

**GENERATION OF THE SPECIAL LINEAR GROUP
BY ELEMENTARY MATRICES
IN SOME MEASURE BANACH ALGEBRAS**

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ABSTRACT. For a commutative unital ring R , and $n \in \mathbb{N}$, let $\mathrm{SL}_n(R)$ denote the special linear group over R , and $\mathrm{E}_n(R)$ the subgroup of elementary matrices. Let \mathcal{M}^+ be the Banach algebra of all complex Borel measures on $[0, +\infty)$ with the norm given by the total variation, the usual operations of addition and scalar multiplication, and with convolution. It is first shown that $\mathrm{SL}_n(A) = \mathrm{E}_n(A)$ for Banach subalgebras A of \mathcal{M}^+ that are closed under the operation $\mathcal{M}^+ \ni \mu \mapsto \mu_t$, $t \in [0, 1]$, where $\mu_t(E) := \int_E (1-t)^x d\mu(x)$ for $t \in [0, 1)$, and Borel subsets E of $[0, +\infty)$, and $\mu_1 := \mu(\{0\})\delta$, where $\delta \in \mathcal{M}^+$ is the Dirac measure. Using this, and with auxiliary results established in the article, many illustrative examples of such Banach algebras A are given, including several well-studied classical Banach algebras such as the class of analytic almost periodic functions. An example of a Banach subalgebra $A \subset \mathcal{M}^+$, that does not possess the closure property above, but for which $\mathrm{SL}_n(A) = \mathrm{E}_n(A)$ nevertheless holds, is also constructed.

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2010 *Mathematics Subject Classification*. Primary 15A23; Secondary 43A10, 46J10, 20H25.

Key words and phrases. Convolution Banach algebras of Borel measures, factorisation by elementary matrices, commutative unital Banach algebras, K -theory.

1. INTRODUCTION

The aim of this article is to establish $\mathrm{SL}_n(R) = \mathrm{E}_n(R)$, where $\mathrm{SL}_n(R)$ is the special linear group over R , $\mathrm{E}_n(R)$ is the group of elementary matrices, and R is a certain Banach algebra of measures on the half-line $[0, +\infty)$. We elaborate on this below.

Definition 1.1 ($\mathrm{SL}_n(R)$ and $\mathrm{E}_n(R)$).

Let R be a commutative unital ring with multiplicative identity 1 and additive identity element 0. Let $n \in \mathbb{N} = \{1, 2, 3, \dots\}$.

- Let $R^{n \times n}$ denote the set of matrices over R with n rows and n columns. We denote by I_n the identity matrix in $R^{n \times n}$, i.e., the matrix with all diagonal entries equal to $1 \in R$ and off-diagonal entries $0 \in R$.
- The *special linear group* $\mathrm{SL}_n(R)$ denotes the group (with matrix multiplication) of all matrices $M \in R^{n \times n}$ whose entries belong to R and the determinant $\det M$ of M is 1. The *general linear group* $\mathrm{GL}_n(R)$ consists of all invertible matrices in $R^{n \times n}$.
- An *elementary matrix* $E_{ij}(\alpha)$ over R has the form $E_{ij} = I_n + \alpha \mathbf{e}_{ij}$, where $i \neq j$, $\alpha \in R$, and \mathbf{e}_{ij} is the $n \times n$ matrix whose entry in the i th row and j th column is 1, and all the other entries of \mathbf{e}_{ij} are zeros.
- $\mathrm{E}_n(R)$ is the subgroup of $\mathrm{SL}_n(R)$ generated by the elementary matrices.

A classical question in algebra is:

(**Q**) For all $n \in \mathbb{N}$, is $\mathrm{SL}_n(R) = \mathrm{E}_n(R)$?

The answer depends on the ring R . For example:

- If $R = \mathbb{C}$, then the answer to (**Q**) is ‘Yes’, and is an exercise in linear algebra; see for instance [2, Chap.2, §2, Exercise 18(c), p.71].
- Let $R = \mathbb{C}[z_1, \dots, z_d]$.
If $d = 1$, then the answer to (**Q**) is ‘Yes’: This follows from the Euclidean division algorithm in $\mathbb{C}[z]$.
If $d = 2$, then the answer to (**Q**) is ‘No’: A counterexample is ([6]):

$$\begin{bmatrix} 1 + z_1 z_2 & z_1^2 \\ -z_2^2 & 1 - z_1 z_2 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{C}[z_1, z_2]) \setminus \mathrm{E}_2(\mathbb{C}[z_1, z_2]).$$

For $d \geq 3$, the answer to (**Q**) is ‘Yes’: This is the K_1 -analogue of Serre’s conjecture, which is the Suslin stability theorem [26].

- Rings of continuous real- or complex-valued functions on a topological space X were considered in [27].

- For the ring \mathcal{O}_X of holomorphic functions on Stein spaces in \mathbb{C}^d , the question **(Q)** was posed as an explicit open problem by Gromov in [11], and was solved in [14]. See also [13] and [15].
- The question **(Q)** for some rings of ‘sequences’ (with termwise operations) such as

$$\begin{aligned} \ell^\infty(\mathbb{Z}^d) & \text{ (bounded),} \\ c_0(\mathbb{Z}^d) & \text{ (converging to 0), and} \\ s'(\mathbb{Z}^d) & \text{ (at most polynomially growing),} \end{aligned}$$

were considered in [25]. The ring $s'(\mathbb{Z}^d)$ is isomorphic to the ring of periodic distributions on \mathbb{R}^d with the multiplication operation given by convolution.

- In [8], it was shown that for a unital commutative ring R , if $n \geq 2$ and if the Bass stable rank of R is 1, then the answer to **(Q)** is ‘Yes’. Using this result, and a result from [17, §7] (Proposition 1.3 below), many examples of Banach algebras of holomorphic functions in one and several variables, like the polydisc/ball algebras, were considered for which the answer to **(Q)** is ‘Yes’.

In this article, we will consider certain Banach algebras of measures. Our main tool will be the known result stated as Proposition 1.3 below (see [17, §7]). Before stating this result, we elaborate on the Banach algebra structure of $A^{n \times n}$ for a Banach algebra A .

Let $(A, \|\cdot\|)$ be a commutative unital Banach algebra. Then $A^{n \times n}$ is a complex algebra with the usual matrix operations. Let $A^{n \times 1}$ denote the vector space of all column vectors of size n with entries from A and componentwise operations. Then $A^{n \times 1}$ is a normed space with the ‘Euclidean norm’ defined by $\|\mathbf{v}\|_2^2 := \|v_1\|^2 + \cdots + \|v_n\|^2$ for all \mathbf{v} in $A^{n \times 1}$, where \mathbf{v} has components denoted by $v_1, \dots, v_n \in A$. If $M \in A^{n \times n}$, then the matrix multiplication map, $A^{n \times 1} \ni \mathbf{v} \mapsto M\mathbf{v} \in A^{n \times 1}$, is a continuous linear transformation, and we equip $A^{n \times n}$ with the induced operator norm, denoted by $\|\cdot\|$ again. Then $A^{n \times n}$ with this operator norm is a unital Banach algebra. Subsets of $A^{n \times n}$ are given the induced subspace topology.

Definition 1.2 (Null-homotopic element of $\mathrm{SL}_n(A)$).

Let A be a commutative unital Banach algebra, and $n \in \mathbb{N}$. An element $M \in \mathrm{SL}_n(A)$ is *null-homotopic* if M is homotopic to the identity matrix I_n , that is there exists a continuous map $H : [0, 1] \rightarrow \mathrm{SL}_n(A)$ such that $H(0) = M$ and $H(1) = I_n$.

We now state the result from [17, §7].

Proposition 1.3.

Let A be a commutative unital Banach algebra, $n \in \mathbb{N}$, and $M \in \text{SL}_n(A)$. Then the following are equivalent:

- $M \in \text{E}_n(A)$.
- M is null-homotopic.

The statement of our first result (Theorem 2.3) will be given in the following section, where we introduce the Banach algebras of complex Borel measures on $[0, +\infty)$ that we will consider. We will prove Theorem 2.3 in Section 3. In Section 4, using Theorem 2.3, and with auxiliary results established there, many illustrative examples of such Banach algebras A are given, including several classical Banach algebras such as the class of holomorphic almost periodic functions. An example of a Banach subalgebra $A \subset \mathcal{M}^+$, that does not possess the closure property specified in Theorem 2.3, but for which $\text{SL}_n(A) = \text{E}_n(A)$ nevertheless holds, is also constructed in Section 4.

2. BANACH ALGEBRAS OF MEASURES

We recall the following classical Banach algebra \mathcal{M}^+ of measures on a half-line; see for example [12, §4, pp.141-150] or [21, Chapter 6].

Definition 2.1 (The Banach algebra \mathcal{M}^+).

Let \mathcal{M}^+ denote the set of all complex Borel measures on \mathbb{R} with support contained in $[0, +\infty)$. Then \mathcal{M}^+ is a complex vector space with addition $+$ and scalar multiplication \cdot defined as usual, and it becomes a complex algebra if we take the operation $*$ of convolution of measures as the operation of multiplication.

Let $|\mu|$ denote the *total variation measure* of $\mu \in \mathcal{M}^+$; see for example [21, Chap.6, p.117]. Recall that $|\mu|$ is a positive measure defined on all Borel subsets of $[0, +\infty)$, and that $|\mu|(E) \geq |\mu(E)|$ for all Borel subsets $E \subset [0, \infty)$. The norm of an element $\mu \in \mathcal{M}^+$ is taken as the *total variation measure* $|\mu|$ of $[0, +\infty)$, i.e., $\|\mu\| = |\mu|([0, +\infty))$. Then $(\mathcal{M}^+, +, \cdot, *, \|\cdot\|)$ is a commutative unital complex Banach algebra. The unit element (i.e., identity with respect to convolution) is the *Dirac measure* δ , given by

$$\delta(E) = \begin{cases} 1 & \text{if } 0 \in E, \\ 0 & \text{if } 0 \notin E. \end{cases}$$

Definition 2.2. If $\mu \in \mathcal{M}^+$ and $t \in [0, 1)$, define $\mu_t \in \mathcal{M}^+$ by

$$\mu_t(E) := \int_E (1-t)^x d\mu(x), \text{ for all Borel sets } E \subset [0, +\infty).$$

If $t = 1$, then define $\mu_1 \in \mathcal{M}^+$ by $\mu_1 := \mu(\{0\}) \cdot \delta$.

We provide motivation for Definition 2.2 in Remark 3.3. Our first result is the following.

Theorem 2.3. *Let A be a Banach subalgebra of \mathcal{M}^+ (with the induced norm), such that it has the following property:*

$$\text{(P)} \text{ For all } \mu \in A, \text{ and all } t \in [0, 1], \mu_t \in A.$$

Then for all $n \in \mathbb{N}$, $\text{SL}_n(A) = \text{E}_n(A)$.

3. PROOF OF THEOREM 2.3

For $\mu \in \mathcal{M}^+$, it can be seen that $\|\mu_t\| \leq \|\mu\|$ for all $t \in [0, 1]$. Also, $\delta_t = \delta$ for all $t \in [0, 1]$. For $\mu, \nu \in \mathcal{M}^+$, and $t \in [0, 1]$, we have

$$(\mu + \nu)_t = \mu_t + \nu_t. \quad (1)$$

The following two results were shown in [23].

Lemma 3.1. *Let $\mu, \nu \in \mathcal{M}^+$ and $t \in [0, 1]$. Then $(\mu * \nu)_t = \mu_t * \nu_t$.*

Proof. See [23, Lemma 3.1]. \square

Recall the following consequence of the Radon-Nikodym theorem (see for example [21, Theorem 6.12]): If $\mu \in \mathcal{M}^+$, then there exists ‘polar decomposition’ of μ , that is, there exists a measurable function h such that $|h(x)| = 1$ for all $x \in [0, +\infty)$, and $d\mu = h d|\mu|$. Using this, the following result can be established.

Lemma 3.2. *Let $\mu \in \mathcal{M}^+$ and $t_0 \in [0, 1]$. Then $\lim_{t \rightarrow t_0} \mu_t = \mu_{t_0}$.*

Proof. See [23, Lemma 3.2]. \square

In the final section, we will use the Laplace transform of measures $\mu \in \mathcal{M}^+$: For $\mu \in \mathcal{M}^+$, we define for $s \in \mathbb{C}_{\geq 0} := \{s \in \mathbb{C} : \text{Re}(s) \geq 0\}$,

$$\hat{\mu}(s) = \int_{[0, +\infty)} e^{-sx} d\mu(x).$$

Let $d\mu = e^{iw(x)} d|\mu|(x)$ for a measurable function w . For all $s \in \mathbb{C}_{\geq 0}$,

$$|\hat{\mu}(s)| = \left| \int_{[0, \infty)} e^{-sx} d\mu(x) \right| = \left| \int_{[0, \infty)} e^{-iw(x)} e^{-sx} d|\mu|(x) \right| \leq \int_{[0, \infty)} 1 d|\mu|(x) = \|\mu\|.$$

For $t \in [0, 1]$, $s \in \mathbb{C}_{\geq 0}$ and $\mu \in \mathcal{M}^+$, $\hat{\mu}_t(s) = \hat{\mu}(s - \log(1-t))$. Indeed,

$$\hat{\mu}_t(s) = \int_{[0, \infty)} e^{-sx} (1-t)^x d\mu(x) = \int_{[0, \infty)} e^{-sx} e^{x \log(1-t)} d\mu = \hat{\mu}(s - \log(1-t)).$$

Recall that $\mu_1 = \mu(\{0\})\delta$. As $\hat{\delta}(s) = 1$ for all $s \in \mathbb{C}_{\geq 0}$, $\hat{\mu}_1(s) = \mu(\{0\})$. As with the Laplace transforms of L^1 functions, the same proof, mutatis mutandis, shows $\widehat{\mu * \nu}(s) = \hat{\mu}(s)\hat{\nu}(s)$ for $s \in \mathbb{C}_{\geq 0}$ and $\mu, \nu \in \mathcal{M}^+$.

We provide some motivation for Definition 2.2 in the following remark.

Remark 3.3. We want to create a homotopy taking $\mu \in \mathrm{SL}_n(\mathcal{M}^+)$ to I_n (with all diagonal entries equal to δ). In the case of disc algebra, one uses dilations $f \mapsto f(t \cdot)$, $t \in [0, 1]$ to take I_n to $f \in \mathrm{SL}_n(A(\mathbb{D}))$. Motivated by this, and bearing in mind that the Laplace transform $\widehat{\mu}$ of $\mu \in \mathcal{M}^+$ satisfies the following analogue of the Riemann-Lebesgue lemma (we include a proof of this in the Appendix)

$$\lim_{\mathbb{R} \ni s \rightarrow +\infty} \widehat{\mu}(s) = \mu(\{0\}),$$

it is natural to try and construct a homotopy by ‘translating’ the Laplace transform and then taking the inverse Laplace transform. As $\lim_{t \rightarrow 1^-} -\log(1-t) = +\infty$, we try

$$H(t) := \mu_t := \text{Inverse Laplace transform of } (s \mapsto \widehat{\mu}(s - \log(1-t))).$$

To determine μ_t , we write

$$\int_{[0, +\infty)} e^{-(s - \log(1-t))x} d\mu(x) = \int_{[0, +\infty)} e^{-sx} (1-t)^x d\mu(x) = \int_{[0, +\infty)} e^{-sx} d\mu_t(x),$$

where μ_t is as in Definition 2.2. *

Finally, we are ready to prove Theorem 2.3.

Proof. (of Theorem 2.3): We need to show $\mathrm{SL}_n(A) \subset \mathrm{E}_n(A)$. Let

$$M = \begin{bmatrix} \mu_{11} & \cdots & \mu_{1n} \\ \vdots & \ddots & \vdots \\ \mu_{n1} & \cdots & \mu_{nn} \end{bmatrix} \in \mathrm{SL}_n(A).$$

For $t \in [0, 1]$, define

$$H(t) = M_t = \begin{bmatrix} (\mu_{11})_t & \cdots & (\mu_{1n})_t \\ \vdots & \ddots & \vdots \\ (\mu_{n1})_t & \cdots & (\mu_{nn})_t \end{bmatrix}.$$

Thanks to (1), as $M \in \mathrm{SL}_n(A)$, we have $M_t \in \mathrm{SL}_n(A)$ for all $t \in [0, 1]$, since $\delta = \delta_t = (\det M)_t = \det(M_t)$. The continuity of $[0, 1] \ni t \mapsto M_t \in \mathrm{SL}_n(A)$ follows from Lemma 3.2. We note that $M_1 = C\delta$, where $C \in \mathrm{SL}_n(\mathbb{C})$ is the constant matrix

$$C = \begin{bmatrix} \mu_{11}(\{0\}) & \cdots & \mu_{1n}(\{0\}) \\ \vdots & \ddots & \vdots \\ \mu_{n1}(\{0\}) & \cdots & \mu_{nn}(\{0\}) \end{bmatrix}.$$

But $\mathrm{SL}_n(\mathbb{C})$ is path connected, and so there exists a homotopy, say $h : [0, 1] \rightarrow \mathrm{SL}_n(\mathbb{C})$, taking C to the identity matrix $I_n \in \mathrm{SL}_n(\mathbb{C})$.

Combining H with h , we obtain a homotopy $\tilde{H} : [0, 1] \rightarrow \mathrm{SL}_n(A)$ that takes M to $I_n \in \mathrm{SL}_n(A)$:

$$\tilde{H}(t) = \begin{cases} H(2t) & \text{if } t \in \left[0, \frac{1}{2}\right], \\ h(2t-1)\delta & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

So $M \in \mathrm{SL}_n(A)$ is null-homotopic. By Proposition 1.3, $M \in \mathrm{E}_n(A)$. \square

4. EXAMPLES

Using Theorem 2.3, and with auxiliary results, we give many illustrative examples of classical Banach algebras A for which $\mathrm{SL}_n(A) = \mathrm{E}_n(A)$.

4.1. The measure algebra \mathcal{M}^+ .

Trivially, the full algebra \mathcal{M}^+ has the property **(P)**. So for all $n \in \mathbb{N}$, $\mathrm{SL}_n(\mathcal{M}^+) = \mathrm{E}_n(\mathcal{M}^+)$.

4.2. The Wiener-Laplace algebra $\delta\mathbb{C} + L^1[0, +\infty)$.

Consider the Wiener-Laplace algebra \mathcal{W}^+ of the half plane, of all functions defined in the half plane $\mathbb{C}_{\geq 0} := \{s \in \mathbb{C} : \mathrm{Re}(s) \geq 0\}$ that differ from the Laplace transform of an $L^1[0, +\infty)$ function by a constant. The Wiener-Laplace algebra \mathcal{W}^+ is a Banach algebra with pointwise operations, and the norm $\|\hat{f} + \alpha\|_{\mathcal{W}^+} = \|f\|_1 + |\alpha|$, for all $\alpha \in \mathbb{C}$ and all $f \in L^1[0, +\infty)$, where

$$\|f\|_1 := \int_0^{+\infty} |f(x)| dx.$$

Then \mathcal{W}^+ is precisely the set of Laplace transforms of elements of the Banach subalgebra $A = \delta\mathbb{C} + L^1[0, +\infty)$ of \mathcal{M}^+ consisting of all complex Borel measures $\mu = \mu_a + \alpha\delta$, where μ_a is absolutely continuous (with respect to the Lebesgue measure) and $\alpha \in \mathbb{C}$. Recall that $\mu_a \in \mathcal{M}^+$ is *absolutely continuous* if there exists an $f \in L^1[0, +\infty)$ such that, with the Lebesgue measure on $[0, +\infty)$ denoted by dx , we have

$$\mu_a = \int \cdot f dx, \text{ i.e., } \mu_a(E) = \int_E f(x) dx \text{ for all Borel } E \subset [0, +\infty).$$

The \mathcal{M}^+ -norm of $\mu \in A = \delta\mathbb{C} + L^1[0, +\infty) \subset \mathcal{M}^+$ is

$$\|\mu\| = \|f\|_1 + |\alpha| \text{ for } \mu = \alpha\delta + \int \cdot f dx.$$

This Banach subalgebra $A = \delta\mathbb{C} + L^1[0, +\infty)$ of \mathcal{M}^+ has the property **(P)**, and so for all $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathrm{SL}_n(\delta\mathbb{C} + L^1[0, +\infty)) &= \mathrm{E}_n(\delta\mathbb{C} + L^1[0, +\infty)), \text{ and} \\ \mathrm{SL}_n(\mathcal{W}^+) &= \mathrm{E}_n(\mathcal{W}^+). \end{aligned}$$

4.3. Measures without a singular nonatomic part.

If $\lambda \geq 0$, then we use the notation $\delta_{\{\lambda\}} \in \mathcal{M}^+$ to denote the Dirac measure with support $\{\lambda\}$, that is, for all Borel subsets $E \subset [0, +\infty)$,

$$\delta_{\{\lambda\}}(E) = \begin{cases} 1 & \text{if } \lambda \in E, \\ 0 & \text{if } \lambda \notin E. \end{cases}$$

(So $\delta = \delta_{\{0\}}$.) Define the subalgebra \mathcal{A}^+ of \mathcal{M}^+ consisting of all complex Borel measures that do not have a singular non-atomic part. The algebra \mathcal{A}^+ is the set of all $\mu \in \mathcal{M}^+$ that have a decomposition

$$\mu = \int \cdot f dx + \sum_{0=\lambda_0 < \lambda_1, \lambda_2, \lambda_3, \dots} f_n \delta_{\{\lambda_n\}},$$

where

$$f \in L^1[0, \infty),$$

dx is the Lebesgue measure on $[0, +\infty)$,

$$\lambda_n \in [0, +\infty) \text{ for all } n \in \mathbb{Z}_{\geq 0} := \{0, 1, 2, 3, \dots\},$$

$(f_n)_{n \geq 0}$ is an absolutely summable sequence of complex numbers,

$$\text{i.e., } \|(f_n)_{n \geq 0}\|_1 := \sup_{\substack{F \text{ finite} \\ F \subset \mathbb{Z}_{\geq 0}}} \sum_{n \in F} |f_n| < \infty.$$

The \mathcal{M}^+ -norm of $\mu \in \mathcal{A}^+ \subset \mathcal{M}^+$ reduces to

$$\|\mu\| = \|f\|_1 + \|(f_n)_{n \geq 0}\|_1 \text{ for } \mu = \int \cdot f dx + \sum_{0=\lambda_0 < \lambda_1, \lambda_2, \lambda_3, \dots} f_n \delta_{\{\lambda_n\}} \in \mathcal{A}^+.$$

The Banach subalgebra \mathcal{A}^+ of \mathcal{M}^+ possesses the property **(P)**. Hence $\mathrm{SL}_n(\mathcal{A}^+) = \mathrm{E}_n(\mathcal{A}^+)$ for all $n \in \mathbb{N}$. We remark that the Bass stable rank of \mathcal{A}^+ is ∞ (see [16]).

4.4. The analytic almost periodic Wiener algebra APW^+ .

Define the subalgebra \mathcal{A}_0^+ of \mathcal{M}^+ to be the set of all measures $\mu \in \mathcal{M}^+$ that are of the form

$$\mu = \sum_{0=\lambda_0 < \lambda_1, \lambda_2, \lambda_3, \dots} f_n \delta_{\{\lambda_n\}},$$

where the sequence of coefficients $(f_n)_{n \geq 0}$ is absolutely summable. The \mathcal{M}^+ -norm of $\mu \in \mathcal{A}_0^+ \subset \mathcal{M}^+$ is given by $\|\mu\| = \|(f_n)_{n \geq 0}\|_1$ for $\mu \in \mathcal{A}_0^+$. The algebra \mathcal{A}_0^+ is a Banach subalgebra of \mathcal{M}^+ , and has the property **(P)**. Hence for all $n \in \mathbb{N}$, $\mathrm{SL}_n(\mathcal{A}_0^+) = \mathrm{E}_n(\mathcal{A}_0^+)$.

Recall the classical Banach algebra APW^+ of almost periodic functions f , whose Bohr-Fourier coefficients $(f_\lambda)_{\lambda \in \mathbb{R}}$, are summable, and $f_\lambda = 0$ for $\lambda < 0$. The algebra operations in APW^+ are defined pointwise, and the norm is given by

$$\|f\|_1 = \sum_{\lambda \geq 0} |f_\lambda|.$$

(See the following Subsection 4.5 for an explanation of the key terms.) The Banach algebra \mathcal{A}_0^+ is isomorphic as a Banach algebra to APW^+ via the isomorphism

$$\mathcal{A}_0^+ \ni \mu = \sum_{0=\lambda_0 < \lambda_1, \lambda_2, \lambda_3, \dots} f_n \delta_{\{\lambda_n\}} \mapsto \sum_{0=\lambda_0 < \lambda_1, \lambda_2, \lambda_3, \dots} f_n e^{i\lambda_n x} \in \text{APW}^+.$$

So for all $n \in \mathbb{N}$, $\text{SL}_n(\text{APW}^+) = \text{E}_n(\text{APW}^+)$.

4.5. The algebra AP^+ of analytic almost periodic functions.

We refer the reader to [7] and [5] for details on almost periodic functions. For $\lambda \in \mathbb{R}$, let $\mathbf{e}_\lambda := e^{i\lambda \cdot} \in L^\infty(\mathbb{R})$. Let \mathcal{T} be the space of trigonometric polynomials, i.e., \mathcal{T} is the linear span of $\{\mathbf{e}_\lambda : \lambda \in \mathbb{R}\}$. Define AP to be the closure of \mathcal{T} with respect to the $L^\infty(\mathbb{R})$ -norm $\|\cdot\|_\infty$. Then AP is a Banach algebra with pointwise operations, and the norm $\|\cdot\|_\infty$. The space \mathcal{T} is equipped with the inner product

$$\langle p, q \rangle = \lim_{R \rightarrow +\infty} \frac{1}{2R} \int_{-R}^R p(x) \overline{q(x)} dx \quad (p, q \in \mathcal{T}),$$

where $\bar{\cdot}$ denotes complex conjugation. The limit exists since

$$\langle \mathbf{e}_\lambda, \mathbf{e}_{\lambda'} \rangle = \begin{cases} 1 & \text{if } \lambda = \lambda', \\ 0 & \text{if } \lambda \neq \lambda'. \end{cases}$$

For $\lambda \in \mathbb{R}$ and $f \in \text{AP}$, the *Bohr-Fourier coefficient* f_λ is defined as follows: If $(p_n)_n$ is a sequence in \mathcal{T} converging to f in AP , then

$$f_\lambda = \lim_{n \rightarrow \infty} \langle p_n, \mathbf{e}_{-\lambda} \rangle.$$

Define the *Bohr spectrum* of f to be the set $\sigma(f) = \{\lambda \in \mathbb{R} : f_\lambda \neq 0\}$, which is at most countable. Let AP^+ be the subspace of AP given by

$$\text{AP}^+ = \{f \in \text{AP} : \sigma(f) \subset [0, +\infty)\}.$$

Each $f \in \text{AP}^+$ has a holomorphic extension to the upper half-plane

$$\mathbb{U} := \{z \in \mathbb{C} : \text{Im } z > 0\}.$$

The set APW^+ is a dense subset of AP^+ (since the trigonometric polynomials are dense in AP^+). We will use this, and the following consequence Proposition 1.3, to show that for all $n \in \mathbb{N}$ we have $\text{SL}_n(\text{AP}^+) = \text{E}_n(\text{AP}^+)$. We remark that the Bass stable rank of AP^+ is ∞ ; see [18]. We have the following result.

Theorem 4.1.

Let A be a commutative unital Banach algebra such that for all $n \in \mathbb{N}$, $\mathrm{SL}_n(A) = \mathrm{E}_n(A)$. Let S be a Banach algebra containing the unit of A such that

- S is a full subalgebra of A , (i.e., if $S \cap \mathrm{GL}_1(A) = \mathrm{GL}_1(S)$),
- the inclusion map is continuous, and
- S is dense in A .

Then for all $n \in \mathbb{N}$, $\mathrm{SL}_n(S) = \mathrm{E}_n(S)$.

Proof. Let $f \in \mathrm{SL}_n(A)$. As S is dense in A , we can find an element $g \in S$ such that

$$\|g - f\| < \frac{1}{\|f^{-1}\|}.$$

Then for all $t \in [0, 1]$,

$$\|tf^{-1}(g - f)\| < 1 \cdot \|f^{-1}\| \cdot \frac{1}{\|f^{-1}\|} = 1.$$

So $I_n + tf^{-1}(g - f) \in \mathrm{GL}_n(A)$. As

$$g = f + g - f = f(I_n + f^{-1}(g - f)),$$

$g \in \mathrm{GL}_n(A)$. Thus $\det g \in S \cap \mathrm{GL}_1(A)$, giving $\det g \in \mathrm{GL}_1(S)$. Also, $(I_n + tf^{-1}(g - f))^{-1}g \in \mathrm{GL}_n(A)$ for all $t \in [0, 1]$, which implies

$$\Delta(t) := \det((I_n + tf^{-1}(g - f))^{-1}g) \in \mathrm{GL}_1(A)$$

for all $t \in [0, 1]$. For $t \in [0, 1]$, define the matrix $H(t) \in A^{n \times n}$ to be the matrix obtained by scaling (any one, say) the first column of $(I_n + tf^{-1}(g - f))^{-1}g$ by $(\Delta(t))^{-1}$. Then

$$\begin{aligned} \det(H(t)) &= (\Delta(t))^{-1} \det((I_n + tf^{-1}(g - f))^{-1}g) \\ &= (\Delta(t))^{-1} \Delta(t) = 1. \end{aligned}$$

Hence $H(t) \in \mathrm{SL}_n(A)$, and the map $[0, 1] \ni t \mapsto H(t) \in \mathrm{SL}_n(A)$ is continuous. We have $H(1) = f$. Also, $H(0)$ is the matrix obtained by scaling the first column of g by $(\det g)^{-1}$. So $H(0) \in \mathrm{SL}_n(S) = \mathrm{E}_n(S)$. By Proposition 1.3, there exists a homotopy $h : [0, 1] \rightarrow \mathrm{SL}_n(S)$ such that $h(0) = I_n$ and $h(1) = H(0)$. Thanks to the fact that the inclusion map for $S \subset A$ is continuous, $h : [0, 1] \rightarrow \mathrm{SL}_n(A)$ is also continuous. By combining h and H , we get a homotopy $\tilde{H} : [0, 1] \rightarrow \mathrm{SL}_n(A)$ such that $\tilde{H}(0) = I_n$ and $\tilde{H}(1) = H(1) = f$. By Proposition 1.3, $f \in \mathrm{E}_n(A)$. Consequently, $\mathrm{SL}_n(A) = \mathrm{E}_n(A)$ for all $n \in \mathbb{N}$. \square

Corollary 4.2. For all $n \in \mathbb{N}$, $\mathrm{SL}_n(\mathrm{AP}^+) = \mathrm{E}_n(\mathrm{AP}^+)$.

Proof. For $f \in \text{APW}^+$, we have $\|f\|_\infty \leq \|f\|_1$, showing that the inclusion map is continuous. We have APW^+ is a dense subset of AP^+ . The full subalgebra assumption can be seen to hold by using the corona theorem for APW^+ (see e.g. [20, Theorem 2.4]). Moreover, we have seen in the previous subsection that $\text{SL}_n(\text{APW}^+) = \text{E}_n(\text{APW}^+)$ for all $n \in \mathbb{N}$. \square

4.6. APW_S^+ and AP_S^+ . Recall that an *additive sub-semigroup of the group* $(\mathbb{R}, +)$ is a subset $S \subset \mathbb{R}$ with the properties $0 \in S$, and $\lambda + \lambda' \in S$ whenever $\lambda, \lambda' \in S$. Given an additive sub-semigroup $S \subset [0, \infty)$, we define the Banach subalgebra APW_S^+ of APW^+ by

$$\text{APW}_S^+ := \{f \in \text{APW}^+ : \sigma(f) \subset S\},$$

with the induced norm $\|\cdot\|_1$ from APW^+ . Similarly, we define the Banach subalgebra AP_S^+ of AP^+ by

$$\text{AP}_S^+ := \{f \in \text{AP}^+ : \sigma(f) \subset S\},$$

with the induced norm $\|\cdot\|_\infty$ from AP^+ . Thus if $S = [0, \infty)$, the APW_S^+ and AP_S^+ are APW^+ and AP^+ , respectively.

The Banach algebra APW_S^+ is isomorphic to the Banach subalgebra $\mathcal{A}_{0,S}^+$ of \mathcal{M}^+ consisting of all measures

$$\mu = \sum_{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \dots \in S} f_n \delta_{\{\lambda_n\}},$$

where the sequence of coefficients $(f_n)_{n \geq 0}$ is absolutely summable. The Banach subalgebra $\mathcal{A}_{0,S}^+$ of \mathcal{M}^+ has the property **(P)**. So for all $n \in \mathbb{N}$, $\text{SL}_n(\mathcal{A}_{0,S}^+) = \text{E}_n(\mathcal{A}_{0,S}^+)$, and $\text{SL}_n(\text{APW}_S^+) = \text{E}_n(\text{APW}_S^+)$.

To get the result for AP_S^+ , we again use Theorem 4.1. The algebra APW_S^+ is a dense subset of AP_S^+ . Next we show that APW_S^+ is a full subalgebra of APW_S^+ . Let $f \in \text{APW}_S^+$ be such that there exists a $g \in \text{AP}_S^+$ such that $f \cdot g = \mathbf{1}$, where $\mathbf{1}$ is the constant function taking value 1 everywhere on \mathbb{R} . Taking the Gelfand transform, we obtain $\hat{f} \cdot \hat{g} = \mathbf{1}$, where $\mathbf{1}$ is the constant function taking value 1 everywhere on the maximal ideal space $M(\text{AP}_S^+)$ of AP_S^+ . In particular, \hat{f} does not vanish anywhere on $M(\text{AP}_S^+)$. By the Arens-Singer theorem [1, Theorem 4.1] (see also [4, Theorem 3.1]), the maximal ideal spaces of AP_S^+ and APW_S^+ are homeomorphic. Thus the Gelfand transform of $f \in \text{APW}_S^+$, namely

$$\hat{f} : M(\text{APW}_S^+) \rightarrow \mathbb{C},$$

is also nowhere zero on $M(\text{APW}_S^+)$. By the elementary theory of Banach algebras (see e.g. [9, Theorem 2.7]), it follows that f is invertible as an element of APW_S^+ . So APW_S^+ is a full subalgebra of APW_S^+ . By Theorem 4.1, we conclude that for all $n \in \mathbb{N}$, $\text{SL}_n(\text{AP}_S^+) = \text{E}_n(\text{AP}_S^+)$.

4.7. Nonexample: $A \subset \mathcal{M}^+$ failing (\mathbf{P}) , but $\mathrm{SL}_n(A) = \mathrm{E}_n(A)$.

We show that while the property (\mathbf{P}) is sufficient for the Banach subalgebra $A \subset \mathcal{M}^+$ for having $\mathrm{SL}_n(A) = \mathrm{E}_n(A)$ for all $n \in \mathbb{N}$, it is not necessary. We do this by constructing a Banach subalgebra A of \mathcal{M}^+ for which the property (\mathbf{P}) does not hold, but for which $\mathrm{SL}_n(A) = \mathrm{E}_n(A)$ for all $n \in \mathbb{N}$. This Banach subalgebra was also considered in [24], but here we will show that its maximal ideal space has topological dimension 2, which will be used to show $\mathrm{SL}_n(A) = \mathrm{E}_n(A)$ for all $n \in \mathbb{N}$.

Let dx denote the Lebesgue measure, and let $\mu \in \mathcal{M}^+$ be given by

$$\mu = \delta + \int_0^1 (2x - 3)e^{-x} dx.$$

Then for $s \in \mathbb{C}_{\geq 0}$,

$$\hat{\mu}(s) = \frac{s(s-1)}{(s+1)^2}.$$

For $m \in \mathbb{N}$, we denote the m -fold convolution $\mu * \cdots * \mu$ by μ^{*m} . Let $A \subset \mathcal{M}^+$ be the Banach subalgebra of \mathcal{M}^+ generated by δ, μ , that is, A is the closure of all ‘polynomials’ in μ :

$$p = c_0\delta + c_1\mu + c_2\mu^{*2} + \cdots + c_d\mu^{*d},$$

where d is any nonnegative integer, and $c_0, c_1, \dots, c_d \in \mathbb{C}$ are arbitrary.

Let us first show that A does not have the property (\mathbf{P}) . We show that for a certain t , $\mu_t \notin A$. In fact, set $t = 1 - \frac{1}{e} \in (0, 1)$. Since the polynomials in μ are dense in A , given $\epsilon = \frac{1}{10} > 0$, there exists a $d \in \mathbb{N}$, and complex numbers c_0, c_1, \dots, c_d , such that

$$\|\mu_t - (c_0\delta + c_1\mu + c_2\mu^{*2} + \cdots + c_d\mu^{*d})\| < \epsilon.$$

But if $\nu := \mu_t - (c_0\delta + c_1\mu + c_2\mu^{*2} + \cdots + c_d\mu^{*d})$, then for $s \in \mathbb{C}_{\geq 0}$,

$$|\hat{\mu}_t(s) - (c_0 + c_1\hat{\mu}(s) + c_2(\hat{\mu}(s))^2 + \cdots + c_d(\hat{\mu}(s))^d)| = |\hat{\nu}(s)| \leq \|\nu\| < \epsilon.$$

We have $-\log(1-t) = -\log\frac{1}{e} = 1$, so that the above gives

$$|\hat{\mu}(s+1) - (c_0 + c_1\hat{\mu}(s) + c_2(\hat{\mu}(s))^2 + \cdots + c_d(\hat{\mu}(s))^d)| < \epsilon.$$

Setting $s = 0$, this gives

$$\left| \frac{(0+1)(0+1-1)}{(0+1+1)^2} - c_0 \right| = |c_0| < \epsilon.$$

On the other hand, with $s = 1$, we obtain

$$\left| \frac{(1+1)(1+1-1)}{(1+1+1)^2} - c_0 \right| = \left| \frac{2}{9} - c_0 \right| < \epsilon.$$

Adding the last two inequalities gives the contradiction that

$$\frac{2}{9} \leq \left| \frac{2}{9} - c_0 \right| + |c_0| < 2\epsilon = \frac{2}{10}.$$

Hence A does not have the property (\mathbf{P}) .

Finally, we show that for the Banach algebra A , $\mathrm{SL}_n(A) = \mathrm{E}_n(A)$ for all $n \in \mathbb{N}$. To do this, we will use the result from [3, Theorem 1.1(3)], which says that if the topological dimension $\dim M(R)$ of the maximal ideal space $M(R)$ of a commutative unital complex Banach algebra R is 0, 1 or 2, then $\mathrm{SL}_n(R) = \mathrm{E}_n(R)$. We will show that the topological dimension of the maximal ideal space $M(A)$ of our Banach algebra A is equal to 2. (Recall that for a normal space X , $\dim X \leq d$ if every finite open cover of X can be refined by an open cover whose order $\leq d + 1$. If $\dim X \leq d$ and the statement $\dim X \leq d - 1$ is false, then we say that the *topological dimension of X* is $\dim X = d$. Also, for a commutative complex semisimple unital Banach algebra R , the maximal ideal space $M(R)$ (set of all complex homomorphisms) is equipped with the Gelfand topology, which is the induced subset topology from the dual space $\mathcal{L}(R; \mathbb{C})$ equipped with the weak-* topology.) We will need the following result; see [9, Theorem 1.4, page 68].

Proposition 4.3. *Let B be a finitely generated Banach algebra, generated by b_1, \dots, b_m . Then the joint spectrum of b_1, \dots, b_m in B , namely the set*

$$\sigma_B(b_1, \dots, b_m) = \{(\widehat{b}_1(\varphi), \dots, \widehat{b}_m(\varphi)) : \varphi \in M(B)\} \subset \mathbb{C}^m,$$

is homeomorphic to the maximal ideal space $M(B)$ of the Banach algebra B . (Here $\widehat{\cdot} : B \rightarrow C(M(B); \mathbb{C})$ denotes the Gelfand transform.)

So in our case, it suffices to show that the joint spectrum of δ and μ in A has topological dimension 2. We observe that

$$\begin{aligned} \sigma_A(\delta, \mu) &= \{(1, \widehat{\mu}(\varphi)) : \varphi \in M(A)\} = \{1\} \times \{\widehat{\mu}(\varphi) : \varphi \in M(A)\} \\ &= \{1\} \times \sigma_A(\mu). \end{aligned} \tag{2}$$

Hence it is enough to show that $\sigma_A(\mu) \subset \mathbb{C} \simeq \mathbb{R}^2$ has topological dimension equal to 2. We recall the following result, which relates the spectrum of an element x of a subalgebra of a Banach algebra with the spectrum of x in the Banach algebra; see [22, Theorem 10.18, p. 238].

Proposition 4.4. *Let B be a unital Banach algebra, and S be a Banach subalgebra of B that contains the unit of B . If $x \in S$, then $\sigma_S(x)$ is the union of $\sigma_B(x)$ and a (possibly empty) collection of bounded components of the complement of $\sigma_B(x)$.*

We apply this with $B := \mathbb{C}\delta + L^1[0, +\infty)$ (the Wiener-Laplace algebra), $S := A$, and $x := \mu \in A \subset B = \mathbb{C}\delta + L^1[0, +\infty)$. The maximal ideal space of $B = \mathbb{C}\delta + L^1[0, +\infty)$ can be identified (see e.g. [10, pp. 112-113]) as a topological space with

$$\{s \in (\mathbb{C} \cup \{\infty\}) : \text{and } \mathrm{Re}(s) \geq 0\}.$$

(Here we identify $\mathbb{C} \cup \{\infty\}$ with the Riemann sphere.) The identification is done via the Laplace transform: For

$$\nu = \alpha \delta + \int_0^\infty f(x) dx \in \mathbb{C} \delta + L^1[0, \infty),$$

and $s \in \mathbb{C}_{\geq 0}$,

$$\hat{\nu}(s) := \alpha + \int_0^\infty f(x)e^{-sx} dx, \quad \text{and} \quad \hat{\nu}(\infty) := \alpha.$$

Consequently,

$$\sigma_B(\mu) = \left\{ \frac{s(s-1)}{(s+1)^2} : s \in (\mathbb{C} \cup \{\infty\}), \operatorname{Re}(s) \geq 0 \right\} \stackrel{s = \frac{1+z}{1-z}}{=} \left\{ \frac{z+z^2}{2} : |z| \leq 1 \right\}.$$

It can be shown that this last set is the interior of a simple closed curve C shown as the bold curve in Figure 1. Thus the complement of $\sigma_B(\mu)$ has no bounded components. By Proposition 4.4 we conclude that $\sigma_A(\mu) = \sigma_B(\mu)$. So $\sigma_A(\mu)$ has topological dimension equal to 2. From (2), also $\sigma_A(\delta, \mu) = \{1\} \times \sigma_A(\mu)$. So $\sigma_A(\delta, \mu)$ has topological dimension equal to 2 too. By Proposition 4.3, $\dim M(A)$ is equal to 2.

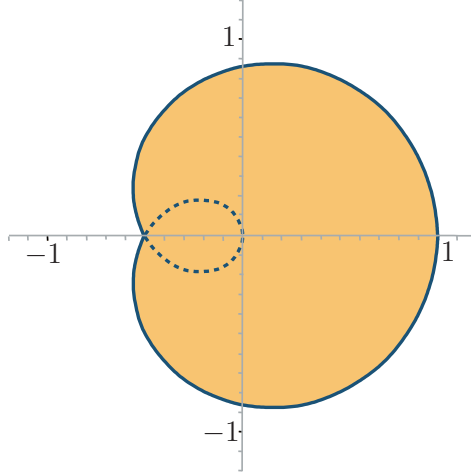


FIGURE 1. The simple closed curve C is depicted by the bold line. The bold curve C , with the dotted curve, is together the curve $\theta \mapsto (e^{i\theta} + e^{2i\theta})/2 : [0, 2\pi] \rightarrow \mathbb{C}$.

4.8. Subalgebras A_d of \mathcal{M}^+ with $\dim M(A_d) = 2d$. In the above, we constructed a Banach sub-algebra A of \mathcal{M}^+ whose maximal ideal space $M(A)$ has topological dimension equal to 2. Now, for each $d \in \mathbb{N}$, we construct a Banach subalgebra A_d of \mathcal{M}_+ such that the topological dimension of their maximal ideal space $M(A_d)$ is $2d$, and for which there also holds $\operatorname{SL}_n(A_d) = \operatorname{E}_n(A_d)$ for all $n \in \mathbb{N}$.

Set $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathbb{D}^d := \mathbb{D} \times \cdots \times \mathbb{D}$ (d times) be the d -dimensional polydisc in \mathbb{C}^d . Every holomorphic $f : \mathbb{D}^d \rightarrow \mathbb{C}$ has a Taylor expansion, and we denote the Taylor coefficients of f by $f_{\mathbf{n}}$:

$$f(\mathbf{z}) = \sum_{\mathbf{n} \in (\mathbb{Z}_{\geq 0})^d} f_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \quad (\mathbf{z} = (z_1, \dots, z_d) \in \mathbb{D}^d),$$

where $\mathbb{Z}_{\geq 0} := \{0, 1, 2, 3, \dots\}$, and we use the usual multi-index notation

$$\mathbf{z}^{\mathbf{n}} = z_1^{n_1} \cdots z_d^{n_d}$$

for $\mathbf{z} = (z_1, \dots, z_d) \in \mathbb{D}^d$ and $\mathbf{n} = (n_1, \dots, n_d) \in (\mathbb{Z}_{\geq 0})^d$. The *Wiener algebra* is the set $W^+(\mathbb{D}^d)$ of all holomorphic $f : \mathbb{D}^d \rightarrow \mathbb{C}$ such that its Taylor coefficients are summable:

$$\|f\|_1 := \sum_{\mathbf{n} \in (\mathbb{Z}_{\geq 0})^d} |f_{\mathbf{n}}| < \infty.$$

With pointwise operations, and the norm $\|\cdot\|_1$, $W^+(\mathbb{D}^d)$ is a Banach algebra. The maximal ideal space $M(W^+(\mathbb{D}^d))$ of $W^+(\mathbb{D}^d)$ is homeomorphic to the the closure $\overline{\mathbb{D}^d}$ of \mathbb{D}^d in \mathbb{C}^d via the map

$$\overline{\mathbb{D}^d} \ni \mathbf{z} \mapsto (W^+(\mathbb{D}^d) \ni f \mapsto f(\mathbf{z}) \in \mathbb{C}) \in M(W^+(\mathbb{D}^d)).$$

Hence $\dim(M(W^+(\mathbb{D}^d))) = 2d$.

Let the set $B = \{e_i : i \in I\}$ be a Hamel basis for the vector space \mathbb{R} over \mathbb{Q} (with vector space addition and scalar multiplication given by the usual arithmetic operations). We can always replace e_i by $-e_i$ to ensure that all the e_i are strictly positive. For $d \in \mathbb{N}$, take any distinct e_{i_1}, \dots, e_{i_d} belonging to B , and set

$$S = \{n_1 e_{i_1} + \cdots + n_d e_{i_d} : n_1, \dots, n_d \in \mathbb{Z}_{\geq 0}\} \subset [0, \infty).$$

Then S is an additive sub-semigroup of the group $(\mathbb{R}, +)$, and the Banach algebra APW_S^+ is isomorphic as a Banach algebra to $W^+(\mathbb{D}^d)$ via the map

$$W^+(\mathbb{D}^d) \ni f = \sum_{\mathbf{n} \in (\mathbb{Z}_{\geq 0})^d} f_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \mapsto \sum_{\mathbf{n} \in (\mathbb{Z}_{\geq 0})^d} f_{\mathbf{n}} e^{i(n_1 e_{i_1} + \cdots + n_d e_{i_d})x} \in \text{APW}_S^+.$$

We have seen that the Banach algebra $\mathcal{A}_{0,S}^+$ is isomorphic as a Banach algebra to APW_S^+ . We take $A_d = \mathcal{A}_{0,S}^+$. We have

$$\dim(M(\mathcal{A}_{0,S}^+)) = \dim(M(\text{APW}_S^+)) = \dim(M(W^+(\mathbb{D}^d))) = 2d.$$

We have already seen that for all $n \in \mathbb{N}$, $\text{SL}_n(\mathcal{A}_{0,S}^+) = \text{E}_n(\mathcal{A}_{0,S}^+)$.

APPENDIX

Proposition 4.5. *Let $\mu \in \mathcal{M}^+$. Then*

$$\lim_{\mathbb{R} \ni s \rightarrow +\infty} \widehat{\mu}(s) = \mu(\{0\}).$$

Proof. First suppose that μ satisfies $\mu(\{0\}) = 0$. Let $\epsilon > 0$. There exists an $r > 0$ such that $|\mu|([0, r]) < \epsilon$, thanks to the assumption that $\mu(\{0\}) = 0$. Let w be a Borel measurable function such that μ has the polar decomposition $d\mu(x) = e^{-iw(x)}d|\mu|(x)$. Then

$$\begin{aligned} |\widehat{\mu}(s)| &= \left| \int_{[0, +\infty)} e^{-sx} e^{-iw(x)} d|\mu|(x) \right| \\ &\leq \left| \int_{[0, r]} e^{-sx} e^{-iw(x)} d|\mu|(x) \right| + \left| \int_{(r, +\infty)} e^{-sx} e^{-iw(x)} d|\mu|(x) \right| \\ &\leq \int_{[0, r]} 1 d|\mu|(x) + e^{-sr} \int_{(r, +\infty)} 1 d|\mu|(x) \leq \epsilon + e^{-sr} |\mu|((r, +\infty)). \end{aligned}$$

So $\limsup_{s \rightarrow +\infty} |\widehat{\mu}| \leq \epsilon$. As ϵ was arbitrary, $\limsup_{s \rightarrow +\infty} |\widehat{\mu}| \leq 0$. As $|\widehat{\mu}(s)| \geq 0$,

$$\lim_{s \rightarrow +\infty} |\widehat{\mu}(s)| = 0, \text{ and } \lim_{s \rightarrow +\infty} \widehat{\mu}(s) = 0.$$

If $\mu(\{0\}) \neq 0$, we complete the proof by considering $\nu := \mu - \mu(\{0\})\delta$, which satisfies $\nu(\{0\}) = 0$ and $\widehat{\nu} = \widehat{\mu} - \mu(\{0\})$. \square

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