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Fractional power series and the method of dominant balances

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This paper derives an explicit formula for a type of fractional power series, known as a Puiseux series, arising in a wide class of applied problems in the physical sciences and engineering. Detailed consideration is given to the gaps which occur in these series ('lacunae'); they are shown to be determined by a number-theoretic argument involving the greatest common divisor of a set of exponents appearing in the Newton polytope of the problem, and by two number-theoretic objects, called here Sylvester sets, which are complements of Frobenius sets. A key tool is Faà di Bruno's formula for high derivatives, as implemented by Bell polynomials. Full account is taken of repeated roots, of arbitrary multiplicity, in the leading-order polynomial which determines a fractional-power expansion, namely the facet polynomial. For high multiplicity, the fractional powers are shown to have large denominators and contain irregularly spaced gaps. The orientation and methods of the paper are those of applications, but in a concluding section we draw attention to a more abstract approach, which is beyond the scope of the paper.

1. Introduction

An approximation method which finds wide application in the physical sciences and engineering is 'balancing as many terms as possible at leading order'. Formally, it is called the method of dominant balances (or distinguished limits), and is expounded in many survey articles and texts on mathematical methods [1–5]. Examples of its use in fluid dynamics, acoustics, and structural vibrations, for example, may be found in [6–10]. To describe the method, let us suppose that a polynomial relation $P(x_1, \ldots, x_d) = 0$ is known to hold between d real or complex quantities $x_1, \ldots, x_d$, where

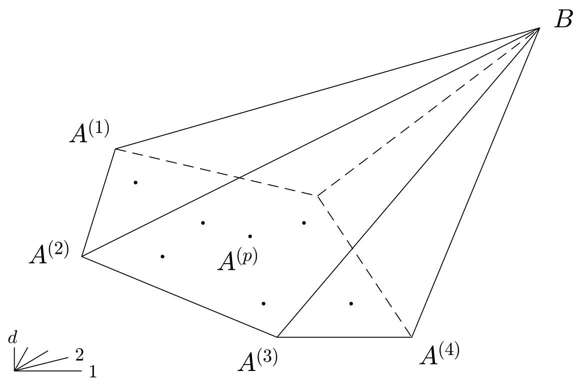

Figure 1. A facet and vertex in d-dimensional exponent space. The facet is the convex hull of $A^{(1)}, \ldots, A^{(p)}$, with coordinate vectors $a^{(1)}, \ldots, a^{(p)}$, and the vertex is B, with coordinate vector b. The facet is a surface of dimension $d-1$, with boundary determined by a subset of $A^{(1)}, \ldots, A^{(p)}$, so that the figure as a whole is a d-dimensional pyramid.

$|x_1| \ll 1$. Full use is assumed to have been made of dynamical similarity, so that all variables and coefficients are dimensionless, i.e. are pure numbers. In applications, a distinction is made between state variables and parameters; here, a uniform notation $x_1, \ldots, x_d$ encompasses both, and x_1 would be a 'small parameter' or the reciprocal of a 'large parameter'. We consider just one polynomial relation, though the method may be extended to a set of simultaneous polynomial relations. The polynomial may be of high degree in any or all of $x_1, \ldots, x_d$, and d may be large.

The aim of the investigation is to exploit the smallness of $|x_1|$ in determining the dependence of x_d on $x_1, \ldots, x_{d-1}$. Let us write the polynomial relation as $P(x) = 0$, with $x = (x_1, \ldots, x_d)$, and put $|x_1| = \eta \ll 1$. The idea of the method of dominant balances is to introduce scalings $|x_j| \sim \eta^{\gamma_j}$, and determine the exponents γ_j so that as many terms as possible in $P(x)$ have the same order of magnitude, as measured by their power of η, while the remaining terms involve only higher powers of η. Here $\gamma_1 = 1$ by definition. This gives an approximation procedure, in which at first only the largest terms in order of magnitude are retained, and then systematically the other terms, to give a series expansion of x_d in powers of η. The result is a fractional power series. In general, there are several dominant balances (many if d is large), giving different series expansions corresponding to different scaling regimes of the roots of $P(x) = 0$.

As just described, the method is classical; indeed, it goes back to Newton. The leading order terms of a dominant balance correspond to facets of the Newton polytope of $P(x)$, defined in a d-dimensional exponent space to be the convex hull of the d-tuples of exponents of the terms in $P(x)$; and the fractional power series obtained are Puiseux series for the roots of $P(x) = 0$ in various asymptotic regimes. A copious literature exists on the subject, much of which cites [11]. Here, we address the two most basic questions which arise when using the method in practice. The first is: which fractional powers actually occur in the complete series? This question is non-trivial, because the exponents, arranged in increasing order, can form a highly irregular sequence of fractions, with unpredictable gaps ('lacunae'), and no obvious pattern. The second question is: how does the form of the series change when the leading-order equation has repeated roots? Examples show that the irregularities just mentioned, and the denominators of the exponents, can become progressively greater as the multiplicity of a repeated root increases.

This paper answers both of the above questions, by giving explicit formulae for the complete fractional power series. The formulae appear to be new, notwithstanding the literature just mentioned, and they involve two number-theoretic objects which are complements of Frobenius sets and which we call Sylvester sets. There is no formula for the size of a Frobenius set, or for the distribution of its elements (except in a few special cases); i.e. its irregularity cannot be described

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by a formula. Thus the reduction to an expression involving Frobenius or Sylvester sets is in general the furthest one can go.

The structure of the paper is that in §§2–5 the theory is developed for a basic type of Newton polytope, derived from one facet and one other point (called a vertex). In §6 the theory is extended to allow for arbitrarily many vertices; this involves number-theoretic ideas, and is where Frobenius and Sylvester sets enter. In §7, full numerical details are given for an example involving a parametric family of octahedral Newton polytopes. The example is rich enough to demonstrate all the possibilities which can occur, but simple enough that the calculations can be carried out explicitly. Readers may find it helpful to refer forward on occasion to this example, as it illustrates the definitions introduced in the paper, and may clarify the nature of the series we are aiming at. Conclusions are presented in §8, together with brief mention of a more algebraic approach to the work, which is beyond the scope of the paper.

2. Geometry of a facet and single vertex

Figure 1 shows schematically, in d-dimensional exponent space, the convex hull of p points $A^{(1)},\ldots,A^{(p)}$ with coordinate vectors $a^{(1)},\ldots,a^{(p)}$, where $p\geqslant d$, lying in an affine hyperplane of dimension $d-1$, but in no subspace of lower dimension, together with a single point B with coordinate vector b, not lying in the hyperplane. We assume that $d\geqslant 2$. The convex hull of the p points is called the facet, and the single point B is called the vertex, so that the configuration defines a d-dimensional pyramid in which the base (i.e. the facet) lies in the affine subspace spanned by $a^{(2)}-a^{(1)},\ldots,a^{(p)}-a^{(1)}$. The corresponding polynomial equation is

$$\sum_{i=1}^{p}\alpha_i x^{a^{(i)}}+\beta x^b=0, \tag{2.1}$$

where the coefficients α_i and β are non-zero scalars, and multi-indices are used for powers, so that $a^{(i)}=(a_1^{(i)},\ldots,a_d^{(i)})$ and $x^{a^{(i)}}=x_1^{a_1^{(i)}}\ldots x_d^{a_d^{(i)}}$. It is assumed that no positive power of any x_j is common to all the terms in (2.1), i.e. there is no root of the form $x_j=0$ for all values of the other variables.

For definiteness, we assume that the facet hyperplane intersects the exponent axes at positive values, and the vertex is on the opposite side of the hyperplane from the origin. Thus the relative positions of the origin, the facet, and vertex are as shown in the figure, and the components of a vector normal to the facet hyperplane, pointing into the half-space containing the vertex, are all positive.

(a) The scaling determined by a facet

Our starting-point is that $|x_1|\ll 1$, and we seek powers of $x_1,\ldots,x_d$ so that the facet terms in (2.1), i.e. the terms with coefficients α_i, have the same order of magnitude, and thereby form a dominant balance. In the geometric approach which follows, it is convenient to write these powers in homogeneous form, by taking

$$x_2\propto x_1^{m_2/m_1},\quad \ldots,\quad x_d\propto x_1^{m_d/m_1}, \tag{2.2}$$

i.e.

$$x_1^{1/m_1}\propto x_2^{1/m_2}\propto\quad\cdots\quad\propto x_d^{1/m_d}, \tag{2.3}$$

in which the m_j are positive real numbers to be determined, up to an arbitrary multiplicative factor, with only their ratios of significance. With this scaling, the facet terms are proportional to powers of x_1 with exponents

$$a_1^{(i)} + \frac{m_2}{m_1} a_2^{(i)} + \cdots + \frac{m_d}{m_1} a_d^{(i)} \qquad (i = 1, \ldots, p). \tag{2.4}$$

The facet terms are said to be balanced if these exponents are equal, i.e.

$$m \cdot a^{(1)} = m \cdot a^{(2)} = \cdots = m \cdot a^{(p)}, \tag{2.5}$$

where $m = (m_1, \ldots, m_d)$. In our use of vector and matrix algebra, vectors are regarded as column vectors, and the scalar product is represented by a dot. On writing (2.5) in the equivalent form

$$m \cdot (a^{(2)} - a^{(1)}) = \cdots = m \cdot (a^{(p)} - a^{(1)}) = 0, \tag{2.6}$$

a geometrical result emerges, that for the facet terms to be in balance, the vector m must be perpendicular to the facet. The homogeneous form of the exponents in (2.2) corresponds to the fact that applying a positive scalar factor to m does not alter its direction, so that it remains perpendicular to the facet. The fact that the numbers m_j are all positive corresponds to the assumed orientation of the facet hyperplane and that m points towards the half-space containing the vertex.

In the three-dimensional case $d = 3$, the vector m is proportional to the vector area of a triangle formed by three points chosen from among $a^{(1)}, \ldots, a^{(p)}$. Three such points can always be found, by our assumption that the facet does not lie in any subspace of lower dimension than $d - 1$, and without loss of generality the three points may be labelled $a^{(1)}, a^{(2)}, a^{(3)}$, so that

$$m = a^{(1)} \wedge a^{(2)} + a^{(2)} \wedge a^{(3)} + a^{(3)} \wedge a^{(1)}. \tag{2.7}$$

In d dimensions, we may likewise, after relabelling if necessary, take $a^{(1)}, \ldots, a^{(d)}$ to be linearly independent (recall that $p \geqslant d$), and construct a $d \times d$ matrix A from their transposes. Then (2.5) is

$$Am = \lambda e, \tag{2.8}$$

where $\lambda \neq 0$ is the common value of the scalar products, and e is a $d \times 1$ column vector of 1's. Thus

$$m = \frac{\lambda \, \mathrm{adj}(A) e}{\det(A)} \propto \mathrm{adj}(A) e \propto \begin{pmatrix} B_{11} \\ \cdots \\ B_{d1} \end{pmatrix} + \cdots + \begin{pmatrix} B_{1d} \\ \cdots \\ B_{dd} \end{pmatrix}, \tag{2.9}$$

where $B = (B_{rs}) = \mathrm{adj}(A)$, the adjugate of A. For $d = 3$, the three vectors on the right-hand side of (2.9) are the vector products in (2.7).

(b) The vertex

On applying the scaling (2.2) to the vertex term βx^b in (2.1), the exponent of x_1 becomes

$$b_1 + \frac{m_2}{m_1} b_2 + \cdots + \frac{m_d}{m_1} b_d, \tag{2.10}$$

or equivalently $(m \cdot b)/m_1$. Let us now divide every term in (2.1) by a common power of x_1, in which the exponent is the common value (2.4) obtained when m satisfies (2.5). Then the facet terms all become of order one, and the vertex term becomes of order x_1^c, where, by (2.10) and (2.5), the exponent c may be written in any of the equivalent ways

$$c = \frac{m \cdot (b - a^{(1)})}{m_1} = \cdots = \frac{m \cdot (b - a^{(p)})}{m_1}. \tag{2.11}$$

A crucial point here is that $c > 0$, by our assumption that the vertex is on the opposite side of the facet hyperplane from the origin. This follows by interpreting (2.11) as projections, taking account of the direction of m.

In three dimensions, expression (2.7) for m suggests that $a^{(1)}, a^{(2)}, a^{(3)}$ is a convenient basis for exponent space. Let square brackets denote the scalar triple product, so that $[a^{(1)}a^{(2)}a^{(3)}]$, for example, denotes a 3×3 determinant, and let $e^{(1)} = (1, 0, 0)$. Then

$$m \cdot a^{(1)} = m \cdot a^{(2)} = m \cdot a^{(3)} = [a^{(1)}a^{(2)}a^{(3)}], \tag{2.12}$$

and with components defined by

$$b = \lambda_1 a^{(1)} + \lambda_2 a^{(2)} + \lambda_3 a^{(3)}, \quad e^{(1)} = \mu_1 a^{(1)} + \mu_2 a^{(2)} + \mu_3 a^{(3)}, \tag{2.13}$$

we obtain at once

$$m \cdot b = (\lambda_1 + \lambda_2 + \lambda_3)[a^{(1)}a^{(2)}a^{(3)}], \quad m_1 = m \cdot e^{(1)} = (\mu_1 + \mu_2 + \mu_3)[a^{(1)}a^{(2)}a^{(3)}], \tag{2.14}$$

so that

$$c = \frac{\lambda_1 + \lambda_2 + \lambda_3 - 1}{\mu_1 + \mu_2 + \mu_3}. \tag{2.15}$$

From (2.13) we also obtain the components as determinant ratios $\lambda_1 = [b\, a^{(2)}a^{(3)}]/[a^{(1)}a^{(2)}a^{(3)}]$, etc, and hence the alternative form

$$c = \frac{[b\, a^{(2)}a^{(3)}] + [a^{(1)}b\, a^{(3)}] + [a^{(1)}a^{(2)}b] - [a^{(1)}a^{(2)}a^{(3)}]}{[e^{(1)}a^{(2)}a^{(3)}] + [a^{(1)}e^{(1)}a^{(3)}] + [a^{(1)}a^{(2)}e^{(1)}]}. \tag{2.16}$$

All the determinants have integer entries. Expression (2.16) is used in the example in §7, because it gives very simply the dependence of c on b.

The result of the scaling is that the facet terms form a dominant balance for equation (2.1), because the vertex term βx^b is smaller than every facet term by a relative amount of order $|x_1|^c$, with c given by the positive rational number (2.11), or (2.16) when $d = 3$. This provides the groundwork for a fractional power series solution of (2.1), which we start to construct in the next section. In figure 1, the exponent c is the distance in the 1-direction from the vertex B to the hyperplane containing the facet. If the expansion were in powers of x_2, for example, then c would be the corresponding distance in the 2-direction. The authors have determined how the formulae in the paper are transformed under a non-singular linear transformation of the variables $x_1, \ldots, x_d$, so that c becomes the distance measured in an arbitrary direction, but we do not present this theory here.

3. Derivation of the facet equation

Our aim is to find a series expansion of x_d in the scaling regime (2.2), and the form of the scaling suggests that we use the quantities $x_2/x_1^{m_2/m_1}, \ldots, x_d/x_1^{m_d/m_1}$. Accordingly, we define scaled versions $(s_1, \ldots, s_{d-1})$ of $(x_1, \ldots, x_{d-1})$ by

$$x_j = s_j x_1^{m_j/m_1} \quad (j = 1, \ldots, d-1) \tag{3.1}$$

and seek a fractional power series for x_d in the form

$$x_d = s_d x_1^{m_d/m_1} + \cdots \quad (|x_1| \ll 1), \tag{3.2}$$

where s_d and all subsequent terms of the series are to be found. From (3.1), we have $s_1 = 1$ by definition, but it is convenient notationally to include s_1 in subsequent formulae. Note that $s_1, \ldots, s_{d-1}$ are to be regarded as given, but not s_d. It is not possible to say at the outset what powers of x_1 will occur in the series (3.2); they must be determined as part of finding the solution of (2.1), in conjunction with finding the coefficient of each power of x_1 which occurs.

The reason for using the scaled variables s_j rather than the original variables x_j is that they are all of order one, so that any set of terms involving x_1 and the s_j can be arranged in order of magnitude simply according to the powers of x_1 they contain. Thus (3.2) is to be regarded as a series in powers of x_1 in which the exponents steadily increase, and the coefficients are functions of $s_1, \ldots, s_{d-1}$. Note also that the ratios $(m_2/m_1, \ldots, m_d/m_1)$ are to be regarded as given; they are determined by (2.9), based on the facet geometry of figure 1 and §2. Only with these exponents do we balance as many terms as possible at leading order, i.e. ensure that all of the facet terms are in balance, and not just some of them. Finally, note that the use of the variables s_j is a means to an end; any formula in $s_1,\dots,s_{d-1}$ and x_1 may be written ultimately in terms of $x_1,\dots,x_{d-1}$ by substituting $s_j = x_j/x_1^{m_j/m_1}$ $(j=1,\dots,d-1)$, so that (3.2) gives a well-ordered series expression for a solution x_d of (2.1) in which each term of the series is a function of $x_1,\dots,x_{d-1}$.

Instead of (3.2), let us write

$$x_d = s_d x_1^{m_d/m_1}(1+z). \tag{3.3}$$

Here $|z| \ll 1$, and the task is to find the fractional power series of z. Substitution of (3.3) into (2.1) leads us to define the facet polynomial

$$F(s_d) = \sum_{i=1}^{p} \alpha_i s_1^{a_1^{(i)}} \dots s_d^{a_d^{(i)}} \tag{3.4}$$

and the vertex polynomial

$$G(s_d) = -\beta s_1^{b_1} \dots s_d^{b_d}, \tag{3.5}$$

in terms of which (2.1) becomes, exactly,

$$F(s_d(1+z)) = G(s_d(1+z))\, x_1^c, \tag{3.6}$$

with c as defined in (2.11). Here $s_d(1+z)$ denotes multiplication, not a function. This equation for z is the main equation used henceforth in solving (2.1). It will be called the facet equation, because its left-hand side, involving F, is specific to the facet with which we started. To clarify the notation, F and G in (3.4) and (3.5) are regarded as functions of the single argument s_d, in which $s_1,\dots,s_{d-1}$ and the coefficients α_i and β held fixed as known constants. The form of (3.6) lends itself to an iterative method, as we shall see in the next section.

4. Solution of the facet equation

(a) Leading-order terms

In the facet equation (3.6), we have $c>0$ and z is to be found as a series of positive fractional powers of x_1. On putting $x_1=0$, we obtain

$$F(s_d) = 0, \tag{4.1}$$

where $F(s_d)$ is the facet polynomial defined by (3.4). Thus (4.1) is of degree $\max_i(a_d^{(i)})$ in s_d, and this is positive, since otherwise we would have $a_d^{(i)}=0$ for all i, and the normal to the facet would be parallel to the d-axis in figure 1, contradicting our orientation assumption. Hence s_d cannot be absent from (4.1). Note also that $s_d \neq 0$, since otherwise (3.3) would imply that x_d is identically zero, in contradiction to our assumption in §2 that the original equation (2.1) does not have any positive powers of a single variable common to all the terms. Equation (4.1) is fully scaled, in that it only involves order-one quantities. Thus for our purposes, we regard it as not capable of further approximation, so that, except in low-order cases, it must be solved numerically.

Higher-order equations are obtained by writing $s_d(1+z) = s_d + s_d z$ and expanding F and G in (3.6) as Taylor series in $s_d z$ about the point s_d, henceforth regarded as a known root of (4.1). This gives

$$s_d F' z + \frac{1}{2} s_d^2 F'' z^2 + \frac{1}{6} s_d^3 F^{(3)} z^3 + \dots = \left(G + s_d G' z + \frac{1}{2} s_d^2 G'' z^2 + \dots\right) x_1^c, \tag{4.2}$$

in which $F(s_d)=0$ has been used on the left-hand side. Functions and their derivatives are assumed evaluated at s_d where this is clear. If the root s_d of (4.1) has multiplicity n, then the complete set of conditions at s_d is

$$F^{(0)}(s_d) = F^{(1)}(s_d) = \cdots = F^{(n-1)}(s_d) = 0, \quad F^{(n)}(s_d) \neq 0, \tag{4.3}$$

where $n \geqslant 1$ and $F^{(0)} \equiv F$. In what follows, all formulae are given for arbitrary multiplicity n of s_d, and the case of a simple root is recovered by taking $n = 1$. Thus the leading terms of (4.2) give $s_d^n F^{(n)} z^n / n! \simeq G x_1^c$, so that

$$z \simeq \frac{1}{s_d} \left(\frac{n!\, G x_1^c}{F^{(n)}} \right)^{\frac{1}{n}}. \tag{4.4}$$

There are n choices for z, corresponding to the nth roots of unity, and the following analysis applies to each of them. Note from (3.5) that if $G(s_d) = 0$, then G is identically zero, and by (3.6) the problem reduces to an equation in F. Henceforth, we therefore assume that $G(s_d) \neq 0$.

(b) Derivation of the recursion relation

From (4.2), the series expansion of z may be written

$$z = z_0 \delta (1 + z_1 \delta + z_2 \delta^2 + \cdots), \tag{4.5}$$

where $\delta = x_1^{c/n}$ and $z_0, z_1, \ldots$ are constants. The series is in integer powers of $x_1^{c/n}$, i.e. is a Taylor series in $x_1^{c/n}$, and it begins with the first power. Here any of the possible values of $x_1^{c/n}$ may be used; when x_1 is real and positive (a common occurrence), the same can be assumed true of δ, because $c > 0$ and $n > 0$. From (4.4), we already have the leading coefficient

$$z_0 = \frac{1}{s_d} \left(\frac{n!\, G}{F^{(n)}} \right)^{\frac{1}{n}}, \tag{4.6}$$

with a suitable choice of nth root, and $z_1, z_2, \ldots$ are obtained by substituting (4.5) into (4.2), followed by equating to zero the coefficients of successive powers of δ. The result is that for $k = 1, 2, \ldots$, the coefficient of δ^k gives an equation of the form

$$\frac{1}{n!} s_d^n F^{(n)} z_0^n \,.\, n z_k = \mathcal{F}_k(z_0, z_1, \ldots, z_{k-1}), \tag{4.7}$$

where the functions $\mathcal{F}_k$ are readily found by hand up to about $k = 3$, and by Maple or Mathematica, for example, up to arbitrary k. Thus $z_1, z_2, \ldots$, can be found recursively from what is in effect a triangular system of equations. By (4.6), an equivalent way of writing (4.7) is

$$n G z_k = \mathcal{F}_k(z_0, z_1, \ldots, z_{k-1}). \tag{4.8}$$

This is a practical form of the recursion equation, and for $k = 1, 2, 3$ gives

$$z_1 = \frac{s_d z_0}{n} \left(\frac{G'}{G} - \frac{1}{n+1} \frac{F^{(n+1)}}{F^{(n)}} \right), \tag{4.9}$$

$$z_2 = \frac{s_d z_0 z_1}{n} \left(\frac{G'}{G} - \frac{F^{(n+1)}}{F^{(n)}} \right) + \frac{s_d^2 z_0^2}{2n} \left(\frac{G''}{G} - \frac{2}{(n+1)(n+2)} \frac{F^{(n+2)}}{F^{(n)}} \right) - \frac{1}{2}(n-1) z_1^2, \tag{4.10}$$

$$\begin{aligned} z_3 = {} & \frac{s_d z_0 z_2}{n} \left(\frac{G'}{G} - \frac{F^{(n+1)}}{F^{(n)}} \right) + \frac{s_d^2 z_0^2 z_1}{n} \left(\frac{G''}{G} - \frac{1}{n+1} \frac{F^{(n+2)}}{F^{(n)}} \right) - \frac{1}{2} s_d z_0 z_1^2 \frac{F^{(n+1)}}{F^{(n)}} \\ & + \frac{s_d^3 z_0^3}{6n} \left(\frac{G^{(3)}}{G} - \frac{6}{(n+1)(n+2)(n+3)} \frac{F^{(n+3)}}{F^{(n)}} \right) - (n-1) z_1 z_2 - \frac{1}{6}(n-1)(n-2) z_1^3. \end{aligned} \tag{4.11}$$

An important fact about (4.7) is that it is linear in z_k, and moreover the factor multiplying z_k is always the same, i.e. is independent of k. Thus in solving (4.7) recursively, zero denominators cannot occur (recall that $G(s_d) \neq 0$), and in fact the value of n was defined so that $F^{(n)}(s_d) \neq 0$. In the recursive solution of (4.7), an extra power of $F^{(n)}(s_d)$ is introduced into the denominators at each step, so that the combined term $z_k \delta^k$ contains within it expressions proportional to

$\{x_1^{c/n}/F^{(n)}(s_d)\}^k$. Thus if $|x_1|^{c/n}$ is small enough compared with $|F^{(n)}(s_d)|$, then not only does the series for z converge, but the convergence is rapid, by comparison with a geometric series.

We may also track the dependence of $z_0, z_1, \dots$ on the coefficient β in the definition (3.5) of G. From (4.6), we have $z_0 \propto \beta^{1/n}$, and a feature of (4.9)–(4.11) is that β cancels from the ratios $G^{(m)}/G$. Thus (4.9) gives $z_1 \propto \beta^{1/n}$, and then (4.10) gives $z_2 \propto \beta^{2/n}$, and so on. This suggests that $z_k \propto \beta^{k/n}$ for $k = 1, 2, \dots$, a result which is readily proved inductively from the Taylor series equation (4.2) with ansatz (4.5). This relation implies that z_k is proportional to an integer power of β if k is a positive multiple of n, a fact we shall need later.

5. Bell symbols and the complete single-vertex series

The method of the previous section gives the early terms in the series for z. However, a more powerful method is available, based on Faà di Bruno's formula for an arbitrary derivative of the composition of two functions, expressed in terms of Bell polynomials [12]. We now show that this method generalises (4.9)–(4.11).

(a) The facet equation

We begin by writing the facet equation (3.6) in the form

$$F(u) = G(u)\,\delta^n, \tag{5.1}$$

where $\delta = x_1^{c/n}$ as before, and u is to be found as the series expansion

$$u = u(\delta) = u_0 + u_1\delta + \frac{1}{2!}u_2\delta^2 + \cdots. \tag{5.2}$$

Thus $u_n = u^{(n)}(0)$, and since $u = s_d(1+z)$ we have from (4.5) the identification

$$u_0 = s_d, \qquad u_1 = s_d z_0, \qquad \frac{u_k}{k!} = s_d z_0 z_{k-1} = u_1 z_{k-1} \quad (k \geqslant 2), \tag{5.3}$$

or equivalently

$$s_d = u_0, \qquad z_0 = \frac{u_1}{s_d}, \qquad z_m = \frac{1}{(m+1)!}\frac{u_{m+1}}{u_1} \quad (m \geqslant 1). \tag{5.4}$$

Hence u-coefficients may be converted to z-coefficients, and vice versa. The aim of this and the following sub-sections is to derive the recursion relation (5.25) for $u_2, u_3, \dots$, from which the complete series (5.2) may readily be calculated.

Substitution of (5.2) into $F(u)$, making use of (4.3) and Faà di Bruno's formula [12], gives the expansion

$$\begin{aligned} F(u) = {} & \frac{1}{n!}\delta^n \left\langle {n \atop 1} \right\rangle F^{(n)} + \frac{1}{(n+1)!}\delta^{n+1}\left(\left\langle {n+1 \atop 2} \right\rangle F^{(n)} + \left\langle {n+1 \atop 1} \right\rangle F^{(n+1)} \right) \\ & + \frac{1}{(n+2)!}\delta^{n+2}\left(\left\langle {n+2 \atop 3} \right\rangle F^{(n)} + \left\langle {n+2 \atop 2} \right\rangle F^{(n+1)} + \left\langle {n+2 \atop 1} \right\rangle F^{(n+2)} \right) + \cdots, \end{aligned} \tag{5.5}$$

in which the derivatives of F are evaluated at $u = u_0$, corresponding to $\delta = 0$, i.e. $z = 0$ by (4.5), and angle-bracket expressions of the general form $\left\langle {r \atop k} \right\rangle$ are functions of $u_1, \dots, u_k$ involving Bell polynomials which we define shortly. There can be no confusion here with Euler symbols, which have the same notation. In the general term of order δ^{n+m} in (5.5), note that u_{m+1} occurs only in the term proportional to $F^{(n)}$, while u_m occurs only in the terms proportional to $F^{(n)}$ and $F^{(n+1)}$, and so on down to u_1, which can occur in all of the terms. This fact is crucial in what follows. Note also that u_0 does not occur in any of the angle-bracket expressions, but only as the argument of F and its derivatives.

(b) The Bell symbol and its properties

The expressions $\left\langle{r \atop k}\right\rangle$ are defined in terms of the partial exponential Bell polynomials $B_{r,s}(u_1, u_2, \dots, u_{r-s+1})$ by

$$\left\langle{r \atop k}\right\rangle = B_{r,r-k+1}(u_1, u_2, \dots, u_k), \tag{5.6}$$

where $r \geqslant 1$ and $1 \leqslant k \leqslant r$. All the properties of Bell polynomials which we use may be found in [12,13]. Indices such as $r, k, s, m, \dots$ are assumed to be non-negative integers. The form (5.6), with the emphasis on simple labelling of the arguments rather than on the degree, appears to be new, and we shall call $\left\langle{r \atop k}\right\rangle$ the Bell symbol. It is homogeneous of degree $r-k+1$. The early symbols, i.e. for k not too large, are

$$\left\langle{r \atop 1}\right\rangle = u_1^r, \qquad \left\langle{r \atop 2}\right\rangle = \binom{r}{2} u_1^{r-2} u_2, \qquad \left\langle{r \atop 3}\right\rangle = \binom{r}{3} u_1^{r-3} u_3 + 3\binom{r}{4} u_1^{r-4} u_2^2, \tag{5.7}$$

$$\left\langle{r \atop 4}\right\rangle = \binom{r}{4} u_1^{r-4} u_4 + 10\binom{r}{5} u_1^{r-5} u_2 u_3 + 15\binom{r}{6} u_1^{r-6} u_2^3, \tag{5.8}$$

$$\left\langle{r \atop 5}\right\rangle = \binom{r}{5} u_1^{r-5} u_5 + 5\binom{r}{6} u_1^{r-6}(3u_2u_4 + 2u_3^2) + 105\binom{r}{7} u_1^{r-7} u_2^2 u_3 + 105\binom{r}{8} u_1^{r-8} u_2^4, \tag{5.9}$$

and the late symbols, i.e. for k at its largest allowed values for given r, are

$$\left\langle{r \atop r-2}\right\rangle = \frac{1}{3!}\sum_{l=1}^{r-2}\binom{r}{l} u_l \left(\sum_{s=1}^{r-1-l}\binom{r-l}{s} u_s u_{r-l-s}\right), \tag{5.10}$$

$$\left\langle{r \atop r-1}\right\rangle = \frac{1}{2!}\sum_{l=1}^{r-1}\binom{r}{l} u_l u_{r-l} = \frac{r!}{2!}\sum_{l=1}^{r-1}\frac{u_l}{l!}\frac{u_{r-l}}{(r-l)!}, \qquad \left\langle{r \atop r}\right\rangle = u_r. \tag{5.11}$$

In using (5.7)–(5.9), note that the Bell symbol $\left\langle{r \atop k}\right\rangle$ is defined only for $r \geqslant k$, and binomial coefficients of the form $\binom{r}{k+s}$ on the right are zero for $k+s>r$; this is taken care of by such relations as $1/(r-s-k)! = 0$ for $k+s>r$ when all quantities are integers [14], or by the formula $\binom{r}{k} = (r)_k/k!$, where $(r)_k$ denotes the falling factorial $r(r-1)\cdots(r-k+1)$.

(c) Recursion relation in Bell symbol form

Analogously to (5.5), we have

$$\begin{aligned}\delta^n G(u) = \delta^n G &+ \frac{1}{1!}\delta^{n+1}\left\langle{1 \atop 1}\right\rangle G^{(1)} + \frac{1}{2!}\delta^{n+2}\left(\left\langle{2 \atop 2}\right\rangle G^{(1)} + \left\langle{2 \atop 1}\right\rangle G^{(2)}\right)\\ &+ \frac{1}{3!}\delta^{n+3}\left(\left\langle{3 \atop 3}\right\rangle G^{(1)} + \left\langle{3 \atop 2}\right\rangle G^{(2)} + \left\langle{3 \atop 1}\right\rangle G^{(3)}\right) + \cdots,\end{aligned} \tag{5.12}$$

where the pattern of Bell symbols is that of (5.5) but with $n=1$. Hence equating successive powers $\delta^n, \delta^{n+1}, \dots$ in the facet equation (5.1) gives

$$\frac{1}{n!}\left\langle{n \atop 1}\right\rangle F^{(n)} = G, \tag{5.13}$$

$$\frac{1}{(n+1)!}\left(\left\langle{n+1 \atop 2}\right\rangle F^{(n)} + \left\langle{n+1 \atop 1}\right\rangle F^{(n+1)}\right) = \frac{1}{1!}\left\langle{1 \atop 1}\right\rangle G^{(1)}, \tag{5.14}$$

$$\frac{1}{(n+2)!}\left(\left\langle{n+2 \atop 3}\right\rangle F^{(n)} + \left\langle{n+2 \atop 2}\right\rangle F^{(n+1)} + \left\langle{n+2 \atop 1}\right\rangle F^{(n+2)}\right) = \frac{1}{2!}\left(\left\langle{2 \atop 2}\right\rangle G^{(1)} + \left\langle{2 \atop 1}\right\rangle G^{(2)}\right), \tag{5.15}$$

and so on. These equations can be written in matrix form involving the lower triangular matrices

$$\begin{pmatrix} \left\langle {n \atop 1} \right\rangle & & & \\ \left\langle {n+1 \atop 2} \right\rangle & \left\langle {n+1 \atop 1} \right\rangle & & \\ \left\langle {n+2 \atop 3} \right\rangle & \left\langle {n+2 \atop 2} \right\rangle & \left\langle {n+2 \atop 1} \right\rangle & \\ \vdots & & & \ddots \end{pmatrix}, \qquad \begin{pmatrix} 1 & & & \\ \left\langle {1 \atop 1} \right\rangle & 0 & & \\ \left\langle {2 \atop 2} \right\rangle & \left\langle {2 \atop 1} \right\rangle & 0 & \\ \vdots & & & \ddots \end{pmatrix} \tag{5.16}$$

and the diagonal matrices

$$\mathrm{diag}\left(\frac{1}{n!}, \frac{1}{(n+1)!}, \dots\right), \qquad \mathrm{diag}\left(1, \frac{1}{1!}, \frac{1}{2!}, \dots\right), \tag{5.17}$$

but this representation will not be used here because for our purposes the unknowns lie in the matrices themselves rather than the column vectors $(F^{(n)}, F^{(n+1)}, \dots)$ and $(G, G^{(1)}, \dots)$. From (5.13) we obtain

$$\left\langle {n \atop 1} \right\rangle = \frac{n!\, G}{F^{(n)}}, \qquad \text{or} \qquad \frac{1}{F^{(n)}} = \frac{\left\langle {n \atop 1} \right\rangle}{n!\, G}, \tag{5.18}$$

and for $n \geqslant 1$ and $m \geqslant 1$ the subsequent equations give

$$\left\langle {n+m \atop m+1} \right\rangle = \binom{n+m}{m+1} \left\langle {n \atop 1} \right\rangle \sum_{l=1}^{m} \left\langle {m \atop l'} \right\rangle \left(\frac{G^{(l)}}{G} - c_{nm}^{l'} \frac{F^{(n+l)}}{F^{(n)}} \right). \tag{5.19}$$

Here $l = 1, 2, \dots, m$ is an increasing index, and $l' = m, m-1, \dots, 1$ is a decreasing index, defined by $l' = m + 1 - l$. The coefficient $c_{nm}^{l'}$ is given by

$$c_{nm}^{l'} = \frac{1}{\binom{n+m}{m}} \frac{\binom{n+m}{l'}}{\left\langle {n \atop 1} \right\rangle \left\langle {m \atop l'} \right\rangle}. \tag{5.20}$$

(d) Recursion relation in Taylor coefficient form

The next stage is to extract the variable u_{m+1} from the left-hand side of (5.19), because the right-hand side involves only $u_1, \dots, u_m$ and the result will be a recursion relation for u_{m+1}. It is convenient to define a reduced Bell symbol $\left\langle {r \atop k} \right\rangle'$ to denote what is left in $\left\langle {r \atop k} \right\rangle$ when the term containing u_k removed. From (5.6)–(5.9), this gives

$$\left\langle {r \atop 1} \right\rangle = u_1^r + \left\langle {r \atop 1} \right\rangle', \qquad \left\langle {r \atop k} \right\rangle = \binom{r}{k} u_1^{r-k} u_k + \left\langle {r \atop k} \right\rangle' \quad (k \geqslant 2), \tag{5.21}$$

where $\left\langle {r \atop 1} \right\rangle' = 0$, $\left\langle {r \atop 2} \right\rangle' = 0$, and, perhaps unexpectedly, $\left\langle {r \atop r} \right\rangle' = 0$. Note that $\left\langle {r \atop k} \right\rangle'$ can depend only on $u_1, \dots, u_{k-1}$. Some other reduced symbols are

$$\left\langle {r \atop 3} \right\rangle' = 3\binom{r}{4} u_1^{r-4} u_2^2, \qquad \left\langle {r \atop 4} \right\rangle' = 10\binom{r}{5} u_1^{r-5} u_2 u_3 + 15\binom{r}{6} u_1^{r-6} u_2^3, \tag{5.22}$$

and

$$\left\langle {r \atop r-1} \right\rangle' = \frac{1}{2!} \sum_{l=2}^{r-2} \binom{r}{l} u_l u_{r-l} = \frac{r!}{2!} \sum_{l=2}^{r-2} \frac{u_l}{l!} \frac{u_{r-l}}{(r-l)!}. \tag{5.23}$$

The most useful form of the above equations is

$$\left\langle {n+m \atop m+1} \right\rangle = \binom{n+m}{m+1} \left\langle {n \atop 1} \right\rangle \frac{u_{m+1}}{u_1} + \left\langle {n+m \atop m+1} \right\rangle' \qquad (m \geqslant 1). \tag{5.24}$$

Substitution of this expression into (5.19) gives

$$\frac{u_{m+1}}{u_1} = -\frac{1}{\left\langle {n \atop 1} \right\rangle} \frac{\left\langle {n+m \atop m+1} \right\rangle'}{\left\langle {n+m \atop m+1} \right\rangle} + \frac{m+1}{n} \sum_{l=1}^{m} \left\langle {m \atop l'} \right\rangle \left(\frac{G^{(l)}}{G} - c_{nm}^{l'} \frac{F^{(n+l)}}{F^{(n)}} \right), \tag{5.25}$$

with l' and $c_{nm}^{l'}$ defined as before. Recall that on the right-hand side here, the reduced symbol $\left\langle {n+m \atop m+1} \right\rangle'$ does not depend on u_{m+1}. The value of u_1 is known from (5.18), since $\left\langle {n \atop 1} \right\rangle = u_1^n$; hence $u_1 = (n!\, G/F^{(n)})^{1/n}$, and so (5.25) gives all subsequent u_k recursively. As in §4, the coefficient β in the definition of G cancels out in terms of the form $G^{(l)}/G$, and an inductive argument using (5.25) shows that $u_k \propto \beta^{k/n}$ for $k = 1, 2, \ldots$, equivalent to the previous relation $z_k \propto \beta^{k/n}$. Equation (5.25) is the main result of this section. It gives a practical way of calculating $u_2, u_3, \ldots$ as far as required.

For small and medium m, the right-hand side of (5.25) may quickly be evaluated using the formulae in (5.7)–(5.9) for the Bell symbols, and for arbitrary n and m the first few coefficients $c_{nm}^{l'}$ are

$$c_{nm}^{1} = \binom{n+m}{m}^{-1}, \qquad c_{nm}^{2} = \binom{n+m-2}{m-2}^{-1} = \frac{(m-2)!}{(n+m-1)_{m-2}}, \tag{5.26}$$

and

$$c_{nm}^{3} = \binom{n+m-3}{m-3}^{-1} \left(\frac{1 + \frac{3}{4}(n+m-3)u_2^2/(u_1u_3)}{1 + \frac{3}{4}(m-3)u_2^2/(u_1u_3)} \right). \tag{5.27}$$

The recursion relation (5.25) then gives

$$\frac{u_2}{u_1} = \frac{2}{n} u_1 \left(\frac{G'}{G} - \frac{1}{n+1} \frac{F^{(n+1)}}{F^{(n)}} \right), \tag{5.28}$$

$$\frac{u_3}{u_1} = \frac{3}{n} \left\{ u_2 \left(\frac{G'}{G} - \frac{F^{(n+1)}}{F^{(n)}} \right) + u_1^2 \left(\frac{G''}{G} - \frac{2}{(n+2)_2} \frac{F^{(n+2)}}{F^{(n)}} \right) \right\} - \frac{3}{4}(n-1)\frac{u_2^2}{u_1^2}, \tag{5.29}$$

and

$$\begin{aligned} \frac{u_4}{u_1} = \frac{4}{n} \Bigg\{ & u_3 \left(\frac{G^{(1)}}{G} - \left(1 + \frac{3n}{4} \frac{u_2^2}{u_1u_3} \right) \frac{F^{(n+1)}}{F^{(n)}} \right) + 3u_1u_2 \left(\frac{G^{(2)}}{G} - \frac{1}{n+1} \frac{F^{(n+2)}}{F^{(n)}} \right) \\ & + u_1^3 \left(\frac{G^{(3)}}{G} - \frac{6}{(n+3)_3} \frac{F^{(n+3)}}{F^{(n)}} \right) \Bigg\} - 2(n-1)\frac{u_2u_3}{u_1^2} - \frac{1}{2}(n-1)(n-2)\frac{u_2^3}{u_1^3}. \end{aligned} \tag{5.30}$$

When $u_1, u_2, \ldots$ are written in terms of $z_0, z_1, \ldots$ by means of (5.3), these expressions are found to agree with (4.9)–(4.11). This is a powerful check of the formalism we have developed.

(e) Single-vertex series

The results of this section give a complete solution to the problem of finding the fractional power-series solutions of the single-vertex equation (2.1), and we now summarise the main steps involved. Each root s_d of the facet equation (4.1) gives rise to a series of the form (5.2), where $\delta = x_1^{c/n}$ and c is determined by the geometrical argument leading to (2.11); the positive integer n is the multiplicity of the root s_d, and gives rise to n choices for δ, corresponding to the different branches of $x_1^{c/n}$, so that the total number of series obtained from all the roots s_d is the degree of the facet polynomial $F(s_d)$ defined in (3.4). The first two coefficients $u_0, u_1, \ldots$ in (5.2) are given by $u_0 = s_d$ and $u_1 = (n!\, G/F^{(n)})^{1/n}$, evaluated at s_d, and the remaining coefficients are determined by the recursion relation (5.25), which we have been able to write compactly by introducing a new notation, the Bell symbol, in Faà di Bruno's formula (5.5) for an arbitrary derivative of the composition of two functions.

The next stage is the multi-vertex extension of (2.1). This requires new ideas, because the regularity of the series (5.2) is destroyed by the addition of extra vertices.

6. Multi-vertex series

(a) Derivation of the facet equation

Instead of the single vertex B in figure 1, we now take q vertices $B^{(1)}, \ldots, B^{(q)}$ with coordinate vectors $b^{(1)}, \ldots, b^{(q)}$. The configuration of facet plus vertices is represented by the polynomial equation

$$\sum_{i=1}^{p} \alpha_i x^{a^{(i)}} + \sum_{r=1}^{q} \beta_r x^{b^{(r)}} = 0, \tag{6.1}$$

where the coefficients β_r are non-zero scalars and $x^{b^{(r)}} = x_1^{b_1^{(r)}} \ldots x_d^{b_d^{(r)}}$; as before, we assume that the terms in (6.1) do not all contain a common monomial factor. The vertices are on the opposite side of the facet hyperplane from the origin, so that the facet is part of the Newton polytope of of (6.1).

There are now q vertex polynomials

$$G_r(s_d) = -\beta_r s_1^{b_1^{(r)}} \ldots s_d^{b_d^{(r)}} \qquad (r = 1, \ldots, q), \tag{6.2}$$

and the facet equation (5.1) becomes

$$F(u) = \sum_{r=1}^{q} G_r(u)\, x_1^{c_r}, \tag{6.3}$$

where c_r is defined by (2.11) with $b^{(r)}$ instead of b. The vectors m and $a^{(1)}, \ldots, a^{(p)}$ in (2.11) are the same as before, because the facet polynomial $F(s_d)$ is unchanged by the addition of out-of-facet vertices.

The exponents $c_1, \ldots, c_q$ are positive and rational. They have a greatest common divisor (gcd) c defined so that $c_r = c g_r$ where the g_r are co-prime positive integers, i.e.

$$\gcd(g_1, \ldots, g_q) = 1. \tag{6.4}$$

Although c is rational, it is not in general an integer, and it may have a large denominator. Without loss of generality, we assume that $c_1 = \min_r(c_r)$, so that $g_1 = \min_r(g_r)$. As before, we take s_d to be a root of multiplicity n of the equation $F(s_d) = 0$. We also define $\delta = x_1^{c/n}$, which reduces to the previous definition when $q = 1$ because one may then take $g_1 = 1$ and $c = c_1$. With $u = s_d(1+z)$ as before, the facet equation (6.3) expressed in terms of δ is

$$F(u) = \sum_{r=1}^{q} G_r(u)\delta^{n g_r} = \left\{ \sum_{r=1}^{q} G_r(u)\delta^{n(g_r - g_1)} \right\} \delta^{n g_1}. \tag{6.5}$$

(b) Solution of the facet equation

The solution u of equation (6.5) has a series expansion

$$u = u(\delta) = u_0 + u_1\delta + \frac{1}{2!}u_2\delta^2 + \cdots \tag{6.6}$$

with δ as just defined. However, in general a very large number of the coefficients u_k are zero, with no obvious pattern. The aim of what follows is to determine which coefficients are non-zero.

Our approach is first to give (6.5) a formally identical structure to the single vertex equation (5.1) by writing it as

$$F(u) = G(u)\,\tilde{\delta}^n, \tag{6.7}$$

where $\tilde{\delta} = \delta^{g_1} = x_1^{c g_1/n} = x_1^{c_1/n}$ and

$$G(u) = G(u, \delta) = \sum_{r=1}^{q} G_r(u)\delta^{n(g_r - g_1)}. \tag{6.8}$$

We now proceed in two stages. The first is to regard (6.7) as determining a series expansion in $\tilde{\delta}$ of the form

$$u = \tilde{u}_0 + \tilde{u}_1\tilde{\delta} + \frac{1}{2!}\tilde{u}_2\tilde{\delta}^2 + \cdots, \tag{6.9}$$

in which the coefficients $\tilde{u}_k$ depend on δ as a parameter, and $\tilde{u}_0 = u_0$. Thus the $\tilde{u}_k$ are determined by the formulae of §5, with

$$G^{(s)} = \sum_{r=1}^{q} G_r^{(s)}(u)\delta^{n(g_r-g_1)} \qquad (s = 0, 1, \ldots), \tag{6.10}$$

evaluated at $u = u_0 = s_d$. The second stage is to expand the $\tilde{u}_k$ in powers of δ and group (6.9) into increasing powers of δ, to obtain the series (6.6).

In carrying out the above procedure, the key question is: which powers of δ occur in the coefficients $\tilde{u}_k$? To answer this question, it is convenient to define

$$\eta_r = \delta^{n(g_r-g_1)} \qquad (r = 1, \ldots, q). \tag{6.11}$$

Here $\eta_r = 1$ for those r for which $g_r = g_1$, but this does not affect the argument which follows. An inductive argument based on (5.25) shows that if k is a multiple of n, say $k = ln$ where $l = 0, 1, \ldots$, then $\tilde{u}_k$ is a polynomial of degree l in the η_r; but if k is not a multiple of n, then all non-negative powers of the η_r occur in $\tilde{u}_k$. This is a consequence of the observation made after (5.25) that $u_k \propto \beta^{k/n}$ in the single-vertex problem, where β is the coefficient appearing in G in (3.5).

(c) Sylvester sets

We now determine the powers of δ appearing in $\tilde{u}_k\tilde{\delta}^k$, first when k is a multiple of n, and second when it is not, where n is the multiplicity of the root s_d of the facet polynomial (3.4).

(i) Powers divisible by the multiplicity

We take $k = ln$ where $l = 0, 1, \ldots$. Then the powers of δ in $u_k\tilde{\delta}^k$ are of the form

$$n\Big[lg_1 + \sum_{r=2}^{q} l_r(g_r - g_1)\Big] \tag{6.12}$$

where the l_r are non-negative integers satisfying $\sum_{r=2}^{q} l_r \leqslant l$. Here the terms for which $g_r = g_1$ are zero. When $q = 1$, a sum over r from 2 to q is always taken to be zero. The total coefficient of g_1 in the expression in brackets is $l - \sum_{r=2}^{q} l_r \geqslant 0$. It follows that the union over all l of powers of the form (6.12) is

$$\Big\{n\sum_{r=1}^{q} l_r g_r : \ l_r \in \mathbb{N}_0\Big\}, \tag{6.13}$$

where $\mathbb{N}_0$ is the set of non-negative integers, i.e. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let us define the Sylvester set $\mathcal{S}$ by

$$\mathcal{S} = \mathrm{Syl}(g_1, \ldots, g_q), \tag{6.14}$$

where Syl denotes the set of linear combinations, with non-negative integer coefficients, of a set of co-prime non-negative integers as arguments. Then (6.13) is the set $n\mathcal{S}$, where a factor multiplying a set is to be applied to all the elements of the set. Thus we have shown that, in the series (6.9), if we retain all the terms $\tilde{u}_k\tilde{\delta}^k$ for which k is a multiple of n, and express them in terms of δ, then the complete set of powers of δ occurring in these terms is $n\mathcal{S}$, where $\mathcal{S}$ is the Sylvester set (6.14).

A Sylvester set is the complement in $\mathbb{N}_0$ of a Frobenius set, defined as the set of positive integers which cannot be expressed as a linear combination of the type we have specified. The literature on Frobenius sets is large: this area of mathematics began with [15], and the standard reference is now [16]; two other relevant works are [17,18]. A Frobenius set is finite, and except in simple cases (e.g. $q = 2$ or special functional forms of g_r such as arithmetic or geometric progressions) there is no formula for its size or the distribution of its elements, which is irregular. This means that the set $\mathcal{S}$ defined by (6.14) is irreducible, in the sense that we cannot hope in general to find a simpler representation of it than the one given, and its elements do not have a pattern.

Since a Frobenius set is finite, it follows that a Sylvester set includes all the integers from some point on. Thus although the early elements of $\mathcal{S}$ are irregular in their distribution and spacing, the later elements are not only regular but complete, in that beyond a certain point there are no more gaps.

(ii) Powers not divisible by the multiplicity

We now turn to the powers of δ appearing in $\tilde{u}_k\tilde{\delta}^k$ when k is not a multiple of n. These are of the form

$$l_1' g_1 + n\sum_{r=2}^{q} l_r(g_r - g_1), \tag{6.15}$$

where $l_1', l_2, \ldots, l_q \in \mathbb{N}_0$, subject only to the restriction that l_1' is not a multiple of n. As before, when $q = 1$, the sum over r is taken to be zero. The most general l_1' is of the form $l_1 n + t$, where $l_1 \in \mathbb{N}_0$ and $t = 1, \ldots, n-1$.

Therefore the set of powers (6.15) may be written

$$\mathcal{T} = g_1\mathcal{I}_{n-1} \oplus n\mathcal{R}, \tag{6.16}$$

where $\mathcal{I}_{n-1} = \{1, 2, \ldots, n-1\}$ and $\mathcal{R}$ is the Sylvester set

$$\mathcal{R} = \mathrm{Syl}(g_1, g_2 - g_1, \ldots, g_q - g_1). \tag{6.17}$$

(In the special case $n = 1$ we define $\mathcal{I}_0 = \emptyset$, the empty set.) Thus in the series (6.9), if we retain all the terms $\tilde{u}_k\tilde{\delta}^k$ for which k is not a multiple of n, then the complete set of powers of δ occurring in these terms is $\mathcal{T}$.

The symbol $\oplus$ denotes the Minkowski sum, i.e. all possible sums of an element of the first set and an element of the second set. Note that $\mathcal{A} \oplus \mathcal{B}$ does not necessarily contain $\mathcal{A}$ or $\mathcal{B}$, and $n\mathcal{A} \neq \mathcal{A} \oplus \cdots \oplus \mathcal{A}$. If $\mathcal{A}$ or $\mathcal{B}$ is the empty set, then so is $\mathcal{A} \oplus \mathcal{B}$. Thus the formulae in this subsection are vacuous when $n = 1$, because then $\mathcal{T} = \emptyset$. The Sylvester sets $\mathcal{S}$ and $\mathcal{R}$ satisfy the inclusion relation $\mathcal{S} \subseteq \mathcal{R}$, because the total multiple of g_1 in $\mathcal{R}$ can be negative, so long as the sum of the coefficients multiplying $g_1, \ldots, g_q$ is non-negative. In (6.16) we may extract a common factor $a = \gcd(n, g_1)$. On writing $n = a\tilde{n}$ and $g_1 = a\tilde{g}_1$, this gives $\mathcal{T} = a\tilde{\mathcal{T}}$, where

$$\tilde{\mathcal{T}} = \tilde{g}_1\mathcal{I}_{n-1} \oplus \tilde{n}\mathcal{R}. \tag{6.18}$$

(iii) Complete set of powers

The set of powers of δ in the solution u of a problem of type (6.5) will be called a Puiseux set and written

$$\mathcal{P} = \mathrm{Pu}(g_1, \ldots, g_q|n). \tag{6.19}$$

That is, if the series (6.9) is expanded into the form (6.6), then $\mathcal{P}$ is the set of integers k for which the coefficient of δ^k is generically non-zero. Thus (i) and (ii) above establish the explicit formula

$$\mathcal{P} = n\mathcal{S} \cup \mathcal{T} = n\mathcal{S} \cup a\tilde{\mathcal{T}} = a\tilde{\mathcal{P}}, \tag{6.20}$$

where the reduced Puiseux set is defined by

$$\tilde{\mathcal{P}} = \tilde{n}\mathcal{S} \cup \tilde{\mathcal{T}}. \tag{6.21}$$

Equation (6.20) is the main result of the paper, and is believed to be new.

(d) Properties of Puiseux sets

Puiseux sets may be classified into three types, according to the values of n and a. The first type is defined by $n = 1$, the second by $n > 1$, $a = 1$, and the third by $n > 1$, $a > 1$. We shall call these simple, irreducible, and reducible Puiseux sets, respectively.

(i) Simple Puiseux sets

These sets correspond to a simple root of the facet equation. Since $n = 1$, we have $\mathcal{I}_{n-1} = \emptyset$, and so $\mathcal{T} = \emptyset$, whence $\mathcal{P} = \mathcal{S} = \mathrm{Syl}(g_1, \ldots, g_q)$. Although this appears elementary, recall that $\mathcal{S}$ is in general a complicated object, because of the irregularity of a general Frobenius set based on the q integers $g_1, \ldots, g_q$. However, the irregularity ultimately disappears, because $\mathcal{S}$ includes all the integers from some point on. Symbolically, we may write $\mathcal{S} \xrightarrow{L} \mathbb{N}$ and $\mathcal{P} \xrightarrow{L} \mathbb{N}$, in which the symbol L stands for 'late terms' or 'late elements'. Generally, if $\mathcal{A}$ and $\mathcal{B}$ are any two infinite sets of integers, the notation $\mathcal{A} \xrightarrow{L} \mathcal{B}$ will be used to mean that if each set is arranged in non-decreasing order, then from some point on the two sets have identical elements. If the set $\mathcal{A}$ is finite, it is convenient to write $\mathcal{A} \xrightarrow{L} \emptyset$.

(ii) Irreducible Puiseux sets

The facet equation now has a multiple root ($n > 1$), but n and g_1 are co-prime ($a = \gcd(n, g_1) = 1$). Thus the distinction between $(\mathcal{P}, \mathcal{T}, \mathcal{R})$ and $(\tilde{\mathcal{P}}, \tilde{\mathcal{T}}, \tilde{\mathcal{R}})$ disappears, and also $n\mathcal{S} \cap \mathcal{T} = \emptyset$. No element of $\mathcal{T}$ is divisible by n, and the late behaviour is

$$n\mathcal{S} \xrightarrow{L} n\mathbb{N}, \qquad \mathcal{T} \xrightarrow{L} \mathbb{N} \setminus n\mathbb{N}, \qquad \mathcal{P} \xrightarrow{L} \mathbb{N}. \tag{6.22}$$

Thus although the early behaviour of $\mathcal{P}$ is very different from that for a simple root, the late behaviour is the same.

(iii) Reducible Puiseux sets

Now $a > 1$ and $n > 1$, and the elements of $(n\mathcal{S}, n\mathcal{R}, \mathcal{P}, \mathcal{T})$ are all divisible by a. We therefore work with the reduced sets $(\tilde{n}\mathcal{S}, \tilde{n}\mathcal{R}, \tilde{\mathcal{P}}, \tilde{\mathcal{T}})$, in which $n = a\tilde{n}$ and $g_1 = a\tilde{g}_1$. We always have $\tilde{n}\mathcal{S} \cap \tilde{\mathcal{T}} \neq \emptyset$, and hence $n\mathcal{S} \cap \mathcal{T} \neq \emptyset$, in contrast to the irreducible case. The late behaviour of the reduced sets is

$$\tilde{n}\mathcal{S} \xrightarrow{L} \tilde{n}\mathbb{N}, \qquad \tilde{\mathcal{T}} \xrightarrow{L} \mathbb{N}, \qquad \tilde{\mathcal{P}} \xrightarrow{L} \mathbb{N}, \tag{6.23}$$

and hence for the original sets

$$n\mathcal{S} \xrightarrow{L} n\mathbb{N}, \qquad \mathcal{T} \xrightarrow{L} a\mathbb{N}, \qquad \mathcal{P} \xrightarrow{L} a\mathbb{N}. \tag{6.24}$$

The relation $\mathcal{P} = n\mathcal{S} \cup \mathcal{T}$ implies that $\mathcal{P}$ may be decomposed into the three disjoint sets $n\mathcal{S} \cap \mathcal{T}$, $n\mathcal{S} \setminus \mathcal{T}$, and $\mathcal{T} \setminus n\mathcal{S}$. From (6.24), their late behaviour is

$$n\mathcal{S} \cap \mathcal{T} \xrightarrow{L} n\mathbb{N}, \qquad n\mathcal{S} \setminus \mathcal{T} \xrightarrow{L} \emptyset, \qquad \mathcal{T} \setminus n\mathcal{S} \xrightarrow{L} a\mathbb{N} \setminus n\mathbb{N}. \tag{6.25}$$

The second of these relations is the statement that $n\mathcal{S} \setminus \mathcal{T}$ is finite; this set can never be empty, because $n\mathcal{S}$ contains 0, but $\mathcal{T}$ does not. The third relation implies that $\mathcal{T} \setminus n\mathcal{S}$ may or may not be finite, according to whether $a = n$ or $a < n$; both are possible, depending on whether $\tilde{n} = 1$ or $\tilde{n} > 1$. The finite case occurs, for example, if $n = 2$ and $g_1 = 6$, for which $a = 2$.

7. Example: a parametric family of octahedra

In figure 1 for three dimensions ($d = 3$) let us take a triangular facet, with $A^{(1)}$ in the (23) plane, $A^{(2)}$ in the (31) plane, and $A^{(3)}$ in the (12) plane. The facet is otherwise arbitrary, i.e. is specified by six integer parameters. We also place three vertices on the axes, i.e. $B^{(1)}$ on the 1 axis, $B^{(2)}$ on the 2 axis, $B^{(3)}$ on the 3 axis; for the analysis of this paper, they must be on the opposite side of the facet hyperplane from the origin. This gives three more integer parameters. Thus in the notation of (6.1), we obtain the polynomial

$$\alpha_1 x_2^{a_2^{(1)}} x_3^{a_3^{(1)}} + \alpha_2 x_1^{a_1^{(2)}} x_3^{a_3^{(2)}} + \alpha_3 x_1^{a_1^{(3)}} x_2^{a_2^{(3)}} + \beta_1 x_1^{b_1^{(1)}} + \beta_2 x_2^{b_2^{(2)}} + \beta_3 x_3^{b_2^{(3)}} = 0, \tag{7.1}$$

for which the Newton polytope belongs to a nine-parameter family of octahedra. This is readily checked visually, as there are eight triangular faces, namely $A^{(1)}A^{(2)}A^{(3)}$, $B^{(1)}B^{(2)}B^{(3)}$, and six further faces connecting these two triangles.

We have found that the family (7.1) includes all the types of fractional power series which can arise, and we regard it as a suitable canonical example for study. In this paper we have restricted ourselves to series expansions in positive powers of x_1, but this is purely for ease of exposition. If $b_1^{(1)}$, $b_2^{(2)}$, and $b_3^{(3)}$ are allowed to take unrestricted positive integer values, then expansions in negative powers of x_1 also arise, for example when the points $B^{(1)}$, $B^{(2)}$, $B^{(3)}$ are on the side of the facet hyperplane containing the origin. Moreover, degenerate cases can be studied, for example if any or all of $B^{(1)}$, $B^{(2)}$, $B^{(3)}$ lie in the facet hyperplane.

In the notation of (6.1), we have

$$a^{(1)} = (0, a_2^{(1)}, a_3^{(1)}), \quad a^{(2)} = (a_1^{(2)}, 0, a_3^{(2)}), \quad a^{(3)} = (a_1^{(3)}, a_2^{(3)}, 0), \tag{7.2}$$

and $b^{(r)} = b_r^{(r)} e^{(r)}$, where $e^{(r)}$ is a unit vector in the r-direction. Then by (2.16), the exponents in (6.3) are

$$c_r = \frac{b_r^{(r)}\big([e^{(r)}a^{(2)}a^{(3)}] + [a^{(1)}e^{(r)}a^{(3)}] + [a^{(1)}a^{(2)}e^{(r)}]\big) - [a^{(1)}a^{(2)}a^{(3)}]}{[e^{(1)}a^{(2)}a^{(3)}] + [a^{(1)}e^{(1)}a^{(3)}] + [a^{(1)}a^{(2)}e^{(1)}]}. \tag{7.3}$$

Here the last determinant in the numerator has just two terms,

$$[a^{(1)}a^{(2)}a^{(3)}] = a_2^{(1)}a_3^{(2)}a_1^{(3)} + a_3^{(1)}a_1^{(2)}a_2^{(3)}, \tag{7.4}$$

and the other determinants have one term each, for example $[e^{(3)}a^{(2)}a^{(3)}] = a_1^{(2)}a_2^{(3)}$ and $[a^{(1)}e^{(2)}a^{(3)}] = -a_3^{(1)}a_1^{(3)}$.

To illustrate the theory in the paper, a suitable choice of (7.1) is

$$\alpha_1 x_2 x_3^3 + \alpha_2 x_1^3 x_3^2 + \alpha_3 x_1 x_2^3 + \beta_1 x_1^{b_1^{(1)}} + \beta_2 x_2^{b_2^{(2)}} + \beta_3 x_3^{b_2^{(3)}} = 0, \tag{7.5}$$

for which the direction (2.7) of the facet is $m = (5, 8, 7)$ and

$$c_1 = \frac{5b_1^{(1)} - 29}{5}, \qquad c_2 = \frac{8b_2^{(2)} - 29}{5}, \qquad c_3 = \frac{7b_3^{(3)} - 29}{5}. \tag{7.6}$$

Thus for $B^{(1)}$, $B^{(2)}$, $B^{(3)}$ to be on the opposite side of the facet hyperplane from the origin, the smallest possible values of $(b_1^{(1)}, b_2^{(2)}, b_3^{(3)})$ are $(6, 4, 5)$. A rich set of examples can be constructed with $b_1^{(1)}$, $b_2^{(2)}$, and $b_3^{(3)}$ in this range, demonstrating all the possibilities listed in §6(*d*).

We shall analyse in detail just one example, $(b_1^{(1)}, b_2^{(2)}, b_3^{(3)}) = (7, 6, 6)$, for which $(c_1, c_2, c_3) = (6/5, 19/5, 13/5)$. Thus the greatest common divisor of the c_r is $c = 1/5$, and we have $c_r = cg_r$ with $(g_1, g_2, g_3) = (6, 19, 13)$; by definition of c, we have $\gcd(g_1, g_2, g_3) = 1$. The last three terms of (7.5) are $\beta_1 x_1^7 + \beta_2 x_2^6 + \beta_3 x_3^6$. Since our notation is such that $c_1 = \min_r(c_r)$, and hence $g_1 = \min_r(g_r)$, these terms would need to be re-labelled in general, but it is not necessary here.

The dominant-balance scaling relation (2.2) is $x_j \propto x_1^{m_j/m_1}$, $(j = 1, 2, 3)$, and since $m = (5, 8, 7)$, this gives $x_2 \propto x_1^{8/5}$ and $x_3 \propto x_1^{7/5}$ (the scaling for $j = 1$ is an identity). For the next stage, it is essential to remember that the ultimate aim of the investigation is to obtain an expansion for x_3 in terms of x_1 and x_2 in the scaling regime represented by the dominant balance; the expansion is to be for $|x_1| \ll 1$. To this end, we define scaled quantities as follows. First, we put $u = x_3/x_1^{7/5}$, and aim to solve for u, an order-one quantity by construction. Second, we define $s_2 = x_2/x_1^{8/5}$. The point here is that if we work with s_2 instead of x_2, then in any expansion in powers of x_1 the coefficients, which are now functions of s_2, are all of order one, and therefore the expansion is well-ordered. Without this scaling step, the expansion would be disordered, and so would not be useful. Third, we put $s_1 = 1$. Strictly speaking, this is not necessary, but it is notationally convenient to regard the definition $s_j = x_j / x_1^{m_j/m_1}$ as holding for $j = 1$ as well as $j = 2$, and then necessarily $s_1 = 1$.

On using the above definitions in (7.5) with $(b_1^{(1)}, b_2^{(2)}, b_3^{(3)}) = (7, 6, 6)$, the result is

$$\alpha_1 s_2 u^3 \ + \ \alpha_2 s_1^3 u^2 \ + \ \alpha_3 s_1 s_2^3 \ = \ -\beta_1 s_1^7 x_1^{6/5} - \beta_2 s_2^6 x_1^{19/5} - \beta_3 u^6 x_1^{13/5}. \tag{7.7}$$

This is the facet equation (6.3), in which the facet polynomial is

$$F(u) = \alpha_1 s_2 u^3 \ + \ \alpha_2 s_1^3 u^2 \ + \ \alpha_3 s_1 s_2^3 \tag{7.8}$$

and the vertex polynomials are

$$G_1(u) = -\beta_1 s_1^7, \quad G_2(u) = -\beta_2 s_2^6, \quad G_3(u) = -\beta_3 u^6. \tag{7.9}$$

The next stage is to seek a series solution u of (7.7) in which the first term u_0 is a root of the facet polynomial, i.e. $F(u_0) = 0$, and the series is in fractional powers of x_1. Although (7.7) is a sextic equation in u, the facet polynomial is only a cubic and so may be written exactly in the Taylor series form

$$F(u) = F'(u_0)(u - u_0) + \frac{1}{2} F''(u_0)(u - u_0)^2 + \frac{1}{6} F^{(3)}(u_0)(u - u_0)^3, \tag{7.10}$$

where

$$F'(u_0) = 3\alpha_1 s_2 u_0^2 + 2\alpha_2 s_1^3 u_0, \quad F''(u_0) = 6\alpha_1 s_2 u_0 + 2\alpha_2 s_1^3, \quad F^{(3)}(u_0) = 6\alpha_1 s_2. \tag{7.11}$$

On casting the left-hand side of (7.7) into the form (7.10), it can be seen that the nature of the series for u depends on the multiplicity n of the root u_0, because this determines which derivative in (7.10) is the first to be non-zero. The derivatives given in (7.11) show in detail how this multiplicity depends on the coefficients of the facet polynomial, including the value of s_2. The analysis of §6 shows that a suitable expansion variable is $\delta = x_1^{c/n} = x_1^{1/(5n)}$, and then only integer powers of δ occur in the series for u. Thus u has a Taylor series in δ, and it may be constructed explicitly by means of the recursion relation (5.25).

We shall consider two cases in detail, the simple root ($n = 1$), for which $F'(u_0) \neq 0$, and the double root ($n = 2$), for which $F'(u_0) = 0$, but $F''(u_0) \neq 0$. In the first case, we put $\delta = x_1^{1/5}$, and in the second case $\delta = x_1^{1/10}$, so that (7.7) becomes

$$\alpha_1 s_2 u^3 \ + \ \alpha_2 s_1^3 u^2 \ + \ \alpha_3 s_1 s_2^3 = \begin{cases} -\beta_1 s_1^7 \delta^6 - \beta_2 s_2^6 \delta^{19} - \beta_3 u^6 \delta^{13} & (n = 1) \\ -\beta_1 s_1^7 \delta^{12} - \beta_2 s_2^6 \delta^{38} - \beta_3 u^6 \delta^{26} & (n = 2). \end{cases} \tag{7.12}$$

In both cases the expansion of u is of the form

$$u = u_0 + u_1 \delta + \frac{1}{2!} u_2 \delta^2 + \cdots, \tag{7.13}$$

but the Puiseux sets are different. Recall that the Puiseux set $\mathcal{P}$ for the series solution (7.13) is the set of integers s for which the coefficient of δ^s is generically non-zero; see the explicit formula (6.20). The long-run properties of the constituent sets determining $\mathcal{P}$ are listed in (6.22)–(6.25). In the next two subsections we give complete details for the two cases. The key parameters are (n, g_1), their greatest common divisor $a = \gcd(n, g_1)$, and the reduced quantities $(\tilde{n}, \tilde{g_1})$ defined by $n = a\tilde{n}_1$ and $g_1 = a\tilde{g_1}$. Recall that $(g_1, g_2, g_3) = (6, 19, 13)$, and $\mathbb{N}_0$ denotes the set of non-negative integers (i.e. it includes 0).

(a) The simple root

We have $n = 1$, and so $a = 1$. The case is covered by §6*d*(*i*), and $\mathcal{P} = \mathcal{S} = \mathrm{Syl}(6, 19, 13)$. Thus $\mathcal{P}$ is the set of all linear combinations of 6, 19, and 13 with non-negative integer coefficients, and is the complement in $\mathbb{N}_0$ of the Frobenius set $\mathcal{F} = \mathrm{Fr}(6, 19, 13)$. The first few elements of $\mathcal{S}$ are $\{0, 6, 12, 13, 18, 19, 24, 25, \ldots\}$ and the last few elements of $\mathcal{F}$ are $\{\ldots, 35, 40, 41, 46, 47, 53, 59\}$.

Note that if the g_r are not too small, it often happens that the early elements of $\mathcal{S}$ are quite sparse, but the gaps decrease (though irregularly); conversely, the later elements of $\mathcal{F}$ tend to be sparse, and this set is always bounded. Here, $\mathcal{F}$ contains 30 elements altogether, and the largest element, i.e. the Frobenius number, is 59. Thus for $n=1$, the series (7.13) contains all the powers of $\delta = x_1^{1/5}$ from δ^{60} onwards, and does not contain the power δ^{59}.

These results have been checked in Mathematica by substituting the series (7.13), up to the term in δ^{80}, into the first of equations (7.12), and using the inbuilt series commands to solve for the coefficients up to u_{80}. The non-zero coefficients match exactly those corresponding to the Sylvester set. We suggest that this verification may serve a useful purpose in the development of specialised computer algebra code in the area.

(b) The double root

Now $n=2$, and so $a=\gcd(2,6)=2$. This gives the more complex case §6d(iii), a reducible Puiseux set, and $(\tilde{n}, \tilde{g_1}) = (1,3)$. The set $\mathcal{S}$ is the same as for the single root, and $\tilde{\mathcal{T}}$, defined by (6.18), contains all the integers from 33 onwards; it contains 15 integers less than this, beginning with $\{3, 9, 10, 15, \ldots, \}$ and ending with $\{\ldots, 28, 29, 30, 31\}$. The reduced Puiseux set $\tilde{\mathcal{P}}$, defined by (6.21), contains all the integers from 21 onwards, and twelve integers less than this, from $\{0, 3, 6, 9, 10, \ldots\}$ to $\{\ldots, 17, 18, 19\}$. From (6.20), the Puiseux set is $\mathcal{P} = 2\tilde{\mathcal{P}}$, which therefore consists of all the even numbers from 42 onwards, together with twelve even numbers less than this, beginning with $\{0, 6, \ldots\}$ and ending with $\{\ldots, 36, 38\}$.

Let us check the last two of the relations (6.25). The first of these states that $n\mathcal{S} \setminus \mathcal{T}$ is always finite, and the second implies that $\mathcal{T} \setminus n\mathcal{S}$ is finite if $a=n$, which is the case here, because $a=n=2$. We find that $n\mathcal{S} \setminus \mathcal{T}$ contains 9 terms, namely $\{0, 12, 24, 26, 36, 38, 50, 52, 64\}$, and $\mathcal{T} \setminus n\mathcal{S}$ contains 21 terms, from $\{6, 18, 20, \ldots\}$ to $\{\ldots, 94, 106, 118\}$, confirming the relations. The set $n\mathcal{S} \cap \mathcal{T}$ is infinite, because it must contain all the even numbers from some point on, in fact from 120 (it does not contain 118). This is in sharp contrast to the multiple-root case when n and g_1 are co-prime ($a=1$), when necessarily $n\mathcal{S} \cap \mathcal{T} = \emptyset$, as described in §6$d$($ii$) for irreducible Puiseux sets.

These results too have been checked using a Mathematica code based on (7.13) and the second of (7.12), but no theory. All the results are confirmed up to high order.

8. Conclusions and further work

The results in this paper have been derived by elementary methods starting from the original polynomial equation, using an equal mix of geometry, algebra, and number theory. It has been shown that with these methods, very complete results can be obtained for a single equation of arbitrary degree, number of variables, and multiplicity at leading order. These results are summarised in the explicit formulae (6.20)–(6.21) for the Puiseux set $\mathcal{P}$ and its reduced form $\tilde{\mathcal{P}}$, which encapsulate the work of the paper and are its most important result.

The method of dominant balances applies to many problems in mathematical physics, including a wide range in both ordinary and partial differential equations [5,7]. A natural extension of the results obtained here would be to a system of simultaneous polynomial equations. However, this would almost certainly involve advanced methods from pure mathematics which are beyond the scope of the paper. The algebraic geometry of the Newton polytope, including its application to Puiseux expansions, has been developed extensively starting from the seminal papers [19,20], and now includes the theory of tropical geometry [21,22]. Our strategy has been to approach the area from the point of view of dominant balance as developed in physical applications, demonstrating how to obtain the expansions using a perturbation technique, but also drawing on some of the geometric ideas in these papers. We see opportunities to combine our methods with computational algebra technology to further contribute to, and perhaps help unify, a branch of symbolic-numerics of considerable scientific importance.

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