# Perpetual American standard and lookback options with event risk and asymmetric information

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We derive closed-form solutions to the perpetual American standard and lookback put and call options in an extension of the Black-Merton-Scholes model with event risk and incomplete information. It is assumed that the contracts are terminated with linear recoveries at the last hitting times for the underlying asset price process of its running maximum or minimum over the infinite time interval which are not stopping times with respect to the observable filtration. We show that the optimal exercise times are the first times at which the asset price reaches some lower or upper stochastic boundaries depending on the current values of its running maximum or minimum. The proof is based on the reduction of the original optimal stopping problems to the associated free-boundary problems and the solution of the latter problems by means of the smooth-fit and normal-reflection conditions. The optimal exercise boundaries are proven to be the maximal or minimal solutions of some first-order nonlinear ordinary differential equations.

## 1. Introduction

Inspired by game options, we study a situation in which financial contracts can be terminated or cancelled prematurely due to certain (insider) information which is not available to the holders of the contracts. More precisely, we suppose that the contracts are terminated by the writers at the last times at which the underlying stock reaches its running maximum or minimum and the linear and fractional recovery amounts are paid to the holders. These particular choices of the termination times are motivated by the studies of the so-called optimal buyback times for short sellers in the face of bubble formations or recall risk from market

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insiders. The framework we use can find interpretation within the recent GameStop saga, where subreddit users colluded to buy and hold GameStop shares in order to bet against short sellers from hedge funds. As a consequence of this act of collusion, the price of GameStop shares was driven up and the timing of the last maximum was effectively when the subreddit users could no longer collude (Social platform Discord Ban WallStreetBets Server on 27th Jan 2021) and trading restrictions were placed by TDAmeritrade (on 27th of Jan 2021), Robinhood and WeBull (on 28th of Jan 2021), both of which were representing not public information. More specifically, we may assume that a short seller initiates a short sale of a risky asset at time zero with the aim to close their position at some random time in the future. The cash flow related to this operation is equal to the difference between the initial market price of the asset and its price at this random time discounted by the value of a riskless asset with prevailing interest rate which is assumed to be identical to the asset lending fee. The short seller's objective is therefore to search for an optimal time to repurchase the asset before the lender (who might have some extra insider information) recalls the asset at its historic maximum or the bubble bursts and the trading is then stopped. For further related studies on the optimal buyback times in faces of recall risk, we refer to Glover and Hulley [25].

For a precise formulation of the problems, we consider a probability space $(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $B = (B_t)_{t \geq 0}$. Assume that the process $X = (X_t)_{t \geq 0}$ describing the price of a risky asset in a financial market is given by:

$$X_t = x \, \exp\Big( \big( r - \delta - \sigma^2/2 \big) \, t + \sigma \, B_t \Big) \tag{1.1}$$

so that it solves the stochastic differential equation:

$$dX_t = (r - \delta) \, X_t \, dt + \sigma \, X_t \, dB_t \quad (X_0 = x) \tag{1.2}$$

where $x > 0$ is fixed, and $r > 0$, $\delta > 0$, and $\sigma > 0$ are some given constants. Here, $r$ is the riskless interest rate, $\delta$ is the dividend rate paid to the asset holders, and $\sigma$ is the volatility rate. Let the processes $S = (S_t)_{t \geq 0}$ and $Q = (Q_t)_{t \geq 0}$ be the *running maximum and minimum* of $X$ defined by:

$$S_t = s \vee \Big( \max_{0 \leq u \leq t} X_u \Big) \quad \text{and} \quad Q_t = q \wedge \Big( \min_{0 \leq u \leq t} X_u \Big) \tag{1.3}$$

for some arbitrary $0 < q \leq x \leq s$. To model the event horizon, we also introduce the random times $\theta$ and $\eta$ by:

$$\theta = \sup\{ t \geq 0 \, | \, X_t = S_t \} \quad \text{and} \quad \eta = \sup\{ t \geq 0 \, | \, X_t = Q_t \} \tag{1.4}$$

which are not stopping times with respect to the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ of the process $X$, but they are honest times in the sense of Barlow [5] and Nikeghbali and Yor [36].

The main aim of this paper is to compute closed-form expressions for the values of the discounted optimal stopping problems:

$$\overline{V}_i = \sup_{\tau} E\big[ e^{-r\tau} \, G_{i,1}(X_\tau, S_\tau) \, I(\tau < \theta) + e^{-r\theta} \, (\varphi_i + \psi_i \, X_\theta) \, I(\theta \leq \tau) \big] \tag{1.5}$$

https://en.wikipedia.org/wiki/GameStop_short_squeeze and

$$\overline{U}_i = \sup_{\zeta} E\big[e^{-r\zeta}\, G_{i,2}(X_\zeta, Q_\zeta)\, I(\zeta < \eta) + e^{-r\eta}\,(\xi_i + \chi_i\, X_\eta)\, I(\eta \le \zeta)\big] \tag{1.6}$$

with

$$G_{1,1}(x,s) = L_1 - x, \quad G_{2,1}(x,s) = s - L_2\, x, \quad G_{3,1}(x,s) = s - L_3 \tag{1.7}$$

and

$$G_{1,2}(x,q) = x - K_1, \quad G_{2,2}(x,q) = K_2\, x - q, \quad G_{3,2}(x,q) = K_3 - q \tag{1.8}$$

for some $L_i, K_i > 0$, $\varphi_i, \xi_i \in \mathbb{R}$, and $\psi_i, \chi_i \in (-1,1)$, for $i = 1,2,3$, fixed, where $I(\cdot)$ denotes the indicator function. Suppose that the suprema in (1.5) and (1.6) are taken over all stopping times $\tau$ and $\zeta$ with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$, and the expectations there are taken with respect to the risk-neutral probability measure $P$. In this view, the values $\overline{V}_i$ and $\overline{U}_i$, for $i = 1,2,3$, in (1.5) and (1.6) provide the rational (no-arbitrage) prices of the perpetual American defaultable standard and lookback options in an extension of the Black-Merton-Scholes model with event risk and asymmetric information, when we formally set $s = x$ and $q = x$ in (1.3) (see, e.g. [49; Chapter VII, Section 3g]). In particular, the functions $G_{1,1}(x,s)$ and $G_{1,2}(x,q)$ are the payoffs of *standard* put and call options, the functions $G_{2,1}(x,s)$ and $G_{2,2}(x,q)$ are the payoffs of put and call *lookback* options with *floating* strikes, while the functions $G_{3,1}(x,s)$ and $G_{3,2}(x,q)$ are the payoffs of put and call *lookback* options with *fixed* strikes. Some extensive overviews of the perpetual American options in diffusion models of financial markets and other related results in the area are provided in Shiryaev [49; Chapter VIII; Section 2a], Peskir and Shiryaev [44; Chapter VII; Section 25], and Detemple [14] among others. Note that, since the contracts are considered on the infinite time horizon, we may skip imposing the positive parts on the appropriate payoffs. This property follows from the comparison of the associated results in the case of complete information presented in Shiryaev [49; Chapter VIII; Sections 2a-2b] and Øksendal [37; Chapter X, Section 10.2] for the standard options and in Beibel and Lerche [11] as well as in Pedersen [39] and Guo and Shepp [29] for the lookback options with floating and fixed strikes, respectively.

From the point of view of financial mathematics and credit risk theory, the models in which the event or default times happen at the last passage times do not fall into the classical reduced form framework. More precisely, unlike in the existing models studied in Szimayer [50], Gapeev and Al Motairi [21], Glover and Hulley [25], Dumitrescu, Quenez, and Sulem [16], and Grigorova, Quenez, and Sulem [28], neither the immersion hypothesis nor the density hypothesis is satisfied (see Aksamit and Jeanblanc [1; Remark 5.31]), so that the default intensity process simply does not exist in our setting (see, e.g. Bielecki and Rutkowski [12; Chapter VIII] and Jeanblanc and Li [31] for the description of these concepts). We can see from the expressions of (2.2) and (2.3) below that, in the case of zero recovery, the diversion from the immersion hypothesis leads to the appearance of modified discounting factors which are no longer functions of the sum of the interest rate and the event time intensity rate but result in an adjusted dividend rate. Finally, if we were to study the finite horizon version of the optimal stopping problem from the point of view of the backward stochastic differential equations (BSDEs), as in Dumitrescu, Quenez, and Sulem [16] and Grigorova, Quenez, and Sulem [28], then it could be shown that the dynamics of the no-arbitrage (pre-default) price will no longer satisfy a reflected BSDE but rather a reflected generalised BSDE in which the generalised driver is related to the running maximum or minimum of the underlying asset. For other work in this direction, we refer to the recent paper by Aksamit, Li, and Rutkowski [2].

We further consider the problems of (1.5) and (1.6) as the associated optimal stopping problems of (2.17) and (2.18) for the two-dimensional continuous Markov processes having the underlying risky asset price $X$ and its running maximum $S$ or minimum $Q$ as their state space components. The resulting problems turn out to be necessarily two-dimensional in the sense that they cannot be reduced to optimal stopping problems for one-dimensional Markov processes. Note that the integrals in the reward functionals of the optimal stopping problems in (2.17) and (2.18) contain complicated integrands depending on the asset price as well as its running maximum and minimum processes. This challenge initiates further developments of techniques to determine the structure of the associated continuation and stopping regions as well as appropriate modifications of the normal-reflection conditions in the equivalent free-boundary problems. In particular, we show that the perpetual American defaultable lookback put and call options may be exercised when the processes $(X, S)$ or $(X, Q)$ start in certain subsets of the edges of their state spaces, under specific relations on the parameters of the model. These properties represent new features of the optimal stopping problems for the running maximum and minimum processes. Note that, in the paper by Shepp, Shiryaev, and Sulem [48] on the barrier lookback options as well as in the paper by Ott [38] on the lookback options with upper and lower caps, the upper bounds for the maxima processes were given *endogenously*. In this work, the upper bounds for the maximum process as well as the lower bounds for the minimum process are given *exogenously*, by virtue of the presence of the linear recovery amounts in the appropriate reward functionals. The case of perpetual American defaultable standard options in models with last passage times of constant levels for the underlying asset prices and zero recoveries was recently considered in Gapeev, Li, and Wu [20].

Discounted optimal stopping problems for the running maxima and minima of the initial continuous (diffusion-type) processes were initiated by Shepp and Shiryaev [47] and further developed by Pedersen [39], Guo and Shepp [29], Shepp, Shiryaev, and Sulem [48], Gapeev [18], Guo and Zervos [30], Peskir [42]-[43], Glover, Hulley, and Peskir [26], Gapeev and Rodosthenous [22]-[24], Rodosthenous and Zervos [46], and Gapeev, Kort, and Lavrutich [19] among others. It was shown, by means of the maximality principle for solutions of optimal stopping stopping problems established by Peskir [40], which is equivalent to the superharmonic characterisation of the value functions, that the optimal stopping boundaries are given by the appropriate extremal solutions of certain (systems of) first-order nonlinear ordinary differential equations. Other optimal stopping problems in more complicated models with spectrally negative Lévy processes and their running maxima were studied by Asmussen, Avram, and Pistorius [3], Avram, Kyprianou, and Pistorius [4], Ott [38], and Kyprianou and Ott [33] among others.

The rest of the paper is organised as follows. In Section 2, we embed the original problems of (1.5) and (1.6) into the optimal stopping problems of (2.17) and (2.18) for the two-dimensional continuous Markov processes $(X, S)$ and $(X, Q)$ defined in (1.1) and (1.3). It is shown that the optimal exercise times $\tau_i^*$ and $\zeta_i^*$ are the first times at which the process $X$ reaches some lower or upper boundaries $a_i^*(S)$ or $b_i^*(Q)$ depending on the current values of the processes $S$ or $Q$, for $i = 1, 2, 3$, respectively. In Section 3, we derive closed-form expressions for the associated value functions $V_i^*(x, s)$ and $U_i^*(x, q)$ as solutions to the equivalent free-boundary problems and apply the modified normal-reflection conditions at the edges of the two-dimensional state space for $(X, S)$ or $(X, Q)$ to characterise the optimal stopping boundaries $a_i^*(S)$ and $b_i^*(Q)$, for $i = 1, 2, 3$, as the maximal or minimal solutions to the resulting first-order nonlinear ordinary differential equations on the appropriate admissible intervals. In Section 4, by using the change-of-variable formula with local time on surfaces from Peskir [41], we verify that the solutions of the free-boundary problems provide the solutions of the original optimal stopping problems. The main results of the paper are stated in Theorems 2.1 and 4.1.

## 2. Preliminaries

In this section, we introduce the setting and notation of the two-dimensional optimal stopping problems which are related to the pricing of perpetual American standard and lookback put and call options with linear recoveries and formulate the equivalent free-boundary problems.

**2.1 The optimal stopping problems.** Let us first transform the rewards in the expressions of (1.5) and (1.6) with the aim to formulate the associated optimal stopping problems. For this purpose, we introduce the conditional survival processes or the Azéma supermartingales $Z = (Z_t)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ of the random times $\theta$ and $\eta$ defined by $Z_t = P(\theta > t \,|\, \mathcal{F}_t)$ and $Y_t = P(\eta > t \,|\, \mathcal{F}_t)$, for all $t \geq 0$, respectively. It is shown that the processes $Z$ and $Y$ have the form:

$$Z_t = \begin{cases} (S_t/X_t)^{\alpha}, & \text{if } \alpha < 0 \\ 1, & \text{if } \alpha \geq 0 \end{cases} \quad \text{and} \quad Y_t = \begin{cases} (Q_t/X_t)^{\alpha}, & \text{if } \alpha > 0 \\ 1, & \text{if } \alpha \leq 0 \end{cases} \tag{2.1}$$

for all $t \geq 0$, under $s = x$ and $q = x$, where we set $\alpha = 2(r - \delta)/\sigma^2 - 1$, respectively. More precisely, since the process $X^{\alpha} = (X_t^{\alpha})_{t \geq 0}$ is a positive martingale which converges to zero as $t$ tends to infinity, under $\alpha \neq 0$, we may conclude from the structure of random times $\theta$ and $\eta$ in (1.4) and using the result of [35; Example 1.3], which is a consequence of the Doob's maximal equality from [35; Lemma 0.1], that the processes $Z$ and $Y$ are given by (2.1), for $\alpha < 0$ and $\alpha > 0$, under $s = x$ and $q = x$, respectively. Similarly, it can be deduced from the law of iterated logarithms for standard Brownian motions that the properties $\limsup_{t \to \infty} X_t = \infty$ and $\liminf_{t \to \infty} X_t = 0$ hold, for $\alpha = 0$, implying that $\theta = \infty$ and $\eta = \infty$, and thus, $Z_t = 1$ and $Y_t = 1$, for all $t \geq 0$, under $s = x$ and $q = x$, respectively. Finally, we observe that the property $\lim_{t \to \infty} X_t = \infty$ holds, so that $Z_t = 1$, for $\alpha > 0$, while the property $\lim_{t \to \infty} X_t = 0$ holds, so that $Y_t = 1$, for $\alpha < 0$, under $s = x$ and $q = x$, for all $t \geq 0$, respectively.

Then, it follows from a direct application of the tower property for conditional expectations that the first terms in the right-hand sides of the expressions in (1.5) and (1.6) have the form:

$$E\big[e^{-r\tau}\, G_{i,1}(X_\tau, S_\tau)\, I(\tau < \theta)\big] = E\big[e^{-r\tau}\, G_{i,1}(X_\tau, S_\tau)\, (S_\tau/X_\tau)^{\alpha}\big] \tag{2.2}$$

when $\alpha < 0$, under $s = x$, and

$$E\big[e^{-r\zeta}\, G_{i,2}(X_\zeta, Q_\zeta)\, I(\zeta < \eta)\big] = E\big[e^{-r\zeta}\, G_{i,2}(X_\zeta, Q_\zeta)\, (Q_\zeta/X_\zeta)^{\alpha}\big] \tag{2.3}$$

when $\alpha > 0$, under $q = x$, for any stopping times $\tau$ and $\zeta$ of the process $X$, respectively. Moreover, it follows from standard applications of Itô's formula (see, e.g. [34; Theorem 4.4] or

[45; Chapter IV, Theorem 3.3]) and the properties that the processes $S$ and $Q$ may change their values only when $X_t = S_t$ and $X_t = Q_t$, for $t \geq 0$, respectively, that the Azéma supermartingales $Z$ and $Y$ from (2.1) admit the stochastic differentials:

$$dZ_t = -\alpha \left(\frac{S_t}{X_t}\right)^{\alpha} \sigma \, dB_t + \alpha \, I(X_t = S_t) \left(\frac{S_t}{X_t}\right)^{\alpha} \frac{dS_t}{S_t} = -\alpha \left(\frac{S_t}{X_t}\right)^{\alpha} \sigma \, dB_t + \alpha \, \frac{dS_t}{S_t} \tag{2.4}$$

when $\alpha < 0$, and

$$dY_t = -\alpha \left(\frac{Q_t}{X_t}\right)^{\alpha} \sigma \, dB_t + \alpha \, I(X_t = Q_t) \left(\frac{Q_t}{X_t}\right)^{\alpha} \frac{dQ_t}{Q_t} = -\alpha \left(\frac{Q_t}{X_t}\right)^{\alpha} \sigma \, dB_t + \alpha \, \frac{dQ_t}{Q_t} \tag{2.5}$$

when $\alpha > 0$, respectively. Hence, it follows from Doob-Meyer decompositions for the processes $Z$ and $Y$ in (2.4) and (2.5) and applications of the dual predictable projection property (see, e.g. [36; Corollary 2.4]) that the second terms in the right-hand sides of the expressions in (1.5) and (1.6) admit the representations:

$$E\big[e^{-r\theta} \, (\varphi_i + \psi_i \, X_\theta) \, I(\theta \leq \tau)\big] = -E\bigg[\int_0^{\tau} e^{-ru} \, (\varphi_i + \psi_i \, X_u) \, \alpha \, \frac{dS_u}{S_u}\bigg] \tag{2.6}$$

when $\alpha < 0$, under $s = x$, and

$$E\big[e^{-r\eta} \, (\xi_i + \chi_i \, X_\eta) \, I(\eta \leq \zeta)\big] = -E\bigg[\int_0^{\zeta} e^{-ru} \, (\xi_i + \chi_i \, X_u) \, \alpha \, \frac{dQ_u}{Q_u}\bigg] \tag{2.7}$$

when $\alpha > 0$, under $q = x$, for any stopping times $\tau$ and $\zeta$, and every $i = 1, 2, 3$, respectively.

By means of standard applications of Itô's formula to the process $e^{-rt} G_{i,1}(X_t, S_t)(S_t/X_t)^{\alpha}$, taking into account the facts that $\partial_{xx} G_{i,1}(x, s) = \partial_{xs} G_{i,1}(x, s) = \partial_{ss} G_{i,1}(x, s) = 0$ and $\alpha$ is selected such that the process $X^{-\alpha} = (X_t^{-\alpha})_{t \geq 0}$ is a positive continuous martingale, using the property that the process $S$ may change its value only when $X_t = S_t$, for $t \geq 0$, we obtain the representation:

$$\begin{aligned} e^{-rt} \, G_{i,1}(X_t, S_t) \, (S_t/X_t)^{\alpha} &= G_{i,1}(x, s) \, (s/x)^{\alpha} \\ &+ \int_0^t e^{-ru} \Big(\partial_x G_{i,1}(X_u, S_u) \, (r - \delta') \, X_u - r \, G_{i,1}(X_u, S_u)\Big) \left(\frac{S_u}{X_u}\right)^{\alpha} du \\ &+ \int_0^t e^{-ru} \Big(\partial_s G_{i,1}(X_u, S_u) \, S_u + \alpha \, G_{i,1}(X_u, S_u)\Big) \frac{dS_u}{S_u} + N_t^{i,1} \end{aligned} \tag{2.8}$$

when $\alpha < 0$, for each $0 < x \leq s$, and all $t \geq 0$, where we set $\delta' = \delta + \alpha\sigma^2 \equiv 2r - \delta - \sigma^2$, that can be considered as a *default adjusted dividend rate*. Here, by virtue of the structure of the integrands as well as the explicit forms of the densities of the marginal distributions of the two-dimensional process $(X, S)$, the processes $N^{i,1} = (N_t^{i,1})_{t \geq 0}$, for $i = 1, 2, 3$, defined by:

$$N_t^{i,1} = \int_0^t e^{-ru} \Big(\partial_x G_{i,1}(X_u, S_u) - \alpha \, G_{i,1}(X_u, S_u)\Big) \left(\frac{S_u}{X_u}\right)^{\alpha} \sigma \, dB_u \tag{2.9}$$

are continuous square integrable martingales under the probability measure $P$, when $\alpha < 0$. Then, by means of Doob's optional sampling theorem (see, e.g. [34; Chapter III, Theorem 3.6] or [45; Chapter II, Theorem 3.2]), we get:

$$
\begin{aligned}
&E\big[e^{-r\tau}\, G_{i,1}(X_\tau, S_\tau)\, (S_\tau/X_\tau)^\alpha\big] = G_{i,1}(x,s)\,(s/x)^\alpha \\
&+ E\bigg[\int_0^\tau e^{-ru}\Big(\partial_x G_{i,1}(X_u,S_u)\,(r-\delta')\,X_u - r\,G_{i,1}(X_u,S_u)\Big)\left(\frac{S_u}{X_u}\right)^\alpha du \\
&\quad + \int_0^\tau e^{-ru}\Big(\partial_s G_{i,1}(X_u,S_u)\,S_u + \alpha\, G_{i,1}(X_u,S_u)\Big)\frac{dS_u}{S_u}\bigg]
\end{aligned} \tag{2.10}
$$

when $\alpha < 0$, for any stopping time $\tau$ with respect to $(\mathcal{F}_t)_{t\ge 0}$. Hence, getting the expressions in (2.10) together with the ones in (2.6) above, we may conclude that the value of (1.5) is given by:

$$
\overline{V}_i = G_{i,1}(x,x) + \sup_\tau E\bigg[\int_0^\tau e^{-ru}\, H_{i,1}(X_u,S_u)\,du + \int_0^\tau e^{-ru}\,\frac{F_{i,1}(S_u)}{S_u}\,dS_u\bigg] \tag{2.11}
$$

when $\alpha < 0$, under $s = x$, where the supremum is taken over all stopping times $\tau$ of the process $(X,S)$. Here, we set:

$$
H_{i,1}(x,s) = \big(\partial_x G_{i,1}(x,s)\,(r-\delta')\,x - r\,G_{i,1}(x,s)\big)\,(s/x)^\alpha \tag{2.12}
$$

for all $0 < x \le s$, and

$$
F_{i,1}(s) = \partial_s G_{i,1}(s,s)\,s + \alpha\,\big(G_{i,1}(s,s) - \varphi_i - \psi_i\, s\big) \tag{2.13}
$$

for all $s > 0$. Thus, applying the arguments similar to the ones used above together with the expressions in (2.7), we may conclude that the value of (1.6) is given by:

$$
\overline{U}_i = G_{i,2}(x,x) + \sup_\zeta E\bigg[\int_0^\zeta e^{-ru}\, H_{i,2}(X_u,Q_u)\,du + \int_0^\zeta e^{-ru}\,\frac{F_{i,2}(Q_u)}{Q_u}\,dQ_u\bigg] \tag{2.14}
$$

when $\alpha > 0$, under $q = x$, where the supremum is taken over all stopping times $\zeta$ of the process $(X,Q)$. Here, we have:

$$
H_{i,2}(x,q) = \big(\partial_x G_{i,2}(x,q)\,(r-\delta')\,x - r\,G_{i,2}(x,q)\big)\,(q/x)^\alpha \tag{2.15}
$$

for all $0 < q \le x$, and

$$
F_{i,2}(q) = \partial_q G_{i,2}(q,q)\,q + \alpha\,\big(G_{i,2}(q,q) - \xi_i - \chi_i\, q\big) \tag{2.16}
$$

for all $q > 0$.

Therefore, we see that the problems in (2.11) and (2.14) can be naturally embedded into the optimal stopping problems for the (time-homogeneous strong) Markov processes $(X,S) = (X_t,S_t)_{t\ge 0}$ and $(X,Q) = (X_t,Q_t)_{t\ge 0}$ with the value functions:

$$
V_i^*(x,s) = \sup_\tau E_{x,s}\bigg[\int_0^\tau e^{-ru}\, H_{i,1}(X_u,S_u)\,du + \int_0^\tau e^{-ru}\,\frac{F_{i,1}(S_u)}{S_u}\,dS_u\bigg] \tag{2.17}
$$

when $\alpha < 0$, and

$$U_i^*(x,q) = \sup_{\zeta} E_{x,q}\bigg[\int_0^{\zeta} e^{-ru}\, H_{i,2}(X_u,Q_u)\, du + \int_0^{\zeta} e^{-ru}\, \frac{F_{i,2}(Q_u)}{Q_u}\, dQ_u\bigg] \tag{2.18}$$

when $\alpha > 0$, for every $i = 1,2,3$, respectively. Here, $E_{x,s}$ and $E_{x,q}$ denote the expectations with respect to the probability measures $P_{x,s}$ and $P_{x,q}$ under which the two-dimensional Markov processes $(X,S)$ and $(X,Q)$ defined in (1.1) and (1.3) start at $(x,s) \in E_1 = \{(x,s) \in \mathbb{R}^2 \,|\, 0 < x \le s\}$ and $(x,q) \in E_2 = \{(x,q) \in \mathbb{R}^2 \,|\, 0 < q \le x\}$, respectively. We further obtain solutions to the optimal stopping problems in (2.17) and (2.18) and verify below that the value functions $V_i^*(x,s)$ and $U_i^*(x,q)$, for $i = 1,2,3$, are the solutions of the problems in (2.11) and (2.14), and thus, give the solutions of the original problems in (1.5) and (1.6), under $s = x$ and $q = x$, respectively.

It follows from the arguments of [44; Chapter III, Section 6] that the continuation regions in the optimal stopping problems of (2.17) and (2.18) have the form:

$$C_{i,1}^* = \big\{(x,s) \in E_1 \,\big|\, V_i^*(x,s) > 0\big\} \quad \text{and} \quad C_{i,2}^* = \big\{(x,q) \in E_2 \,\big|\, U_i^*(x,q) > 0\big\} \tag{2.19}$$

so that the corresponding stopping regions in those problems are given by:

$$D_{i,1}^* = \big\{(x,s) \in E_1 \,\big|\, V_i^*(x,s) = 0\big\} \quad \text{and} \quad D_{i,2}^* = \big\{(x,q) \in E_2 \,\big|\, U_i^*(x,q) = 0\big\} \tag{2.20}$$

for every $i = 1,2,3$, respectively. It is seen from the results of Theorem 4.1 proved below that the value functions $V_i^*(x,s)$ and $U_i^*(x,q)$ are continuous, so that the sets $C_{i,1}^*$ and $C_{i,2}^*$ in (2.19) are open, while the sets $D_{i,1}^*$ and $D_{i,2}^*$ in (2.20) are closed, for every $i = 1,2,3$.

**2.2 The structure of optimal exercise times.** Let us now determine the structure of the optimal stopping times at which the holders should exercise the contracts.

**Theorem 2.1** *Let the processes $(X,S)$ and $(X,Q)$ be given by (1.1) and (1.3), with some $r > 0$, $\delta > 0$, and $\sigma > 0$ fixed, and the inequality $\delta' \equiv 2r - \delta - \sigma^2 > 0$ be satisfied. Suppose that the random times $\theta$ and $\eta$ are defined by (1.4). Then, the optimal exercise times for the perpetual American standard and lookback put and call options with the values in (2.17) and (2.18) have the structure:*

$$\tau_i^* = \inf\{t \ge 0 \,|\, X_t \le a_i^*(S_t)\} \quad \text{and} \quad \zeta_i^* = \inf\{t \ge 0 \,|\, X_t \ge b_i^*(Q_t)\} \tag{2.21}$$

*under $\alpha < 0$ and $\alpha > 0$, for $i = 1,2,3$, respectively. The optimal exercise boundaries $a_i^*(s)$ and $b_i^*(q)$ in (2.21) satisfy the inequalities $\underline{a}_i(s) < a_i^*(s) < \overline{a}_i(s) \wedge s$, for $\underline{s}_i < s < \overline{s}_i$, and $\underline{b}_i(q) \vee q < b_i^*(q) < \overline{b}_i(q)$, for $\underline{q}_i < q < \overline{q}_i$, as well as the equalities $a_1^*(s) = s$, $a_3^*(s) = 0$, for all $0 < s \le \underline{s}_i$, and $b_1^*(q) = q$, $b_3^*(q) = \infty$, for all $q \ge \overline{q}_i$, for every $i = 1,2,3$. Here, under certain relations between the parameters of the model, the boundary estimates and related numbers are specified as follows:*

*(i) in the case $i = 1$, that is, for $G_{1,1}(x,s) = L_1 - x$ and $G_{1,2}(x,q) = x - K_1$, we have:*

*• when $L_1 > \varphi_1$ and $\psi_1 > -1$ as well as $\alpha < 0$, we have $0 \le \underline{s}_1 \le \overline{a}_1 \wedge s_1^*$ with $\overline{a}_1 = rL_1/\delta'$ and $s_1^* = (L_1 - \varphi_1)/(1 + \psi_1)$, as well as $\overline{s}_1 = \infty$ and $\underline{a}_1 = rL_1\alpha/(\delta'(\alpha - 1))$,*

• *when* $K_1 > -\xi_1$ *and* $\chi_1 < 1$ *as well as* $\alpha > 0$, *we have* $\overline{q}_1 \geq \underline{b}_1 \vee q_1^*$ *with* $\underline{b}_1 = rK_1/\delta'$ *and* $q_1^* = (K_1 + \xi_1)/(1 - \chi_1)$, *as well as* $\underline{q}_1 = 0$ *[in addition, when* $\alpha > 1$, *we also have* $\overline{b}_1 = rK_1\alpha/(\delta'(\alpha - 1))$, *while when* $0 < \alpha \leq 1$, *we also have* $\overline{b}_1 = \infty$*];*

*(ii) in the case* $i = 2$, *that is, for* $G_{2,1}(x, s) = s - L_2 x$ *and* $G_{2,2}(x, q) = K_2 x - q$, *we have:*

• *when* $\varphi_2 \geq 0$ *and* $1+\alpha(1-L_2-\psi_2) > 0$ *as well as* $\alpha < 0$, *we have* $\underline{a}_2(s) = rs\alpha/(\delta' L_2(\alpha - 1))$ *and* $\overline{a}_2(s) = rs/(\delta' L_2)$ *as well as* $\underline{s}_2 = 0$ *and* $\overline{s}_2 = \infty$,

• *when* $\xi_2 \geq 0$ *and* $1 + \alpha(1 - K_2 + \chi_2) > 0$ *as well as* $\alpha > 0$, *we have* $\underline{b}_2(q) = rq/(\delta' K_2)$ *as well as* $\underline{q}_2 = 0$ *and* $\overline{q}_2 = \infty$ *[in addition, when* $\alpha > 1$, *we also have* $\overline{b}_2(q) = rq\alpha/(\delta' K_2(\alpha - 1))$, *while when* $0 < \alpha \leq 1$, *we also have* $\overline{b}_2 = \infty$*];*

*(iii) in the case* $i = 3$, *that is, for* $G_{3,1}(x, s) = s - L_3$ *and* $G_{3,2}(x, q) = K_3 - q$, *we have:*

• *when* $L_3 > -\varphi_3$ *and* $1 + \alpha(1 - \psi_3) < 0$ *as well as* $\alpha < 0$, *we have* $\underline{a}_3 = 0$ *and* $\overline{a}_3(s) = s$ *as well as* $\underline{s}_3 = L_3 \wedge s_3^*$ *and* $\overline{s}_3 = s_3^*$ *with* $s_3^* = (L_3 + \varphi_3)\alpha/(1 + \alpha(1 - \psi_3))$,

• *when* $K_3 > \xi_3$ *and* $1 + \alpha(1 + \chi_3) > 0$ *as well as* $\alpha > 0$, *we have* $\underline{b}_3(q) = q$ *and* $\overline{b}_3 = \infty$ *as well as* $\underline{q}_3 = q_3^*$ *and* $\overline{q}_3 = K_3 \vee q_3^*$ *with* $q_3^* = (K_3 - \xi_3)\alpha/(1 + \alpha(1 + \chi_3))$.

Observe that, when either $\alpha \geq 0$ or $\alpha \leq 0$ holds, the perpetual American standard and lookback option pricing problems of either (2.17) or (2.18) are reduced to the ones with complete information, respectively. We also note that the assertions stated above may also hold under the relations between the parameters of the model other than the ones considered above. However, the solutions to the problems of (2.17) and (2.18) might also be either trivial or non-transparent under the conditions on the parameters of the model different to the ones mentioned in the assertions above.

**Proof (a)** In order to clarify the structure of the continuation and stopping regions in (2.19)-(2.20), we first note that, by virtue of properties of the running maximum $S$ and minimum $Q$ from (1.3) of the geometric Brownian motion $X$ from (1.1) (see, e.g. [15; Subsection 3.3] for similar arguments applied to the running maxima of the Bessel processes), it is seen that, for any $s > 0$ and $q > 0$ fixed and an infinitesimally small deterministic time interval $\Delta$, we have:

$$S_\Delta = s \vee \max_{0 \leq u \leq \Delta} X_u = s \vee (s + \Delta X) + o(\Delta) \quad \text{as} \;\; \Delta \downarrow 0 \tag{2.22}$$

and

$$Q_\Delta = q \wedge \min_{0 \leq u \leq \Delta} X_u = q \wedge (q + \Delta X) + o(\Delta) \quad \text{as} \;\; \Delta \downarrow 0 \tag{2.23}$$

where we set $\Delta X = X_\Delta - s$ and $\Delta X = X_\Delta - q$, respectively. Observe that $\Delta S = o(\Delta)$ when $\Delta X \leq 0$, $\Delta S = \Delta X + o(\Delta)$ when $\Delta X > 0$, $\Delta Q = o(\Delta)$ when $\Delta X \geq 0$, and $\Delta Q = \Delta X + o(\Delta)$ when $\Delta X < 0$, where we set $\Delta S = S_\Delta - s$ and $\Delta Q = Q_\Delta - q$, and recall that $o(\Delta)$ denotes a random function satisfying $o(\Delta)/\Delta \to 0$ as $\Delta \downarrow 0$ ($P$-a.s.). In this case, using the asymptotic formulas:

$$E_{s,s}\big[\Delta X \,;\, \Delta X > 0\big] \equiv E_{s,s}\big[\Delta X \, I(\Delta X > 0)\big] \sim s\sqrt{\frac{\Delta}{2\pi}} \quad \text{as} \;\; \Delta \downarrow 0 \tag{2.24}$$

and

$$E_{q,q}\big[\Delta X \,;\, \Delta X < 0\big] \equiv E_{q,q}\big[\Delta X \, I(\Delta X < 0)\big] \sim -q\sqrt{\frac{\Delta}{2\pi}} \quad \text{as} \;\; \Delta \downarrow 0 \tag{2.25}$$

as well as taking into account the structure of the rewards in (2.17) and (2.18), we get:

$$E_{s,s}\Big[e^{-r\Delta}\, H_{i,1}(s,s)\,\Delta + e^{-r\Delta}\, F_{i,1}(s)\,\Delta S\Big] \tag{2.26}$$
$$\sim e^{-r\Delta}\, H_{i,1}(s,s)\,\Delta + e^{-r\Delta}\, F_{i,1}(s)\, s\,\sqrt{\frac{\Delta}{2\pi}} \quad \text{as} \;\; \Delta \downarrow 0$$

and

$$E_{q,q}\Big[e^{-r\Delta}\, H_{i,2}(q,q)\,\Delta + e^{-r\Delta}\, F_{i,2}(q)\,\Delta Q\Big] \tag{2.27}$$
$$\sim e^{-r\Delta}\, H_{i,2}(q,q)\,\Delta - e^{-r\Delta}\, F_{i,2}(q)\, q\,\sqrt{\frac{\Delta}{2\pi}} \quad \text{as} \;\; \Delta \downarrow 0$$

for each $s > 0$ and $q > 0$ fixed.

**(b)** Let us first consider the cases of $G_{1,1}(x,s)$ and $G_{1,2}(x,q)$ from (1.7)-(1.8), so that the functions $F_{1,1}(s)$ and $F_{1,2}(q)$ from (2.13) and (2.16) take the form:

$$F_{1,1}(s) = \alpha\,(L_1 - \varphi_1) - \alpha\,(1+\psi_1)\, s, \quad F_{1,2}(q) = -\alpha\,(K_1 + \xi_1) + \alpha\,(1-\chi_1)\, q \tag{2.28}$$

for $s > 0$, under $\alpha < 0$, and for $q > 0$, under $\alpha > 0$, respectively. Then, we see that the resulting coefficients by the terms of order $\sqrt{\Delta}$ in the expressions of (2.26) and (2.27) are strictly positive, when $s > s_1^*$ with $s_1^* = (L_1 - \varphi_1)/(1+\psi_1)$, under $L_1 > \varphi_1$ and $\psi_1 > -1$ (or when $s > 0$, under $L_1 \le \varphi_1$ and $\psi_1 > -1$), as well as when $q < q_1^*$ with $q_1^* = (K_1 + \xi_1)/(1-\chi_1)$, under $K_1 > -\xi_1$ and $\chi_1 < 1$. Hence, taking into account the facts that the process $S$ is positive and increasing and the process $Q$ is positive and decreasing, we may therefore conclude from the structure of integrands in the second integrals in the expressions of (2.17) and (2.18) with (2.28) as well as the heuristic arguments presented in (2.26) and (2.27) above that it is not optimal to exercise the standard put option with event risk when $s_1^* < S_t = X_t$ with $s_1^* = (L_1 - \varphi_1)/(1+\psi_1)$, under $L_1 > \varphi_1$ and $\psi_1 > -1$ (or when $0 < S_t = X_t$, under either $L_1 \le \varphi_1$ and $\psi_1 > -1$), while it is not optimal to exercise the standard call option with event risk when $X_t = Q_t < q_1^*$ with $q_1^* = (K_1 + \xi_1)/(1-\chi_1)$, under $K_1 > -\xi_1$ and $\chi_1 < 1$, for any $t \ge 0$, respectively. In other words, these facts mean that the set $d'_{1,1} = \{(x,s) \in E_1 \,|\, x = s > s_1^*\}$, under $L_1 > \varphi_1$ and $\psi_1 > -1$ (which becomes the whole diagonal $d_1 = \{(x,s) \in E_1 \,|\, x = s\}$, under $L_1 \le \varphi_1$ and $\psi_1 > -1$), surely belongs to the continuation region $C_{1,1}^*$ in (2.19) above, while the set $d'_{1,2} = \{(x,q) \in E_2 \,|\, x = q < q_1^*\}$, under $K_1 > -\xi_1$ and $\chi_1 < 1$ (which becomes an empty set, under $K_1 \le -\xi_1$ and $\chi_1 < 1$), surely belongs to the continuation region $C_{1,2}^*$ in (2.19) above. Here, we recall that $E_1 = \{(x,s) \in \mathbb{R}^2 \,|\, 0 < x \le s\}$ and $E_2 = \{(x,q) \in \mathbb{R}^2 \,|\, 0 < q \le x\}$ are the state spaces of the processes $(X,S)$ and $(X,Q)$, respectively. In particular, for the case of a fractional recovery with $\varphi_1 = \beta L_1$ and $\psi_1 = -\beta$, as well as $\xi_1 = -\beta K_1$ and $\chi_1 = \beta$, for some $\beta \in (0,1)$, the inequalities above hold with $s_1^* = L_1$ and $q_1^* = K_1$.

Let us now consider the cases of $G_{2,1}(x,s)$ and $G_{2,2}(x,q)$ from (1.7)-(1.8), so that the functions $F_{2,1}(s)$ and $F_{2,2}(q)$ from (2.13) and (2.16) take the form:

$$F_{2,1}(s) = -\alpha\,\varphi_2 + (1 + \alpha\,(1 - L_2 - \psi_2))\, s, \quad F_{2,2}(q) = -\alpha\,\xi_2 - (1 + \alpha\,(1 - K_2 + \chi_2))\, q \tag{2.29}$$

for $s > 0$, under $\alpha < 0$, and for $q > 0$, under $\alpha > 0$, respectively. Then, we see that the resulting coefficients by the terms of order $\sqrt{\Delta}$ in the expressions of (2.26) and (2.27) are strictly positive, when either $s > s_2^*$ with $s_2^* = \alpha\varphi_2/(1+\alpha(1-L_2-\psi_2))$, under $\varphi_2 \le 0$ and $1+\alpha(1-L_2-\psi_2) > 0$ (or when $s > 0$, under $\varphi_2 > 0$ and $1+\alpha(1-L_2-\psi_2) > 0$), while when $q > q_2^*$ with $q_2^* = -\alpha\xi_2/(1+\alpha(1-K_2+\chi_2))$, under $\xi_2 \le 0$ and $1+\alpha(1-K_2+\chi_2) > 0$ (or when $q > 0$, under $\xi_2 > 0$ and $1+\alpha(1-K_2+\chi_2) > 0$). In other words, these facts mean that the set $d'_{2,1} = \{(x,s) \in E_1 \,|\, x = s > s_2^*\}$, under $\varphi_2 \le 0$ and $1+\alpha(1-L_2-\psi_2) > 0$ (which becomes the whole diagonal $d_1 = \{(x,s) \in E_1 \,|\, x = s\}$, under $\varphi_2 > 0$ and $1+\alpha(1-L_2-\psi_2) > 0$), surely belongs to the continuation region $C_{2,1}^*$ in (2.19). Also, the set $d''_{2,2} = \{(x,q) \in E_2 \,|\, x = q > q_2^*\}$, under $\xi_2 \le 0$ and $1+\alpha(1-K_2+\chi_2) > 0$ (which becomes the whole diagonal $d_2 = \{(x,q) \in E_2 \,|\, x = q\}$, under $\xi_2 > 0$ and $1+\alpha(1-K_2+\chi_2) > 0$), surely belongs to the continuation region $C_{2,2}^*$ in (2.19) above, and thus, the complement $d_2 \setminus d''_{2,2}$ surely belongs to the stopping region $D_{2,2}^*$ in (2.20) above. The latter property occurs, because of the fact that the value $F_{2,2}(Q)$ in the expression of (2.29) remains positive once the decreasing process $Q$ passes through the point $q_2^*$. In particular, for the case of a fractional recovery with $\varphi_2 = 0$ and $\psi_2 = \beta(1-L_2)$, as well as $\xi_2 = 0$ and $\chi_2 = \beta(K_2 - 1)$, the inequalities above hold with $s_2^* = 0$ and $q_2^* = 0$.

Let us now consider the cases of $G_{3,1}(x,s)$ and $G_{3,2}(x,q)$ from (1.7)-(1.8), so that the functions $F_{3,1}(s)$ and $F_{3,2}(q)$ from (2.13) and (2.16) take the form:

$$F_{3,1}(s) = -\alpha\,(L_3+\varphi_3) + (1+\alpha\,(1-\psi_3))\,s, \quad F_{3,2}(q) = \alpha(K_3-\xi_1) - (1+\alpha(1+\chi_3))q \tag{2.30}$$

for $s > 0$, under $\alpha < 0$, and for $q > 0$, under $\alpha > 0$, respectively. Then, we see that the resulting coefficients by the terms of order $\sqrt{\Delta}$ in the expressions of (2.26) and (2.27) are strictly positive, when either $s > s_3^*$ with $s_3^* = \alpha(L_3+\varphi_3)/(1+\alpha(1-\psi_3))$, under $L_3 \le -\varphi_3$ and $1+\alpha(1-\psi_3) > 0$ (or when $s > 0$, under $L_3 > -\varphi_3$ and $1+\alpha(1-\psi_3) > 0$), or $s < s_3^*$, under $L_3 \ge -\varphi_3$ and $1+\alpha(1-\psi_3) < 0$, while when $q > q_3^*$ with $q_3^* = \alpha(K_3-\xi_3)/(1+\alpha(1+\chi_3))$, under $K_3 \ge \xi_3$ and $1+\alpha(1+\chi_3) > 0$ (or when $q > 0$, under $K_3 < \xi_3$ and $1+\alpha(1+\chi_3) > 0$). In other words, these facts mean that the set $d'_{3,1} = \{(x,s) \in E_1 \,|\, x = s > s_3^*\}$, under $L_3 \le -\varphi_3$ and $1+\alpha(1-\psi_3) > 0$ (which becomes the whole diagonal $d_1 = \{(x,s) \in E_1 \,|\, x = s\}$, under $L_3 > -\varphi_3$ and $1+\alpha(1-\psi_3) > 0$), or the set $d''_{3,1} = \{(x,s) \in E_1 \,|\, x = s < s_3^*\}$, under $L_3 \ge -\varphi_3$ and $1+\alpha(1-\psi_3) < 0$, surely belongs to the continuation region $C_{3,1}^*$ in (2.19) above, and thus, the complement $d_1 \setminus d''_{3,1}$ surely belongs to the stopping region $D_{3,1}^*$ in (2.20) above. The latter property occurs, because of the fact that the value $F_{3,1}(S)$ in the expression of (2.30) remains negative once the increasing process $S$ passes through the point $s_3^*$. Also, the set $d''_{3,2} = \{(x,q) \in E_2 \,|\, x = q > q_3^*\}$, under $K_3 \ge \xi_3$ and $1+\alpha(1+\chi_3) > 0$ (which becomes the whole diagonal $d_2 = \{(x,q) \in E_2 \,|\, x = q\}$, under $K_3 < \xi_3$ and $1+\alpha(1+\chi_3) > 0$), surely belongs to the continuation region $C_{3,2}^*$ in (2.19) above, and thus, the complement $d_2 \setminus d''_{3,2}$ surely belongs to the stopping region $D_{3,2}^*$ in (2.20) above. The latter property occurs, because of the fact that the value $F_{3,2}(Q)$ in the expression of (2.30) remains positive once the decreasing process $Q$ passes through the point $q_3^*$. it follows In particular, for the case of a fractional recovery with $\varphi_3 = -\beta L_3$ and $\psi_3 = \beta$, as well as $\xi_3 = \beta K_3$ and $\chi_3 = -\beta$, for some $\beta \in (0,1)$, the inequalities above hold with

$$s_3^* = L_3\alpha(1-\beta)/(1+\alpha(1-\beta)), \quad q_3^* = K_3\alpha(1-\beta)/(1+\alpha(1-\beta)). \tag{2.31}$$

**(c)** We now observe from the structure of the integrands in the first integrals of (2.17) and (2.18) that it is not optimal to exercise the perpetual American defaultable standard or lookback put option when $H_{i,1}(X_t, S_t) \ge 0$ and $X_t < S_t$, while it is not optimal to exercise the appropriate standard or lookback call option when $H_{i,2}(X_t, Q_t) \ge 0$ and $X_t > Q_t$, for any $t \geq 0$ and every $i = 1, 2, 3$. In other words, these facts mean that the set $\{(x,s) \in E_1 \setminus d_1 \,|\, H_{i,1}(x,s) \geq 0\}$ belongs to the continuation region $C^*_{i,1}$ in (2.19) above, while the set $\{(x,q) \in E_2 \setminus d_2 \,|\, H_{i,2}(x,q) \geq 0\}$ belongs to the continuation region $C^*_{i,2}$ in (2.19) above, for every $i = 1, 2, 3$. For simplicity of presentation, we further assume that $\delta' \equiv 2r - \delta - \sigma^2 > 0$ holds, as well as note that the fact that $\alpha \equiv 2(r-\delta)/\sigma^2 - 1 > 0$ holds obviously implies that $\delta' \equiv 2r - \delta - \sigma^2 > 0$ holds. In this case, the inequalities $H_{1,1}(x,s) = (\delta' x - rL_1)(s/x)^\alpha \geq 0$ and $x < s$ are satisfied if and only if $\overline{a}_1 \leq x < s$ holds with $\overline{a}_1 = rL_1/\delta'$, the inequalities $H_{2,1}(x,s) = (\delta' L_2 x - rs)(s/x)^\alpha \geq 0$ and $x < s$ are satisfied if and only if $\overline{a}_2(s) \leq x < s$ holds with $\overline{a}_2(s) = rs/(\delta' L_2)$, while the inequalities $H_{3,1}(x,s) = r(L_3 - s)(s/x)^\alpha \geq 0$ and $x < s$ are satisfied if and only if $0 < x < s \leq L_3$ holds. Furthermore, the inequalities $H_{1,2}(x,q) = (rK_1 - \delta' x)(q/x)^\alpha \geq 0$ and $x > q$ are satisfied if and only if $q < x \leq \underline{b}_1$ holds with $\underline{b}_1 = rK_1/\delta'$, the inequalities $H_{2,2}(x,q) = (rq - \delta' K_2 x)(q/x)^\alpha \geq 0$ and $x > q$ are satisfied if and only if $q < x \leq \underline{b}_2(q)$ holds with $\underline{b}_2(q) = rq/(\delta' K_2)$, while the inequalities $H_{3,2}(x,q) = r(q - K_3)(q/x)^\alpha \geq 0$ and $x > q$ are satisfied if and only if $x > q \geq K_3$ holds.

**(d)** Let us now specify the structure of the regions in (2.19)-(2.20). For this purpose, we provide an analysis of the reward functionals of the problems in (2.17)-(2.18). On one hand, we observe that the function $H_{1,1}(x,s) = (\delta' x - rL_1)(s/x)^\alpha$ decreases in $x$ on the interval $(0, \underline{a}_1)$, and then, it increases in $x$ on the interval $(\underline{a}_1, s)$ with $\underline{a}_1 = rL_1\alpha/(\delta'(\alpha-1)) < rL_1/\delta' = \overline{a}_1$, under $\alpha < 0$, for each $s > \underline{s}_1$ fixed and some $0 \leq \underline{s}_1 \leq \overline{a}_1 \wedge s^*_1$. In this case, the function $H_{1,1}(x,s)$ attains its global minimum at $x = \underline{a}_1$, for any $s > \underline{s}_1$. According to the comparison results for strong solutions of (one-dimensional) stochastic differential equations (see, e.g. [17; Theorem 1]), this fact means that the process $(H_{1,1}(X_t, S_t))_{t \geq 0}$ started at the point $H_{1,1}(\underline{a}_1, s)$ has the smallest sample paths than the one started at any other point $H_{1,1}(x,s)$, for any $0 < x < s$ such that $x \neq \underline{a}_1$ and $s > \underline{s}_1$. In this respect, we may conclude that the point $(\underline{a}_1, s)$ belongs to the stopping region $D^*_{1,1}$ from (2.20) above, since otherwise, all the points $(x,s)$ such that $0 < x < s$, for any $s > \underline{s}_1$, would belong to the continuation region $C^*_{1,1}$ from (2.19) too. The latter fact contradicts the obvious property that it is better to stop the process $(X,S)$ at time zero than not to stop the process at all during the infinite time interval, under the assumption that $\alpha < 0$. Therefore, taking into account the fact that the function $H_{1,1}(x,s)$ is negative on the interval $(0, \underline{a}_1)$, we see that all the points $(x,s)$ such that $0 < x \leq \underline{a}_1 \wedge s$, for any $s > \underline{s}_1$, belong to the stopping region $D^*_{1,1}$ from (2.20) as well.

Note that similar arguments applied for the function $H_{2,1}(x,s) = (\delta' L_2 x - rs)(s/x)^\alpha$ show that all the points $(x,s)$ such that $0 < x \leq \underline{a}_2(s) \wedge s$, with $\underline{a}_2(s) = rs\alpha/(\delta' L_2(\alpha-1)) < rs/(\delta' L_2) = \overline{a}_2(s)$, under $\alpha < 0$, for each $s > 0$ fixed, belong to the stopping region $D^*_{2,1}$ from (2.20). Moreover, it follows from the property that the function $H_{3,1}(x,s) = r(L_3 - s)(s/x)^\alpha$ is negative and decreasing in $x$ on the interval $(0,s)$, under $\alpha < 0$, that, for each $s > L_3$ fixed, there exists a sufficiently small $x > 0$ such that the point $(x,s)$ belongs to the stopping region $D^*_{3,1}$ from (2.20). According to arguments similar to the ones applied in [15; Subsection 3.3] and [40; Subsection 3.3], the latter properties can be explained by the fact that the costs of waiting until the process $X$ comes from such a small $x > 0$ to the current value of the maximum $S$ may be too high, due to the presence of the discounting factor in the reward functional of (2.17), one should stop at this $x > 0$ immediately.

On the other hand, we observe that the function $H_{1,2}(x,q) = (rK_1 - \delta' x)(q/x)^\alpha$ decreases in $x$ on the interval $(q, \overline{b}_1)$, and then, it increases in $x$ on the interval $(\overline{b}_1, \infty)$ with $\overline{b}_1 = rK_1\alpha/(\delta'(\alpha-1)) > rK_1/\delta' = \underline{b}_1$, under $\alpha > 1$, for each $0 < q < \overline{q}_1$ fixed and some $\overline{q}_1 \geq \underline{b}_1 \vee q^*_1$.

In this case, the function $H_{1,2}(x,q)$ attains its global minimum at $x = \overline{b}_1$, for any $0 < q < \overline{q}_1$. According to the comparison results for strong solutions of (one-dimensional) stochastic differential equations, this fact means that the process $(H_{1,2}(X_t, Q_t))_{t \ge 0}$ started at the point $H_{1,2}(\overline{b}_1, q)$ has the smallest sample paths than the one started at any other point $H_{1,2}(x,q)$, for any $x > q$ such that $x \neq \overline{b}_1$ and $0 < q < \overline{q}_1$. In this respect, we may conclude that the point $(\overline{b}_1, q)$ belongs to the stopping region $D^*_{1,2}$ from (2.20) above, since otherwise, all the points $(x,q)$ such that $x > q$, for any $0 < q < \overline{q}_1$, would belong to the continuation region $C^*_{1,2}$ from (2.19) too. The latter fact contradicts the obvious property that it is better to stop the process $(X,Q)$ at time zero than not to stop the process at all during the infinite time interval, under the assumption that $\alpha > 1$. Therefore, taking into account the fact that the function $H_{1,2}(x,q)$ is negative on the interval $(\overline{b}_1, \infty)$, we see that all the points $(x,q)$ such that $x \ge \overline{b}_1 \vee q$, for any $0 < q < \overline{q}_1$, belong to the stopping region $D^*_{1,2}$ from (2.20) as well.

Note that similar arguments applied for the function $H_{2,2}(x,q) = (rq - \delta' K_2 x)(q/x)^\alpha$ show that all the points $(x,q)$ such that $x \ge \overline{b}_2(q) \vee q$, with $\overline{b}_2(q) = rq\alpha/(\delta' K_2(\alpha - 1)) > rq/(\delta' K_2) = \underline{b}_2(q)$, under $\alpha > 1$, for each $q > 0$ fixed, belong to the stopping region $D^*_{2,2}$ from (2.20). Moreover, it follows from the fact that the function $H_{3,2}(x,q) = r(q - K_3)(q/x)^\alpha$ is negative and increasing in $x$ on the interval $(q, \infty)$, under $\alpha > 0$, that, for each $0 < q < K_3$ fixed, there exists a sufficiently large $x > 0$ such that the point $(x,q)$ belongs to the stopping region $D^*_{3,2}$ from (2.20). The same arguments based on the strict increase of the functions $H_{i,2}(x,q)$, for $i = 1,2$, in $x$ on the interval $(q, \infty)$, under $0 < \alpha \le 1$, for each $0 < q < \overline{q}_i$ fixed, for $i = 1,2$, with some $\overline{q}_1 \ge \underline{b}_1 \vee q^*_1$ and $\overline{q}_2 = \infty$, show that, there exists a sufficiently large $x > 0$ such that the point $(x,q)$ belongs to the stopping regions $D^*_{i,2}$, for $i = 1,2$, from (2.20). The latter properties can be explained by the fact that the costs of waiting until the process $X$ comes from such a large $x > 0$ to the current value of the minimum $Q$ may be too high, due to the presence of the discounting factor in the reward functional of (2.18), one should stop at this $x > 0$ immediately. In this view, we can set $\overline{b}_1 = \overline{b}_2 = \infty$, under $0 < \alpha \le 1$.

**(e)** Now, let us take some $(x,s) \in D^*_{i,1}$ from (2.20) such that $x > \underline{a}_i(s)$ with $\underline{a}_i(s)$ specified above. Then, using the fact that the process $(X,S)$ started at some $(x',s)$ such that $\underline{a}_i(s) \le x' < x$ passes through the point $(x,s)$ before hitting the diagonal $d_1 = \{(x,s) \in E_1 \,|\, x = s\}$, according to the explicit structure of the reward functional in (2.17), we conclude that the inequality $V^*_i(x',s) \le V^*_i(x,s) = 0$ holds, so that $(x',s) \in D^*_{i,1}$, for $i = 1,2,3$. Also, let us take some $(x,q) \in D^*_{i,2}$ from (2.20) such that $x < \overline{b}_i(q)$ with $\overline{b}_i(q)$ specified above. Hence, using the fact that the process $(X,Q)$ started at some $(x',q)$ such that $\overline{b}_i(q) \ge x' > x$ passes through the point $(x,q)$ before hitting the diagonal $d_2 = \{(x,q) \in E_2 \,|\, x = q\}$, taking into account the explicit structure of the reward functional in (2.18), we conclude that the inequality $U^*_i(x',q) \le U^*_i(x,q) = 0$ holds, so that $(x',q) \in D^*_{i,2}$, for $i = 1,2,3$.

Finally, let us take some $(x,s) \in C^*_{i,1}$ from (2.19). Then, using the fact that the process $(X,S)$ started at $(x,s)$ passes through some point $(x'',s)$ such that $x'' > x$ before hitting the diagonal $d_1$, according to the explicit structure of the reward functional in (2.17), we conclude that the inequality $V^*_i(x'',s) \ge V^*_i(x,s) > 0$ holds, so that $(x'',s) \in C^*_{i,1}$, for $i = 1,2,3$. Also, let us take some $(x,q) \in C^*_{i,2}$ from (2.19). Hence, using the fact that the process $(X,Q)$ started at $(x,q)$ passes through some point $(x'',q)$ such that $x'' < x$ before hitting the diagonal $d_2$, taking into account the explicit structure of the reward functional in (2.18), we conclude that the inequality $U^*_i(x'',q) \ge U^*_i(x,q) > 0$ holds, so that $(x'',q) \in C^*_{i,2}$, for $i = 1,2,3$.

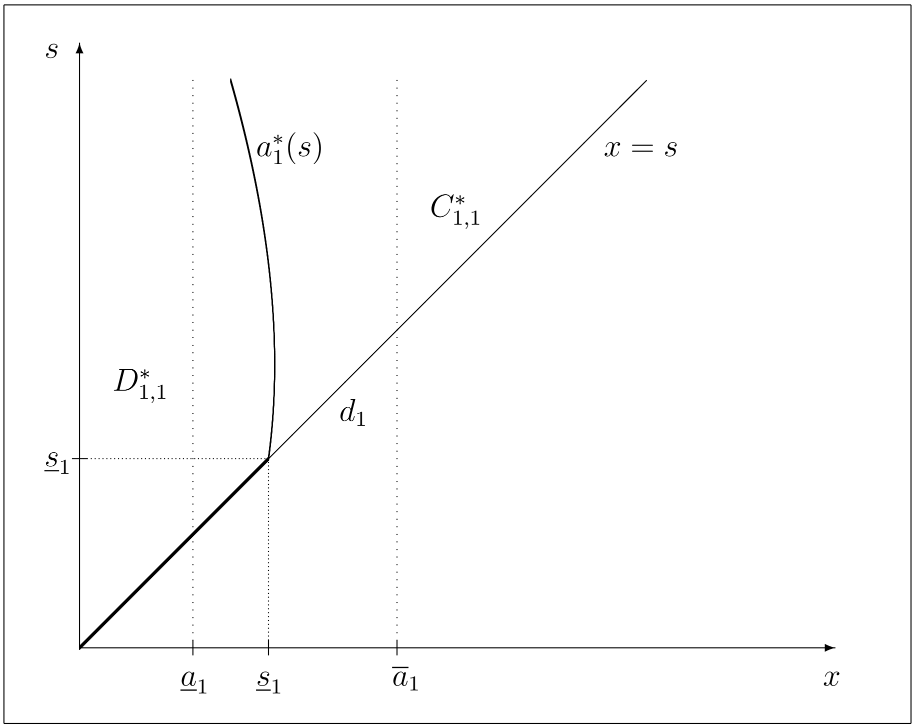


**Figure 1.** A computer drawing of the optimal exercise boundary $a_1^*(s)$.

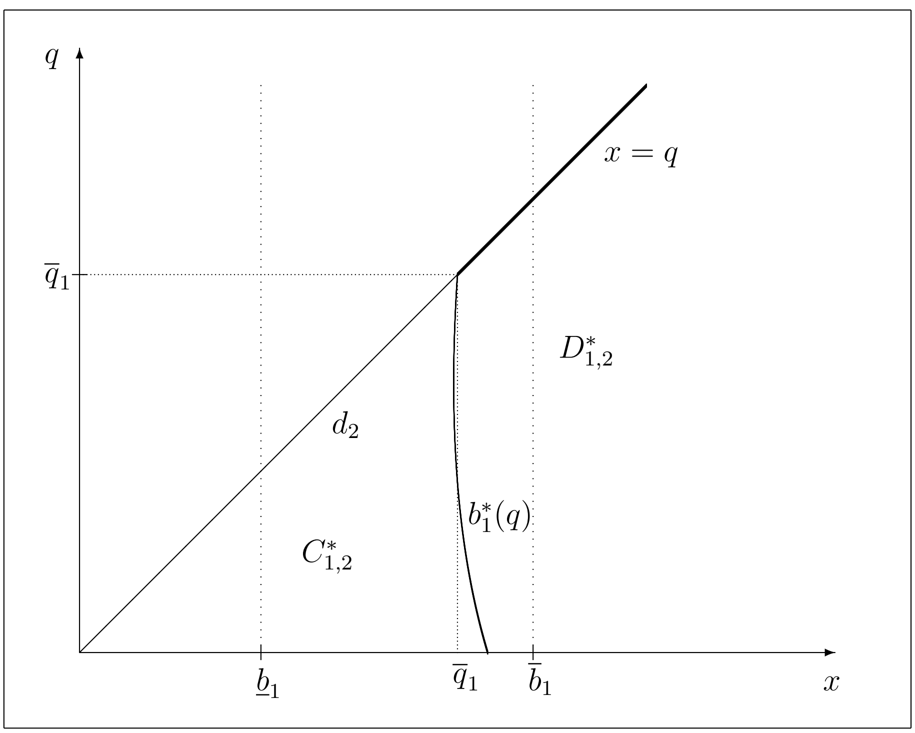


**Figure 2.** A computer drawing of the optimal exercise boundary $b_1^*(q)$.

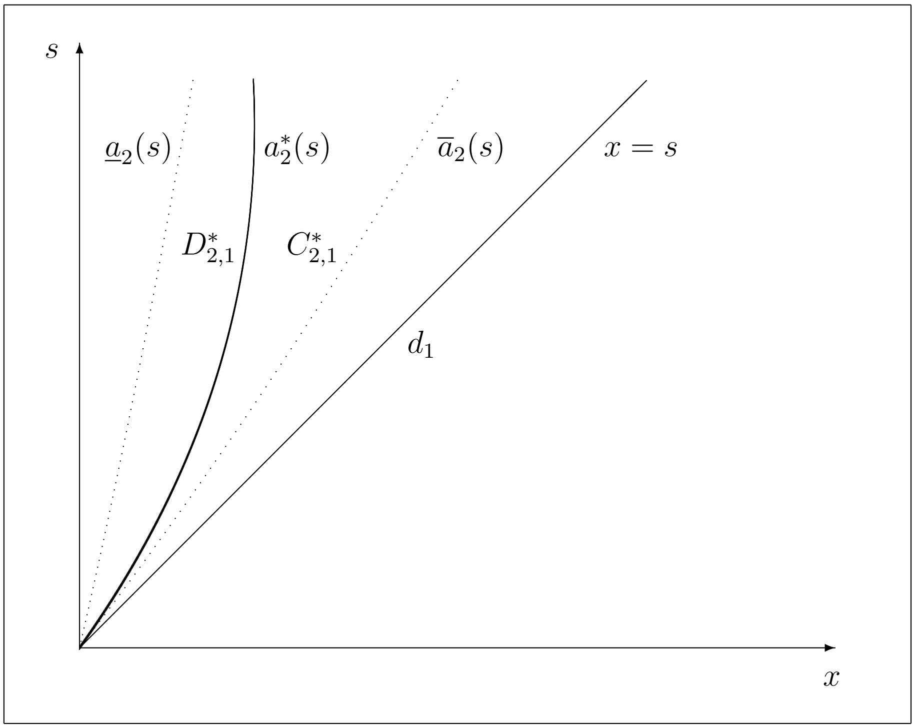


**Figure 3.** A computer drawing of the optimal exercise boundary $a_2^*(s)$.

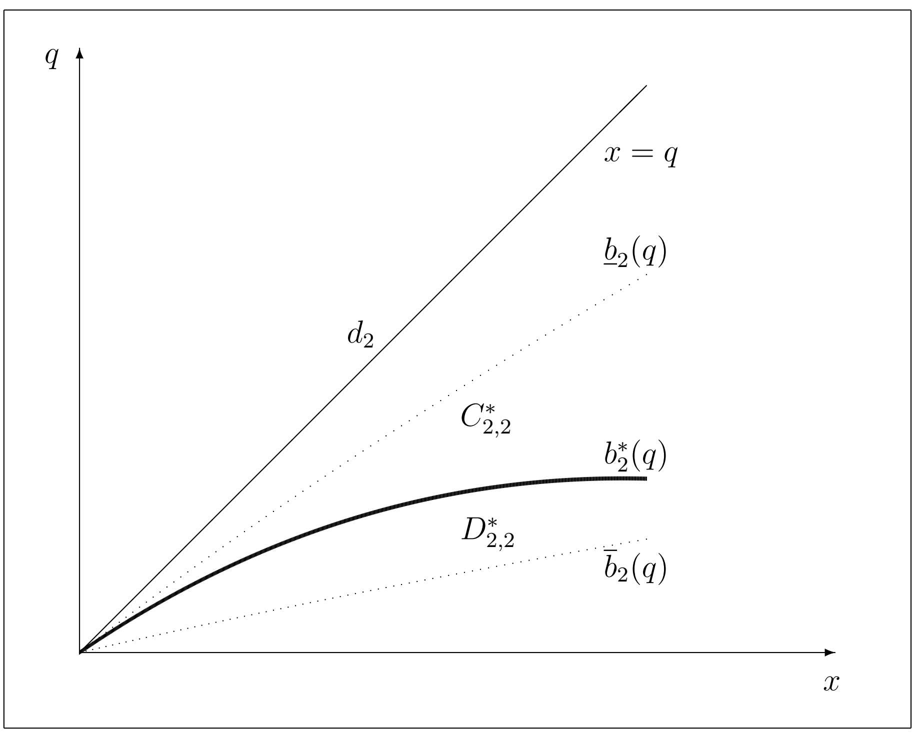


**Figure 4.** A computer drawing of the optimal exercise boundary $b_2^*(q)$.

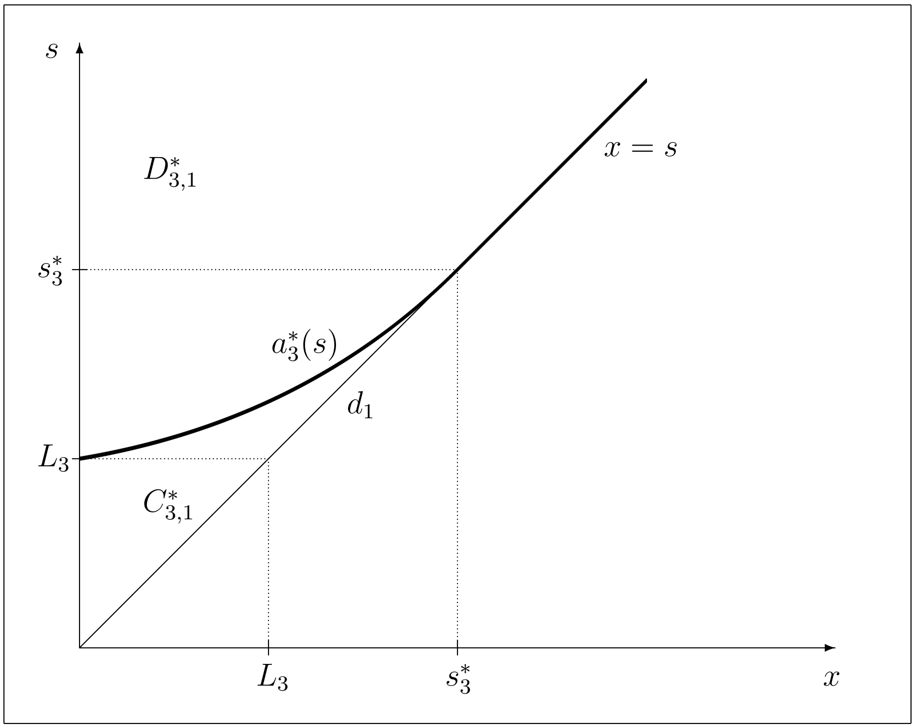


**Figure 5.** A computer drawing of the optimal exercise boundary $a^*_3(s)$.

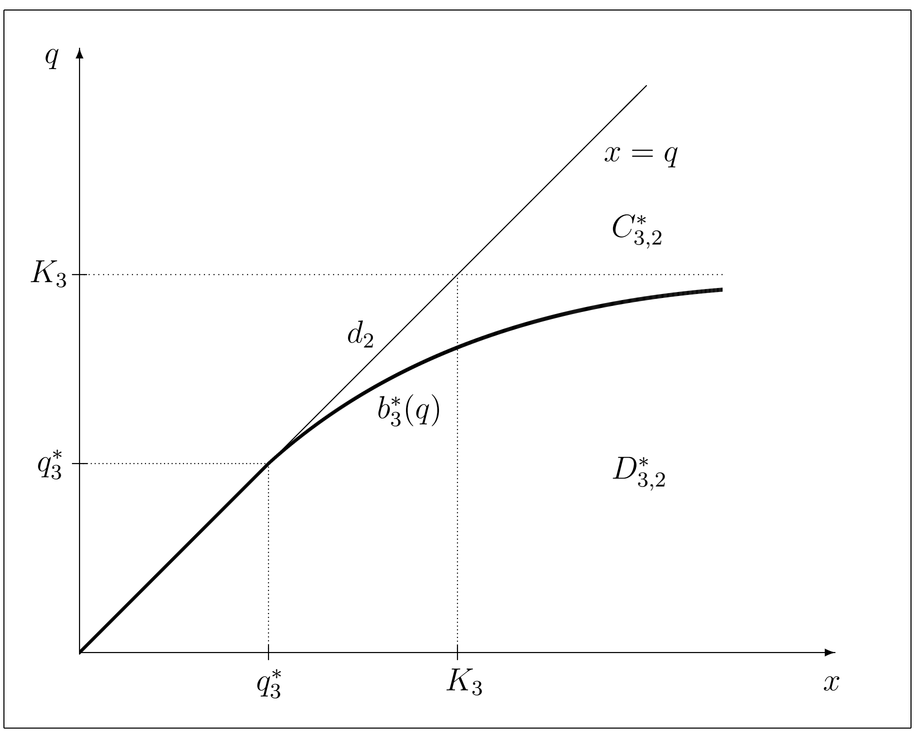


**Figure 6.** A computer drawing of the optimal exercise boundary $b^*_3(q)$.

We also recall that one should start with $s = x$ and $q = x$ in the original optimal stopping problems of (1.5) and (1.6), which are equivalent to the ones of (2.11) and (2.14), in order to obtain the values of the associated perpetual American defaultable standard and lookback put and call option pricing problems. In this respect, in the cases in which the complements $d_j \setminus d''_{i,j}$ considered in part (b) above belong to the stopping regions $D^*_{i,j}$ from (2.20), for $i = 1, 2, 3$ and $j = 1, 2$, we may declare that all the points $(x, s) \in E_1$ or $(x, q) \in E_2$ such that $(s, s) \in d_1 \setminus d''_{i,1}$ or $(q, q) \in d_2 \setminus d''_{i,2}$ belong to the stopping regions $D^*_{i,j}$, for $i = 1, 2, 3$ and $j = 1, 2$, respectively.

Summarising all these arguments, we may conclude that there exist functions $a^*_i(s)$ and $b^*_i(q)$ satisfying the inequalities $a^*_i(s) < \overline{a}_i(s) \wedge s$, for all $\underline{s}_i < s < \overline{s}_i$, and $b^*_i(q) > \underline{b}_i(q) \vee q$, for all $\underline{q}_i < q < \overline{q}_i$, as well as the equalities $a^*_1(s) = s$, $a^*_3(s) = 0$, for all $0 < s \leq \underline{s}_i$, and $b^*_1(q) = q$, $b^*_3(q) = \infty$, for all $q \geq \overline{q}_i$, such that the continuation regions $C^*_{i,j}$, for $j = 1, 2$, in (2.19) have the form:

$$C^*_{i,1} = \big\{ (x, s) \in E_1 \, \big| \, a^*_i(s) < x \leq s \big\} \quad \text{and} \quad C^*_{i,2} = \big\{ (x, q) \in E_2 \, \big| \, q \leq x < b^*_i(q) \big\} \tag{2.32}$$

while the stopping regions $D^*_{i,j}$, for $j = 1, 2$, in (2.20) are given by:

$$D^*_{i,1} = \big\{ (x, s) \in E_1 \, \big| \, x \leq a^*_i(s) \big\} \quad \text{and} \quad D^*_{i,2} = \big\{ (x, q) \in E_2 \, \big| \, x \geq b^*_i(q) \big\} \tag{2.33}$$

for every $i = 1, 2, 3$, respectively (see Figures 1-6 above for computer drawings of the optimal stopping boundaries $a^*_i(s)$ and $b^*_i(q)$, for $i = 1, 2, 3$). $\square$

We now summarise the properties proved above for the case of fractional recoveries.

**Corollary 2.2** *Suppose that the assumptions of Theorem 2.1 are satisfied with $\varphi_1 = \beta L_1$ and $\psi_1 = -\beta$, as well as $\xi_1 = -\beta K_1$ and $\chi_1 = \beta$, $\varphi_2 = 0$ and $\psi_2 = \beta(1 - L_2)$, as well as $\xi_2 = 0$ and $\chi_2 = \beta(K_2 - 1)$, and $\varphi_3 = -\beta L_3$ and $\psi_3 = \beta$, as well as $\xi_3 = \beta K_3$ and $\chi_3 = -\beta$, for some $\beta \in (0, 1)$. In these cases, the boundary estimates in parts (i)-(iii) of Theorem 2.1 are specified as follows:*

*(i) for $i = 1$, we have $0 \leq \underline{s}_1 \leq \overline{a}_1 \wedge s^*_1$ with $\overline{a}_1 = rL_1/\delta'$ and $s^*_1 = L_1$, as well as $\overline{s}_1 = \infty$ and $\underline{a}_1 = rL_1\alpha/(\delta'(\alpha - 1))$, under $\alpha < 0$, while we have $\overline{q}_1 \geq \underline{b}_1 \vee q^*_1$ with $\underline{b}_1 = rK_1/\delta'$ and $q^*_1 = K_1$, as well as $\underline{q}_1 = 0$, under $\alpha > 0$, where, additionally, $\overline{b}_1 = rK_1\alpha/(\delta'(\alpha - 1))$, for $\alpha > 1$, and $\overline{b}_1 = \infty$, for $0 < \alpha \leq 1$;*

*(ii) for $i = 2$, we have $\underline{a}_2(s) = rs\alpha/(\delta' L_2(\alpha - 1))$ and $\overline{a}_2(s) = rs/(\delta' L_2)$ as well as $\underline{s}_2 = 0$ and $\overline{s}_2 = \infty$, under $\alpha < 0$, while $\underline{b}_2(q) = rq/(\delta' K_2)$ as well as $\underline{q}_2 = 0$ and $\overline{q}_2 = \infty$, under $\alpha > 0$, where, additionally, $\overline{b}_2(q) = rq\alpha/(\delta' K_2(\alpha - 1))$, for $\alpha > 1$, and $\overline{b}_2 = \infty$, for $0 < \alpha \leq 1$;*

*(iii) for $i = 3$, we have $\underline{a}_3 = 0$ and $\overline{a}_3(s) = s$ as well as $\underline{s}_3 = L_3 \wedge s^*_3$ and $\overline{s}_3 = s^*_3$ with $s^*_3$ given by (2.31), under $1 + \alpha(1 - \beta) < 0$ and $\alpha < 0$, while $\underline{b}_3(q) = q$ and $\overline{b}_3 = \infty$ as well as $\underline{q}_3 = q^*_3$ and $\overline{q}_3 = K_3 \vee q^*_3$ with $q^*_3$ given by (2.31), under $\alpha > 0$.*

**2.3 The free-boundary problems.** By means of standard arguments based on the application of Itô's formula, it is shown that the infinitesimal operator $\mathbb{L}$ of the process $(X, S)$ or $(X, Q)$ from (1.2) and (1.3) has the form:

$$\mathbb{L} = (r - \delta)\, x\, \partial_x + \frac{\sigma^2 x^2}{2}\, \partial_{xx} \quad \text{in} \quad 0 < x < s \quad \text{or} \quad 0 < q < x \tag{2.34}$$

$$\partial_s = 0 \quad \text{at} \quad 0 < x = s \quad \text{or} \quad \partial_q = 0 \quad \text{at} \quad 0 < x = q \tag{2.35}$$

(see, e.g. [40; Subsection 3.1]). In order to find analytic expressions for the unknown value functions $V_i^*(x,s)$ and $U_i^*(x,q)$ from (2.17) and (2.18) and the unknown boundaries $a_i^*(s)$ and $b_i^*(q)$ from (2.32) and (2.33), for every $i = 1, 2, 3$, we apply the results of general theory for solving optimal stopping problems for Markov processes presented in [44; Chapter IV, Section 8] among others (see also [44; Chapter V, Sections 15-20] for optimal stopping problems for maxima processes and other related references). More precisely, for the original optimal stopping problems in (2.17) and (2.18), we formulate the associated free-boundary problems (see, e.g. [44; Chapter IV, Section 8]) and then verify in Theorem 4.1 below that the appropriate candidate solutions of the latter problems coincide with the solutions of the original problems. In other words, we reduce the optimal stopping problems of (2.17) and (2.18) to the following equivalent free-boundary problems:

$$(\mathbb{L}V_i - rV_i)(x,s) = -H_{i,1}(x,s) \quad \text{for} \quad (x,s) \in C_{i,1} \setminus \{(x,s) \in E_1 \,|\, x = s < \overline{s}_i\} \tag{2.36}$$

$$(\mathbb{L}U_i - rU_i)(x,q) = -H_{i,2}(x,q) \quad \text{for} \quad (x,q) \in C_{i,2} \setminus \{(x,q) \in E_2 \,|\, x = q > \underline{q}_i\} \tag{2.37}$$

$$V_i(x,s)\big|_{x=a_i(s)+} = 0 \quad \text{and} \quad U_i(x,q)\big|_{x=b_i(q)-} = 0 \tag{2.38}$$

$$\partial_x V_i(x,s)\big|_{x=a_i(s)+} = 0 \quad \text{and} \quad \partial_x U_i(x,q)\big|_{x=b_i(q)-} = 0 \tag{2.39}$$

$$\partial_s V_i(x,s)\big|_{x=s-} = -F_{i,1}(s)/s \quad \text{and} \quad \partial_q U_i(x,q)\big|_{x=q+} = -F_{i,2}(q)/q \tag{2.40}$$

$$V_i(x,s) = 0 \quad \text{for} \quad (x,s) \in D_{i,1} \quad \text{and} \quad U_i(x,q) = 0 \quad \text{for} \quad (x,q) \in D_{i,2} \tag{2.41}$$

$$V_i(x,s) > 0 \quad \text{for} \quad (x,s) \in C_{i,1} \quad \text{and} \quad U_i(x,q) > 0 \quad \text{for} \quad (x,q) \in C_{i,2} \tag{2.42}$$

$$(\mathbb{L}V_i - rV_i)(x,s) < -H_{i,1}(x,s) \quad \text{for} \quad (x,s) \in D_{i,1} \tag{2.43}$$

$$(\mathbb{L}U_i - rU_i)(x,q) < -H_{i,2}(x,q) \quad \text{for} \quad (x,q) \in D_{i,2} \tag{2.44}$$

where $C_{i,j}$ and $D_{i,j}$ are defined as $C_{i,j}^*$ and $D_{i,j}^*$, for $j = 1, 2$, in (2.32) and (2.33) with $a_i(s)$ and $b_i(q)$ instead of $a_i^*(s)$ and $b_i^*(q)$, where the functions $H_{i,1}(x,s)$ and $H_{i,2}(x,q)$ have the form of (2.12) and (2.15) and the functions $F_{i,1}(s)$ and $F_{i,2}(q)$ are given by (2.13) and (2.16), for every $i = 1, 2, 3$, respectively. Here, the instantaneous-stopping as well as the smooth-fit and normal-reflection conditions of (2.38)-(2.40) are satisfied, for all $\underline{s}_i < s < \overline{s}_i$ and $\underline{q}_i < q < \overline{q}_i$, where the end points of the admissible intervals $(\underline{s}_i, \overline{s}_i)$ and $(\underline{q}_i, \overline{q}_i)$, for $i = 1, 2, 3$, are specified in parts (a)-(d) of the proof of Theorem 2.1 above, under certain relations between the parameters of the model. Observe that the superharmonic characterisation of the value function (see, e.g. [44; Chapter IV, Section 9]) implies that $V_i^*(x,s)$ and $U_i^*(x,q)$ are the smallest functions satisfying (2.36)-(2.38) and (2.41)-(2.42) with the boundaries $a_i^*(s)$ and $b_i^*(q)$, for every $i = 1, 2, 3$, respectively. Note that the inequalities in (2.43) and (2.44) follow directly from the arguments of parts (c)-(d) of the proof of Theorem 2.1 above.

## 3. Solutions to the free-boundary problems

In this section, we obtain solutions to the free-boundary problems in (2.36)-(2.44) and derive first-order nonlinear ordinary differential equations for the candidate optimal stopping boundaries on the appropriate admissible intervals specified above.

**3.1 The candidate value functions.** It is shown that the second-order ordinary differential equations in (2.36) and (2.37) have the general solutions:

$$V_i(x,s) = C_{i,1}(s)\,x^{\gamma_1} + C_{i,2}(s)\,x^{\gamma_2} + A_{i,1}(s)\,x^{1-\alpha}\,s^{\alpha} + A_{i,2}(s)\,x^{-\alpha}\,s^{\alpha} \tag{3.1}$$

for $0 < x \le s$ such that $\underline{s}_i < s < \overline{s}_i$, for $i = 1, 2, 3$, when $\alpha < 0$, and

$$U_i(x,q) = D_{i,1}(q)\,x^{\gamma_1} + D_{i,2}(q)\,x^{\gamma_2} + B_{i,1}(q)\,x^{1-\alpha}\,q^{\alpha} + B_{i,2}(q)\,x^{-\alpha}\,q^{\alpha} \tag{3.2}$$

for $0 < q \le x$ such that $\underline{q}_i < q < \overline{q}_i$, for $i = 1, 2, 3$, when $\alpha > 0$, respectively. Here, we assume that $C_{i,j}(s)$ and $D_{i,j}(q)$, for $i = 1, 2, 3$ and $j = 1, 2$, are some arbitrary (continuously differentiable) functions, and $\gamma_j$, for $j = 1, 2$, are given by:

$$\gamma_j = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - (-1)^j \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \tag{3.3}$$

so that $\gamma_2 < 0 < 1 < \gamma_1$ holds. The functions $A_{i,j}(s)$ and $B_{i,j}(q)$, for $i = 1, 2, 3$ and $j = 1, 2$, are specified by $A_{1,1}(s) = 1$, $A_{1,2}(s) = -L_1$, $A_{2,1}(s) = L_2$, $A_{2,2}(s) = -s$, $A_{3,1}(s) = 0$, $A_{3,2}(s) = L_3 - s$, and $B_{1,1}(q) = -1$, $B_{1,2}(q) = K_1$, $B_{2,1}(q) = -K_2$, $B_{2,2}(q) = q$, $B_{3,1}(q) = 0$, $B_{3,2}(q) = q - K_3$. Then, by applying the conditions of (2.38)-(2.40) to the functions in (3.1), we obtain the equalities:

$$C_{i,1}(s)\,a_i^{\gamma_1}(s) + C_{i,2}(s)\,a_i^{\gamma_2}(s) + A_{i,1}(s)\,a_i^{1-\alpha}(s)\,s^{\alpha} + A_{i,2}(s)\,a_i^{-\alpha}(s)\,s^{\alpha} = 0 \tag{3.4}$$

$$\gamma_1\,C_{i,1}(s)\,a_i^{\gamma_1}(s) + \gamma_2\,C_{i,2}(s)\,a_i^{\gamma_2}(s) + A_{i,1}(s)\,(1-\alpha)\,a_i^{1-\alpha}(s)\,s^{\alpha} - A_{i,2}(s)\,\alpha\,a_i^{-\alpha}(s)\,s^{\alpha} = 0 \tag{3.5}$$

$$C'_{i,1}(s)\,s^{\gamma_1} + C'_{i,2}(s)\,s^{\gamma_2} + A'_{i,1}(s)\,s + A_{i,1}(s)\,\alpha + A'_{i,2}(s) + A_{i,2}(s)\,\alpha/s = -F_{i,1}(s)/s \tag{3.6}$$

for all $\underline{s}_i < s < \overline{s}_i$, and

$$D_{i,1}(q)\,b_i^{\gamma_1}(q) + D_{i,2}(q)\,b_i^{\gamma_2}(q) + B_{i,1}(q)\,b_i^{1-\alpha}(q)\,q^{\alpha} + B_{i,2}(q)\,b_i^{-\alpha}(q)\,q^{\alpha} = 0 \tag{3.7}$$

$$\gamma_1\,D_{i,1}(q)\,b_i^{\gamma_1}(q) + \gamma_2\,D_{i,2}(q)\,b_i^{\gamma_2}(q) + B_{i,1}(q)\,(1-\alpha)\,b_i^{1-\alpha}(q)\,q^{\alpha} - B_{i,2}(q)\,\alpha\,b_i^{-\alpha}(q)\,q^{\alpha} = 0 \tag{3.8}$$

$$D'_{i,1}(q)\,q^{\gamma_1} + D'_{i,2}(q)\,q^{\gamma_2} + B'_{i,1}(q)\,q + B_{i,1}(q)\,\alpha + B'_{i,2}(q) + B_{i,2}(q)\,\alpha/q = -F_{i,2}(q)/q \tag{3.9}$$

for all $\underline{q}_i < q < \overline{q}_i$, respectively. Hence, by solving the systems of equations in (3.4)-(3.5) and (3.7)-(3.8), we obtain that the candidate value functions admit the representations:

$$V_i(x,s;a_i(s)) = C_{i,1}(s;a_i(s))\,x^{\gamma_1} + C_{i,2}(s;a_i(s))\,x^{\gamma_2} + A_{i,1}(s)\,x^{1-\alpha}\,s^{\alpha} + A_{i,2}(s)\,x^{-\alpha}\,s^{\alpha} \tag{3.10}$$

for $a_i(s) < x \le s$ such that $\underline{s}_i < s < \overline{s}_i$, with

$$C_{i,j}(s;a_i(s)) = \frac{A_{i,1}(s)(\gamma_{3-j}+\alpha-1)a_i(s) + A_{i,2}(s)(\gamma_{3-j}+\alpha)}{(\gamma_j - \gamma_{3-j})a_i^{\gamma_j+\alpha}(s)s^{-\alpha}} \tag{3.11}$$

for $j = 1, 2$, and

$$U_i(x,q;b_i(q)) = D_{i,1}(q;b_i(q))\,x^{\gamma_1} + D_{i,2}(q;b_i(q))\,x^{\gamma_2} + B_{i,1}(q)\,x^{1-\alpha}\,q^{\alpha} + B_{i,2}(q)\,x^{-\alpha}\,q^{\alpha} \tag{3.12}$$

for $q \le x < b_i(q)$ such that $\underline{q}_i < q < \overline{q}_i$, with

$$D_{i,j}(q; b_i(q)) = \frac{B_{i,1}(q)(\gamma_{3-j} + \alpha - 1)b_i(q) + B_{i,2}(q)(\gamma_{3-j} + \alpha)}{(\gamma_j - \gamma_{3-j})b_i^{\gamma_j+\alpha}(q)q^{-\alpha}} \tag{3.13}$$

for $i = 1, 2, 3$ and $j = 1, 2$, respectively. Moreover, by means of straightforward computations, it can be deduced from the expressions in (3.10) and (3.12) that the first-order and second-order partial derivatives $\partial_x V_i(x, s; a_i(s))$ and $\partial_{xx} V_i(x, s; a_i(s))$ of the function $V_i(x, s; a_i(s))$ take the form:

$$\begin{aligned} \partial_x V_i(x, s; a_i(s)) = {} & C_{i,1}(s; a_i(s))\, \gamma_1\, x^{\gamma_1 - 1} + C_{i,2}(s; a_i(s))\, \gamma_2\, x^{\gamma_2 - 1} \\ & + A_{i,1}(s)\, (1-\alpha)\, x^{-\alpha}\, s^{\alpha} - A_{i,2}(s)\, \alpha\, x^{-\alpha-1}\, s^{\alpha} \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} \partial_{xx} V_i(x, s; a_i(s)) = {} & C_{i,1}(s; a_i(s))\, \gamma_1(\gamma_1 - 1)\, x^{\gamma_1 - 2} + C_{i,2}(s; a_i(s))\, \gamma_2(\gamma_2 - 1)\, x^{\gamma_2 - 2} \\ & - A_{i,1}(s)\, (1-\alpha)\alpha\, x^{-\alpha-1}\, s^{\alpha} + A_{i,2}(s)\, \alpha(\alpha+1)\, x^{-\alpha-2}\, s^{\alpha} \end{aligned} \tag{3.15}$$

on the interval $a_i(s) < x \le s$, for each $\underline{s}_i < s < \overline{s}_i$ and every $i = 1, 2, 3$ fixed, while the first-order and second-order partial derivatives $\partial_x U_i(x, q; b_i(q))$ and $\partial_{xx} U_i(x, q; b_i(q))$ of the function $U_i(x, q; b_i(q))$ take the form:

$$\begin{aligned} \partial_x U_i(x, q; b_i(q)) = {} & D_{i,1}(q; b_i(q))\, \gamma_1\, x^{\gamma_1 - 1} + D_{i,2}(q; b_i(q))\, \gamma_2\, x^{\gamma_2 - 1} \\ & + B_{i,1}(q)\, (1-\alpha)\, x^{-\alpha}\, q^{\alpha} - B_{i,2}(q)\, \alpha\, x^{-\alpha-1}\, q^{\alpha} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \partial_{xx} U_i(x, q; b_i(q)) = {} & D_{i,1}(q; b_i(q))\, \gamma_1(\gamma_1 - 1)\, x^{\gamma_1 - 2} + D_{i,2}(q; b_i(q))\, \gamma_2(\gamma_2 - 1)\, x^{\gamma_2 - 2} \\ & - B_{i,1}(q)\, (1-\alpha)\alpha\, x^{-\alpha-1}\, q^{\alpha} + B_{i,2}(q)\, \alpha(\alpha+1)\, x^{-\alpha-2}\, q^{\alpha} \end{aligned} \tag{3.17}$$

on the interval $q \le x < b_i(q)$, for each $\underline{q}_i < q < \overline{q}_i$ and every $i = 1, 2, 3$ fixed.

**3.2 The candidate stopping boundaries.** By applying the conditions of (3.6) and (3.9) to the functions in (3.11) and (3.13), we conclude that the candidate boundaries satisfy the first-order nonlinear ordinary differential equations:

$$a_i'(s) = \frac{\Psi_{i,1,1}(s, a_i(s)) s^{\gamma_1} + \Psi_{i,1,2}(s, a_i(s)) s^{\gamma_2} - \Xi_{i,1}(s)}{\Phi_{i,1,1}(s, a_i(s)) s^{\gamma_1} + \Phi_{i,1,2}(s, a_i(s)) s^{\gamma_2}} \tag{3.18}$$

for $\underline{s}_i < s < \overline{s}_i$, and

$$b_i'(q) = \frac{\Psi_{i,2,1}(q, b_i(q)) q^{\gamma_1} + \Psi_{i,2,2}(q, b_i(q)) q^{\gamma_2} - \Xi_{i,2}(q)}{\Phi_{i,2,1}(q, b_i(q)) q^{\gamma_1} + \Phi_{i,2,2}(q, b_i(q)) q^{\gamma_2}} \tag{3.19}$$

for $\underline{q}_i < q < \overline{q}_i$, respectively. Here, the functions $\Phi_{1,j}(s, a_i(s))$, $\Psi_{1,j}(s, a_i(s))$ and $\Phi_{2,j}(q, b_i(q))$, $\Psi_{2,j}(q, b_i(q))$ are defined by:

$$\Phi_{i,1,j}(s, a_i(s)) = \frac{(\gamma_j + \alpha - 1)(\gamma_{3-j} + \alpha - 1)A_{i,1}(s)a_i(s) + (\gamma_j + \alpha)(\gamma_{3-j} + \alpha)A_{i,2}(s)}{(\gamma_j - \gamma_{3-j})a_i^{\gamma_j+\alpha+1}(s)s^{-\alpha}} \tag{3.20}$$

$$\Psi_{i,1,j}(s, a_i(s)) = \frac{(A'_{i,1}(s)s + A_{i,1}(s)\alpha)(\gamma_{3-j} + \alpha - 1)a_i(s) + (A'_{i,2}(s)s + A_{i,2}(s)\alpha)(\gamma_{3-j} + \alpha)}{(\gamma_j - \gamma_{3-j})a_i^{\gamma_j+\alpha}(s)s^{1-\alpha}} \tag{3.21}$$

$$\Xi_{i,1}(s) = F_{i,1}(s)/s + A'_{i,1}(s)\, s + A_{i,1}(s)\, \alpha + A'_{i,2}(s) + A_{i,2}(s)\, \alpha/s \tag{3.22}$$

for $\underline{s}_i < s < \overline{s}_i$, and

$$\Phi_{i,2,j}(q, b_i(q)) = \frac{(\gamma_j + \alpha - 1)(\gamma_{3-j} + \alpha - 1)B_{i,1}(q)b_i(q) + (\gamma_j + \alpha)(\gamma_{3-j} + \alpha)B_{i,2}(q)}{(\gamma_j - \gamma_{3-j})b_i^{\gamma_j+\alpha+1}(q)q^{-\alpha}} \tag{3.23}$$

$$\Psi_{i,2,j}(q, b_i(q)) = \frac{(B'_{i,1}(q)q + B_{i,1}(q)\alpha)(\gamma_{3-j} + \alpha - 1)b_i(q) + (B'_{i,2}(q)q + B_{i,2}(q)\alpha)(\gamma_{3-j} + \alpha)}{(\gamma_j - \gamma_{3-j})b_i^{\gamma_j+\alpha}(q)q^{1-\alpha}} \tag{3.24}$$

$$\Xi_{i,2}(q) = F_{i,2}(q)/q + B'_{i,1}(q)\, q + B_{i,1}(q)\, \alpha + B'_{i,2}(q) + B_{i,2}(q)\, \alpha/q \tag{3.25}$$

for $\underline{q}_i < q < \overline{q}_i$, and every $i = 1, 2, 3$ and $j = 1, 2$.

**3.3 The maximal and minimal admissible solutions $a_i^*(s)$ and $b_i^*(q)$, $i = 1, 2, 3$.** We further consider the *maximal and minimal admissible* solutions of first-order nonlinear ordinary differential equations as the largest and smallest possible solutions $a_i^*(s)$ and $b_i^*(q)$ of the equations in (3.18) and (3.19) with (3.20)-(3.21) and (3.23)-(3.24) which satisfy the inequalities $a_i^*(s) < s \wedge \overline{a}_i(s)$ and $b_i^*(q) > q \vee \underline{b}_i(q)$, for all $\underline{s}_i < s < \overline{s}_i$ and $\underline{q}_i < q < \overline{q}_i$, and every $i = 1, 2, 3$. Here, we recall that the end points of the admissible intervals $(\underline{s}_i, \overline{s}_i)$ and $(\underline{q}_i, \overline{q}_i)$, for $i = 1, 2, 3$, are specified in parts (a)-(c) of the proof of Theorem 2.1 above, under certain relations between the parameters of the model. By virtue of the classical results on the existence and uniqueness of solutions for first-order nonlinear ordinary differential equations, we may conclude that these equations admit (locally) unique solutions, in view of the facts that the right-hand sides in (3.18) and (3.19) with (3.20)-(3.22) and (3.23)-(3.25) are (locally) continuous in $(s, a_i(s))$ and $(q, b_i(q))$ and (locally) Lipschitz in $a_i(s)$ and $b_i(q)$, for each $\underline{s}_i < s < \overline{s}_i$ and $\underline{q}_i < q < \overline{q}_i$ fixed, and every $i = 1, 2, 3$ (see also [40; Subsection 3.9] for similar arguments based on the analysis of other first-order nonlinear ordinary differential equations). Then, it is shown by means of technical arguments based on Picard's method of successive approximations that there exist unique solutions $a_i(s)$ and $b_i(q)$ to the equations in (3.18) and (3.19) with (3.20)-(3.21) and (3.23)-(3.24), for $\underline{s}_i < s < \overline{s}_i$ and $\underline{q}_i < q < \overline{q}_i$, started at some points $(\overline{a}_i(s_{i,0}), s_{i,0})$ and $(\underline{b}_i(q_{i,0}), q_{i,0})$, for $i = 1, 2, 3$, such that $\underline{s}_i < s_{i,0} < \overline{s}_i$ and $\underline{q}_i < q_{i,0} < \overline{q}_i$, for every $i = 1, 2, 3$ (see also [27; Subsection 3.2] and [40; Example 4.4] for similar arguments based on the analysis of other first-order nonlinear ordinary differential equations).

Hence, in order to construct the appropriate functions $a_i^*(s)$ and $b_i^*(q)$ which satisfy the equations in (3.18) and (3.19) and stays strictly above and below the appropriate diagonal, for $\underline{s}_i < s < \overline{s}_i$ and $\underline{q}_i < q < \overline{q}_i$, and every $i = 1, 2, 3$, respectively, we can follow the arguments from [43; Subsection 3.5] (among others) which are based on the construction of sequences of the so-called bad-good solutions which intersect the upper or lower bounds or diagonals. For this purpose, for any sequences $(s_{i,l})_{l \in \mathbb{N}}$ and $(q_{i,l})_{l \in \mathbb{N}}$ such that $\underline{s}_i < s_{i,l} < \overline{s}_i$ and $\underline{q}_i < q_{i,l} < \overline{q}_i$ as well as $s_{i,l} \uparrow \overline{s}_i$ and $q_{i,l} \downarrow \underline{q}_i$ as $l \to \infty$, we can construct the sequence of solutions $a_{i,l}(s)$ and $b_{i,l}(q)$, $l \in \mathbb{N}$, to the equations (3.18) and (3.19), for all $\underline{s}_i < s < \overline{s}_i$ and $\underline{q}_i < q < \overline{q}_i$ such that $a_{i,l}(s_{i,l}) = \overline{a}_i(s_{i,l})$ and $b_{i,l}(q_{i,l}) = \underline{b}_i(q_{i,l})$ holds, for every $i = 1, 2, 3$ and each $l \in \mathbb{N}$. It follows from the structure of the equations in (3.18) and (3.19) as well as the functions in (3.20)-(3.21) and (3.23)-(3.24) that the inequalities $a'_{i,l}(s_{i,l}) > \overline{a}'_i(s_{i,l}) \wedge 1$ and $b'_{i,l}(q_{i,l}) < \underline{b}'_i(q_{i,l}) \vee 1$ should hold for the derivatives of the corresponding functions, for each $l \in \mathbb{N}$ (see also [39; pages 979-982] for the analysis of solutions of another first-order nonlinear differential equation). Observe that, by virtue of the uniqueness of solutions mentioned above, we know that each two curves $s \mapsto a_{i,l}(s)$ and $s \mapsto a_{i,m}(s)$ as well as $q \mapsto b_{i,l}(q)$ and $q \mapsto b_{i,m}(q)$ cannot intersect, for $l, m \in \mathbb{N}$ such that $l \neq m$, and thus, we see that the sequence $(a_{i,l}(s))_{l \in \mathbb{N}}$ is increasing and the sequence $(b_{i,l}(q))_{l \in \mathbb{N}}$ is decreasing, so that the limits $a^*_i(s) = \lim_{l \to \infty} a_{i,l}(s)$ and $b^*_i(q) = \lim_{l \to \infty} b_{i,l}(q)$ exist, for each $\underline{s}_i < s < \overline{s}_i$ and $\underline{q}_i < q < \overline{q}_i$, and every $i = 1, 2, 3$, respectively. We may therefore conclude that $a^*_i(s)$ and $b^*_i(q)$ provides the maximal and minimal solutions to the equations in (3.18) and (3.19) such that $a^*_i(s) < \overline{a}_i(s) \wedge s$ and $b^*_i(q) > \underline{b}_i(q) \vee q$ holds, for all $\underline{s}_i < s < \overline{s}_i$ and $\underline{q}_i < q < \overline{q}_i$, and every $i = 1, 2, 3$.

Moreover, since the right-hand sides of the first-order nonlinear ordinary differential equations in (3.18) and (3.19) with (3.20)-(3.21) and (3.23)-(3.24) are (locally) Lipschitz in $s$ and $q$, respectively, one can deduce by means of Gronwall's inequality that the functions $a_{i,l}(s)$ and $b_{i,l}(q)$, for each $l \in \mathbb{N}$, are continuous, so that the functions $a^*_i(s)$ and $b^*_i(q)$ are continuous too, for every $i = 1, 2, 3$. The corresponding *maximal admissible* solutions of first-order nonlinear ordinary differential equations and the associated maximality principle for solutions of optimal stopping problems which is equivalent to the superharmonic characterisation of the payoff functions were established in [40] and further developed in [27], [39], [29], [18], [8], [30], [42]-[43], [26], [38], [33], [22]-[24], [46], and [19] among other subsequent papers (see also [44; Chapter I; Chapter V, Section 17] for other references).

## 4. Main results (Verification)

In this section, based on the expressions computed above, we formulate and prove the main results of the paper.

**Theorem 4.1** *Let the processes $(X, S)$ and $(X, Q)$ be given by (1.1) and (1.3), with some $r > 0$, $\delta > 0$, and $\sigma > 0$, and the inequality $\delta' \equiv 2r - \delta - \sigma^2 > 0$ be satisfied. Suppose that the random times $\theta$ and $\eta$ are defined by (1.4). Then, the value functions of the perpetual American standard and lookback put and call options with event risk from (2.17) and (2.18) admit the expressions:*

$$V^*_i(x, s) = \begin{cases} V_i(x, s; a^*_i(s)), & \text{if} \quad a^*_i(s) < x \le s \quad \text{and} \quad \underline{s}_i < s < \overline{s}_i \\ 0, & \text{if} \quad 0 < x \le a^*_i(s) \quad \text{and} \quad \underline{s}_i < s < \overline{s}_i \\ 0, & \text{if} \quad 0 < x \le s \le \underline{s}_i \quad \text{or} \quad s \ge \overline{s}_i \end{cases} \tag{4.1}$$

*whenever* $\alpha \equiv 2(r-\delta)/\sigma^2 - 1 < 0$, *and*

$$U_i^*(x,q) = \begin{cases} U_i(x,q;b_i^*(q)), & \text{if} \quad q \le x < b_i^*(q) \quad \text{and} \quad \underline{q}_i < q < \overline{q}_i \\ 0, & \text{if} \quad x \ge b_i^*(q) \quad \text{and} \quad \underline{q}_i < q < \overline{q}_i \\ 0, & \text{if} \quad x \ge q \ge \overline{q}_i \quad \text{or} \quad 0 < q \le \underline{q}_i \end{cases} \tag{4.2}$$

*whenever* $\alpha > 0$. *Here, the function* $V_i(x,s;a_i(s))$ *is given by (3.10) with (3.11), whenever* $\alpha < 0$, *and the optimal exercise boundary* $a_i^*(s)$ *provides the maximal solution of the first-order nonlinear ordinary differential equation in (3.18) with (3.20)-(3.22) satisfying the inequalities* $[\underline{a}_i(s) <] a_i^*(s) < \overline{a}_i(s) \wedge s$, *for all* $\underline{s}_i < s < \overline{s}_i$ *and every* $i = 1,2,3$, *where the boundary estimates and related numbers are given in the beginnings of parts (i)-(iii) of Theorem 2.1 above, under the specified relations between the parameters of the model. The function* $U_i(x,q;b_i(q))$ *is given by (3.12) with (3.13), whenever* $\alpha > 0$, *and the optimal exercise boundary* $b_i^*(q)$ *provides the minimal solution of the first-order nonlinear ordinary differential equation in (3.19) with (3.23)-(3.25) satisfying the inequalities* $\underline{b}_i(q) \vee q < b_i^*(q) [< \overline{b}_i(q)]$, *for all* $\underline{q}_i < q < \overline{q}_i$ *and every* $i = 1,2,3$, *where the boundary estimates and related numbers are given in the ends of parts (i)-(iii) of Theorem 2.1 above, under the specified relations between the parameters of the model.*

Observe that we can put $s = x$ and $q = x$ to obtain the values of the original perpetual American standard and lookback put and call option pricing problems of (2.11) and (2.14), which are equivalent to the ones of (1.5) and (1.6), from the values of the optimal stopping problems of (2.17) and (2.18). Note that, since both parts of the assertion stated above are proved using similar arguments, we may only give a proof for the case of the two-dimensional optimal stopping problem of (2.18) related to the perpetual American standard and lookback call options with event risk and asymmetric information.

**Proof** In order to verify the assertion stated above, it remains for us to show that the function defined in (4.1) coincides with the value function in (2.17) and that the stopping time $\tau_i^*$ in (2.21) is optimal with the boundary $a_i^*(s)$ specified above. For this purpose, let $a_i(s)$ be any solution of the ordinary differential equation in (3.18) satisfying the inequality $a_i(s) < \overline{a}_i(s) \wedge s$, for all $\underline{s}_i < s < \overline{s}_i$ and every $i = 1,2,3$. Here, $\overline{a}_1(s) \equiv \overline{a}_1 = rL_1/\delta'$, $\overline{a}_2(s) = rs/(\delta' L_2)$, and $\overline{a}_3(s) = s$, with some $0 \le \underline{s}_1 \le \overline{a}_1 \wedge s_1^*$, $\overline{s}_1 = \infty$ as well as $\underline{s}_2 = 0$, $\overline{s}_2 = \infty$ and $\underline{s}_3 = L_3 \wedge s_3^*$, $\overline{s}_3 = s_3^*$, where $s_1^*$ and $s_3^*$ are specified in part (b) of the proof of Theorem 2.1 above, under certain relations between the parameters of the model. Let us also denote by $V_i^{a_i}(x,s)$ the right-hand side of the expression in (4.1) associated with $a_i(s)$, for every $i = 1,2,3$. Then, it is shown by means of straightforward calculations from the previous section that the function $V_i^{a_i}(x,s)$ solves the system of (2.36) with the left-hand sides of (2.41)-(2.42) and (2.43) as well as satisfies the left-hand conditions of (2.38)-(2.40). Recall that the function $V_i^{a_i}(x,s)$ is $C^{2,1}$ on the closure $\overline{C}_{i,1}$ of $C_{i,1}$ and is equal to zero on $D_{i,1}$, which are defined as $\overline{C}_{i,1}^*$, $C_{i,1}^*$ and $D_{i,1}^*$ in (2.32) and (2.33) with $a_i(s)$ instead of $a_i^*(s)$, for $i = 1,2,3$, respectively. Hence, taking into account the assumption that the boundary $a_i(s)$ is continuously differentiable, for all $\underline{s}_i < s < \overline{s}_i$, by applying the change-of-variable formula from [41; Theorem 3.1] to the process $e^{-rt} V_i^{a_i}(X_t, S_t)$ (see also [44; Chapter II, Section 3.5] for a summary of the related results and further references), we obtain the expression:

$$
\begin{aligned}
& e^{-rt}\, V_i^{a_i}(X_t,S_t) = V_i^{a_i}(x,s) \\
& + \int_0^t e^{-ru}\, (\mathbb{L}V_i^{a_i} - rV_i^{a_i})(X_u,S_u)\, I(X_u \neq a_i(S_u), X_u \neq S_u)\, du \\
& + \int_0^t e^{-ru}\, \partial_s V_i^{a_i}(X_u,S_u)\, I(X_u = S_u)\, dS_u + M_t^i
\end{aligned}
\tag{4.3}
$$

for all $t \geq 0$, for every $i = 1,2,3$. Here, the process $M^i = (M_t^i)_{t \geq 0}$ defined by:

$$
M_t^i = \int_0^t e^{-ru}\, \partial_x V_i^{a_i}(X_u,S_u)\, I(X_u \neq S_u)\, \sigma\, X_u\, dB_u \tag{4.4}
$$

is a continuous local martingale with respect to the probability measure $P_{x,s}$. Note that, since the time spent by the process $(X,S)$ at the boundary surface $\partial C_{i,1} = \{(x,s) \in E_1 \,|\, x = a_i(s)\}$, for every $i = 1,2,3$, as well as at the diagonal $d_1 = \{(x,s) \in E_1 \,|\, x = s\}$ is of the Lebesgue measure zero (see, e.g. [13; Chapter II, Section 1]), the indicators in the second line of the formula in (4.3) as well as in the expression of (4.4) can be ignored. Moreover, since the component $S$ decreases only when the process $(X,S)$ is located on the diagonal $d_1 = \{(x,s) \in E_1 \,|\, x = s\}$, the indicator in the third line of (4.3) can also be set equal to one. Observe that the integral in the third line of (4.3) will actually be compensated accordingly, due to the fact that the candidate value function $V_i^{a_i}(x,s)$ satisfies the modified normal-reflection condition of the left-hand part of (2.40) at the subset of the diagonal $\{(x,s) \in E_1 \,|\, x = s < \overline{s}_i\}$, for every $i = 1,2,3$.

It follows from straightforward calculations and the arguments of the previous section that the function $V_i^{a_i}(x,s)$ satisfies the second-order ordinary differential equation in (2.36), which together with the left-hand conditions of (2.38)-(2.39) and (2.41) as well as the fact that the inequality in (2.43) holds imply that the inequality $(\mathbb{L}V_i^{a_i} - rV_i^{a_i})(x,s) \leq -H_{i,1}(x,s)$ is satisfied with $H_{i,1}(x,s)$ given by (2.12), for all $0 < x < s$ such that $x \neq a_i(s)$ and $\underline{s}_i < s < \overline{s}_i$, for every $i = 1,2,3$. Moreover, we observe directly from the expressions in (3.10) as well as (3.14) and (3.15) with (3.11) that the function $V_i^{a_i}(x,s)$ is convex and increases from zero, because its first-order partial derivative $\partial_x V_i^{a_i}(x,s)$ is positive and increases from zero, while its second-order partial derivative $\partial_{xx} V_i^{a_i}(x,s)$ is positive, on the interval $a_i(s) < x \leq s$, under $\alpha < 0$, for each $\underline{s}_i < s < \overline{s}_i$ and every $i = 1,2,3$ fixed. Thus, we may conclude that the left-hand inequality in (2.42) holds, which together with the left-hand conditions of (2.38)-(2.39) and (2.41) imply that the inequality $V_i^{a_i}(x,s) \geq 0$ is satisfied, for all $0 < x \leq s$ such that $\underline{s}_i < s < \overline{s}_i$, and every $i = 1,2,3$. Let $(\varkappa_{i,n})_{n \in \mathbb{N}}$ be the localising sequence of stopping times for the process $M^i$ from (4.4) such that $\varkappa_{i,n} = \inf\{t \geq 0 \,|\, |M_t^i| \geq n\}$, for each $n \in \mathbb{N}$ and every $i = 1,2,3$. It therefore follows from the expression in (4.3) that the inequalities:

$$
\begin{aligned}
& \int_0^{\tau \wedge \varkappa_{i,n}} e^{-ru}\, H_{i,1}(X_u,S_u)\, du + \int_0^{\tau \wedge \varkappa_{i,n}} e^{-ru}\, \frac{F_{i,1}(S_u)}{S_u}\, dS_u \\
& \leq e^{-r(\tau \wedge \varkappa_{i,n})}\, V_i^{a_i}(X_{\tau \wedge \varkappa_{i,n}}, S_{\tau \wedge \varkappa_{i,n}}) \\
& \quad + \int_0^{\tau \wedge \varkappa_{i,n}} e^{-ru}\, H_{i,1}(X_u,S_u)\, du + \int_0^{\tau \wedge \varkappa_{i,n}} e^{-ru}\, \frac{F_{i,1}(S_u)}{S_u}\, dS_u \\
& \leq V_i^{a_i}(x,s) + M_{\tau \wedge \varkappa_{i,n}}^i
\end{aligned}
\tag{4.5}
$$

hold with any stopping time $\tau$ of the process $X$ and for each $n \in \mathbb{N}$ fixed. Then, taking the expectation with respect to $P_{x,s}$ in (4.5), by means of Doob's optional sampling theorem, we get:

$$
\begin{aligned}
& E_{x,s}\bigg[\int_0^{\tau\wedge\varkappa_{i,n}} e^{-ru}\, H_{i,1}(X_u, S_u)\, du + \int_0^{\tau\wedge\varkappa_{i,n}} e^{-ru}\, \frac{F_{i,1}(S_u)}{S_u}\, dS_u\bigg] \\
& \le E_{x,s}\bigg[e^{-r(\tau\wedge\varkappa_{i,n})}\, V_i^{a_i}(X_{\tau\wedge\varkappa_{i,n}}, S_{\tau\wedge\varkappa_{i,n}}) \\
& \qquad + \int_0^{\tau\wedge\varkappa_{i,n}} e^{-ru}\, H_{i,1}(X_u, S_u)\, du + \int_0^{\tau\wedge\varkappa_{i,n}} e^{-ru}\, \frac{F_{i,1}(S_u)}{S_u}\, dS_u\bigg] \\
& \le V_i^{a_i}(x,s) + E_{x,s}\big[M^i_{\tau\wedge\varkappa_{i,n}}\big] = V_i^{a_i}(x,s)
\end{aligned}
\tag{4.6}
$$

for all $0 < x \le s$ such that $\underline{s}_i < s < \overline{s}_i$, for each $n \in \mathbb{N}$ and every $i = 1, 2, 3$. Hence, letting $n$ go to infinity and using Fatou's lemma, we obtain from the expressions in (4.6) that the inequalities:

$$
\begin{aligned}
& E_{x,s}\bigg[\int_0^{\tau} e^{-ru}\, H_{i,1}(X_u, S_u)\, du + \int_0^{\tau} e^{-ru}\, \frac{F_{i,1}(S_u)}{S_u}\, dS_u\bigg] \\
& \le E_{x,s}\bigg[e^{-r\tau}\, V_i^{a_i}(X_\tau, S_\tau) + \int_0^{\tau} e^{-ru}\, H_{i,1}(X_u, S_u)\, du + \int_0^{\tau} e^{-ru}\, \frac{F_{i,1}(S_u)}{S_u}\, dS_u\bigg] \\
& \le V_i^{a_i}(x,s)
\end{aligned}
\tag{4.7}
$$

are satisfied with any stopping time $\tau$, for all $0 < x \le s$ such that $\underline{s}_i < s < \overline{s}_i$, for each $n \in \mathbb{N}$ and every $i = 1, 2, 3$. Thus, taking the supremum over all stopping times $\tau$ and then the infimum over all boundaries $a_i$ in the expressions of (4.7), we may therefore conclude that the inequalities:

$$
\begin{aligned}
& \sup_\tau E_{x,s}\bigg[\int_0^{\tau} e^{-ru}\, H_{i,1}(X_u, S_u)\, du + \int_0^{\tau} e^{-ru}\, \frac{F_{i,1}(S_u)}{S_u}\, dS_u\bigg] \\
& \le \inf_{a_i} V_i^{a_i}(x,s) = V_i^{a_i^*}(x,s)
\end{aligned}
\tag{4.8}
$$

hold, for all $0 < x \le s$ such that $\underline{s}_i < s < \overline{s}_i$, where $a_i^*(s)$ is the maximal solution of the ordinary differential equation in (3.19) as well as satisfying the inequality $a_i^*(s) < \overline{a}_i(s) \wedge s$, for all $\underline{s}_i < s < \overline{s}_i$ and every $i = 1, 2, 3$. By using the fact that the function $V_i^{a_i}(x,s)$ is (strictly) increasing in the value $a_i(s)$, for each $\underline{s}_i < s < \overline{s}_i$ fixed, we see that the infimum in (4.8) is attained over any sequence of solutions $(a_{i,m}(s))_{m\in\mathbb{N}}$ to (3.18) satisfying the inequality $a_{i,m}(s) < \overline{a}_i(s) \wedge s$, for all $\underline{s}_i < s < \overline{s}_i$, for each $m \in \mathbb{N}$ and every $i = 1, 2, 3$, and such that $a_{i,m}(s) \uparrow a_i^*(s)$ as $m \to \infty$, for each $\underline{s}_i < s < \overline{s}_i$ fixed, and every $i = 1, 2, 3$. It follows from the (local) uniqueness of the solutions to the first-order (nonlinear) ordinary differential equations in (3.18) that no distinct solutions intersect, so that the sequence $(a_{i,m}(s))_{m\in\mathbb{N}}$ is increasing and the limit $a_i^*(s) = \lim_{m\to\infty} a_{i,m}(s)$ exists, for each $\underline{s}_i < s < \overline{s}_i$ fixed and every $i = 1, 2, 3$. Since the inequalities in (4.7) hold for $a_i^*(s)$ too, we see that the expression in (4.8) holds, for $a_i^*(s)$ and all $0 < x \le s$ such that $\underline{s}_i < s < \overline{s}_i$, for every $i = 1, 2, 3$, as well. We also note from the inequality in (4.6) that the function $V_i^{a_i}(x,s)$ is superharmonic for the Markov process $(X, S)$ on the state space $E_1$. Hence, taking into account the facts that $V_i^{a_i}(x,s)$ is increasing in $a_i(s) < \overline{a}_i(s) \wedge s$ and the inequality $V_i^{a_i}(x,s) \geq 0$ holds, for all $0 < x \leq s$ such that $\underline{s}_i < s < \overline{s}_i$, we observe that the selection of the maximal solution $a_i^*(s)$ which stays strictly below the part of the diagonal $\{(x,s) \in E_1 \,|\, x = s < \overline{s}_i\}$ and the curve $x = \underline{a}_i(s)$, for every $i = 1,2,3$, is equivalent to the implementation of the superharmonic characterisation of the value function as the smallest superharmonic function dominating the payoff function (cf. [40] or [44; Chapter I and Chapter V, Section 17]).

In order to prove the fact that the boundary $a_i^*(s)$ is optimal, we consider the sequence of stopping times $\tau_{i,m}$, $m \in \mathbb{N}$, defined as in the left-hand part of (2.21) with $a_{i,m}(s)$ instead of $a_i^*(s)$, where $a_{i,m}(s)$ is a solution to the first-order ordinary differential equation in (3.18) and such that $a_{i,m}(s) \uparrow a_i^*(s)$ as $m \to \infty$, for each $\underline{s}_i < s < \overline{s}_i$ and every $i = 1,2,3$ fixed. Then, by virtue of the fact that the function $V_i^{a_{i,m}}(x,s)$ from the right-hand side of the expression in (4.1) associated with the boundary $a_{i,m}(s)$ satisfies the conditions of (2.36) and the left-hand part of (2.38), and taking into account the structure of $\tau_i^*$ in (2.21), it follows from the expression which is equivalent to the one in (4.3) that the equalities:

$$\begin{aligned} &\int_0^{\tau_{i,m} \wedge \varkappa_{i,n}} e^{-ru} \, H_{i,1}(X_u, S_u) \, du + \int_0^{\tau_{i,m} \wedge \varkappa_{i,n}} e^{-ru} \, \frac{F_{i,1}(S_u)}{S_u} \, dS_u \\ &= e^{-r(\tau_{i,m} \wedge \varkappa_{i,n})} \, V_i^{a_{i,m}}(X_{\tau_{i,m} \wedge \varkappa_{i,n}}, S_{\tau_{i,m} \wedge \varkappa_{i,n}}) \\ &\quad + \int_0^{\tau_{i,m} \wedge \varkappa_{i,n}} e^{-ru} \, H_{i,1}(X_u, S_u) \, du + \int_0^{\tau_{i,m} \wedge \varkappa_{i,n}} e^{-ru} \, \frac{F_{i,1}(S_u)}{S_u} \, dS_u \\ &= V_i^{a_{i,m}}(x,s) + M^i_{\tau_{i,m} \wedge \varkappa_{i,n}} \end{aligned} \tag{4.9}$$

hold, for all $0 < x \leq s$ such that $\underline{s}_i < s < \overline{s}_i$, for each $n, m \in \mathbb{N}$ and every $i = 1,2,3$. Observe that, by virtue of the arguments from [49; Chapter VIII, Section 2a], the property:

$$E_{x,s} \bigg[ \sup_{t \geq 0} \bigg( \int_0^{\tau_i^* \wedge t} e^{-ru} \, H_{i,1}(X_u, S_u) \, du + \int_0^{\tau_i^* \wedge t} e^{-ru} \, \frac{F_{i,1}(S_u)}{S_u} \, dS_u \bigg) \bigg] < \infty \tag{4.10}$$

holds, for all $0 < x \leq s$ such that $\underline{s}_i < s < \overline{s}_i$ and every $i = 1,2,3$, under $\alpha < 0$. Hence, letting $m$ and $n$ go to infinity and using the condition of (2.38) as well as the property $\tau_{i,m} \downarrow \tau_i^*$ ($P_{x,s}$-a.s.) as $m \to \infty$, we can apply the Lebesgue dominated convergence theorem to the appropriate (diagonal) subsequence in the expression of (4.9) to obtain the equality:

$$E_{x,s} \bigg[ \int_0^{\tau_i^*} e^{-ru} \, H_{i,1}(X_u, S_u) \, du + \int_0^{\tau_i^*} e^{-ru} \, \frac{F_{i,1}(S_u)}{S_u} \, dS_u \bigg] = V_i^{a_i^*}(x,s) \tag{4.11}$$

for all $0 < x \leq s$ such that $\underline{s}_i < s < \overline{s}_i$ and every $i = 1,2,3$, which together with the inequalities in (4.8) directly implies the desired assertion. We finally recall that the results of parts (c) and (d) of the proof of Theorem 2.1 above, which are obtained by standard comparison arguments applied to the value functions of the appropriate optimal stopping problems, show that the inequality $a_i^*(s) > \underline{a}_i(s)$, for all $\underline{s}_i < s < \overline{s}_i$ and every $i = 1,2,3$, should hold for the optimal exercise boundary, that completes the verification. □

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