THE SIZE-RAMSEY NUMBER OF POWERS OF BOUNDED DEGREE TREES

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Abstract. Given a positive integer s, the *s-colour size-Ramsey number* of a graph H is the smallest integer m such that there exists a graph G with m edges with the property that, in any colouring of $E(G)$ with s colours, there is a monochromatic copy of H. We prove that, for any positive integers k and s, the s-colour size-Ramsey number of the kth power of any n-vertex bounded degree tree is linear in n. As a corollary we obtain that the s-colour size-Ramsey number of n-vertex graphs with bounded treewidth and bounded degree is linear in n, which answers a question raised by Kamčev, Liebenau, Wood and Yepremyan.

§1. Introduction

Given graphs G and H and a positive integer s, we denote by $G \to (H)_s$ the property that any s-colouring of the edges of G contains a monochromatic copy of H. We are interested in the problem proposed by Erdős, Faudree, Rousseau and Schelp [13] of determining the minimum integer m for which there is a graph G with m edges such that property $G \to (H)_2$ holds. Formally, the *s-colour size-Ramsey number* $\hat{r}_s(H)$ of a graph H is defined as follows:

$$\hat{r}_s(H) = \min\{e(G) \colon G \to (H)_s\}.$$

Answering a question posed by Erdős [12], Beck [3] showed that $\hat{r}_2(P_n) = O(n)$ by means of a probabilistic proof. Alon and Chung [1] proved the same fact by explicitly constructing a graph G with $O(n)$ edges such that $G \to (P_n)_2$. In the last decades many successive improvements were obtained in order to determine the size-Ramsey number of paths (see, e.g., [3, 5, 11] for lower bounds, and [3, 10, 11, 25] for upper bounds). The best known bounds for paths are $5n/2 - 15/2 \leqslant \hat{r}_2(P_n) \leqslant 74n$ from [11]. For any $s \geqslant 2$ colours, Dudek and Prałat [11] and Krivelevich [24] proved that there are positive constants c and C such that $cs^2 n \leqslant \hat{r}_s(P_n) \leqslant Cs^2(\log s)n$.

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Moving away from paths, Beck [3] asked whether $\hat{r}_2(H)$ is linear for any bounded degree graph. This question was later answered negatively by Rödl and Szemerédi [30], who constructed a family $\{H_n\}_{n\in\mathbb{N}}$ of n-vertex graphs of maximum degree $\Delta(H_n) \leqslant 3$ such that $\hat{r}_2(H_n) = \Omega(n \log^{1/60} n)$. The current best upper bound for the size-Ramsey number of graphs with bounded degree was obtained in [22] by Kohayakawa, Rödl, Schacht and Szemerédi, who proved that for any positive integer Δ there is a constant c such that, for any graph H with n vertices and maximum degree Δ, we have

$$\hat{r}_2(H) \leqslant cn^{2-1/\Delta} \log^{1/\Delta} n.$$

For more results on the size-Ramsey number of bounded degree graphs see [8, 14, 16, 17, 20, 21].

Let us turn our attention to powers of bounded degree graphs. Let H be a graph with n vertices and let k be a positive integer. The *kth power* H^k of H is the graph with vertex set $V(H)$ in which there is an edge between distinct vertices u and v if and only if u and v are at distance at most k in H. Recently it was proved that the 2-colour size-Ramsey number of powers of paths and cycles is linear [6]. This result was extended to any fixed number s of colours in [15], i.e.,

$$\hat{r}_s(P_n^k) = O_{k,s}(n) \quad \text{and} \quad \hat{r}_s(C_n^k) = O_{k,s}(n). \tag{1}$$

In our main result (Theorem 1) we extend (1) to bounded powers of bounded degree trees. We prove that for any positive integers k and s, the s-colour size-Ramsey number of the kth power of any n-vertex bounded degree tree is linear in n.

Theorem 1. *For any positive integers k, Δ and s and any n-vertex tree T with $\Delta(T) \leqslant \Delta$, we have*

$$\hat{r}_s(T^k) = O_{k,\Delta,s}(n).$$

We remark that Theorem 1 is equivalent to the following result for the 'general' or 'off-diagonal' size-Ramsey number $\hat{r}(H_1, \ldots, H_s) = \min\{e(G)\colon G \to (H_1, \ldots, H_s)\}$: if $H_i = T_i^k$ for $i = 1, \ldots, s$ where $T_1, \ldots, T_s$ are bounded degree trees, then $\hat{r}(H_1, \ldots, H_s)$ is linear in $\max_{1\leqslant i\leqslant s} v(H_i)$. To see this, it is sufficient to apply Theorem 1 to a tree containing the disjoint union of $T_1, \ldots, T_s$.

The graph that we present to prove Theorem 1 does not depend on T, but only on Δ, k and n. Moreover, our proof not only gives a monochromatic copy of T^k for a given T, but a monochromatic subgraph that contains a copy of the kth power of every n-vertex tree with maximum degree at most Δ. That is, we prove the existence of so called 'partition universal graphs' with $O_{k,\Delta,s}(n)$ edges for the family of powers T^k of n-vertex trees with $\Delta(T) \leqslant \Delta$.

Theorem 1 was announced in the extended abstract [4]. While finalizing this paper, we learned that Kamčev, Liebenau, Wood, and Yepremyan [19] proved, among other things, that the 2-colour size-Ramsey number of an n-vertex graph with bounded degree and bounded treewidth is $O(n)$[1]. This is equivalent to our result for $s = 2$. Indeed, any graph with bounded treewidth and bounded maximum degree is contained in a suitable blow-up of some bounded degree tree [9, 31] and a blow-up of a bounded degree tree is contained in the power of another bounded degree tree. Conversely, bounded powers of bounded degree trees have bounded treewidth and bounded degree. Therefore, we obtain the following equivalent version of Theorem 1, which generalises the result from [19] and answers one of their main open questions (Question 5.2 in [19]).

[1]They in fact formulate this for the general 2-colour size-Ramsey number $\hat{r}(H_1, H_2)$.

Corollary 2. *For any positive integers k, Δ and s and any n-vertex graph H with treewidth k and $\Delta(H) \leqslant \Delta$, we have*

$$\hat{r}_s(H) = O_{k,\Delta,s}(n).$$

The proof of Theorem 1 follows the strategy developed in [15], proving the result by induction on the number of colours s. Very roughly speaking, we start with a graph G with suitable properties and, given any s-colouring of the edges of G ($s \geqslant 2$), either we obtain a monochromatic copy of the power of the desired tree in G, or we obtain a large subgraph H of G that is coloured with at most $s-1$ colours; moreover, the graph H that we obtain is such that we can apply the induction hypothesis on it. Naturally, we design the requirements on our graphs in such a way that this induction goes through. As it turns out, the graph G will be a certain blow-up of a random-like graph. While this approach seems uncomplicated upon first glance, the proof requires a variety of additional ideas and technical details.

To implement the above strategy, we need, among other results, two new and key ingredients which are interesting on their own: (i) a result that states that for any sufficiently large graph G, either G contains a large expanding subgraph or there is a given number of reasonably large disjoint subsets of $V(G)$ without any edge between any two of them (see Lemma 9[2]); (ii) an embedding result that states that in order to embed a power T^k of a tree T in a certain blow-up of a graph G it is enough to find an embedding of an auxiliary tree T' in G (see Lemma 11).

§2. Auxiliary results

In this section we state a few results which will be needed in the proof of our main theorem. The first lemma guarantees that, in a graph G that has edges between large subsets of vertices, there exists a long "transversal" path along a constant number of large subsets of vertices of G. Denote by $e_G(X,Y)$ the number of edges between two disjoint sets X and Y in a graph G.

Lemma 3 ([6, Lemma 3.5])**.** *For every integer $\ell \geqslant 1$ and every $\gamma > 0$ there exists $d_0 = 2 + 4/(\gamma(\ell+1))$ such that the following holds for any $d \geqslant d_0$. Let G be a graph on dn vertices such that for every pair of disjoint sets $X, Y \subseteq V(G)$ with $|X|, |Y| \geqslant \gamma n$ we have $e_G(X,Y) > 0$. Then for every family $V_1, \ldots, V_\ell \subseteq V(G)$ of pairwise disjoint sets each of size at least γdn, there is a path $P_n = (x_1, \ldots, x_n)$ in G with $x_i \in V_j$ for all $1 \leqslant i \leqslant n$, where $j \equiv i \pmod{\ell}$.*

We will also use the classical Chernoff's inequality and Kővári–Sós–Turán theorem.

Theorem 4 (Chernoff's inequality)**.** *Let $0 < \varepsilon \leqslant 3/2$. If X is a sum of independent Bernoulli random variables then*

$$\mathbb{P}(|X - \mathbb{E}[X]| > \varepsilon\mathbb{E}[X]) \leqslant 2 \cdot e^{-(\varepsilon^2/3)\mathbb{E}[X]}.$$

Theorem 5 (Kővári–Sós–Turán [23])**.** *Let $k \geqslant 1$ and let G be a bipartite graph with x vertices in each vertex class. If G contains no copy of $K_{2k,2k}$, then G has at most $4x^{2-1/(2k)}$ edges.*

§3. Bijumbledness, expansion and embedding of trees

In this section we provide the necessary tools to obtain the desired monochromatic embedding of a power of a tree in the proof of Theorem 1. We start by defining the expanding property of a graph.

[2]We are grateful to the authors of [19], who pointed out to us that similar lemmas have been proved in [28, 29].

Property 6 (Expanding)**.** *A graph G is* (n, a, b)-expanding *if for all $X \subseteq V(G)$ with $|X| \leqslant a(n-1)$, we have $|N_G(X)| \geqslant b|X|$.*

Here $N_G(X)$ is the set of neighbours of X, i.e. all vertices in $V(G)$ that share an edge with some vertex from X. The following embedding result due to Friedman and Pippenger [14] guarantees the existence of copies of bounded degree trees in expanding graphs.

Lemma 7. *Let n and Δ be positive integers and G a non-empty graph. If G is $(n, 2, \Delta + 1)$-expanding, then G contains any n-vertex tree with maximum degree Δ as a subgraph.*

Owing to Lemma 7, we are interested in graph properties that guarantee expansion. One such property is bijumbledness, defined below.

Property 8 (Bijumbledness)**.** *A graph G on N vertices is* (p, θ)-bijumbled *if, for all disjoint sets X and $Y \subseteq V(G)$ with $\theta/p < |X| \leqslant |Y| \leqslant pN|X|$, we have $\big|e_G(X, Y) - p|X||Y|\big| \leqslant \theta\sqrt{|X||Y|}$.*

We remark that, in the definition above, we restrict our sets X and Y not to be too small; such a restriction is not usually imposed when defining bijumbledness, but we have to do so here for certain technical reasons.

Note that bijumbledness immediately implies that

$$\text{for all disjoint sets } X,\, Y \subseteq V(G) \text{ with } |X|,\, |Y| > \theta/p \text{ we have } e_G(X, Y) > 0. \tag{2}$$

Moreover, a simple averaging argument guarantees that in a (p, θ)-bijumbled graph G on N vertices we have

$$\left| e(G) - p\binom{N}{2} \right| \leqslant \theta N. \tag{3}$$

We now state the first main novel ingredient in the proof of our main result (Theorem 1). The following lemma ensures that in a sufficiently large graph we get an expanding subgraph with appropriate parameters or we get reasonably large disjoint subsets of vertices that span no edges between them. This result was inspired by [27, Theorem 1.5]. Furthermore, we remark that similar results have been proved in [28, 29].

Lemma 9. *Let $f \geqslant 0$, $D \geqslant 0$, $\ell \geqslant 2$ and $\eta > 0$ be given and let $A = (\ell - 1)(D + 1)(\eta + f) + \eta$. If G is a graph on at least An vertices, then*

(i) there is a non-empty set $Z \subseteq V(G)$ such that $G[Z]$ is (n, f, D)-expanding, or

(ii) there exist $V_1, \ldots, V_\ell \subseteq V(G)$ such that $|V_i| \geqslant \eta n$ for $1 \leqslant i \leqslant \ell$ and $e_G(V_i, V_j) = 0$ for $1 \leqslant i < j \leqslant \ell$.

Proof. Let us assume that (i) does not hold. Since G is not (n, f, D)-expanding, we can take $V_1 \subseteq V(G)$ of maximum size satisfying that $|V_1| \leqslant (\eta + f)n$ and $|N_G(V_1)| < D|V_1|$. We claim that $|V_1| \geqslant \eta n$. Assume, for the sake of contradiction that $|V_1| < \eta n$. Let

$$W_1 = V(G) \setminus (V_1 \cup N_G(V_1)).$$

Then $|W_1| > An - (D + 1)\eta n > 0$. Applying that (i) does not hold, we get $X \subseteq W_1$ such that $|X| \leqslant f(n - 1)$ and $|N_{G[W_1]}(X)| < D|X|$. Note that $N_G(X) \subseteq N_{G[W_1]}(X) \cup N_G(V_1)$. Thus

$$\begin{aligned} |N_G(X \dot\cup V_1)| &= |N_{G[W_1]}(X) \cup N_G(V_1)| \\ &< D(|X| + |V_1|). \end{aligned}$$

Also $|X \dot\cup V_1| \leqslant (\eta + f)n$, deriving a contradiction to the maximality of V_1.

Let $1 \leqslant k \leqslant \ell - 2$ and suppose we have $(V_1, \dots, V_k)$ such that

(I) $|V_i| \geqslant \eta n$, for $1 \leqslant i \leqslant k$;

(II) $e(V_i, V_j) = 0$, for $1 \leqslant i < j \leqslant k$;

(III) $|\bigcup_{i=1}^{k} (V_i \cup N_G(V_i))| < k(D+1)(\eta + f)n$.

We can increase this sequence in the following way. Let $W_k = V(G) \smallsetminus \bigcup_{i=1}^{k} (V_i \cup N_G(V_i))$ and note that

$$\begin{aligned} |W_k| &\overset{\text{(III)}}{\geqslant} An - (\ell - 2)(D+1)(\eta + f)n \\ &\geqslant (D+1)(\eta + f)n + \eta n \\ &> 0. \end{aligned}$$

Since (i) does not hold, there exists $V_{k+1} \subseteq W_k$ of maximum size with $|V_{k+1}| \leqslant (\eta + f)n$ such that $|N_{G[W_k]}(V_{k+1})| < D|V_{k+1}|$. Note that $e_G(V_i, V_{k+1}) \leqslant e_G(V_i, W_{k+1}) = 0$, for every $1 \leqslant i \leqslant k$. Therefore we have that (II) holds for the sequence $(V_1, \dots, V_{k+1})$. Furthermore, note that

$$N_G(V_{k+1}) \subseteq \bigcup_{i=1}^{k} N_G(V_i) \cup N_{G[W_k]}(V_{k+1})\,. \tag{4}$$

This gives us (III) for the sequence $(V_1, \dots, V_{k+1})$, since

$$\begin{aligned} \left|\bigcup_{i=1}^{k+1} (V_i \cup N_G(V_i))\right| &\overset{(4)}{=} \left|\bigcup_{i=1}^{k} (V_i \cup N_G(V_i)) \cup V_{k+1} \cup N_{G[W_k]}(V_{k+1})\right| \\ &< (k+1)(D+1)(\eta + f)n. \end{aligned}$$

To see that $(V_1, \dots, V_{k+1})$ satisfies (I), define

$$W_{k+1} = V(G) \smallsetminus \bigcup_{i=1}^{k+1} (V_i \cup N_G(V_i)) \overset{(4)}{=} W_k \smallsetminus (V_{k+1} \cup N_{G[W_k]}(V_{k+1})).$$

Assume that $|V_{k+1}| < \eta n$ and derive a contradiction as before.

Therefore, when $k = \ell - 2$, we generate a sequence $(V_1, \dots, V_{\ell-1})$ with the properties required by (ii). To complete the sequence, note that (III) gives that $|W_{\ell-1}| \geqslant \eta n$ and set $V_\ell = W_{\ell-1}$. $\square$

As a corollary of the previous lemma, we get the following lemma that says that sufficiently large bijumbled graphs contain a non-empty expanding subgraph.

Lemma 10 (Bijumbledness implies expansion)**.** *Let f, θ, D and $c \geqslant 1$ be positive numbers with $c \geqslant 4(D+2)\theta$ and $a \geqslant 2(D+1)f$. If G is a $(c/(an), \theta)$-bijumbled graph with an vertices, then there exists a non-empty subgraph H of G that is (n, f, D)-expanding.*

Proof. Let $p = c/(an)$ and let G be a (p, θ)-bijumbled graph. Suppose for a contradiction that no subgraph of G is (n, f, D)-expanding. We apply Lemma 9 with $\ell = 2$ and $\eta = 2\theta a/c$. Note that, since $a \geqslant 2(D+1)f$ and $c \geqslant 4(D+2)\theta$, from the choice of η we have

$$a \geqslant (D+1)f + \frac{a}{2} \geqslant (D+1)f + \frac{2(D+2)\theta a}{c} \geqslant (D+1)f + (D+2)\eta = (D+1)(f + \eta) + \eta.$$

Then, we get two disjoint sets $V_1, V_2 \subseteq V(G)$ with $|V_1| = |V_2| = \eta n > \theta/p$ such that $e_G(V_1, V_2) = 0$. On the other hand, by (2), we have $e_G(V_1, V_2) > 0$, a contradiction. Therefore, there is some subgraph of G that is (n, f, D)-expanding. $\square$

The next lemma is crucial for embedding the desired power of a tree. Let G be a graph and $\ell \geqslant r$ be positive integers. An (ℓ, r)-*blow-up* of G is a graph obtained from G by replacing each vertex of G by a clique of size ℓ and for every edge of G arbitrarily adding a complete bipartite graph $K_{r,r}$ between the cliques corresponding to the vertices of this edge.

Lemma 11 (Embedding lemma for powers of trees)**.** *Given positive integers k and Δ, there exists r_0 such that the following holds for every n-vertex tree T with maximum degree Δ. There is a tree $T' = T'(T,k)$ on at most $n+1$ vertices and with maximum degree at most Δ^{2k} such that for every graph J with $T' \subseteq J$ and any (ℓ, r)-blow-up J' of J with $\ell \geqslant r \geqslant r_0$ we have $T^k \subseteq J'$.*

Proof. Given positive integers k, Δ, take $r_0 = \Delta^{4k}$. Let T be an n-vertex tree with maximum degree Δ. Let x_0 be any vertex in $V(T)$ and consider T as rooted at x_0. For each vertex $v \in V(T)$, let $D(v)$ denote the set of *descendants* of v in T (including v itself). Let $D^i(v)$ be the set of vertices $u \in D(v)$ at distance at most i from v in T.

Let T' be a tree with vertex set consisting of a special vertex x^* and the vertices $x \in V(T)$ such that the distance between x and x_0 is a multiple of $2k$. The edge set of T' consists of the edge x^*x_0 and the pairs of vertices $x, y \in V(T') \smallsetminus \{x^*\}$ for which $x \in D^{2k}(y)$ or $y \in D^{2k}(x)$. That is,

$$V(T') = \{x \in V(T) \colon \mathrm{dist}_T(x_0, x) \equiv 0 \pmod{2k}\} \cup \{x^*\}$$

$$E(T') = \left\{ xy \in \binom{V(T') \smallsetminus \{x^*\}}{2} : x \in D^{2k}(y) \text{ or } y \in D^{2k}(x) \right\} \cup \{x^* x_0\}.$$

In particular, note that $\Delta(T') \leqslant \Delta^{2k}$ and $|V(T')| \leqslant n+1$. Let us consider T' as a tree rooted at x^*.

Now suppose that J is a graph such that $T' \subseteq J$ and J' is an (ℓ, r)-blow-up of J with $\ell \geqslant r \geqslant r_0$. Our goal is to show that $T^k \subseteq J'$. First, since J' is an (ℓ, r)-blow-up of J, there is a collection $\{K(x) : x \in V(J)\}$ of disjoint ℓ-cliques in J' such that for each edge $xy \in E(J)$, there is a copy of $K_{r,r}$ between the vertices of $K(x)$ and $K(y)$. Let us denote by $K(x,y)$ such copy of $K_{r,r}$.

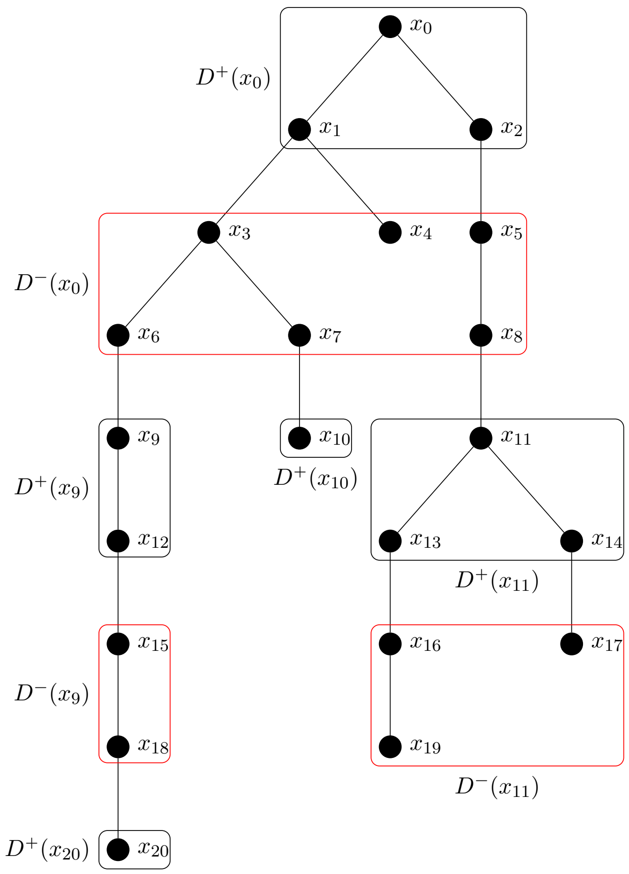
For each $x \in V(T') \smallsetminus \{x^*\}$, let $D^+(x) = D^{k-1}(x)$ and $D^-(x) = D^{2k-1}(x) \smallsetminus D^{k-1}(x)$. In order to fix the notation, it helps to think in $D^+(x)$ and $D^-(x)$ as the *upper* and *lower half of close descendants* of x, respectively. We denote by x^+ the parent of x in T'. Suppose that there exists an injective map $\phi : V(T) \to V(J')$ such that for every $x \in V(T') \smallsetminus \{x^*\}$, we have

(1) $\phi(D^+(x)) \subseteq K(x, x^+) \cap K(x^+)$;
(2) $\phi(D^-(x)) \subseteq K(x, x^+) \cap K(x)$.

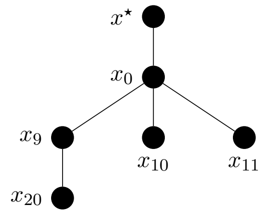
Then we claim that such map is in fact an embedding of T^k into J'. Figure 1 should help to visualize the concepts developed so far.

Claim 12. *If $\phi : V(T) \to V(J')$ is an injective map such that for all $x \in V(T') \smallsetminus \{x^*\}$ the properties (1) and (2) hold, then ϕ is an embedding of T^k into J'.*

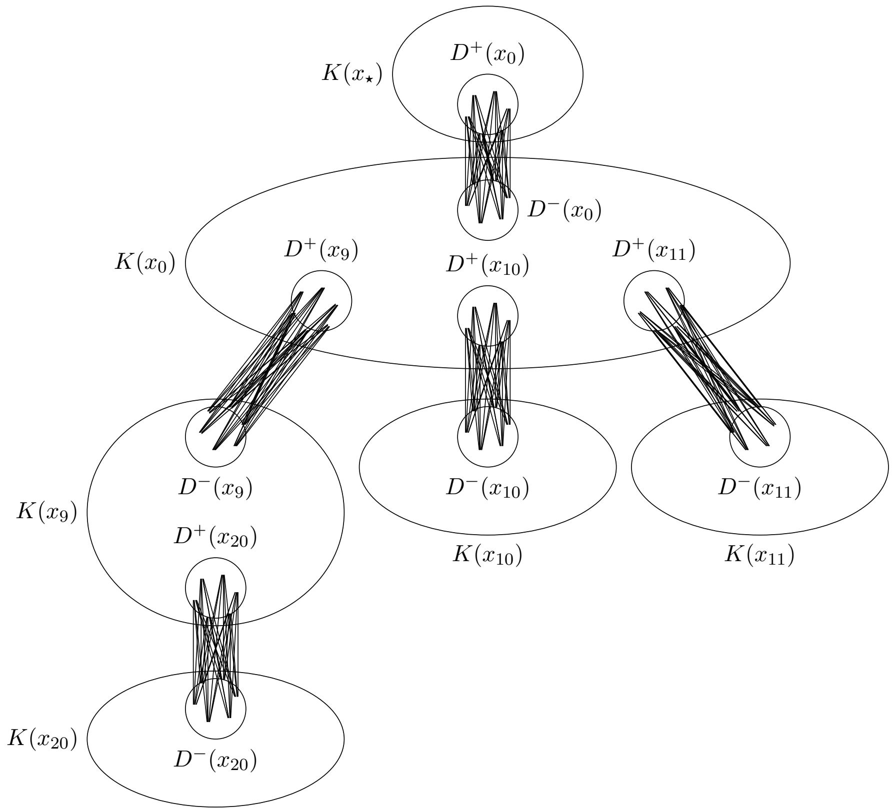
Proof. We want to show that if u and v are distinct vertices in T at distance at most k, then $\phi(u)\phi(v)$ is an edge in J'. Let $\tilde{u}$ and $\tilde{v}$ be vertices in $V(T') \smallsetminus \{x^*\}$ with $u \in D^{2k-1}(\tilde{u})$ and $v \in D^{2k-1}(\tilde{v})$. If $\tilde{u} = \tilde{v}$, then by properties (1) and (2), we have $\phi(u)$ and $\phi(v)$ adjacent in J', once all the vertices in $\phi(D^{2k-1}(\tilde{u}))$ are adjacent in J' either by edges from $K(\tilde{u})$, $K(\tilde{u}^+)$ or $K(\tilde{u}, \tilde{u}^+)$. If $\tilde{u} = \tilde{v}^+$, then we must have $u \in D^-(\tilde{u})$ and $v \in D^+(\tilde{v})$ and properties (1) and (2) give us $\phi(u), \phi(v) \in K(\tilde{u})$. Analogously, if $\tilde{v} = \tilde{u}^+$, then $v \in D^-(\tilde{v})$ and $u \in D^+(\tilde{u})$ and



(A) Tree T.



(B) Corresponding T'.



(C) Embedding T^k into an (ℓ, r)-blow-up of T'.

FIGURE 1. Illustration of the concepts and notation used throughout the proof of Lemma 11 when $\Delta = 3$ and $k = 2$.

properties (1) and (2) imply that $\phi(u), \phi(v) \in K(\tilde{v})$. If $\tilde{u}^+ = \tilde{v}^+$ (with $\tilde{u} \neq \tilde{v}$), then we have $u \in D^+(\tilde{u})$ and $v \in D^+(\tilde{v})$ and property (1) give us $\phi(u), \phi(v) \in K(\tilde{u}^+)$.

Therefore we may assume that $\tilde{u}$ and $\tilde{v}$ are at distance at least 2 in T' and do not share a parent. But this implies that

$$\min\{\mathrm{dist}_T(x, y) : x \in D^{2k-1}(\tilde{u}), y \in D^{2k-1}(\tilde{v})\} \geqslant 2k + 1,$$

contradicting the fact that u and v are at distance at most k in T. □

We conclude the proof by showing that such a map exists.

Claim 13. *There is an injective map $\phi : V(T) \to V(J')$ for which (1) and (2) hold for every $x \in V(T') \setminus \{x^*\}$.*

Proof. We just need to show that for every $x \in V(T')$, there is enough room in $K(x)$ and in $K(x, x^+)$ to guarantee that (1) and (2) hold. In order to do so, $K(x)$ should be large enough to accommodate the set

$$D^-(x) \cup \bigcup_{\substack{y \in V(T') \\ y^+ = x}} D^+(y). \tag{5}$$

Since T' has maximum degree at most Δ^{2k} and T has maximum degree Δ, we have that the set in (5) has at most Δ^{4k} vertices. Since $|K(x)| = \ell \geqslant r_0 = \Delta^{4k}$, the set $K(x)$ is indeed large enough to accommodate the set in (5). Finally, since $|K(x, x^+) \cap K(x)| = |K(x, x^+) \cap K(x)| = r \geqslant r_0 = \Delta^{4k}$ the set $K(x, x^+)$ is also large enough to accommodate $D^-(x)$ or $D^+(x)$ as in properties (1) and (2). □

□

We end this section discussing a graph property that needs to be inherited by some subgraphs when running the induction in the proof of Theorem 1.

Definition 14. *For positive numbers n, a, b, c, ℓ and θ, let $\mathcal{P}_n(a, b, c, \ell, \theta)$ denote the class of all graphs G with the following properties, where $p = c/(an)$.*

(*i*) *$|V(G)| = an$,*
(*ii*) *$\Delta(G) \leqslant b$,*
(*iii*) *G has no cycles of length at most 2ℓ,*
(*iv*) *G is (p, θ)-bijumbled.*

Only mild conditions on a, b, c, ℓ and θ are necessary to guarantee the existence of a graph in $\mathcal{P}_n(a, b, c, \ell, \theta)$ for sufficiently large n. These conditions can be seen in (*i*)–(*iii*) in Definition 15 below. In order to keep the induction going in our main proof we also need a condition relating k and Δ, which represents, respectively, the power of the tree T we want to embed and the maximum degree of T (see (*iv*) in the next definition).

Definition 15. *A 7-tuple $(a, b, c, \ell, \theta, \Delta, k)$ is* good *if*

(*i*) *$a \geqslant 3$,*
(*ii*) *$c \geqslant \theta\ell$,*
(*iii*) *$b \geqslant 9c$,*
(*iv*) *$\ell \geqslant 21\Delta^{2k}$.*

Next we prove that conditions *(i)*–*(iii)* in Definition 15 together with $\theta \geqslant 32\sqrt{c}$ are enough to guarantee that there are graphs in $\mathcal{P}_n(a,b,c,\ell,\theta)$ as long as n is large enough. We remark that next lemma is stated for a good 7-tuple, but condition *(iv)* of Definition 15 is not necessary and, therefore, also Δ and k are irrelevant.

Lemma 16. *If $(a,b,c,\ell,\theta,\Delta,k)$ is a good 7-tuple with $\theta \geqslant 32\sqrt{c}$, then for sufficiently large n the family $\mathcal{P}_n(a,b,c,\ell,\theta)$ is non-empty.*

Proof. Let $(a,b,c,\ell,\theta,\Delta,k)$ be a good 7-tuple with $\theta \geqslant 32\sqrt{c}$ and let n be sufficiently large. Put $N = an$ and let $G^* = G(3N,p)$ be the binomial random graph with $3N$ vertices and edge probability $p = c/N$. From Chernoff's inequality (Theorem 4) we know that almost surely

$$e(G^*) \leqslant 2p\binom{3N}{2} \leqslant 9cN. \tag{6}$$

From [17, Lemma 8], we know that almost surely G^* is $(p, e^2\sqrt{6p(3N)})$-bijumbled, i.e. the following holds almost surely: for all disjoint sets X and $Y \subseteq V(G^*)$ with $e^2\sqrt{18N}/\sqrt{p} < |X| \leqslant |Y| \leqslant p(3N)|X|$, we have

$$\big|e_{G^*}(X,Y) - p|X||Y|\big| \leqslant (e^2\sqrt{6})\sqrt{p(3N)|X||Y|}. \tag{7}$$

The expected number of cycles of length at most 2ℓ in G^* is given by $\mathbb{E}(C_{\leqslant 2\ell}) = \sum_{i=3}^{2\ell}\mathbb{E}(C_i)$, where C_i is the number of cycles of length i. Then,

$$\mathbb{E}(C_{\leqslant 2\ell}) = \sum_{i=3}^{2\ell}\binom{3an}{i}\frac{(i-1)!}{2}p^i \leqslant \sum_{i=3}^{2\ell}(3c)^i \leqslant 2\ell(3c)^{2\ell}.$$

Then, from Markov's inequality, we have

$$\mathbb{P}\big(C_{\leqslant 2\ell} \geqslant 4\ell(3c)^{2\ell}\big) \leqslant \frac{1}{2}. \tag{8}$$

Since (6) and (7) hold almost surely and the probability in (8) is at most $1/2$, for sufficiently large n there exists a $(p, e^2\sqrt{18c})$-bijumbled graph G' with $3N$ vertices that contains less than $4\ell(3c)^{2\ell}$ cycles of length at most 2ℓ and $e(G') \leqslant 2p\binom{3N}{2} \leqslant 9cN$. Then, by removing $4\ell(3c)^{2\ell}$ vertices we obtain a graph G'' with no such cycles such that

$$|V(G'')| = 3an - 4\ell(3c)^{2\ell} \geqslant 2an \quad \text{and} \quad e(G'') \leqslant 9cN.$$

To obtain the desired graph G in $\mathcal{P}_n(a,b,c,\ell,\theta)$, we repeatedly remove vertices of highest degree in G'' until N vertices are left, obtaining a subgraph $G \subseteq G''$ such that $\Delta(G) \leqslant 9c \leqslant b$, as otherwise we would have deleted more than $e(G'')$ edges. Note that deleting vertices preserves the bijumbledness. Therefore, for all disjoint sets X and $Y \subseteq V(G)$ with $e^2\sqrt{18N}/\sqrt{p} < |X| \leqslant |Y| \leqslant p(3N)|X|$ we have

$$\big|e_G(X,Y) - p|X||Y|\big| \leqslant (e^2\sqrt{6})\sqrt{p(3N)|X||Y|} \leqslant (32\sqrt{pN})\sqrt{|X||Y|} \leqslant \theta\sqrt{|X||Y|}. \tag{9}$$

We obtained a graph G on N vertices and maximum degree $\Delta(G) \leqslant b$ such that G contains no cycles of length at most 2ℓ and is (p,θ)-bijumbled, for $p = c/N$. Therefore, the proof of the lemma is complete. □

§4. Proof of the main result

We derive Theorem 1 from Proposition 17 below. Before continuing, given an integer $\ell \geqslant 1$, let us define what we mean by a *sheared complete blow-up* $H\{\ell\}$ of a graph H: this is any graph obtained by replacing each vertex v in $V(H)$ by a complete graph $C(v)$ with ℓ vertices, and by adding all edges *but a perfect matching* between $C(u)$ and $C(v)$, for each $uv \in E(H)$. We also define the *complete blow-up* $H(\ell)$ of a graph H analogously, but by adding all the edges between $C(u)$ and $C(v)$, for each $uv \in E(H)$.

Proposition 17. *For all integers $k \geqslant 1$, $\Delta \geqslant 2$, and $s \geqslant 1$ there exists r_s and a good 7-tuple $(a_s, b_s, c_s, \ell_s, \theta_s, \Delta, k)$ with $\theta_s \geqslant 32\sqrt{c_s}$ for which the following holds. If n is sufficiently large and $G \in \mathcal{P}_n(a_s, b_s, c_s, \ell_s, \theta_s)$ then, for any tree T on n vertices with $\Delta(T) \leqslant \Delta$, we have*

$$G^{r_s}\{\ell_s\} \to (T^k)_s.$$

Theorem 1 follows from Proposition 17 applied to a certain subgraph of a random graph.

Proof of Theorem 1. Fix positive integers k, Δ and s and let T be an n-vertex tree with maximum degree Δ. Proposition 17 applied with parameters k, Δ and s gives r_s and a good 7-tuple $(a_s, b_s, c_s, \ell_s, \theta_s, \Delta, k)$ with $\theta_s \geqslant 32\sqrt{c_s}$.

Let n be sufficiently large. By Lemma 16, since $\theta_s \geqslant 32\sqrt{c_s}$, there exists a graph $G \in \mathcal{P}_n(a_s, b_s, c_s, \ell_s, \theta_s)$. Let χ be an arbitrary s-colouring of $E(G^{r_s}\{\ell_s\})$. Then, Proposition 17 gives that $G^{r_s}\{\ell_s\} \to (T^k)_s$. Since $|V(G)| = a_s n$, the maximum degree of G is bounded by the constant b_s, and since r_s and ℓ_s are constants, we have $e(G^{r_s}\{\ell_s\}) = O_{k,\Delta,s}(n)$, which concludes the proof of Theorem 1. □

The proof of Proposition 17 follows by induction in the number of colours. Before we give this proof, let us state the results for the base case and the induction step.

Lemma 18 (Base Case)**.** *For all integers $h \geqslant 1$, $k \geqslant 1$ and $\Delta \geqslant 2$ there is an integer r and a good 7-tuple $(a, b, c, \ell, \theta, \Delta, k)$ with $\theta \geqslant 2^{h-1}32\sqrt{c}$ such that if n is sufficiently large, then the following holds for any $G \in \mathcal{P}_n(a, b, c, \ell, \theta)$. For any n-vertex tree T with $\Delta(T) \leqslant \Delta$, the graph $G^r\{\ell\}$ contains a copy of T^k.*

Lemma 19 (Induction Step)**.** *For any positive integers $\Delta \geqslant 2$, $s \geqslant 2$, k, r, $h \geqslant 1$ and any good 7-tuple $(a, b, c, \ell, \theta, \Delta, k)$ with $\theta \geqslant 2^h 32\sqrt{c}$, there is a positive integer r' and a good 7-tuple $(a', b', c', \ell', \theta', \Delta, k)$ with $\theta' \geqslant 2^{h-1}32\sqrt{c'}$ such that the following holds. If n is sufficiently large then for any graph $G \in \mathcal{P}_n(a', b', c', \ell', \theta')$ and any s-colouring χ of $E(G^{r'}\{\ell'\})$*

- *(i) there is a monochromatic copy of T^k in $G^{r'}\{\ell'\}$ for any n-vertex tree T with $\Delta(T) \leqslant \Delta$, or*
- *(ii) there is $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$ such that $H^r\{\ell\} \subseteq G^{r'}\{\ell'\}$ and $H^r\{\ell\}$ is coloured with at most $s-1$ colours under χ.*

Now we are ready to prove Proposition 17.

Proof of Proposition 17. Fix integers $k \geqslant 1$, $\Delta \geqslant 2$ and $s \geqslant 1$ and define $h_i = s - i$ for $1 \leqslant i \leqslant s$. Let r_1 and a good 7-tuple $(a_1, b_1, c_1, \ell_1, \theta_1, \Delta, k)$ with $\theta_1 \geqslant 2^{h_1}32\sqrt{c_1}$ be given by Lemma 18 applied with s, k and Δ.

We will prove the proposition by induction on the number of colours $i \in \{1, \ldots, s\}$ with the additional property that if the colouring has i colours then $\theta_i \geqslant 2^{h_i}32\sqrt{c_i}$.

Notice that Lemma 18 implies that for sufficiently large n, if $G \in \mathcal{P}_n(a_1, b_1, c_1, \ell_1, \theta_1)$, then $G^{r_1}\{\ell_1\} \to (T^k)_1$. Therefore, since $\theta_1 \geqslant 2^{h_1}32\sqrt{c_1}$, if $i = 1$, we are done.

Assume $2 \leqslant i \leqslant s$ and suppose the statement holds for $i-1$ colours with the additional property that $\theta_{i-1} \geqslant 2^{h_{i-1}}32\sqrt{c_{i-1}}$, where r_{i-1} and a good 7-tuple $(a_{i-1}, b_{i-1}, c_{i-1}, \ell_{i-1}, \theta_{i-1}, \Delta, k)$ are given by the induction hypothesis. Therefore, for any tree T on n vertices with $\Delta(T) \leqslant \Delta$, we know that for a sufficiently large n

$$H^{r_{i-1}}\{\ell_{i-1}\} \to (T^k)_{i-1} \quad \text{for any} \quad H \in \mathcal{P}_n(a_{i-1}, b_{s-1}, c_{i-1}, \ell_{i-1}, \theta_{i-1}). \tag{10}$$

Note that since $i \leqslant s$, we have $h_{i-1} = s - (i-1) \geqslant 1$. Then, since $\theta_{i-1} \geqslant 2^{h_{i-1}} 32\sqrt{c_{i-1}}$, we can apply Lemma 19 with parameters $\Delta, s, k, r_{i-1}, h_{i-1}$ and $(a_{i-1}, b_{i-1}, c_{i-1}, \ell_{i-1}, \theta_{i-1}, \Delta, k)$, obtaining r_i and $(a_i, b_i, c_i, \ell_i, \theta_i, \Delta, k)$ with $\theta_i \geqslant 2^{h_i} 32\sqrt{c_i}$.

Let $G \in \mathcal{P}_n(a_i, b_i, c_i, \ell_i, \theta_i)$ and let n be sufficiently large. Now let χ be an arbitrary i-colouring of $E(G^{r_i}\{\ell_i\})$. From Lemma 19, we conclude that either *(i)* there is a monochromatic copy of T^k in $G^{r_i}\{\ell_i\}$ for any tree T on n vertices with $\Delta(T) \leqslant \Delta$, in which case the proof is finished, or *(ii)* there exists a graph $H \in \mathcal{P}_n(a_{i-1}, b_{i-1}, c_{i-1}, \ell_{i-1}, \theta_{i-1})$ such that $H^{r_{i-1}}\{\ell_{i-1}\} \subseteq G^{r_i}\{\ell_i\}$ and $H^{r_{i-1}}\{\ell_{i-1}\}$ is coloured with at most $i-1$ colours under χ. In case *(ii)*, the induction hypothesis (10) implies that we find the desired monochromatic copy of T^k in $H^{r_{i-1}}\{\ell_{i-1}\} \subseteq G^{r_i}\{\ell_i\}$. $\square$

The proof of Lemma 18 follows by proving that for a good 7-tuple $(a, b, c, \ell, \theta, \Delta, k)$ with $\theta \geqslant 2^{h-1} 32\sqrt{c}$, large graphs G in $\mathcal{P}_n(a, b, c, \ell, \theta)$ are expanding (using Lemma 10). Then, we use Lemma 7 to conclude that G contains the desired tree T. After this step we greedily find an embedding of T^k in $G\{\ell\}^k$.

Proof of the base case (Lemma 18). Let $h \geqslant 1$, $k \geqslant 1$ and $\Delta \geqslant 2$ be integers. Let

$$r = k, \quad \ell = 21\Delta^{2k}, \quad \theta = 4^h 256\ell, \quad c = \theta\ell, \quad b = 9c$$

and put $D = \Delta + 1$. Note that $\theta \geqslant 2^{h-1} 32\sqrt{c}$ and let

$$a \geqslant 4(D+1).$$

Since $\ell \geqslant 4(\Delta + 3)$, we have $c \geqslant 4(D+2)\theta$. From the lower bounds on c and a we know that we can use the conclusion of Lemma 10 applying it with $f = 2$, θ, $D = \Delta + 1$ and c.

Note that from our choice of constants, $(a, b, c, \ell, \theta, \Delta, k)$ is a good tuple. Let n be sufficiently large and let T be a tree on n vertices with $\Delta(T) \leqslant \Delta$. Let $G \in \mathcal{P}_n(a, b, c, \ell, \theta)$. From Lemma 10 we know that G has an $(n, 2, \Delta + 1)$-expanding subgraph and, therefore, from Lemma 7 we conclude that G contains a copy of T. Clearly, the graph G^k contains a copy of T^k. It remains to prove that the graph $G^k\{\ell\}$ also contains a copy of T^k.

Let $\{v_1, \ldots, v_n\}$ be the vertices of T_n and denote by T_j the subgraph of T induced by $\{v_1, \ldots, v_j\}$. Given a vertex $v \in V(G)$, let $C(v)$ denote the ℓ-clique in $G^k\{\ell\}$ that corresponds to v. Suppose that for some $1 \leqslant j < k$ we have embedded T_j^k in $G^k\{\ell\}$ where, for each $1 \leqslant i \leqslant j$, the vertex v_i was mapped to some $w_i \in C(v_i)$.

By the definition of $G^k\{\ell\}$, every neighbour v of v_{j+1} in G^k is adjacent to all but one vertex of $C(v_{j+1})$. Therefore, since $\Delta(T^k) \leqslant \Delta^k$ and $|C(v_{j+1})| = \ell \geqslant \Delta^k + 1$, we may thus find a vertex $w_{j+1} \in C(v_{j+1})$ such that w_{j+1} is adjacent in $G^k\{\ell\}$ to every w_i with $1 \leqslant i \leqslant j$ such that $v_i v_{j+1} \in E(T_{j+1}^k)$. From that we obtain a copy of T_{j+1}^k in $G^k\{\ell\}$ where $w_i \in C(v_i)$ for $1 \leqslant i \leqslant j+1$. Therefore, starting with any vertex w_1 in $C(v_1)$, we may obtain a copy of T^k in $G^k\{\ell\}$ inductively, which proves the lemma. $\square$

The core of the proof of Theorem 1 is the induction step (Lemma 19). We start by presenting a sketch of its proof.

Sketch of the induction step (Lemma 19). We start by fixing suitable constants r', a', b', c', ℓ' and θ'. Let n be sufficiently large and let $G \in \mathcal{P}_n(a', b', c', \ell', \theta')$ be given. Consider an arbitrary colouring χ of the edges of a sheared complete blow-up $G^{r'}\{\ell'\}$ of $G^{r'}$ with s colours. We shall prove that either there is a monochromatic copy of T^k in $G^{r'}\{\ell'\}$, or there is a graph $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$ such that a sheared complete blow-up $H^r\{\ell\}$ of H^r is a subgraph of $G^{r'}\{\ell'\}$ and this copy of $H^r\{\ell\}$ is coloured with at most $s-1$ colours under χ.

First, note that, by Ramsey's theorem, if ℓ' is large then each ℓ'-clique $C(v)$ of $G^{r'}\{\ell'\}$ contains a large monochromatic clique. Let us say that blue is the most common colour of these monochromatic cliques. Let these blue cliques be $C'(v) \subseteq C(v)$. Then we consider a graph $J \subseteq G^{r'}$ induced by the vertices v corresponding to the blue cliques $C'(v)$ and having only the edges $\{u, v\}$ such that there is a blue copy of a large complete bipartite graph under χ in the bipartite graph induced between the blue cliques $C'(u)$ and $C'(v)$ in $G^{r'}\{\ell'\}$.

Then, by Lemma 9 applied to J, either there is a set $\varnothing \neq Z \subseteq V(J)$ such that $J[Z]$ is expanding, or there are large disjoint sets $V_1, \ldots, V_\ell$ with no edges between them in J. In the first case, Lemma 11 guarantees that there is a tree T' such that, if $T' \subseteq J[Z]$, then there is a blue copy of T^k in $G^{r'}\{\ell'\}$. To prove that $T' \subseteq J[Z]$, we recall that $J[Z]$ is expanding and use Lemma 7. This finishes the proof of the first case.

Now let us consider the second case, in which there are large disjoint sets $V_1, \ldots, V_\ell$ with no edges between them in J. The idea is to obtain a graph $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$ such that $H^r\{\ell\} \subseteq G^{r'}\{\ell'\}$ and, moreover, $H^r\{\ell\}$ does not have any blue edge. For that we first obtain a path Q in G with vertices $(x_1, \ldots, x_{2a\ell n})$ such that $x_i \in V_j$ for all i where $i \equiv j \bmod \ell$. Then we partition Q into $2an$ paths $Q_1, \ldots, Q_{2an}$ with ℓ vertices each, and consider an auxiliary graph H' on $V(H') = \{Q_1, \ldots, Q_{2an}\}$ with $Q_iQ_j \in E(H')$ if and only $E_G(V(Q_i), V(Q_j)) \neq \varnothing$. To ensure that H' inherits properties from G we use that there can bet at most one edge between Q_i and Q_j in G, because there are no cycles of length less than 2ℓ in G.

We obtain a subgraph $H'' \subseteq H'$ by choosing edges of H' uniformly at random with a suitable probability p. Then, successively removing vertices of high degree, we obtain a graph $H \subseteq H''$ with $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$. It now remains to find a copy of $H^r\{\ell\}$ in $G^{r'}\{\ell'\}$ with no blue edges. To do so, we first observe that the paths $Q_i \in V(H')$ give rise to ℓ-cliques in $G^{r'}$ ($r' \geqslant \ell$). One can then prove that there is a copy of $H^r\{\ell\}$ in $G^{r'}$ that avoids the edges of J. By applying the Lovász local lemma we can further deduce that there is a copy of $H^r\{\ell\}$ in $G^{r'}\{\ell'\}$ with no blue edges. $\square$

Proof of the induction step (Lemma 19). We start by fixing positive integers $\Delta \geqslant 2$, $s \geqslant 2$, k, r, h and a good 7-tuple $(a, b, c, \ell, \theta, \Delta, k)$ with

$$\theta \geqslant 2^h 32\sqrt{c}.$$

Recall that from the definition of good 7-tuple, we have

$$b \geqslant 9c.$$

Let d_0 be obtained from Lemma 3 applied with ℓ and $\gamma = 1/(2\ell)$ (note that $d_0 \leqslant 10$). Further let

$$a'' = \ell(\Delta^{2k} + 2)(2a \cdot d_0 + 2).$$

Notice that a'' is an upper bound on the value A given by Lemma 9 applied with $f = 2$, $D = \Delta^{2k} + 1$, ℓ and $\eta = 2a \cdot d_0$.

Let r_0 be given by Lemma 11 on input Δ and k. We may assume r_0 is even. Furthermore, let

$$t = \max\{r_0, \big(40(\ell b^{r+1} + \ell)\big)^{r_0}\} \quad \text{and} \quad \ell' = \max\{4s\ell^2, r_s(t)\},$$

where $r_s(t) = r(t, \dots, t) = r(K_t, \dots, K_t)$ denotes the s-colour Ramsey number for cliques of order t. Let $a' = \ell' a$ and note that $a'/s \geqslant 2a''$ because $\ell \geqslant 21\Delta^{2k}$. Define constants c^*, c' and r' as follows.

$$c^* = 2\ell' c, \quad c' = \frac{\ell'}{2\ell^2} c^* = \frac{\ell'^2}{\ell^2} c, \quad r' = \ell r. \tag{11}$$

Put

$$b' = 9c' \quad \text{and} \quad \theta' = \frac{c^*}{4c\ell}\theta = \frac{\ell'}{2\ell}\theta$$

Claim 20. *$(a', b', c', \ell', \theta', \Delta, k)$ is a good 7-tuple and $\theta' \geqslant 2^{h-1} 32\sqrt{c'}$.*

Proof. We have to check all conditions in Definition 15. Clearly $a' \geqslant 3$, $b' \geqslant 9c'$ and $\ell' \geqslant \ell \geqslant 21\Delta^{2k}$. Below we prove that the other conditions hold

- $c' \geqslant \theta' \ell'$:
$$c' = \frac{\ell'^2}{\ell^2} c \geqslant \frac{\ell'^2}{\ell}\theta = 2\theta'\ell' > \theta'\ell'.$$
- $\theta' \geqslant 2^{h-1} 32\sqrt{c'}$:
$$\theta' = \frac{\ell'}{2\ell}\theta \geqslant \frac{\ell'}{2\ell} 2^h 32\sqrt{c} = 2^{h-1} 32\sqrt{c'}.$$

□

Let G be a graph in $\mathcal{P}_n(a', b', c', \ell', \theta')$. Assume

$$N_G = a'n \quad \text{and} \quad p_G = c'/N_G$$

and let T be an arbitrary tree with n vertices and maximum degree Δ and consider an arbitrary s-colouring $\chi\colon E(G^{r'}\{\ell'\}) \to [s]$ of the edges of $G^{r'}\{\ell'\}$. We shall prove that either there is a monochromatic copy of T^k in $G^{r'}\{\ell'\}$, or there is a graph $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$ such that a sheared complete blow-up $H^r\{\ell\}$ of H^r is a subgraph of $G^{r'}\{\ell'\}$ and this copy of $H^r\{\ell\}$ is coloured with at most $s-1$ colours under χ.

By Ramsey's theorem (see, for example, [7]), since $\ell' \geqslant r_s(t)$, each ℓ'-clique $C(w)$ in $G^{r'}\{\ell'\}$ (for $w \in V(G)$) contains a monochromatic clique of size at least t. Without lost of generality, let us assume that most of those monochromatic cliques are blue. Let $W \subseteq V(G)$ be the set of vertices w such that there is a blue t-clique $C'(w) \subseteq C(w)$. We have

$$|W| \geqslant \frac{|V(G)|}{s} = \frac{a'n}{s} \geqslant 2a''n. \tag{12}$$

Define J as the subgraph of $G^{r'}$ with vertex set W and edge set

$$E(J) = \Big\{ uv \in E(G^{r'}[W]) : \text{ there is a blue copy of } K_{r_0,r_0} \text{ in } G^{r'}\{\ell'\}[C'(u), C'(v)] \Big\}.$$

That is, J is the subgraph of $G^{r'}$ induced by W and the edges uv such that there is a blue copy of K_{r_0,r_0} under χ in the bipartite graph induced by $G^{r'}\{\ell'\}$ between the vertex sets of the blue cliques $C'(u)$ and $C'(v)$.

We now apply Lemma 9 with $f = 2$, $D = \Delta^{2k} + 1$, ℓ, and $\eta = 2a \cdot d_0$ to the graph J (notice that $|V(J)| \geqslant 2a''n$ is large enough so we can apply Lemma 9), splitting the proof into two cases:

(i) there is $\varnothing \neq Z \subseteq V(J)$ such that $J[Z]$ is $(n+1, 2, \Delta^{2k}+1)$-expanding,

(ii) there exist $V_1, \dots, V_\ell \subseteq V(J)$ such that $|V_i| \geqslant 2ad_0 n$ for $1 \leqslant i \leqslant \ell$ and $J[V_i, V_j]$ is empty for any $1 \leqslant i < j \leqslant \ell$.

In case $J[Z]$ is $(n+1, 2, \Delta^{2k}+1)$-expanding, we first notice that Lemma 11 applied to the graph $J[Z]$ implies the existence of a tree $T' = T'(T, \Delta, k)$ of maximum degree at most Δ^{2k} with at most $n+1$ vertices such that if $J[Z]$ contains T', then $T^k \subseteq J'$ for any (r_0, r_0)-blow-up J' of J. But since $J[Z]$ is $(n+1, 2, \Delta^{2k}+1)$-expanding, Lemma 7 implies that $J[Z]$ contains a copy of T'. Therefore, the graph $G^{r'}\{\ell'\}$ contains a blue copy of T^k, as we can consider J' as the subgraph of $G^{r'}\{\ell'\}$ containing only edges inside the blue cliques $C'(u)$ (which have size $t \geqslant r_0$) and the edges of the complete blue bipartite graphs K_{r_0,r_0} between the blue cliques $C'(u)$. This finishes the proof of the first case.

We may now assume that there are subsets $V_1, \ldots, V_\ell \subseteq V(J)$ with $|V_i| \geqslant 2ad_0n$ for $1 \leqslant i \leqslant \ell$ and $J[V_i, V_j]$ is empty for any $1 \leqslant i < j \leqslant \ell$. We want to obtain a graph $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$ such that $H^r\{\ell\} \subseteq G^{r'}\{\ell'\}$ and contains no blue edges.

Let $J' = J[V_1 \cup \cdots \cup V_\ell]$, $G' = G[V_1 \cup \cdots \cup V_\ell]$ and note that $|V(G')| = |V(J')| \geqslant d_0 \cdot 2a\ell n$, where we recall that d_0 is the constant obtained by applying Lemma 3 with ℓ and $\gamma = 1/(2\ell)$. We want to use the assertion of Lemma 3 to obtain a transversal path of length $2a\ell n$ in G' and so we have to check the conditions adjusted to this parameter.

First note, that we have $|V_i| \geqslant 2ad_0n \geqslant \gamma d_0 \cdot 2a\ell n$ for $1 \leqslant i \leqslant \ell$. Moreover, since G' is an induced subgraph of G and $G \in \mathcal{P}_n(a', b', c', \ell, \theta')$, we know by (2) that for all $X, Y \subseteq V(G')$ with $|X|, |Y| > \theta' a' n/c'$ we have $e_{G'}(X, Y) > 0$. Observe that $\theta' a' n/c' < an = \gamma \cdot 2a\ell n$ once $a' = \ell' a$ and $c' > \theta' \ell'$. Therefore, we may use Lemma 3 to conclude that G' contains a path $P_{2a\ell n} = (x_1, \ldots, x_{2a\ell n})$ with $x_i \in V_j$ for all i, where $j \equiv i \pmod{\ell}$.

We split the obtained path $P_{2a\ell n}$ of G' into consecutive paths $Q_1, \ldots, Q_{2an}$ each on ℓ vertices. More precisely, we let $Q_i = (x_{(i-1)\ell+1}, \ldots, x_{i\ell})$ for $i = 1, \ldots, 2an$. The following auxiliary graph is the base of our desired graph $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$.

> H' is the graph on $V(H') = \{Q_1, \ldots, Q_{2an}\}$ such that $Q_iQ_j \in E(H')$ if and only if there is an edge in G between the vertex sets of Q_i and Q_j.

Claim 21. *$H' \in \mathcal{P}_n(2a, \ell b', c^*, \ell, \ell\theta')$.*

Proof. We verify the conditions of Definition 14. Since H' has $2an$ vertices, condition *(i)* clearly holds. Since $\Delta(G) \leqslant b'$ and for any $Q_i \in V(H')$ we have $|Q_i| = \ell$ (as a subset of $V(G)$), there are at most $\ell b'$ edges in G with an endpoint in Q_i. Then, $\Delta(H') \leqslant \ell b'$.

For condition *(iii)*, recall that any vertex of H' corresponds to a path on ℓ vertices in G. Thus, a cycle of length at most 2ℓ in H' implies the existence of a cycle of length at most $2\ell^2$ in G. Since $2\ell' \geqslant 2\ell^2$ and G has no cycles of length at most $2\ell'$, we conclude that H' contains no cycle of length at most 2ℓ, which verifies condition *(iii)*.

Let $N_{H'} = 2an$ and

$$p_{H'} = \frac{c^*}{N_{H'}} = \frac{c^*}{2an}. \tag{13}$$

Let us verify condition *(iv)*, i.e., we shall prove that H' is $(p_{H'}, \ell\theta')$-bijumbled.

Consider arbitrary sets X and Y of $V(H')$ with $\ell\theta'/p_{H'} < |X| \leqslant |Y| \leqslant p_{H'}N_{H'}|X|$. For simplicity, we may assume that $X = \{Q_1, \ldots, Q_x\}$ and $Y = \{Q_{x+1}, \ldots, Q_{x+y}\}$. Let $X_G = \bigcup_{j=1}^{x} Q_j \subseteq V(G)$ and $Y_G = \bigcup_{j=x+1}^{x+y} Q_j \subseteq V(G)$. Note that $|X_G| = \ell|X|$ and $|Y_G| = \ell|Y|$. As there are no cycles of length smaller than 2ℓ in G, we only have at most one edge between the vertex sets of Q_i and Q_j. Therefore we have

$$e_{H'}(X, Y) = e_G(X_G, Y_G). \tag{14}$$

We shall prove that $|e_{H'}(X,Y) - p_{H'}|X||Y|| \leqslant \ell\theta'\sqrt{|X||Y|}$. From the choice of c', we have

$$p_{H'}|X||Y| = \frac{c^*}{2an}|X||Y| = \frac{c'}{a'n}\ell|X|\ell|Y| = \frac{c'}{a'n}|X_G||Y_G| = p_G|X_G||Y_G|. \tag{15}$$

From the choice of θ', c', and $p_{H'}$, since $\ell\theta'/p_{H'} < |X| \leqslant |Y| \leqslant p_{H'}N_{H'}|X|$, we obtain

$$\frac{\theta'}{p_G} < |X_G| \leqslant |Y_G| \leqslant p_G N_G |X_G|.$$

Combining (15) with (14) and the fact that G is (p_G, θ')-bijumbled, we get that

$$|e_{H'}(X,Y) - p_{H'}|X||Y|| = |e_G(X_G, Y_G) - p_G|X_G||Y_G|| \leqslant \theta'\sqrt{|X_G||Y_G|} = \ell\theta'\sqrt{|X||Y|}. \tag{16}$$

Therefore, H' is $(p_{H'}, \ell\theta')$-bijumbled, which verifies condition *(iv)*. □

The parameters for $\mathcal{P}_n(2a, \ell b', c^*, \ell, \ell\theta')$ are tightly fitted such that we can find the following subgraph of H'.

Claim 22. *There exists $H \subseteq H'$ such that $H \in \mathcal{P}_n(a, b, c, \ell, \theta)$.*

Proof. We first obtain $H'' \subseteq H'$ by picking each edge of H' with probability

$$p = \frac{2c}{c^*} = \frac{1}{\ell'}$$

independently at random. Note that $p \leqslant 1/2$.

From (3), we get

$$e(H') \leqslant p_{H'}\binom{2an}{2} + \ell\theta' 2an \leqslant (c^* + 2\ell\theta')an \leqslant (c^* + 2\ell\frac{c'}{\ell'})an \leqslant 2c^*an$$

From Chernoff's inequality, we then know that almost surely we have

$$e(H'') \leqslant 2p \cdot e(H') \leqslant 2 \cdot \left(\frac{2c}{c^*}\right) \cdot 2c^*an \leqslant 8acn \leqslant abn. \tag{17}$$

Let $N_{H''} = 2an$ and

$$p_{H''} = p \cdot p_{H'} = \frac{c}{an}.$$

We shall prove that H'' is $(p_{H''}, \theta)$-bijumbled almost surely. For that, we will first prove by using Chernoff's inequality (Theorem 4) that, for any disjoint sets X and Y of $V(H')$ with $\theta/p_{H''} < |X| \leqslant |Y| \leqslant p_{H'}N_{H'}|X|$, we have

$$|e_{H''}(X,Y) - p \cdot e_{H'}(X,Y)| \leqslant \frac{\theta}{2}\sqrt{|X||Y|}. \tag{18}$$

Note that for such sets X and Y, since $|X| > \theta/p_{H''} \geqslant \ell\theta'/p_{H'}$, we can use (16).

Since $|X|, |Y| > \theta/p_{H''}$, we have $\sqrt{|X||Y|} > \theta an/c$. From $\sqrt{|X||Y|} > \theta an/c$, we obtain that $\ell'\theta < \frac{2\ell' c\sqrt{|X||Y|}}{2an}$ from which we can conclude that $2\ell\theta' < p_{H'}\sqrt{|X||Y|}$. Thus, we get $\ell\theta'\sqrt{|X||Y|} < p_{H'}|X||Y|/2$. Therefore, combining this with (16) we have

$$\frac{p_{H'}|X||Y|}{2} < e_{H'}(X,Y) < 2p_{H'}|X||Y|. \tag{19}$$

Let $\varepsilon = \theta\sqrt{|X||Y|}/(2p \cdot e_{H'}(X,Y))$ and note that from (19) we have $\varepsilon < 1$. Since $\theta \geqslant 10\sqrt{c}$, also from (19) we obtain

$$\frac{\varepsilon^2 p \cdot e_{H'}(X,Y)}{3} = \frac{|X||Y|\ell'\theta^2}{12 \cdot e_{H'}(X,Y)} > 4an.$$

Therefore, by using Chernoff's inequality, since there are at most 2^{4an} choices of pairs of sets $\{X,Y\}$, almost surely we have that for any disjoint subsets X and Y of vertices of H'' with $\theta/p_{H''} < |X| \leqslant |Y| \leqslant p_{H'}N_{H'}|X|$, inequality (18) holds.

Observe that $p_{H''}N_{H''}|X| = 2c|X| \leqslant c^*|X| = p_{H'}N_{H'}|X|$. Therefore, H'' is almost surely $(p_{H''},\theta)$-bijumbled, as by (16) and (18) we get

$$\begin{aligned}
|e_{H''}(X,Y) - p_{H''}|X||Y|| &\leqslant |e_{H''}(X,Y) - p\cdot e_{H'}(X,Y)| + |p\cdot e_{H'}(X,Y) - p_{H''}|X||Y|| \\
&\overset{(18)}{\leqslant} \frac{\theta}{2}\sqrt{|X||Y|} + p(|e_{H'}(X,Y) - p_{H'}|X||Y||) \\
&\overset{(16)}{\leqslant} \frac{\theta}{2}\sqrt{|X||Y|} + \frac{\ell\theta'}{\ell'}\sqrt{|X||Y|} \\
&= \theta\sqrt{|X||Y|}.
\end{aligned}$$

Therefore, there exists a $(p_{H''},\theta)$-bijumbled graph H'' as above. We fix such a graph and construct the desired graph H from this H'' by sequentially removing the an vertices of highest degree. Notice that H has maximum degree at most b, otherwise this would imply that H'' has more than abn edges, contradicting (17). Since H is a subgraph of H', and H' does not contain cycles of length at most 2ℓ, the same holds for H. Finally, since deleting vertices preserves the bijumbledness property, we conclude that $H \in \mathcal{P}_n(a,b,c,\ell,\theta)$. □

Recall that J is the subgraph of $G^{r'}$ induced by W, with $|W| \geqslant a'n/s$ and edges uv such that there is a blue copy of K_{r_0,r_0} under χ in the bipartite graph induced by the vertex sets of blue cliques $C'(u)$ and $C'(v)$ in $G^{r'}\{\ell'\}$. Furthermore, recall that there are subsets $V_1,\ldots,V_\ell \subseteq V(J)$ with $|V_i| \geqslant 2ad_0n$ for $1 \leqslant i \leqslant \ell$ and $J[V_i,V_j]$ is empty for any $1 \leqslant i < j \leqslant \ell$, and we defined $J' = J[V_1 \cup \cdots \cup V_\ell]$ and $G' = G[V_1 \cup \cdots \cup V_\ell]$. Lastly, recall that $Q_i = (x_{(i-1)\ell+1},\ldots,x_{i\ell})$ for $i = 1,\ldots,2an$, where the vertices x_i belong to G'. Assume, without loss of generality, $V(H) = \{Q_1,\ldots,Q_{an}\}$. In what follows, when considering the graph $H^r(\ell)$, the ℓ-clique corresponding to Q_i is composed of the vertices $x_{(i-1)\ell+1},\ldots,x_{i\ell}$, and hence one can view $V\big(H^r(\ell)\big)$ as a subset of $V(G')$.

Claim 23. *$H^r(\ell) \subseteq G^{r'}$. Moreover, $G^{r'}$ contains a copy of $H^r\{\ell\}$ that avoids the edges of J.*

Proof. We will prove that $H^r(\ell) \subseteq G^{r'}$ where $Q_1,\ldots,Q_{an} \subseteq V(J)$ are the ℓ-cliques of $H^r(\ell)$. Suppose that Q_i and Q_j are at distance at most r in the graph H. Without loss of generality, let $Q_i = Q_1$ and $Q_j = Q_m$ for some $m \leqslant r$. Moreover, let $(Q_1,Q_2,\ldots,Q_m)$ be a path in H. Note that there exist vertices $u_1,\ldots,u_{m-1}$ and $u'_2,\ldots,u'_m$ in $V(G')$ such that $u_1 \in Q_1$, $u'_m \in Q_m$, $u_j, u'_j \in Q_j$ for all $j = 2,\ldots,m-1$ and $\{u_i,u'_{i+1}\}$ is an edge of G' for $i = 1,\ldots,m-1$.

Let $u'_1 \in Q_1$ and $u_m \in Q_m$ be arbitrary vertices. Since for any j, the set Q_j is spanned by a path on ℓ vertices in G', it follows that u_j and u'_j are at distance at most $\ell-1$ in G' for all $1 \leqslant j \leqslant m$. Therefore, u'_1 and u_m are at distance at most $(\ell-1)m+(m-1) < \ell r \leqslant r'$ in G' and hence u'_1u_m is an edge in $G[V_1 \cup \ldots \cup V_\ell]^{r'} \subseteq G^{r'}$. Since the vertices u'_1 and u_m were arbitrary, we have shown that if Q_i and Q_j are adjacent in H^r (i.e., Q_i and Q_j are at distance at most r in H) then (Q_i,Q_j) gives a complete bipartite graph $C(Q_i,Q_j)$ in $G^{r'}$. Moreover, taking $i = j$ we see that each Q_i in $G^{r'}$ must be complete. This implies that $H^r(\ell)$ is a subgraph of $G^{r'}$.

For the second part of the claim we consider which of the edges of this copy of $H^r(\ell)$ can also be edges of J. Recall from the definition of J' that we found subsets $V_1,\ldots,V_\ell \subseteq J$ such that no edge of J lies between different parts. Moreover each set $Q_i \subseteq J$ takes precisely one vertex from each set $V_1,\ldots,V_\ell$. It follows that each Q_i is independent in J. Now let us say we have

$x \in Q_i$ and $y \in Q_j$ ($i \neq j$) that are adjacent in J. We can not have x and y in different parts of the partition $\{V_1, \ldots, V_\ell\}$. Thus x and y lie in the same part. Therefore edges from J between Q_i and Q_j must form a matching. Then we can find a copy of $H^r\{\ell\}$ that avoids J by removing a matching between the l-cliques from $H^r(\ell)$.

□

To complete the proof of Lemma 19, we will embed a copy of the graph $H^r\{\ell\} \subseteq G^{r'}$ found in Claim 23 in $G^{r'}\{\ell'\}$ in such a way that $H^r\{\ell\}$ uses at most $s-1$ colours.

Claim 24. *$G^{r'}\{\ell'\}$ contains a copy of $H^r\{\ell\}$ with no blue edges.*

Proof. Recall that each vertex u in J corresponds to a clique $C'(u) \subseteq G^{r'}\{\ell'\}$ of size t and that this clique is monochromatic in blue in the original colouring χ of $E(G^{r'}\{\ell'\})$. Recall also that if an edge $\{u, v\}$ of $G^{r'}[W]$ is not in J, then there is no blue copy of K_{r_0,r_0} in the bipartite graph between $C'(u)$ and $C'(v)$ in $G^{r'}\{\ell'\}$. By the Kővári–Sós–Turán theorem (Theorem 5), there are at most $4t^{2-1/r_0}$ blue edges between $C'(u)$ and $C'(v)$. Recall further that $C'(u)$ and $C'(v)$ are, respectively, subcliques of the ℓ'-cliques $C(u)$ and $C(v)$ in $G^{r'}\{\ell'\}$. Since $\{u, v\}$ is an edge of $G^{r'}$, there is a complete bipartite graph with a matching removed between $C(u)$ and $C(v)$ in $G^{r'}\{\ell'\}$ and so there is a complete bipartite graph with at most a matching removed for $C'(u)$ and $C'(v)$. It follows that there are at least

$$t^2 - t - 4t^{2-1/r_0}$$

non-blue edges between $C'(u)$ and $C'(v)$.

Using the copy of $H^r\{\ell\} \subseteq G^{r'}$ avoiding edges of J obtained in Claim 23 as a 'template', we will embed a copy of $H^r\{\ell\}$ in $G^{r'}\{\ell'\}$ with no blue edges. For each vertex $u \in V(H^r\{\ell\}) \subseteq V(J)$ we will pick precisely one vertex from $C'(u) \subseteq G^{r'}\{\ell'\}$ in our embedding. The argument proceeds by the Lovász Local Lemma.

For each $u \in V(H^r\{\ell\}) \subseteq V(J)$ let us choose $x_u \in C'(u)$ uniformly and independently at random. Let $e = \{u, v\}$ be an edge of our copy of $H^r\{\ell\}$ in $G^{r'}$ that is not in J. As pointed out above, we know that there are at least $t^2 - t - 4t^{2-1/r_0}$ non-blue edges between $C'(u)$ and $C'(v)$. Letting A_e be the event that $\{x_u, x_v\}$ is a blue edge or a non-edge in $G^{r'}\{\ell'\}$, we have that

$$\mathbb{P}[A_e] \leqslant \frac{t + 4t^{2-1/r_0}}{t^2} \leqslant 5t^{-1/r_0}.$$

The events A_e are not independent, but we can define a dependency graph D for the collection of events A_e by adding an edge between A_e and A_f if and only if $e \cap f \neq \varnothing$. Then, $\Delta = \Delta(D) \leqslant 2\Delta(H^r\{\ell\}) \leqslant 2(b^{r+1}\ell + \ell)$. From our choice of t we get that

$$4\Delta\mathbb{P}[A_e] \leqslant 40(b^{r+1}\ell + \ell^2)t^{-1/r_0} \leqslant 1$$

for all e. Then the Local Lemma [2, Lemma 5.1.1] tells us that $\mathbb{P}\big[\bigcap_e \bar{A}_e\big] > 0$, and hence a simultaneous choice of the x_u's ($u \in V(H^r\{\ell\})$) is possible, as required. This concludes the proof of Claim 24. □

The proof of Lemma 19 is now complete. □

§5. Concluding remarks

To construct our graphs we need that $\mathcal{P}_n(a, b, c, \ell, \theta)$ is non-empty given a good 7-tuple $(a, b, c, \ell, \theta, \Delta, k)$ with $\theta \geqslant 32\sqrt{c}$. We prove this in Lemma 16 using the binomial random graph. Alternatively, it is possible to replace this by using explicit constructions of high girth expanders.

For example, the Ramanujan graphs constructed by Lubotzky, Phillips, and Sarnak [26] can be used to prove Lemma 16.

We now discuss further connections between powers of trees and graph parameters related to treewidth. As pointed out in the introduction, every graph with maximum degree and bounded treewidth is contained in some bounded power of a bounded degree tree and vice versa. This implies that Corollary 2 is equivalent to Theorem 1. For bounded degree graphs, bounded treewidth is equivalent to bounded cliquewidth and also to bounded rankwidth [18]. Therefore, Corollary 2 also holds with treewidth replaced by any of these parameters. Finally, an obvious direction for further research is to investigate the size-Ramsey number of powers T^k of trees T when k and $\Delta(T)$ are no longer bounded.

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References

[1] N. Alon and F. R. K. Chung, *Explicit construction of linear sized tolerant networks*, Discrete Math. **72** (1988), no. 1-3, 15–19. ↑1

[2] N. Alon and J. H Spencer, *The probabilistic method*, 2nd ed., John Wiley & Sons, 2004. ↑18

[3] J. Beck, *On size Ramsey number of paths, trees, and circuits. I*, J. Graph Theory **7** (1983), no. 1, 115–129. ↑1, 2

[4] S. Berger, Y. Kohayakawa, G. Maesaka, T. Martins, W. Mendonça, G. Mota, and O. Parczyk, *The size-ramsey number of powers of bounded degree trees*, Acta Mathematica Universitatis Comenianae **88** (2019), no. 3, 451–456. ↑2

[5] B. Bollobás, *Extremal graph theory with emphasis on probabilistic methods*, CBMS Regional Conference Series in Mathematics, vol. 62, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1986. ↑1

[6] D. Clemens, M. Jenssen, Y. Kohayakawa, N. Morrison, G. O. Mota, D. Reding, and B. Roberts, *The size-Ramsey number of powers of paths*, J. Graph Theory **91** (2019), no. 3, 290–299. ↑2, 3

[7] D. Conlon, J. Fox, and B. Sudakov, *Recent developments in graph Ramsey theory*, Surveys in combinatorics 2015, 2015, pp. 49–118. ↑13

[8] D. Dellamonica Jr., *The size-Ramsey number of trees*, Random Structures Algorithms **40** (2012), no. 1, 49–73. ↑2

[9] G. Ding and B. Oporowski, *Some results on tree decomposition of graphs*, J. Graph Theory **20** (1995), no. 4, 481–499. ↑2

[10] A. Dudek and P. Prałat, *An alternative proof of the linearity of the size-Ramsey number of paths*, Combin. Probab. Comput. **24** (2015), no. 3, 551–555. ↑1

[11] ———, *On some multicolor Ramsey properties of random graphs*, SIAM J. Discrete Math. **31** (2017), no. 3, 2079–2092. ↑1, 2

[12] P. Erdős, *On the combinatorial problems which I would most like to see solved*, Combinatorica **1** (1981), no. 1, 25–42. ↑1

[13] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp, *The size Ramsey number*, Period. Math. Hungar. **9** (1978), no. 1-2, 145–161. ↑1

[14] J. Friedman and N. Pippenger, *Expanding graphs contain all small trees*, Combinatorica **7** (1987), no. 1, 71–76. ↑2, 4

[15] J. Han, M. Jenssen, Y. Kohayakawa, G. O. Mota, and B. Roberts, *The multicolour size-ramsey number of powers of paths*, J. Comb. Theory Ser. B **145** (2020), 359–375. ↑2, 3

[16] P. E. Haxell and Y. Kohayakawa, *The size-Ramsey number of trees*, Israel J. Math. **89** (1995), no. 1-3, 261–274. ↑2

[17] P. E. Haxell, Y. Kohayakawa, and T. Łuczak, *The induced size-Ramsey number of cycles*, Combin. Probab. Comput. **4** (1995), no. 3, 217–239. ↑2, 9

[18] M. Kamiński, V. V. Lozin, and M. Milanič, *Recent developments on graphs of bounded clique-width*, Discrete Appl. Math. **157** (2009), no. 12, 2747–2761. ↑18

[19] N. Kamčev, A. Liebenau, D. R. Wood, and L. Yepremyan, *The size Ramsey number of graphs with bounded treewidth* (2019), available at arXiv:1906.09185. ↑2, 3

[20] X. Ke, *The size Ramsey number of trees with bounded degree*, Random Structures Algorithms **4** (1993), no. 1, 85–97. ↑2

[21] Y. Kohayakawa, T. Retter, and V. Rödl, *The size-Ramsey number of short subdivisions of bounded degree graphs*, Random Structures Algorithms **54** (2019), no. 2, 304–339. ↑2

[22] Y. Kohayakawa, V. Rödl, M. Schacht, and E. Szemerédi, *Sparse partition universal graphs for graphs of bounded degree*, Adv. Math. **226** (2011), no. 6, 5041–5065. ↑2

[23] T. Kővári, V. T. Sós, and P. Turán, *On a problem of K. Zarankiewicz*, Colloq. Math. **3** (1954), 50–57. ↑3

[24] M. Krivelevich, *Long cycles in locally expanding graphs, with applications*, Combinatorica **39** (2019), no. 1, 135–151. ↑2

[25] S. Letzter, *Path Ramsey number for random graphs*, Combin. Probab. Comput. **25** (2016), no. 4, 612–622. ↑1

[26] A. Lubotzky, R. Phillips, and P. Sarnak, *Ramanujan graphs*, Combinatorica **8** (1988), no. 3, 261–277. ↑18

[27] A. Pokrovskiy, *Calculating Ramsey numbers by partitioning colored graphs*, J. Graph Theory **84** (2017), no. 4, 477–500. ↑4

[28] A. Pokrovskiy and B. Sudakov, *Ramsey goodness of paths*, J. Combin. Theory Ser. B **122** (2017), 384–390↑3, 4

[29] ———, *Ramsey goodness of cycles*, SIAM J. Discrete Math. **34** (2020), no. 3, 1884–1908. ↑3, 4

[30] V. Rödl and E. Szemerédi, *On size Ramsey numbers of graphs with bounded degree*, Combinatorica **20** (2000), no. 2, 257–262. ↑2

[31] D. R Wood, *On tree-partition-width*, Eur. J. Comb. **30** (2009), no. 5, 1245–1253. ↑2

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