A Law of Conservation of Symbols

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable, and $\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \equiv 0$.
Then $\frac{\partial^2 f^2}{\partial x\,\partial y} \equiv 0$.

Proof: We first show $\frac{\partial^2 f}{\partial x\,\partial y} \equiv 0$. Let $p \in \mathbb{R}^2$. We consider the following cases:

(1) If $\frac{\partial f}{\partial x}(p) \neq 0$, then by continuity $\frac{\partial f}{\partial x} \neq 0$ in a neighbourhood N of p. In N, $\frac{\partial f}{\partial y} \equiv 0$, and so $\frac{\partial^2 f}{\partial x \partial y} \equiv 0$, giving in particular $\frac{\partial^2 f}{\partial x \partial y}(p) = 0$.

(2) If $\frac{\partial f}{\partial x} \equiv 0$ in a neighbourhood N of p, then $\frac{\partial^2 f}{\partial y \partial x} \equiv 0$ in N. As $f \in C^2$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \equiv 0$ in N. In particular $\frac{\partial^2 f}{\partial x \partial y}(p) = 0$.

(3) If $\frac{\partial f}{\partial x}(p) = 0$, but $\frac{\partial f}{\partial x}$ does not identically vanish in a neighbourhood of p, then there exists a sequence of points $(p_n)_n$ that converges to p, such that $\frac{\partial f}{\partial x}(p_n) \neq 0$. By (1), $\frac{\partial^2 f}{\partial x \partial y}(p_n) = 0$. As f is twice continuously differentiable, $\frac{\partial^2 f}{\partial x \partial y}(p) = \lim\limits_{n\to\infty} \frac{\partial^2 f}{\partial x \partial y}(p_n) = \lim\limits_{n\to\infty} 0 = 0$.

Moreover, $\frac{\partial^2 f^2}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f^2}{\partial y}\right) = \frac{\partial}{\partial x}\left(2f\frac{\partial f}{\partial y}\right) = 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + 2f\frac{\partial^2 f}{\partial x\,\partial y} = 0 + 0 = 0.$ ■

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