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# When does portfolio compression reduce systemic risk?

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**Abstract**

We analyze the consequences of portfolio compression for systemic risk. Portfolio compression is a post-trade netting mechanism that reduces gross positions while keeping net positions unchanged and it is part of the financial legislation in the United States (Dodd–Frank Act) and in Europe (European Market Infrastructure Regulation). We derive necessary structural conditions for portfolio compression to be harmful and discuss policy implications. We show that any potential harmfulness of portfolio compression arises from contagion effects. We show how portfolio compression affects systemic risk depends on the resilience of nodes taking part in compression, on the proportion of debt that they can repay, and on the recovery rates in case of default. In particular, the potential danger of portfolio compression comes from defaults of firms that conduct portfolio compression. If no defaults occur among the firms that engage in compression, then portfolio compression always reduces systemic risk.

**KEYWORDS**

cycles, financial networks, netting, portfolio compression, systemic risk

**JEL codes**

D85, G01, G28, G33

## 1 INTRODUCTION

Portfolio compression is a mechanism in which multiple offsetting contracts are replaced by fewer contracts to reduce the gross exposure of each institution while keeping its net exposure unchanged. Reducing gross exposure is beneficial for a wide range of reasons including complying with regulatory requirements such as the minimum leverage ratio introduced under the Basel III regulation and margin requirements (Duffie, 2017). The new contracts that replace the old contracts, however, lead to a new network structure of exposures between the market participants. It is not clear, a priori, what the consequences for systemic risk are—this is what we analyze here.

The main contribution of this paper is to derive general theoretical results on the consequences of portfolio compression for systemic risk. To the best of our knowledge, these considerations have been absent from the literature so far. The European Securities and Markets Authority (ESMA) has published a consultation paper in 2020, see European Securities and Market Authority (2020), on post-trade risk reduction services (PTRR) of which portfolio compression is an important example. They ask: "Would you agree with the description of the benefits (i.e., reduced risks) derived from PTRR services? Are there any missing? Could PTRR services instead increase any of those risks? Are there any other risks you see involved in using PTRR services?" Hence, there remains uncertainty about the risks of PTRR services such as portfolio compression.

In this paper, we derive structural conditions for portfolio compression to be harmful or to reduce systemic risk in a sense that we will formally define in Definition 3.11. First, we show that portfolio compression cannot cause new fundamental defaults (Proposition 4.6). Hence, any potential harmfulness of portfolio compression arises from contagion effects. Theorem 4.7 contains the main results. It establishes a relationship between systemic risk and the financial resilience of those nodes taking part in compression, the proportion of debt that they can repay and recovery rates in case of default. In particular, it states that as long as only nodes that would not default in the non-compressed system conduct portfolio compression, portfolio compression always reduce systemic risk. Furthermore, if every node that defaults in the compressed network, can still repay at least the same proportion of its debt in the compressed network as in the non-compressed network, then portfolio compression reduces systemic risk. Furthermore, under zero recovery rates portfolio compression always reduces systemic risk. We derive and discuss more fine-tuned results as well. We also show both theoretically and in numerical case studies that there are situations under which portfolio compression can indeed be harmful.

We will proceed as follows. In Section 2, we introduce the theoretical model for the financial market and formally define what we refer to as *portfolio compression* (Definitions 2.1 and 2.3) building on D'Errico and Roukny (2021). We only consider portfolio compression that (potentially repeatedly) removes one cycle from a network and also consider an optimization framework for portfolio compression in this context. In Section 3, we explain how we measure systemic risk. We use the framework by Veraart (2020), which generalizes the approaches by Eisenberg and Noe (2001) and Rogers and Veraart (2013). We assess how different types of payments obligations associated with derivative positions arise and might lead to loss cascades if a node fails to satisfy their payment obligations. Our analysis includes variation margins becoming due building on the models by Paddrik et al. (2020) and Ghamami et al. (2021). Section 4 contains all results on the consequences of portfolio compression for systemic risk. Theorem 4.7 is the main result, providing structural conditions that are necessary for portfolio compression to be harmful. We discuss these conditions in detail, derive some additional results that contribute to our understanding of the consequences of portfolio compression and use them to discuss policy implications. In Section 4.6, we illustrate the results using some example networks. Section 5 concludes.

## 1.1 | Policy framework and related literature

Understanding the consequences of portfolio compression for systemic risk is of fundamental importance since it is used both in Europe under the European Market Infrastructure Regulation on derivatives, central counterparties and trade repositories (EMIR), and in the United States under the Dodd–Frank Wall Street Reform and Consumer Protection Act (Dodd–Frank Act). Under EMIR, portfolio compression is one of the risk mitigation mechanisms for non-centrally cleared OTC derivative contracts (European Union, 2012). In the United States, portfolio compression is used as a risk management tool in the swap market (Commodities Futures Trading Commission, 2012).

Portfolio compression plays an important role in today's financial markets. It is performed by private providers—a well-known one is the company TriOptima. It states that "OTC derivative market participants have eliminated more than $973 trillion in notional principal through April 2017" (TriOptima, 2017). Portfolio compression is currently available for "cleared and uncleared interest rate swaps in 28 currencies, cross currency swaps, credit default swaps (CDS), FX forwards, and commodity swaps," (TriOptima, 2017). Its compression service *triReduce* is currently used by over 260 institutions worldwide.

Regulatory reforms such as the Basel III minimum leverage ratio provide strong incentives for market participants to engage in compression activities, see, for example, Duffie (2017). The Basel III leverage ratio is defined as tier 1 capital divided by the total exposure. Since compression reduces the exposure, compression increases the leverage ratio making it easier to satisfy the lower bound, see also Haynes and McPhail (2021) for further discussions and Remark A.1.

The introduction of margin requirements (also for non-centrally cleared derivatives, see BCBS IOSCO (2015, 2020)) provides incentives for market participants to engage in portfolio compression since lower total exposures are associated with both lower initial margin and also lower variation margin requirements (Duffie, 2017). Despite margin requirements, risk of contagion in derivative markets remains as demonstrated empirically in Paddrik et al. (2020) and Bardoscia et al. (2019).

The literature on regulatory reforms and their consequences for systemic risk in the derivatives markets has mainly focused on the role of central counterparties, see, for example, Duffie and Zhu (2011); Cont and Kokholm (2014). Amini et al. (2016b) analyzed different netting mechanisms but not in the context of portfolio compression in centrally cleared markets.

The literature on portfolio compression is still in its infancy. In particular, it mainly focuses on the actual algorithms to perform the portfolio compression rather than the potential consequences of portfolio compression. O' Kane (2017) proposes and analyzes different multilateral netting algorithms. He shows that an algorithm based on the $L_1$-norm is particularly beneficial for eliminating a high fraction of bilateral connections and for retaining the greatest common divisor of existing positions.

D'Errico and Roukny (2021) introduce different types of portfolio compression mechanisms to show theoretically how the size of over-the-counter markets can be reduced without affecting the net positions of the market participants. In addition, they show empirically the large potential for compression to reduce exposure size using a transaction-level data set for CDS derivatives. They do not use any risk measures to study the effect of compression on systemic risk.

Schuldenzucker et al. (2018) provide one example that shows that portfolio compression can be harmful for the system but they do not provide any general results.

Duffie (2018) used ideas from portfolio compression in his new auction mechanism (*compression auctions*) that convert centrally cleared contracts on the London Interbank Offer Rate to contracts on the Secured Overnight Financing Rate.

Building on the work presented here, Amini and Feinstein (2021) recently introduced an optimization problem for portfolio compression that uses a systemic risk measure in the objective function. They analyzed the related computational complexity and showed that such an optimization problem is generically NP-hard.

# 2 | PORTFOLIO COMPRESSION

## 2.1 | The network of liabilities

We consider a financial network consisting of $N$ financial institutions with indices in $\mathcal{N} = \{1, 2, \ldots, N\}$ representing the nodes. We denote by $X_{ij}$ the nominal liability of financial institution $i$ to financial institution $j$ and write $X = (X_{ij}) \in [0, \infty)^{N \times N}$ for the corresponding *liabilities matrix*. Furthermore, we assume that $X_{ii} = 0$ for all $i \in \mathcal{N}$, that is, nodes do not have liabilities to themselves. The set of edges is given by $\mathcal{E} = \{(i, j) \in \mathcal{N}^2 \mid X_{ij} > 0\}$.

The liabilities can arise due to entering into derivative contracts such as interest rate swaps or CDS, see, for example, Schuldenzucker et al. (2018) [1]. We assume that all these positions are fungible. Our framework would also apply to other types of liabilities such as interbank lending.

We assume that all contracts are established at time $t = 0$ and have the same maturity date $t = T > 0$. A generalization to the situation with multiple maturities in the spirit of Kusnetsov and Veraart (2019) would also be possible.

We denote by $\bar{L}_i^{(X)} = \sum_{j=1}^N X_{ij}$ the total nominal liabilities of node $i$ and write $\bar{L}^{(X)} \in [0, \infty)^N$ for the vector of total liabilities arising within the network. Similarly, we write $\bar{A}_i^{(X)} = \sum_{j=1}^N X_{ji}$ for the total assets of financial institution $i$ from within the network and write $\bar{A}^{(X)} \in [0, \infty)^N$ for the vector of these total assets. Then we refer to $\bar{A}^{(X)} + \bar{L}^{(X)}$ as gross positions in the network and to $\eta = \bar{A}^{(X)} - \bar{L}^{(X)} \in \mathbb{R}^N$ as net positions in the network.

## 2.2 | Defining portfolio compression

Portfolio compression is a mechanisms that nets trades between two or more counterparties such that the net positions stay the same for all nodes but the gross positions decrease for all market participants. We only consider a method of compression that would be referred to as *conservative portfolio compression* in D'Errico and Roukny (2021). Intuitively, conservative compression is a mechanism that eliminates cycles in networks.

Figure 1 provides an example of a network consisting of four nodes in which three perform multilateral portfolio compression by reducing their exposures along a cycle.

**Definition 2.1** (Portfolio compression). Consider a liabilities matrix $X \in [0, \infty)^{N \times N}$ with corresponding nodes $\mathcal{N} = \{1, \ldots, N\}$ and edges $\mathcal{E} = \{(i, j) \in \mathcal{N}^2 \mid X_{ij} > 0\}$.

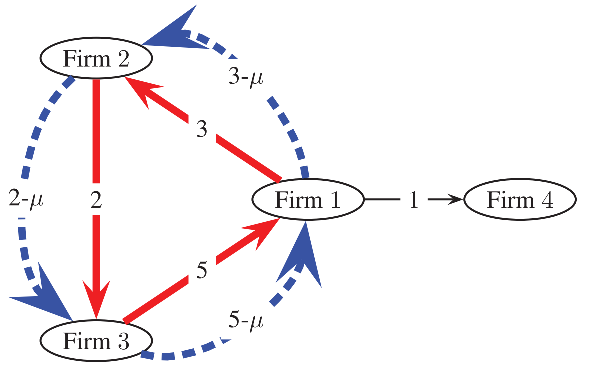



**FIGURE 1** Example of compressing a cycle in a network: the cycle in red with solid lines is replaced by the cycle in blue with dashed lines where $\mu \in (0, 2]$ [Color figure can be viewed at wileyonlinelibrary.com]

1. A *cycle* is a sequence of distinct vertices $\mathcal{C}_{\text{nodes}} = \{i_1, \ldots, i_n\} \subseteq \mathcal{N}$ with $n \leq N$ with $(i_\nu, i_{\nu+1}) \in \mathcal{E}$ for all $\nu \in \{1, \ldots, n-1\}$ and $(i_n, i_1) \in \mathcal{E}$. We denote the corresponding set of edges by $\mathcal{C}_{\text{edges}} = \{(i_1, i_2), \ldots, (i_{n-1}, i_n), (i_n, i_1)\}$ and write $\mathcal{C} = (\mathcal{C}_{\text{nodes}}, \mathcal{C}_{\text{edges}})$.
2. Let $\mathcal{C}_{\text{nodes}}$ be a cycle and let $\mathcal{C}_{\text{edges}}$ be the corresponding set of edges such that

$$\mu^{\max} = \min_{(i,j)\in\mathcal{C}_{\text{edges}}} X_{ij} > 0. \tag{1}$$

   We then refer to $\mathcal{C} = (\mathcal{C}_{\text{nodes}}, \mathcal{C}_{\text{edges}}, \mu^{\max})$ as a *conservative compression network cycle of* $X$ with maximal capacity $\mu^{\max}$.
3. Let $\mathcal{C} = (\mathcal{C}_{\text{nodes}}, \mathcal{C}_{\text{edges}}, \mu^{\max})$ be a conservative compression network cycle of $X$ with maximal capacity $\mu^{\max}$. For any $0 < \mu \leq \mu^{\max}$, we refer to the matrix $X^{\mathcal{C},\mu}$ with

$$X_{ij}^{\mathcal{C},\mu} = \begin{cases} X_{ij} - \mu & \text{if } (i,j) \in \mathcal{C}_{\text{edges}}, \\ X_{ij} & \text{otherwise,} \end{cases} \tag{2}$$

   as the *$\mu$-compressed liabilities matrix* (using cycle $\mathcal{C}$). We refer to $\bar{L}_i^{(X),\mathcal{C},\mu} = \sum_{j\in\mathcal{N}} X_{ij}^{\mathcal{C},\mu}$ as the *total $\mu$-compressed nominal obligations* of node $i$ (using cycle $\mathcal{C}$).

Conservative compression network cycles may or may not exist for a given liability matrix. Throughout this paper, we assume that for any liabilities matrix that we analyze, there exists at least one conservative compression network cycle.

We show in Lemma B.1 in the Appendix that conservative compression does indeed reduce gross positions while keeping net positions fixed. Portfolio compression reduces the size of the balance sheet of a participating node by reducing its total assets and its total liabilities by $\mu$. The resulting net worth, that is, the difference between total assets and total liabilities, remains the same.

*Remark* 2.2 (Compression tolerances). In practice firms provide so-called *compression tolerances* to the compression provider (D'Errico and Roukny, 2021). These are restrictions on the changes that can be made to the original positions. Mathematically, they can be characterized by requiring that any new position $X_{ij}^{\mathcal{C},\mu}$ that might replace the original position $X_{ij}$ needs to satisfy

$$a_{ij} \leq X_{ij}^{\mathcal{C},\mu} \leq b_{ij}, \tag{3}$$

for some $0 \leq a_{ij} \leq b_{ij}$, see (D'Errico and Roukny, 2021, Definition 3). For example, if a firm does not want to change a particular position it could set $a_{ij} = b_{ij} = X_{ij}$. In such a case, we could just set the corresponding $\mu^{\max} = 0$.

Since we only consider conservative compression in this paper, we will assume that $b_{ij} = X_{ij}$ for all $i, j \in \mathcal{N}$. This in particular implies that no new edges can be created as part of a compression exercise (since if $X_{ij} = 0$ then under this assumption $X_{ij} = b_{ij} = 0$). It might be the case that a node does not want to remove an edge completely but only wants to reduce the weight of an edge, that is, this would correspond to setting $a_{ij} > 0$ as lower bound for the weight of this particular edge. It is clear from the definition of $\mu^{\max}$ that if one sets $\mu = \mu^{\max}$ in a compression exercise then at least one edge (and possibly more) would be removed. To avoid this, one could change the definition of $\mu^{\max}$ by setting $\widetilde{\mu^{\max}} = \min_{(i,j)\in C_{\text{edges}}}(X_{ij} - a_{ij}) > 0$. Obviously, $\widetilde{\mu^{\max}} \leq \mu^{\max}$. Since all our results will hold for all choices of $\mu \in (0, \mu^{\max}]$ they in particular hold for all $\mu \in (0, \widetilde{\mu^{\max}}]$. Therefore, there is no need for us to explicitly add such constraints in our analysis.

These compression tolerances are not considered to be functions of any other parameters of the network and are restrictions on the individual positions. Neither D'Errico and Roukny (2021) nor O' Kane (2017) indicate any global constraints when considering the actual compression mechanism. We will come back to this when we discuss systemic risk in compressed networks.

## 2.3 | Portfolio compression as an optimization problem

Next, we consider conservative portfolio compression as an optimization problem as described in D'Errico and Roukny (2021). Its objective is to minimize the total gross exposures of all nodes participating in the compression exercise while satisfying some constraints.

**Definition 2.3** (Conservative compression optimization problem). Let $X \in [0, \infty)^{N\times N}$ be a liability matrix. We refer to the following optimization problem as the *conservative compression optimization problem*. It is given by

$$\min_{\tilde{X}_{ij}, i,j\in\mathcal{N}} \sum_{i=1}^{N}\sum_{j=1}^{N} \tilde{X}_{ij}, \tag{4}$$

subject to

$$\sum_{j=1}^{N}(\tilde{X}_{ji} - \tilde{X}_{ij}) = \sum_{j=1}^{N}(X_{ji} - X_{ij}) \quad \forall i \in \mathcal{N}, \tag{5}$$

$$0 \leq \tilde{X}_{ij} \leq X_{ij} \quad \forall i, j \in \mathcal{N}. \tag{6}$$

This is a linear programming problem and can be solved using standard methods. Since $\tilde{X} = X$ satisfies both constraints ((5) and (6)), a feasible solution to these constraints exists. Furthermore, since the constraint set is bounded, it is clear that a solution exists.

Constraint (5) says that the net exposures of the nodes are not allowed to change by compression. Constraint (6) ensures that the compression is indeed conservative. Since in the new network, the value of any edge is between 0 and its original value, this means that this type of compression can only reduce liabilities along existing edges but cannot create new edges. As pointed out in D'Errico and Roukny (2021), the resulting network is therefore a subnetwork of the original one.

One could replace the condition (6) by a condition representing the compression tolerance of the market participants by, for example, requiring that

$$a_{ij} \leq \tilde{X}_{ij} \leq b_{ij} \quad \forall i, j \in \mathcal{N}, \text{for } 0 \leq a_{ij} \leq b_{ij}. \tag{7}$$

Setting, for example $a_{ij} = 0$ and $b_{ij} = +\infty$ would correspond to nonconservative compression, see D'Errico and Roukny (2021), which would in principle allow for the underlying network to be rewired. Therefore new trading relationships could be established, which is not possible under conservative compression. As argued in D'Errico and Roukny (2021, Section 3.1.2), setting $a_{ij} = 0$ and $b_{ij} = X_{ij}$ for all $i, j \in \mathcal{N}$, is "arguably close to the way most compression cycles take place in derivatives markets. The authors thank Per Sjöberg, founder and former CEO of TriOptima, for fruitful discussion on these particular points." We will therefore focus on this setting.

*Remark* 2.4 (Relationship of solution to optimization problem and compressing one cycle). As shown in D'Errico and Roukny (2021, Proposition 7), a solution to the optimization problem in Definition 2.3 is a *directed acyclic graph*, that is a graph that does not contain any cycles. In particular, one can obtain a solution by repeatedly compressing along one compression network cycle, see D'Errico and Roukny (2021, EC.5., e-companion). As discussed there it will matter in which order this is done, since it is possible that some edges are part of several cycles. An algorithm for choosing the ordering is given in D'Errico and Roukny (2021, Algorithm 2, e-companion). They always choose $\mu = \mu^{\max}$ in each compression step.

## 3 | MEASURING SYSTEMIC RISK

We consider different types of payment obligations that arise from the network of liabilities $X$ and describe how we measure the systemic risk associated with them.

### 3.1 | Payment obligations, margins, and liquidity buffer

We assume that all payment obligations that arise from the original liabilities matrix $X$ can be expressed by using a suitable function $f^V$ that maps the original liabilities matrix $X$ into payment obligations $L = f^V(X)$.

**Definition 3.1** (Payment function and payment obligation matrix). Let $X \in [0, \infty)^{N \times N}$ be a liabilities matrix. Consider a function $f^V : [0, \infty)^{N \times N} \to [0, \infty)^{N \times N}$ where $f^V(x) = Vx$ for $V \in [0, \infty)$, $x \in [0, \infty)^{N \times N}$. We refer to $f^V$ as *payment function*.

We define a matrix $L = (L_{ij}) \in [0, \infty)^{N\times N}$ where each element $L_{ij} = f^V_{ij}(X) = VX_{ij}$ represents the payments that are due from $i$ to $j$, $i, j \in \mathcal{N}$ at a given point in time. We refer to $L$ as *payment obligation matrix*. We refer to $\bar{L} \in [0, \infty)^N$, where $\bar{L}_i = \sum_{j=1}^{N} L_{ij}$, as the *total payment obligations*.

Payment obligations can in principle arise throughout the lifetime of the contract. We restrict our analysis to only one point in time at which payments are due. In principle, our analysis could be extended to allow for multiple points in times at which payments are due.

If we set $V = 1$ in the definition of $f^V$, then $L = X$ and hence the payments due are the original liabilities.[2] In practice, this does not need to be the case. Payment obligations can for example arise from variation margins becoming due (BCBS IOSCO, 2015; Paddrik et al., 2020). One distinguishes between variation and initial margins. Variation margins reflect current exposures and are settled regularly, initial margins reflect potential future exposures and are usually required at the outset of a derivatives transaction. By choosing an appropriate payment function $f^V$, our payment obligation matrix can represent variation margins that are due on a given day.

Consider, for example, a situation in which the original network $X$ represents CDS contracts. In particular, $X_{ij}$ represents the amount of protection sold from $i$ to $j$ in case of a credit event occurring to the underlying reference entity over a given time period. If there is shock to this reference entity that increases its probability of default, for example, variation margins will be due from the seller of the CDS protection to the buyer of the protection since the value of the CDS contracts becomes more valuable to the protection buyer and increases the liabilities of the protection seller. This change is reflected in the variation margin that is then due from the protection seller to the protection buyer, see Paddrik et al. (2020) who discussed such a situation in detail. As in their model, we will also allow for the existence of initial margins.

**Definition 3.2** (Initial margins). Let $X$ be a liabilities matrix and let $J \in [0, \infty)$. The initial margin that node $i$ sets aside to protect its liabilities to node $j$ is given by $JX_{ij}$, where $i, j \in \mathcal{N}$.

Setting $J = 0$, would imply that there are no initial margins available, and for $J > 0$ initial margins are available, which are proportional to the notional size of the contract. This proportionality assumption is referred to as the *standard schedule* and was introduced in BCBS IOSCO (2015). There have been alternative proposals since then, see, for example, Cont (2018) who highlighted that the standard schedule typically overestimates margin requirements. For tractability purposes, we will still consider the proportional case.[3]

To complete our modeling framework, we assume that at the time when payment obligations become due each node is equipped with external assets, that is, assets from outside the network, only a part of which, the liquidity buffer, is available to satisfy any payment obligations.

**Definition 3.3** (External assets and liquidity buffer). We denote by $A^{(e)} \in [0, \infty)^N$ the vector of *external assets* and by $b \in [0, A^{(e)}]$ the liquidity buffer.

We analyze how portfolio compression affects payment obligations and liquidity buffers.

**Definition 3.4** (Payment obligations, initial margins, liquidity buffer under compression). Let $X$ be a liabilities matrix for which there exists a conservative compression network cycle $\mathcal{C} = (\mathcal{C}_{\text{nodes}}, \mathcal{C}_{\text{edges}})$ of $X$ with maximal capacity $\mu^{\max}$. Let $0 < \mu \leq \mu^{\max}$ and let $X^{\mathcal{C},\mu}$ be the $\mu$-

compressed liabilities matrix. Let $L = f^V(X) = VX$ be the payment obligation matrix, where $V \in [0, \infty)$.

1. We refer to the matrix $L^{C,\mu}$ with

$$L_{ij}^{C,\mu} = f_{ij}^V(X^{C,\mu}) = \begin{cases} VX_{ij} - V\mu & \text{if } (i,j) \in C_{\text{edges}}, \\ VX_{ij} & \text{otherwise,} \end{cases} = \begin{cases} L_{ij} - V\mu & \text{if } (i,j) \in C_{\text{edges}}, \\ L_{ij} & \text{otherwise,} \end{cases} \tag{8}$$

   as the *$\mu$-compressed payment obligation matrix* (using cycle $C$). We refer to $\bar{L}_i^{C,\mu} = \sum_{j\in\mathcal{N}} f_{ij}^V(X^{C,\mu}) = V\sum_{j\in\mathcal{N}} X_{ij}^{C,\mu}$ as the *total $\mu$-compressed payment obligations* of node $i \in \mathcal{N}$ (using cycle $C$).
2. The *$\mu$-compressed initial margins* are given by $JX^{C,\mu}$, where

$$JX_{ij}^{C,\mu} = \begin{cases} JX_{ij} - J\mu & \text{if } (i,j) \in C_{\text{edges}}, \\ JX_{ij} & \text{otherwise.} \end{cases} \tag{9}$$

3. The *$\mu$-compressed liquidity buffer* $b^{C,\mu,\gamma} \in [0,\infty)^N$, where $\gamma \in [0,1]$, is given by

$$b_i^{C,\mu,\gamma} = \begin{cases} b_i + \gamma J\mu & \text{if } i \in C_{\text{nodes}}, \\ b_i, & \text{if } i \in \mathcal{N} \setminus C_{\text{nodes}}. \end{cases} \tag{10}$$

Hence, we see that portfolio compression reduces the payment obligations, and therefore variation margins, since the payment function $f^V$ is nondecreasing. Furthermore, portfolio compression also reduces the initial margins. Therefore there are strong incentives for market participants to engage in portfolio compression. This is particularly relevant for initial margins, which can never be netted.

Regarding the liquidity buffer we allow for different effects of compression. If $\gamma = 0$, then the liquidity buffer is not affected by portfolio compression, which can be interpreted as the corresponding assets that are no longer tied up in initial margins are considered as illiquid assets. If $\gamma = 1$, then we assume that the liquidity buffer of those nodes taking part in portfolio compression increases by exactly the amount that is no longer required as initial margins since the position was reduced. For $\gamma \in (0,1)$, we have some increase of the liquidity buffer for those nodes taking part in portfolio compression but this is less than the corresponding reduction in initial margins. In particular, the $\mu$-compressed liquidity buffer is monotonically nondecreasing in $\gamma$.

We now formally define a payment system in which we will analyze systemic risk.[4]

**Definition 3.5** (Payment system). Let $X$ be a liabilities matrix for which there exists a conservative compression network cycle $C = (C_{\text{nodes}}, C_{\text{edges}})$ of $X$ with maximal capacity $\mu^{\max}$. Let $0 < \mu \leq \mu^{\max}$ and let $X^{C,\mu}$ be the $\mu$-compressed liabilities matrix. Let $V \in [0,\infty)$ and let $L = f^V(X) = VX$ be the corresponding payment obligation matrix and $b$ the liquidity buffer. We will refer to $(L, b)$ as *payment system* and to $(L^{C,\mu}, b^{C,\mu,\gamma})$ as *$\mu$-compressed* payment system, where $L^{C,\mu}$ and $b^{C,\mu,\gamma}$ are defined in Equations (8) and (10), respectively, and $\mu \in [0, \mu^{\max}]$.

## 3.2 | Clearing the payment obligations

To measure systemic risk, we consider a suitable extension of the Eisenberg and Noe (2001) framework for clearing payments in financial networks. In particular, we incorporate ideas developed by Paddrik et al. (2020) and Ghamami et al. (2021) for clearing with collateral (i.e., initial margins) into the the framework developed by Veraart (2020) to measure systemic risk.

In the following, we define the notion of equity revaluation, which is a slightly modified version of Veraart (2020, Definition 2.4), which is related to the approach developed in Barucca et al. (2020).

**Definition 3.6** (Re-evaluated equity). Consider a payment system $(L, b)$ and let $(L^{C,\mu}, b^{C,\mu,\gamma})$ be the corresponding *$\mu$-compressed* payment system where $\mu \in (0, \mu^{\max}]$.

1. A *valuation function* is a function $\mathbb{V} : \mathbb{R} \to [0,1]$, given by

$$\mathbb{V}(y) = \begin{cases} 1, & \text{if } y \geq 1+k, \\ r(y), & \text{if } y < 1+k, \end{cases} \tag{11}$$

   where $k \geq 0$ and $r : (-\infty, 1+k) \to [0,1]$ is a nondecreasing and right-continuous function.
2. Consider a valuation function $\mathbb{V}$. We define a function $\Phi = \Phi(\cdot; \mathbb{V}) : \mathcal{G} \to \mathcal{G}$ where

$$\Phi_i(E) = \Phi_i(E; \mathbb{V}) = b_i + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i, \tag{12}$$

   $\mathcal{M} = \{i \in \mathcal{N} \mid \bar{L}_i > 0\}$, $\mathcal{G} = [b - \bar{L}, b + \bar{A} - \bar{L}]$, $\bar{A}_i = \sum_{j=1}^N L_{ji}$, $\bar{L}_i = \sum_{j=1}^N L_{ij}$ $\forall i \in \mathcal{N}$. The *re-evaluated equity in the non-compressed network* is a vector $E \in \mathcal{G}$ satisfying

$$E = \Phi(E). \tag{13}$$

3. Consider a valuation function $\mathbb{V}$. We define a function $\Phi^{C,\mu,\gamma} = \Phi(\cdot; \mathbb{V}) : \mathcal{G}^C \to \mathcal{G}^C$ where

$$\Phi_i^{C,\mu,\gamma}(E) = \Phi_i^{C,\mu,\gamma}(E; \mathbb{V}) = b_i^{C,\mu,\gamma} + \sum_{j \in \mathcal{M}^C} L_{ji}^{C,\mu} \mathbb{V}\left(\frac{E_j + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i^{C,\mu}, \tag{14}$$

   $\mathcal{M}^C = \{i \in \mathcal{N} \mid \bar{L}_i^{C,\mu} > 0\}$, $\mathcal{G}^C = [b^{C,\mu,\gamma} - \bar{L}^{C,\mu}, b^{C,\mu,\gamma} + \bar{A}^C - \bar{L}^{C,\mu}]$, $\bar{A}_i^C = \sum_{j=1}^N L_{ji}^{C,\mu}$, $\bar{L}_i^{C,\mu} = \sum_{j=1}^N L_{ij}^{C,\mu}$ $\forall i \in \mathcal{N}$. The *re-evaluated equity in the compressed network* is a vector $E \in \mathcal{G}^C$ satisfying

$$E = \Phi^{C,\mu,\gamma}(E). \tag{15}$$

Since $r$ is nondecreasing and right-continuous, $\mathbb{V}$ is also nondecreasing and right-continuous. Therefore, Veraart (2020, Theorem 2.5) guarantees the existence of the re-evaluated equities in Equations (13) and (15) as fixed points of $\Phi$ and $\Phi^{C,\mu,\gamma}$, respectively.

Similar to Veraart (2020), for a given node $i \in \mathcal{N}$, the function $\Phi_i$ models the difference between the liquid assets $b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}(\frac{E_j+\bar{L}_j}{\bar{L}_j})$ and the total payment obligations $\bar{L}_i$. The liquid assets consist of the liquidity buffer $b_i$ and the payments received from the other financial institutions given by $\sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}(\frac{E_j+\bar{L}_j}{\bar{L}_j})$. If the function value of $\mathbb{V}$ is strictly less than 1, this implies that not the full amount of payment obligations is paid by $j$ to $i$, which reduces the liquid assets that $i$ has. The interpretation for the terms appearing in $\Phi^{C,\mu,\gamma}$ is the same as for $\Phi$ with the only exceptions that the compressed network is considered.

If $L = X$ and $A^{(e)} = b$, then the positive part of the re-evaluated equity would correspond to the *equity* of the node, as in Eisenberg and Noe (2001).

In this re-evaluation approach, all payment obligations are treated equally. In particular, no netting takes place prior to clearing. Any type of netting (such as bilateral netting, compression, etc.) would effectively introduce a seniority structure, where liabilities that are netted have implicitly a higher seniority than those that are not netted, see Elsinger (2009) for a clearing approach with different seniorities of debt.

As described earlier, we refer to a tuple $(L, b)$ as a payment system, where $L$ is a payment obligation matrix and $b$ is a vector of liquidity buffers. If we use such a system to make any statements about its associated systemic risk based on a valuation function $\mathbb{V}$, we write $(L, b; \mathbb{V})$ and also refer to it as a *payment system*.

We consider special choices of valuation functions next to give some intuition on what they can represent. All results we derive, however, will hold for general valuation functions defined in Equation (11).

*Remark* 3.7 (Special choices for $\mathbb{V}$ from the literature).

1. The Eisenberg and Noe (2001) model can be recovered[5] by setting $k = 0$ and

$$\mathbb{V}^{\mathrm{EN}}(y) = \min\{1, y^+\}. \tag{16}$$

2. The special case of the model by Rogers and Veraart (2013) with bankruptcy costs parameters $\alpha = \beta \in [0, 1]$ can be recovered by setting $k = 0$ and

$$\mathbb{V}^{\mathrm{RV}}(y) = \begin{cases} 1 & \text{if } y \geq 1, \\ \beta y^+ & \text{if } y < 1. \end{cases} \tag{17}$$

3. In Veraart (2020)and Glasserman and Young (2015), it was argued that contagion can be triggered prior to the point where the equity of an institution is zero and it was proposed to consider .
4. The *zero recovery rate valuation function* is defined by

$$\mathbb{V}^{\mathrm{zero}}(y) = \mathbb{I}_{\{y \geq 1+k\}}, \text{ where } k \geq 0. \tag{18}$$

All special choices of valuation functions mentioned so far do not account for initial margins. In the following, we define valuation functions that incorporate initial margins.

First, we consider a valuation function that extends the zero recovery rate valuation function to a situation with initial margins, by setting

$$\mathbb{V}^{\text{zero};\tilde{J}}(y) = \mathbb{I}_{\{y \geq 1+k\}} + \tilde{J}\mathbb{I}_{\{y<1+k\}} \text{ where } k \geq 0, \tilde{J} \in [0,1]. \tag{19}$$

Hence, this function captures the effects that a proportion $\tilde{J} \in [0,1]$ of the position will always be guaranteed by initial margins. Note that $\mathbb{V}^{\text{zero};0} = \mathbb{V}^{\text{zero}}$. Throughout this paper, we will also refer to $\mathbb{V}^{\text{zero};\tilde{J}}$ as *zero recovery rate valuation function* even though the function returns $\tilde{J}$ and not necessarily 0 in the case of default. The reason for this is that the parameter $\tilde{J}$ captures the effect that some payments are guaranteed because initial margins were posted and not because they were recovered from any payments from other nodes in the system.

Second, we consider the situation where we can have both initial margins and non-negative recovery rates. To do so, we generalize the definition of $\mathbb{V}^{\text{RV}}$ to include initial margins.

**Definition 3.8** (Valuation function accounting for initial margins). Let $\tilde{J} \in [0,\infty)$ and $\beta \in [0,1]$. We define the *initial margin valuation function* by $\mathbb{V}^{\text{InitialMargin}} : \mathbb{R} \to [0,1]$, where $\forall y \in \mathbb{R}$

$$\mathbb{V}^{\text{InitialMargin}}(y) = \begin{cases} 1, & \text{if } y \geq 1, \\ \min\{1, \tilde{J} + \beta y^+\}, & \text{if } y < 1. \end{cases} \tag{20}$$

One can easily check that $\mathbb{V}^{\text{InitialMargin}}$ is a valuation function. If we set the bankruptcy costs parameter to $\beta = 0$ in $\mathbb{V}^{\text{InitialMargin}}$, then this captures exactly the situation of zero recovery rates and $\mathbb{V}^{\text{InitialMargin}} = \mathbb{V}^{\text{zero};\tilde{J}}$ for $k = 0$. Using Veraart (2020, Theorem 2.11), we conclude that higher values of initial margins lead to a better outcome for the system in the following sense: let $0 \leq J_1 \leq J_2$ be two possible parameters for $\tilde{J}$ in Equation (20), then the greatest re-evaluated equity corresponding to parameter $J_2$ would be greater or equal than the greatest re-evaluated equity corresponding to $J_1$. In particular, a system with initial margins has better outcomes than a system without initial margins.

The choice of $\mathbb{V}^{\text{InitialMargin}}$ is motivated by the approaches developed in Paddrik et al. (2020) and Ghamami et al. (2021) for clearing in collateralised networks. We are essentially using these ideas but rewrite them to fit a slightly different mathematical framework that is more tractable for the purpose of our analysis. The first difference between Paddrik et al. (2020) and Ghamami et al. (2021) and our formulation here is that we express the clearing problem in terms of the re-evaluated equity, that is, within the framework of Veraart (2020), and not in terms of the clearing payments, since this makes the analysis in the compression context more tractable.[6] The second difference is that we include bankruptcy costs modeled in terms of the parameter $\beta \in [0,1]$, whereas the other approaches mainly focus on $\beta = 0$ and $\beta = 1$.

We provide some intuition for the choice of $\mathbb{V}^{\text{InitialMargin}}$ next. If we set $\tilde{J} = 0$, $\mathbb{V}^{\text{InitialMargin}}$ reduces to $\mathbb{V}^{\text{RV}}$. We therefore consider for now. Let $E^*$ be a fixed point of $\Phi$. Suppose $y = \frac{E_j^* + \bar{L}_j}{\bar{L}_j} < 1$ for a $j \in \mathcal{N}$. Then, the payment that node $j$ makes to node $i$ is given by

$$L_{ji}\mathbb{V}^{\text{InitialMargin}}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) = \min\left\{L_{ji}, \tilde{J}L_{ji} + \beta L_{ji}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right)^+\right\} =: (\star). \tag{21}$$

Now if $\tilde{J} \leq 1$, then $(\star) \geq \tilde{J}L_{ji}$ implying that node $j$ will always at least pay the amount corresponding to the initial margin to $i$ but possibly even more if $\beta L_{ji}(\frac{E_j^*+\bar{L}_j}{\bar{L}_j})^+ > 0$. Under no circumstances can $i$ receive more than $L_{ji}$ from $j$. If $\tilde{J} \geq 1$, then the initial margins guarantee full payment of $L_{ji}$.

Note that initial margins are only available in the case of default. Recall that $JX_{ji}$ is the initial margin that $j$ posted to protect its liabilities to $i$. But node $i$ cannot use $JX_{ji}$ to make any payments if $j$ does not default. If node $j$ does default, however, then $i$ can seize (parts of) the initial margin $JX_{ji}$. This effect is captured by the parameter $\tilde{J}$ appearing in the default branch of the valuation function $\mathbb{V}^{\text{InitialMargin}}$. If we set $\tilde{J} = J/V$ in $\mathbb{V}^{\text{InitialMargin}}$ (assuming that $V > 0$ otherwise there are no payment obligations), then $\tilde{J}L_{ji} = \tilde{J}VX_{ji} = JX_{ji}$ is exactly the payment obligation from $j$ to $i$ that is guaranteed by the initial margins.

## 3.3 | Definition of default, reduction of systemic risk, and harmfulness of portfolio compression

In the following, we will compute the greatest re-evaluated equity both in the original network and in the compressed network, that is, we will always consider the greatest fixed point of $\Phi$ and $\Phi^{C,\mu,\gamma}$ in Equations (13) and (15), respectively. They correspond to the best possible outcome for the economy. Based on these quantities, we can then infer which nodes are in default in the network with compression and in the network without compression. Hence, we take an ex post point of view. We ask what would happen if no compression takes place and we then evaluate the network at a point in time when payments are due. Then we consider the case where compression has taken place and we then evaluate the network when payments are due and compare the outcome to the situation without compression. We summarize the mathematical setting as follows.

**Assumption 3.9** (Market setting).

- Let $X$ be a liabilities matrix for which there exists a conservative compression network cycle $\mathcal{C} = (\mathcal{C}_{\text{nodes}}, \mathcal{C}_{\text{edges}}, \mu^{\max})$ with maximal capacity $\mu^{\max} > 0$.
- Let $\mathbb{V}$ be a valuation function and $k \geq 0$.
- Let $(L, b; \mathbb{V})$ be the corresponding payment system with total payment obligations $\bar{L}$.
- Let $0 < \mu \leq \mu^{\max}$. Let $L^{C,\mu}$ be the $\mu$-compressed payment obligation matrix. Let $\bar{L}^{C,\mu}$ be the total $\mu$-compressed payment obligations.
- Let $E^*$ be the greatest re-evaluated equity in the non-compressed network.
- Let $E^{C,\mu;\gamma;*}$ be the greatest re-evaluated equity in the compressed network with $\gamma \in [0, 1]$.

We can now define what it means for an institution to be in default.

**Definition 3.10** (Definition of default). Consider the market setting of Assumption 3.9. Then, the set of *defaulting financial institutions in the non-compressed system* is

$$\mathcal{D}(L, b; \mathbb{V}) = \{i \in \mathcal{N} \mid E_i^* < k\bar{L}_i\} \tag{22}$$

and the set of *defaulting financial institutions in the compressed system* is

$$\mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V}) = \{i \in \mathcal{N} \mid E_i^{C,\mu;\gamma;*} < k\bar{L}_i^{C,\mu}\}. \tag{23}$$

The definition above defines *default* (in the non-compressed system) as the point when the quantity $(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}) < 1 + k$. For $k = 0$, this is equivalent to saying that the available liquid assets are strictly smaller than the payment obligations, which is equivalent to $E_j^* < 0$. This is the situation we have in mind when considering variation margin payments, that is, for $\mathbb{V} = \mathbb{V}^{\text{InitialMargin}}$, as in Paddrik et al. (2020) and Ghamami et al. (2021).

If $k > 0$, then the default condition is equivalent to saying that the available liquid assets are strictly less than the required payment obligations plus an additionally required buffer. In Remark A.1, we show that to be able to account for certain capital requirements, it is sometimes beneficial to allow for an earlier start point of default, that is, $k > 0$.

Recall that the payments from $j$ to $i$ are $L_{ji}\mathbb{V}(\frac{E_j^*+\bar{L}_j}{\bar{L}_j})$. If $j$ is in default, then we are in the default branch of the valuation function[7], that is, $\mathbb{V}(y) = r(y)$ and this value can be strictly less than 1 implying that payment obligations from $j$ to $i$ are no longer satisfied completely. In the case of initial margins, that is, if $\mathbb{V} = \mathbb{V}^{\text{InitialMargin}}$, then it is possible (but this will depend on the magnitude of the initial margins) that $L_{ji}\mathbb{V}(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}) = L_{ji}$ even though $j$ defaults.

We are now in a position to formally define what we mean by saying that a particular compression reduces systemic risk or is harmful. We do this by comparing the defaults in the non-compressed network to the defaults in the compressed network (see Definition 3.10).

**Definition 3.11** ((Strong) reduction of systemic risk and harmfulness). Consider the market setting of Assumption 3.9. We say that the compression network cycle $C$ *reduces systemic risk* if $\mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V}) \subseteq \mathcal{D}(L, b; \mathbb{V})$. We say that the compression network cycle $C$ *strongly reduces systemic risk* if $\mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V}) \subsetneq \mathcal{D}(L, b; \mathbb{V})$. We say that the compression network cycle $C$ is *harmful* if $\mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V}) \setminus \mathcal{D}(L, b; \mathbb{V}) \neq \emptyset$.

Based on this definition, we say that compression reduces systemic risk if every node that defaults in the compressed network also defaults in the non-compressed network. In the same spirit, we classify compression as harmful if there exists at least one node in the network that defaults in the compressed network that does not default in the non-compressed network.

These definitions can be interpreted in a Pareto sense as follows. Our notion of portfolio compression to strongly reduce systemic risk can be interpreted as a Pareto improvement: if portfolio compression strongly reduces systemic risk, this means that at least one institution is strictly better off under compression (since it no longer defaults) than without compression. At the same time, no other institution is harmed, since there are no new defaults caused by compression.

The same spirit applies to our definition of harmfulness. In particular, we do not allow for a trade-off between some institutions being better off and some being worse off (measured in terms of defaults). As soon as there exist one node that is worse off by compression in the sense that it defaults only in the compressed but not in the non-compressed network, then we say that this compression is harmful.

We use the set of defaulting institutions to characterize systemic risk here. The advantage of this measure is that it allows for a direct comparison between a compressed and a non-compressed system. Since portfolio compression reduces the gross positions, we need a normalized measure for comparing outcomes in a compressed and a non-compressed financial network and cannot just consider, for example, losses directly. We will later see that the proportion of debt that a node repays either in the non-compressed or in the compressed network, plays an important role when characterizing potential harmfulness of portfolio compression. These repayment proportions are therefore another normalized quantity that we consider in this context.

## 4 | CONSEQUENCES OF PORTFOLIO COMPRESSION FOR SYSTEMIC RISK

We now analyze the consequences of portfolio compression for systemic risk. One might think that portfolio compression reduces systemic risk since it reduces gross exposures while not changing the net exposures. Indeed we will show that in many realistic scenarios portfolio compression reduces or even strongly reduces systemic risk.

There are circumstances, however, in which compression can be harmful. Portfolio optimization is an optimization problem that aims to reduce gross exposures subject to some constraints such as keeping net exposures unchanged (O' Kane, 2017; D'Errico and Roukny, 2021). As long as these constraints do not explicitly account for systemic risk, there is no reason why a solution to such an optimization problem should automatically reduce systemic risk.

### 4.1 | Who can be affected by portfolio compression?

We identify those nodes that can in principle be affected by portfolio compression by defining a compression risk orbit[8].

**Definition 4.1** (Compression risk orbit). Consider the market setting of Assumption 3.9. The *compression risk orbit of $C$* is

$$\mathcal{O} = C_{\text{nodes}} \cup \{j \in \mathcal{N} \mid \exists i \in C_{\text{nodes}} \text{ and } \exists \text{ a directed path from } i \text{ to } j \text{ in } \hat{G}\}, \tag{24}$$

where $\hat{G}$ is the graph with nodes $\mathcal{N}$ and edges $\hat{\mathcal{E}} = \{(i,j) \in \mathcal{N}^2 \mid L_{ij} > 0\}$.

The compression risk orbit contains all nodes on the compression network cycle and all nodes that can be reached from nodes on the compression network cycle; compression could in principle affect their outcome (both positively or negatively). All nodes in $\mathcal{N} \setminus \mathcal{O}$ cannot be affected (positively or negatively) by compression, that is, the greatest re-evaluated equity with or without compression coincides for those nodes.

**Proposition 4.2.** *Consider the market setting of Assumption 3.9. Then, $E_i^* = E_i^{C,\mu;\gamma;*} \quad \forall i \in \mathcal{N} \setminus \mathcal{O}$, where $\mathcal{O}$ is given in Equation (24).*

The proof of this proposition and proofs of all following results are provided in Appendix B.

### 4.2 | Fundamental versus contagious defaults

We will distinguish between two types of default: fundamental default and contagious default. Fundamental defaults are defaults that occur even if all nodes pay their payment obligations in full.[9] Contagious defaults are all defaults that are not fundamental defaults. In order to formally define these two types of default, we consider the difference between nominal liquid assets and total payment obligations and refer to it as initial equity (even though the assets and liabilities considered here might not reflect the full balance sheet).

**Definition 4.3** (Initial equity). Consider the market setting of Assumption 3.9. For all $i \in \mathcal{N}$ define the *initial equity* in the non-compressed and in the compressed network (for parameter $\gamma = 0$ $\gamma \in [0,1]$) by

$$E_i^{(0)} = b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i, \qquad E_i^{C(0);\gamma} = b_i^{C,\mu,\gamma} + \sum_{j\in\mathcal{N}} L_{ji}^{C,\mu} - \bar{L}_i^C. \tag{25}$$

Hence, the initial equity corresponds to the best possible situation, namely all nodes repaying their payment obligations in full. Recall, that for $\gamma = 0$, it holds that $b_i^{C,\mu,\gamma} = b$.

**Definition 4.4** (Fundamental and contagious defaults). Consider the market setting of Assumption 3.9. Let $E^{(0)}$and $E^{C(0);\gamma}$ be the initial equity defined in Equation (25). We refer to $\mathcal{F} = \{i \in \mathcal{N} \mid E_i^{(0)} < k\bar{L}_i\}$ and $\mathcal{D}(L, b; \mathbb{V}) \setminus \mathcal{F}$ as the fundamental defaults and contagious defaults in the non-compressed network, respectively. Similarly, we refer to $\mathcal{F}^C = \{i \in \mathcal{N} \mid E^{C(0);\gamma} < k\bar{L}_i^{C,\mu}\}$ and $\mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V}) \setminus \mathcal{F}^C$ as the fundamental defaults and contagious defaults in the compressed network (where $\gamma \in [0,1]$), respectively.

We will show in Lemma B.4 that $\mathcal{F} \subseteq \mathcal{D}(L, b; \mathbb{V})$ and $\mathcal{F}^C \subseteq \mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V})$. We show how portfolio compression affects fundamental defaults using properties of the initial equity.

**Lemma 4.5.** *Consider the market setting of Assumption 3.9. The initial equities $E_i^{(0)}, E_i^{C(0);\gamma}, i \in \mathcal{N}$, are as in Equation (25). Then, $E_i^{(0)} = E_i^{C(0);0} \leq E_i^{C(0);\gamma} \quad \forall i \in \mathcal{N}, \forall \gamma \in [0,1]$.*

**Proposition 4.6** (Fundamental defaults and compression). *Consider the market setting of Assumption 3.9. Then, $\mathcal{F}^C \subseteq \mathcal{F}$. If $\mathcal{F} \setminus \mathcal{F}^C \neq \emptyset$, then $kV + \gamma J > 0$ and $\mathcal{F} \setminus \mathcal{F}^C \subseteq C_{nodes}$.*

Hence, this proposition shows, that compression can only improve fundamental defaults, in the sense that every node that is in fundamental default in the compressed network is also in fundamental default in the non-compressed network. Furthermore, it states that any strict shrinkage of the set of fundamental defaults under compression, is due to the fact that portfolio compression has avoided some fundamental defaults among nodes that took part in compression, that is, are in $C_{\text{nodes}}$. For this to happen, some parameter constraints are required. They capture two types of effects: first, portfolio compression can increase the available liquidity buffer. In particular, if we set $\gamma > 0$ and assume positive initial margins by setting $J > 0$, then $\gamma J > 0$ and nodes on the $\mu$-compression network cycle then have a strictly larger liquidity buffer for $\mu > 0$ in the compressed than in the non-compressed network, which can potentially avoid fundamental defaults. Second, portfolio compression could move a node further away from the default boundary if $k > 0$, in which case portfolio compression could also potentially avoid a fundamental default.

For compression to be harmful, we need at least one firm that defaults in the compressed network that does not default in the non-compressed network. Since according to Proposition 4.6, portfolio compression cannot cause any fundamental defaults, such an additional default would have to be a contagious default.

## 4.3 | Structural conditions for the consequences of portfolio compression

The following theorem contains the main theoretical results of this paper. It identifies three key structural conditions that are necessary for portfolio compression to be harmful.

**Theorem 4.7** (Necessary conditions for compression to be harmful). *Consider the market setting of Assumption 3.9. Suppose that compressing cycle $C$ is harmful. Then,*

1.
$$\mathcal{D}(L, b; \mathbb{V}) \cap C_{nodes} \neq \emptyset; \tag{26}$$

2. *there exists an $i \in \mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V}) \cap C_{nodes}$ such that*
$$\mathbb{V}\left(\frac{E_i^{C,\mu;\gamma;*} + \bar{L}_i^{C,\mu}}{\bar{L}_i^{C,\mu}}\right) < \mathbb{V}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right); \tag{27}$$

3. *the valuation function satisfies*
$$\mathbb{V} \neq \mathbb{V}^{zero;\tilde{J}}. \tag{28}$$

In the following, we discuss the three structural conditions (26)–(28) in more detail.

### 4.3.1 | Defaults on the compression network cycle in the non-compressed network

Condition (26) tells us, that for portfolio compression to be potentially harmful, one needs at least one default on the compression network cycle in the non-compressed network. Compressing such a cycle can be harmful. The following proposition is used to prove part 1. of Theorem 4.7 and identifies the relationship between the re-evaluated equity in the compressed and non-compressed network.

**Proposition 4.8.** *Consider the market setting of Assumption 3.9 and let $\mathcal{D}(L, b; \mathbb{V}) \cap C_{nodes} = \emptyset$. Then, $E_i^* = E_i^{C,\mu;0;*} \leq E_i^{C,\mu;\gamma;*} \quad \forall i \in \mathcal{N}$ and $\mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V}) \subseteq \mathcal{D}(L^{C,\mu}, b^{C,\mu,0}; \mathbb{V}) = \mathcal{D}(L, b; \mathbb{V})$.*

Hence, compression can only increase the re-evaluated equity if there are no defaults on the compression network cycle in the non-compressed system. Under this assumption, if additionally $\gamma = 0$, implying that compression does not increase the liquidity buffer, then the re-evaluated equities with and without compression coincide. In both cases, systemic risk is reduced.

We do an ex post analysis here. In practice, firms would conduct portfolio compression prior to payment obligations becoming due. Hence, at the time compression is done, Equation (26) is a condition on the future state of the network. In this spirit, we conclude that if the probability that condition (26) is satisfied in the future is low, meaning that it is unlikely for firms who took part in compression to default in the future, then it is likely that this compression reduces systemic risk.

### 4.3.2 | Repayment proportions

The condition (27) in Theorem 26, is a statement about repayment proportions of nodes on the compression network cycle that default in the compressed network. The total payments that node $i$ makes to the other nodes in the non-compressed network are

$$\sum_{j\in\mathcal{M}} L_{ij}\mathbb{V}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right) = \mathbb{V}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right)\bar{L}_i, \tag{29}$$

and hence it repays the proportion $\mathbb{V}(\frac{E_i^*+\bar{L}_i}{\bar{L}_i})\bar{L}_i/\bar{L}_i = \mathbb{V}(\frac{E_i^*+\bar{L}_i}{\bar{L}_i})$ of its total payment obligations if no compression is used. Similarly, the repayment proportion of node $i$ in the compressed network is $\mathbb{V}(\frac{E_i^{C,\mu;\gamma;*}+\bar{L}_i^{C,\mu}}{\bar{L}_i^{C,\mu}})$. Condition (27), therefore, says that there exists a node $i \in \mathcal{C}_{\text{nodes}}$ that repays a smaller proportion of its total payment obligations in the compressed network compared to the non-compressed network. Any node $i \in \mathcal{N}$ satisfying Equation (27) is in $\mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V})$, since from Equation (27), $\mathbb{V}(\frac{E_i^{C,\mu;\gamma;*}+\bar{L}_i^{C,\mu}}{\bar{L}_i^{C,\mu}}) < \mathbb{V}(\frac{E_i^*+\bar{L}_i}{\bar{L}_i}) \leq 1$ and hence $\frac{E_i^{C,\mu;\gamma;*}+\bar{L}_i^{C,\mu}}{\bar{L}_i^{C,\mu}} < 1 + k$ by Equation (11).

Hence condition (27) says, that for portfolio compression to be potentially harmful, one needs at least one node on the compression network cycle that repays a strictly smaller proportion of its total payment obligations in the compressed network than in the non-compressed network.

Consider such an $i \in \mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V}) \cap \mathcal{C}_{\text{nodes}}$ satisfying Equation (27). Since $i$ repays a smaller proportion of its debt after compression, it can transmit larger losses to other nodes in the network. The payment that node $i$ makes to any node $j \in \mathcal{N}$ without compression is $L_{ij}\mathbb{V}(\frac{E_i^*+\bar{L}_i}{\bar{L}_i})$ and with compression it is $L_{ij}^{C,\mu}\mathbb{V}(\frac{E_i^{C,\mu;\gamma;*}+\bar{L}_i^{C,\mu}}{\bar{L}_i^{C,\mu}})$. Then, for all $j \in \mathcal{N}$ with $L_{ij}^{C,\mu} > 0$, it holds that $L_{ij} > 0$ and hence

$$L_{ij}^{C,\mu}\mathbb{V}\left(\frac{E_i^{C,\mu;\gamma;*} + \bar{L}_i^{C,\mu}}{\bar{L}_i^{C,\mu}}\right) \leq L_{ij}\mathbb{V}\left(\frac{E_i^{C,\mu;\gamma;*} + \bar{L}_i^{C,\mu}}{\bar{L}_i^{C,\mu}}\right) < L_{ij}\mathbb{V}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right). \tag{30}$$

Hence, node $j$ receives less from $i$ in the compressed network than in the non-compressed network.

Therefore, as long as all nodes on the compression network cycle repay a greater or equal proportion of their debt in the compressed network compared to the non-compressed network (as stated in Equation (31)), then compression reduces systemic risk. We formulate several related results, which are of interest for interpreting the results.

**Proposition 4.9.** *Consider the market setting of Assumption 3.9. Suppose that at least one of the following three conditions is satisfied:*

*1.*

$$\mathbb{V}\left(\frac{E_i^{C,\mu;\gamma;*} + \bar{L}_i^{C,\mu}}{\bar{L}_i^{C,\mu}}\right) \geq \mathbb{V}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right) \quad \forall i \in \mathcal{C}_{nodes}; \tag{31}$$

*2.*

$$\mathbb{V}\left(\frac{E_i^{C,\mu;\gamma;*} + \bar{L}_i^{C,\mu}}{\bar{L}_i^{C,\mu}}\right) = 1 \quad \forall i \in C_{nodes}; \tag{32}$$

*3.*

$$\mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V}) \cap C_{nodes} = \emptyset. \tag{33}$$

*Then,* $E_i^* \leq E_i^{C,\mu;\gamma;*}$ *for all* $i \in \mathcal{N}$ *and* $\mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V}) \subseteq \mathcal{D}(L, b; \mathbb{V})$*.*

Note that condition (33) implies condition (32), which implies condition (31).

Condition (33) says that for compression to be potentially harmful, one needs to have a default on the compression network cycle not just in the non-compressed network (condition (26)) but also in the compressed network. Hence, there cannot be a situation in which financial institutions engage in conservative compression such that none of them defaults in the compressed network but a financial institution outside the compression network cycle is worse off (in the sense that it defaults only in the compressed network). Hence, if compression is harmful for a node outside the compression network cycle, then there must exist a defaulting node on the compression network cycle.

Now consider the situation where at least one node on the compression network cycle defaults in the compressed network. Then condition (32) implies that as long as all nodes on the compression network cycle repay their debt in full—this could happen due to sufficient initial margins—then compression cannot be harmful.

A direct consequence of Proposition 4.9 is that if we consider the valuation function $\mathbb{V}^{\text{InitialMargin}}$ with $\tilde{J} \geq 1$ (which reduces to $\mathbb{V}^{\text{InitialMargin}}(y) = 1\ \forall y \in \mathbb{R}$ for $\tilde{J} \geq 1$) or any other constant valuation function, then compression reduces systemic risk.

**Corollary 4.10.** *Consider the market setting of Assumption 3.9. Consider a valuation function* $\mathbb{V}(y) = A$, $\forall y \in \mathbb{R}$, *for some* $A \in [0, 1]$*. Then,* $E_i^* \leq E_i^{C,\mu;\gamma;*}$ *for all* $i \in \mathcal{N}$ *and* $\mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V}) \subseteq \mathcal{D}(L, b; \mathbb{V})$*.*

Next we provide an intuitive explanation how portfolio compression can change the distribution of losses in the network. In several approaches in the literature such as Eisenberg and Noe (2001), Rogers and Veraart (2013), and Veraart (2020), the valuation function $\mathbb{V}$ is a capped piecewise linear function. Also, our newly introduced function $\mathbb{V}^{\text{InitialMargin}}$ falls in this class. A key idea of the Eisenberg and Noe (2001) clearing approach (which also applies to more general capped piecewise linear functions) is that all defaulting nodes repay their debt according to the proportions according to which their nominal payment obligations are distributed. These proportions are specified in terms of a relative liabilities matrix.

**Proposition 4.11.** *Consider the market setting of Assumption 3.9. Consider the relative payment obligation matrices* $\Pi, \Pi^{C,\mu} \in \mathbb{R}^{N\times N}$, *where*

$$\Pi_{ij} = \begin{cases} \frac{L_{ij}}{\bar{L}_i}, & \text{if } \bar{L}_i > 0, \\ 0, & \text{if } \bar{L}_i = 0, \end{cases} \qquad \Pi^{C,\mu}_{ij} = \begin{cases} \frac{L^{C,\mu}_{ij}}{\bar{L}^{C,\mu}_i}, & \text{if } \bar{L}^{C,\mu}_i > 0, \\ 0, & \text{if } \bar{L}^{C,\mu}_i = 0. \end{cases} \tag{34}$$

*For* $i \in C_{nodes}$, *we denote by* $suc(i)$ *(successor) the node in* $C_{nodes}$ *that satisfies* $(i, suc(i)) \in C_{edges}$. *Then, for all* $i \in C_{nodes}$

$$\Pi^{C,\mu}_{isuc(i)} \leq \Pi_{isuc(i)}, \tag{35}$$

$$\Pi^{C,\mu}_{ij} \geq \Pi_{ij} \quad \forall j \in \mathcal{N} \setminus \{suc(i)\}; \tag{36}$$

*and for all* $i \in \mathcal{N} \setminus C_{nodes}$ *and for all* $j \in \mathcal{N}\Pi^{C,\mu}_{ij} = \Pi_{ij}$.

We see that for nodes that are not on the compression network cycle, the proportions according to which they distribute their payments to the other nodes in the system do not change. For the nodes on the compression network cycle, these proportions do change: smaller (or equal) proportions are paid to the immediate successor of a node on the compression network cycle. To all other nodes, larger (or equal) proportions are used to allocate the payments.

Note that if proportions increase, this can also imply that a larger proportion of losses hits neighboring nodes and this is where the danger is coming from. As long as there are no defaults on the compression network cycle, the fact that the proportions change for nodes on the compression network cycle is irrelevant because they still satisfy the required payment obligations. As soon as that is no longer the case, and the proportions determine how losses are spread, the change in these proportions starts to matter.

### 4.3.3 | Recovery rates

According to part 3. of Theorem 4.7 one needs nonzero recovery rates for portfolio compression to be potentially harmful.

**Proposition 4.12.** *Consider the market setting of Assumption 3.9 and assume that* $\mathbb{V} = \mathbb{V}^{zero;\tilde{J}}$. *Then,* $E^*_i \leq E^{C,\mu;\gamma;*}_i$ *for all* $i \in \mathcal{N}$ *and* $\mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V}^{zero;\tilde{J}}) \subseteq \mathcal{D}(L, b; \mathbb{V}^{zero;\tilde{J}})$.

Hence, under zero recovery rates portfolio compression leads to a greater re-evaluated equity. In practice, the recovery rates will depend on the time-horizon considered. Assuming a zero recovery rate is reasonable when considering short-term consequences of default, see, for example, Amini et al. (2016a) and the references therein for a discussion. For mid- to long-term consequences of default, it is important to consider models that allow for positive recovery rates.

If recovery rates are positive, we can have a *worse default* of a node on the compression cycle meaning that it defaults both in the non-compressed and in the compressed network but it repays a strictly smaller proportion of its debt in the compressed network (satisfying Equation (27)). If recovery rates are zero, we cannot have such a worse default.

## 4.4 | Compressing multiple cycles

All results so far (Theorem 4.7, Propositions 4.8, 4.9, Proposition 4.12) are statements about compressing a single cycle. In practice, multiple cycles would/could be compressed. The results, nevertheless, carry over to the multiple cycle case in the following sense. Suppose there are multiple cycles $\mathcal{C}_1, \ldots, \mathcal{C}_m$ such that conservative compression could be carried out along those cycles sequentially starting from $\mathcal{C}_1$ and finishing at $\mathcal{C}_m$. This in particular implies that $\mathcal{C}_i,\ i \in \{1, \ldots, m\}$ is still a possible compression network cycle after the cycles $\mathcal{C}_j,\ j = 1, \ldots, i-1$ have been compressed. If $\mathbb{V} = \mathbb{V}^{\text{zero};\tilde{J}}$, then we know from Proposition 4.12 that compressing one cycle after the other cannot be harmful. Suppose now that $\mathbb{V} \neq \mathbb{V}^{\text{zero};\tilde{J}}$. Then according to Theorem 4.7 (parts 1. and 2.), we need to check properties of the nodes on the compression cycle. Suppose that at least one of the conditions in parts 1. and 2. is not satisfied for compression cycle $\mathcal{C}_1$, then compressing this cycle cannot be harmful. Next one would need to check the conditions for the nodes on the compression network cycle $\mathcal{C}_2$ after $\mathcal{C}_1$ has been compressed. Again, if at least one of the conditions in parts 1. and 2. is not satisfied, then compressing $\mathcal{C}_2$ cannot be harmful and so forth. This can be formalized as follows.

**Proposition 4.13** (Compressing multiple cycles). *Let $X$ be a liabilities matrix. Suppose there exist $m \in \mathbb{N}$ compression network cycles $\mathcal{C}^{(1)}, \ldots, \mathcal{C}^{(m)}$ such that conservative compression can be carried out along those cycles sequentially starting from $\mathcal{C}_1$ and finishing at $\mathcal{C}_m$. This in particular implies that $\mathcal{C}^{(i)}, i \in \{1, \ldots, m\}$ is still a possible compression network cycle after the cycles $\mathcal{C}^{(j)},\ j = 1, \ldots, i-1$ have been compressed. We assume that each compression network cycle $i \in \{1, \ldots, m\}$ is compressed by a quantity $\mu_i \in (0, \mu_i^{\max}]$ where $\mu_i^{\max}$ is the maximal compression capacity on cycle $i$ after the cycles $\mathcal{C}^1, \ldots, \mathcal{C}^{i-1}$ have been compressed. Let $\mathcal{C}^{all}_{nodes} = \mathcal{C}^{(1)}_{nodes} \cup \ldots \cup \mathcal{C}^{(m)}_{nodes}$. Let $(L, b; \mathbb{V})$ denote the corresponding payment system (without compression) and denote by $E^*$ the greatest re-evaluated equity in the non-compressed system. We denote by $E^{\mathcal{C}^1,\ldots,\mathcal{C}^i,\star}$ the greatest re-evaluated equity that corresponds to the payment system in which the cycles $\mathcal{C}^1, \ldots, \mathcal{C}^i, i \in \{1, \ldots, m\}$ have been compressed. The total payment obligation of node $i$ in this system is denoted by $\bar{L}_i^{\mathcal{C}^1,\ldots,\mathcal{C}^i}$. Suppose at least one of the following three conditions is satisfied:*

*1.*

$$\mathcal{D}(L, b; \mathbb{V}) \cap \mathcal{C}^{all}_{nodes} = \emptyset; \tag{37}$$

*2.*

$$\mathbb{V}\left(\frac{E_i^{C^{(1)},\star}+\bar{L}_i^{C^{(1)}}}{\bar{L}_i^{C^{(1)}}}\right)\geq\mathbb{V}\left(\frac{E_i^{*}+\bar{L}_i}{\bar{L}_i}\right)\quad\forall i\in C^{(1)}_{nodes};$$

$$\forall n\in\{2,\dots,m\}\text{ it holds that}\tag{38}$$

$$\mathbb{V}\left(\frac{E_i^{C^{(1)},\dots,C^{(n)}\star}+\bar{L}_i^{C^{(1)},\dots,C^{(n)}}}{\bar{L}_i^{C^{(1)},\dots,C^{(n)}}}\right)\geq\mathbb{V}\left(\frac{E_i^{C^{(1)},\dots,C^{(n-1)}\star}+\bar{L}_i^{C^{(1)},\dots,C^{(n-1)}}}{\bar{L}_i^{C^{(1)},\dots,C^{(n-1)}}}\right)\quad\forall i\in C^{(n)}_{nodes}.$$

*3.* $\mathbb{V}=\mathbb{V}^{zero;\tilde{J}}$.

*Then, compressing sequentially $C^{(1)},\dots,C^{(m)}$ reduces systemic risk.*

By combining the results from Proposition 4.13 above with the results derived in D'Errico and Roukny (2021, EC.5., e-companion) for the characterization of $\tilde{X}$, we immediately obtain the following corollary (of which statement 1 can be found in D'Errico and Roukny (2021, EC.5., e-companion).

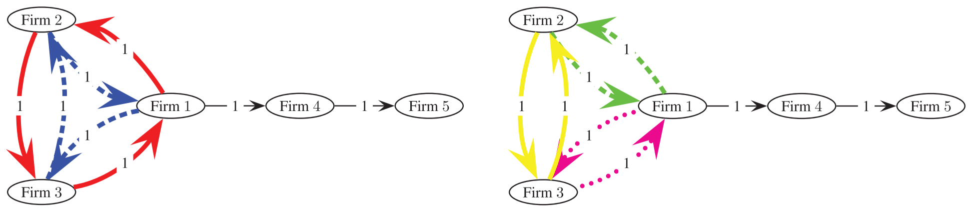
**Corollary 4.14** (Compression as optimization problem). *Let $X$ be a liabilities matrix and let $\tilde{X}$ be a solution to the conservative compression optimization problem defined in Definition 2.3. Let $(L,b;\mathbb{V})$ and $(\tilde{L},\tilde{b};\mathbb{V})$ be the payment systems corresponding to $X$ and $\tilde{X}$, respectively.*

1. *There exists a finite sequence of conservative compression network cycles $C^{(1)},\dots,C^{(m)}$, $m\in\mathbb{N}$, such that conservative compression can be carried out along those cycles sequentially starting from $C_1$ and finishing at $C_m$, such that $\tilde{X}$ is obtained by sequentially compressing $C^{(1)},\dots,C^{(m)}$ starting from the liabilities matrix $X$.*
2. *Consider the $m$ conservative compression network cycles from part 1. of this Corollary in Proposition 4.13. If at least one of the three conditions in Proposition 4.13 is satisfied, then the systemic risk in the payment system $(\tilde{L},\tilde{b};\mathbb{V})$ is reduced compared to the systemic risk in the payment system $(L,b;\mathbb{V})$.*
3. *Part 2. of this Corollary remains valid if condition (6) in the Definition 2.3 of $\tilde{X}$ is replaced by $a_{ij}\leq\tilde{X}_{ij}\leq X_{ij}\quad\forall i,j\in\mathcal{N}$, where $a_{ij}\in[0,X_{ij}]\ \forall i,j\in\mathcal{N}$.*

## 4.5 | Policy implications

We have provided necessary conditions for portfolio compression to be harmful. Since we have shown that portfolio compression cannot cause fundamental defaults, these are necessary conditions for portfolio compression to cause contagious defaults. This implies that policy measures that reduce the likelihood and severity of financial contagion automatically mitigate potentially negative effects of portfolio compression.

Key mitigation mechanism for financial contagion is, for example, sufficient liquidity buffers and sufficient collateral in form of initial margins. Higher levels of liquidity buffers and initial margins would make it less likely that condition (31) would be satisfied in practice, decreasing the probability of portfolio compression having negative consequences.




(a) Cycles involving 3 financial institutions. (b) Cycles involving 2 financial institutions.

**FIGURE 2** Illustration of possible compression network cycles [Color figure can be viewed at wileyonlinelibrary.com]

Could one address possible negative consequences from portfolio compression more directly? We have shown that for portfolio compression to be potentially harmful, we need to have at least one node defaulting in the non-compressed network that takes part in compression (see conditions (26) and (37)). A possible conclusion from this result would be to exclude firms with high default risk from compression activities. While there is currently no regulatory framework to do this, it might not even be desirable. This would severely restrict the possible reduction in gross exposure that can be achieved, which would lead to other disadvantages such as operational risks and so forth. We will show that allowing high risk firms to participate in portfolio compression can sometimes even strongly reduce systemic risk.

So a more nuanced approached might be more promising. As discussed in Remark 2.2, institutions participating in compression provide compression tolerances to manage their risk associated with portfolio compression. Currently, these tolerances are specified on the individual contract level as in Equation (3), and hence do not account for network spillover effects. To mitigate systemic risk, it would be beneficial to take a network perspective when deciding on compression tolerances and setting constraints in portfolio compression exercises. This is something that usually cannot be done by the individual institution requesting portfolio compression.

We will show in our case study that portfolio compression can be harmful for nodes not taking part in portfolio compression. These nodes would never provide any information or compression tolerances to the compression provider, which shows that there is a need for a financial regulator to oversee such an exercise or to provide a suitable framework for it. This could involve, for example, stress testing exercises, checking the validity of conditions like Equations (31) and (38). Alternatively, one could add conditions of this nature to the portfolio compression optimization problem.

## 4.6 | Illustration of the theoretical results

We illustrate our theoretical results by considering a network that allows for different conservative compressions. Figure 2 highlights the nine different cycles. The liabilities matrix $X$ is defined as

$$X = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{39}$$

**TABLE 1** Results for $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$. Comments: (sr) = strong reduction of systemic risk, (h) = harmful. No comment means that there is no difference between compression and no compression in terms of defaults (reduction of systemic risk). $\nu \in \{1, 2, 3\}$ represents three different liquidity buffers $b = A^{(\nu)}$: $A^{(1)} = (1, 0, 0, 0, 0)^{\top}$, $A^{(2)} = (0, 0.25, 0.25, 0.5, 0)^{\top}$, $A^{(3)} = (0.25, 0.25, 0.25, 0.25, 0)^{\top}$

| $\nu$ | Defaulting financial institutions when the following cycles are removed | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | None | Red | Blue | r & b | Green | Yellow | Pink | g & y | g & p | y & p |
| 1 | | | | | | | | | | |
| 2 | 1,2,3 | 1,3 (sr) | 1,2 (sr) | 1,4 (h) | 1,3,4 (h) | 1,2,3 | 1,2,4 (h) | 1,3,4 (h) | 1,4 (h) | 1,2,4 (h) |
| 3 | 1 | 1,3,4 (h) | 1,2,4 (h) | 1,4 (h) | 1,3,4 (h) | 1 | 1,2,4 (h) | 1,3,4 (h) | 1,4 (h) | 1,2,4 (h) |

**TABLE 2** Results for $\mathbb{V} = \mathbb{V}^{\mathrm{zero};\tilde{J}}$ with $k = 0$ and $\tilde{J} = 0$. Comments as in Table 1

| $\nu$ | Defaulting financial institutions when the following cycles are removed | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | None | Red | Blue | r & b | Green | Yellow | Pink | g & y | g & p | y & p |
| 1 | | | | | | | | | | |
| 2 | 1,2,3,4 | 1,2,3,4 | 1,2,3,4 | 1,4 (sr) | 1,2,3,4 | 1,2,3,4 | 1,2,3,4 | 1,3,4 (sr) | 1,4 (sr) | 1,2,4 (sr) |
| 3 | 1,2,3,4 | 1,2,3,4 | 1,2,3,4 | 1,4 (sr) | 1,2,3,4 | 1,2,3,4 | 1,2,3,4 | 1,3,4 (sr) | 1,4 (sr) | 1,2,4 (sr) |

The only node that will never default is node 5 since it does not have any liabilities. Whether any of the nodes $1, \dots, 4$ default will depend on the liquidity buffer $b$, the actual payment obligations arising from $X$ and the choice of compression network cycles. In the following, we will assume that the corresponding payment obligations are given by $L = X$ (i.e., $V = 1$ in the definition of $f^V$). For this liabilities matrix, the total net positions are $\bar{A}^{(X)} - \bar{L}^{(X)} = (-1, 0, 0, 0, 1)^{\top}$. Hence, any compressed network will have the same net positions. In particular, we see, that there are nine different ways of how conservative compression could be applied to this particular liabilities matrix. Formally, these compression network cycles are given by:

- Red cycle (red solid line in Figure 2(a)): $\mathcal{C}_{\mathrm{nodes}} = \{1, 2, 3\}$, $\mathcal{C}_{\mathrm{edges}} = \{(1, 2), (2, 3), (3, 1)\}$;
- Blue cycle (blue dashed line in Figure 2(a)): $\mathcal{C}_{\mathrm{nodes}} = \{1, 2, 3\}$, $\mathcal{C}_{\mathrm{edges}} = \{(1, 3), (3, 2), (2, 1)\}$;
- Green cycle (green dashed line in Figure 2(b)): $\mathcal{C}_{\mathrm{nodes}} = \{1, 2\}$, $\mathcal{C}_{\mathrm{edges}} = \{(1, 2), (2, 1)\}$;
- Yellow cycle (yellow solid line in Figure 2(b)): $\mathcal{C}_{\mathrm{nodes}} = \{2, 3\}$, $\mathcal{C}_{\mathrm{edges}} = \{(2, 3), (3, 2)\}$;
- Pink cycle (pink dotted line in Figure 2(b)): $\mathcal{C}_{\mathrm{nodes}} = \{1, 3\}$, $\mathcal{C}_{\mathrm{edges}} = \{(1, 3), (3, 1)\}$.

We can compress one or more of these cycles. We only consider compression with the maximal capacity, which is $\mu = \mu^{\max} = 1$ for all cycles. We can see that node 1 has fewer internetwork assets than liabilities. Hence, in the absence of any liquidity buffer for node 1, it will default.

We will now show that compression can have very different consequences depending on the liquidity buffer and depending on the recovery rates. Hence, an *optimal compression* in the sense of reducing the number of defaults will not just depend on the network structure but also on quantities outside the network, for example, the liquidity buffer. We will highlight the effects of the different structural conditions identified in the previous section.

We will assume that the liquidity buffer corresponds to the external assets in the compressed and non-compressed network, that is, $b = b^{\mathcal{C},\mu,\gamma} = A^{(e)}$ (where $\gamma = 0$).

Tables 1 and 2 show which financial institutions default for different liquidity buffers $b$ (corresponding to the rows in the tables) and different choices of compression network cycles (corresponding to the different columns in the table). Furthermore, Table 1 reports the results for the Eisenberg and Noe (2001) contagion mechanism, that is, $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$, whereas Table 2 reports the results for the Rogers and Veraart (2013) contagion mechanism with $\alpha = \beta = 0$, that is, zero recovery rate in case of default ($\mathbb{V} = \mathbb{V}^{\mathrm{zero};\tilde{J}}$ with $k = 0$ and $\tilde{J} = 0$). Hence, these two tables allow us to compare the effect of the third structural condition—the role of the recovery rate.

The first two structural conditions are concerned with nodes on the compression cycle. We underline all nodes that are on a compression network cycle and default both with and without compression, for example, $\underline{1},\underline{2}$,3 indicates that nodes 1 and 2 are on the compression cycle and default with and without compression. Node 3 is either not on the compression network cycle or does not default without compression.

All cells that correspond to compressed networks with no comment indicate that exactly the same financial institutions default for the compressed network as for the uncompressed network. Cells with the comment (sr) indicate that the corresponding compression mechanism strongly reduces systemic risk. Cells with the comment (h) indicate situations under which compression is harmful.

We consider three different vectors of liquidity buffers and assume that the total liquidity buffer aggregated over all nodes remains the same, that is, $\sum_{i \in \mathcal{N}} A_i^{(\nu)} = 1$ for all $\nu \in \{1, 2, 3\}$.

Only in the first row corresponding to $b = A^{(1)}$ the liquidity buffer is distributed such that no default occurs (for any choice of compression or no compression). In all other cases, node 1 will always default. Since node 5 does not have any liabilities, it will never default. For nodes 2, 3, 4, it depends on the distribution of the liquidity buffer, the recovery rates and the choice of compression whether they default or not. For nodes 1, 2, 3, there exist cycles that can be used to compress their portfolios whereas for nodes 4, 5, no such cycles exist.

We observe the following consequences of compression in line with our theoretical results:

**Reduction of systemic risk for zero recovery rates**: Table 2 contains the results for $\mathbb{V} = \mathbb{V}^{\mathrm{zero};\tilde{J}}$ with $k = 0$ and $\tilde{J} = 0$. In line with Proposition 4.12, we see a reduction in systemic risk throughout and many examples of a strong reduction in systemic risk indicated by (sr).

**Reduction of systemic risk without defaults on compression network cycle in non-compressed financial system**: consistent with Proposition 4.8, for $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$, $b = A^{(3)}$ compressing the yellow cycle (consisting of nodes 2 and 3, which both do not default) makes no difference to the set of defaults and hence reduces systemic risk.

**Compression can be harmful for nodes outside the compression network cycle**: let $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$ and $A^{(e)} = A^{(2)}$. Then, nodes 1,2,3 default without compression. When both the red and the blue cycles are compressed, nodes 1 and 4 default. This observation and the example is very similar to the example considered in Schuldenzucker et al. (2018).

**Compression can be harmful for nodes on the compression network cycle**: when $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$, $A^{(e)} = A^{(3)}$, only node 1 defaults without compression, but node 3 on the compression network cycle defaults if the red cycle is compressed and node 4 outside the cycle defaults too.

**Different choices of compression network cycles can lead to different outcomes**: let $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$ and $A^{(e)} = A^{(2)}$. Then some compression cycles strongly reduce systemic risk (e.g., compressing only the blue or only the red cycle) whereas other compression cycles are harmful (e.g., compressing both the red and the blue cycle or compressing the green cycle) or make no difference in terms of defaults (e.g., compressing the yellow cycle).

**Different distribution of liquidity buffer can lead to different outcomes**: let $\mathbb{V} = \mathbb{V}^{\mathrm{EN}}$ and consider compressing the red cycle. For some liquidity buffers (e.g., $\nu = 2$), we observe a strong reduction of systemic risk whereas for others (e.g., $\nu = 3$) this compression is harmful. There are other cases (e.g., $\nu = 1$) where compression makes no difference in terms of defaults.

**Consequences of compression depends on recovery rates**: let $b = A^{(2)}$. By comparing Tables 1 and 2, we find that compressing both the blue and the red cycle is harmful under positive recovery rates, but strongly reduces systemic risk under zero recovery rates. Furthermore compressing only the red or only the blue cycle strongly reduces systemic risk under positive recovery rates (where as it makes no difference under zero recovery rates).

**Strong reduction of systemic risk**: Let $\mathbb{V} = \mathbb{V}^{\text{EN}}$ and $b = A^{(2)}$. Compressing only the red cycle strongly reduces systemic risk since node 2 no longer defaults.

In the following, we illustrate the effects of different valuation functions in more detail.

**Example 4.15.** We set $b = A^{(2)}$ and compress sequentially first the red cycle referred to as $C^{(1)}$ and then the blue cycle referred to as $C^{(2)}$. We assume that $\gamma = 0$, that is, compression does not increase the liquidity buffer. We consider three different valuation functions $\mathbb{V}^{\text{EN}}$, $\mathbb{V}^{\text{RV}}$, and $\mathbb{V}^{\text{InitialMargin}}$.

First, let $\mathbb{V} = \mathbb{V}^{\text{EN}}$, that is, we consider the Eisenberg and Noe (2001) model. Table 1 shows that $\mathcal{D}(L, b; \mathbb{V}^{\text{EN}}) = \{1, 2, 3\}$. Compressing the red cycle yields $\mathcal{D}(L^{C^{(1)}}, b; \mathbb{V}^{\text{EN}}) = \{1, 3\}$, hence, a strong reduction in systemic risk. The repayment proportions for $i \in C_{\text{nodes}}$ satisfy

$$\mathbb{V}\left(\frac{E_1^* + \bar{L}_1}{\bar{L}_1}\right) = 0.5 = \mathbb{V}\left(\frac{E_1^{C^{(1)}\star} + \bar{L}_1^{C^{(1)}}}{\bar{L}_1^{C^{(1)}}}\right), \tag{40}$$

$$\mathbb{V}\left(\frac{E_2^* + \bar{L}_2}{\bar{L}_2}\right) = 0.75 < 1 = \mathbb{V}\left(\frac{E_2^{C^{(1)}\star} + \bar{L}_2^{C^{(1)}}}{\bar{L}_2^{C^{(1)}}}\right), \tag{41}$$

$$\mathbb{V}\left(\frac{E_3^* + \bar{L}_3}{\bar{L}_3}\right) = 0.75 = \mathbb{V}\left(\frac{E_3^{C^{(1)}\star} + \bar{L}_3^{C^{(1)}}}{\bar{L}_3^{C^{(1)}}}\right). \tag{42}$$

Hence, from Proposition 4.9, we know that this compression reduces systemic risk and here it even strongly reduces systemic risk. Even though nodes 1 and 3 default both in the compressed and the non-compressed network they repay the same relative proportion of their debt in both situations (0.5 and 0.75, respectively) and that is why compression cannot be harmful. Without compression, node 1 has a shortfall of $(1 - \mathbb{V}(\frac{E_1^* + \bar{L}_1}{\bar{L}_1}))\bar{L}_1 = 1.5$ and losses of $1.5/3 = 0.5$ hit the three creditors of node 1 (nodes 2, 3, 4). With compression node 1 has a shortfall of 1 and losses of 1/2 hit its two creditors (nodes 3 and 4), that is, even though the repayment proportions change (see Proposition 4.11), the absolute losses transmitted to nodes 3 and 4 remain the same in this example.

Suppose that after compressing the red cycle, we compress the blue cycle. Then,

$\mathcal{D}(L^{\mathcal{C}^{(1)},\mathcal{C}^{(2)}}, b; \mathbb{V}^{\mathrm{EN}}) = \{1,4\}$, hence node 4 is a new default, which is not on any of the compression cycles. Furthermore,

$$\mathbb{V}\left(\frac{E_1^{\mathcal{C}^{(1)}\mathcal{C}^{(2)}\star} + \bar{L}_1^{\mathcal{C}^{(1)}\mathcal{C}^{(2)}}}{\bar{L}_1^{\mathcal{C}^{(1)}\mathcal{C}^{(2)}}}\right) = 0 < 0.5 = \mathbb{V}\left(\frac{E_1^{\mathcal{C}^{(1)}\star} + \bar{L}_1^{\mathcal{C}^{(1)}}}{\bar{L}_1^{\mathcal{C}^{(1)}}}\right), \tag{43}$$

$$\mathbb{V}\left(\frac{E_4^{\mathcal{C}^{(1)}\mathcal{C}^{(2)}\star} + \bar{L}_4^{\mathcal{C}^{(1)}\mathcal{C}^{(2)}}}{\bar{L}_4^{\mathcal{C}^{(1)}\mathcal{C}^{(2)}}}\right) = 0.5 < 1 = \mathbb{V}\left(\frac{E_4^{\mathcal{C}^{(1)}\star} + \bar{L}_4^{\mathcal{C}^{(1)}}}{\bar{L}_4^{\mathcal{C}^{(1)}}}\right). \tag{44}$$

Hence, node 1 always defaults. It repays a strictly smaller proportion of its debt when $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ (the red and blue cycle) are compressed than when only $\mathcal{C}^{(1)}$ (the red cycle) is compressed. Since $\mathbb{V} \neq \mathbb{V}^{\mathrm{zero};\tilde{J}}$ all three necessary conditions for compression to be harmful are satisfied. Node 1 pays 0 to node 4 if both the red and the blue cycles are compressed since it has no longer any income. Node 4 cannot cope with this and defaults. When only the red cycle was compressed node, 1 was still able to pay 0.5 to node 4, which was just enough for it not to default.

Second, let $\mathbb{V} = \mathbb{V}^{\mathrm{RV}}$ with $\beta = 0.99$, which is the Rogers and Veraart (2013) model. We find that $\mathcal{D}(L, b; \mathbb{V}^{\mathrm{RV}}) = \{1,2,3,4\}$. Even the small bankruptcy costs modeled by $\beta = 0.99 < 1$ cause the total collapse of the non-compressed financial system. Compressing the red cycle yields $\mathcal{D}(L^{\mathcal{C}^{(1)}}, b; \mathbb{V}^{\mathrm{RV}}) = \{1,2,3,4\}$, that is, the same default set.

If both the blue and the red cycles are compressed, then $\mathcal{D}(L^{\mathcal{C}^{(1)},\mathcal{C}^{(2)}}, b; \mathbb{V}^{\mathrm{RV}}) = \{1,4\}$. Nodes 2 and 3 can no longer default because they do not have any liabilities any more. Hence, compressing these two cycles strongly reduces systemic risk.

Third, we repeat the analysis with $\mathbb{V} = \mathbb{V}^{\mathrm{InitialMargin}}$ where the parameter for the initial margins is $\tilde{J} = 0.1$. We consider $\beta = 1.0$ (no bankruptcy costs) and $\beta = 0.99$ (small bankruptcy costs). For both choices of $\beta$, the default sets coincide. They are given by

$$\mathcal{D}(L, b; \mathbb{V}^{\mathrm{InitialMargin}}) = \{1\}, \tag{45}$$

$$\mathcal{D}(L^{\mathcal{C}^{(1)}}, b; \mathbb{V}^{\mathrm{InitialMargin}}) = \{1,3\}, \tag{46}$$

$$\mathcal{D}(L^{\mathcal{C}^{(1)},\mathcal{C}^{(2)}}, b; \mathbb{V}^{\mathrm{InitialMargin}}) = \{1,4\}. \tag{47}$$

In line with the ordering results in Veraart (2020), we see that adding initial margins to the Eisenberg and Noe (2001) and the Rogers and Veraart (2013) models yields a better outcome for the system. But even when initial margins are available, that is, $\tilde{J} > 0$, we find that compressing first the red cycle is harmful since node 3 is a new default and then compressing the blue cycle is also harmful since node 4 is a new default. By increasing $\tilde{J}$, we could avoid all contagious defaults, but node 1 remains in default since it is a fundamental default.

*Remark* 4.16 (An optimization perspective on the numerical example). The optimal solution to the conservative compression optimization problem (Definition 2.3) for the given example corresponds to the network in which both the red and the blue cycles are removed. There exists liquidity buffers, for example, $b = A^{(2)}$ or $b = A^{(3)}$ for which this compression is harmful, since node 4 is a new default under compression in the optimally compressed network.

If we consider the non-conservative compression optimization in this example, that is, the optimization problem that has the same objective as the conservative compression optimization problem and also constraint (5) but does not have constraint (6), then the optimal solution is a network that consists of exactly one edge, the edge from node 1 to node 5, with weight 1. Node 1 remains in default (for all choices of $b$ considered in the example), hence it pays less than 1 to node 5. But node 5 cannot default since its payment obligations are zero, so technically this compression is not harmful. As discussed in D'Errico and Roukny (2021), the non-conservative compression optimization problem is solved by a bipartite graph, that is, the nodes can be split into two sets where nodes in one set have only outgoing edges and nodes in the other set have only incoming edges. (It is possible to have nodes that do not have any in- or outgoing edges in which case they can be assigned to any of the two groups.) This is exactly what we get here. Hence, losses can spread from node 1 to node 5, but node 5 cannot transmit them further.

## 5 | CONCLUSION

When does portfolio compression reduce systemic risk? We have identified three structural conditions that imply a reduction in systemic risk: no defaults on a compression network cycle in the non-compressed financial system, all nodes on the compression network cycle repay a larger proportion of their total payment obligations in the compressed system than in the non-compressed system and zero recovery rates.

Even though there are many situations under which portfolio compression reduces systemic risk, we have shown that there are circumstances under which compression can be harmful. Ultimately the danger from portfolio compression comes from firms at risk of default engaging in portfolio compression. If they then default, losses are spread in a network that now has a different structure compared to the original non-compressed network. In particular, since compression has implicitly changed the seniority structure of the debt, all those debts that were compressed prior to payment obligations becoming due have effectively been paid in full, which is obviously not the case for still outstanding debt. For nodes that do not default this change in seniority structure does not matter, since they continue to be able to satisfy all their payment obligations. For nodes that do default (and who have not posted sufficient initial margins to cover the payment shortfall) compression can imply that they spread losses now differently and some counterparties might be hit by larger losses in the compressed network.

We have shown that portfolio compression cannot cause any fundamental defaults. Portfolio compression might even potentially avoid some fundamental defaults. For compression to be harmful, we need at least one firm that defaults in the compressed network that would not have defaulted in the non-compressed network. Hence, such a default would have to be a contagious default. These findings imply that any mechanisms that reduce the likelihood of contagion in financial markets also reduce the likelihood of portfolio compression having a negative outcome. Requiring collateral (initial margins) is an obvious mechanism, which reduces the probability of contagion. Nevertheless, a residual risk remains for all not fully collateralised trades and in such a situation portfolio compression can change the outcome for the system. Another mechanism would be to require larger liquidity buffers as they fundamentally determine the likelihood of contagion, see Glasserman and Young (2015) and Paddrik et al. (2020).

Our analysis shows that classical compression tolerances that are meant to provide a safety net for compression to not increase risk, cannot fully achieve this as long as they do not account for network effects. The paths that can transmit losses from a compression network cycle to other nodes in the system are not directly observable for the participants making it difficult for them to assess potential risks from portfolio compression themselves and including them in a meaningful way as part of their compression tolerances.

In general, we find that if only firms with low default risk engage in compression activities, then this does not give cause for concern. Whether one should restrict portfolio compression services to low-risk firms is a different question. Any restrictions on who can participate would significantly limit the reduction in gross exposure that can be achieved and the associated benefits that come with it, such as operational benefits. In practice, portfolio compression is done for a wide range of reasons, and we have only considered it from a systemic risk point of view. Even then, we have found situations under which allowing high risk firms to compress their portfolio can sometimes strongly reduce systemic risk.

Ultimately, one would need to conduct a cost–benefit analysis of portfolio compression to decide whether one might want to use such a technology on a large scale or not. Using our analysis within such a cost–benefit analysis would be an interesting avenue for future research.

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## ENDNOTES

1 Note that in the context of CDS exposures, we exclude the situation in which nodes write CDSs on each other as in Schuldenzucker et al. (2018). We refer to Schuldenzucker et al. (2020) for more details on clearing in a network with CDSs if this situation is not excluded.

2 Such a setting would correspond to the classical setting of Eisenberg and Noe (2001).

3 When modeling initial and variation margins, we should keep in mind that for the purpose of this analyses, we consider fungible derivative positions, meaning portfolio compression can actually be done since these contracts are completely comparable. Hence, assuming that margin requirements are proportional to exposure size is reasonable. Initial margins are often set as 99% loss quantile for a 10-day period and hence represent a value-at-risk, see, for example, Cont (2018); BCBS IOSCO (2020), which is known to be positive homogeneous, that is, scales with position size. (This would also apply if other risk measures were used such as the expected shortfall.)

4 The payment system characterizes all payments due and hence serves as the basis for analyzing systemic risk. It is related to the original liabilities $X$ via the payment function $f^V$. We assumed that $L_{ij} = f^V(X) = VX_{ij}$ for all $i, j \in \mathcal{N}$ and $V \in [0, \infty)$. The analysis on systemic risk does not rely on this proportionality assumption, since it is conducted on the payment system directly. Therefore, one could consider more general functions $f^V$ as long as they are meaningful from an economic perspective. Since we assume that the liabilities $X$ are fungible, the proportionality assumption makes sense and is consistent with approaches used to derive initial margins as outlined before.

5 In Veraart (2020), it was shown how the corresponding clearing vector considered in Eisenberg and Noe (2001) and Rogers and Veraart (2013) can be derived from the re-evaluated equity and how the re-evaluated equity can be derived from the clearing vector. In particular, if $E^*$ is the greatest re-evaluated equity, then

$L_i^* = \mathbb{V}(\frac{E_i^* + \bar{L}_i}{\bar{L}_i})\bar{L}_i \quad \forall i \in \mathcal{M}$ and $L_i^* = 0$ for all $i \in \mathcal{N} \setminus \mathcal{M}$ denotes the corresponding clearing payments, that is, these are the total payments that node $i$ makes, which ideally would be its total nominal obligations $\bar{L}_i$ but it could be less than that.

6 When initial margins are used, it is possible that a defaulting node satisfies its payment obligations in full by covering a potential shortfall with the initial margins, see also Ghamami et al. (2021). Therefore, one cannot infer who defaults from the payments made. In the classical Eisenberg and Noe (2001) framework, this is indeed possible: there, a node defaults if and only if it does not pay its liabilities in full. By analyzing the re-evaluated equity, we can distinguish between defaulting and non-defaulting nodes and the corresponding payments follow from there.

7 Veraart (2020) distinguishes between default and distress, but we do not make this distinction here.

8 A risk orbit for an individual node has been considered in Eisenberg and Noe (2001).

9 The fundamental defaults correspond to the so-called "first-order" defaults as defined by the fictitious default algorithm of Eisenberg and Noe (2001).

10 When "regulatory capital charges are aligned with the counterparty exposure risk, the capital charge should not change. However, if cruder approaches are being used that do not accurately capture offsetting risks, such as the current exposure method (CEM) or leverage ratio approach, compression will tend to reduce the capital charge" (O' Kane, 2017).

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## APPENDIX A: COMPRESSION AND CAPITAL REQUIREMENTS

*Remark* A.1 (Compression and capital requirements). We show that portfolio compression can be beneficial for complying with the minimum leverage ratio under Basel III[10] and this effect can be captured within or model. Under Basel III, the leverage ratio is defined as (tier 1 capital)/(balance sheet and off-balance sheet exposures) and is required to be larger than 3% in Europe (slightly higher in the United States). Since the leverage ratio uses gross exposures, compression reduces the denominator of the leverage ratio and hence increases it.

We assume for now that $L = X$ and $b = A^{(e)}$. Let $(L, b; \mathbb{V})$ be the corresponding payment system and let $E^*$ be the greatest re-evaluated equity that corresponds to using the valuation function $\mathbb{V}$. By using $E^*$ as approximation of the tier 1 capital and $E^* + \bar{L}$ as approximation of the exposures, and requiring that the corresponding leverage ratio is larger than 3%, we obtain $\frac{E_i^*}{E_i^* + \bar{L}_i} \geq 0.03 \Leftrightarrow E_i^* \geq \frac{0.03}{0.97}\bar{L}_i \approx 0.031\bar{L}_i$. Any breach of this inequality could cause default. Within our model, recall that $i \in \mathcal{D}(L, b; \mathbb{V}) \Leftrightarrow E_i^* < k\bar{L}_i$, and therefore, we can set $k = \frac{3}{97}$ as threshold in Equation (11) to define such a default event demonstrating the benefit of allowing for $k > 0$. Note, that in the compressed network, the corresponding default threshold would satisfy $k\bar{L}_i^{C,\mu} \leq k\bar{L}_i$ for all $i \in \mathcal{N}$ and would therefore be lower (and hence better) than in the non-compressed network. In models with $k = 0$, compression does not affect the default threshold and one can, therefore, not capture the advantage of portfolio compression for capital requirements.

## APPENDIX B: PROOFS

### B.1 | Additional notation

Let $\mathcal{C} = (\mathcal{C}_{\text{nodes}}, \mathcal{C}_{\text{edges}}, \mu^{\max})$ be a compression network cycle with maximal capacity $\mu^{\max}$. We will use the notation $\text{pred}(i)$ for the node in $\mathcal{C}_{\text{nodes}}$ that is the predecessor of $i$ on the cycle $\mathcal{C}_{\text{nodes}}$, that is, $\text{pred}(i)$ is the node that satisfies $(\text{pred}(i), i) \in \mathcal{C}_{\text{edges}}$. Similarly, $\text{suc}(i)$ is the successor of $i$ on the cycle, that is, it is the node in $\mathcal{C}_{\text{nodes}}$ that satisfies $(i, \text{suc}(i)) \in \mathcal{C}_{\text{edges}}$.

### B.2 | Proofs of the results in Section 2

The following result formalizes the claim made that conservative compression keeps net exposures fixed while reducing gross exposures and is, therefore, in line with the corresponding definition in D'Errico and Roukny (2021).

**Lemma B.1.** *Let $X$ be a liabilities matrix for which there exists a conservative compression network cycle $\mathcal{C}$ with maximal capacity $\mu^{\max} > 0$. Let $0 < \mu \leq \mu^{\max}$ and let $X^{\mathcal{C},\mu}$ be the $\mu$-compressed liabilities matrix using cycle $\mathcal{C}$. Then,*

1. *$X^{\mathcal{C},\mu}$ is a liabilities matrix, that is, $X_{ij}^{\mathcal{C},\mu} \geq 0 \; \forall i, j \in \mathcal{N}$ and $X_{ii}^{\mathcal{C},\mu} = 0 \; \forall i \in \mathcal{N}$;*
2. *$X_{ij}^{\mathcal{C},\mu} \leq X_{ij}$ for all $i, j \in \mathcal{N}$;*
3. 
$$\bar{L}_i^{(X),\mathcal{C},\mu} = \begin{cases} \bar{L}_i^{(X)}, & \text{if } i \notin \mathcal{C}_{nodes}, \\ \bar{L}_i^{(X)} - \mu, & \text{if } i \in \mathcal{C}_{nodes}; \end{cases} \tag{B.1}$$

4. *the net positions in the compressed network $X^{\mathcal{C},\mu}$ coincide with the net positions in the original network $X$, that is, $\eta_i^{\mathcal{C},\mu} = \sum_{j\in\mathcal{N}} X_{ji}^{\mathcal{C},\mu} - \bar{L}_i^{(X),\mathcal{C},\mu} = \sum_{j\in\mathcal{N}} X_{ji} - \bar{L}_i^{(X)} = \eta_i \; \forall i \in \mathcal{N}$;*
5. *the gross positions in the compressed network $X^{\mathcal{C},\mu}$ are less than or equal to the gross positions in the original network $X$, that is, $\sum_{j\in\mathcal{N}} X_{ji}^{\mathcal{C},\mu} + \bar{L}_i^{(X),\mathcal{C},\mu} \leq \sum_{j\in\mathcal{N}} X_{ji} + \bar{L}_i^{(X)} \; \forall i \in \mathcal{N}$;*
6. *compression strictly reduces gross positions of all $i \in \mathcal{C}_{nodes}$, that is, $\sum_{j\in\mathcal{N}} X_{ji}^{\mathcal{C},\mu} + \bar{L}_i^{(X),\mathcal{C},\mu} < \sum_{j\in\mathcal{N}} X_{ji} + \bar{L}_i^{(X)}$.*

*Proof of Lemma* B.1.

1. By definition if $(i, j) \notin \mathcal{C}_{\text{edges}}$ then $X_{ij}^{\mathcal{C},\mu} = X_{ij} \geq 0$ and if $(i, j) \in \mathcal{C}_{\text{edges}}$ then $X_{ij}^{\mathcal{C},\mu} = X_{ij} - \mu \geq X_{ij} - \min_{(\nu,\mu)\in\mathcal{C}_{\text{edges}}} X_{\nu\mu} \geq 0$. Since $X_{ii} = 0$ for all $i \in \mathcal{N}$ also $X_{ii}^{\mathcal{C},\mu} = 0$ for all $i \in \mathcal{N}$.
2. This is obvious from the definition of $X^{\mathcal{C},\mu}$.
3. It follows immediately from the definition that if $i \notin \mathcal{C}_{\text{nodes}}$, then $\bar{L}_i^{(X),\mathcal{C},\mu} = \bar{L}_i^{(X)}$. Now let $i \in \mathcal{C}_{\text{nodes}}$. Then,

$$\bar{L}_i^{(X),\mathcal{C},\mu} = \sum_{j\in\mathcal{N}} X_{ij}^{\mathcal{C},\mu} = \underbrace{X_{i\text{suc}(i)}^{\mathcal{C},\mu}}_{=X_{i,\text{suc}(i)}-\mu} + \sum_{j\in\mathcal{N}\setminus\{\text{suc}(i)\}} \underbrace{X_{ij}^{\mathcal{C},\mu}}_{=X_{ij}} = \sum_{j\in\mathcal{N}} X_{ij} - \mu = \bar{L}_i^{(X)} - \mu. \tag{B.2}$$

4. It is obvious that for $i \notin C_{\text{nodes}}$ that the net position for the non-compressed network and the compressed network coincides. For $i \in C_{\text{nodes}}$, this also holds since

$$\eta_i^{C,\mu} = \sum_{j\in\mathcal{N}} X_{ji}^{C,\mu} - \bar{L}_i^{(X),C,\mu} = \underbrace{X_{\text{pred}(i)i}^{C,\mu}}_{X_{\text{pred}(i)i}-\mu} + \sum_{j\in\mathcal{N}\setminus\{\text{pred}(i)\}} \underbrace{X_{ji}^{C,\mu}}_{=X_{ji}} - \underbrace{\bar{L}_i^{(X),C,\mu}}_{\bar{L}_i^{(X)}-\mu} = \sum_{j\in\mathcal{N}} X_{ji} - \bar{L}_i^{(X)} = \eta_i. \tag{B.3}$$

5. From parts 2. and 3., we immediately get that for all $i \in \mathcal{N}$

$$\sum_{j\in\mathcal{N}} \underbrace{X_{ji}^{C,\mu}}_{\leq X_{ji}} + \underbrace{\bar{L}_i^{(X),C,\mu}}_{\leq \bar{L}_i^{(X)}} \leq \sum_{j\in\mathcal{N}} X_{ji} + \bar{L}_i^{(X)}. \tag{B.4}$$

6. Let $i \in C_{\text{nodes}}$. Since $\mu > 0$, we get

$$\begin{aligned}\sum_{j\in\mathcal{N}} X_{ji}^{C,\mu} + \underbrace{\bar{L}_i^{(X),C,\mu}}_{=\bar{L}_i^{(X)}-\mu} &= \underbrace{X_{\text{pred}(i)i}^{C,\mu}}_{=X_{\text{pred}(i)i}-\mu} + \sum_{j\in\mathcal{N}\setminus\{\text{pred}(i)\}} \underbrace{X_{ji}^{C,\mu}}_{=X_{ji}} + \bar{L}_i^{(X)} - \mu = \sum_{j\in\mathcal{N}} X_{ji} + \bar{L}_i^{(X)} - 2\mu \\ &< \sum_{j\in\mathcal{N}} X_{ji} + \bar{L}_i^{(X)}.\end{aligned} \tag{B.5}$$

□

### B.3 | Proofs of the results in Section 4

The following lemma will be used in several proofs below.

**Lemma B.2.** *Let $X$ be a liabilities matrix for which there exists a conservative compression network cycle of $X$ with maximal capacity $\mu^{\max}$. Let $(L, b; \mathbb{V})$ be the corresponding payment system, let $0 < \mu \leq \mu^{\max}$ and let $L^{C,\mu}$ be the $\mu$-compressed liabilities matrix. Set*

$$\mathcal{M} = \{i \in \mathcal{N} \mid \bar{L}_i > 0\}, \qquad \mathcal{M}^C = \{i \in \mathcal{N} \mid \bar{L}_i^C > 0\}. \tag{B.6}$$

*Let $j \in \mathcal{M} \setminus \mathcal{M}^C$. Then the following holds. First, $j \in C_{nodes}$ and $\bar{L}_j = \mu V$. Second,*

$$L_{ji} = \begin{cases} \mu V, & \text{if } i = suc(j), \\ 0, & \text{otherwise.} \end{cases} \tag{B.7}$$

*Proof of Lemma* B.2. First, let $j \in \mathcal{M} \setminus \mathcal{M}^C$. Then by the definition of the sets, it holds that $\bar{L}_j > 0$ and $\bar{L}_j^{C,\mu} = 0$. Hence $\bar{L}_j \neq \bar{L}_j^{C,\mu}$, which implies that $j \in C_{\text{nodes}}$. Since then $\bar{L}_j^{C,\mu} = \bar{L}_j - \mu V = 0$, we immediately get that $\bar{L}_j = \mu V$.

Second, from part 1. of this lemma, we know that $j \in C_{\text{nodes}}$ and $\bar{L}_j = \mu V$. For fixed $j \in \mathcal{M} \setminus \mathcal{M}^C$, we have by definition

$$L_{ji}^{C,\mu} = \begin{cases} L_{ji} - \mu V, & \text{if } i = \text{suc}(j), \\ L_{ji}, & \text{if } i \in \mathcal{N} \setminus \{\text{suc}(j)\}. \end{cases} \tag{B.8}$$

Since $L_{ji}^{C,\mu} \geq 0$ for all $i \in \mathcal{N}$, it holds that in particular $L_{j\text{suc}(j)} - \mu V \geq 0$, and hence $L_{j\text{suc}(j)} \geq \mu V$. Since,

$$0 = \bar{L}_j^{C,\mu} = \sum_{i \in \mathcal{N}} L_{ji} - \mu V = \underbrace{L_{j\text{suc}(j)}}_{\geq \mu V} + \sum_{i \in \mathcal{N} \setminus \{\text{suc}(j)\}} L_{ji} - \mu V \geq \mu V + \sum_{i \in \mathcal{N} \setminus \{\text{suc}(j)\}} L_{ji} - \mu V$$

$$= \sum_{i \in \mathcal{N} \setminus \{\text{suc}(j)\}} L_{ji}, \tag{B.9}$$

we see that since $L_{ji} \geq 0$ for all $i \in \mathcal{N} \setminus \{\text{suc}(j)\}$, it holds that $L_{ji} = 0$ for all $i \in \mathcal{N} \setminus \{\text{suc}(j)\}$. Furthermore, since $\bar{L}_j = \mu V$, we must have that $L_{j\text{suc}(j)} = \mu V$. □

*Proof of Proposition* 4.2. Recall from the definition of $E^*$ that

$$E_i^* = \Phi_i(E^*) = b_i + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i, \tag{B.10}$$

for all $i \in \mathcal{N}$. We denote by $\Phi^{C,\mu,\gamma}$ the function that corresponds to the compressed network, that is, $E^{C,\mu;\gamma;*}$ is the greatest fixed point of $\Phi^{C,\mu,\gamma}$, that is,

$$E_i^{C,\mu;\gamma;*} = \Phi_i^{C,\mu,\gamma}(E^{C,\mu;\gamma;*}) = b_i^{C,\mu,\gamma} + \sum_{j \in \mathcal{M}^C} L_{ji}^{C,\mu} \mathbb{V}\left(\frac{E_j^{C,\mu;\gamma;*} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i^{C,\mu} \tag{B.11}$$

for all $i \in \mathcal{N}$. In the following, we show that $E_i^* = E_i^{C,\mu;\gamma;*}$ for all $i \in \mathcal{N} \setminus \mathcal{O}$.

Let $i \in \mathcal{N} \setminus \mathcal{O}$. From (B.12),

$$\begin{aligned} E_i^* &= \Phi_i(E^*) = b_i + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\ &= b_i + \sum_{j \in \mathcal{M} \setminus \mathcal{O}} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) + \underbrace{\sum_{j \in \mathcal{M} \cap \mathcal{O}} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right)}_{\overset{(\star)}{=} 0} - \bar{L}_i \\ &= b_i + \sum_{j \in \mathcal{M} \setminus \mathcal{O}} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i =: f_i(E^*_{\mathcal{M} \setminus \mathcal{O}}), \end{aligned} \tag{B.12}$$

where $(\star)$ holds because by assumption $i \in \mathcal{N} \setminus \mathcal{O}$ hence there cannot be a $j \in \mathcal{M} \cap \mathcal{O}$ with $L_{ji} > 0$ otherwise this would imply that $i \in \mathcal{O}$.

Similarly, from Equation (B.13),

$$
\begin{aligned}
E_i^{C,\mu;\gamma;*} &= \Phi_i^{C,\mu,\gamma}(E^{C,\mu;\gamma;*}) = b_i^{C,\mu,\gamma} + \sum_{j\in\mathcal{M}^C} L_{ji}^{C,\mu}\mathbb{V}\left(\frac{E_j^{C,\mu;\gamma;*}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i^{C,\mu} \\
&= b_i + \sum_{j\in\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^{C,\mu;\gamma;*}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i \\
&= b_i + \sum_{j\in\mathcal{M}^C\setminus\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^{C,\mu;\gamma;*}+\bar{L}_j}{\bar{L}_j}\right) + \underbrace{\sum_{j\in\mathcal{M}^C\cap\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^{C,\mu;\gamma;*}+\bar{L}_j}{\bar{L}_j}\right)}_{\overset{(\star)}{=}0} - \bar{L}_i \\
&= b_i + \sum_{j\in\mathcal{M}^C\setminus\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^{C,\mu;\gamma;*}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \\
&\overset{(\star\star)}{=} b_i + \sum_{j\in\mathcal{M}\setminus\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^{C,\mu;\gamma;*}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i = f_i(E_{\mathcal{M}\setminus\mathcal{O}}^{C,\mu;\gamma;*}),
\end{aligned} \tag{B.13}
$$

where the justification for $(\star)$ is as before and the justification for $(\star\star)$ is that $(\mathcal{M}\setminus\mathcal{M}^C)\setminus\mathcal{O}=\emptyset$ by Lemma B.2.

Let $i\in\mathcal{O}$. From Equation (B.12) and using ideas from Equation (B.14),

$$
E_i^* = \Phi_i(E^*) = b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \tag{B.14}
$$

$$
= \underbrace{b_i + \sum_{j\in\mathcal{M}\setminus\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i}_{=f_i(E_{\mathcal{M}\setminus\mathcal{O}}^*)} + \underbrace{\sum_{j\in\mathcal{M}\cap\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right)}_{=:g_i(E_{\mathcal{M}\cap\mathcal{O}}^*)} = f_i(E_{\mathcal{M}\setminus\mathcal{O}}^*) + g_i(E_{\mathcal{M}\cap\mathcal{O}}^*). \tag{B.15}
$$

Similarly, from Equation (B.13) and using ideas from Equation (B.15),

$$
E_i^{C,\mu;\gamma;*} = \Phi_i^{C,\mu,\gamma}(E^{C,\mu;\gamma;*}) = b_i^{C,\mu,\gamma} + \sum_{j\in\mathcal{M}^C} L_{ji}^{C,\mu}\mathbb{V}\left(\frac{E_j^{C,\mu;\gamma;*}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i^{C,\mu} \tag{B.16}
$$

$$
= b_i + \sum_{j\in\mathcal{M}^C\setminus\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^{C,\mu;\gamma;*}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i + \tag{B.17}
$$

$$+\underbrace{\sum_{j\in\mathcal{M}^C\cap\mathcal{O}} L_{ji}^{C,\mu}\mathbb{V}\left(\frac{E_j^{C,\mu;\gamma;*}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right)+\mu(\gamma J+V)\mathbb{1}_{\{i\in\mathcal{C}_{\text{nodes}}\}}}_{=:g_i^C(E^{C,\mu;\gamma;*}_{\mathcal{M}^C\cap\mathcal{O}})} \tag{B.18}$$

$$=b_i+\sum_{j\in\mathcal{M}\setminus\mathcal{O}} L_{ji}\mathbb{V}\left(\frac{E_j^{C,\mu;\gamma;*}+\bar{L}_j}{\bar{L}_j}\right)-\bar{L}_i+g_i^C(E^{C,\mu;\gamma;*}_{\mathcal{M}^C\cap\mathcal{O}})=f_i(E^{C,\mu;\gamma;*}_{\mathcal{M}\setminus\mathcal{O}})+g_i^C(E^{C,\mu;\gamma;*}_{\mathcal{M}^C\cap\mathcal{O}}). \tag{B.19}$$

The function $f_{\mathcal{N}\setminus\mathcal{O}}:\mathcal{G}_{\mathcal{N}\setminus\mathcal{O}}\to\mathcal{G}_{\mathcal{N}\setminus\mathcal{O}}$ is nondecreasing and its greatest fixed point exists by Tarksi's fixed point theorem. In particular, it coincides with $E^*_{\mathcal{N}\setminus\mathcal{O}}$ and $E^{C,\mu;\gamma;*}_{\mathcal{N}\setminus\mathcal{O}}$, since we have seen that the fixed points $E^*$ and $E^{C,\mu;\gamma;*}$ can be decomposed into a component characterized by $f$ and a component characterized by $g$ or $g^C$ with nonoverlapping arguments. □

The following lemma is a reformulated version of Veraart (2020, Theorem 2.6) for the situation with compression. We will use the sequences defined in there in several proofs about the main results (i.e., in the proofs of Propositions 4.8, 4.9, 4.12).

**Lemma B.3.** *Consider the market setting of Assumption 3.9. Define, the initial equity as in Definition 4.3, that is, for all $i\in\mathcal{N}$*

$$E_i^{(0)}=b_i+\sum_{j\in\mathcal{N}}L_{ji}-\bar{L}_i \qquad E_i^{C(0);\gamma}=b_i^{C,\mu,\gamma}+\sum_{j\in\mathcal{N}}L_{ji}^{C,\mu}-\bar{L}_i^C. \tag{B.20}$$

*We define recursively the ($N$-dimensional) sequences*

$$E^{(n)}=\Phi(E^{(n-1)}), \qquad E^{C(n);\gamma}=\Phi^{C,\mu,\gamma}(E^{C(n-1);\gamma}), \tag{B.21}$$

*where $n\in\mathbb{N}$. The functions $\Phi$ and $\Phi^{C,\mu,\gamma}$ are defined in Equations (12) and (14), respectively.*

*Then,*

1. *The sequences $(E^{(n)})$ and $(E^{C(n);\gamma})$ are nonincreasing, that is, for all $i\in\mathcal{N}$ and for all $n\in\mathbb{N}_0$, it holds that*

$$E_i^{(n)}\ge E_i^{(n+1)}, \qquad E_i^{C(n);\gamma}\ge E_i^{C(n+1);\gamma}. \tag{B.22}$$

2. *The sequences defined in Equation (B.23) converge to the corresponding greatest re-evaluated equities, that is, for all $i\in\mathcal{N}$*

$$\lim_{n\to\infty}E_i^{(n)}=E_i^*, \qquad \lim_{n\to\infty}E_i^{C(n);\gamma}=E_i^{C;\gamma;*}. \tag{B.23}$$

*Proof of Lemma* B.3. First note that $\Phi$ and $\Phi^{C,\mu,\gamma}$ are nondecreasing, see Veraart (2020, Lemma A.1). The statements follow directly from Veraart (2020, Theorem 2.6). $\square$

**Lemma B.4.** *Consider the market setting of Assumption 3.9. Let $\mathcal{F}$ be the fundamental defaults in the non-compressed network and let $\mathcal{F}^C$ be the fundamental defaults in the compressed network (see Definition 4.4). Then, $\mathcal{F} \subseteq \mathcal{D}(L, b; \mathbb{V})$ and $\mathcal{F}^C \subseteq \mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V})$.*

*Proof of Lemma* B.4. Recall, that $\mathcal{F} = \{i \in \mathcal{N} \mid E_i^{(0)} < k\bar{L}_i\}$ and $\mathcal{F}^C = \{i \in \mathcal{N} \mid E^{C(0);\gamma} < k\bar{L}_i^{C,\mu}\}$. We consider the sequences $(E^{(n)})$ and $(E^{C(n);\gamma})$ defined in Equation (B.23). Let $i \in \mathcal{F}$. Then, by Lemma B.3, $\forall m \in \mathbb{N}$: $k\bar{L}_i > E_i^{(0)} \geq E_i^{(m)} \geq \lim_{n\to\infty} E_i^{(n)} = E_i^*$ and hence $i \in \mathcal{D}(L, b; \mathbb{V})$. Similarly, let $i \in \mathcal{F}^C$. Then, by Lemma B.3, $\forall m \in \mathbb{N}$: $k\bar{L}_i^{C,\mu} > E_i^{C(0);\gamma} \geq E_i^{C(m);\gamma} \geq \lim_{n\to\infty} E_i^{C(n);\gamma} = E_i^{C,\mu;\gamma;*}$ and hence $i \in \mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V})$. $\square$

*Proof of Lemma* 4.5. Recall that $b_i^{C,\mu,0} = b_i$ for all $i \in \mathcal{N}$ and from the definition of $b^{C,\mu,\gamma}$, it follows immediately that $b_i^{C,\mu,0} \leq b_i^{C,\mu,\gamma}$ for all $i \in \mathcal{N}$ and for all $\gamma \in [0, 1]$. If $i \notin \mathcal{C}_{\text{nodes}}$, then one immediately sees that $E_i^{(0)} = E_i^{C(0);0} \leq E_i^{C(0);\gamma}$. If $i \in \mathcal{C}_{\text{nodes}}$, then

$$E_i^{C(0);0} = b_i^{C,\mu,0} + \sum_{j\in\mathcal{N}} L_{ji}^{C,\mu} - \bar{L}_i^{C,\mu} = b_i^{C,\mu,0} + \sum_{j\in\mathcal{N}} L_{ji}^{C,\mu} - (\bar{L}_i - \mu V) \tag{B.24}$$

$$= b_i^{C,\mu,0} + \sum_{j\in\mathcal{N}\setminus\{\text{pred}(i)\}} \underbrace{L_{ji}^{C,\mu}}_{=L_{ji}} + \underbrace{L_{\text{pred}(i)i}^{C,\mu}}_{=L_{\text{pred}(i)i}-\mu V} - \bar{L}_i + \mu V \tag{B.25}$$

$$= \underbrace{b_i^{C,\mu,0}}_{=b_i} + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i = E_i^{(0)}. \tag{B.26}$$

Now let $\gamma \in [0, 1]$, then

$$E_i^{C(0);\gamma} = b_i^{C,\mu,\gamma} + \sum_{j\in\mathcal{N}} L_{ji}^{C,\mu} - \bar{L}_i^{C,\mu} \geq b_i^{C,\mu,0} + \sum_{j\in\mathcal{N}} L_{ji}^{C,\mu} - \bar{L}_i^{C,\mu} = E_i^{C(0);0}. \tag{B.27}$$

$\square$

*Proof of Proposition* 4.6. To prove the first statement, let $i \in \mathcal{F}^C \setminus \mathcal{C}_{\text{nodes}}$. Then,

$$E_i^{C(0);\gamma} = \underbrace{b_i^{C,\mu,\gamma}}_{=b_i} + \sum_{j\in\mathcal{N}} \underbrace{L_{ji}^{C,\mu}}_{=L_{ij}} - \underbrace{\bar{L}_i^{C,\mu}}_{=\bar{L}_i} < k \underbrace{\bar{L}_i^{C,\mu}}_{=\bar{L}_i} \iff E_i^{(0)} = b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i < k\bar{L}_i, \tag{B.28}$$

and hence $i \in \mathcal{F}$. Let $i \in \mathcal{F}^C \cap \mathcal{C}_{\text{nodes}}$. Then,

$$E_i^{C(0);\gamma} = \underbrace{b_i^{C,\mu,\gamma}}_{=b_i+\gamma\mu J} + \underbrace{\sum_{j\in\mathcal{N}} L_{ji}^{C,\mu}}_{=\sum_{j\in\mathcal{N}} L_{ji}-\mu V} - \underbrace{\bar{L}_i^{C,\mu}}_{=\bar{L}_i-\mu V} < k \underbrace{\bar{L}_i^{C,\mu}}_{=\bar{L}_i-\mu V} \tag{B.29}$$

$$\Longleftrightarrow b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i < k\bar{L}_i \underbrace{-k\mu V - \gamma\mu J}_{=-\mu(kV+\gamma J)} \tag{B.30}$$

$$\Longleftrightarrow b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i < k\bar{L}_i \underbrace{-\mu(kV+\gamma J)}_{\leq 0} \leq k\bar{L}_i, \tag{B.31}$$

and hence $i \in \mathcal{F}$.

To prove the second statement, let $i \in \mathcal{F} \setminus \mathcal{F}^C$. From the arguments used in part 1., it is clear that $i \in \mathcal{C}_{\text{nodes}}$. Furthermore, since $i \notin \mathcal{F}^C$

$$E_i^{C(0);\gamma} = \underbrace{b_i^{C,\mu,\gamma}}_{=b_i+\gamma\mu J} + \underbrace{\sum_{j\in\mathcal{N}} L_{ji}^{C,\mu}}_{=\sum_{j\in\mathcal{N}} L_{ji}-\mu V} - \underbrace{\bar{L}_i^{C,\mu}}_{=\bar{L}_i-\mu V} \geq k \underbrace{\bar{L}_i^{C,\mu}}_{=\bar{L}_i-\mu V} \tag{B.32}$$

$$\Longleftrightarrow b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i \geq k\bar{L}_i \underbrace{-k\mu V - \gamma\mu J}_{=-\mu(kV+\gamma J)} \tag{B.33}$$

$$\Longleftrightarrow b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i \geq k\bar{L}_i \underbrace{-\mu(kV+\gamma J)}_{\leq 0}. \tag{B.34}$$

Since $i \in \mathcal{F}$, it holds that $E_i^{(0)} = b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i < k\bar{L}_i$. Combining these two inequalities gives $k\bar{L}_i > b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i \geq k\bar{L}_i - \mu(kV+\gamma J)$. For this to hold, we need $(kV+\gamma J) > 0$. □

*Proof of Theorem* 4.7.

1. Assume that $\mathcal{D}(L, b; \mathbb{V}) \cap \mathcal{C}_{\text{nodes}} = \emptyset$. Then by Proposition 4.8, this compression reduces systemic risk, which is a contradiction to it being harmful.
2. Suppose compression is harmful. This means that there exists a node $\nu \in \mathcal{N}$ such that $\nu \in \mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V})$ and $\nu \in \mathcal{N} \setminus \mathcal{D}(L, b; \mathbb{V})$, that is, node $\nu$ defaults if compression is done but does not default without compression. This in particular implies that $\nu \in \mathcal{M}^C = \{i \in \mathcal{N} \mid \bar{L}_i^{C,\mu} > 0\}$ and that $E_\nu^{C,\mu;\gamma;*} < k\bar{L}_\nu^{C,\mu} \leq k\bar{L}_\nu \leq E_\nu^*$. Therefore $E_\nu^{C,\mu;\gamma;*} < E_\nu^*$. Then by Propo-

sition 4.9 part 1., this implies that there exists an $i \in \mathcal{C}_{\text{nodes}}$ satisfying

$$\mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*} + \bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right) < \mathbb{V}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right). \tag{B.35}$$

Hence, there exists an $i \in \mathcal{C}_{\text{nodes}}$ satisfying

$$\mathbb{V}\left(\frac{E_i^{\mathcal{C},\mu;\gamma;*} + \bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}\right) < \mathbb{V}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right) \leq 1, \tag{B.36}$$

which implies that $\mathbb{V}(\frac{E_i^{\mathcal{C},\mu;\gamma;*} + \bar{L}_i^{\mathcal{C},\mu}}{\bar{L}_i^{\mathcal{C},\mu}}) < 1$, which implies that $i \in \mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V})$.

3. Assume that $\mathbb{V} = \mathbb{V}^{\text{zero};\tilde{J}}$. By Proposition 4.12, we know that this compression reduces systemic risk, which is a contradiction to the assumption that it is harmful.

□

*Proof of Proposition* 4.8. To prove this statement, we consider a fixed point iteration as in Veraart (2020). We consider the sequences $(E^{(n)})$ and $(E^{\mathcal{C}(n);\gamma})$ defined in Equation (B.23). By Lemma 4.5, we know that $E_i^{\mathcal{C}(0);\gamma} \geq E_i^{\mathcal{C}(0);0} = E_i^{(0)}$ for all $i \in \mathcal{N}$ and for all $\gamma \in [0, 1]$. We will prove by induction that if $\{i \in \mathcal{C}_{\text{nodes}} \mid E_i^* < k\bar{L}_i\} = \emptyset$ then

$$E_i^{\mathcal{C}(n);\gamma} \geq E_i^{\mathcal{C}(n);0} = E_i^{(n)} \text{ for all } i \in \mathcal{N}, \tag{B.37}$$

holds for all $n \in \mathbb{N}_0$. Once this has been shown it follows that

$$E_i^{\mathcal{C};\gamma;*} = \lim_{n\to\infty} E_i^{\mathcal{C}(n);\gamma} \geq E_i^{\mathcal{C};0;*} = \lim_{n\to\infty} E_i^{\mathcal{C}(n);0} = \lim_{n\to\infty} E_i^{(n)} = E_i^*, \tag{B.38}$$

for all $i \in \mathcal{N}$, which is the statement of the theorem.

By Lemma B.3, $(E^{(n)})$ and $(E^{\mathcal{C}(n);\gamma})$ are nonincreasing and converge to the greatest re-evaluated equity in the non-compressed and in the compressed network, respectively. This implies that in particular, $E_i^{(n)} \geq \lim_{m\to\infty} E_i^{(m)} = E_i^*$ for all $i \in \mathcal{N}$ and for all $n \in \mathbb{N}_0$ and hence for all $n \in \mathbb{N}_0$, it holds that $E_i^{(n)} \geq E_i^* \geq k\bar{L}_i \quad \forall i \in \mathcal{C}_{\text{nodes}}$ and hence

$$\{i \in \mathcal{C}_{\text{nodes}} \mid E_i^{(n)} < k\bar{L}_i\} = \emptyset. \tag{B.39}$$

We now start our proof of Equation (B.39) by induction. Let $n = 0$. Since

$$E_i^{(0)} = b_i + \sum_{j\in\mathcal{N}} L_{ji} - \bar{L}_i, \qquad E_i^{\mathcal{C}(0);\gamma} = b_i^{\mathcal{C},\mu,\gamma} + \sum_{j\in\mathcal{N}} L_{ji}^{\mathcal{C},\mu} - \bar{L}_i^{\mathcal{C}}, \tag{B.40}$$

we are in exactly the same situation as in Lemma 4.5 in which it was shown that indeed $E_i^{\mathcal{C}(0);\gamma} \geq E_i^{\mathcal{C}(0);0} = E_i^{(0)}$ for all $i \in \mathcal{N}$.

Suppose Equation (B.39) holds for a fixed $n \in \mathbb{N}_0$. We show that it also holds for $n+1$. Then, by the definition of the sequences

$$E_i^{(n+1)} = \Phi_i(E^{(n)}) = b_i + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i, \tag{B.41}$$

$$E_i^{C(n+1);\gamma} = \Phi_i^{C;\gamma}(E^{C(n);\gamma}) = b_i^{C,\mu,\gamma} + \sum_{j \in \mathcal{M}^C} L_{ji}^{C,\mu} \mathbb{V}\left(\frac{E_j^{C(n);\gamma} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i^{C,\mu}. \tag{B.42}$$

First note that by the monotonicity of $\mathbb{V}$, the definition of $b^{C,\mu,\gamma}$, and the induction hypothesis that $E_i^{C(n);\gamma} \geq E_i^{C(n);0}$ for all $i \in \mathcal{N}$, we immediately see that

$$E_i^{C(n+1);\gamma} = \Phi_i^{C;\gamma}(E^{C(n);\gamma}) = b_i^{C,\mu,\gamma} + \sum_{j \in \mathcal{M}^C} L_{ji}^{C,\mu} \mathbb{V}\left(\frac{E_j^{C(n);\gamma} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i^{C,\mu} \tag{B.43}$$

$$\geq b_i^{C,\mu,0} + \sum_{j \in \mathcal{M}^C} L_{ji}^{C,\mu} \mathbb{V}\left(\frac{E_j^{C(n);0} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i^{C,\mu} = \Phi_i^{C;0}(E^{C(n);0}) \tag{B.44}$$

$$= E_i^{C(n+1);0}, \tag{B.45}$$

holds for all $i \in \mathcal{N}$. Hence, it remains to show that $E_i^{C(n+1);0} = E_i^{(n+1)}$ for all $i \in \mathcal{N}$.

Let $i \in \mathcal{C}_{\text{nodes}}$. Then,

$$\begin{aligned} E_i^{C(n+1);0} &= b_i^{C,\mu,0} + \sum_{j \in \mathcal{M}^C} L_{ji}^{C,\mu} \mathbb{V}\left(\frac{E_j^{C(n);0} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \underbrace{\bar{L}_i^{C,\mu}}_{=\bar{L}_i - \mu V} \\ &= b_i^{C,\mu,0} + \sum_{j \in \mathcal{M}^C, (j,i) \in \mathcal{C}_{\text{edges}}} L_{ji}^{C,\mu} \mathbb{V}\left(\frac{E_j^{C(n);0} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) \\ &\quad + \sum_{j \in \mathcal{M}^C, (j,i) \notin \mathcal{C}_{\text{edges}}} \underbrace{L_{ji}^{C,\mu}}_{=L_{ji}} \mathbb{V}\left(\frac{E_j^{C(n);0} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i + \mu V. \end{aligned} \tag{B.46}$$

Note that there exists at most one $j \in \mathcal{M}^C$ with $(j, i) \in \mathcal{C}_{\text{edges}}$. As before we write pred$(i)$ for the predecessor of $i$ on the cycle $\mathcal{C}_{\text{edges}}$, that is, pred$(i)$ is the index of the node that satisfies $(\text{pred}(i), i) \in \mathcal{C}_{\text{edges}}$.

We distinguish between two cases. First, suppose that $\text{pred}(i) \in \mathcal{M}^C$. Then,

$$\sum_{j\in\mathcal{M}^C,(j,i)\in\mathcal{C}_{\text{edges}}} L_{ji}^{C,\mu}\mathbb{V}\left(\frac{E_j^{C(n);0}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) = (L_{\text{pred}(i),i}-\mu V)\mathbb{V}\left(\frac{E_{\text{pred}(i)}^{C(n);0}+\bar{L}_{\text{pred}(i)}^{C,\mu}}{\bar{L}_{\text{pred}(i)}^{C,\mu}}\right). \tag{B.47}$$

By the induction hypothesis $E_{\text{pred}(i)}^{C(n);0} = E_{\text{pred}(i)}^{(n)}$ and by Equation (B.41), it holds that $E_{\text{pred}(i)}^{(n)} \geq k\bar{L}_{\text{pred}(i)}$ since $\text{pred}(i) \in \mathcal{C}_{\text{nodes}}$. By the definition of $\mathbb{V}$ this implies that

$$\mathbb{V}\left(\frac{E_{\text{pred}(i)}^{C(n);0}+\bar{L}_{\text{pred}(i)}^{C,\mu}}{\bar{L}_{\text{pred}(i)}^{C,\mu}}\right) = 1 = \mathbb{V}\left(\frac{E_{\text{pred}(i)}^{(n)}+\bar{L}_{\text{pred}(i)}}{\bar{L}_{\text{pred}(i)}}\right). \tag{B.48}$$

Hence,

$$\begin{aligned}\sum_{j\in\mathcal{M}^C,(j,i)\in\mathcal{C}_{\text{edges}}} L_{ji}^{C,\mu}\mathbb{V}\left(\frac{E_j^{C(n);0}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) &= (L_{\text{pred}(i)i}-\mu V) \\ &= L_{\text{pred}(i)i}\mathbb{V}\left(\frac{E_{\text{pred}(i)}^{C(n);0}+\bar{L}_{\text{pred}(i)}^{C,\mu}}{\bar{L}_{\text{pred}(i)}^{C,\mu}}\right)-\mu V.\end{aligned} \tag{B.49}$$

Furthermore, since $\text{pred}(i) \in \mathcal{M}^C$, we obtain by Lemma B.2 part 2. that $L_{ji} = 0$ for all $j \in \mathcal{M}\setminus\mathcal{M}^C$ and hence

$$\sum_{j\in\mathcal{M}\setminus\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right) = 0. \tag{B.50}$$

By plugging Equation (B.51) into (B.48), we immediately obtain that

$$\begin{aligned}E_i^{C(n+1);0} &= b_i^{C,\mu,0} + \sum_{j\in\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^{C(n);0}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i + \mu V - \mu V \\ &= b_i^{C,\mu,0} + \sum_{j\in\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i \quad \text{(by induction hypothesis)}\end{aligned} \tag{B.51}$$

$$= b_i^{C,\mu,0} + \sum_{j\in\mathcal{M}^C,E_j^{(n)}\geq k\bar{L}_j} L_{ji}\underbrace{\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right)}_{=1=\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)} + \sum_{j\in\mathcal{M}^C,E_j^{(n)}<k\bar{L}_j} L_{ji}\underbrace{\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right)}_{\overset{(\star)}{=}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)} - \bar{L}_i \tag{B.52}$$

$$= b_i^{C,\mu,0} + \sum_{j\in\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \tag{B.53}$$

$$= \underbrace{b_i^{C,\mu,0}}_{=b_i} + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right) - \underbrace{\sum_{j\in\mathcal{M}\setminus\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)}_{=0 \text{ by (B.52)}} - \bar{L}_i = E_i^{(n+1)}, \tag{B.54}$$

where $(\star)$ holds because if for an $j \in \mathcal{N}$ it holds that $E_j^{(n)} < k\bar{L}_j$ then $j \in \mathcal{N} \setminus \mathcal{C}_{\text{nodes}}$ since by assumption no defaults occur on the compression cycle. Hence, $\bar{L}_j^{C,\mu} = \bar{L}_j$.

Second, suppose that $\text{pred}(i) \in \mathcal{M} \setminus \mathcal{M}^C$. Then, by Lemma B.2 part 2.

$$\sum_{j\in\mathcal{M}^C,(j,i)\in\mathcal{C}_{\text{edges}}} L_{ji}^{C,\mu}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) = 0. \tag{B.55}$$

Furthermore, again from Lemma B.2 part 2. and using the assumption that no node on the compression network cycle defaults we get

$$\begin{aligned}\sum_{j\in\mathcal{M}\setminus\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right) &= L_{\text{pred}(i)i}\underbrace{\mathbb{V}\left(\frac{E_{\text{pred}(i)}^{(n)}+\bar{L}_{\text{pred}(i)}}{\bar{L}_{\text{pred}(i)}}\right)}_{=1} \\ &= L_{\text{pred}(i)i} = \mu V,\end{aligned} \tag{B.56}$$

where we used the fact that $\text{pred}(i) \in \mathcal{M} \setminus \mathcal{M}^C$.

By plugging Equation (B.59) into (B.48), we obtain

$$E_i^{C(n+1);0} = \underbrace{b_i^{C,\mu,0}}_{=b_i} + \underbrace{\sum_{j\in\mathcal{M}^C,(j,i)\in\mathcal{C}_{\text{edges}}} L_{ji}^{C,\mu}\mathbb{V}\left(\frac{E_j^{C(n);0}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right)}_{=0} \tag{B.57}$$

$$+ \sum_{j\in\mathcal{M}^C,(j,i)\notin\mathcal{C}_{\text{edges}}} L_{ji}\mathbb{V}\left(\frac{E_j^{C(n);0}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i + \mu V \tag{B.58}$$

$$= b_i + \sum_{j\in\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^{C(n);0} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i + \mu V \quad (\text{since}\,\text{pred}(i) \in \mathcal{M}\setminus\mathcal{M}^C) \tag{B.59}$$

$$= b_i + \sum_{j\in\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i + \mu V \quad (\text{by induction hypothesis}) \tag{B.60}$$

$$= b_i + \sum_{j\in\mathcal{M}^C, E_j^{(n)}\geq k\bar{L}_j} L_{ji}\underbrace{\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right)}_{=1=\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)} \tag{B.61}$$

$$+ \sum_{j\in\mathcal{M}^C, E_j^{(n)}< k\bar{L}_j} L_{ji}\underbrace{\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right)}_{\overset{=}{(\star)}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)} - \bar{L}_i + \mu V \tag{B.62}$$

$$= b_i + \sum_{j\in\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i + \mu V \tag{B.63}$$

$$= b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \underbrace{\sum_{j\in\mathcal{M}\setminus\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right)}_{=\mu V \text{ by } (B.60)} - \bar{L}_i + \mu V \tag{B.64}$$

$$= b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i = \Phi_i(E^{(n)}) = E_i^{(n+1)}, \tag{B.65}$$

where the same argument was used in $(\star)$ as before, namely that nodes with $E_j^{(n)} < k\bar{L}_j$ cannot be on the compression network cycle.

Let $i \notin \mathcal{C}_{\text{nodes}}$. Then, using the induction hypothesis in the second line we get

$$E_i^{C(n+1);0} = \underbrace{b_i^{C,\mu,0}}_{=b_i} + \sum_{j\in\mathcal{M}^C} \underbrace{L_{ji}^{C,\mu}}_{=L_{ji}} \mathbb{V}\left(\frac{E_j^{C(n);0} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \underbrace{\bar{L}_i^{C,\mu}}_{=\bar{L}_i} \tag{B.66}$$

$$= b_i + \sum_{j\in\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i \tag{B.67}$$

$$= b_i + \sum_{j\in\mathcal{M}^C, E_j^{(n)} \geq k\bar{L}_j} L_{ji}\, \underbrace{\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right)}_{=1=\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)} \tag{B.68}$$

$$+ \sum_{j\in\mathcal{M}^C, E_j^{(n)} < k\bar{L}_j} L_{ji}\, \underbrace{\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right)}_{\underset{(\star)}{=}\mathbb{V}\left(\frac{E_j^{(n)}+\bar{L}_j}{\bar{L}_j}\right)} - \bar{L}_i \tag{B.69}$$

$$= b_i + \sum_{j\in\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \tag{B.70}$$

$$= b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \underbrace{\sum_{j\in\mathcal{M}\setminus\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right)}_{=0 (\text{by Lemma B.2 part 2.})} - \bar{L}_i = \Phi_i(E^{(n)}) = E_i^{(n+1)}, \tag{B.71}$$

using again the fact in $(\star)$ that nodes with $E_j^{(n)} < k\bar{L}_j$ cannot be on the compression network cycle.

Hence, we have shown that indeed for all $n \in \mathbb{N}_0$ and for all $i \in \mathcal{N} E_i^{C(n+1);\gamma} \geq E_i^{C(n+1);0} = E_i^{(n+1)}$, which completes the induction. Hence, for all $i \in \mathcal{N}$

$$E_i^{C;\gamma;*} = \lim_{n\to\infty} E_i^{C(n);\gamma} \geq E_i^{C;0;*} = \lim_{n\to\infty} E_i^{C(n);0} = \lim_{n\to\infty} E_i^{(n)} = E_i^*. \tag{B.72}$$

From this, it follows immediately that $\mathcal{D}(L^{C,\mu}, b^{C,\mu,\gamma}; \mathbb{V}) \subseteq \mathcal{D}(L^{C,\mu}, b^{C,\mu,0}; \mathbb{V}) = \mathcal{D}(L, b; \mathbb{V})$. □

*Proof of Proposition* 4.9. We will prove part 1. first and will show that parts 2. and 3. are essentially corollaries of part 1.

1. Suppose that condition (31) is satisfied. We will prove now that compression can only increase the re-evaluated equity. This proof uses similar arguments as in the proof of Proposition 4.8. Again we consider the sequences $(E^{(n)})$ and $(E^{C(n);\gamma})$ defined in Equation (B.23).
   Using the same argument as in the proof of Proposition 4.8, we know from Lemma B.3 that $\lim_{n\to\infty} E_j^{(n)} = E_j^*$ and $\lim_{n\to\infty} E_j^{C(n);\gamma} = E_j^{C,\mu;\gamma;*}$ exist for all $j \in \mathcal{N}$. Furthermore, $(E^{(n)})$ and $(E^{C(n);\gamma})$ are decreasing sequences, that is, they converge to their limits from above. In particular, $E_j^{(n)} \geq E_j^*$ and $E_j^{C(n);\gamma} \geq E_j^{C,\mu;\gamma;*}$ for all $j \in \mathcal{N}$ and for all $n \in \mathbb{N}_0$ and since $\mathbb{V}$ is nondecreasing this implies that
$$\mathbb{V}\left(\frac{E_j^{C(n);\gamma} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) \geq \mathbb{V}\left(\frac{E_j^{C,\mu;\gamma;*} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) \quad \forall j \in \mathcal{N}. \tag{B.73}$$
   Combining this with Equation (31), we obtain that for all $n \in \mathbb{N}_0$
$$\mathbb{V}\left(\frac{E_j^{C(n);\gamma} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) \geq \mathbb{V}\left(\frac{E_j^{C,\mu;\gamma;*} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) \geq \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) \quad \forall j \in \mathcal{C}_{\text{nodes}}. \tag{B.74}$$
   We will prove by induction that for all $n \in \mathbb{N}_0$
$$E_i^{C(n);\gamma} \geq E_i^* \qquad \forall i \in \mathcal{N}. \tag{B.75}$$
   Once this has been shown, it follows that $E_i^{C,\mu;\gamma;*} = \lim_{n\to\infty} E_i^{C(n);\gamma} \geq E_i^* \quad \forall i \in \mathcal{N}$, which is the statement of the proposition.
   For the start of the induction, we consider $n = 0$. By Lemma 4.5, we know that $E_i^{C(0);\gamma} \geq E_i^{(0)}$ for all $i \in \mathcal{N}$ and for all $\gamma \in [0,1]$. Since $(E^{(n)})$ is converging to $E^*$ from above this implies that $E_i^{C(0);\gamma} \geq E_i^{(0)} \geq E_i^*$ for all $i \in \mathcal{N}$.
   Next assume that the statement (B.79) holds for an $n \in \mathbb{N}_0$. We will show that it holds for $n+1$. We distinguish between two cases:
   Let $i \in \mathcal{N} \setminus \mathcal{C}_{\text{nodes}}$. Then,
$$E_i^{C(n+1);\gamma} = \Phi_i^C(E^{C(n);\gamma}) = \underbrace{b_i^{C,\mu,\gamma}}_{=b_i} + \sum_{j\in\mathcal{M}^C} \underbrace{L_{ji}^{C,\mu}}_{=L_{ji}} \mathbb{V}\left(\frac{E_j^{C(n);\gamma} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \underbrace{\bar{L}_i^{C,\mu}}_{=\bar{L}_i} \tag{B.76}$$

$$= b_i + \sum_{j\in\mathcal{M}^C\cap\mathcal{C}_{\text{nodes}}} L_{ji} \underbrace{\mathbb{V}\left(\frac{E_j^{C(n);\gamma}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right)}_{\geq\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right)\quad\text{by (B.78)}} + \sum_{j\in\mathcal{M}^C\setminus\mathcal{C}_{\text{nodes}}} L_{ji} \underbrace{\mathbb{V}\left(\frac{E_j^{C(n);\gamma}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right)}_{=\mathbb{V}\left(\frac{E_j^{C(n);\gamma}+\bar{L}_j}{\bar{L}_j}\right)} - \bar{L}_i \tag{B.77}$$

$$\geq b_i + \sum_{j\in\mathcal{M}^C\cap\mathcal{C}_{\text{nodes}}} L_{ji}\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right) + \sum_{j\in\mathcal{M}^C\setminus\mathcal{C}_{\text{nodes}}} L_{ji} \underbrace{\mathbb{V}\left(\frac{E_j^{C(n);\gamma}+\bar{L}_j}{\bar{L}_j}\right)}_{\geq\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right)\quad\text{by ind. hyp. }\&\mathbb{V}\text{monotone}} - \bar{L}_i \tag{B.78}$$

$$\geq b_i + \sum_{j\in\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \tag{B.79}$$

$$= \underbrace{b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i}_{=\Phi_i(E^*)=E_i^*} - \sum_{j\in\mathcal{M}\setminus\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right) \tag{B.80}$$

$$= E_i^* - \underbrace{\sum_{j\in\mathcal{M}\setminus\mathcal{M}^C} L_{ji}\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right)}_{=0} = E_i^*. \tag{B.81}$$

Note that $\sum_{j\in\mathcal{M}\setminus\mathcal{M}^C} L_{ji}\mathbb{V}(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}) = 0$, since $i \in \mathcal{N}\setminus\mathcal{C}_{\text{nodes}}$ and by Lemma B.2 $L_{ji} = 0$ for all $j \in \mathcal{M}\setminus\mathcal{M}^C$.

Let $i \in \mathcal{C}_{\text{nodes}}$. Then,

$$E_i^{C(n+1);\gamma} = \Phi_i^C(E^{C(n);\gamma}) = \underbrace{b_i^{C,\mu,\gamma}}_{\geq b_i} + \sum_{j\in\mathcal{M}^C} L_{ji}^{C,\mu}\mathbb{V}\left(\frac{E_j^{C(n);\gamma}+\bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \underbrace{\bar{L}_i^{C,\mu}}_{=\bar{L}_i-\mu V} \tag{B.82}$$

$$\geq b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}\cap\mathcal{C}_{\text{nodes}}} L_{ji}^{\mathcal{C},\mu} \underbrace{\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right)}_{\geq\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right)\quad\text{by (B.78)}} \tag{B.83}$$

$$+ \sum_{j\in\mathcal{M}^{\mathcal{C}}\setminus\mathcal{C}_{\text{nodes}}} L_{ji}^{\mathcal{C},\mu} \underbrace{\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j^{\mathcal{C},\mu}}{\bar{L}_j^{\mathcal{C},\mu}}\right)}_{=\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma}+\bar{L}_j}{\bar{L}_j}\right)\quad\text{since } j\notin\mathcal{C}_{\text{nodes}}} - \bar{L}_i + \mu V \tag{B.84}$$

$$\geq b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}\cap\mathcal{C}_{\text{nodes}}} L_{ji}^{\mathcal{C},\mu} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) \tag{B.85}$$

$$+ \sum_{j\in\mathcal{M}^{\mathcal{C}}\setminus\mathcal{C}_{\text{nodes}}} \underbrace{L_{ji}^{\mathcal{C},\mu}}_{=L_{ji}} \underbrace{\mathbb{V}\left(\frac{E_j^{\mathcal{C}(n);\gamma} + \bar{L}_j}{\bar{L}_j}\right)}_{\geq\mathbb{V}\left(\frac{E_j^*+\bar{L}_j}{\bar{L}_j}\right)\quad\text{by ind. hyp. and} \mathbb{V} \text{ nondecreasing}} - \bar{L}_i + \mu V \tag{B.86}$$

$$\geq b_i + \sum_{j\in\mathcal{M}^{\mathcal{C}}\cap\mathcal{C}_{\text{nodes}}\setminus\{\text{pred}(i)\}} \underbrace{L_{ji}^{\mathcal{C},\mu}}_{L_{ji}} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) \tag{B.87}$$

$$+ (L_{\text{pred}(i)i} - \mu V)\mathbb{V}\left(\frac{E_{\text{pred}(i)}^* + \bar{L}_{\text{pred}(i)}}{\bar{L}_{\text{pred}(i)}}\right)\mathbb{I}_{\{\text{pred}(i)\in\mathcal{M}^{\mathcal{C}}\}} \tag{B.88}$$

$$+ \sum_{j\in\mathcal{M}^{\mathcal{C}}\setminus\mathcal{C}_{\text{nodes}}} L_{ji}\mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i + \mu V \tag{B.89}$$

$$=: (\star\star). \tag{B.90}$$

Let $\mathrm{pred}(i) \in \mathcal{M}^C$. Then,

$$(\star\star) = b_i + \sum_{j \in \mathcal{M}^C} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i + \mu V\left(1 - \mathbb{V}\left(\frac{E^*_{\mathrm{pred}(i)} + \bar{L}_{\mathrm{pred}(i)}}{\bar{L}_{\mathrm{pred}(i)}}\right)\right) \tag{B.91}$$

$$= \underbrace{b_i + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i}_{=\Phi_i(E^*)=E_i^*} + \underbrace{\mu V\left(1 - \mathbb{V}\left(\frac{E^*_{\mathrm{pred}(i)} + \bar{L}_{\mathrm{pred}(i)}}{\bar{L}_{\mathrm{pred}(i)}}\right)\right)}_{\geq 0} \tag{B.92}$$

$$- \underbrace{\sum_{j \in \mathcal{M} \setminus \mathcal{M}^C} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right)}_{=0 \text{ since } \mathrm{pred}(i) \in \mathcal{M}^C} \geq E_i^*. \tag{B.93}$$

Let $\mathrm{pred}(i) \in \mathcal{M} \setminus \mathcal{M}^C$. Then,

$$(\star\star) = b_i + \sum_{j \in \mathcal{M}^C} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i + \mu V \tag{B.94}$$

$$= \underbrace{b_i + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i}_{=\Phi_i(E^*)=E_i^*} - \underbrace{\sum_{j \in \mathcal{M} \setminus \mathcal{M}^C} L_{ji} \mathbb{V}\left(\frac{E_j^* + \bar{L}_j}{\bar{L}_j}\right)}_{=\mu V \mathbb{V}\left(\frac{E^*_{\mathrm{pred}(i)} + \bar{L}_{\mathrm{pred}(i)}}{\bar{L}_{\mathrm{pred}(i)}}\right)} + \mu V \tag{B.95}$$

$$= E_i^* + \underbrace{\mu V\left(1 - \mathbb{V}\left(\frac{E^*_{\mathrm{pred}(i)} + \bar{L}_{\mathrm{pred}(i)}}{\bar{L}_{\mathrm{pred}(i)}}\right)\right)}_{\geq 0} \geq E_i^*. \tag{B.96}$$

Hence, this completes the induction and the result follows.

2. Suppose that Equation (32) holds. Then by the definition of $\mathbb{V}$, we immediately get that

$$1 = \mathbb{V}\left(\frac{E_i^{C,\mu;\gamma;*} + \bar{L}_i^{C,\mu}}{\bar{L}_i^{C,\mu}}\right) \geq \mathbb{V}\left(\frac{E_i^* + \bar{L}_i}{\bar{L}_i}\right) \quad \forall i \in \mathcal{C}_{\mathrm{nodes}}. \tag{B.97}$$

Hence, the result follows with part 1. of this Proposition.

3. Suppose that Equation (33) holds, then $\mathbb{V}(\frac{E_i^{C,\mu;\gamma;*} + \bar{L}_i^{C,\mu}}{\bar{L}_i^{C,\mu}}) = 1 \quad \forall i \in \mathcal{C}_{\mathrm{nodes}}$ and hence the statement follows directly from part 2. of this Proposition since condition (32) is satisfied.

□

*Proof of Proposition* 4.11. Let $i \in \mathcal{C}_{\text{nodes}}$. Hence, $\bar{L}_i > 0$ and $\bar{L}_i^{C,\mu} = \bar{L}_i - \mu$. Let $j = \text{suc}(i) \in \mathcal{N}$ and first suppose that $\bar{L}_i^{C,\mu} > 0$. Then,

$$\Pi_{ij}^{C,\mu} = \Pi_{i\text{suc}(i)}^{C,\mu} = \frac{L_{i\text{suc}(i)}^{C,\mu}}{\bar{L}_i^{C,\mu}} = \frac{L_{i\text{suc}(i)} - \mu}{\bar{L}_i - \mu} \leq \frac{L_{i\text{suc}(i)}}{\bar{L}_i} = \Pi_{i\text{suc}(i)}, \tag{B.98}$$

since

$$\frac{L_{i\text{suc}(i)} - \mu}{\bar{L}_i - \mu} \leq \frac{L_{i\text{suc}(i)}}{\bar{L}_i} \quad \Leftrightarrow L_{i\text{suc}(i)}\bar{L}_i - \mu\bar{L}_i \quad \leq L_{i\text{suc}(i)}\bar{L}_i - L_{i\text{suc}(i)}\mu \Leftrightarrow \quad 0 \leq \mu(\bar{L}_i - L_{i\text{suc}(i)}), \tag{B.99}$$

is always satisfied. Second, suppose that $\bar{L}_i^{C,\mu} = 0$. Then, $\Pi_{i\text{suc}(i)}^{C,\mu} = 0 \leq \frac{L_{i\text{suc}(i)}}{\bar{L}_i} = \Pi_{i\text{suc}(i)}$. Now let $j \in \mathcal{N} \setminus \{\text{suc}(i)\}$. Then, $\Pi_{ij}^{C,\mu} = \frac{L_{ij}^{C,\mu}}{\bar{L}_i^{C,\mu}} = \frac{L_{ij}}{\bar{L}_i} = \Pi_{ij}$.

Let $i \in \mathcal{N} \setminus \mathcal{C}_{\text{nodes}}$ and $j \in \mathcal{N}$. Then, $\bar{L}_i^{C,\mu} = \bar{L}_i$. If $\bar{L}_i > 0$, then $\bar{L}_i^{C,\mu} = \bar{L}_i > 0$ and $\Pi_{ij}^{C,\mu} = \frac{L_{ij}^{C,\mu}}{\bar{L}_i^{C,\mu}} = \frac{L_{ij}}{\bar{L}_i} = \Pi_{ij}$; and if $\bar{L}_i = 0$, then $\bar{L}_i^{C,\mu} = \bar{L}_i = 0$ and $\Pi_{ij}^{C,\mu} = 0 = \Pi_{ij}$. □

We will use the following Lemma to prove Theorem 4.12.

**Lemma B.5.** *Let $E_i^{C(n);\gamma}, E_i^{(n)}, \bar{L}_i^{C,\mu}, \bar{L}_i \in \mathbb{R}$, $E_i^{C(n);\gamma} \geq E_i^{(n)}$, $\bar{L}_i^{C,\mu} \leq \bar{L}_i$, $\tilde{J} \in [0,1]$ and $k \geq 0$. Then,*

$$\mathbb{1}_{\{E_i^{C(n);\gamma} \geq k\bar{L}_i^{C,\mu}\}} + \tilde{J}\mathbb{1}_{\{E_i^{C(n);\gamma} < k\bar{L}_i^{C,\mu}\}} \geq \mathbb{1}_{\{E_i^{(n)} \geq k\bar{L}_i\}} + \tilde{J}\mathbb{1}_{\{E_i^{(n)} < k\bar{L}_i\}}. \tag{B.100}$$

*Proof of Lemma* B.5. For $\tilde{J} = 1$, the result follows directly. Let $\tilde{J} \in [0, 1)$.

First, consider the case that $\mathbb{1}_{\{E_i^{C(n);\gamma} \geq k\bar{L}_i^{C,\mu}\}} + \tilde{J}\mathbb{1}_{\{E_i^{C(n);\gamma} < k\bar{L}_i^{C,\mu}\}} = 1$. Then, $1 \geq \mathbb{1}_{\{E_i^{(n)} \geq k\bar{L}_i\}} + \tilde{J}\mathbb{1}_{\{E_i^{(n)} < k\bar{L}_i\}}$ by the definition of the indicator function.

Second, suppose that $\mathbb{1}_{\{E_i^{C(n);\gamma} \geq k\bar{L}_i^{C,\mu}\}} + \tilde{J}\mathbb{1}_{\{E_i^{C(n);\gamma} < k\bar{L}_i^{C,\mu}\}} = \tilde{J}$. Then, $E_i^{C(n);\gamma} < k\bar{L}_i^{C,\mu}$ and since $k\bar{L}_i \geq k\bar{L}_i^{C,\mu}$, we obtain that $E_i^{(n)} \leq E_i^{C(n);\gamma} < k\bar{L}_i^{C,\mu} \leq k\bar{L}_i$. Hence, $\mathbb{1}_{\{E_i^{(n)} \geq k\bar{L}_i\}} + \tilde{J}\mathbb{1}_{\{E_i^{(n)} < k\bar{L}_i\}} = \tilde{J} = \mathbb{1}_{\{E_i^{C(n);\gamma} \geq k\bar{L}_i^{C,\mu}\}} + \tilde{J}\mathbb{1}_{\{E_i^{C(n);\gamma} < k\bar{L}_i^{C,\mu}\}}$. □

*Proof of Proposition* 4.12. We proceed similarly as in the proof of Proposition 4.8. We consider two sequences $(E^{(n)})$ and $(E^{C(n);\gamma})$ defined in Equation (B.23) but now assume that $\mathbb{V} = \mathbb{V}^{\text{zero};\tilde{J}}$.

We will prove by induction that

$$E_i^{C(n);\gamma} \geq E_i^{(n)} \text{ for all } i \in \mathcal{N}, \tag{B.101}$$

holds for all $n \in \mathbb{N}_0$. Once this has been shown, it follows that $E_i^{C,\mu;\gamma;*} = \lim_{n\to\infty} E_i^{C(n);\gamma} \geq \lim_{n\to\infty} E_i^{(n)} = E_i^*$ for all $i \in \mathcal{N}$, which is the statement of the theorem.

Let $n = 0$. Then the result follows directly from Lemma 4.5.

We now assume that Equation (B.105) holds true for an $n \in \mathbb{N}$. We show that Equation (B.105) is true for $n+1$. Consider

$$E_i^{(n+1)} = \Phi_i(E^{(n)}) = b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}^{\text{zero};\tilde{J}}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i, \tag{B.102}$$

$$E_i^{C(n+1);\gamma} = \Phi_i^{C;\gamma}(E^{C(n);\gamma}) = b_i^{C,\mu,\gamma} + \sum_{j\in\mathcal{M}^C} L_{ji}^{C,\mu}\mathbb{V}^{\text{zero};\tilde{J}}\left(\frac{E_j^{C(n);\gamma} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i^{C,\mu}. \tag{B.103}$$

By the induction hypothesis (B.105) $E_i^{C(n);\gamma} \geq E_i^{(n)}$ for all $i \in \mathcal{N}$ and hence by Lemma (B.5)

$$\mathbb{I}_{\{E_j^{C(n);\gamma} \geq k\bar{L}_j^{C,\mu}\}} + \tilde{J}\mathbb{I}_{\{E_j^{C(n);\gamma} < k\bar{L}_j^{C,\mu}\}} \geq \mathbb{I}_{\{E_j^{(n)} \geq k\bar{L}_j\}} + \tilde{J}\mathbb{I}_{\{E_j^{(n)} < k\bar{L}_j\}}. \tag{B.104}$$

Hence,

$$E_i^{C(n+1);\gamma} = b_i^{C,\mu,\gamma} + \sum_{j\in\mathcal{M}^C} L_{ji}^{C,\mu}\mathbb{V}^{\text{zero};\tilde{J}}\left(\frac{E_j^{C(n);\gamma} + \bar{L}_j^{C,\mu}}{\bar{L}_j^{C,\mu}}\right) - \bar{L}_i^{C,\mu} \tag{B.105}$$

$$= \underbrace{b_i^{C,\mu,\gamma}}_{\geq b_i} + \sum_{j\in\mathcal{N}} L_{ji}^{C,\mu}\left(\mathbb{I}_{\{E_j^{C(n);\gamma} \geq k\bar{L}_j^{C,\mu}\}} + \tilde{J}\mathbb{I}_{\{E_j^{C(n);\gamma} < k\bar{L}_j^{C,\mu}\}}\right) - \bar{L}_i^{C,\mu} \tag{B.106}$$

$$\geq b_i + \sum_{j\in\mathcal{N}} L_{ji}^{C,\mu}\left(\mathbb{I}_{\{E_j^{(n)} \geq k\bar{L}_j\}} + \tilde{J}\mathbb{I}_{\{E_j^{(n)} < k\bar{L}_j\}}\right) - \bar{L}_i^{C,\mu} = (*), \tag{B.107}$$

where we used Equation (B.104) to derive the inequality. If $i \notin C_{\text{nodes}}$, then

$$(*) = b_i + \sum_{j\in\mathcal{N}} L_{ji}\left(\mathbb{I}_{\{E_j^{(n)} \geq k\bar{L}_j\}} + \tilde{J}\mathbb{I}_{\{E_j^{(n)} < k\bar{L}_j\}}\right) - \bar{L}_i \tag{B.108}$$

$$= b_i + \sum_{j\in\mathcal{M}} L_{ji}\mathbb{V}^{\text{zero};\tilde{J}}\left(\frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j}\right) - \bar{L}_i \tag{B.109}$$

$$= E_i^{(n+1)}. \tag{B.110}$$

If $i \in \mathcal{C}_{\text{nodes}}$, then

$$(*) = b_i + \sum_{j \in \mathcal{N} \setminus \{\text{pred}(i)\}} L_{ji} \left( \mathbb{I}_{\{E_j^{(n)} \geq k\bar{L}_j\}} + \tilde{J} \mathbb{I}_{\{E_j^{(n)} < k\bar{L}_j\}} \right) \tag{B.111}$$

$$+ (L_{\text{pred}(i)i} - \mu V) \left( \mathbb{I}_{\{E_{\text{pred}(i)}^{(n)} \geq k\bar{L}_{\text{pred}(i)}\}} + \tilde{J} \mathbb{I}_{\{E_{\text{pred}(i)}^{(n)} < k\bar{L}_{\text{pred}(i)}\}} \right) - (\bar{L}_i - \mu V) \tag{B.112}$$

$$= b_i + \sum_{j \in \mathcal{N}} L_{ji} \left( \mathbb{I}_{\{E_j^{(n)} \geq k\bar{L}_j\}} + \tilde{J} \mathbb{I}_{\{E_j^{(n)} < k\bar{L}_j\}} \right) - \bar{L}_i \tag{B.113}$$

$$+ \underbrace{\mu V \left( 1 - \left( \mathbb{I}_{\{E_{\text{pred}(i)}^{(n)} \geq k\bar{L}_{\text{pred}(i)}\}} + \tilde{J} \mathbb{I}_{\{E_{\text{pred}(i)}^{(n)} < k\bar{L}_{\text{pred}(i)}\}} \right) \right)}_{\geq 0 \text{ since } \tilde{J} \in [0,1]} \tag{B.114}$$

$$\geq b_i + \sum_{j \in \mathcal{N}} L_{ji} \left( \mathbb{I}_{\{E_j^{(n)} \geq k\bar{L}_j\}} + \tilde{J} \mathbb{I}_{\{E_j^{(n)} < k\bar{L}_j\}} \right) - \bar{L}_i \tag{B.115}$$

$$= b_i + \sum_{j \in \mathcal{M}} L_{ji} \mathbb{V}^{\text{zero};\tilde{J}} \left( \frac{E_j^{(n)} + \bar{L}_j}{\bar{L}_j} \right) - \bar{L}_i = E_i^{(n+1)}. \tag{B.116}$$

Hence, $E_i^{\mathcal{C}(n+1)} \geq E_i^{(n+1)}$ for all $i \in \mathcal{N}$, which completes the induction.

To see that indeed systemic risk is reduced by compression here, we use the results of the first part, namely $E_i^{\mathcal{C},\mu;\gamma;*} \geq E_i^*$ for all $i \in \mathcal{N}$. Suppose $\mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}^{\text{zero};\tilde{J}}) \neq \emptyset$, otherwise there is nothing to show. Let $i \in \mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}^{\text{zero};\tilde{J}})$. Then, $E_i^{\mathcal{C},\mu;\gamma;*} < k\bar{L}_i^{\mathcal{C},\mu}$ and hence $E_i^* \leq E_i^{\mathcal{C},\mu;\gamma;*} < k\bar{L}_i^{\mathcal{C},\mu} \leq k\bar{L}_i$. This implies that $i \in \mathcal{D}(L, b; \mathbb{V}^{\text{zero};\tilde{J}})$. Hence,

$$\mathcal{D}(L^{\mathcal{C},\mu}, b^{\mathcal{C},\mu,\gamma}; \mathbb{V}^{\text{zero};\tilde{J}}) = \{i \in \mathcal{N} \mid E_i^{\mathcal{C},\mu;\gamma;*} < k\bar{L}_i^{\mathcal{C},\mu}\} \subseteq \{i \in \mathcal{N} \mid E_i^* < k\bar{L}_i\} \tag{B.117}$$

$$= \mathcal{D}(L, b; \mathbb{V}^{\text{zero};\tilde{J}}). \tag{B.118}$$

□

*Proof of Proposition* 4.13. Suppose, condition 1., that is, formula (37), is satisfied, that is, $\mathcal{D}(L, b; \mathbb{V}) \cap \mathcal{C}_{\text{nodes}}^{\text{all}} = \emptyset$. Then, in particular, $\mathcal{D}(L, b; \mathbb{V}) \cap \mathcal{C}_{\text{nodes}}^{(1)} = \emptyset$. Then, by Proposition 4.8, compressing $\mathcal{C}^{(1)}$ reduces systemic risk. In particular, $\mathcal{D}(L^{\mathcal{C}^{(1)}}, b^{\mathcal{C}^{(1)}}; \mathbb{V}) \subseteq \mathcal{D}(L, b; \mathbb{V})$. Combining this results with Equation (37) implies that $\mathcal{D}(L^{\mathcal{C}^{(1)}}, b^{\mathcal{C}^{(1)}}; \mathbb{V}) \cap \mathcal{C}^{(2)}_{\text{nodes}} = \emptyset$. Then, applying Proposition 4.8 to the system $\mathcal{D}(L^{\mathcal{C}^{(1)}}, b^{\mathcal{C}^{(1)}}; \mathbb{V})$ by compressing cycle $\mathcal{C}^{(2)}$ yields $\mathcal{D}(L^{\mathcal{C}^{(1)},\mathcal{C}^{(2)}}, b^{\mathcal{C}^{(1)},\mathcal{C}^{(2)}}; \mathbb{V}) \subseteq \mathcal{D}(L^{\mathcal{C}^{(1)}}, b^{\mathcal{C}^{(1)}}; \mathbb{V})$. By repeating these arguments, we obtain that

$$\mathcal{D}(L^{\mathcal{C}^{(1)},\dots,\mathcal{C}^{(m)}}, b^{\mathcal{C}^{(1)},\dots,\mathcal{C}^{(m)}}; \mathbb{V}) \subseteq \mathcal{D}(L^{\mathcal{C}^{(1)},\dots,\mathcal{C}^{(m-1)}}, b^{\mathcal{C}^{(1)},\dots,\mathcal{C}^{(m-1)}}; \mathbb{V}) \tag{B.119}$$

$$\subseteq \dots \subseteq \mathcal{D}(L^{\mathcal{C}^{(1)}}, b^{\mathcal{C}^{(1)}}; \mathbb{V}) \subseteq \mathcal{D}(L, b; \mathbb{V}), \tag{B.120}$$

and hence indeed compressing sequentially $\mathcal{C}^{(1)}, \dots, \mathcal{C}^{(m)}$ reduces systemic risk.

Suppose the second condition, that is, Equation (38) holds, then Proposition 4.9 yields the statement. If the third condition, that is, $\mathbb{V} = \mathbb{V}^{\text{zero};\tilde{J}}$ holds, the statement follows from Proposition 4.12. □

*Proof of Corollary* 4.14.

1. This statement and its proof is given in D'Errico & Roukny (2021, EC.5., e-companion).
2. This statement follows directly from Proposition 4.13 by using the sequence of cycles to obtain $\tilde{X}$ that is guaranteed to exist from part 1. of this Corollary.
3. The algorithm developed in D'Errico & Roukny (2021, Algorithm 2, e-companion) to determine the sequence of cycles $\mathcal{C}^{(1)}, \dots, \mathcal{C}^{(m)}$ can still be used if a lower bound $a_{ij} \geq 0$ is introduced. The results derived in Proposition 4.13 hold for all possible compression volumes and not just for the original $\mu_i^{\max}$, $i \in \{1, \dots, m\}$. Hence, in line with Remark 2.2, the results remain valid for the case of a lower bound that is not necessarily 0.

□