INFERENCE ON INCOMPLETE INFORMATION GAMES WITH MULTI-DIMENSIONAL ACTIONS

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Abstract. By extending de Paula and Tang (2012) and Aradillas-López and Gandhi (2016), we derive testable restrictions for uniqueness of equilibrium in games with multi-dimensional actions. We discuss two models of payoff functions which imply certain covariance restrictions for players' actions. These restrictions can be used to construct an identified set of strategic parameters under multiple equilibria.

1. Introduction

Multiplicity of equilibria often causes problems when researchers estimate game theoretic models. de Paula and Tang (2012) proposed a test for uniqueness of equilibrium and derived partial identification results for incomplete information games where players take binary actions. Aradillas-López and Gandhi (2016) extended their results to games where players have ordered choices. Their tests and identification strategies are based on covariance restrictions between actions and strategic parts of players' payoff functions. Although their results are quite insightful, they focus on the case where each player's choice set is one-dimensional. This note extends Aradillas-López and Gandhi's (2016) analysis and derives covariance restrictions in games where players take multi-dimensional actions.

2. Main results

There are two players $p = 1, 2$, and each player has two-dimensional action space $\mathcal{A}^p = \mathcal{A}_1^p \times \mathcal{A}_2^p \subseteq \mathbb{R}^2$. Let $Y^p = (Y_1^p, Y_2^p) \in \mathcal{A}^p$ be player p's action. A lowercase y^p represents p's potential action. Let Ξ^p be a random vector of player p's payoff shifter, which can be decomposed into observable exogenous variables X and p's private payoff shock $\mathcal{E}^p$, i.e., $\Xi^p = (X, \mathcal{E}^p)$. X and $\mathcal{E}^p$ can be correlated in an arbitrary way, and the dimension of $\mathcal{E}^p$ is unrestricted. A lowercase $\xi^p = (x, \varepsilon^p)$ represents a potential value of Ξ^p. Player p's payoff function is given by $\nu^p(y^p, y^q; \xi^p)$, where y^q is another player q's action.

We impose the following assumption on players' information structure.

Assumption 1. *X is public information, $\mathcal{E}^1$ is observed only by player 1, and $\mathcal{E}^2$ is observed only by player 2. $\mathcal{E}^1$ and $\mathcal{E}^2$ are independent conditional on X. The distribution of $(X, \mathcal{E}^1, \mathcal{E}^2)$ and payoff structures are common knowledge for the players 1 and 2.*

The conditional independence assumption on the private shocks is prevalent in the literature on estimation of games using covariance restrictions (see, de Paula and Tang, 2012; Aradillas-López and Ghandi, 2016). We note that the elements of $\mathcal{E}^1$ or $\mathcal{E}^2$ can be arbitrarily correlated.

Player p's expected payoff is written as

$$\bar{\nu}^p_\sigma(y^p;\xi^p) = \sum_{y^q\in\mathcal{A}^q} \sigma^q(y^q)\cdot\nu^p(y^p,y^q;\xi^p),$$

where q is another player and $\sigma^q : \mathcal{A}^q \to [0,1]$ is p's belief over q's action. A typical solution concept of static incomplete information games is Bayesian Nash equilibrium (BNE), in which each player chooses an action that maximizes its expected utility given the equilibrium belief. Given that players' shocks are independent (Assumption 1), BNE can be characterized as a collection of choice probabilities $\sigma_*(x) = \{\sigma^p_*(\cdot|x) : \mathcal{A}^p \to [0,1]|p = 1,2\}$ conditional on $X = x$, where

$$\sigma^p_*(y^p|x) = E_{\Xi^p|X}\left[\mathbb{I}\left\{y^p = \arg\max_{y\in\mathcal{A}^p} \bar{\nu}^p_{\sigma_*}(y;\Xi^p)\right\}\middle| X = x\right], \tag{1}$$

for each $y^p \in \mathcal{A}^p$ and x, where $\mathbb{I}\{\cdot\}$ is the indicator function. Hereafter, we assume that $\arg\max_{y\in\mathcal{A}^p} \bar{\nu}^p_{\sigma_*}(y;\Xi^p)$ is singleton with probability one. This assumption is widely employed in the literature.

The next assumption requires that the observed data are generated according to some BNE.

Assumption 2. *Each observation is generated according to a BNE, i.e., for $p = 1,2$,*

$$Y^p = \arg\max_{y\in\mathcal{A}^p} \bar{\nu}^p_{\sigma_*}(y;\Xi^p) \quad \text{for some BNE } \sigma_*(X),$$

with probability one.

We allow that observations are generated from multiple equilibria after conditioning on X. We note that this assumption does not impose any equilibrium selection mechanisms.

Aradillas-López and Gandhi (2016) considered the case of a univariate action variable with an ordinal structure on the action set, and derived a covariance restriction between p's action and some strategic component of p's payoff function, which can be used for inference on the strategic component. Their key idea for deriving testable implications is to explore certain separability and monotonicity conditions for the payoff function. This paper extends their analysis to the case of multi-dimensional action variables, where it is not trivial how to extend shape constraints on the payoff functions, such as separability and monotonicity.

2.1. **First model: Multi-dimensional separability.** We first consider an extension of the separability assumption in Aradillas-López and Gandhi (2016, Assumption 1) to the multi-dimensional case. In particular, we impose the following assumption on p's payoff.

Assumption 3. *The payoff function ν^p can be expressed as*

$$\nu^p(y^p,y^q;\xi^p) = \nu^{p,a}(y^p;\xi^p) - \sum_{k=1}^{2} \nu^{p,b}_k(y^p_k;\xi^p)\cdot\eta^p_k(y^q;x),$$

for some $\nu^{p,a}(\cdot)$, $\nu^{p,b}_k(\cdot)$, and $\eta^p_k(\cdot)$ with $k = 1,2$.

In words, the payoff function can be decomposed so that the strategic part (i.e., the second term) is additively separable with respect to each dimension of the action. For the univariate case, this assumption reduces to Aradillas-López and Gandhi (2016, Assumption 1).

For each belief σ^q, the expected payoff for p from choosing y^p can be expressed as

$$\bar{\nu}^p_\sigma(y^p;\xi^p) = \sum_{y^q \in \mathcal{A}^q} \sigma^q(y^q) \cdot \nu^p(y^p, y^q; \xi^p) = \nu^{p,a}(y^p;\xi^p) - \sum_{k=1}^{2} \nu^{p,b}_k(y^p_k;\xi^p) \cdot \bar{\eta}^p_{\sigma,k}(x),$$

where $\bar{\eta}^p_{\sigma,k}(x) = \sum_{y^q \in \mathcal{A}^q} \sigma^q(y^q) \cdot \eta^p_k(y^q;x)$. Hereafter we focus on inference for parameters contained in the component $\eta^p_1(\cdot)$. Then for each $y^p_2 \in \mathcal{A}^p_2$, ξ^p, pair of actions $v > u$ in $\mathcal{A}^p_1$, and pair of beliefs σ and σ', we obtain the following characterization for the changes in the expected payoff between (v, y^p_2) and (u, y^p_2):

$$\begin{aligned} & [\bar{\nu}^p_\sigma(v, y^p_2;\xi^p) - \bar{\nu}^p_\sigma(u, y^p_2;\xi^p)] - [\bar{\nu}^p_{\sigma'}(v, y^p_2;\xi^p) - \bar{\nu}^p_{\sigma'}(u, y^p_2;\xi^p)] \\ = \; & [\bar{\eta}^p_{\sigma',1}(x) - \bar{\eta}^p_{\sigma,1}(x)] \cdot [\nu^{p,b}_1(v;\xi^p) - \nu^{p,b}_1(u;\xi^p)]. \end{aligned} \tag{2}$$

We note that due to separability in Assumption 3, the right hand side of this expression is independent of y^p_2.

To derive moment inequalities from this characterization, we impose monotonicity of $\nu^{p,b}_1(\cdot)$ with respect to the first argument.

Assumption 4. *For each $v > u$ in $\mathcal{A}^p_1$ and ξ^p, it holds $\nu^{p,b}_1(v;\xi^p) \geq \nu^{p,b}_1(u;\xi^p)$.*

Under this assumption and (2), the inequality $\bar{\eta}^p_{\sigma,1}(x) \geq \bar{\eta}^p_{\sigma',1}(x)$ implies

$$\bar{\nu}^p_\sigma(v, y^p_2;\xi^p) - \bar{\nu}^p_\sigma(u, y^p_2;\xi^p) \leq \bar{\nu}^p_{\sigma'}(v, y^p_2;\xi^p) - \bar{\nu}^p_{\sigma'}(u, y^p_2;\xi^p),$$

for each $v > u$ in $\mathcal{A}^p_1$ and ξ^p. Based on this, we obtain the following lemma for optimal choices under given beliefs. Let $y^p_\sigma(\xi^p) = (y^p_{\sigma,1}(\xi^p), y^p_{\sigma,2}(\xi^p)) = \arg\max_{y \in \mathcal{A}^p} \bar{\nu}^p_\sigma(y;\xi^p)$, which is assumed to be singleton in $\mathbb{R}^2$.

Lemma 1. *Suppose Assumptions 1-4 hold true. Pick any x, σ, σ', and ξ^p such that $\bar{\eta}^p_{\sigma,1}(x) \geq \bar{\eta}^p_{\sigma',1}(x)$ and $y^p_{\sigma,2}(\xi^p) = y^p_{\sigma',2}(\xi^p)$. Then it holds $\mathbb{I}\{y^p_{\sigma,1}(\xi^p) \leq y^p_1\} \geq \mathbb{I}\{y^p_{\sigma',1}(\xi^p) \leq y^p_1\}$ for each $y^p_1 \in \mathcal{A}^p_1$.*

Based on this lemma, we can derive the covariance restrictions (or moment inequalities) for observables.

Theorem 1. *Suppose Assumptions 1-4 hold. Then, for each $y^p_1 \in \mathcal{A}^p_1$, it holds*

$$E[\mathbb{I}\{Y^p_1 \leq y^p_1\} \cdot \eta^p_1(Y^q;X)|X, Y^p_2] \geq E[\mathbb{I}\{Y^p_1 \leq y^p_1\}|X, Y^p_2] \cdot E[\eta^p_1(Y^q;X)|X, Y^p_2],$$

with probability one.

Intuitively Lemma 1 characterizes co-movement of $\bar{\eta}^p_{s,1}(x)$ and $\mathbb{I}\{y^p_{s,1}(\xi^p) \leq y^p_1\}$ across different equilibria $s = \sigma, \sigma'$ given that $y^p_{\sigma,2}(\xi^p) = y^p_{\sigma',2}(\xi^p)$, and this co-movement implies non-zero correlation for observables $\mathbb{I}\{Y^p_1 \leq y^p_1\}$ and $\eta^p_1(Y^q;X)$ given (X, Y^p_2). This intuition is analogous to the ones in de Paula and Tang (2012) and Aradillas-López and Gandhi (2016) except for conditioning on Y^p_2 due to multi-dimensional actions.

We can also show that if the BNE is unique, then the above moment inequalities become equalities. This is an immediate implication of the assumption of conditional independence between $\mathcal{E}^1$ and $\mathcal{E}^2$. Thus, we can conduct a statistical test for uniqueness of the BNE by testing the zero covariance restrictions. Such a test is considered as a multi-dimensional version of de Paula and Tang's (2012) test for uniqueness of the BNE.

If there are multiple equilibria, we can use these moment inequalities to conduct inference on parameters that specify $\eta_1^p(Y^q;X) = \eta_1^p(Y^q;X|\theta_1^p)$. To implement inference on θ_1^p, we can employ several existing econometric methods for (conditional) moment inequalities, such as adaptations of Andrews and Shi (2013) and Chernozhukov, Lee and Rosen (2011), and the method proposed in Aradillas-López and Gandhi (2016). Since our moment inequalities are conditional on Y_2^p, richer support of Y_2^p will lead to more restrictions which can help to obtain more informative identified regions for θ_1^p.

Remark 1. Assumptions 3 and 4 are needed to derive the restriction on η_1^p. However, we do not have to impose any restrictions on the shape of another player q's payoff function $\nu^q(y^q, y^p; \xi^p)$. We also allow that $\nu_2^{p,b}(y_2^p;\xi)$ is not monotone with respect to y_2^p.

Remark 2. The results in this paper can be extended to more general setups with multi-players and multi-dimensional actions.[1] Each player $p \in \{1, \ldots, P\}$ has K^p-dimensional action space $\mathcal{A}^p = \prod_{k=1}^{K^p} \mathcal{A}_k^p$, where $\mathcal{A}_k^p$ denotes player p's action set for the k-th dimension. Let $Y^p = (Y_1^p, \ldots, Y_{K^p}^p) \in \mathcal{A}^p$ be player p's action variable. We use $\mathcal{A}^{-p} = \prod_{q \neq p} \mathcal{A}^q$ and $Y^{-p} = (Y^q)_{q \neq p}$ to denote the action space and a profile of action variables of all players other than p, respectively. Let Y_{-k}^p be p's action variable other than k-th dimension. Suppose p's payoff function ν^p can be expressed as

$$\nu^p(y^p, y^{-p};\xi^p) = \nu^{p,a}(y^p;\xi^p) - \sum_{k=1}^{K^p} \nu_k^{p,b}(y_k^p;\xi^p) \cdot \eta_k^p(y^{-p};x),$$

where the lowercase letters represent potential actions. Then, under similar assumptions as above, we can derive the following covariance restriction

$$E[\mathbb{I}\{Y_k^p \leq y_k^p\} \cdot \eta_k^p(Y^{-p};X)|X, Y_{-k}^p] \geq E[\mathbb{I}\{Y_k^p \leq y_k^p\}|X, Y_{-k}^p] \cdot E[\eta_k^p(Y^{-p};X)|X, Y_{-k}^p].$$

2.2. **Second model: Strategic interaction through one channel.** As another example, this subsection considers the situation where only one channel directly affects the strategic interaction term. We now impose the following assumption.

Assumption 5. ν^p *can be expressed as*

$$\nu^p(y^p, y^q;\xi^p) = \nu^{p,a}(y^p;\xi^p) - \nu^{p,b}(y_1^p;\xi^p) \cdot \eta^p(y_2^p, y^q;x),$$

for some $\nu^{p,a}(\cdot)$, $\nu^{p,b}(\cdot)$, *and* $\eta^p(\cdot)$.

In this case, the expected payoff for player p of choosing y^p under belief σ can be written as

$$\bar{\nu}_\sigma^p(y^p;\xi^p) = \sum_{y^q \in \mathcal{A}^q} \sigma^q(y^q) \cdot \nu^p(y^p, y^q;\xi^p) = \nu^{p,a}(y^p;\xi^p) - \nu^{p,b}(y_1^p;\xi^p) \cdot \bar{\eta}_\sigma^p(y_2^p, x),$$

[1]Details are available in the working paper version of this paper: https://www.sanken.keio.ac.jp/publication/KEO-dp/161/KEO-DP161.pdf.

where $\bar{\eta}^p_\sigma(y^p_2, x) = \sum_{y^q \in \mathcal{A}^q} \sigma^q(y^q) \cdot \eta^p(y^p_2, y^q; x)$. Then for each $y^p_2 \in \mathcal{A}^p_2$, ξ^p, pair of actions $v > u$ in $\mathcal{A}^p_1$, and pair of beliefs σ and σ', we obtain the following characterization for the changes in the expected payoff between (v, y^p_2) and (u, y^p_2):

$$\begin{aligned} & [\bar{\nu}^p_\sigma(v, y^p_2; \xi^p) - \bar{\nu}^p_\sigma(u, y^p_2; \xi^p)] - [\bar{\nu}^p_{\sigma'}(v, y^p_2; \xi^p) - \bar{\nu}^p_{\sigma'}(u, y^p_2; \xi^p)] \\ = \; & [\bar{\eta}^p_{\sigma'}(y^p_2, x) - \bar{\eta}^p_\sigma(y^p_2, x)] \cdot [\nu^{p,b}(v; \xi^p) - \nu^{p,b}(u; \xi^p)]. \end{aligned} \tag{3}$$

In addition, we maintain the assumption on monotonicity of $v^{p,b}(\cdot)$ with respect to the first argument.

Assumption 6. *For each $v > u$ in $\mathcal{A}^p_1$ and ξ^p, it holds $\nu^{p,b}(v; \xi^p) \geq \nu^{p,b}(u; \xi^p)$.*

Under this assumption (3), the event $\bar{\eta}^p_\sigma(y^p_2, x) \geq \bar{\eta}^p_{\sigma'}(y^p_2, x)$ implies

$$\bar{\nu}^p_\sigma(v, y^p_2; \xi^p) - \bar{\nu}^p_\sigma(u, y^p_2; \xi^p) \leq \bar{\nu}^p_{\sigma'}(v, y^p_2; \xi^p) - \bar{\nu}^p_{\sigma'}(u, y^p_2; \xi^p),$$

for each $v > u$ and ξ^p. Based on this, we obtain the following lemma for optimal choices under given beliefs. Let $y^p_\sigma(\xi^p) = (y^p_{\sigma,1}(\xi^p), y^p_{\sigma,2}(\xi^p)) = \arg\max_{y \in \mathcal{A}^p} \bar{\nu}^p_\sigma(y; \xi^p)$ which is assumed to be singleton in $\mathbb{R}^2$.

Lemma 2. *Suppose Assumptions 1-2 and 5-6 hold true. Pick any x, σ, σ', and ξ^p such that $\bar{\eta}^p_\sigma(y^p_2, x) \geq \bar{\eta}^p_{\sigma'}(y^p_2, x)$ and $y^p_{\sigma,2}(\xi^p) = y^p_{\sigma',2}(\xi^p) = y^p_2$. Then it holds $\mathbb{I}\{y^p_{\sigma,1}(\xi^p) \leq y^p_1\} \geq \mathbb{I}\{y^p_{\sigma',1}(\xi^p) \leq y^p_1\}$ for each $y^p_1 \in \mathcal{A}^p_1$.*

Based on this lemma, we can derive covariance restrictions (or moment inequalities) for observables.

Theorem 2. *Suppose the Assumptions 1-2 and 5-6 hold. Then, for each $y^p_1 \in \mathcal{A}^p_1$, it holds*

$$E[\mathbb{I}\{Y^p_1 \leq y^p_1\} \cdot \eta^p(Y^p_2, Y^q; X)|X, Y^p_2] \geq E[\mathbb{I}\{Y^p_1 \leq y^p_1\}|X, Y^p_2] \cdot E[\eta^p(Y^p_2, Y^q; X)|X, Y^p_2],$$

with probability one.

Similar comments to Theorem 1 apply. A test for the uniqueness of the BNE and inference on parameters to specify $\eta^p(Y^p_2, Y^q; X)$ can be conducted by the existing econometric methods.

Remark 3. As with the first model, we can consider more general games with multi-players and multi-dimensional actions. Suppose ν^p can be expressed as

$$\nu^p(y^p, y^{-p}; \xi^p) = \nu^{p,a}(y^p; \xi^p) - \nu^{p,b}(y^p_1; \xi^p) \cdot \eta^p(y^p_{-1}, y^{-p}; x).$$

In words, there exists only one channel y^p_1 which directly affects the strategic interaction term $\nu^{p,b}$. Then, under the similar conditions, we can derive the following covariance restrictions.

$$E[\mathbb{I}\{Y^p_1 \leq y^p_1\} \cdot \eta^p(Y^p_{-1}, Y^{-p}; X)|X, Y^p_{-1}] \geq E[\mathbb{I}\{Y^p_1 \leq y^p_1\}|X, Y^p_{-1}] \cdot E[\eta^p(Y^p_{-1}, Y^{-p}; X)|X, Y^p_{-1}].$$

3. Supplementary material: numerical illustration

In the supplementary material, we provide some numerical examples to illustrate identified sets of payoff parameters for multi-dimensional action games obtained from our theorems. We present the identified sets for two-player games with two-dimensional binary actions and payoff functions satisfying Assumption 3. Our examples illustrate situations where the sign of an interaction effect can be identified and also variations in x can help to shrink the identified sets. Also in our examples, we find that the equilibrium selection mechanism has a negligible impact on the identified set as long as it selects each equilibrium with strictly positive probability.

Although it is beyond the scope of this paper, it is interesting to further investigate how the identified sets changes (or collapses to singleton) by various elements to specify the multi-dimensional action games, such as equilibrium selection mechanisms (including the case of unique equilibrium), support of covariates and actions, number of players, dimensions of actions, and shapes of payoff functions.

Appendix A. Mathematical appendix

Since the proofs of Lemma 2 and Theorem 2 are similar to those of Lemma 1 and Theorem 1, respectively, here we only present the proofs for Lemma 1 and Theorem 1.

A.1. **Proof of Lemma 1.** Pick any x, σ, σ', and ξ^p such that $\bar{\eta}^p_{\sigma,1}(x) \geq \bar{\eta}^p_{\sigma',1}(x)$ and $y^p_{\sigma,2}(\xi^p) = y^p_{\sigma',2}(\xi^p) = y^p_2$. Furthermore, take any $y^p_1 \in \mathcal{A}^p_1$. Then define

$$\mathbb{I}^p_\sigma(y^p_1, y^p_2; \xi^p) = \max_{u \leq y^p_1} \min_{v > y^p_1} \mathbb{I}\{\bar{\nu}^p_{\sigma,1}(v, y^p_2; \xi^p) - \bar{\nu}^p_{\sigma,1}(u, y^p_2; \xi^p) \leq 0\}.$$

By Assumption 4, the inequality $\bar{\eta}^p_{\sigma,1}(x) \geq \bar{\eta}^p_{\sigma',1}(x)$ implies

$$\mathbb{I}^p_\sigma(y^p_1, y^p_2; \xi^p) \geq \mathbb{I}^p_{\sigma'}(y^p_1, y^p_2; \xi^p). \tag{4}$$

Also, when $y^p_{\sigma,2}(\xi^p) = y^p_{\sigma',2}(\xi^p) = y^p_2$, the definition of $\mathbb{I}^p_\sigma(\cdot)$ yields

$$\mathbb{I}\{y^p_{\sigma,1}(\xi^p) \leq y^p_1\} = \mathbb{I}^p_\sigma(y^p_1, y^p_2; \xi^p), \quad \mathbb{I}\{y^p_{\sigma',1}(\xi^p) \leq y^p_1\} = \mathbb{I}^p_{\sigma'}(y^p_1, y^p_2; \xi^p). \tag{5}$$

Combining (4) and (5), the conclusion follows.

A.2. **Proof of Theorem 1.** The proof is analogous to that of Aradillas-López and Gandhi (2016, Theorem 1). Given $X = x$, let $\{\sigma_{*j}(x)\}_{j=1}^J$ be the set of BNE and $P^S_j(x)$ be the probability that the equilibrium $\sigma_{*j}(x)$ is selected. The probability that equilibrium $\sigma_{*j}(x)$ is selected conditional on $Y^p_2 = y^p_2$ is written as

$$P^S_j(x, y^p_2) = \frac{P^S_j(x) \cdot \sigma^p_{*j}(y^p_2|x)}{\sum_{j'=1}^J P^S_{j'}(x) \cdot \sigma^p_{*j'}(y^p_2|x)},$$

where with slight abuse of notation, $\sigma^p_{*j}(\cdot|x)$ represents the conditional density function of $Y^p_2|X = x$ under the equilibrium $\sigma_{*j}(x)$. Pick any $y^p_1 \in \mathcal{A}^p_1$. Observe that for almost every

x and y_2^p,

$$\begin{aligned}
& E[\mathbb{I}\{Y_1^p \leq y_1^p\} \cdot \eta_1^p(Y^q; X)|X = x, Y_2^p = y_2^p] \\
= & \sum_{j=1}^{J} P_j^S(x, y_2^p) \cdot E_{\Xi^p|X,Y_2^p}[\mathbb{I}\{y_{\sigma*j,1}^p(\Xi^p) \leq y_1^p\} \cdot \eta_1^p(y_{\sigma*j}^q(\Xi^q); X)|X = x, Y_2^p = y_2^p] \\
= & \sum_{j=1}^{J} P_j^S(x, y_2^p) \cdot E_{\Xi^p|X,Y_2^p}[\mathbb{I}\{y_{\sigma*j,1}^p(\Xi^p) \leq y_1^p\}|X = x, Y_2^p = y_2^p] \cdot E_{\Xi^q|X}[\eta_1^p(y_{\sigma*j}^q(\Xi^q); X)|X = x] \\
= & E_{\Xi^p|X,Y_2^p}\left[\sum_{j=1}^{J} P_j^S(X, Y_2^p) \cdot \mathbb{I}\{y_{\sigma*j}^p(\Xi^p) \leq y_1^p\} \cdot \bar{\eta}_{\sigma*j,1}^p(X)\right| X = x, Y_2^p = y_2^p\Bigg],
\end{aligned}$$

where the second equality follows from $\Xi^p \perp \Xi^q|X$ and $Y_2^p \perp \Xi^q|X$ (by Assumption 1). We also have that for almost every x and y_2^p,

$$\begin{aligned}
& E[\mathbb{I}\{Y_1^p \leq y_1^p\}|X = x, Y_2^p = y_2^p] \cdot E[\eta_1^p(Y^q; X)|X = x, Y_2^p = y_2^p] \\
= & \sum_{j=1}^{J} P_j^S(x, y_2^p) \cdot E_{\Xi^p|X,Y_2^p}[\mathbb{I}\{y_{\sigma*j,1}^p(\Xi^p) \leq y_1^p\}|X = x, Y_2^p = y_2^p] \\
& \times \sum_{j=1}^{J} P_j^S(x, y_2^p) \cdot E_{\Xi^q|X}[\eta_1^p(y_{\sigma*j}^q(\Xi^q); X)|X = x] \\
= & \sum_{j=1}^{J} P_j^S(x, y_2^p) \cdot E_{\Xi^p|X,Y_2^p}[\mathbb{I}\{y_{\sigma*j,1}^p(\Xi^p) \leq y_1^p\}|X = x, Y_2^p = y_2^p] \times \sum_{j=1}^{J} P_j^S(x, y_2^p) \cdot \bar{\eta}_{\sigma*j,1}^p(x) \\
= & E_{\Xi^p|X,Y_2^p}\left[\left(\sum_{j=1}^{J} P_j^S(X, Y_2^p) \cdot \mathbb{I}\{y_{\sigma*j,1}^p(\Xi^p) \leq y_1^p\}\right) \times \left(\sum_{j=1}^{J} P_j^S(X, Y_2^p) \cdot \bar{\eta}_{\sigma*j,1}^p(X)\right)\right| X = x, Y_2^p = y_2^p\Bigg].
\end{aligned}$$

Combining these equations, we obtain

$$\begin{aligned}
& E[\mathbb{I}\{Y_1^p \leq y_1^p\} \cdot \eta_1^p(Y^q; X)|X = x, Y_2^p = y_2^p] \\
& -E[\mathbb{I}\{Y_1^p \leq y_1^p\}|X = x, Y_2^p = y_2^p] \cdot E[\eta_1^p(Y^q; X)|X = x, Y_2^p = y_2^p] \\
= & E_{\Xi^p|X,Y_2^p}\Bigg[\sum_{j=1}^{J} P_j^S(X, Y_2^p) \cdot \mathbb{I}\{y_{\sigma*j,1}^P(\Xi^p) \leq y_1^p\} \cdot \bar{\eta}_{\sigma*j,1}^p(X) \\
& - \left(\sum_{j=1}^{J} P_j^S(X, Y_2^p) \cdot \mathbb{I}\{y_{\sigma*j,1}^p(\Xi^p) \leq y_1^p\}\right) \times \left(\sum_{j=1}^{J} P_j^S(X, Y_2^p) \cdot \bar{\eta}_{\sigma*j,1}^p(X)\right)\right| X = x, Y_2^p = y_2^p\Bigg],
\end{aligned}$$

for almost every x and y_2^p. Note that conditioning on $Y_2^p = y_2^p$ is equivalent to conditioning on the event $\{y_{\sigma,2}^p(\Xi^p) = y_{\sigma',2}^p(\Xi^p)\}$. Now the object inside the above conditional expectation is nonnegative since for each (x, y_1^p, y_2^p) and each ξ^p such that $y_{\sigma,2}^p(\xi^p) = y_{\sigma',2}^p(\xi^p)$, we have

$$\begin{aligned}
& \sum_{j=1}^{J} P_j^S(x, y_2^p) \cdot \mathbb{I}\{y_{\sigma*j,1}^p(\xi^p) \leq y_1^p\} \cdot \bar{\eta}_{\sigma*j1}^p(x) \\
& - \left(\sum_{j=1}^{J} P_j^S(x, y_2^p) \cdot \mathbb{I}\{y_{\sigma*j,1}^p(\xi^p) \leq y_1^p\} \right) \times \left(\sum_{j=1}^{J} P_j^S(x, y_2^p) \cdot \bar{\eta}_{\sigma*j,1}^p(x) \right) \\
= & \sum_{\ell=1}^{J} \sum_{j=1}^{J} P_\ell^S(x, y_2^p) P_j^S(x, y_2^p) \cdot \mathbb{I}\{y_{\sigma*j,1}^p(\xi^p) \leq y_1^p\} \cdot (1 - \mathbb{I}\{y_{\sigma*\ell,1}^p(\xi^p) \leq y_1^p\}) \cdot \{\bar{\eta}_{\sigma*j,1}^p(x) - \bar{\eta}_{\sigma*\ell,1}^p(x)\} \\
\geq & \ 0,
\end{aligned}$$

where the inequality follows from Lemma 1. More specifically, if $\bar{\eta}_{\sigma*j,1}^p(x) - \bar{\eta}_{\sigma*\ell,1}^p(x) < 0$ and $\mathbb{I}\{y_{\sigma*j,1}^p(\xi^p) \leq y_1^p\} = 1$, then Lemma 1 implies $1 - \mathbb{I}\{y_{\sigma*\ell,1}^p(\xi^p) \leq y_1^p\} = 0$, and thus $\mathbb{I}\{y_{\sigma*j,1}^p(\xi^p) \leq y_1^p\} \cdot (1 - \mathbb{I}\{y_{\sigma*\ell,1}^p(\xi^p) \leq y_1^p\}) \cdot \{\bar{\eta}_{\sigma*j,1}^p(x) - \bar{\eta}_{\sigma*\ell,1}^p(x)\}$ cannot be negative.

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SUPPLEMENTARY MATERIAL FOR "INFERENCE ON INCOMPLETE INFORMATION GAMES WITH MULTI-DIMENSIONAL ACTIONS"

HIDEYUKI TOMIYAMA AND TAISUKE OTSU

1. Numerical example

In this supplement, we provide some numerical examples to illustrate identified sets for multi-dimensional action games obtained from our theorems. There are two players, p and q. For the sake of simplicity, we assume both players are symmetric. Player p makes a two-dimensional choice, $y^p = (y_1^p, y_2^p) \in \{0,1\} \times \{0,1\}$, and hence, there are 16 possible outcomes. We assume the following payoff functions

$$\nu^p(y^p, y^q; \xi^p) = y_1^p(x_1^p + \alpha_1^p y_2^p + \beta_{11}^p y_1^q + \beta_{12}^p y_2^q) + y_2^p(x_2^p + \alpha_2^p y_1^p + \beta_{21}^p y_1^q + \beta_{22}^p y_2^q) + \varepsilon(y^p),$$

where $\varepsilon(y^p)$ follows the (i.i.d.) Type-I extreme value distribution, $\alpha = (\alpha_1^p, \alpha_2^p)$ and $\beta = (\beta_{11}^p, \beta_{12}^p, \beta_{21}^p, \beta_{22}^p)$ are parameters, and x_k^p (for $k = 1, 2$) is a covariate that affects only player p's payoff from the k-th dimension action. This model fits into Assumption 3 in the paper by setting

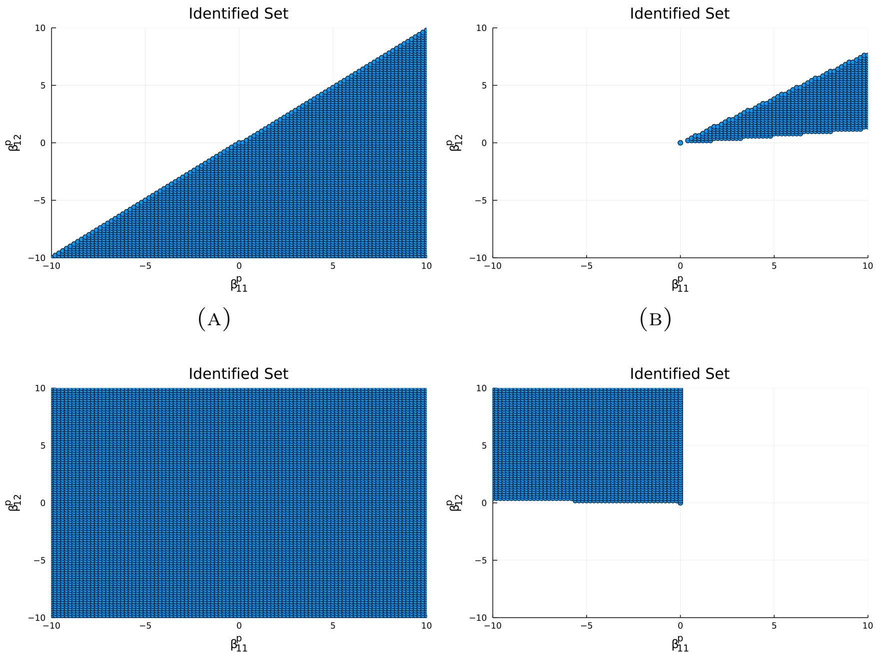
$$\begin{aligned} \nu^{p,a}(y^p; \xi^p) &= y_1^p x_1^p + y_2^p x_2^p + (\alpha_1^p + \alpha_2^p) y_1^p y_2^p + \varepsilon(y^p), \\ \nu_k^{p,b}(y_k^p; \xi^p) &= y_k^p, \\ \eta_k^p(y^p; x) &= -(\beta_{k1}^p y_1^p + \beta_{k2}^p y_2^p), \end{aligned}$$

for $k = 1, 2$. Since the payoff function is symmetric, hereafter, we focus on the identified set of β_{11}^p and β_{12}^p. To derive the identified set, we implement the following simulation procedure.

- For each value of $x = (x_1^p, x_2^p, x_1^q, x_2^q)$ with support $\mathcal{X}$, we characterize the BNE (i.e., 4×2 choice probabilities $\sigma_*(x)$) by using 1000 randomly drawn starting points for the fixed point iteration based on eq. (1) in the paper.
- We assume the true equilibrium selection probabilities are consistent with the 1000 generated equilibria. For instance, suppose we obtain $\sigma_*(x) = ((0.2, 0.4, 0.2, 0.2), (0.2, 0.4, 0.2, 0.2))$ 250 times and $\sigma^*(x) = ((0.2, 0.2, 0.4, 0.2), (0.2, 0.2, 0.4, 0.2))$ 750 times. Then, we assume the equilibrium selection mechanism picks $\sigma_*(x) = ((0.2, 0.4, 0.2, 0.2), (0.2, 0.4, 0.2, 0.2))$ with probability 0.25 and $\sigma_*(x) = ((0.2, 0.2, 0.4, 0.2), (0.2, 0.2, 0.4, 0.2))$ with probability 0.75.
- Based on the simulated BNE and equilibrium selection probabilities, we search $(\beta_{11}^p, \beta_{12}^p)$ that satisfies the inequality constraint in Theorem 1 for every $x \in \mathcal{X}$ and $y_2^p \in \{0,1\}$.

Figure 1 depicts the identification sets using different parameter values and cardinalities of $\mathcal{X}$. Note that $(\beta_{11}^p, \beta_{12}^p)$ can be identified up to scale. In Figure (A), we set $\alpha_1^p = \alpha_2^p = -2$, $\beta_{11}^p = \beta_{22}^p = 5$, $\beta_{12}^p = \beta_{21}^p = 1$, and $\mathcal{X} = \{(0,0,0,0)\}$ (i.e., $\mathcal{X}$ is singleton). Then, we can identify

FIGURE 1. Identified sets



(A) (B) (C) (D)

Note: (A) sets $\alpha^p_1 = \alpha^p_2 = -2$, $\beta^p_{11} = \beta^p_{22} = 5$, $\beta^p_{12} = \beta^p_{21} = 1$, and $x \in \{(0,0,0,0)\}$; (B) sets $\alpha^p_1 = \alpha^p_2 = -2$, $\beta^p_{11} = \beta^p_{22} = 5$, $\beta^p_{12} = \beta^p_{21} = 1$, and $x \in \{-2,-1,0,1,2\}^4$; (C) sets $\alpha^p_1 = \alpha^p_2 = -1$, $\beta^p_{11} = -5$, $\beta^p_{12} = 1$, $\beta^p_{21} = -1$, $\beta^p_{22} = 5$, and $x \in \{(0,0,0,0)\}$; and (D) sets $\alpha^p_1 = \alpha^p_2 = -1$, $\beta^p_{11} = -5$, $\beta^p_{12} = 1$, $\beta^p_{21} = -1$, $\beta^p_{22} = 5$, and $x \in \{-2,-1,0,1,2\}^4$.

$\beta^p_{11} \geq \beta^p_{12}$. This is consistent with the result of de Paula and Tang (2012), which identifies the sign of interaction effect using only one value of x. Since $\eta^p_k(y^p; x)$ contains two parameters $(\beta^p_{11}, \beta^p_{12})$, we can identify only the sign of $\beta^p_{11} - \beta^p_{12}$. However, we can shrink the identified set using variation of x. In Figure (B), we use the same parameter value but set $\mathcal{X} = \{-2,-1,0,1,2\}^4$ (i.e., cardinality of $\mathcal{X}$ is 5^4). Then, we can identify $\beta^p_{11} \geq \beta^p_{12}$, $\beta^p_{11} \geq 0$, and $\beta^p_{12} \geq 0$. Figure (C) sets $\alpha^p_1 = \alpha^p_2 = -1$, $\beta^p_{11} = -5$, $\beta^p_{12} = 1$, $\beta^p_{21} = -1$, $\beta^p_{22} = 5$ and $\mathcal{X} = \{(0,0,0,0)\}$. The identified set is not informative because the equilibrium is unique. However, as with the case of Figure (A) and (B), we can shrink the identified set using a variation of X. In Figure (D), we set $\mathcal{X} = \{-2,-1,0,1,2\}^4$ and can identify the sign of β^p_{11} and β^p_{12}. In any setting, we obtain the same identified set even when we try different sets of 1000 starting points, which typically yield different equilibrium selection mechanisms. This indicates that in this numerical example, the equilibrium selection mechanism has a negligible impact on the identified set as long as it selects each equilibrium with strictly positive probability.

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