n$, many of the steps outlined above are not necessary. In this case indeed we do not have to cover all graph with triangles and we only want to find αn pairwise vertex-disjoint triangles in $G \cup G(n, p)$. We will see that an application of Lemmas 4.2 and 4.3 to the cherries and the matching edges found at the beginning (after having made them super-regular and suitable for Lemma 4.2) is already enough to find these αn triangles.

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[2]We remark that covers of the reduced graph by stars were used in [4, 24], and this inspired our approach. For our purposes, it is necessary that we cover the reduced graph by cherries and matching edges. Furthermore, in contrast to [4] we cannot rely on any triangles in the random graph but need to use the edges of G to build them.

4.3 Sublinear case

Assume G is an n-vertex graph with minimum degree $\delta(G) \geq m$ and let $p \geq C \log n/n$. We want to show that a.a.s. there exist m pairwise vertex-disjoint triangles in $G \cup G(n, p)$. Any vertex of large degree in G can easily be covered by a triangle later, so we assume an upper bound on the maximum degree $\Delta(G)$ of the graph G. With this condition, we split the proof in three ranges for the value of m: $1 \leq m \leq (\log n)^3$, $(\log n)^3 \leq m \leq \sqrt{n}$, and $\sqrt{n} \leq m \leq n/256$. If $1 \leq m \leq (\log n)^3$ a.a.s. m pairwise vertex-disjoint triangles already exist in $G(n, p)$. If $(\log n)^3 \leq m \leq \sqrt{n}$ we will find many large enough vertex-disjoint stars in G (see Lemma 7.3) and a.a.s. at least m of them will be completed to triangles using edges of $G(n, p)$ (see Proposition 7.1). However if $m > \sqrt{n}$ we cannot hope to find m large enough vertex-disjoint stars and instead we will apply a greedy strategy using that a.a.s. every vertex has an edge in its neighbourhood (see Proposition 7.2).

5. Proof of the extremal theorem

Proof of Theorem 2.3. Let $0 < \alpha_0 \leq 1/3$ and choose $d = 1/2$. Let $\varepsilon > 0$ and $C_{4.1} > 0$ be given by Lemma 4.1 on input d. We can assume $\varepsilon < 1/5$ and then choose $0 < \beta < \alpha_0 \varepsilon/36$ and $0 < \gamma < \beta/11$. With $C_{2.4}$ given by Theorem 2.4, let $C \geq 2C_{2.4} + 1 + 2C_{4.1}/\alpha_0$. Finally, let $\alpha_0 \leq \alpha \leq 1/3$.

Given n, let $p \geq C \log n/n$. With our choice of C, we can reveal $G(n, p)$ in three rounds $G_1 \sim G(n, 2C_{2.4} \log n/n)$, $G_2 \sim G(n, \log n/n)$, and $G_3 \sim G(n, \frac{2C_{4.1}}{\alpha_0} \log n/n)$. We will only know later in which subset we will use G_1 and G_3, but we have that a.a.s. there is an edge of G_2 between any two not necessarily disjoint sets of size βn. Indeed, fixed two such sets, the probability that there is no edge of G_2 is at most $(1 - \log n/n)^{(\beta n)^2} \leq \exp(-\beta^2 n \log n)$, and we conclude by an union bound over the at most 2^{2n} choices for the two sets. Now let G be an n-vertex graph with minimum degree $\delta(G) \geq (\alpha - \gamma)n$ that is (α, β)-stable and define $m_0 = \max\{n/3 - \delta(G), n/3 - \lfloor \alpha n \rfloor\}$. Our goal is to a.a.s. find pairwise vertex-disjoint triangles in $G \cup G(n, p)$ such that at most $3m_0$ vertices are left uncovered.

To aid with calculations we let $\kappa = n/3 - \lceil \alpha n \rceil$ and observe that $\kappa \in \{0, -1/3, -2/3\}$ if $\alpha = 1/3$ and that $\kappa > 0$ if $\alpha < 1/3$ and n large enough. Also note that $m_0 - \kappa$ is an integer and that $3\kappa = \lfloor (1 - \alpha)n \rfloor - 2\lceil \alpha n \rceil$. With this we set $w = \max\{3\kappa, 0\}$. As G is (α, β)-stable we get a partition of $V(G)$ into sets A and B satisfying the conditions of Definition 2.1.

Claim 5.1. *There a.a.s. are a collection of triangles $\mathcal{T}_1$ in $G \cup G_1 \cup G_2$ with $|\mathcal{T}_1| \leq \beta n$ and a set $W \subseteq V(G) \setminus V(\mathcal{T}_1)$ with $|W| \leq 3m_0 - w$ such that the following holds. For $A_1 = A \setminus (V(\mathcal{T}_1) \cup W)$ and $B_1 = B \setminus (V(\mathcal{T}_1) \cup W)$, we have that $|A_1| \leq \lceil \alpha n \rceil$, $|B_1| = 2|A_1| + w$, the minimum degree between A_1 and B_1 is at least $\alpha n/5$, all but at most βn vertices of A_1 have degree at least $|B_1| - \beta n$ into B_1, and all but at most βn vertices of B_1 have degree at least $|A_1| - \beta n$ into A_1.*

The sets A_1 and B_1 partition $V(G) \setminus (V(\mathcal{T}_1) \cup W)$ and, after proving Claim 5.1, we will cover all but w vertices from $A_1 \cup B_1$ with additional triangles. Hence, if we manage to find these triangles, we have covered all but $|W| + w \leq 3m_0$ vertices, as desired. We remark for later that $|W| \leq 3m_0 - w \leq 4\gamma n$.

Proof of Claim 5.1. We have either $|B| > \lfloor (1 - \alpha)n \rfloor$ or $|A| \geq \lceil \alpha n \rceil$. First suppose that we are in the first case, where $|B| = \lfloor (1 - \alpha)n \rfloor + m$ for some $1 \leq m \leq \beta n$ (and $|A| = \lceil \alpha n \rceil - m$), and note that

$$|B| - 2|A| = n - 3\lceil \alpha n \rceil + 3m = 3m + 3\kappa > 0.$$

If $1 \leq m \leq m_0 - \kappa$, then $0 < 3m \leq 3m_0 - 3\kappa$ and we let W be any set with $\min\{3m, 3m + 3\kappa\} \leq 3m_0 - w$ vertices from B. Then with the choice of $\mathcal{T}_1 = \emptyset$, we have that the sets $A_1 = A$ and $B_1 = B \setminus W$ partition $V(G) \setminus W$, and $|A_1| = \lceil \alpha n \rceil - m$ and $|B_1| = |B| - \min\{3m, 3m + 3\kappa\} = 2|A_1| + w$.

If on the other hand, $m_0 < m + \kappa$, then

$$\delta(G[B]) \geq \delta(G) - |A| \geq (n/3 - m_0) - (\lceil \alpha n \rceil - m) = m - m_0 + \kappa > 0$$

and we observe that $m - m_0 + \kappa$ is an integer. Moreover, $m - m_0 + \kappa \leq m \leq |B|/256$, where we use $m_0 - \kappa \geq 0$, $m \leq \beta n \leq \alpha_0 \varepsilon n/36 \leq n/(3 \cdot 5 \cdot 36)$ and $|B| = \lfloor (1 - \alpha)n \rfloor + m \geq n/2 + m$. Thus, by Theorem 2.4 and as $2C_{2.4} \log n/n \geq C_{2.4} \log |B|/|B|$, we a.a.s. find $m - m_0 + \kappa$ pairwise vertex-disjoint triangles in $(G \cup G_1)[B]$. Denote by $\mathcal{T}_1$ the collection of these $m - m_0 + \kappa$ triangles. Let W be any set of $3m_0 - w$ vertices from B not covered by any triangle in $\mathcal{T}_1$. Then the sets $A_1 = A$ and $B_1 = B \setminus (V(\mathcal{T}_1) \cup W)$ partition $V(G) \setminus (V(\mathcal{T}_1) \cup W)$, and we have $|A_1| = \lceil \alpha n \rceil - m$ and

$$|B_1| = |B| - 3(m - m_0 + \kappa) - (3m_0 - w) = 2|A_1| + w.$$

It remains to consider the second case, where $|A| = \lceil \alpha n \rceil + m$ for some $0 \leq m \leq \beta n$. First, we greedily pick m pairwise vertex-disjoint triangles in $G \cup G_2$ each with two vertices in A and one vertex in B. Indeed during the process, there is always a vertex v in B, not yet contained in a triangle, with at least $\deg(v, A) - 2m \geq (\alpha/4 - 2\beta)n \geq \beta n$ uncovered neighbours in A in the graph G. By the property assumed in G_2, we can then find an edge within these neighbours of v to get a triangle. Denote by $\mathcal{T}_1$ the collection of these m triangles.

If $\kappa \geq 0$, then, with the choice of $W = \emptyset$, we have that $A_1 = A \setminus V(\mathcal{T}_1)$ and $B_1 = B \setminus V(\mathcal{T}_1)$ partition $V(G) \setminus V(\mathcal{T}_1)$ and

$$|B_1| = |B| - |\mathcal{T}_1| = \lfloor (1 - \alpha)n \rfloor - 2m = 2(\lceil \alpha n \rceil - m) + 3\kappa = 2|A_1| + w.$$

If $\kappa < 0$, we additionally pick a set W of vertices not covered by triangles from $\mathcal{T}_1$, such that $|W| = 1$, $|W \cap A| = 1$, $|W \cap B| = 0$ if $\kappa = -2/3$, and $|W| = 2$, $|W \cap A| = |W \cap B| = 1$ if $\kappa = -1/3$. Then, the sets $A_1 = A \setminus (V(\mathcal{T}_1) \cup W)$ and $B_1 = B \setminus (V(\mathcal{T}_1) \cup W)$ partition $V(G) \setminus (V(\mathcal{T}_1) \cup W)$, and $|B_1| = 2|A_1| + w$. Indeed, if $\kappa = -2/3$, we have $|A_1| = |A| - 2|\mathcal{T}_1| - 1 = \lceil \alpha n \rceil - m - 1$ and

$$|B_1| = |B| - |\mathcal{T}_1| = \lfloor (1 - \alpha)n \rfloor - 2m = 2(\lceil \alpha n \rceil - m) + 3\kappa = 2|A_1|$$

and if $\kappa = -1/3$ we have $|A_1| = |A| - 2|\mathcal{T}_1| - 1 = \lceil \alpha n \rceil - m - 1$ and

$$|B_1| = |B| - |\mathcal{T}_1| - 1 = \lfloor (1 - \alpha)n \rfloor - 2m - 1 = 2(\lceil \alpha n \rceil - m) + 3\kappa - 1 = 2|A_1|.$$

Observe, that in both the first and the second case $|B_1| = 2|A_1| + w$ and $|W| \leq 3m_0 - w$. Moreover, as we remove at most $3m_0 - w \leq 4\gamma n \leq \alpha n/20$ vertices from each A and B, the minimum degree between A_1 and B_1 is at least $\alpha n/4 - \alpha n/20 = \alpha n/5$. The other conditions on the degrees between A_1 and B_1 are clearly satisfied, because for all but at most βn vertices from each set there are still at most βn non-neighbours in the other set. The bounds $|\mathcal{T}| \leq \beta n$ and $|A_1| \leq \lceil \alpha n \rceil$ also hold in all cases. □

We want to cover all but w vertices in $A_1 \cup B_1$ and we start from those vertices in A_1 and B_1 that do not have a high degree to the other part. We will always cover them with triangles with one vertex in A_1 and two vertices in B_1 to ensure that the relation between the number of vertices remaining in A_1 and B_1 does not change. Let

$$\tilde{A}_1 = \{v \in A_1 : \deg(v, B_1) \leq |B_1| - 9\beta n\} \text{ and } \tilde{B}_1 = \{v \in B_1 : \deg(v, A_1) \leq |A_1| - 9\beta n\},$$

and observe that $|\tilde{A}_1|, |\tilde{B}_1| \leq \beta n$.

We claim that a.a.s. we can greedily pick pairwise vertex-disjoint triangles in $(G \cup G_2)[A_1 \cup B_1]$ that cover all vertices of $\tilde{A}_1$, with each triangle having one vertex in $\tilde{A}_1$ and two vertices in $B_1 \setminus \tilde{B}_1$. Indeed, at each step during the process, an uncovered vertex v in $\tilde{A}_1$ has at least $\deg(v, B_1) - |\tilde{B}_1| - 2|\tilde{A}_1| \geq (\alpha/5 - 3\beta)n \geq \beta n$ uncovered neighbours in $B_1 \setminus \tilde{B}_1$ in the graph G. We then find an edge of G_2 within these neighbours of v and build a triangle. Denote by $\mathcal{T}_2$ the collection of these triangles and note that $|\mathcal{T}_2| \leq \beta n$.

Observe that at this point $2|\mathcal{T}_2| \leq 2\beta n$ vertices of $B_1 \setminus \tilde{B}_1$ have already been covered. We claim that a.a.s. we can greedily pick pairwise vertex-disjoint triangles in $(G \cup G_2)[(A_1 \cup B_1) \setminus V(\mathcal{T}_2)]$ that cover all vertices of $\tilde{B}_1$, where each triangle has one vertex in $A_1 \setminus \tilde{A}_1$, one vertex in $\tilde{B}_1$, and one vertex in $B_1 \setminus \tilde{B}_1$. Indeed, at each step during the process, an uncovered vertex v in $\tilde{B}_1$ has at least $\deg(v, A_1) - |\tilde{A}_1| - |\tilde{B}_1| \geq (\alpha/5 - 2\beta)n \geq \beta n$ uncovered neighbours in $A_1 \setminus \tilde{A}_1$ in the graph G and at least

$$\delta(G) - 3|\mathcal{T}_1| - |W| - \deg(v, A_1) - 2|\mathcal{T}_2| - 2|\tilde{B}_1|$$
$$\geq (\alpha - \gamma)n - 3\beta n - 4\gamma n - (\lceil \alpha n \rceil - 9\beta n) - 4\beta n \geq \beta n$$

uncovered neighbours in $B_1 \setminus \tilde{B}_1$ in the graph G. We then find an edge of G_2 between these two neighbourhood sets to get a triangle. Denote by $\mathcal{T}_3$ the collection of these triangles and note that $|\mathcal{T}_3| \leq \beta n$.

The sets $A_2 = A_1 \setminus V(\mathcal{T}_2 \cup \mathcal{T}_3)$ and $B_2 = B_1 \setminus V(\mathcal{T}_2 \cup \mathcal{T}_3)$ give a partition of the remaining vertices in $V(G) \setminus (V(\mathcal{T}_1) \cup V(\mathcal{T}_2) \cup V(\mathcal{T}_3) \cup W)$. We have

$$|A_2| \geq |A| - 2|\mathcal{T}_1| - |\mathcal{T}_2| - |\mathcal{T}_3| - |W| \geq \alpha n - 5\beta n - 4\gamma n \geq \alpha n/2$$

and $|B_2| = 2|A_2| + w$. Moreover, the degree from A_2 to B_2 is at least $|B_1| - 9\beta n - 2|\mathcal{T}_2 \cup \mathcal{T}_3| = |B_2| - 9\beta n$ and the degree from B_2 to A_2 is at least $|A_1| - 9\beta n - |\mathcal{T}_2 \cup \mathcal{T}_3| = |A_2| - 9\beta n$. We partition B_2 arbitrarily into three subsets B_1', B_2'', and W' of size $|B_1'| = |B_2''| = |A_2|$ and $|W'| = |B_2| - 2|A_2| = w$. Then the degree from A_2 to B_1' and the degree from A_2 to B_2'' are at least $|B_1'| - 9\beta n$.

We claim that the pair (A_2, B_1') is $(\varepsilon, 1/2)$-super-regular, with ε being chosen as stated at the beginning of the proof. Indeed for all $X \subseteq A_2$ and $Y \subseteq B_1'$ with $|X| \geq \varepsilon|A_2|$ and $|Y| \geq \varepsilon|B_1'|$, we have

$$e(X, Y) \geq |X|(|Y| - 9\beta n) \geq \frac{1}{2}|X||Y|$$

and $\deg(a, B_1') \geq |B_1'| - 9\beta n \geq |B_1'|/2$ for all $a \in A_2$, and $\deg(b, A_2) \geq |A_2| - 9\beta n \geq |A_2|/2$ for all $b \in B_1'$, where for all inequalities we use $\beta \leq \alpha/36$. For the same reason, the pair (A_2, B_2'') is $(\varepsilon, 1/2)$-super-regular as well.

Now, as $\frac{2C_{4.1}}{\alpha_0} \log n/n \geq C_{4.1} \log |A_2|/|A_2|$, we can apply Lemma 4.1 to A_2, B_1' and B_2'', with $d = 1/2$ and ε, and a.a.s. get a triangle factor $\mathcal{T}_4$ in $(G \cup G_3)[V']$, where $V' = V(G) \setminus (V(\mathcal{T}_1) \cup V(\mathcal{T}_2) \cup V(\mathcal{T}_3) \cup W \cup W')$. Then $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4$ contains at least

$$(n - |W| - |W'|)/3 \geq n/3 - m_0 \geq \min\{\delta(G), \lfloor \alpha n \rfloor\}$$

pairwise vertex-disjoint triangles covering $V(G) \setminus (W \cup W')$. $\qquad \square$

We point out that under certain conditions our proof of Theorem 2.3 gives more triangles. When $\alpha < 1/3$ and $|A| \geq \alpha n$, as $|W| \leq 3m_0 - w$, we get $\lceil \alpha n \rceil$ pairwise vertex-disjoint triangles in $G \cup G(n, p)$, even when $\delta(G) < \alpha n$. Similarly, when $\alpha = 1/3$ and $|A| \geq n/3$, as $|W| \leq 2$, we get $\lfloor n/3 \rfloor$ pairwise vertex-disjoint triangles in $G \cup G(n, p)$, even when $\delta(G) < n/3$. Moreover, for any value of α, when $|A| - n/3 > 0$ is linear in n, we could use Lemma 4.2 instead of Lemma 4.1 to avoid the log n-factor in the probability.

6. Proof of the stability theorem

Proof of Theorem 2.2. We start by defining necessary constants. Given $0 < \beta < 1/12$, let $d > 0$ be obtained from Lemma 4.4, and set $\gamma = d/2$ and $t_0 = 11/d$. Next, we take any $0 < \delta' < 160^{-2}d^2$ and use Lemma 4.2 on input $d/2$ and δ' to obtain $\delta_0, \delta, \varepsilon'$ with $\delta' \geq \delta_0 > \delta > \varepsilon' > 0$ and C_1. Additionally, we assume that C_1 is large enough and ε' is small enough for Lemma 4.3 to hold with input $d/2$. Finally, let C_2 be given by Lemma 3.2 on input $d/2$. We let $0 < \varepsilon \leq \varepsilon'/2$. In summary,

the dependencies between our constants are as follows:

$$\varepsilon \ll \varepsilon' < \delta < \delta_0 \le \delta' \ll d \ll \beta < \frac{1}{12} \quad \text{and} \quad \frac{1}{t_0}, \gamma \ll d.$$

We apply Lemma 3.5 with ε and t_0 to obtain T. We take C large enough such that, for $p \ge C/n$, the random graph $G(n, p)$ contains the union $G_1 \cup G_2 \cup G_3$, where $G_1 \sim G(n, 2C_1 T/n)$, $G_2 \sim G(n, 4C_2 T/(dn))$, and $G_3 \sim G(n, 96T^2/(d^2 n))$.

Now, for any α with $4\beta \le \alpha \le 1/3$, let G be an n-vertex graph on the vertex set V with minimum degree $\delta(G) \ge (\alpha - \gamma)n$ that is not (α, β)-stable. With the regularity lemma (Lemma 3.5) applied to G, we get G', $t_0 < t + 1 \le T$ and a partition $V_0, \ldots, V_t$ of $V(G)$ such that (P1)–(P4) hold. Define $n_0 = |V_1| = |V_2| = \cdots = |V_t|$ and observe that $(1 - \varepsilon)n/t \le n_0 \le n/t$. We denote by R the (ε, d)-reduced graph for G, that is, the graph on the vertex set $[t]$ with edges ij corresponding to ε-regular pairs (V_i, V_j) of density at least d in G'. We observe that the minimum degree of R satisfies $\delta(R) \ge (\alpha - 2d)t$ because, otherwise, thre would be vertices with degree at most $(\alpha - 2d)t(n/t) + \varepsilon n < (\alpha - \gamma)n - (d + \varepsilon)n$ in G', contradicting (P2).

The purpose of G_1 will become clear later, but we describe some useful properties of G_2 and G_3 now. Let U and W be any two clusters that give an edge in R, V any cluster, and $U' \subseteq U$, $W' \subseteq W$, $V' \subseteq V$ three pairwise disjoint subsets each of size $dn_0/2$. Then, with G_2 and as $4C_2 T/(dn) \ge 2C_2/(dn_0)$, by Lemma 3.2 we have that with probability at least $1 - 2^{-4(dn_0/2)/(d/2)} = 1 - 2^{-4n_0}$

$$\text{there is a triangle in } G \cup G_2 \text{ with one vertex in each set } U', W', V'. \tag{1}$$

With a union bound over the at most $t^3 2^{3n_0}$ choices for U, W, V and U', W', V', we conclude that a.a.s. (1) holds for all choices as above.

With G_3 we a.a.s. have that

$$\text{any set } A \text{ of size at least } dn_0/2 \text{ contains an edge of } G_3. \tag{2}$$

In fact, given any set A of size at least $dn_0/2$, the expected number of edges of G_3 in A is

$$\binom{|A|}{2} \cdot \frac{96T^2}{d^2 n} \ge \frac{1}{3} \cdot \frac{d^2 n_0^2}{4} \cdot \frac{96T^2}{d^2 n} = 8T^2 \frac{n_0^2}{n} \ge 2n,$$

where we used that $n/n_0 \le t/(1 - \varepsilon) \le 2T$. Therefore, the probability that the set A does not contain an edge of G_3 is at most $(1 - \frac{96T^2}{d^2 n})^{\binom{|A|}{2}} \le \exp\left(-\binom{|A|}{2} \cdot \frac{96T^2}{d^2 n}\right) \le \exp\left(-2n\right)$ and (2) follows from a union bound over the at most 2^n choices for A.

Now let M_1 be a largest matching in R. Since G is not (α, β)-stable, using Lemma 4.4, we conclude that $|M_1| \ge (\alpha + 2d)t$. At this point, for the sake of clarity, we split our proof into two cases – $0 < \alpha < 1/3 - d/3$ and $1/3 - d/3 \le \alpha \le 1/3$ – although some steps will be the same. The first case is indeed much easier, as we do not need to cover all the graph with triangles, while in the second case we are looking for a spanning structure and we want to find $\lfloor n/3 \rfloor$ pairwise vertex-disjoint triangles.

Case $0 < \alpha < 1/3 - d/3$

As M_1 is a largest matching in R, the set $V(R) \setminus V(M_1)$ is independent and only one endpoint of each edge of M_1 can be adjacent to more than one vertex from $V(R) \setminus V(M_1)$. Therefore, we can greedily pick a second matching M_2 such that each edge of M_2 contains a vertex of $V(R) \setminus V(M_1)$ and a vertex of $V(M_1)$, and M_2 covers at least $\min\{|V(R) \setminus V(M_1)|, \delta(R)\}$ vertices of $V(R) \setminus V(M_1)$. The two matchings M_1 and M_2 together cover a subset $V(M_1 \cup M_2) \subseteq V(R)$ of

$$2|M_1| + \min\{|V(R) \setminus V(M_1)|, \delta(R)\} \ge \min\{t, (3\alpha + 2d)t\} \ge (3\alpha + d)t$$

vertices, and we can extract a collection of $|M_2|$ vertex-disjoint cherries and a disjoint matching that cover such vertices. This gives a subgraph $R' \subseteq R$ consisting of cherries and a matching such that for all edges $ij \in E(R')$ the pair (V_i, V_j) is ε-regular of density at least d in G', and therefore in G as well. We denote by $\mathcal{J} \subseteq [t]$ the indices of the clusters V_i of the cherries and the matching

edges in R' and we observe from above that $|\mathcal{J}| \geq (3\alpha + d)t$. We add to V_0 all the vertices of G that are in the clusters V_j for $j \notin \mathcal{J}$.

Then we make all pairs associated with the edges of R' super-regular. Given a pair (A, B), by Lemma 3.3, all but at most εn_0 vertices of A (resp. B) have degree at least $(d - \varepsilon)n_0$ to B (resp. A). For every such pair we remove these vertices from A and B, and remove additional vertices to ensure all clusters have the same size. As R' only contains vertex-disjoint cherries and a disjoint matching, we can achieve that by removing a total of at most $2\varepsilon n_0$ vertices from each cluster. We add all the removed vertices to V_0. Observe that afterwards all the pairs (A, B) associated with the edges of R' are $(2\varepsilon, d - 3\varepsilon)$-super-regular, because every vertex $a \in A$ has degree at least $(d - \varepsilon)n_0 - 2\varepsilon n_0 \geq (d - 3\varepsilon)|B|$ into B, and every vertex $b \in B$ has degree at least $(d - 3\varepsilon)|A|$ into A.

Recall that for a later application of Lemma 4.2, we need that for each cherry the sizes of the leaf-clusters are smaller than the size of the centre-cluster. Thus, for each cherry ijk of R', with j being the centre, we additionally remove $\delta|V_j| \leq \delta n_0$ vertices from the leaves V_i and V_k, and add them to V_0. We have $|V_i| = |V_k| = (1 - \delta)|V_j|$ that implies $|V_i| = |V_k| \geq (1 - \delta_0)|V_j|$, as $\delta_0 > \delta$. We have that all edges of R' still give $(2\varepsilon, d - 3\varepsilon - \delta)$-super-regular pairs. Moreover,

$$\left| \bigcup_{j \in \mathcal{J}} V_j \right| \geq (1 - 2\varepsilon - \delta)n_0|\mathcal{J}| \geq (1 - \varepsilon)(1 - 2\varepsilon - \delta)(3\alpha + d)n \geq 3\alpha n \, .$$

We can assume (by moving only a few additional vertices to V_0 that do not harm the bounds above) that for all cherries and matching edges in R' the number of vertices in the clusters together is divisible by three.

For each such super-regular cherry ijk of R', after revealing $G_1[V_i \cup V_j \cup V_k]$ we find by Lemma 4.2 a.a.s. a triangle factor covering all the vertices in $V_i \cup V_j \cup V_k$. Similarly for any matching edge ij of R', after revealing $G_1[V_i \cup V_j]$ we find by Lemma 4.3 a.a.s. a triangle factor covering all the vertices in $V_i \cup V_j$. Note that we apply Lemmas 4.2 and 4.3 only constantly many times and thus a.a.s. we get a triangle factor in all such applications. Let $\mathcal{T}$ be the union of all such triangle factors. Then $\mathcal{T}$ covers $\left| \bigcup_{j \in \mathcal{J}} V_j \right| \geq 3\alpha n$ vertices and gives at least $\alpha n = \min\{\alpha n, \lfloor n/3 \rfloor\}$ pairwise vertex-disjoint triangles in $G \cup G(n, p)$.

Case $1/3 - d/3 \leq \alpha \leq 1/3$

As discussed in the overview, here we cannot directly apply Lemmas 4.2 and 4.3 as in the case $0 < \alpha < 1/3 - d/3$, but we need additional steps. However, even with a lower minimum degree, we will cover all vertices of G and find $\lfloor n/3 \rfloor$ pairwise vertex-disjoint triangles. Recall that M_1 is a largest matching and that $|M_1| \geq (\alpha + 2d)t$. Then the set $V(R) \setminus V(M_1)$ is independent, has size

$$|V(R) \setminus V(M_1)| = t - 2|M_1| \leq (1 - 2\alpha - 4d)t \leq (\alpha - 3d)t$$

and only one endpoint of each edge of M_1 can be adjacent to more than one vertex from $V(R) \setminus V(M_1)$. Given that $\delta(R) \geq (\alpha - 2d)t$, we can greedily pick a second matching M_2 such that each edge of M_2 contains a vertex of $V(R) \setminus V(M_1)$ and a vertex of $V(M_1)$, and M_2 covers the remaining vertices $V(R) \setminus V(M_1)$ completely. Therefore, the two matchings M_1 and M_2 together cover the vertex set $V(R)$ and we can extract a collection of $\ell = |M_2| \leq (\alpha - 3d)t$ vertex-disjoint cherries and a disjoint matching that cover $V(R)$.

This gives a spanning subgraph $R' \subseteq R$ on vertex set $[t]$ containing $\ell \leq (\alpha - 3d)t$ cherries and a matching of size $(t - 3\ell)/2 \geq (1 - 3\alpha + 9d)t/2 \geq 9dt/2$ such that for all edges $ij \in E(R')$ the pair (V_i, V_j) is ε-regular of density at least d in G', and therefore in G as well. We denote by $\mathcal{I} \subseteq [t]$ the indices of the clusters V_i that are not the centre of a cherry in R'. As above, with Lemma 3.3, we can make the pairs associated with the edges of R' $(2\varepsilon, d - 3\varepsilon)$-super-regular, while keeping the clusters all of the same size. For this we have to remove at most $t2\varepsilon n_0 \leq t2\varepsilon n/t = 2\varepsilon n$ vertices, which we add to V_0. Next, as for a later application of Lemma 4.2 we need that for each cherry

the sizes of the leaf-clusters are smaller than the size of the centre-cluster, we remove for each $i \in \mathcal{I}$ additionally $\delta|V_i| \leq \delta n_0 \leq \delta n/t$ vertices from V_i and add them to V_0. Note that we remove vertices from the clusters of matching edges as well, although this is not necessary. We then get $|V_0| \leq \varepsilon n + 2\varepsilon n + t\delta n/t \leq 2\delta n$. We can assume (by moving only a few additional vertices to V_0 that do not harm the bounds above) that for all cherries and matching edges in R' the number of vertices in the clusters together is divisible by three. By removing $n \pmod 3 \in \{0, 1, 2\}$ vertices from V_0, we also have $|V_0| \equiv 0 \pmod 3$; note that this only happens when n is not divisible by 3 and we can discard these vertices.

Covering V_0 with triangles. We now want to cover the exceptional vertices in V_0 by triangles. It would be easy to do this greedily by just using (2), but it might happen that afterwards in many of the cherries the number of vertices is not divisible by three or that the centre cluster gets too small. To avoid both these issues, we will cover V_0 while using the same number of vertices from clusters that are together in a cherry or matching edge. For this we will always cover three vertices at a time and combine (1) with (2) to find additional triangles. Observe that $|V_0 \cup \bigcup_{i \notin \mathcal{I}} V_i| \leq 2\delta n + \ell n/t \leq (\alpha - \gamma)n - 2dn$ and, therefore, any $v \in V_0$ has at least $2dn$ neighbours in $\bigcup_{i \in \mathcal{I}} V_i$.

Assume we have already covered $V' \subseteq V_0$ vertices of V_0 using at most $5|V'|$ triangles in total. Let W' be the set of vertices from $\bigcup_{i \in \mathcal{I}} V_i$ used for the triangles covering V' and note that $|W'| \leq 30\delta n$. Then let $\mathcal{I}' \subseteq \mathcal{I}$ be the set of indices of clusters V_i with $i \in \mathcal{I}$ which intersect W' in at least $\sqrt{\delta}n_0$ vertices and note that $|\mathcal{I}'| \leq |W'|/(\sqrt{\delta}n_0) \leq 30\sqrt{\delta}t/(1-\varepsilon) \leq 40\sqrt{\delta}t \leq dt/4$. Moreover, notice that as for each $v \in V_0$ we have $\deg_G(v, \bigcup_{i \in \mathcal{I}} V_i) \geq 2dn$, there are at least dt indices $i \in \mathcal{I}$ such that v has at least dn_0 neighbours in V_i. In particular, as $|\mathcal{I}'| \leq dt/4$ and $t \geq 10/d$, there are at least $dt - |\mathcal{I}'| \geq 3dt/4 \geq 7$ indices $i \in \mathcal{I} \setminus \mathcal{I}'$ such that v has at least dn_0 neighbours in V_i. Therefore, we can pick three vertices $v_1, v_2, v_3 \in V_0 \setminus V'$ and three indices i_1, i_2, i_3 in $\mathcal{I} \setminus \mathcal{I}'$ such that v_j has dn_0 neighbours in V_{i_j} for $j = 1, 2, 3$ and the clusters $V_{i_1}, V_{i_2}, V_{i_3}$ belong to pairwise different cherries or matching edges. For $j = 1, 2, 3$ with (2), we find an edge e_j in $G_3[N(v_j, V_{i_j}) \setminus W']$ and we cover the three vertices with triangles. It is easy to show that we can find at most 10 additional triangles with the help of (1) and (2), in such a way that, overall, for each cherry and matching edge, we use the same number of vertices from each of their clusters; in particular, the number of vertices used from each cherry and matching edge is divisible by three. The clusters $V_{i_1}, V_{i_2}, V_{i_3}$ can belong to three cherries, two cherries and one matching edge, one cherry and two matching edges, or three matching edges. We give details in the case where they are all leaves of (different) cherries, and we refer to Figure 1 for the other three cases. With (1) we find four triangles: two with a vertex in each of the other cluster of the cherry containing V_{i_1} and the third vertex in one of the other clusters of the cherry containing V_{i_2}, and other two triangles with one vertex in each of the other cluster of the cherry containing V_{i_3} and the third vertex in the remaining cluster of the cherry containing V_{i_2}. When a V_{i_j} belongs to a matching edge of R', we first find with (2) two triangles inside this matching edge each with one vertex in the cluster V_{i_j} and the other two vertices in the other cluster of the matching edge, then we proceed as before (see Figure 1). Note that we cover three vertices of V_0 using at most 13 triangles, and thus to cover $V' \subseteq V_0$ we use at most $13|V'|/3 \leq 5|V'|$, as claimed above. Therefore we can repeat this procedure until $V' = V_0$.

Let $\mathcal{T}_1$ be the set of triangles we found above to cover V_0 and keep the divisibility condition. We now update the regularity partition by deleting $V(\mathcal{T}_1)$ from each V_i for $i \in [t]$ and note that for all cherries and matchings from R' the number of vertices in the clusters together is divisible by three. We recall that so far we removed at most $(2\varepsilon + \delta + 2\sqrt{\delta})n_0$ vertices from each cluster, where the first (resp. second, third) term bounds the number of vertices removed for making each pair super-regular (resp. for a later application of Lemma 4.2, for covering V_0).

Balancing the partition. Now the matching edges in R' are already ready for an application of Lemma 4.3, and we will not modify the corresponding clusters anymore. However, before an application of Lemma 4.2 to the cherries in R', we need to ensure that the ratio between their size and the size of the centre-cluster satisfies the hypotheses of the lemma. This is what we are
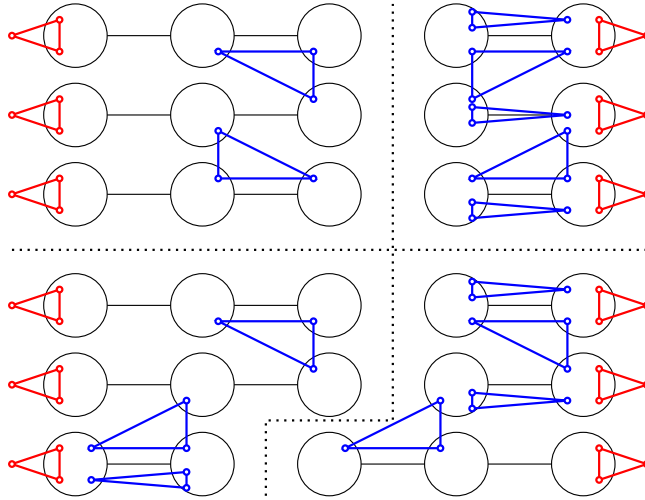
Figure 1. Embeddings of triangles for absorbing V_0 while using the same number of vertices from each cluster within a cherry or a matching edge. Each red triangle covers a vertex of V_0. Each blue triangle stands for two triangles with end-points in the same clusters; we only draw one for simplicity.

going to do now. For $i \notin \mathcal{I}$ we denote by W_i and U_i the leaf clusters of the cherry centred at V_i. Before covering the vertices of V_0, we had $|W_i| = |U_i| \leq (1 - \delta)|V_i|$ for $i \notin \mathcal{I}$, which still holds as we removed the same number of vertices from each cluster of a cherry.

However we still need to guarantee the other inequality $|W_i|, |U_i| \geq (1 - \delta_0)|V_i|$. For that, we find $2m$ triangles with two vertices in V_i, of which one half has the third vertex in U_i and the other half in W_i, where m is the smallest integer such that

$$U_i| - m \geq (1 - \delta_0)(|V_i| - 4m). \tag{3}$$

Then after removing these $2m$ triangles, we will have precisely $(1 - \delta_0)|V_i| \leq |U_i| = |W_i|$. Observe that the inequality (3) implies that $m \geq \frac{(1-\delta_0)|V_i| - |U_i|}{4(1-\delta_0) - 1}$ and, as we chose the smallest such m, we get $m \leq \lceil \frac{(1-\delta_0)|V_i| - |U_i|}{4(1-\delta_0) - 1} \rceil$. Moreover, as $\delta < \delta_0$, $|V_i| \leq n_0$ and $|U_i| \geq (1 - 2\varepsilon - \delta - 2\sqrt{\delta})n_0$, we have $\frac{(1-\delta_0)|V_i| - |U_i|}{4(1-\delta_0) - 1} < \frac{(1-\delta_0) - (1-2\varepsilon - \delta - 2\sqrt{\delta})}{2} n_0 < 2\sqrt{\delta} n_0$. Therefore, for n (and thus n_0) large enough, $m \leq 2\sqrt{\delta} n_0$. We can find these at most $4\sqrt{\delta} n_0$ triangles, by iteratively picking them with (2) and removing the corresponding vertices from U_i, W_i, and V_i. Indeed, for any $v \in W_i \cup U_i$ we have degree into V_i at least $(d - 3\varepsilon - \delta - 10\sqrt{\delta})n_0 \geq dn_0/2$, as we started from $(2\varepsilon, d - 3\varepsilon)$-super-regular pairs and $\delta < \delta' < 160^{-2}d^2$.

Note that afterwards we still have $|U_i| \leq (1 - \delta)|V_i|$ as for large enough n and with $\delta < \delta_0$ we have $m \leq \lceil \frac{(1-\delta_0)|V_i| - |U_i|}{4(1-\delta_0) - 1} \rceil \leq \frac{(1-\delta)|V_i| - |U_i|}{4(1-\delta) - 1}$. Therefore, we have $(1 - \delta_0)|V_i| \leq |U_i| = |W_i| \leq (1 - \delta)|V_i|$. Moreover with $d - 3\varepsilon - \delta - 10\sqrt{\delta} \geq d/2$ and $2\varepsilon \leq \varepsilon'$, we get that the pairs (U_i, V_i) and (W_i, V_i) are $(\varepsilon', d/2)$-super-regular. Let $\mathcal{T}_2$ be the set of triangles we removed during this phase.

Completing the triangles. Now for any $i \notin \mathcal{I}$, after revealing $G_1[V_i \cup W_i \cup U_i]$, we a.a.s. find a triangle factor covering the vertices of U_i, W_i, and V_i by Lemma 4.2. Similarly for any matching edge ij of R' observe that (V_i, V_j) is a $(\varepsilon', d/2)$-super-regular pair. Then after revealing $G_1[V_i \cup V_j]$, we a.a.s. find a triangle factor covering the vertices of V_i and V_j by Lemma 4.3. Note that we apply Lemma 4.2 and Lemma 4.3 only constantly many times and thus a.a.s. we get a triangle factor in all such applications. Let $\mathcal{T}_3$ be the union of the triangle factors we obtain for each $i \notin \mathcal{I}$ and each matching edge ij from R'. Then $\mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ gives $\lfloor n/3 \rfloor$ pairwise vertex-disjoint triangles in $G \cup G(n, p)$. $\qquad\square$

7. Proof of the sublinear theorem

As outlined in the overview, we use the following two Propositions to prove Theorem 2.4.

Proposition 7.1. *For any $0 < \gamma < 1/2$ there exists $C > 0$ such that for any $(\log n)^3 \leq m \leq \sqrt{n}$ and any n-vertex graph G with maximum degree $\Delta(G) \leq \gamma n$ and minimum degree $\delta(G) \geq m$ the following holds. With $p \geq C/n$ there are a.a.s. at least m pairwise vertex-disjoint triangles in $G \cup G(n, p)$.*

Proposition 7.2. *There exists $C > 0$ such that for any $\sqrt{n} \leq m \leq n/32$ and any n-vertex graph G with maximum degree $\Delta(G) \leq n/32$ and minimum degree $\delta(G) \geq m$ the following holds. With $p \geq C \log n/n$ there are a.a.s. at least m pairwise vertex-disjoint triangles in $G \cup G(n, p)$.*

With this at hand we can prove Theorem 2.4.

Proof of Theorem 2.4. Let $1 \leq m \leq n/256$ and let G be an n-vertex graph on vertex set V with minimum degree $\delta(G) \geq m$. We let C be large enough such that with $p \geq C \log n/n$ we can expose $G(n, p)$ in four rounds as $\bigcup_{i=1}^{4} G_i$ with $G_i \sim G(n, C_i \log n/n)$ for $i = 1, \ldots, 4$ such that the following hold. We let $C_1 = 1$ and observe that, by a union bound, a.a.s. for any set of vertices U of size at least $n/256$ there is at least one edge in $G_1[U]$. Next, we let C_2 be large enough such that for a set of vertices U of size at least $n/2$ there are a.a.s. at least $\log^3 n$ pairwise vertex-disjoint triangles in $G_2[U]$ [19, Theorem 3.29]. Finally, let $C_3 = 1$ (this is sufficient for our application of Proposition 7.1 because we do not need the $\log n$ term) and C_4 be such that we can apply Proposition 7.2 with $C_4/2$. We expose G_1 already now and assume that the described property holds, while we leave G_2, G_3, and G_4 until we need them.

To apply one of the two propositions to a large subgraph G' of G, we need $\Delta(G') \leq v(G')/32$. For this let V' be the set of vertices from G of degree at least $n/64$. If $|V'| \geq m$, then we let V'' be any subset of V' of size m and we greedily find m pairwise vertex-disjoint triangles in $G \cup G_1$, each containing exactly one vertex from V''. Indeed, as long as we have less than m triangles, there is a vertex $v \in V''$ not yet contained in a triangle. Then there is a set $U \subseteq N_G(v) \setminus V''$ of at least $n/64 - 3m \geq n/256$ vertices not covered by triangles, and we can find an edge within $G_1[U]$ that gives us a triangle containing v and two vertices from U.

Otherwise, $|V'| < m$ and we remove V' from G to obtain $G' = G[V \setminus V']$. Note that we have $v(G') = n - |V'| \geq n/2$, minimum degree $\delta(G') \geq m' = m - |V'|$, and maximum degree $\Delta(G') < n/64 \leq v(G')/32$. If $m' < (\log n)^3$, then we a.a.s. find m' pairwise vertex-disjoint triangles within $G_2[V(G')]$. If $(\log v(G'))^3 \leq (\log n)^3 \leq m' \leq \sqrt{v(G')}$, then by Proposition 7.1 and as $\log n/n = \omega(1/v(G'))$ there are a.a.s. at least m' pairwise vertex-disjoint triangles in $G' \cup G_3[V(G')]$. Finally, if $\sqrt{v(G')} \leq m' \leq n/256 \leq v(G')/32$, then by Proposition 7.2 and as $C_4 \log n/n \geq \frac{C_4}{2} \log v(G')/v(G')$, there are a.a.s. at least m' pairwise vertex-disjoint triangles in $G' \cup G_4[V(G')]$.

Now, that we found m' pairwise vertex-disjoint triangles, we can greedily add triangles by using the $m - m'$ vertices from V' and an edge in their neighbourhood until we have m triangles. Analogous to above, as long as we have less than m triangles, for each available vertex $v \in V'$, there is a set $U \subseteq N_G(v) \setminus V'$ of at least $n/256$ vertices not covered by triangles, and we find an edge within $G_1[U]$. □

It remains to prove Propositions 7.1 and 7.2. For Proposition 7.1, which deals with the cases $(\log n)^3 \leq m \leq \sqrt{n}$, we first need to find many large enough vertex-disjoint stars in G. These can be found deterministically with Lemma 7.3 below and afterwards we will show that a.a.s. at least m of them can be completed to triangles with the help of $G(n, p)$.

For any integer $g \geq 2$, we define the *star* on $g + 1$ vertices as the graph with one vertex of degree g (this vertex is called the *centre*) and the other vertices of degree one (these vertices are called *leaves*). Given a star K, we denote the number of its leaves by g_K. Moreover, given a family

of vertex-disjoint stars $\mathcal{K}$, we denote the set of all their centre vertices by $\mathcal{K}_C$ and the set of all their leaf vertices by $\mathcal{K}_L$.

Lemma 7.3. *For every $0 < \gamma < 1/2$ and integer s there exists an $\varepsilon > 0$ such that for n large enough and any m with $2/\varepsilon \leq m \leq \sqrt{n}$ the following holds. In every n-vertex graph G with minimum degree $\delta(G) \geq m$ and maximum degree $\Delta(G) \leq \gamma n$, there exists a family $\mathcal{K}$ of vertex-disjoint stars in G such that every $K \in \mathcal{K}$ has g_K leaves with $\varepsilon m \leq g_K \leq \varepsilon\sqrt{n}$ and*

$$\sum_{K \in \mathcal{K}} g_K^2 \geq s\varepsilon^2 nm.$$

Proof of Lemma 7.3. Given $0 < \gamma < 1/2$ and an integer s we let $\varepsilon > 0$ such that $\varepsilon \leq 1/(6s)$ and $\varepsilon < 1/2 - s\gamma$. Moreover, we let n be large enough for our calculations and, for simplicity, assume that $\varepsilon\sqrt{n}$ is an integer. Then let $2/\varepsilon \leq m \leq \sqrt{n}$ and G be an n-vertex graph on vertex set V with $\delta(G) \geq m$ and $\Delta(G) \leq \gamma n$.

Let $\mathcal{K}$ be a family of vertex-disjoint stars in G with $\varepsilon m \leq g_K \leq \varepsilon\sqrt{n}$ for all $K \in \mathcal{K}$, that maximizes the sum

$$\sum_{K \in \mathcal{K}} g_K^2 \tag{4}$$

among all such families. Note that each star in $\mathcal{K}$ has at least 2 leaves because $g_K \geq \varepsilon m$ and $m \geq 2/\varepsilon$.

If the sum in (4) is bigger than $s\varepsilon^2 nm$ we are done. So we assume the family $\mathcal{K}$ satisfies

$$\sum_{K \in \mathcal{K}} g_K^2 < s\varepsilon^2 nm. \tag{5}$$

We are going to prove that then there exists a vertex of degree larger than γn, contradicting our assumption on the maximum degree.

For this we split $\mathcal{K}$ into two subfamilies

$$\mathcal{M} = \left\{ K \in \mathcal{K} : \varepsilon m \leq g_K < \varepsilon\sqrt{n} \right\} \quad \text{and} \quad \mathcal{H} = \left\{ K \in \mathcal{K} : g_K = \varepsilon\sqrt{n} \right\}$$

and we let R be the set of vertices not covered by the stars in $\mathcal{K}$, that is $R = V(G) \setminus (\mathcal{H}_C \cup \mathcal{H}_L \cup \mathcal{M}_C \cup \mathcal{M}_L)$, where $\mathcal{H}_C$, $\mathcal{H}_L$, $\mathcal{M}_C$, and $\mathcal{M}_L$ are obtained from $\mathcal{M}$ and $\mathcal{H}$ as defined above.

For all stars $K \in \mathcal{H}$ we have $g_K^2 = \varepsilon^2 n$. From (5), we get that the subfamily $\mathcal{H}$ contains at most sm stars and hence

$$|\mathcal{H}_C| \leq sm \leq s\sqrt{n} \quad \text{and} \quad |\mathcal{H}_L| = |\mathcal{H}_C|\varepsilon\sqrt{n} \leq sm\varepsilon\sqrt{n} \leq s\varepsilon n, \tag{6}$$

because $m \leq \sqrt{n}$.

As each star in $\mathcal{M}$ has at least εm leaves, we have $\sum_{K \in \mathcal{M}} g_K \geq |\mathcal{M}|\varepsilon m$. Using the Cauchy-Schwarz inequality, we then get

$$\left(\sum_{K \in \mathcal{M}} g_K \right)^2 \leq \left(\sum_{K \in \mathcal{M}} g_K^2 \right) |\mathcal{M}| \overset{(5)}{\leq} s\varepsilon^2 nm |\mathcal{M}| \leq \left(\sum_{K \in \mathcal{M}} g_K \right) s\varepsilon n,$$

which implies

$$|\mathcal{M}_L| = \sum_{K \in \mathcal{M}} g_K \leq s\varepsilon n. \tag{7}$$

Therefore, $|\mathcal{M}_C| \leq |\mathcal{M}_L|/2 \leq s\varepsilon n/2$ since each star has at least 2 leaves.

These bounds on $\mathcal{M}_L$ and $\mathcal{M}_C$ together with (6) immediately imply that $|R| \geq (1 - 3s\varepsilon)n \geq n/2$. We are going to show that there are many edges between R and $\mathcal{H}_C$ and from that we derive the existence of a high degree vertex, giving the desired contradiction.

A vertex in R cannot have at least εm neighbours inside R, because otherwise we could create a new star and increase the sum in (4). Therefore, $e(R) < |R|\varepsilon m/2$. We also have $e(R, \mathcal{M}_C) = 0$ since otherwise we could add an edge to one of the existing stars in $\mathcal{K}$ increasing the sum in (4) (recall that stars in $\mathcal{M}$ have less than $\varepsilon\sqrt{n}$ leaves).

Given a leaf $v \in \mathcal{M}_L$ that belongs to a star K with g_K leaves, we must have $\deg(v, R) < g_K + 1$. Otherwise, we could take $V' \subseteq N_R(v)$ of size $|V'| = g_k + 1 \le \varepsilon\sqrt{n}$ and create a new family of vertex-disjoint stars, given by $\mathcal{K} \setminus \{K\}$ and the star on $v \cup V'$, to increase the sum in (4). Therefore,

$$e(R, \mathcal{M}_L) \le \sum_{K \in \mathcal{M}} g_K(g_K + 1) \le \sum_{K \in \mathcal{K}} g_K^2 + \sum_{K \in \mathcal{M}} g_K \overset{(5),(7)}{\le} s\varepsilon^2 nm + s\varepsilon n.$$

Similarly, given $v \in \mathcal{H}_L$, we must have $\deg(v, R) < \varepsilon\sqrt{n}$. Otherwise, we could take $V' \subseteq N_R(v)$ of size $|V'| = \varepsilon\sqrt{n}$ and create a new family of vertex-disjoint stars, given by $\mathcal{K} \setminus \{K\}$, the star $K \setminus \{v\}$, and the star on $v \cup V'$, to increase the sum in (4). Therefore, $e(R, \mathcal{H}_L) \le |\mathcal{H}_L|\varepsilon\sqrt{n} \le s\varepsilon^2 nm$ by (6).

On the other hand, $\delta(G) \ge m$ implies $e(R, V) \ge m|R|$, where the edges inside of R are counted twice. Then we can lower bound the number of edges between R and $\mathcal{H}_C$ by

$$\begin{aligned} e(R, \mathcal{H}_C) &\ge m|R| - 2e(R) - e(R, \mathcal{M}_C) - e(R, \mathcal{M}_L) - e(R, \mathcal{H}_L) \\ &\ge m|R| - \varepsilon m|R| - 0 - 2s\varepsilon^2 nm - s\varepsilon n \\ &\ge m|R| - \varepsilon m|R| - 4s\varepsilon^2 m|R| - s\varepsilon^2 m|R| \\ &\ge (1 - \varepsilon - 5s\varepsilon^2)|R|m \ge |R|(1 - 2\varepsilon)m. \end{aligned}$$

where we used the bounds on $e(R)$, $e(R, \mathcal{M}_C)$, $e(R, \mathcal{M}_L)$, and $e(R, \mathcal{H}_L)$ we found above, together with $|R| \ge n/2$, $\varepsilon m \ge 2$ and the choice of $\varepsilon < 1/(6s)$. In particular, as $|\mathcal{H}_C| \le sm$ and using $\varepsilon < 1/2 - s\gamma$, and $|R| > n/2$, there exists a vertex $v \in \mathcal{H}_C$ of degree

$$\deg(v) \ge \deg(v, R) \ge \frac{|R|(1 - 2\varepsilon)m}{sm} \ge |R|2\gamma > \gamma n.$$

This contradicts the maximum degree of G.   □

Proof of Proposition 7.1. Let n be sufficiently large for the following arguments. With $0 < \gamma < 1/2$ let G be an n-vertex graph with maximum degree $\Delta(G) \le \gamma n$ and minimum degree $\delta(G) \ge m$. With $(\log n)^3 \le m \le \sqrt{n}$, we first find many vertex-disjoint stars in G and then complete at least m of them to triangles with the help of $G(n, p)$. We apply Lemma 7.3 with γ and $s = 8$ to get $0 < \varepsilon < 1/2$ and, as n is large enough and $m \ge 2/\varepsilon$, we get a family $\mathcal{K}$ of vertex-disjoint stars on $V(G)$ such that $\varepsilon m \le g_K \le \varepsilon\sqrt{n}$ for $K \in \mathcal{K}$ and $\sum_{K \in \mathcal{K}} g_K^2 \ge 8\varepsilon^2 nm$.

As we have stars of different sizes, we split $\mathcal{K}$ into $t = \lceil \log(\sqrt{n}/m)/\log 2 \rceil + 1$ subfamilies

$$\mathcal{K}_i = \{K \in \mathcal{K} : 2^{i-1}\varepsilon m \le g_K < 2^i\varepsilon m\}, \quad 1 \le i \le t,$$

and set $k_i = |\mathcal{K}_i|$.

By deleting leaves, we may assume that all stars in $\mathcal{K}_i$ have exactly $\lceil 2^{i-1}\varepsilon m \rceil$ leaves. Denote by $\mathcal{I}$ the set of indices $i \in [t]$ such that $k_i(2^{i-1}\varepsilon m)^2 \ge \varepsilon^2 nm/t$. Next we prove that $\sum_{i \in \mathcal{I}} k_i(2^{i-1}\varepsilon m)^2 \ge \varepsilon^2 nm$.

Observe first that $\sum_{i \notin \mathcal{I}} k_i(2^{i-1}\varepsilon m)^2 \le t(\varepsilon^2 nm/t) = \varepsilon^2 nm$. It follows that

$$\begin{aligned} \sum_{i \in \mathcal{I}} k_i(2^{i-1}\varepsilon m)^2 &= \frac{1}{4}\sum_{i \in \mathcal{I}} k_i(2^i\varepsilon m)^2 = \frac{1}{4}\sum_{i=1}^t |\mathcal{K}_i|(2^i\varepsilon m)^2 - \sum_{i \notin \mathcal{I}} k_i(2^{i-1}\varepsilon m)^2 \\ &\ge \frac{1}{4}\sum_{i=1}^t \sum_{K \in \mathcal{K}_i} g_K^2 - \varepsilon^2 nm \ge 2\varepsilon^2 nm - \varepsilon^2 nm \ge \varepsilon^2 nm. \end{aligned}$$

Now we reveal random edges on $V(G)$ with probability $p \geq C/n$ where C is large enough for the Chernoff bounds and inequalities below. We shall show that this allows us to find at least m triangles a.a.s.. Indeed, for each $i \in \mathcal{I}$, we find many pairwise vertex-disjoint triangles in $\mathcal{K}_i$ using random edges.

Claim 7.4. *For any $i \in \mathcal{I}$, after revealing edges of $G(n, p)$ with $p \geq C/n$, we have with probability at least $1 - 1/n$ at least $k_i(2^{i-1}m)^2/n$ pairwise vertex-disjoint triangles within $(G \cup G(n, p))[\cup_{K \in \mathcal{K}_i} V(K)]$.*

Having this claim and since $|\mathcal{I}| \leq t = o(n)$, with a union bound over $i \in \mathcal{I}$, there are a.a.s. at least

$$\sum_{i \in \mathcal{I}} k_i(2^{i-1}m)^2/n = \frac{1}{\varepsilon^2 n} \sum_{i \in \mathcal{I}} k_i \left(2^{i-1}\varepsilon m\right)^2 \geq \frac{\varepsilon^2 nm}{\varepsilon^2 n} \geq m$$

pairwise vertex-disjoint triangles in $G \cup G(n, p)$. It remains to prove Claim 7.4.

Proof of Claim 7.4. Fix $i \in \mathcal{I}$ and let $k = k_i$ and $g = \lceil 2^{i-1}\varepsilon m \rceil$. We reveal random edges with probability p within each set of leaves of the k stars in $\mathcal{K}_i$. We recall that these k sets are pairwise disjoint and each has size g. Let X_j be the indicator variable of the event that the jth of these sets contains at least one edge for $1 \leq j \leq k$, and set $X = \sum_{j=1}^{k} X_i$. Then $\mathbb{P}[X_j = 1] = 1 - (1 - p)^{\binom{g}{2}}$ and $\mathbb{E}[X] = k \left(1 - (1 - p)^{\binom{g}{2}}\right)$. We have that $\mathbb{E}[X] \geq 2kg^2/(\varepsilon^2 n)$. Indeed,

$$k \left(1 - (1 - p)^{\binom{g}{2}}\right) \geq 2kg^2/(\varepsilon^2 n) \quad \Leftrightarrow \quad 1 - 2g^2/(\varepsilon^2 n) \geq \left(1 - \frac{C}{n}\right)^{\binom{g}{2}},$$

and the later holds for large enough C and n using the inequality $1 - x \leq e^{-x} \leq 1 - \frac{x}{2}$ valid for $x < 3/2$.

From Chernoff's inequality (Lemma 3.1) and from the fact that $kg^2/(\varepsilon^2 n) \geq m/t$ by the definition of $\mathcal{I}$, it follows that with probability at most

$$2 \exp\left(-\frac{1}{6}\frac{kg^2}{\varepsilon^2 n}\right) \leq 2 \exp\left(-\frac{1}{6}\frac{m}{t}\right) \leq \frac{1}{n}$$

there are less than $kg^2/(\varepsilon^2 n)$ triangles, where the last inequality holds as $t \leq \log n$, $m \geq (\log n)^3$, and n is large enough. $\qquad\square$

Proof of Proposition 7.2. Let G be an n-vertex graph with maximum degree $\Delta(G) \leq n/32$ and minimum degree $\delta(G) \geq m$. With $\sqrt{n} \leq m \leq n/32$, we cannot hope to find sufficiently many large enough vertex-disjoint stars (as we did in the proof of Proposition 7.1). Instead we apply a greedy strategy, using that a.a.s. every vertex of G has random edges in its neighbourhood. We can greedily obtain a spanning bipartite subgraph $G' \subseteq G$ of minimum degree $\delta(G') \geq m/2$ by taking a partition of $V(G)$ into sets A and B such that $e_G(A, B)$ is maximised and letting $G' = G[A, B]$. Indeed, a vertex of degree less than $m/2$ can be moved to the other class to increase $e_G(A, B)$. W.l.o.g. we assume $|B| \geq n/2 \geq |A|$. Moreover, we have $|A| \geq 8m$, as otherwise with $e(A, B) \geq nm/4$ there is a vertex of degree larger than $n/32$, a contradiction.

Claim 7.5. *For every $A' \subseteq A$, $B' \subseteq B$ with $|A'| < 2m$, $|B'| \geq n/4$ we have $e(A \setminus A', B') \geq nm/16$.*

Proof. If $e(A \setminus A', B') < nm/16$, it follows from $e(A, B') \geq |B'|m/2 \geq nm/8$ that we have $e(A', B') \geq nm/16$. Since $|A'| < 2m$, there must be a vertex of degree at least $n/32$ in A', a contradiction. $\qquad\square$

From this claim, it follows that there are many vertices of high degree in $A \setminus A'$.

Claim 7.6. *Suppose that $A' \subseteq A$, $B' \subseteq B$ with $|A'| < 2m$, $|B'| \geq n/4$. Let $A^* = \{v \in A \setminus A' : \deg(v, B') \geq m/16\}$. Then $|A^*| \geq m$.*

Proof. We have $|A^*|n/32 + |A|m/16 \geq e(A^*, B') + e(A \setminus (A' \cup A^*), B') = e(A \setminus A', B') \geq nm/16$, where the last inequality uses Claim 7.5. Since $|A| \leq n/2$, we get

$$|A^*| \geq \frac{nm/16 - nm/32}{n/32} = m.$$

$\square$

Let $s = \lceil \frac{2n}{m} \rceil$ and $t = \lceil \frac{m^2}{2n} \rceil$. We will now iteratively construct our m triangles in t rounds of s triangles each. In each round, we will reveal $G(n, q)$ with $q = \frac{C \log n}{m^2}$, where C is large enough for the Chernoff bound below. For the start, we set $A' = B_0 = \emptyset$.

Let $i = 1, \ldots, t$, suppose that before the ith round we have

$$|A'| = (i-1)s \leq \left(\left\lceil \frac{m^2}{2n} \right\rceil - 1 \right) \left\lceil \frac{2n}{m} \right\rceil \leq \frac{m^2}{2n} \left(\frac{2n}{m} + 1 \right) < 2m$$

and $|B_0| = (i-1)(2s) < 3m$, and note this is true for $i = 1$. In the ith round, we pick vertices $v_1, \ldots, v_s \in A \setminus A'$ and pairwise disjoint sets $B_1, \ldots, B_s \subseteq B \setminus B_0$, each of size $\lceil m/16 \rceil$, such that $B_j \subseteq N_{G'}(v_j)$ for each $j = 1, \ldots, s$. We can do this greedily, where for $j = 1, \ldots, s$ we set $B' = B \setminus (B_0 \cup B_1 \cup \cdots \cup B_{j-1})$ and apply Claim 7.6 to obtain a vertex $v_j \in A \setminus A'$ together with a set $B_j \subseteq B'$ of $\lceil m/16 \rceil$ neighbours of v_j. We can do this as $|A'| < 2m$ and $|B'| \geq n/2 - s\lceil m/16 \rceil - |B_0| \geq n/4$ as $m \leq n/32$.

Now we reveal additional edges at random with probability q. Then with probability at least $1 - 1/n^2$, we have at least one edge in each set $B_1, \ldots, B_s$. Indeed the probability that there is no edge in a set B_i is at most $(1-q)^{\binom{|B_i|}{2}} \leq \exp\left(-C\frac{\log n}{m}\binom{\lceil m/16 \rceil}{2}\right) \leq n^{-3}$ as C is large enough. Therefore, the probability that there is a set without any edge is at most $sn^{-3} \leq n^{-2}$ by a union bound. We fix an arbitrary edge from each B_i and together with $v_1, \ldots, v_s$ this gives us s triangles. We add the vertices $v_1, \ldots, v_s$ to A' and the vertices of the edges that we chose to B_0. Notice that $|A'| = is$ and $|B'| = i(2s)$, as required at the beginning of next round.

We can repeat the above t times because with $m \geq \sqrt{n}$ we get $tq \leq \frac{C \log n}{n} = p$. By a union bound over the $t = \lceil \frac{m^2}{2n} \rceil \leq n$ rounds, we get that we succeed a.a.s. and find $ts \geq m$ triangles. $\square$

8. Proof of the auxiliary lemmas

In this section, we prove Lemmas 4.1, 4.2, and 4.3. For each of them, we first give an overview of the strategy and then a full proof.

8.1 Proof of Lemmas 4.1 and 4.2

We describe the general setup of both Lemmas 4.1 and 4.2. Let G be a graph on $U \cup V \cup W$ with (V, U) and (V, W) being super-regular with respect to G. We will find all/most triangles, respectively, with one vertex in each of the sets U, V, and W, with the edges between U and W coming from the random graph. To find these edges, we consider a random matching M in $G(U, W, p)$ such that each matching edge is contained in many triangles with the third vertex from V. As we later want to match edges from M to vertices from V, in order to get triangles, we consider the following bipartite auxiliary graph. Given a matching M between U and W, the vertex set of the graph $H_G(M, V)$ consists of M and V and there is an edge between $m \in M$ and $v \in V$ if and only if the vertices of m are incident to v in G. The lemma below states that with $p \geq C/n$, we can a.a.s. find

a large matching M such that additionally (M, V) gives a super-regular pair in $H_G(M, V)$. Observe that a matching within $H_G(M, V)$ induces pairwise vertex-disjoint triangles in $G \cup G(U, W, p)$.

Lemma 8.1. *For any $0 < d, \delta, \varepsilon' < 1$ with $2\delta \leq d$ there exist $\varepsilon, C > 0$ such that the following holds. Let G be a graph on $U \cup V \cup W$, with $|V| = n$ and $(1 - 1/2)n \leq |U| = |W| \leq (1 + 1/2)n$, such that (V, U) and (V, W) are (ε, d)-super-regular pairs with respect to G. Further, let $G(U, W, p)$ be a random graph with $p \geq C/n$. Then a.a.s. there exists a matching $M \subseteq G(U, W, p)$ of size $|M| = (1 - \delta)|W|$ such that the pair (M, V) is $(\varepsilon', d^3/32)$-super-regular with respect to the auxiliary graph $H_G(M, V)$.*

We will prove this lemma at the end of this section. For Lemma 4.2, we will use the minimum degree condition and $G(V, p)$ to find additional triangles covering the remaining vertices from U and W that are not covered by M. We now proceed to the details of this proof.

Proof of Lemma 4.2. Given $0 < \delta' \leq d < 1$, let $\varepsilon' > 0$ be given by Lemma 3.4 on input $d^3/64$ and let $0 < \delta < \delta_0 \leq \min\{\delta', d^3/22\}$. Furthermore, let $C \geq 8\delta^{-2}$, let $0 < \varepsilon < \delta/4$ be given by Lemma 8.1 on input d, $\frac{\delta}{3(1-\delta)}$ (in place of δ), and $\varepsilon'/2$ and let $p \geq C/n$.

Suppose U, V, W are disjoint sets of size $|V| = n$ and $(1 - \delta_0)n \leq |U| = |W| \leq (1 - \delta)n$ with $|V| + |U| + |W| \equiv 0 \pmod 3$, and G is a graph with vertex set $U \cup V \cup W$ such that the pairs (V, U) and (V, W) are (ε, d)-super-regular with respect to G. Let δ_1 be such that $|U| = |W| = (1 - \delta_1)n$ and observe that $\delta \leq \delta_1 \leq \delta_0$. We reveal random edges $G_1 \sim G(U, W, p)$ and $G_2 \sim G(V, p)$ and we have that a.a.s. any set of size at least δn in V contains an edge of G_2. Indeed, fixed a set of size at least δn, the probability that it does not contain an edge of G_2 is at most $(1 - p)^{\binom{\delta n}{2}} \leq \exp\left(-p\binom{\delta n}{2}\right) \leq \exp\left(-2n\right)$ since $C \geq 8\delta^{-2}$, and we conclude by a union bound over the at most 2^n choices of such set. Then we apply Lemma 8.1 with G_1 to obtain a matching $M \subseteq G_1$ of size $|M| = \left(1 - \frac{\delta}{3(1-\delta)}\right)|W| = \left(1 - \frac{\delta}{3(1-\delta)}\right)(1 - \delta_1)n$ such that the pair (M, V) is $(\varepsilon'/2, d^3/32)$-super-regular with respect to $H_G(M, V)$. As for $x \in (0, 1)$ the function $x \to x/(1 - x)$ is increasing and $\delta \leq \delta_1$, we have $\left(\frac{\delta_1}{3} - \frac{\delta}{3(1-\delta)}(1 - \delta_1)\right) \geq \left(\frac{\delta_1}{3} - \frac{\delta_1}{3(1-\delta_1)}(1 - \delta_1)\right) \geq 0$. Thus, by ignoring $\left(\frac{\delta_1}{3} - \frac{\delta}{3(1-\delta)}(1 - \delta_1)\right)n \leq d^3 n/64$ edges of M, we get a subset $M' \subseteq M$ with $|M'| = (1 - 4\delta_1/3)n$.

Next, let $U' = U \setminus V(M')$ and $W' = W \setminus V(M')$ be the sets of vertices in U and W, respectively, that are not incident to edges of M'. Note that both U' and W' have size $\delta_1 n/3$. We want to cover these vertices with triangles having the other two vertices in V. Any vertex $v \in U' \cup W'$ has degree at least dn into V and as $d > 2\delta_0 \geq 2\delta_1$, we can pick these triangles greedily for each $v \in U' \cup W'$ using G_2. Let $V' \subseteq V$ be the vertices that were used for these triangles and observe $|V \setminus V'| = |M'| = (1 - 4\delta_1/3)n$.

To obtain the triangle factor it remains to find a perfect matching in $H_G(M', V \setminus V')$. By Lemma 3.4, it is sufficient to observe that the pair $(M', V \setminus V')$ is $(\varepsilon', d^3/64)$-super-regular with respect to $H_G(M', V \setminus V')$, which holds because (M, V) is $(\varepsilon'/2, d^3/32)$-super-regular with respect to $H_G(M, V)$. $\qquad \square$

Now we turn to the overview of the proof of Lemma 4.1, for which $p \geq C \log n/n$. We will rely again on Lemma 8.1, which gives a large matching $M \subseteq G(U, W, p)$ such that the pair (M, V) is super-regular with respect to the auxiliary graph $H_G(M, V)$. Starting from this matching M, we add more matching edges of $G(U, W, p)$ between the vertices not covered by M, and extend M to a perfect matching in $G(U, W, p)$. This will be possible using Lemma 8.2 from below and Lemma 3.6, where we emphasise that the log n-term is essential for this last lemma. It is then easy to find a perfect matching in $H_G(M, V)$ that gives a triangle factor.

Before we come to the proof of Lemma 4.1, we introduce another auxiliary structure to describe in general which potential edges between U and W we would like to use for the matching M. We

define an auxiliary bipartite graph $F = F_{G,V}(U, W)$ with bipartition $U \cup W$, where a pair $(u, w) \in U \times W$ is an edge of F if u and w have at least $d^2 n/2$ common neighbours in G, i.e. $|N_G(u, V) \cap N_G(w, V)| \geq d^2 n/2$. Similarly, for a set $X \subseteq V$, we call an edge $uw \in E(F)$ *good for X* if there are at least $d^2|X|/2$ vertices $x \in X$ that are incident to u and w in G, i.e. uwx is a triangle in $G \cup F$. We denote the spanning subgraph of F with edges that are good for X by F_X. We prove the following lemma, which will be used in the proof of Lemma 8.1 to construct M and in the proof of Lemma 4.1 to extend M.

Lemma 8.2. *For any $\varepsilon, d > 0$ with $\varepsilon \leq d/2$, the following holds. Let G be a graph on $U \cup V \cup W$, with $|V| = n$ and $(1 - 1/2)n \leq |U| = n_0 = |W| \leq (1 + 1/2)n$, such that (V, U) and (V, W) are (ε, d)-super-regular pairs with respect to G. Let $F = F_{G,V}(U, W)$ be the bipartite graph described above. Then F satisfies the following properties.*

  (i) *The minimum degree of F is at least $(1 - \varepsilon)n_0$.*
  (ii) *If $X \subseteq V$ and $|X| \geq 2\varepsilon n/d$, then all but at most εn_0 vertices from U have degree at least $(1 - 2\varepsilon)n_0$ in F_X.*

Proof of Lemma 8.2. Take any $u \in U$. Since (U, V) is an (ε, d)-super-regular pair, we have $|N_G(u, V)| \geq d|V| \geq \varepsilon|V|$ and we can apply Lemma 3.3 with $A = W$, $B = V$, and $Y = N_G(u, V)$ to conclude that the set

$$W_u = \left\{ w \in W : \deg_G(w, Y) \geq (d - \varepsilon)|Y| \right\}$$

has size at least $(1 - \varepsilon)n_0$. Since $(d - \varepsilon)|Y| \geq (d - \varepsilon)dn \geq d^2 n/2$, all the vertices in W_u are neighbours of u in the auxiliary graph F and, therefore, $|N_F(u)| \geq (1 - \varepsilon)n_0$. Analogously, we infer that $|N_F(w)| \geq (1 - \varepsilon)n_0$ for all $w \in W$ and hence $\delta(F) \geq (1 - \varepsilon)n_0$.

If $|X| \geq 2\varepsilon n/d$, then by Lemma 3.3 all but at most εn_0 vertices $u \in U$ have at least $d|X|/2$ neighbours in X. Fixing any $u \in U$ with this property, and again using Lemma 3.3, all but at most εn_0 vertices $w \in W$ have at least $d^2|X|/4$ common neighbours with u in X. From (i) we know that $\delta(F) \geq (1 - \varepsilon)n_0$, therefore all but at most εn_0 vertices from U have degree at least $(1 - 2\varepsilon)n_0$ in F_X. □

Now we prove Lemma 4.1 using Lemmas 8.1 and 8.2.

Proof of Lemma 4.1. Given $d \in (0, 1)$, let $\varepsilon' > 0$ be given by Lemma 3.4 on input $d^3/64$ and let $0 < \delta < d^3/64$. Furthermore, let $0 < \varepsilon < \delta/4$ be given by Lemma 8.1 on input d, δ, and $\varepsilon'/2$, let C' be given by Lemma 3.6 for input $1/4$, set $C = 2C'/\delta$, and let $p \geq C \log n/n$.

Now let G be a graph on $U \cup V \cup W$, with $|U| = |V| = |W| = n$, such that (V, U) and (V, W) are (ε, d)-super-regular pairs with respect to G. We reveal the random edges in $G(U, W, p)$ in two rounds as $G_1 \sim G(U, W, p/2)$ and $G_2 \sim G(U, W, p/2)$. We apply Lemma 8.1 with G_1 to a.a.s. obtain a matching $M \subseteq G_1$ of size $|M| = (1 - \delta)n$ such that the pair (M, V) is $(\varepsilon'/2, d^3/32)$-super-regular with respect to $H_G(M, V)$. Observe that $p \geq C \log n/n$, while the logarithmic factor is not required by Lemma 8.1, and thus the constant C does not need to depend on the constant given by this lemma.

Next, let $U' = U \setminus V(M)$ and $W' = W \setminus V(M)$ be the sets of vertices in U and W respectively that are not incident to edges of M. Note that both U' and W' have size δn. For the auxiliary graph $F = F_{G,V}(U, W)$ defined above, we consider the subgraph $F[U', W']$ induced by U' and W' and note that by Lemma 8.2 (i) it has minimum degree at least $(\delta - \varepsilon)n \geq \frac{3}{4}\delta n$. We use G_2 to reveal the edges of $F[U', W']$ with probability $p/2$ and a.a.s. by Lemma 3.6 with $p/2 \geq C' \log(\delta n)/(\delta n)$ we find a perfect matching M' in $G_2 \cap F[U', W']$.

To obtain the triangle factor, it remains to find a perfect matching in $H_G(M \cup M', V)$. For this we first greedily select a neighbour v from V for each $m \in M'$ and denote the set of vertices v used in this way by V'. Since $|V'| = |M'| = \delta n$, it follows from the choice of δ that this is possible and

that the pair $(M, V \setminus V')$ is $(\varepsilon', d^3/64)$-super-regular with respect to $H_G(M, V \setminus V')$. As $|V \setminus V'| = |M|$, by Lemma 3.4, there is a perfect matching in $H_G(M, V \setminus V')$ and we are done. $\qquad\square$

It remains to prove Lemma 8.1.

Proof of Lemma 8.1. Given $0 < d, \delta, \varepsilon' < 1$ with $2\delta \leq d$, suppose that

$$\varepsilon \ll \eta \ll \gamma \ll d, \delta, \varepsilon'$$

are positive real numbers such that

$$\gamma \leq \varepsilon'/2, \quad \eta \log(1/\eta) \leq \gamma/4, \quad \varepsilon \leq \eta d/4 \text{ and } \varepsilon \leq \gamma\delta/56 \,.$$

Furthermore, let $C > 0$ such that $p \geq C/n$ is large enough for the applications of Chernoff's inequality below.

Let G be a graph on $U \cup V \cup W$, with $|V| = n$ and $|U| = |W| = n_0 = (1 \pm 1/2)n$, such that (V, U) and (V, W) are (ε, d)-super-regular pairs with respect to G. To find the matching M, construct a graph $\tilde{F} \subseteq F = F_{G,V}(U, W)$ by including each edge of F randomly with probability p, independently from all other edges.

Claim 8.3. *A.a.s. for every $X \subseteq V$ of size ηn and $U' \subseteq U$ and $W' \subseteq W$ both of size at least δn_0, the following statements hold.*

$$\text{There are } (1 \pm \varepsilon)e_F(U', W')p \text{ edges in } \tilde{F}[U', W'] \tag{8}$$

and

$$\text{at least } (1 - 7\varepsilon/\delta)|U'||W'|p \text{ edges in } \tilde{F}[U', W'] \text{ are good for } X. \tag{9}$$

Proof. The expected number of edges between U' and W' in $\tilde{F}$ is by Lemma 8.2 (i) at least $e_F(U', W')p \geq |U'|(|W'| - \varepsilon n_0)p \geq C\delta^2 n/8$. Using Chernoff's inequality (Lemma 3.1), we obtain with $C\delta^2\varepsilon^2 \geq 120$ that

$$\mathbb{P}\left[e_{\tilde{F}}(U', W') \neq (1 \pm \varepsilon)e_F(U', W')p\right] \leq 2\exp\left(-\frac{\varepsilon^2}{3}e_F(U', W')p\right) \leq \exp(-4n) \,.$$

From Lemma 8.2 (ii) we get that the expected number of edges between U' and W' in $\tilde{F}$ that are good for X is at least $(|U'| - \varepsilon n_0)(|W'| - 2\varepsilon n_0)p \geq (1 - 3\varepsilon/\delta)|U'||W'|p$. Moreover, observe that

$$\left(1 - 7\frac{\varepsilon}{\delta}\right) = \left(1 - \frac{4\varepsilon/\delta}{1 - 3\varepsilon/\delta}\right)(1 - 3\varepsilon/\delta) \leq \left(1 - \frac{4\varepsilon/\delta}{\sqrt{1 - 3\varepsilon/\delta}}\right)(1 - 3\varepsilon/\delta) \,.$$

Thus, by Chernoff's inequality applied with $\frac{4\varepsilon/\delta}{\sqrt{1-3\varepsilon/\delta}}$ in place of δ, we get with $C\varepsilon^2 \geq 3$ that

$$\mathbb{P}\left[|E(\tilde{F}[U', W']) \cap E(F_X)| \leq \left(1 - 7\frac{\varepsilon}{\delta}\right)|U'||W'|p\right] \leq 2\exp\left(-\frac{16\varepsilon^2}{3\delta^2}|U'||W'|p\right) \leq 2\exp(-4n) \,.$$

Claim 8.3 follows from the union bound over the at most 2^{3n} choices for X, U', and W'. $\qquad\square$

Using a random greedy process, we now choose a matching M of size $(1 - \delta)|W|$ between U and W in $\tilde{F}$ as follows. Having chosen edges $m_1, \ldots, m_t \in \tilde{F}$ with $t < (1 - \delta)|W|$, we pick m_{t+1} uniformly at random from all edges of $\tilde{F}$ that do not share an endpoint with any of $m_1, \ldots, m_t$. This is possible since, by (8) and Lemma 8.2 (i), between any two subsets of U and W of size $\delta|W|$ there is at least one edge of $\tilde{F}$. For $i = 1, \ldots, (1 - \delta)n$ we denote by $\mathcal{H}_i$ the history $m_1, \ldots, m_i$. It remains to show that (M, V) is $(\varepsilon', d^3/32)$-super-regular with respect to the auxiliary graph $H = H_G(M, V)$.

Observe, that any $m \in M$ has $|N_H(m)| \geq d^2 n/2$ by construction. We now show that for any $v \in V$ we have $|N_H(v)| \geq d^3 n_0/32$. For this, it is sufficient to consider the first $dn_0/2$ edges

$m_1, \ldots, m_{dn_0/2}$ that are chosen for M by the random greedy process. For $i = 1, \ldots, dn_0/2$, by (8), there are at most $e(\tilde{F}) \leq (1 + \varepsilon)n_0^2 p$ available edges to chose m_i from. On the other hand, as long as $i \leq dn_0/2$, the vertex v has at least $dn_0/2$ neighbours in each of the sets U and W in G that are not covered by the edges in $\mathcal{H}_{i-1}$. Let $U' \subseteq U$ and $W' \subseteq W$ be the sets of those vertices and observe that $|U'|, |W'| \geq dn_0/2 \geq \delta n_0$. Therefore, by (8) and Lemma 8.2 (i), there are at least

$$(1 - \varepsilon)e_F(U', W')p \geq (1 - \varepsilon)(d/2)(d/2 - \varepsilon)n_0^2 p$$

edges in $\tilde{F}[(U \cup W) \setminus V(\mathcal{H}_{i-1})]$ available to choose m_i from, such that both endpoints of m_i are incident to v in G.

Hence, for $i = 1, \ldots, dn_0/2$, we get

$$\mathbb{P}[m_i \in N_H(v)|\mathcal{H}_{i-1}] \geq \frac{(1 - \varepsilon)(d/2)(d/2 - \varepsilon)n_0^2 p}{(1 + \varepsilon)n_0^2 p} = \frac{(1 - \varepsilon)(d/2)(d/2 - \varepsilon)}{(1 + \varepsilon)} \geq \frac{d^2}{8}.$$

As this holds independently of the history of the process, this process dominates a binomial distribution with parameters $dn_0/2$ and $d^2/8$. Therefore, even though the events are not mutually independent, we can use Chernoff's inequality to infer that $|N_H(v)| \geq d^3 n_0/32$ with probability at least $1 - 1/n^2$. Then, by applying the union bound over all $v \in V$, we obtain that indeed a.a.s. $|N_H(v)| \geq d^3 n_0/32$ for all $v \in V$.

Now we show the regularity of $H = H_G(M, V)$. Let $X \subseteq V$ be given with $|X| = \eta n$. For $i = 0, 1, \ldots, t - 1$, we obtain from (8) that there are at most $(1 + \varepsilon)(n_0 - i)^2 p$ edges in $\tilde{F}[(U \cup W) \setminus V(\mathcal{H}_i)]$ available for choosing m_{i+1}, of which, by (9), at least $(1 - 7\varepsilon/\delta)(n_0 - i)^2 p$ are good for X. Then

$$\mathbb{P}\left[m_{i+1} \text{ bad for } X \middle| \mathcal{H}_{i-1}\right] \leq 1 - \frac{(1 - 7\varepsilon/\delta)(n_0 - i)^2 p}{(1 + \varepsilon)(n_0 - i)^2 p} = 1 - \frac{1 - 7\varepsilon/\delta}{1 + \varepsilon} \leq 8\varepsilon/\delta.$$

As the upper bound on the probability holds independently of the history, this process is dominated by a binomial distribution with parameters $(1 - \delta)n_0$ and $8\varepsilon/\delta$. Observe that the expected value of this distribution is $(1 - \delta)n_0 \, 8\varepsilon/\delta \leq 8\varepsilon n_0/\delta$. Then, with $B_X \subseteq M$ being the edges in M that are not good for X and as $\gamma n_0 \geq 7 \cdot 8\varepsilon n_0/\delta$, we get from Chernoff's inequality (second part of Lemma 3.1) that

$$\mathbb{P}\left[|B_X| > \gamma n_0\right] \leq \exp\left(-\gamma n_0\right).$$

There are at most $\binom{n}{\eta n} \leq (e/\eta)^{\eta n} \leq \exp(\eta \log(1/\eta)n) \leq \exp(\gamma n_0/2)$ choices for X and, thus, with the union bound over all these choices, we obtain that a.a.s. there are at most γn_0 bad edges in M for any $X \subseteq V$ with $|X| = \eta n$.

Therefore for any set $X' \subseteq V$ and $M' \subseteq M$ with $|X'| \geq \varepsilon' n$ and $|M'| \geq \varepsilon'|M|$ we find

$$e_H(M', X') \geq (|M'| - \gamma n_0)\frac{d^2 \eta n}{2} \frac{|X'|}{2\eta n} \geq \frac{d^2}{8}|M'||X'|.$$

Overall, we can conclude the pair (M, V) is $(\varepsilon', d^3/32)$-super-regular with respect to the auxiliary graph $H_G(M, V)$. □

8.2 Proof of Lemma 4.3

Lemma 4.3 easily follows from Lemma 4.2 after splitting the sets U and V appropriately.

Proof of Lemma 4.3. Let $0 < d < 1$, choose δ' with $0 < \delta' \leq d/4$ and apply Lemma 4.2 with δ' and $d/4$ to obtain $\delta_0, \delta, \varepsilon'$ with $\delta' \geq \delta_0 > \delta > \varepsilon' > 0$ and $C' > 0$. Then let ε with $0 < \varepsilon \leq \varepsilon'/8$ and $C \geq 10C'$. Next let U and V be vertex sets of size $|V| = n$ and $3n/4 \leq |U| = n_0 \leq n$ where $n + n_0 \equiv 0 \pmod 3$ and assume that (U, V) is a (ε, d)-super-regular pair.

The idea is to split the regular pair (U, V) into two regular cherries and then use Lemma 4.2 in each cherry. More precisely, we partition V into V_1, U_2, W_2 and U into V_2, U_1, W_1 such that for $i = 1, 2$ the pairs (U_i, V_i) and (W_i, V_i) are $(\varepsilon', d/4)$-super-regular pairs and $(1 - \delta_0)|V_i| \leq |U_i| = |W_i| \leq (1 - \delta)|V_i|$. To obtain this, we split the sets according to the following random process. We put any vertex of V into each of U_2 and W_2 with probability q_1 and into V_1 with probability $1 - 2q_1$. Similarly, we put any vertex of U into each of U_1 and W_1 with probability q_2 and into V_2 with probability $1 - 2q_2$. We choose q_1 and q_2 such that the expected sizes satisfy for $i = 1, 2$

$$\mathbb{E}[|U_i|] = \mathbb{E}[|W_i|] = \left(1 - \frac{\delta_0 + \delta}{2}\right)\mathbb{E}[|V_i|].$$

This is possible since such conditions give a linear system of two equations in two variables q_1 and q_2. As $3n/4 \leq n_0 \leq n$, the solution satisfies $1/7 < q_1, q_2 < 3/7$. Then by Chernoff's inequality (Lemma 3.1) and with n large enough there exists a partition such that for $i = 1, 2$ we have that $|W_i|, |U_i|$, and $|V_i|$ are all within $\pm n^{2/3}$ of their expectation and the minimum degree within both pairs (U_i, V_i) and (W_i, V_i) is at least a $d/2$-fraction of the other set. For $i = 1, 2$ we redistribute $o(n)$ vertices between U_i and W_i and move at most one vertex from or to V_i to obtain

$$(1 - \delta_0)|V_i| \leq |U_i| = |W_i| \leq (1 - \delta)|V_i|$$

with minimum degree within both pairs (U_i, V_i) and (W_i, V_i) at least a $d/4$-fraction of the other set.

From this we get that for $i = 1, 2$ the pairs (U_i, V_i) and (W_i, V_i) are $(\varepsilon', d/4)$-super-regular. Also note that $\min\{|V_1|, |V_2|\} \geq \min\{(1 - 2.1q_1)n, (1 - 2.1q_2)n_0\} \geq n/10$ by the bound on q_1 and q_2 from above. Then for $i = 1, 2$ and with $C/n \geq C'/\min\{|V_1|, |V_2|\}$ we a.a.s. get a triangle factor on V_i, U_i, W_i from Lemma 4.2. Together these give a triangle factor on $U \cup V$. $\quad\square$

9. Concluding remarks and open problems

In this paper, we close the last gap for the existence of a triangle factor in randomly perturbed graphs $G_\alpha \cup G(n, p)$ and now the threshold is determined for any $\alpha \in [0, 1]$ (c.f. Table 1). In this last section, we would like to discuss a possible improvement for Theorem 2.4 and three different directions for generalisations of our results.

9.1 Optimality of the sublinear theorem

The bound on p in Theorem 1.3 is asymptotically optimal. However, as discussed earlier, when m is significantly smaller than $n/3$, then $K_{m,n-m} \cup G(n, p)$ contains m pairwise vertex-disjoint triangles already at $p \geq C/n$. When dealing with the sublinear case $1 \leq m \leq n/256$ in Theorem 2.4, we require $p \geq C \log n/n$; in fact we do not use the log n-term when $(\log n)^3 \leq m \leq \sqrt{n}$ (notice indeed that Proposition 7.1 is stated with $p \geq C/n$). It would be interesting to understand if $p \geq C/n$ also suffices for larger values of m and if Proposition 7.2 can be improved.

Problem 9.1. *Show that there exists $C > 0$ such that for any $\sqrt{n} \leq m \leq n/32$ and any n-vertex graph G with maximum degree $\Delta(G) \leq n/32$ and minimum degree $\delta(G) \geq m$, a.a.s. $G \cup G(n, p)$ contains m pairwise vertex-disjoint triangles, provided $p \geq C/n$.*

Note that this would improve Theorem 2.3 when $\alpha < 1/3$. Indeed in our proof we only need the log n-term as we apply Theorem 2.4, while we could use Lemma 4.2 instead of 4.1 and avoid the log n-term in the rest of the argument.

9.2 Larger cliques

Similarly to a triangle factor, for any integer $r \geq 2$ and any $\alpha \in [0, 1]$, one can look at the threshold for the existence of a K_r-factor in $G_\alpha \cup G(n, p)$. The result of Johannson, Kahn, and Vu [20] determines the threshold in $G(n, p)$ at $n^{-2/r} \log^{2/(r^2-r)} n$ and for $\alpha \geq 1 - 1/r$ the existence in G_α alone is proved by Hajnal-Szemerédi Theorem [16]. For other values of α, the results of Balogh, Treglown, and Wagner [4] and of Han, Morris and Treglown [17] can be summarised in the following theorem generalising Theorem 1.2.

Theorem 9.2. *For any integers $2 \leq k \leq r$ and any $\alpha \in (1 - \frac{k}{r}, 1 - \frac{k-1}{r})$ there exists $C > 0$ such that the following holds. For any n-vertex graph G with minimum degree $\delta(G) \geq \alpha n$ there a.a.s. is a K_r-factor in $G \cup G(n, p)$ provided that $p \geq C n^{-2/k}$.*

However, this leaves open the question for the threshold in the boundary cases, i.e. when $\alpha = 1 - k/r$ for $2 \leq k \leq r - 1$. A natural extremal structure is given by the complete $\lceil r/k \rceil$-partite graph with $\lfloor r/k \rfloor$ classes of size kn/r and possibly one class of size $(1 - \lfloor r/k \rfloor k/r)n$ if $k \nmid r$. This implies that to get a K_r-factor in $G_\alpha \cup G(n, p)$, we need to cover all but polylog n vertices of the sets of size kn/r with vertex-disjoint copies of K_k, i.e. the threshold is at least $n^{-2/k}(\log n)^{2/(k^2-k)}$. Surprisingly, this is not sufficient in the case when $r > 3$ and $k \neq 2$; in fact, for small ε, even $n^{-2/k+\varepsilon}$ is not sufficient.

We briefly explain the counterexample for $r = 4$ and $k = 3$, by constructing an n-vertex graph G with minimum degree $\delta(G) \geq (1 - 3/4)n = n/4$ such that even for small $\varepsilon > 0$ and $p \geq n^{-2/3+\varepsilon}$, a.a.s. the graph $G \cup G(n, p)$ does not contain a K_4-factor. Let $0 < \varepsilon \leq 1/49$, $p \geq n^{-2/3+\varepsilon}$, and $n^{7\varepsilon} \leq m \leq n^{1/7}$. Then, for two sets A and B with $|A| = n/4 - m$ and $|B| = 3n/4 + m$, we let G be the n-vertex graph on $V(G) = A \cup B$ such that A is an independent set, $G[B]$ is given by $|B|/(2m)$ disjoint copies of $K_{m,m}$, and any pair of vertices (a, b) with $a \in A$ and $b \in B$ is an edge. Clearly, G has minimum degree $n/4$. If $G \cup G(n, p)$ contains a K_4-factor, since A only contains $n/4 - m$ vertices, at least m copies of K_4 must lie within B. However we claim that a.a.s. the perturbed graph $G \cup G(n, p)[B]$ contains less than m copies of K_4 and thus a.a.s. $G \cup G(n, p)$ does not contain a K_4-factor. Denote by X the number of K_4's in $G \cup G(n, p)[B]$. It is not hard to see that when m is not too small the best way to build a K_4 in B is to choose a $K_{m,m}$ in B and ask for an edge of $G(n, p)$ on each side of $K_{m,m}$. We get $\mathbb{E}[X] \lesssim \frac{n}{m} m^4 p^2 = o(m)$ and by Markov's inequality a.a.s. $X < m$ as claimed.

Problem 9.3. *Determine the behaviour of the threshold and the extremal graphs at and around minimum degree $n/4$.*

The counterexamples for other values of $r > 3$ and $k \neq 2$ can be constructed in a similar way, by slightly modifying the corresponding extremal graph defined above. In the case when $k = 2$, this construction does not increase the lower bound $n^{-1} \log n$ and, with Theorem 2.4 in mind, we believe that Theorem 1.4 generalises to K_r. However, we believe that in all cases, using our methods, Theorem 2.2 can be extended to K_r: i.e. for all $2 \leq k \leq r - 1$ and with $\alpha = 1 - k/r$, when G_α is not close (with a similar condition as in Definition 2.1) to the extremal graph defined above, then $p \geq C n^{-2/k}$ is sufficient for a K_r-factor in $G_\alpha \cup G(n, p)$.

9.3 Longer cycles

Another possible generalisation is to consider factors of longer cycles C_ℓ. With slight adjustments to our methods (mainly to Theorem 2.4), we are able to show the following theorem.

Theorem 9.4. *For any integer $\ell \geq 3$ there exists $C > 0$ such that for any n-vertex graph G we can a.a.s. find at least $\min\{\delta(G), \lfloor n/\ell \rfloor\}$ pairwise vertex-disjoint C_ℓ's in $G \cup G(n, p)$, provided that $p \geq C \log n/n$.*

We will discuss this in [9, 10], including the corresponding variants for the finer statements, Theorems 2.2–2.4.

9.4 Universality

Even further, we call a graph 2-universal if it contains any n-vertex graph of maximum degree 2 as a subgraph. It is known that the threshold for 2-universality is asymptotically the same as for a triangle factor when $\alpha < 1/3$ or $\alpha \geq 2/3$ [1, 15, 28]. In a follow-up paper, we will expand our approach and prove that this also holds for the remaining cases, i.e. $\log n / n$ gives the threshold for $\alpha = 1/3$ and $1/n$ gives the threshold when $1/3 < \alpha < 2/3$.

Acknowledgements

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A. Proof of Lemmas 3.2, 3.6, and 4.4

For the proof of Lemma 3.2 we use Janson's inequality (see e.g. [19, Theorem 2.14]).

Lemma A.1. *Janson's inequality Let $p \in (0, 1)$ and consider a family $\{H_i\}_{i \in \mathcal{I}}$ of subgraphs of the complete graph on the vertex set $[n] = \{1, \ldots, n\}$. For each $i \in \mathcal{I}$, let X_i denote the indicator random variable for the event that $H_i \subseteq G(n, p)$ and, write $H_i \sim H_j$ for each ordered pair $(i, j) \in \mathcal{I} \times \mathcal{I}$ with $i \neq j$ if $E(H_i) \cap E(H_j) \neq \emptyset$. Then, for $X = \sum_{i \in \mathcal{I}} X_i$, $\mathbb{E}[X] = \sum_{i \in \mathcal{I}} p^{e(H_i)}$,*

$$\delta = \sum_{H_i \sim H_j} \mathbb{E}[X_i X_j] = \sum_{H_i \sim H_j} p^{e(H_i) + e(H_j) - e(H_i \cap H_j)}$$

and any $0 < \gamma < 1$ we have

$$\mathbb{P}[X \leq (1 - \gamma)\mathbb{E}[X]] \leq \exp\left(-\frac{\gamma^2 \mathbb{E}[X]^2}{2(\mathbb{E}[X] + \delta)}\right).$$

Proof of Lemma 3.2. Let $d > 0$ and pick $C > 68/d^3$ and $p > C/n$. Let $\mathcal{I} = E_G(U, W) \times V$ and for each $i = (uw, v) \in \mathcal{I}$ let H_i be the path uvw on the three vertices $u \in U$, $v \in V$ and $w \in W$. We want to apply Lemma A.1 to the family $\{H_i\}_{i \in \mathcal{I}}$. Using the same notation, we have $\mathbb{E}[X] = |\mathcal{I}|p^2 \geq dn^3 p^2$ and $\delta \leq n^2(2n)np^3$, as for $i = (uw, v)$ and $j = (u'w', v')$ with $i, j \in \mathcal{I}$ and $i \neq j$ we have $H_i \sim H_j$ if and only if $v = v'$ and precisely one of the equalities $u = u'$ and $w = w'$ holds. Then the Janson's inequality with $\gamma = 1/2$ gives

$$\mathbb{P}\left[X \leq \frac{\mathbb{E}[X]}{2}\right] \leq \exp\left(-\frac{\gamma^2}{2}\frac{dn^3 p^2}{1 + 2np/d}\right) \leq \exp\left(-\frac{1}{17}d^2 n^2 p\right) \leq 2^{-4n/d},$$

as $p > C/n$ and $C > 68/d^3$. Thus with probability at least $1 - 2^{-4n/d}$ we have $X > \mathbb{E}[X]/2$ and there is at least one path H_i on vertices u, v, w for some $i = (uw, v) \in \mathcal{I}$ in $G(n, p)$. As uw is an edge of G by definition of $\mathcal{I}$, we get a triangle in $G \cup G(U \cup W, V, p)$ with one vertex in each of U, V, W, as required. $\square$

Next, we prove that if a bipartite graph G has large minimum degree, its random subgraph G_p contains a perfect matching if $p \geq C \log n/n$.

Proof of Lemma 3.6. For $\varepsilon > 0$ let $C \geq 3/\varepsilon$. Let G be a bipartite graph with partition classes U and W of size n and minimum degree at least $(1/2 + \varepsilon)n$. Let $p \geq C \log n/n$ and suppose that G_p does not have a perfect matching. Then, by Hall's Theorem, there exists a set $S \subseteq U$ or $S \subseteq W$ with $|S| > |N(S)|$. Let S be a minimal such set, then $|S| = |N(S)| + 1$, $|S| \leq \lceil n/2 \rceil$, and every vertex in $|N(S)|$ is adjacent to at least two vertices of S.

Let $1 \le s \le \lceil n/2 \rceil$ and let A_s denote the event that there is a minimal set S of size s violating Hall's condition. If $s = 1$, then S is an isolated vertex. Let X_n be the number of isolated vertices in G_p, then for n large enough

$$\mathbb{E}[X_n] \le 2n \left(1 - \frac{C \log n}{n} \right)^{(1/2+\varepsilon)n} \le 2n \exp\left(- C(1/2+\varepsilon) \log n \right) \le 2n^{1-C(1/2+\varepsilon)}$$

and $\lim_{n\to\infty} \mathbb{E}[X_n] = 0$. Therefore, a.a.s. G_p does not contain isolated vertices.

If $s \ge 2$, then

$$\mathbb{P}[A_s] \le \binom{n}{s} \binom{n}{s-1} \left(\binom{s}{2} p^2 \right)^{s-1} (1-p)^{s((1/2+\varepsilon)n-s+1)} \le \left(\frac{en}{s} \right)^s \left(\frac{en}{s-1} \right)^{s-1} s^{2(s-1)} (1-p)^{s\varepsilon n}$$

$$\le e^{2s-1} \frac{s^{2(s-1)}}{s^s (s-1)^{s-1}} n^{2s-1} \exp\left(- s\varepsilon C \log n \right) \le e^{2s-1} n^{-s(\varepsilon C-2)-1} = \frac{1}{en} \left(\frac{e^2}{n^{\varepsilon C-2}} \right)^s.$$

We have

$$\mathbb{P}[A_s] \le \frac{1}{en} \left(\frac{e^2}{n} \right)^s.$$

The probability that there is a minimal set S violating Hall's condition is then bounded by

$$\sum_{s=2}^{\lceil n/2 \rceil} \mathbb{P}[A_s] \le \sum_{s=2}^{\lceil n/2 \rceil} \frac{1}{en} \left(\frac{e^2}{n} \right)^s \le \frac{1}{en} \sum_{s \ge 0} \left(\frac{e^2}{n} \right)^s = \frac{1}{en} \frac{1}{1-e^2/n}.$$

As $\lim_{n\to\infty} \frac{1}{en} \frac{1}{1-e^2/n} = 0$, a.a.s. G_p does not violate Hall's condition and, therefore, contains a perfect matching. □

Finally we prove Lemma 4.4 following the argument of [3, Lemma 12 and 13]. For this we consider a largest matching M in the reduced graph R and assume that $|M| < (\alpha + 2d)t$. Then we will find a set $I \subseteq V(R)$ of size roughly $(1-\alpha)t$ which contains very few edges. With the properties of the reduced graph, we conclude that the original graph G is (α, β)-stable.

Proof of Lemma 4.4. Given $0 < \beta < \frac{1}{12}$ let $0 < d < 10^{-4}\beta^6$. Then let $0 < \varepsilon < d/4$, $4\beta \le \alpha \le \frac{1}{3}$, and $t \ge \frac{10}{d}$. Next let G be an n-vertex graph on vertex set V with minimum degree $\delta(G) \ge (\alpha - \frac{1}{2}d)n$ that is not (α, β)-stable and let R be the (ε, d)-reduced graph for some (ε, d)-regular partition $V_0, \ldots, V_t$ of G. We observe for the minimum degree of R that $\delta(R) \ge (\alpha - 2d)t$ because, otherwise, there would be vertices with degree at most $(\alpha - 2d)t(n/t) + \varepsilon n < (\alpha - d/2)n - (d+\varepsilon)n$ in G' contradicting (P2).

Let M be a matching in R of maximal size. Observe that $|M| \ge \min\{\delta(R), \lfloor t/2 \rfloor\} \ge (\alpha - 2d)t$. We assume $|M| < (\alpha + 2d)t$ and show that G must then be (α, β)-stable, which is a contradiction. Let $U = V(R) \setminus V(M)$. We shall first show that there exists a set $I \subseteq V(R)$ of size $|U| + |M|$ that contains only few egdes.

Since M is a matching of maximal size in R, U is independent. Moreover, given an edge $xy \in M$, either x or y has at most one neighbour in U. Then we can split $V(M)$ into two disjoint subsets X and Y by placing for each matching edge xy of M one of its endpoints with at most one neighbour in U into the subset X, and the other endpoint into the subset Y. We claim that $I = U \cup X$ contains only few edges. We have $e(U) = 0$, $e(X, U) \le |X|$, and we can upper bound $e(X)$ as follows. Let $xy \in E(X)$ and denote by x' and y' the vertices matched to x and y in M respectively. Then $x', y' \in Y$ and either x' or y' has at most one neighbour in U. Otherwise, there would be two distinct vertices $x'', y'' \in U$ such that $x'x''$ and $y'y''$ are edges of R, and we could apply the rotation $M \setminus \{xx', yy'\} \cup$

$\{xy, x'x'', y'y''\}$ and get a larger matching, contradicting the maximality of M. Therefore, $e(X) \le |X||Z|$, where $Z = \{v \in Y \mid \deg(v, U) < 2\}$. Observe that

$$e(Y, U) \le (|Y| - |Z|)|U| + |Z|$$

and

$$e(Y, U) \ge |U|\delta(R) - e(X, U) \ge |U|\delta(R) - |X|$$

where we use that since U is independent, a vertex in U can have neighbours only in X and Y. We get

$$|Z| \le \frac{(|Y| - \delta(R))|U| + |X|}{|U| - 1} < \frac{4dt|U| + |X|}{|U| - 1} \le 5dt,$$

where the first inequality comes from the upper and lower bound on $e(Y, U)$, the second one from $|Y| = |M| < (\alpha + 2d)t$ and $\delta(R) \ge (\alpha - 2d)t$, and the last one from $|U| = t - 2|M| \ge t/4$, $|X| = |M| < t/2$ and $10/t \le d$. Hence, $e(X) \le |X|5dt$.

Therefore, the set $I = U \cup X$ has size

$$|I| = |V(R)| - |M| = (1 - \alpha \pm 2d)\,t$$

because $|Y| = |M| = (\alpha \pm 2d)t$ and contains at most

$$e(I) \le e(X) + e(X, U) + e(U) \le |X|5dt + |X| \le (5dt + 1)(\alpha + 2d)t \le 6\alpha dt^2$$

edges, where we use $|X| = |M| = (\alpha \pm 2d)t$, $d \le \alpha/20$ and $10/t \le d$ in the last inequality.

We now move to the original graph G and prove that the existence of such set I in R implies that G is (α, β)-stable. Let $B'' = \bigcup_{i \in I} V_i$ be the union of the clusters I. Then $|B''| = (1 - \alpha \pm 3d)n$ and $e(B'') \le 6\alpha dn^2$. Let

$$B' = \{v \in B'' \mid \deg(v, B'') \le \sqrt{d}n\}.$$

Then $e(B'') \ge (|B''| - |B'|)\sqrt{d}n$ and, therefore, all but at most $6\alpha\sqrt{d}n$ vertices of B'' belong to B' and, thus, $|B'| = (1 - \alpha \pm 4\sqrt{d})n$. Let

$$A' = \{v \in V \mid \deg(v, B') \ge (1 - \beta/4)|B'|\}$$

and note that $A' \cap B' = \emptyset$. Observe that if $v \in B'$, then

$$\deg(v, V \setminus B') \ge \delta(G) - \deg(v, B') \ge (\alpha - d/2 - \sqrt{d})n \ge (\alpha - 2\sqrt{d})n. \tag{10}$$

With $|V \setminus B'| \le (\alpha + 4\sqrt{d})n$ this implies

$$e(B', V \setminus B') \ge (\alpha - 2\sqrt{d})n|B'| \ge (|V \setminus B'| - 6\sqrt{d}n)|B'|$$

and with the definition of A' and the fact that $|V \setminus (B' \cup A')| = |V \setminus B'| - |A'|$, we get

$$e(B', V \setminus B') \le |V \setminus (B' \cup A')|(1 - \beta/4)|B'| + |A'||B'| = (|V \setminus B'| - |V \setminus (B' \cup A')|\beta/4)|B'|.$$

The last two inequalities imply that all but at most $24\sqrt{d}n/\beta$ vertices of $V \setminus B'$ belong to A'. Therefore we can bound the size of A' as follows

$$|A'| \ge |V \setminus B'| - 24\sqrt{d}n/\beta \ge \alpha n - 4\sqrt{d}n - 24\sqrt{d}n/\beta \ge \alpha n - \beta^2 n,$$

and

$$|A'| \le |V \setminus B'| \le \alpha n + 4\sqrt{d}n \le \alpha n + \beta^2 n.$$

where we used in both inequalities that $4\sqrt{d}n + 24\sqrt{d}n/\beta \le \beta^2 n$, as $d \le 10^{-4}\beta^6$.

It follows that we have built two sets A' and B' such that $|A' \cup B'| \ge n - \beta^2 n$, $|A'| = \alpha n \pm \beta^2 n$ and $|B'| = (1 - \alpha)n \pm \beta^2 n$. Moreover each vertex of A' has at least $(1 - \beta/4)|B'|$ neighbours in B'

by the definition of A', and each vertex of B' has at least $(1 - \beta/2)|A'|$ neighbours in A'. This can be justified as follows. Given $v \in B'$,

$$
\begin{aligned}
\deg(v, A') &= \deg(v, V \setminus B') - \deg(v, V \setminus (A' \cup B')) \\
&\geq \deg(v, V \setminus B') - |V \setminus (A' \cup B')| \\
&\geq (\alpha - 2\sqrt{d})n - 24\sqrt{d}n/\beta \geq (\alpha - \beta^2)n \\
&\geq \frac{\alpha - \beta^2}{\alpha + \beta^2}|A'| \geq (1 - \beta/2)|A'|,
\end{aligned}
$$

where we used that A' and B' are disjoint, the inequalities (10), $|V \setminus (A' \cup B')| \leq 24\sqrt{d}n/\beta$ and $2\sqrt{d} + 24\sqrt{d}/\beta \leq \beta^2$, the upper bound on $|A'|$ and the inequality $\alpha \geq 4\beta$.

Now we need to take care of the vertices of G not yet covered by $A' \cup B'$, i.e. the at most $\beta^2 n$ vertices in $V \setminus (A' \cup B')$. Let v be one such vertex. Then $\deg(v, A' \cup B') \geq \delta(G) - |V \setminus (A' \cup B')| \geq (\alpha - d/2)n - \beta^2 n \geq \alpha n/2$. Therefore, it is possible to add these vertices to A' and B' to obtain $A \supseteq A'$ and $B \supseteq A'$ such that each vertex of B has at least $\alpha n/4$ neighbours in A, and each vertex of A has at least $\alpha n/4$ neighbours in B. As we add at most $\beta^2 n$ vertices, we have $|A| = (\alpha \pm 2\beta^2)n$ and $|B| = (1 - \alpha \pm 2\beta^2)n$. Moreover, all but at most $\beta^2 n \leq \beta n$ vertices from A have degree at least

$$
(1 - \beta/4)|B'| \geq (1 - \beta/4)(|B| - \beta^2 n) \geq (1 - \beta)|B|
$$

into B, where we used that $|B| \leq |B'| + \beta^2 n$, $|B| \geq (1 - \alpha - 2\beta^2)n \geq (2/3 - 2\beta^2)n$ and $\beta < 1/12$. Similarly, all but at most $\beta^2 n \leq \beta n$ vertices from B have degree at least $(1 - \beta/2)|A'| \geq (1 - \beta)|A|$ into A. Moreover, as B' is a subset of B'' and we add at most $\beta^2 n$ vertices to B' to get B, we have $e(B) \leq e(B'') + \beta^2 n^2 \leq (6\alpha d + \beta^2)n^2 \leq \beta n^2$. Therefore, G is (α, β)-stable according to Definition 2.1. □