Ann. Henri Poincaré *Online First*

https://doi.org/10.1007/s00023-022-01172-2

Annales Henri Poincaré

Decay of Solutions to the Klein–Gordon Equation on Some Expanding Cosmological Spacetimes

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Abstract. The decay of solutions to the Klein–Gordon equation is studied in two expanding cosmological spacetimes, namely

- the de Sitter universe in flat Friedmann–Lemaître–Robertson–Walker (FLRW) form and
- the cosmological region of the Reissner–Nordström–de Sitter (RNdS) model.

Using energy methods, for initial data with finite higher-order energies, decay rates for the solution are obtained. Also, a previously established decay rate of the time derivative of the solution to the wave equation, in an expanding de Sitter universe in flat FLRW form, is improved, proving Rendall's conjecture. A similar improvement is also given for the wave equation in the cosmological region of the RNdS spacetime.

Mathematics Subject Classification. Primary 58J45; Secondary 35B40, 35L15, 83C57, 83F05.

Contents

Published online: 16 March 2022

Birkhäuser

1. Introduction

The aim of this article is to obtain exact decay rates for solutions to the Klein–Gordon equation in a fixed background of some expanding cosmological spacetimes. The two spacetimes we will consider are the de Sitter universe in flat Friedmann–Lemaître–Robertson–Walker (FLRW) form and the cosmological region of the Reissner–Nordström–de Sitter (RNdS) model. The problem we consider is linear, in that the background is fixed. This constitutes a first step towards understanding the more complicated nonlinear coupled problem, where one also considers the effect of the energy–momentum tensor of the solution to the Klein–Gordon equation on the Einstein equation. This nonlinear coupled problem is much more complicated and usually requires, as a first step, a detailed understanding of our simpler linear problem.

There are several motivations behind the interest in this question. Firstly, one may consider the linear wave equations as a proxy for the Einstein equations, with the ultimate goal of understanding the qualitative behaviour of solutions to the Einstein equations. (The vacuum Einstein equations become wave-like equations in harmonic coordinates; see for example [18, §5.4, p. 110]). After this first step, one may then proceed to consider linearised Einstein equations (which can be reduced to tensor wave-like linear equations) and, finally, the full nonlinear Einstein equations. With the addition of a positive cosmological constant to the Einstein field equations, the expectation is that the resulting accelerated expansion has a dominating effect on the decay of solutions. Precise estimates on solutions may then prove useful in formulating and proving cosmic no-hair theorems (e.g. [3,7]).

The wave equation $\square_g \phi = 0$ in expanding cosmological spacetimes (M, g) has been amply studied in the literature; see for example [5,8,10,23] and the references therein. It is a natural question to also study the Klein–Gordon equation $\square_g \phi - m^2 \phi = 0$, the degenerate version of which, when $m^2 = 0$, is the wave equation. For example, in [23, §6], also the case of the Klein–Gordon equation in the Schwarzschild–de Sitter spacetime is considered. In [21], the asymptotic behaviour of the solutions to the Klein–Gordon equation near the Big Bang singularity is studied, while we investigate the asymptotics of the Klein–Gordon equation in the far future in the case of the de Sitter universe in flat FLRW form, and in the cosmological region of the Reissner–Nordström–de Sitter solution. Recently, in [11] (see also [26] and [22]), among other things, decay estimates for the solutions to the Klein–Gordon equation were obtained in de Sitter models (see in particular, Corollary 2.1 and the less obvious Proposition 3.1 of [11]). However, these results are proved via Fourier transformation (reminiscent of our mode calculation in Appendix A (Section 6)) and do not seem to be as sharp as our Theorem 3.1.

The wave equation in the de Sitter spacetime having flat three-dimensional spatial sections was considered in Rendall [20]. There, it was shown that the time derivative $\partial_t\phi =: \dot\phi$ decays at least as $e^{-Ht} = (a(t))^{-1}$, where $H = \sqrt{\Lambda/3}$ is the Hubble constant and $\Lambda > 0$ is the cosmological constant. Moreover, it was conjectured that the decay is of the order $e^{-2Ht} = (a(t))^{-2}$. The almost-exact conjectured decay rate of $|\dot\phi| \lesssim (a(t))^{-2+\delta}$ (where $\delta > 0$ can be chosen arbitrarily at the outset) follows as a corollary of a result shown recently [8, Remark 1.1]. We improve this result, to obtain full conformity with Rendall's conjecture, in our result Theorem 2.3.

Finally, from the pure mathematical perspective, analysis of linear wave equations on Lorentzian manifolds is a natural topic of study within the realm of hyperbolic partial differential equations and differential geometry; see for example [2], [25, §7, Chap. 2].

A naive heuristic indication of the effect of the accelerated expansion on the decay of the solution, based on physical energy considerations, can be obtained as follows. Considering an expanding FLRW model with flat n-dimensional spatial sections of radius $a(t)$, we have on the one hand that the energy density of a solution ϕ of the Klein–Gordon equation is of the order of $m^2\phi^2$. On the other hand, if the wavelength of the particles associated with ϕ follows the expansion, then it is proportional to $a(t)$, and so the energy varies as $E^2 \sim m^2 + p^2 \sim A + \frac{B}{(a(t))^2}$, where $A, B > 0$ are constants. Thus,

$$m^2\phi^2(a(t))^n \ \propto \ \Big(A + \frac{B}{(a(t))^2}\Big),$$

giving $m^2\phi^2 \sim (a(t))^{-n}(A + \frac{B}{(a(t))^2})$. As $\dot a \geqslant 0$ (expanding FLRW spacetime), the term $A + \frac{B}{(a(t))^2}$ approaches a finite positive value, and so one may expect

$$\phi \sim (a(t))^{-\frac{n}{2}}.$$

We will find out that in fact things are much more complicated: this decay rate is valid only for $|m| \geqslant \frac{n}{2}$. In order to obtain precise conjectures on the expected decay, we will consider Fourier modes for spatially periodic solutions to the Klein–Gordon equation or, equivalently, consider the expanding de Sitter universe in flat FLRW form with toroidal spatial sections. This exercise already demonstrates that the underlying decay mechanism is the cosmological expansion, as opposed to dispersion. The Fourier mode analysis, which is peripheral to the rest of the paper, is relegated to Appendix A (Section 6).

In the cosmological region of the Reissner–Nordström–de Sitter spacetimes, the expanding region is foliated by spacelike hypersurfaces of 'constant r'. One expects the decay rate with respect to r, for the solution to the Klein–Gordon equation, in the cosmological region of the Reissner–Nordström–de Sitter spacetime, to be the same as the one for the de Sitter universe in flat FLRW form, when e^t is replaced by r. We show that this expectation is correct, and a suitable modification of the technique used in the case of the de Sitter universe in flat FLRW form does enable one to obtain the expected decay rates also for the case of the Reissner–Nordström–de Sitter spacetime.

Our main results are as follows:

- Theorem 2.3 considers the $m = 0$ case (wave equation), and we obtain a decay estimate on $\partial_t \phi$, improving a corollary of [8, Theorem 1], and proving the aforementioned Rendall's conjecture.
- Theorem 5.3 improves [8, Theorem 2], and we obtain a decay estimate on $\partial_r \phi$, using a similar method to the one we use for proving Rendall's conjecture.
- Theorem 3.1 gives the decay rate of the solutions ϕ to the Klein–Gordon equation in the de Sitter universe in flat FLRW form.
- Theorem 4.2 gives the decay rate of the solutions ϕ to the Klein–Gordon equation in the cosmological region of the RNdS model.

Theorems 2.3, 3.1, 4.2 and 5.3 are stated and proved in Sects. 2–5, respectively. The Fourier mode analysis for spatially-periodic solutions to the Klein–Gordon equation is given in Appendix A(Section 6), while Appendix B (Section 7) contains a technical lemma which is needed in the proof of Theorem 3.1. Finally, in Appendix C (Section 8), we establish the sharpness of the bound of the $|m| = \frac{n}{2}$ case of Theorem 3.1.

1.1. Relation of Our Results to Previous Work

Our decay rates for the Klein–Gordon equation solutions in the case of the de Sitter universe can be retrieved from the article [26] by setting $x = e^{-t}$, $Y = \mathbb{R}^n$ therein. However, the methods used are entirely different: our proof in this case is more explicit and more elementary (relying on energy methods, rather than technical tools from microlocal analysis of partial differential operators).

In the article [13], the Klein–Gordon equation is studied in the Nariai spacetime using energy methods, and en route it is also established that solutions of the Klein–Gordon equation decay exponentially in the de Sitter case (with spherical spatial sections). However, the decay rates are not given explicitly.

The article [9] contains a general discussion of redshift estimates, which we use to prove our results in the context of the Reissner–Nordström–de Sitter spacetime. Similar estimates are used in the article [23] to study the wave equation in the Schwarzschild–de Sitter spacetime, of which the Reissner–Nordström–de Sitter spacetime is a perturbation for large radius. Nevertheless, we do not appeal to these results, and instead of extracting what we need from these sources, we give a less technical, self-contained derivation for the convenience of the reader in Sect. 4.5. Here, we follow [8] (where a similar derivation was given for the wave equation).

2. Decay in the de Sitter Universe in Flat FLRW Form; $m = 0$

In [8, Theorem 1], the following result was shown:

Theorem 2.1. *Suppose that*

- $\delta > 0$,
- $I \subset \mathbb{R}$ *is an open interval of the form* $(t_*, +\infty)$, $t_0 \in I$,
- $a(\cdot) \in C^1(I)$ *with* $\dot{a}(t) \geqslant 0$ *for* $t \geqslant t_0$, *and* $\epsilon > 0$ *is such that*
$$\int_{t_0}^{\infty} \frac{1}{(a(t))^\epsilon} \mathrm{d}t < +\infty,$$
- $n \geqslant 2$,
- (M, g) *is an expanding* FLRW *spacetime with flat* n*-dimensional sections, given by* $I \times \mathbb{R}^n$, *with the metric*
$$g = -dt^2 + (a(t))^2 \left((dx^1)^2 + \cdots + (dx^n)^2\right), \tag{1}$$
- $k > \frac{n}{2} + 2$, $\quad \phi_0 \in H^k(\mathbb{R}^n)$, $\quad \phi_1 \in H^{k-1}(\mathbb{R}^n)$, *and*
- ϕ *is a smooth solution to the Cauchy problem*
$$\left\{ \begin{array}{rl} \Box_g \phi = 0, & (t \geqslant t_0,\ \mathbf{x} \in \mathbb{R}^n), \\ \phi(t_0, \mathbf{x}) = \phi_0(\mathbf{x}) & (\mathbf{x} \in \mathbb{R}^n), \\ \partial_t \phi(t_0, \mathbf{x}) = \phi_1(\mathbf{x}) & (\mathbf{x} \in \mathbb{R}^n). \end{array} \right.$$

Then
$$\forall t \geqslant t_0, \quad \|\partial_t \phi(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim (a(t))^{-2+\epsilon+\delta}.$$

Here, the symbol $\lesssim$ is used to mean that there exists a constant $C(\delta)$, independent of ϵ, such that
$$\|\partial_t \phi(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \leqslant C(\delta)\, (a(t))^{-2+\epsilon+\delta}.$$
We also use the standard notation $H^k(\mathbb{R}^n)$ for the Sobolev space,
$$\|\phi\|^2_{H^k(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \sum_{|\boldsymbol{\alpha}| \leqslant k} (\partial_{\boldsymbol{\alpha}} \phi)^2 d^n \mathbf{x} < +\infty \quad \text{for } \phi \in H^k(\mathbb{R}^n),$$
where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$, $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$, and $\partial_{\boldsymbol{\alpha}} = (\partial_{x_1})^{\alpha_1} \cdots (\partial_{x_n})^{\alpha_n}$; see for example [27, p. 249] or [25, Chap. 4].

Remark 2.2. (*Smoothness assumption on the solution* ϕ) In Theorem 2.1 (and later also in Theorems 2.3, 3.1, 4.2, 5.1, 5.3), we will assume, for the sake of simplicity of exposition, that the solution ϕ to the wave/Klein–Gordon equation is smooth. However, these theorems are also true without this assumption. To see this, we note that for non-smooth solutions with initial data in $H^k \times H^{k-1}$, we can approximate the initial data by smooth functions in $H^k \times H^{k-1}$, prove the bounds for the H^k norms of the corresponding solutions, and then take limits. Since the solution of the problem with rough initial data is in $C^0(I, H^k) \cap C^1(I, H^{k-1})$, these bounds will continue to be true in the limit, and we can then use the Sobolev embedding theorem. This enables one to drop the smoothness assumption.

In Theorem 2.1, in particular, if $a(t) = e^{Ht}$, where H is the Hubble constant, then since $\epsilon > 0$ can be taken to be arbitrarily small, we obtain

$$\|\partial_t \phi(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim (a(t))^{-2+\delta} = e^{-(2-\delta)Ht},$$

and this is in agreement with Rendall's conjecture up to the small quantity $\delta > 0$. We will show below that in fact one gets the exact rate $(a(t))^{-2}$ when $n > 2$. There is no loss of generality in assuming that $H = 1$. Our result is the following.

Theorem 2.3. *Suppose that*

- $I \subset \mathbb{R}$ *is an open interval of the form* $(t_*, +\infty)$, $t_0 \in I$,
- $n > 2$,
- (M, g) *is the expanding de Sitter universe in flat FLRW form, with flat* n*-dimensional sections, given by* $I \times \mathbb{R}^n$, *with the metric*
$$g = -dt^2 + e^{2t}\left((dx^1)^2 + \cdots + (dx^n)^2\right),$$
- $k > \frac{n}{2} + 2$, $\phi_0 \in H^k(\mathbb{R}^n)$, $\phi_1 \in H^{k-1}(\mathbb{R}^n)$, *and*
- ϕ *is a smooth solution to the Cauchy problem*
$$\begin{cases} \Box_g \phi = 0, & (t \geqslant t_0,\ \mathbf{x} \in \mathbb{R}^n), \\ \phi(t_0, \mathbf{x}) = \phi_0(\mathbf{x}) & (\mathbf{x} \in \mathbb{R}^n), \\ \partial_t \phi(t_0, \mathbf{x}) = \phi_1(\mathbf{x}) & (\mathbf{x} \in \mathbb{R}^n). \end{cases}$$

Then

$$\forall t \geqslant t_0, \quad \|\partial_t \phi(t, \cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim (a(t))^{-2}.$$

Proof. We proceed in several steps.

Step 1: Bound on $\Delta\phi$.

We will follow the preliminary steps of the proof of [8, Theorem 1] in order to obtain a bound on $\Delta\phi$, which will be needed in the proof of our Theorem 2.3. We repeat this preliminary step here from [8, §2.2] for the sake of completeness and for the convenience of the reader.

For a vector field $X = X^\mu \partial_\mu$, it can be shown that

$$\nabla_\mu X^\mu = \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g}\, X^\mu),$$

where $g := \det[g_{\mu\nu}]$ is the determinant of the matrix $[g_{\mu\nu}]$ describing the metric in the chart. Then, it follows that

$$\Box_g \phi = \nabla_\mu(\partial^\mu \phi) = \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g}\, \partial^\mu \phi).$$

Thus, $\Box_g \phi = 0$ can be rewritten as $\partial_\mu(\sqrt{-g}\, \partial^\mu \phi) = 0$. With the metric for the de Sitter universe in flat FLRW form given by

$$g = -dt^2 + (a(t))^2\left((dx^1)^2 + \cdots + (dx^n)^2\right),$$

the wave equation can be rewritten as $\partial_\mu(a^n\partial^\mu\phi)=0$, that is,

$$-\ddot{\phi}-\frac{n\dot{a}}{a}\dot{\phi}+\frac{1}{a^2}\delta^{ij}\partial_i\partial_j\phi=0.$$

We recall (see e.g. [27, Appendix E]) that the energy–momentum tensor for the wave equation is

$$T_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-\frac{g_{\mu\nu}}{2}\partial_\alpha\phi\partial^\alpha\phi. \tag{2}$$

Then, it can be shown that $\nabla_\mu T^{\mu\nu}=0$. From (2), we have in particular that

$$T_{00}=\frac{1}{2}\left(\dot{\phi}^2+a^{-2}\delta^{ij}\partial_i\phi\partial_j\phi\right).$$

Define the vector field

$$X=a^{2-n}\frac{\partial}{\partial t}.$$

Then, X is future-pointing ($g(X,\partial_t)<0$) and causal (X is time-like since $g(X,X)<0$). We form the current J, given by

$$J_\mu=T_{\mu\nu}X^\nu.$$

Then, (set $m=0$ in [18, Ex.5.7(1), p. 116], but a justification is given below)

$$J=(X\cdot\phi)\text{grad }\phi-\frac{1}{2}g(\text{grad }\phi,\text{grad }\phi)X. \tag{$\star$}$$

Here, $X\cdot\phi$ is the application of X on ϕ. To see ($\star$), note that

$$\begin{aligned}\text{LHS}&=g^{\mu\nu}J_\nu=g^{\mu\nu}T_{\nu\sigma}a^{2-n}\delta^{\sigma0}=g^{\mu\nu}\Big((\partial_\nu\phi)(\partial_\sigma\phi)-\frac{g_{\nu\sigma}}{2}(\partial_\alpha\phi)(\partial^\alpha\phi)\Big)a^{2-n}\delta^{\sigma0}\\&=a^{2-n}\dot{\phi}g^{\mu\nu}\partial_\nu\phi-\frac{1}{2}(\partial_\alpha\phi)(\partial^\alpha\phi)a^{2-n}\delta^{\mu0},\ \text{ and}\\\text{RHS}&=a^{2-n}\dot{\phi}g^{\alpha\mu}\partial_\alpha\phi-\frac{1}{2}g_{\alpha\beta}(\partial_\theta\phi)g^{\theta\alpha}(\partial_\sigma\phi)g^{\sigma\beta}a^{2-n}\delta^{\mu0}\\&=a^{2-n}\dot{\phi}g^{\alpha\mu}\partial_\alpha\phi-\frac{1}{2}(\partial_\beta\phi)(\partial^\beta\phi)a^{2-n}\delta^{\mu0}.\end{aligned}$$

It follows from ($\star$) that $g(J,J)\leqslant 0$, so that J is causal. Also, J is past-pointing. To see this, we choose $E_1,\dots,E_n$ orthogonal and spacelike such that $\{X,E_1,\dots,E_n\}$ forms an orthogonal basis in each tangent space. Then, expressing grad $\phi=c^0X+c^1E_1+\cdots c^nE_n$, we obtain

$$\begin{aligned}g(J,X)&=(g(X,\text{grad }\phi))^2-\frac{1}{2}g(\text{grad }\phi,\text{grad }\phi)\cdot g(X,X)\\&=\frac{(c^0)^2}{2}(g(X,X))^2-\frac{1}{2}\big((c^1)^2g(E_1,E_1)+\cdots\\&\qquad+(c^n)^2g(E_n,E_n)\big)\cdot g(X,X)\geqslant 0.\end{aligned}$$

Set $N=\frac{\partial}{\partial t}$, the future unit normal vector field. We define the energy E by

$$E(t)=\int_{\{t\}\times\mathbb{R}^n}J_\mu N^\mu=\int_{\mathbb{R}^n}a^2T_{00}d^n\mathbf{x}=\int_{\mathbb{R}^n}\frac{1}{2}\left(a^2\dot{\phi}^2+\delta^{ij}\partial_i\partial_j\phi\right)d^n\mathbf{x}.$$

We note that $[X, \partial_\mu] = [a^{2-n}\partial_t, \partial_\mu]$, and

$$[a^{2-n}\partial_t, \partial_t] = -(2-n)a^{1-n}\dot{a}\partial_t,$$
$$[a^{2-n}\partial_t, \partial_j] = a^{2-n}\partial_t\partial_j - \partial_j(a^{2-n}\partial_t) = 0,$$

so that $dx^i[X, \partial_\mu] = 0$. The deformation tensor Π associated with the multiplier X is

$$\begin{aligned}\Pi &:= \frac{1}{2}\mathcal{L}_X g = \frac{1}{2}\mathcal{L}_X(-dt^2 + a^2\delta_{ij}dx^i dx^j) = -dt\mathcal{L}_X dt + \frac{a^{2-n}}{2}2a\dot{a}\delta_{ij}dx^i dx^j \\ &:= -dt\mathcal{L}_X dt + \dot{a}a^{3-n}\delta_{ij}dx^i dx^j.\end{aligned}$$

Here, we used the facts $\mathcal{L}_X(a^2) = a^{2-n}\frac{\partial}{\partial t}a^2 = a^{2-n}2a\dot{a}$ and $\mathcal{L}_X dx^i = 0$ (since $0 = \mathcal{L}_X(dx^i(\partial_\mu)) = (\mathcal{L}_X dx^i)_\mu + dx^i([X, \partial_\mu]) = (\mathcal{L}_X dx^i)_\mu + 0$). We have $0 = \mathcal{L}_X(dt(\partial_\mu)) = (\mathcal{L}_X dt)_\mu + dt[X, \partial_\mu]$, and so from the above expression for $[X, \partial_\mu]$, we obtain

$$\mathcal{L}_X dt = (2-n)\dot{a}a^{1-n}dt.$$

Thus, $\Pi = (n-2)\dot{a}a^{1-n}dt^2 + \dot{a}a^{3-n}\delta_{ij}dx^i dx^j$.

It can be shown that $\nabla_\mu J^\mu = T^{\mu\nu}\Pi_{\mu\nu}$. Indeed, from the expression for the Lie derivative of the metric given in [18, Exercise 2, p. 93] and the expression for the divergence of $T^{\mu\nu}X_\nu$ given in [18, §5.2], we have

$$T^{\mu\nu}\Pi_{\mu\nu} = \frac{1}{2}T^{\mu\nu}(\mathcal{L}_X g)_{\mu\nu} = \frac{1}{2}T^{\mu\nu}(\nabla_\mu X_\nu + \nabla_\nu X_\mu) = \frac{1}{2}(\nabla_\mu J^\mu + \nabla_\mu J^\mu) = \nabla_\mu J^\mu.$$

So the ‘bulk term’ is

$$\begin{aligned}\nabla_\mu J^\mu &= T^{\mu\nu}\Pi_{\mu\nu} \\ &= (n-2)\dot{a}a^{1-n}\dot{\phi}^2 + \frac{n-2}{2}\dot{a}a^{1-n}\partial_\alpha\phi\partial^\alpha\phi + \dot{a}a^{-1-n}\delta^{ij}\partial_i\phi\partial_j\phi - \frac{n}{2}\dot{a}a^{1-n}\partial_\alpha\phi\partial^\alpha\phi \\ &= (n-1)\dot{a}a^{1-n}\dot{\phi}^2 \geqslant 0.\end{aligned}$$

For each $R > 0$, define the set $B_0 := \{(t_0, \mathbf{x}) \in I \times \mathbb{R}^n : \langle \mathbf{x}, \mathbf{x} \rangle_{\mathbb{R}^n} \leqslant R^2\}$. The future domain of dependence of B_0 is the set

$$D^+(B_0) := \left\{ p \in M \;\middle|\; \begin{array}{l}\text{Every past inextendible causal curve} \\ \text{through } p \text{ intersects } B_0.\end{array} \right\}.$$

(Here by a *causal curve*, we mean one whose tangent vector at each point is a causal vector. A curve $c : (a, b) \to M$ which is smooth and future directed[1] is called *past inextendible* if $\lim\limits_{t \to a} c(t)$ does not exist.)

Let $t_1 > t_0$. We will now apply the divergence theorem to the region

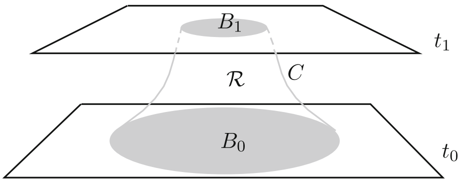
$$\mathcal{R} := D^+(B_0) \cap \{(t, \mathbf{x}) \in M : t \leqslant t_1\}.$$

For preliminaries on the divergence theorem in the context of a time-oriented Lorentzian manifold, we refer the reader to [27, Appendix 7]. We have

$$\int_{\mathcal{R}} (\nabla_\mu J^\mu)\epsilon = \int_{\partial\mathcal{R}} J \lrcorner \epsilon,$$

[1] That is, $\dot{c}$ is future-pointing.

where $\partial\mathcal{R}$ denotes the boundary of $\mathcal{R}$, ϵ is the volume form on M induced by g, and $\lrcorner$ denotes contraction in the first index.



Since J is past-pointing, the boundary integral over the null portion C of the boundary $\partial\mathcal{R}$ is nonpositive. Also, because $\nabla_\mu J^\mu$ is nonnegative, we have that the volume integral over $\mathcal{R}$ is nonnegative. This gives an inequality on the two boundary integrals, one over B_0 and the other over $B_1 := D^+(B_0) \cap \{t = t_1\}$, as follows:

$$\int_{B_0} \frac{1}{2}(a^2\dot{\phi}^2 + \delta^{ij}\partial_i\phi\partial_j\phi)d^n\mathbf{x} \geqslant \int_{B_1} \frac{1}{2}(a^2\dot{\phi}^2 + \delta^{ij}\partial_i\phi\partial_j\phi)d^n\mathbf{x}.$$

Passing the limit $R \to \infty$ yields $E(t_0) \geqslant E(t_1)$. (We note that the radius of B_1 also goes to infinity as $R \to \infty$.) As the choice of $t_1 > t_0$ was arbitrary, we have that for all $t \geqslant t_0$, $E(t) \leqslant E(t_0) < \infty$. The finiteness of $E(t_0)$ follows from our assumption that $\phi_0 \in H^k(\mathbb{R}^n)$ and $\phi_1 \in H^{k-1}(\mathbb{R}^n)$ for a k satisfying $k > \frac{n}{2} + 2 \geqslant 1$. From here, it follows that for all $t \geqslant t_0$,

$$\int_{\mathbb{R}^n} \dot{\phi}^2 d^n\mathbf{x} \lesssim \frac{1}{a^2}, \quad \text{and} \quad \int_{\mathbb{R}^n} \delta^{ij}\partial_i\phi\partial_j\phi d^n\mathbf{x} \lesssim 1.$$

But since each partial derivative $\partial_i\phi$ is also a solution of the wave equation, and as $k \geqslant 2$, we obtain, by applying the above to the partial derivatives $\partial_i\phi$, that also

$$\int_{\mathbb{R}^n} (\Delta\phi)^2 d^n\mathbf{x} \lesssim 1.$$

In fact, since $k > \frac{n}{2} + 2$, we also obtain that for a $k' > \frac{n}{2}$, $\|\Delta\phi\|_{H^{k'}(\mathbb{R}^n)} \lesssim 1$. Finally, by the Sobolev inequality (see e.g. [14, (7.30), p. 158]), we obtain

$$\|\Delta\phi\|_{L^\infty(\mathbb{R}^n)} \lesssim 1. \tag{3}$$

This completes Step 1 of the proof of Theorem 2.3. We note that this step loses two derivatives when we drop the smoothness assumption on ϕ.

Step 2: The wave equation in conformal coordinates.

The key point of departure from the earlier derivation of the estimates from [8] is the usage of 'conformal coordinates', which renders the wave equation in a form where it becomes possible to integrate, leaving essentially just the time derivative of ϕ with other terms (e.g. $\Delta\phi$) for which we have a known bound. An application of the triangle inequality will then deliver the desired bound.

Define $\tau = \displaystyle\int_{t_0}^{t} \frac{1}{a(s)} \mathrm{d}s$. Then we obtain that $\dfrac{\mathrm{d}\tau}{\mathrm{d}t} = \dfrac{1}{a(t)}$ and $a(t)\dfrac{\mathrm{d}}{\mathrm{d}t} = \dfrac{\mathrm{d}}{\mathrm{d}\tau}$.

With a slight abuse of notation, we write $a(\tau) := a(t(\tau))$. Then, $dt = a(\tau)d\tau$. So $g = -dt^2 + (a(t))^2\left((dx^1)^2 + \cdots + (dx^n)^2\right) = (a(\tau))^2\big(-d\tau^2 + \delta_{ij}\, dx^i dx^j\big)$. The wave equation $\Box_g \phi = 0$ can be rewritten as $\partial_\mu(\sqrt{-g}\,\partial^\mu \phi) = 0$, and so we obtain $\partial_\mu(a^{n+1}\partial^\mu\phi) = 0$. Separating the partial derivative operators with respect to the τ and $\mathbf{x}$ coordinates, we obtain the wave equation in conformal coordinates $\partial_\tau(a^{n-1}\partial_\tau\phi) = a^{n-1}\Delta\phi$, where Δ is the usual Laplacian on $\mathbb{R}^n$. This completes Step 2 of the proof of Theorem 2.3.

Step 3: $n > 2$ **and** $a(t) = e^t$. We have

$$\tau = \int_{t_0}^{t} \frac{1}{e^s}\mathrm{d}s = e^{-t_0} - \frac{1}{e^t} = e^{-t_0} - \frac{1}{a}, \tag{4}$$

and so $a(\tau) = \dfrac{1}{e^{-t_0} - \tau}$. We note that $\tau \in [0, e^{-t_0})$. Also,

$$a(\tau = 0) = \frac{1}{e^{-t_0}} = e^{t_0} = a(t = t_0).$$

Integrating $\partial_\tau(a^{n-1}\partial_\tau\phi) = a^{n-1}\Delta\phi$ from $\tau = 0$ to τ, we obtain

$$a^{n-1}\partial_\tau\phi - a(t_0)^{n-1}\, \partial_\tau\phi|_{\tau=0} = \int_0^\tau \Delta\phi \frac{1}{(e^{-t_0} - \tau)^{n-1}}\mathrm{d}\tau,$$

and so $a^{n-1}a\partial_t\phi = a(t_0)^{n-1}a(t_0)\, \partial_t\phi|_{t=t_0} + \displaystyle\int_0^\tau \Delta\phi\frac{1}{(e^{-t_0} - \tau)^{n-1}}\mathrm{d}\tau$, that is,

$$\partial_t\phi = (a(t))^{-n}\Big(a(t_0)^n\phi_1 + \int_0^\tau \Delta\phi\frac{1}{(e^{-t_0} - \tau)^{n-1}}\mathrm{d}\tau\Big).$$

Hence, using the bound from (3), namely $\|\Delta\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leqslant C$ for all $t \geqslant t_0$,

$$\begin{aligned}
\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} &\leqslant (a(t))^{-n}\Big(a(t_0)^n\|\phi_1\|_{L^\infty(\mathbb{R}^n)} + \int_0^\tau \|\Delta\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}\frac{1}{(e^{-t_0}-\tau)^{n-1}}\mathrm{d}\tau\Big)\\
&= (a(t))^{-n}\Big(a(t_0)^n\|\phi_1\|_{L^\infty(\mathbb{R}^n)} + \frac{C}{n-2}\Big((e^{-t_0} - \tau)^{2-n} - (e^{-t_0})^{2-n}\Big)\Big)\\
&= (a(t))^{-n}\Big(a(t_0)^n\|\phi_1\|_{L^\infty(\mathbb{R}^n)} + \frac{C}{n-2}\Big((a(t))^{n-2} - (a(t_0))^{n-2}\Big)\Big)\\
&\leqslant (a(t))^{-n}(a(t))^{n-2}\Big(\frac{a(t_0)^n\|\phi_1\|_{L^\infty(\mathbb{R}^n)}}{(a(t))^{n-2}} + \frac{C}{n-2}\Big(1 - \Big(\frac{a(t_0)}{a(t)}\Big)^{n-2}\Big)\Big)\\
&\leqslant \frac{1}{(a(t))^2}\Big(\frac{a(t_0)^n\|\phi_1\|_{L^\infty(\mathbb{R}^n)}}{(a(t_0))^{n-2}} + \frac{C}{n-2}(1-0)\Big).
\end{aligned}$$

Hence,

$$\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \leqslant \frac{1}{(a(t))^2}\Big((a(t_0))^2\|\phi_1\|_{L^\infty(\mathbb{R}^n)} + \frac{C}{n-2}\Big),$$

and so

$$\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim (a(t))^{-2}.$$

This completes the proof of Theorem 2.3. $\square$

Remark 2.4. **The case when** $n = 2$ **and** $a(t) = e^t$:

Integrating $\partial_\tau(a\partial_\tau\phi) = a\Delta\phi$ from $\tau = 0$ to τ, we obtain

$$a\partial_\tau\phi - a(t_0)\, \partial_\tau\phi|_{\tau=0} = \int_0^\tau \Delta\phi\frac{1}{e^{-t_0} - \tau}\mathrm{d}\tau,$$

and so

$$\partial_t\phi = (a(t))^{-2}\left(a(t_0)^2\phi_1 + \int_0^\tau \Delta\phi\frac{1}{e^{-t_0}-\tau}\mathrm{d}\tau\right).$$

Hence,

$$\begin{aligned}
&\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}\\
&\quad\leqslant (a(t))^{-2}\Big(a(t_0)^2\|\phi_1\|_{L^\infty(\mathbb{R}^2)} + \int_0^\tau \|\Delta\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}\frac{1}{e^{-t_0}-\tau}\mathrm{d}\tau\Big)\\
&\quad= (a(t))^{-2}\Big(a(t_0)^2\|\phi_1\|_{L^\infty(\mathbb{R}^2)} + C\left(-\log(e^{-t_0}-\tau)\Big|_0^\tau\right)\Big)\\
&\quad= (a(t))^{-2}(\log a(t))\Big(\frac{a(t_0)^2\|\phi_1\|_{L^\infty(\mathbb{R}^2)}}{\log a(t)} + C\Big(1-\frac{\log a(t_0)}{\log a(t)}\Big)\Big)\\
&\quad\leqslant (a(t))^{-2}(\log a(t))\Big(\frac{a(t_0)^2\|\phi_1\|_{L^\infty(\mathbb{R}^2)}}{t_0} + C\Big),
\end{aligned}$$

and so

$$\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim (a(t))^{-2}\log a(t).$$

This can be viewed as an improvement to [8, Theorem 1] in the special case when $a(t) = e^t$ and $n = 2$, since

$$\log a(t) = t \lesssim e^{\delta t} = 1 + \delta t + \cdots .$$

Remark 2.5. **The case when** $a(t) = t^p$, $p \geqslant 1$:

One can prove an analogue of Theorem 2.3 when $a(t) = t^p$ as well. In this case, the ϵ from Theorem 2.1 can be chosen to be any number satisfying

$$\epsilon > \frac{1}{p},$$

and so Theorem 2.1 gives the decay estimate

$$\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim (a(t))^{-2+\frac{1}{p}+\delta} = t^{-(2p-1-\delta')},$$

where $\delta' > 0$ can be chosen arbitrarily. We can improve this to the following:

$$\|\partial_t\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim (a(t))^{-2+\frac{1}{p}} = t^{-(2p-1)}.$$

The proof is the same, mutatis mutandis, as that of Theorem 2.3.

Remark 2.6. Using a similar method, one can also obtain an improvement to [8, Theorem 2]. But we will postpone this discussion until after Sect. 4, since we will need some preliminaries about the RNdS spacetime, which will be established in Sect. 4.

3. Decay in the de Sitter Universe in Flat FLRW Form

The Klein–Gordon equation is $\Box_g\phi - m^2\phi = 0$, that is,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,\partial^\mu\phi) - m^2\phi = 0.$$

In the case of the de Sitter universe in flat FLRW form, we obtain

$$-\ddot{\phi}-\frac{n\dot{a}}{a}\dot{\phi}+\frac{1}{a^2}\delta^{ij}\partial_i\partial_j\phi-m^2\phi=0. \tag{5}$$

In this section, we will prove Theorem 3.1. We arrive at the guesses for the specific estimates given in Theorem 3.1, based on an analysis using Fourier modes, assuming spatially periodic solutions. This Fourier mode analysis is given in 'Appendix 6'.

Theorem 3.1. *Suppose that*

- $I\subset\mathbb{R}$ *is an open interval of the form* $(t_*,+\infty)$, $t_0\in I$,
- $m\in\mathbb{R}$,
- $n>2$,
- (M,g) *is the expanding de Sitter universe in flat FLRW form, with flat* n*-dimensional sections, given by* $I\times\mathbb{R}^n$, *with the metric*

$$g=-dt^2+e^{2t}\left((dx^1)^2+\cdots+(dx^n)^2\right),$$

- $k>\frac{n}{2}+2$, $\quad\phi_0\in H^k(\mathbb{R}^n)$, $\quad\phi_1\in H^{k-1}(\mathbb{R}^n)$, *and*
- ϕ *is a smooth solution to the Cauchy problem*

$$\begin{cases}\Box_g\phi-m^2\phi=0, & (t\geqslant t_0,\ \mathbf{x}\in\mathbb{R}^n),\\ \phi(t_0,\mathbf{x})=\phi_0(\mathbf{x}) & (\mathbf{x}\in\mathbb{R}^n),\\ \partial_t\phi(t_0,\mathbf{x})=\phi_1(\mathbf{x}) & (\mathbf{x}\in\mathbb{R}^n).\end{cases}$$

Then, for all $t\geqslant t_0$, *we have*

$$\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}\lesssim\begin{cases}a^{-\frac{n}{2}} & \text{if}\quad |m|>\frac{n}{2},\\ a^{-\frac{n}{2}}\log a & \text{if}\quad |m|=\frac{n}{2},\\ a^{-\frac{n}{2}+\sqrt{\frac{n^2}{4}-m^2}} & \text{if}\quad |m|<\frac{n}{2}.\end{cases}$$

Remark 3.2. We recall that the conformally invariant wave equation in $n+1$ dimensions is

$$\left(\Box_g-\frac{n-1}{4n}R_g\right)\phi=0,$$

where R_g is the scalar curvature of the metric g; see for instance [27]. If g is a FLRW metric with flat n-dimensional spatial sections, having the form given by (1), then

$$R_g=\frac{2n\ddot{a}}{a}+\frac{n(n-1)\dot{a}^2}{a^2}.$$

Thus, in de Sitter space in flat FLRW form, the conformally invariant wave equation can be interpreted as a Klein–Gordon equation, with the mass parameter satisfying $m^2=\frac{n^2-1}{4}$. From [8, Appendix 7], we have

$$\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}\lesssim a^{\frac{1-n}{2}}, \tag{6}$$

which follows from using the fact that the L^∞-norm of $\psi(\cdot,t)$, defined by $\phi=a^{1-\frac{n+1}{2}}\psi$ (see [8, eq. (178)]), is uniformly bounded with respect to t. The estimate (6) is in complete agreement with the result of our Theorem 3.1, since the relation $\frac{n^2}{4}-m^2=\frac{1}{4}$ reduces our bound $a^{-\frac{n}{2}+\sqrt{\frac{n^2}{4}-m^2}}$ precisely to $a^{\frac{1-n}{2}}$.

3.1. Preliminary Energy Function and Estimates

Define the energy–momentum tensor T by

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial_\alpha \phi \partial^\alpha \phi + m^2 \phi^2).$$

Then, $\nabla_\mu T^{\mu\nu} = 0$. Also, in particular,

$$T_{00} = \frac{1}{2} \left(\dot{\phi}^2 + \frac{1}{a^2} |\nabla \phi|^2 + m^2 \phi^2 \right) = T^{00}.$$

Set $X = a^{-n} \frac{\partial}{\partial t}$. Then, X is time-like and hence causal, and X is future-pointing. Define J by $J^\mu = T^{\mu\nu} X_\nu$. Then, J is causal and past-pointing. Let $N = \partial_t$. Define the energy E by

$$E(t) = \int_{\{t\} \times \mathbb{R}^n} J_\mu N^\mu = \int_{\mathbb{R}^n} \frac{1}{2} \left(\dot{\phi}^2 + \frac{1}{a^2} |\nabla \phi|^2 + m^2 \phi^2 \right) d^n \mathbf{x}.$$

Define

$$\Pi = \frac{1}{2} \mathcal{L}_X g = -dt \mathcal{L}_X dt + a^{-n+1} \dot{a} \left((dx^1)^2 + \cdots + (dx^n)^2 \right).$$

As $\mathcal{L}_X dt = -n a^{-n-1} \dot{a} dt$, we have

$$\Pi = n a^{-n-1} \dot{a} dt^2 + a^{-n+1} \dot{a} \left((dx^1)^2 + \cdots + (dx^n)^2 \right).$$

Hence,

$$\nabla_\mu J^\mu = T^{\mu\nu} \Pi_{\mu\nu} = \frac{a^{-n-1} \dot{a}}{2} \left(2n \dot{\phi}^2 + \frac{2}{a^2} |\nabla \phi|^2 \right) \geqslant 0.$$

For $R > 0$, define $B_0 := \{(t_0, \mathbf{x}) \in I \times \mathbb{R}^n : \langle \mathbf{x}, \mathbf{x} \rangle_{\mathbb{R}^n} \leqslant R^2\}$. The future domain of dependence of B_0 is denoted by $D^+(B_0)$. Let $t_1 > t_0$. We will apply the divergence theorem to the region $\mathcal{R} := D^+(B_0) \cap \{(t, \mathbf{x}) \in M : t \leqslant t_1\}$. We have

$$\int_{\mathcal{R}} (\nabla_\mu J^\mu) \epsilon = \int_{\partial \mathcal{R}} J \lrcorner \epsilon.$$

Using

- $\nabla_\mu J^\mu \geqslant 0$, and
- the fact that the boundary contribution on C, the null portion of $\partial \mathcal{R}$, is nonpositive (since J is causal and past-pointing),

we obtain the inequality

$$\int_{B_0} \frac{1}{2} \left(\dot{\phi}^2 + \frac{1}{a^2} |\nabla \phi|^2 + m^2 \phi^2 \right) d^n \mathbf{x} \geqslant \int_{B_1} \frac{1}{2} \left(\dot{\phi}^2 + \frac{1}{a^2} |\nabla \phi|^2 + m^2 \phi^2 \right) d^n \mathbf{x}.$$

Passing the limit $R \to \infty$ yields $E(t_1) \leqslant E(t_0) < +\infty$. As $t_1 > t_0$ was arbitrary, we obtain

$$\forall t \geqslant t_0, \quad E(t) = \int_{\mathbb{R}^n} \frac{1}{2} \left(\dot{\phi}^2 + \frac{1}{a^2} |\nabla \phi|^2 + m^2 \phi^2 \right) d^n \mathbf{x} \leqslant E(t_0) \leqslant \infty.$$

From here, it follows that for all $t \geqslant t_0$,

$$\int_{\mathbb{R}^n} \dot{\phi}^2 d^n \mathbf{x} \lesssim 1, \quad \int_{\mathbb{R}^n} |\nabla \phi|^2 d^n \mathbf{x} \lesssim a^2, \quad \text{and} \quad \int_{\mathbb{R}^n} \phi^2 d^n \mathbf{x} \lesssim 1 \text{ (if } m \neq 0).$$

But since each partial derivative $(\partial_{x^1})^{i_1}\cdots(\partial_{x^n})^{i_n}\phi$ is also a solution of the Klein–Gordon equation, it follows from $\phi_0 \in H^k(\mathbb{R}^n)$ and $\phi_1 \in H^{k-1}(R^n)$ for a $k > \frac{n}{2}+2$, that also $\phi(t,\cdot) \in H^k(\mathbb{R}^n)$ and $\partial_t\phi(t,\cdot) \in H^{k-1}(\mathbb{R}^n)$, and moreover,

$$\|\dot{\phi}\|_{H^{k'}(\mathbb{R}^n)} \lesssim 1, \quad \|\partial_i\phi\|_{H^{k'}(\mathbb{R}^n)} \lesssim a, \quad \text{and} \quad \|\phi\|_{H^{k'}(\mathbb{R}^n)} \lesssim 1 \text{ (if } m \neq 0),$$

where $k' := k-1$.

3.2. The Auxiliary Function ψ and Its PDE

Motivated by the decay rate we anticipate for ϕ, we define the auxiliary function ψ by $\psi := a^\kappa\phi$, where

$$\kappa := \begin{cases} \frac{n}{2} & \text{if } |m| \geqslant \frac{n}{2}, \\ \frac{n}{2} - \sqrt{\frac{n^2}{4} - m^2} & \text{if } |m| \leqslant \frac{n}{2}. \end{cases}$$

Then, using (5), it can be shown that ψ satisfies the equation

$$\ddot{\psi} + (\kappa^2 - n\kappa + m^2)\psi + (n-2\kappa)\dot{\psi} - \frac{1}{a^2}\Delta\psi = 0. \tag{7}$$

3.3. The Case $|m| > \frac{n}{2}$

We have $\kappa = \frac{n}{2}$, so that $n - 2\kappa = 0$, while $\kappa^2 - n\kappa + m^2 = m^2 - \frac{n^2}{4}$, and thus (7) becomes

$$\ddot{\psi} - \frac{1}{a^2}\Delta\psi + \left(m^2 - \frac{n^2}{4}\right)\psi = 0.$$

We note that if $\phi \in H^\ell(\mathbb{R}^n)$ and $\dot{\phi} \in H^{\ell-1}(\mathbb{R}^n)$ for some ℓ, then $\psi \in H^\ell(\mathbb{R}^n)$ too, and also

$$\dot{\psi} = \frac{n}{2}a^{\frac{n}{2}-1}\dot{a}\phi + a^{\frac{n}{2}}\dot{\phi} \in H^{\ell-1}(\mathbb{R}^n).$$

Define the new energy $\mathcal{E}$, associated with the ψ-evolution, by

$$\mathcal{E}(t) := \frac{1}{2}\int_{\mathbb{R}^n}\left(\dot{\psi}^2 + \frac{1}{a^2}|\nabla\psi|^2 + \left(m^2 - \frac{n^2}{4}\right)\psi^2\right)d^n\mathbf{x}.$$

Then, using the fact that $a = e^t = \dot{a} > 0$, and also Eq. (7), we obtain

$$\begin{aligned}
\mathcal{E}'(t) &= \int_{\mathbb{R}^n}\left(\dot{\psi}\ddot{\psi} - \frac{a\dot{a}}{a^4}|\nabla\psi|^2 + \frac{1}{a^2}\langle\nabla\psi,\nabla\dot{\psi}\rangle + \left(m^2 - \frac{n^2}{4}\right)\psi\dot{\psi}\right)d^n\mathbf{x} \\
&\leqslant \int_{\mathbb{R}^n}\left(\dot{\psi}\ddot{\psi} + \frac{1}{a^2}\langle\nabla\psi,\nabla\dot{\psi}\rangle + \left(m^2 - \frac{n^2}{4}\right)\psi\dot{\psi}\right)d^n\mathbf{x} \\
&\leqslant \int_{\mathbb{R}^n}\left(\dot{\psi}\left(\frac{1}{a^2}\Delta\psi - \left(m^2 - \frac{n^2}{4}\right)\psi\right) + \frac{1}{a^2}\langle\nabla\psi,\nabla\dot{\psi}\rangle + \left(m^2 - \frac{n^2}{4}\right)\psi\dot{\psi}\right)d^n\mathbf{x} \\
&= \frac{1}{a^2}\int_{\mathbb{R}^n}\left(\dot{\psi}\Delta\psi + \langle\nabla\psi,\nabla\dot{\psi}\rangle\right)d^n\mathbf{x} = \frac{1}{a^2}\int_{\mathbb{R}^n}\nabla\cdot(\dot{\psi}\nabla\psi)d^n\mathbf{x}.
\end{aligned}$$

For a fixed t, and for a ball $B(\mathbf{0},r)\subset\mathbb{R}^n$, where $r>0$, it follows from the divergence theorem (since $\dot{\psi}$ and $\nabla\psi$ are smooth) that

$$\int_{B(\mathbf{0},r)}\nabla\cdot(\dot{\psi}\nabla\psi)\,d^n\mathbf{x}=\int_{\partial B(\mathbf{0},r)}\dot{\psi}\,\langle\nabla\psi,\mathbf{n}\rangle\,d\sigma_r,$$

where $d\sigma_r$ is the surface area measure on the sphere $S_r=\partial B(\mathbf{0},r)$ and $\mathbf{n}$ is the outward-pointing unit normal. The right-hand side surface integral tends to 0 as $r\to+\infty$, by an application of Lemma 7.1, given in 'Appendix 7'.

So for $t\geqslant t_0$, we have $\mathcal{E}'(t)\leqslant 0$, which yields $\mathcal{E}(t)\leqslant\mathcal{E}(t_0)$. In particular, for all $t\geqslant t_0$, $\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)}\lesssim C$, that is, $\|a^{\frac{n}{2}}\phi(t,\cdot)\|_{L^2(\mathbb{R}^n)}\lesssim C$, and so[2]

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{R}^n)}\lesssim a^{-\frac{n}{2}}. \tag{8}$$

Then, with enough regularity on ϕ_0,ϕ_1 at the outset, that is, if $\phi_0\in H^k(\mathbb{R}^n)$ and $\phi_1\in H^{k-1}(\mathbb{R}^n)$ for a $k>\frac{n}{2}+2$, and by considering $(\partial_{x^1})^{i_1}\cdots(\partial_{x^n})^{i_n}\phi$ as a solution to the Klein–Gordon equation, we arrive at[3]

$$\|\phi(t,\cdot)\|_{H^{k'}(\mathbb{R}^n)}\lesssim a^{-\frac{n}{2}},$$

where $k':=k-2$. As $k'=k-2>\frac{n}{2}$, we have, using the Sobolev inequality, that

$$\forall t\geqslant t_0,\quad \|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}\lesssim a^{-\frac{n}{2}}.$$

This completes the proof of Theorem 3.1 in the case when $|m|>\frac{n}{2}$.

3.4. The Case $|m|<\frac{n}{2}$

We have $\kappa=\frac{n}{2}-\sqrt{\frac{n^2}{4}-m^2}$, $n-2\kappa=2\sqrt{\frac{n^2}{4}-m^2}\ >0$, and $\kappa^2-n\kappa+m^2=0$. Equation (7) becomes

$$\ddot{\psi}+2\left(\sqrt{\frac{n^2}{4}-m^2}\right)\,\dot{\psi}-\frac{1}{a^2}\Delta\psi=0.$$

Defining $\widetilde{\mathcal{E}}(t):=\frac{1}{2}\int_{\mathbb{R}^n}\left(\dot{\psi}^2+\frac{1}{a^2}|\nabla\psi|^2\right)d^n\mathbf{x}$, we obtain

$$\begin{aligned}\widetilde{\mathcal{E}}'(t)&=\int_{\mathbb{R}^n}\left(\dot{\psi}\ddot{\psi}-\frac{\dot{a}}{a^3}|\nabla\psi|^2+\frac{1}{a^2}\langle\nabla\psi,\nabla\dot{\psi}\rangle\right)d^n\mathbf{x}\\&=\int_{\mathbb{R}^n}\left(\dot{\psi}\left(-2\left(\sqrt{\frac{n^2}{4}-m^2}\right)\,\dot{\psi}+\frac{1}{a^2}\Delta\psi\right)-\frac{\dot{a}}{a^3}|\nabla\psi|^2+\frac{1}{a^2}\langle\nabla\psi,\nabla\dot{\psi}\rangle\right)d^n\mathbf{x}\\&=-2\left(\sqrt{\frac{n^2}{4}-m^2}\right)\int_{\mathbb{R}^n}\dot{\psi}^2d^n\mathbf{x}-\frac{\dot{a}}{a^3}\int_{\mathbb{R}^n}|\nabla\psi|^2d^n\mathbf{x}.\end{aligned}$$

---

[2]We note that to reach this conclusion, we used Lemma 7.1, for which we need $\dot{\psi}(t,\cdot),\nabla\psi(t,\cdot)\in H^1(\mathbb{R}^n)$, which means that the initial conditions for ϕ must be such that $\phi_0\in H^2(\mathbb{R}^n)$ and $\phi_1\in H^1(\mathbb{R}^n)$.

[3]Note that in order to use the estimate (8), for $D\phi:=(\partial_{x^1})^{i_1}\cdots(\partial_{x^n})^{i_n}\phi$ replacing ϕ, where $|(i_1,\dots,i_n)|=:k'$, we must ensure that the initial conditions for $D\phi$, namely $(D\phi(t_0,\cdot),D\dot{\phi}(t_0,\cdot))$, are in $(H^2(\mathbb{R}^n),H^1(\mathbb{R}^n))$, which is guaranteed if the initial condition for ϕ, namely (ϕ_0,ϕ_1), is in $(H^k(\mathbb{R}^n),H^{k-1}(\mathbb{R}^n))$, with $k-k'=2$.

Using $a = e^t = \dot{a}$, we obtain

$$\begin{aligned}
\widetilde{\mathcal{E}}'(t) &= -4\left(\sqrt{\frac{n^2}{4} - m^2}\right)\frac{1}{2}\int_{\mathbb{R}^n} \dot{\psi}^2 d^n\mathbf{x} - 2\frac{1}{2}\int_{\mathbb{R}^n} \frac{1}{a^2}|\nabla\psi|^2 d^n\mathbf{x} \\
&\leqslant -\min\left\{4\left(\sqrt{\frac{n^2}{4} - m^2}\right),\ 2\right\}\cdot\frac{1}{2}\int_{\mathbb{R}^n}\left(\dot{\psi}^2 + \frac{1}{a^2}|\nabla\psi|^2\right)d^n\mathbf{x} \\
&= -\,\theta\cdot\widetilde{\mathcal{E}}(t),
\end{aligned}$$

where

$$\theta := \min\left\{4\left(\sqrt{\frac{n^2}{4} - m^2}\right),\ 2\right\} > 0.$$

So $\widetilde{\mathcal{E}}'(t) + \theta\cdot\widetilde{\mathcal{E}}(t) \leqslant 0$. Multiplying throughout by $e^{\theta t} > 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{\theta t}\cdot\widetilde{\mathcal{E}}(t)) \leqslant 0.$$

Integrating from t_0 to t yields $e^{\theta t}\cdot\widetilde{\mathcal{E}}(t) \leqslant e^{\theta t_0}\cdot\widetilde{\mathcal{E}}(t_0)$, that is, $\widetilde{\mathcal{E}}(t) \lesssim e^{-\theta t}$. So

$$\|\dot{\psi}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leqslant \sqrt{2\widetilde{\mathcal{E}}(t)} \lesssim e^{-\frac{\theta}{2}t}.$$

We have $\psi(t,\mathbf{x}) = \psi(t_0,\mathbf{x}) + \int_{t_0}^{t}(\partial_t\psi)(s,\mathbf{x})\mathrm{d}s$, and so

$$\begin{aligned}
\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} &\leqslant \|\psi(t_0,\cdot)\|_{L^2(\mathbb{R}^n)} + \int_{t_0}^{t}\|(\partial_t\psi)(s,\cdot)\|_{L^2(\mathbb{R}^n)}\mathrm{d}s, \\
&\lesssim A + \int_{t_0}^{t} Be^{-\frac{\theta}{2}s}\mathrm{d}s = A + B\frac{e^{-\frac{\theta}{2}t_0} - e^{-\frac{\theta}{2}t}}{\theta/2} \lesssim C.
\end{aligned}$$

Thus, for all $t \geqslant t_0$, we have

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{R}^n)} = a^{-\kappa}\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim a^{-\kappa}.$$

By considering $(\partial_{x^1})^{i_1}\cdots(\partial_{x^n})^{i_n}\phi$ and using the Sobolev inequality, we have

$$\forall t \geqslant t_0,\quad \|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim a^{-\kappa} = a^{-(\frac{n}{2}-\sqrt{\frac{n^2}{4}-m^2})}.$$

This completes the proof of Theorem 3.1 in the case when $|m| < \frac{n}{2}$.

3.5. The Case $|m| = \frac{n}{2}$

We have $\kappa = \frac{n}{2}$, and equation (7) becomes $\ddot{\psi} - \frac{1}{a^2}\Delta\psi = 0$.

Defining the same energy as we used earlier in the case when $|m| < \frac{n}{2}$,

$$\widetilde{\mathcal{E}}(t) := \frac{1}{2}\int_{\mathbb{R}^n}\left(\dot{\psi}^2 + \frac{1}{a^2}|\nabla\psi|^2\right)d^n\mathbf{x},$$

we obtain

$$\widetilde{\mathcal{E}}'(t) = \int_{\mathbb{R}^n}\left(\dot{\psi}\ddot{\psi} - \frac{\dot{a}}{a^3}|\nabla\psi|^2 + \frac{1}{a^2}\langle\nabla\psi,\nabla\dot{\psi}\rangle\right)d^n\mathbf{x}$$

$$= \int_{\mathbb{R}^n} \left(\dot{\psi} \frac{1}{a^2} \Delta\psi - \frac{\dot{a}}{a^3} |\nabla\psi|^2 + \frac{1}{a^2} \langle \nabla\psi, \nabla\dot{\psi} \rangle \right) d^n\mathbf{x}$$
$$= -\frac{\dot{a}}{a^3} \int_{\mathbb{R}^n} |\nabla\psi|^2 d^n\mathbf{x} \leqslant 0.$$

So $\widetilde{\mathcal{E}}(t) \leqslant \widetilde{\mathcal{E}}(t_0)$ for $t \geqslant t_0$. In particular, $\|\dot{\psi}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim B$ for $t \geqslant t_0$. Again,

$$\psi(t,\mathbf{x}) = \psi(t_0,\mathbf{x}) + \int_{t_0}^{t} (\partial_t\psi)(s,\mathbf{x})\mathrm{d}s,$$

gives

$$\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leqslant \|\psi(t_0,\cdot)\|_{L^2(\mathbb{R}^n)} + \int_{t_0}^{t} \|(\partial_t\psi)(s,\cdot)\|_{L^2(\mathbb{R}^n)}\mathrm{d}s,$$
$$\lesssim A' + \int_{t_0}^{t} B\mathrm{d}s \lesssim A + Bt \lesssim \log a.$$

Thus, for all $t \geqslant t_0$, we have

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{R}^n)} = a^{-\kappa}\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim a^{-\kappa}\log a.$$

Hence (by considering $(\partial_{x^1})^{i_1}\cdots(\partial_{x^n})^{i_n}\phi$ and using the Sobolev inequality),

$$\forall t \geqslant t_0, \quad \|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim a^{-\frac{n}{2}}\log a. \tag{9}$$

(One can show that this bound is sharp; see Appendix C (Section 8).)

This completes the proof of Theorem 3.1.

4. Decay in the Cosmological Region of the RNdS Spacetime

The Reissner–Nordström–de Sitter (RNdS) spacetime (M,g) is a solution to the Einstein–Maxwell equations with a positive cosmological constant, and it represents a pair[4] of antipodal charged black holes in a spherical[5] universe which is undergoing accelerated expansion. The Reissner–Nordström–de Sitter metric in $n+1$ dimensions is given by

$$g = -\frac{1}{V}dr^2 + V dt^2 + r^2 d\Omega^2,$$

where

$$V = r^2 + \frac{2M}{r^{n-2}} - \frac{e^2}{r^{n-1}} - 1,$$

[4]We note that there is no solution analogous to RNdS but with only one black hole. This is analogous to (but much more complicated than, and still not fully understood) the fact that one cannot have a single electric charge on a spherical universe. (Gauss's law requires that the total charge must be zero.) In fact, the fundamental solution of the Laplace equation on the sphere gives a unit positive charge at some point and a unit negative charge at the antipodal point. One can have more than two black holes, for instance the so-called Kastor–Traschen solution [17].

[5]'Spherical' here means that the Cauchy hypersurface (that is, 'space') is an n-sphere.

and $d\Omega^2$ is the unit round metric on S^{n-1}. The constants M and e are proportional to the mass and the charge, respectively, of the black holes, and the cosmological constant is chosen to be

$$\Lambda = \frac{n(n-1)}{2}$$

by an appropriate choice of units.

Consider the polynomial

$$p(r) := r^{n-1}V(r) = r^{n+1} - r^{n-1} + 2Mr - e^2.$$

As $p(0) = -e^2 < 0$ and as $p(r) \stackrel{r\to\infty}{\longrightarrow} \infty$, it follows that p will have a real root in $(0, +\infty)$, and the largest real root of p, which we denote by r_c, must be positive. If $r > r_c$, then clearly $p(r) > 0$, and so also $V(r) > 0$.

It can also be seen that p has at most three distinct positive roots. Suppose, on the contrary, that p has more than three distinct positive roots: $r_1 < r_2 < r_3 < r_4$. Applying Rolle's theorem to p on $[r_i, r_{i+1}]$ $(i = 1, 2, 3)$, we conclude that p' must have three distinct roots $r_i' \in (r_i, r_{i+1})$ $(i = 1, 2, 3)$. Applying Rolle's theorem to p' on $[r_i', r_{i+1}']$ $(i = 1, 2)$, we conclude that p'' must have two distinct roots $r_i'' \in (r_i', r_{i+1}')$ $(i = 1, 2)$. But

$$p'' = r^{n-3}n(n+1)\Big(r^2 - \frac{(n-1)(n-2)}{n(n+1)}\Big),$$

which has only one positive root, a contradiction.

The 'subextremality' assumption on the RNdS spacetime made in Theorem 3.1 refers to a nondegeneracy of the positive roots of p: we assume that there are exactly three positive roots, r_-, r_+ and r_c, and $0 < r_- < r_+ < r_c$. These describe the event horizon $r = r_+$, and the Cauchy 'inner' horizon $r = r_-$. It can be seen that the subextremality condition then implies $p'(r_c) > 0$. (Indeed, $p'(r_c)$ cannot be negative, as otherwise p would acquire a root larger than r_c since $p(r) \stackrel{r\to\infty}{\longrightarrow} \infty$. Also, if $p'(r_c) = 0$, then Rolle's theorem implies again that p' would have three positive roots, ones in (r_-, r_+) and (r_+, r_c), and one at r_c, which is impossible, as we had seen above.) $p'(r_c) > 0$ implies that $V'(r_c) > 0$. We will also assume that

$$V''(r_c) > 0.$$

(See [6, Appendix 6] for the range of parameters for which this is guaranteed.) Our assumptions have the following consequence, which will be used in our proof of Theorem 4.2.

Lemma 4.1 (Global redshift) $V'(r) > 0$ *for all* $r \geqslant r_c$.

Proof. We have

$$V'(r) = \frac{rp'(r) - (n-1)p(r)}{r^n} = \frac{2r^{n+1} + 2(3-n)Mr + (n-1)e^2}{r^n} =: \frac{q(r)}{r^n}.$$

As $V'(r_c) > 0$, we have $q(r_c) > 0$. Also, $V''(r_c) > 0$ and so V' is increasing near r_c. But then $q(r) = r^nV'(r)$ is also increasing near r_c, and in particular,

$q'(r_c) \geqslant 0$. Let us suppose that there exists an $r_* > r_c$ such that $V'(r_*) = 0$, and let r_* be the smallest such root. Then, $q(r_*) = 0$ too. We note that

$$q' = 2(n+1)r^n + 2(3-n)M,$$

and so q' can have only one nonnegative root, namely $\left(\frac{(n-3)}{n+1}M\right)^{\frac{1}{n}} \geqslant 0$.

1° r_* is a repeated root of q. Then, $q'(r_*) = 0$.

If in addition $q'(r_c) = 0$, then we arrive at a contradiction, since q' then has two positive roots (at r_c and at r_*), which is impossible.

If $q'(r_c) > 0$, then we arrive at a contradiction as follows. As q is increasing near r_c, and since $q(r_c) > 0 = q(r_*)$, it follows by the intermediate value theorem that there is some $r_c' \in (r_c, r_*)$ such that $q(r_c') = q(r_c)$. But by Rolle's theorem applied to q on $[r_c, r_c']$, there must exist an $r_*' \in (r_c, r_c')$ such that $q'(r_*') = 0$. Again, q' acquires two zeros (at r_* and at r_*'), which is impossible.

2° r_* is a simple root of q. But as $q(r) \stackrel{r\to\infty}{\longrightarrow} \infty$, it follows that there must be at least one more root $r_{**} > r_*$ of q. By Rolle's theorem applied to q on $[r_*, r_{**}]$, it follows that $q'(r_{**}') = 0$ for some $r_{**}' \in (r_*, r_{**})$.

If in addition $q'(r_c) = 0$, then we arrive at a contradiction, since q' then has two positive roots (at r_c and at r_{**}'), which is impossible.

If $q'(r_c) > 0$, then, as in the last paragraph of 1° above, there exists an $r_*' \in (r_c, r_c') \subset (r_c, r_*)$ such that $q'(r_*') = 0$. Thus, q' again gets two positive roots (at r_*' and at r_{**}'), which is impossible.

This shows that our assumption that V' is zero beyond r_c is incorrect. □

We will need the previous result in Sect. 4.5 in the analysis following (12) (where in particular V' appears in the denominator).

The hypersurfaces of constant r are spacelike cylinders with a future-pointing unit normal vector field $N = V^{\frac{1}{2}}\frac{\partial}{\partial r}$ and volume element $dV_n = V^{\frac{1}{2}} r^{n-1} \mathrm{d}t\mathrm{d}\Omega$.

The global structure of a maximal spherically symmetric extension of this metric can be depicted by a conformal Penrose diagram shown below, repeated periodically; see for example [7].

We are interested in the behaviour of the solution to the Klein–Gordon equation in the cosmological region $\mathcal{R}_5$ of this spacetime (see Figure 1), bounded by the cosmological horizon branches $\mathcal{CH}_1^+$, $\mathcal{CH}_2^+$, the future null infinity $\mathscr{I}^+$, and the point i^+. In particular, we want to obtain estimates for the decay rate of ϕ as $r \to \infty$. We guess the decay rates simply by substituting r instead of e^t in the estimates we had obtained for the decay rate of ϕ with respect to t in the case of the de Sitter universe in flat FLRW form from the previous Sect. 3.

We will prove the following result.

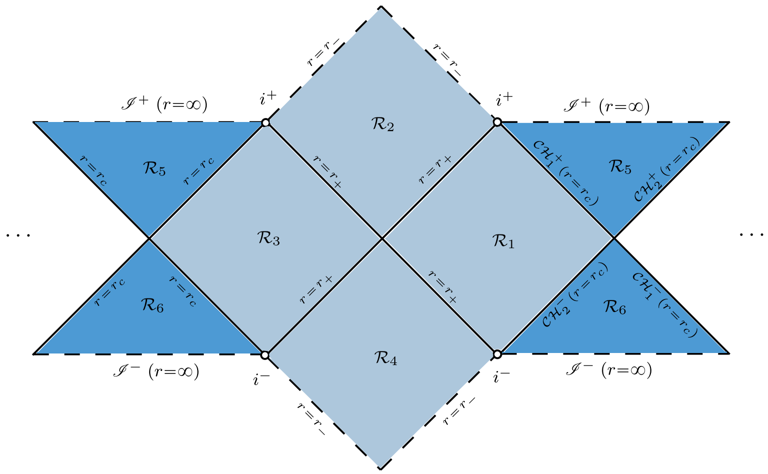


FIGURE 1. Conformal diagram of the Reissner–Nordström–de Sitter spacetime

Theorem 4.2. *Suppose that*

- $\epsilon > 0$,
- $m \in \mathbb{R}$,
- $M > 0$,
- $e > 0$,
- $n > 2$,
- (M, g) *is the* $(n+1)$*-dimensional subextremal Reissner–Nordström–de Sitter solution given by the metric*
$$g = -\frac{1}{V}dr^2 + V dt^2 + r^2 d\Omega^2,$$
where
$$V = r^2 + \frac{2M}{r^{n-2}} - \frac{e^2}{r^{n-1}} - 1,$$
and $d\Omega^2$ *is the metric of the unit* $(n-1)$*-dimensional sphere* S^{n-1},
- $k > \frac{n}{2} + 2$, *and*
- ϕ *is a smooth solution to* $\Box_g \phi - m^2 \phi = 0$ *such that*
$$\|\phi\|_{H^k(\mathcal{CH}_1^+)} < +\infty \quad \text{and} \quad \|\phi\|_{H^k(\mathcal{CH}_2^+)} < +\infty,$$
where $\mathcal{CH}_1^+ \simeq \mathcal{CH}_2^+ \simeq \mathbb{R} \times S^{n-1}$ *are the two components of the future cosmological horizon, parameterised by the flow parameter of the global Killing vector field* $\frac{\partial}{\partial t}$. *(For* $\mathcal{CH}_1^+$ *and* $\mathcal{CH}_2^+$, *we use the usual Sobolev norms for* $\mathbb{R} \times S^{n-1}$*; see e.g. [4].)*

Then, there exists a r_0 *large enough so that for all* $r \geqslant r_0$,
$$\|\phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \lesssim \begin{cases} r^{-\frac{n}{2}+\epsilon} & \text{if} \quad |m| > \frac{n}{2}, \\ r^{-\frac{n}{2}+\sqrt{\frac{n^2}{4}-m^2}+\epsilon} & \text{if} \quad |m| \leqslant \frac{n}{2}. \end{cases}$$

4.1. Preliminary Energy Function

For a ϕ defined in the cosmological region $\mathcal{R}_5$, we define

$$\phi' := \frac{\partial\phi}{\partial r} \quad \text{and} \quad \dot{\phi} := \frac{\partial\phi}{\partial t}.$$

We will also use the following notation:

$\mathring{\nabla}\phi$	gradient of ϕ on S^{n-1} with respect to the unit round metric,
$\lvert\mathring{\nabla}\phi\rvert$	norm with respect to the unit round metric,
$\mathring{\Delta}\phi$	Laplacian of ϕ on S^{n-1} with respect to the unit round metric,
$\mathring{g}$	determinant of the unit round metric.

Suppose that ϕ satisfies the Klein–Gordon equation $\Box_g\phi - m^2\phi = 0$. Recall that the energy–momentum tensor associated with ϕ is given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial_\alpha\partial^\alpha\phi + m^2\phi^2).$$

Thus,

$$\begin{aligned} T(N,N) &= \left(\phi'^2 - \frac{1}{2}\frac{(-1)}{V}\left(\phi'^2(-V) + \dot{\phi}^2\frac{1}{V} + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + m^2\phi^2\right)\right)V \\ &= \frac{1}{2}\left(V\phi'^2 + \frac{1}{V}\dot{\phi}^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + m^2\phi^2\right). \end{aligned}$$

Define $X := \frac{V^{\frac{1}{2}}}{r^{n-1}}N = \frac{V^{\frac{1}{2}}}{r^{n-1}}V^{\frac{1}{2}}\frac{\partial}{\partial r} = \frac{V}{r^{n-1}}\frac{\partial}{\partial r}$. We define the energy

$$\begin{aligned} E(r) &:= \int_{\mathbb{R}\times S^{n-1}} T(X,N)\mathrm{d}V_n \\ &= \frac{1}{2}\int_{\mathbb{R}\times S^{n-1}} \left(V^2\phi'^2 + \dot{\phi}^2 + \frac{V}{r^2}|\mathring{\nabla}\phi|^2 + m^2V\phi^2\right)\mathrm{d}t\mathrm{d}\Omega. \end{aligned}$$

4.2. The Auxiliary Function ψ and Its PDE

The Klein–Gordon equation $\Box_g\phi - m^2\phi = 0$ can be rewritten as:

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,\partial^\mu\phi) - m^2\phi = 0, \\ &\Leftrightarrow \frac{1}{r^{n-1}\sqrt{\mathring{g}}}\partial_\mu(r^{n-1}\sqrt{\mathring{g}}\,g^{\mu\nu}\partial_\nu\phi) - m^2\phi = 0. \end{aligned}$$

This becomes

$$\frac{1}{r^{n-1}\sqrt{\mathring{g}}}\left(\partial_r(r^{n-1}\sqrt{\mathring{g}}\,(-V)\partial_r\phi) + \partial_t(r^{n-1}\sqrt{\mathring{g}}\,\frac{1}{V}\partial_t\phi) + \frac{r^{n-1}\sqrt{\mathring{g}}}{r^2}\,\mathring{\Delta}\phi\right) - m^2\phi = 0$$

that is,

$$\begin{aligned} &-(V\phi')' - \frac{(n-1)}{r}V\phi' + \frac{\ddot{\phi}}{V} + \frac{1}{r^2}\mathring{\Delta}\phi - m^2\phi = 0, \\ &\Leftrightarrow \phi'' + \frac{(n-1)}{r}\phi' + \frac{V'}{V}\phi' - \frac{\ddot{\phi}}{V^2} - \frac{1}{r^2V}\mathring{\Delta}\phi + \frac{m^2}{V}\phi = 0. \end{aligned}$$

Define $\psi := r^\kappa \phi$, where

$$\kappa = \begin{cases} \dfrac{n}{2} & \text{if } |m| \geqslant \dfrac{n}{2}, \\ \dfrac{n}{2} - \sqrt{\dfrac{n^2}{4} - m^2} & \text{if } |m| \leqslant \dfrac{n}{2}. \end{cases}$$

Then, using the PDE for ϕ, it can be shown that

$$\psi'' + \left(\frac{V'}{V} + \frac{n-1}{r} - \frac{2\kappa}{r}\right)\psi' - \frac{\ddot{\psi}}{V^2} - \frac{1}{r^2 V}\mathring{\Delta}\psi + \theta\psi = 0, \tag{10}$$

where $\theta := \dfrac{m^2}{V} + \dfrac{\kappa}{r}\Big(\dfrac{1}{r} - \dfrac{V'}{V}\Big) - \dfrac{\kappa}{r^2}(n-1-\kappa)$.

4.3. The Case $|m| > \frac{n}{2}$

Then, $\kappa = \frac{n}{2}$, and (10) becomes

$$\psi'' + \left(\frac{V'}{V} - \frac{1}{r}\right)\psi' - \frac{\ddot{\psi}}{V^2} - \frac{1}{r^2 V}\mathring{\Delta}\psi + \theta\psi = 0, \tag{11}$$

where

$$\theta := -\frac{n}{2}\left(\frac{n}{2} - 1\right)\frac{1}{r^2} + \frac{m^2}{V} + \frac{n}{2r}\left(\frac{1}{r} - \frac{V'}{V}\right).$$

We will use an energy function to obtain the required decay of ψ for large r, and in order to do so, we will need to keep careful track of the limiting behaviour of the various functions appearing in the expression for θ and the coefficients of the PDE (11). We will do this step-by-step in a sequence of lemmas.

Lemma 4.3. *Given any $\epsilon > 0$, there exists an r_0 large enough so that for all $r \geqslant r_0$,*

$$\frac{2+\epsilon}{r} \geqslant \frac{V'}{V} \geqslant \frac{2-\epsilon}{r}.$$

Proof. This follows immediately from

$$\lim_{r\to\infty} r\frac{V'}{V} = \lim_{r\to\infty} r \cdot \frac{2r - \frac{2(n-2)M}{r^{n-1}} + \frac{e^2(n-1)}{r^n}}{r^2 + \frac{2M}{r^{n-2}} - \frac{e^2}{r^{n-1}} - 1} = \lim_{r\to\infty} \frac{2 - \frac{2(n-2)M}{r^n} + \frac{e^2(n-1)}{r^{n+1}}}{1 + \frac{2M}{r^n} - \frac{e^2}{r^{n+1}} - \frac{1}{r^2}} = 2.$$

□

Lemma 4.4. *There exists r_0 large enough so that for $r \geqslant r_0$, we have $\theta > 0$.*

(We note that the proof uses the fact that $|m| > \frac{n}{2}$, and so this result is specific to this subsection.)

Proof. As $\displaystyle\lim_{r\to\infty}\frac{r^2}{V} = \lim_{r\to\infty}\frac{1}{1+\frac{2M}{r^n} - \frac{e^2}{r^{n+1}} - \frac{1}{r^2}} = 1$, there exists a r_0' such that

$$\frac{r^2}{V} \geqslant 1 - \epsilon$$

for $r \geqslant r_0'$. Also, by the previous lemma, there exists a $r_0 > r_0'$ such that $\frac{V'}{V} \leqslant \frac{2+\epsilon}{r}$ for all $r \geqslant r_0$. Then, we have for $r > r_0$ that

$$\begin{aligned}\theta &= \frac{1}{r^2}\left(-\frac{n}{2}\left(\frac{n}{2}-1\right)+m^2\frac{r^2}{V}+\frac{n}{2}\Big(1-\frac{V'}{V}r\Big)\right)\\ &\geqslant \frac{1}{r^2}\left(-\frac{n^2}{4}+\frac{n}{2}+m^2(1-\epsilon)+\frac{n}{2}\Big(1-\frac{(2+\epsilon)}{r}r\Big)\right)\\ &= \frac{1}{r^2}\left(\delta-\epsilon\Big(\delta+\frac{n}{2}+\frac{n^2}{4}\Big)\right),\end{aligned}$$

where $\delta := m^2 - \frac{n^2}{4} > 0$.

Taking ϵ at the outset small enough so as to satisfy $0 < \epsilon < \frac{\delta}{\delta+\frac{n}{2}+\frac{n^2}{4}}$, we see that $\theta > 0$ for $r \geqslant r_0$. $\square$

Define the energy

$$\mathcal{E}(r) := \frac{1}{2}\int_{\mathbb{R}\times S^{n-1}}\left(\psi'^2+\frac{1}{V^2}\dot{\psi}^2+\frac{1}{r^2V}|\mathring{\nabla}\psi|^2+\theta\psi^2\right)\mathrm{d}t\mathrm{d}\Omega.$$

(We assume for the moment that this is finite for a sufficiently large r_0. Later on, in the subsection on redshift estimates, we will see how our initial finiteness of Sobolev norms of ϕ on the two branches $\mathcal{CH}_1^+$, $\mathcal{CH}_2^+$ of the cosmological horizon guarantees this.)

We now proceed to find an expression for $\mathcal{E}'(r)$, and to simplify it, we will use (11), and the divergence theorem, to get rid of the terms involving $\ddot{\psi}$ and $\mathring{\Delta}\psi$, the spherical Laplacian of ψ:

$$\begin{aligned}\mathcal{E}'(r) = &\int_{\mathbb{R}\times S^{n-1}}\bigg(\psi'\psi''+\frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2+\frac{1}{V^2}\dot{\psi}\dot{\psi}'+\frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2\\ &+\frac{1}{r^2V}\langle\mathring{\nabla}\psi,(\mathring{\nabla}\psi)'\rangle+\frac{\theta'}{2}\psi^2+\theta\psi\psi'\bigg)\,\mathrm{d}t\mathrm{d}\Omega\\ =&\int_{\mathbb{R}\times S^{n-1}}\bigg(\psi'\Big(-\Big(\frac{V'}{V}-\frac{1}{r}\Big)\psi'+\frac{1}{V^2}\ddot{\psi}+\frac{1}{r^2V}\mathring{\Delta}\psi-\cancel{\theta\psi}\Big)\\ &+\frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2+\frac{1}{V^2}\dot{\psi}\dot{\psi}'+\frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2\\ &+\frac{1}{r^2V}\langle\mathring{\nabla}\psi,(\mathring{\nabla}\psi)'\rangle+\frac{\theta'}{2}\psi^2+\cancel{\theta\psi\psi'}\bigg)\,\mathrm{d}t\mathrm{d}\Omega\\ =&\int_{\mathbb{R}\times S^{n-1}}\bigg(-\Big(\frac{V'}{V}-\frac{1}{r}\Big)\psi'^2+\cancel{\frac{1}{V^2}\ddot{\psi}\psi'}+\cancel{\frac{1}{V^2}\dot{\psi}\dot{\psi}'}\\ &+\frac{1}{r^2V}\Big(\cancel{\psi'\mathring{\Delta}\psi}+\cancel{\langle\mathring{\nabla}\psi,(\mathring{\nabla}\psi)'\rangle}\Big)\\ &+\frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2+\frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2+\frac{\theta'}{2}\psi^2\bigg)\,\mathrm{d}t\mathrm{d}\Omega.\end{aligned}$$

We note that in the above, getting rid of the spherical Laplacian by using the divergence theorem is allowed because the compact sphere S^{n-1} has no boundary. For the second time derivative, however, there is a boundary at infinity (with two connected components), namely

$$\lim_{t\to+\infty}\int_{S^{n-1}}\dot{\psi}\psi'\mathrm{d}\Omega-\lim_{t\to-\infty}\int_{S^{n-1}}\dot{\psi}\psi'\mathrm{d}\Omega,$$

which can be seen to be equal to 0, by Lemma 7.3 from 'Appendix B'.

Thus,

$$\mathcal{E}'(r)=\int_{\mathbb{R}\times S^{n-1}}\left(-\Big(\frac{V'}{V}-\frac{1}{r}\Big)\psi'^2+\frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2+\frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2+\frac{\theta'}{2}\psi^2\right)\mathrm{d}t\mathrm{d}\Omega.$$

Let $\epsilon>0$ be given. Then, there exists an r_0 large enough such that:

(a) $\displaystyle\frac{V'}{V}-\frac{1}{r}\geqslant\frac{2-\epsilon}{r}-\frac{1}{r}=\frac{1-\epsilon}{r},$

(b) $\displaystyle\Big(\frac{1}{V^2}\Big)'=-2\frac{V'}{V}\frac{1}{V^2}\leqslant-2\frac{(2-\epsilon)}{r}\frac{1}{V^2},$

(c) $\displaystyle\Big(\frac{1}{r^2V}\Big)'=-\frac{1}{r^2V}\Big(\frac{2}{r}+\frac{V'}{V}\Big)\leqslant-\frac{1}{r^2V}\Big(\frac{2}{r}+\frac{2-\epsilon}{r}\Big)=-\frac{1}{r^2V}\frac{(4-\epsilon)}{r},$

(d) $\displaystyle\frac{\theta'}{\theta}=\frac{1}{r}\left(\frac{-\frac{n}{2}(\frac{n}{2}-1)(-2)-\frac{m^2V'}{r^3V^2}-\frac{n}{2r^5}(\frac{1}{r}-\frac{V'}{V})+\frac{n}{2r^4}(-\frac{1}{r^2}-\frac{V''V-V'^2}{V^2})}{-\frac{n}{2}(\frac{n}{2}-1)+\frac{m^2}{r^2V}+\frac{n}{2r^3}(\frac{1}{r}-\frac{V'}{V})}\right)$

$\displaystyle\leqslant\frac{1}{r}(-2+\epsilon).$

Hence, using (a)–(d) above, we obtain

$$\begin{aligned}\mathcal{E}'(r)&=\int_{\mathbb{R}\times S^{n-1}}\left(-\Big(\frac{V'}{V}-\frac{1}{r}\Big)\psi'^2+\frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2+\frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2+\frac{\theta'}{2}\psi^2\right)\mathrm{d}t\mathrm{d}\Omega\\&\leqslant\int_{\mathbb{R}\times S^{n-1}}\left(-\frac{(1-\epsilon)}{r}\psi'^2+\frac{1}{2}\frac{(-2)(2-\epsilon)}{rV^2}\dot{\psi}^2+\frac{1}{2}\frac{(-1)}{r^2V}\frac{(4-\epsilon)}{r}|\mathring{\nabla}\psi|^2\right.\\&\qquad\left.+\frac{1}{2}\frac{1}{r}(-2+\epsilon)\theta\psi^2\right)\mathrm{d}t\mathrm{d}\Omega\\&=-\frac{1}{r}\frac{1}{2}\int_{\mathbb{R}\times S^{n-1}}\left(2(1-\epsilon)\psi'^2+2(2-\epsilon)\frac{1}{V^2}\dot{\psi}^2+(4-\epsilon)\frac{1}{r^2V}|\mathring{\nabla}\psi|^2\right.\\&\qquad\left.+(2-\epsilon)\theta\psi^2\right)\mathrm{d}t\mathrm{d}\Omega\\&\leqslant-\frac{2(1-\epsilon)}{r}\mathcal{E}(r).\end{aligned}$$

Using Grönwall's inequality (see e.g. [12, Appendix 7(j)]), we obtain

$$\mathcal{E}(r)\leqslant\ \mathcal{E}(r_0)e^{\int_{r_0}^{r}-\frac{2(1-\epsilon)}{r}\mathrm{d}r}\ =\mathcal{E}(r_0)\Big(\frac{r}{r_0}\Big)^{-2(1-\epsilon)}\lesssim r^{-2+2\epsilon}.$$

Thus, $\displaystyle\int_{\mathbb{R}\times S^{n-1}}\theta\psi^2\mathrm{d}t\mathrm{d}\Omega\ \leqslant\ 2\mathcal{E}(r)\ \lesssim\ r^{-2+2\epsilon}$, and so

$$\int_{\mathbb{R}\times S^{n-1}}\psi^2\mathrm{d}t\mathrm{d}\Omega\ \lesssim\ \frac{r^{2\epsilon}}{r^2\theta}\ \lesssim\ \frac{r^{2\epsilon}}{r^2\frac{1}{r^2}}\ =\ r^{2\epsilon}.$$

Hence, $\|\psi(r,\cdot)\|_{L^2(\mathbb{R}\times S^{n-1})}\lesssim r^{\epsilon}$. Consequently,

$$\|\phi(r,\cdot)\|_{L^2(\mathbb{R}\times S^{n-1})}\lesssim r^{-\frac{n}{2}+\epsilon}.$$

Recall that S^{n-1} admits $\frac{n(n-1)}{2}$ independent Killing vectors, given by

$$L_{ij} = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i},$$

for $i < j$ (under the usual embedding $S^{n-1} \subset \mathbb{R}^n$). As $\frac{\partial}{\partial t}$ and L_{ij} are Killing vector fields, it follows that $\dot{\phi}$ and $L_{ij} \cdot \phi$ are also solutions to $\Box_g \phi - m^2 \phi = 0$. Commuting with the Killing vector fields $\frac{\partial}{\partial t}$ and L_{ij}, if we assume at the moment[6] that at r_0 we have $\|\phi(r_0, \cdot)\|_{H^k(\{r=r_0\})} < +\infty$, then we also obtain for all $r \geqslant r_0$ that $\|\phi(r,\cdot)\|_{H^{k'}(\mathbb{R}\times S^{n-1})} \lesssim r^{-\frac{n}{2}+\epsilon}$, where $k' = k-2 > \frac{n}{2}$. By the Sobolev inequality,[7] $\|\phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \lesssim r^{-\frac{n}{2}+\epsilon}$.

This completes the proof of Theorem 4.2 in the case when $|m| > \frac{n}{2}$ (provided we show the aforementioned finiteness of energy, which will be carried out in Sect. 4.5 on redshift estimates).

4.4. The Case $|m| \leqslant \frac{n}{2}$

Let $\epsilon' > 0$ be given. Define

$$\widetilde{\mathcal{E}}(r) = \frac{1}{2}\int_{\mathbb{R}\times S^{n-1}} \left(\psi'^2 + \frac{1}{V^2}\dot{\psi}^2 + \frac{1}{r^2 V}|\mathring{\nabla}\psi|^2 + \frac{\epsilon'}{r^2}\psi^2\right) \mathrm{d}t\mathrm{d}\Omega.$$

We now proceed to find an expression for $\widetilde{\mathcal{E}}'(r)$, and we will simplify it using (10) and the divergence theorem, in order to get rid of the terms involving $\ddot{\psi}$ and the spherical Laplacian of ψ:

$$\begin{aligned}
&\widetilde{\mathcal{E}}'(r) \\
&\quad= \int_{\mathbb{R}\times S^{n-1}} \Bigg(\psi'\psi'' + \frac{1}{2}\left(\frac{1}{V^2}\right)'\dot{\psi}^2 + \frac{1}{V^2}\dot{\psi}\dot{\psi}' + \frac{1}{2}\left(\frac{1}{r^2 V}\right)'|\mathring{\nabla}\psi|^2 \\
&\qquad + \frac{1}{r^2 V}\langle \mathring{\nabla}\psi, (\mathring{\nabla}\psi)'\rangle - \frac{\epsilon'}{r^3}\psi^2 + \frac{\epsilon'}{r^2}\psi\psi'\Bigg)\mathrm{d}t\mathrm{d}\Omega \\
&\quad= \int_{\mathbb{R}\times S^{n-1}} \Bigg(\psi'\Big(-\Big(\frac{V'}{V} + \frac{n-1}{r} - \frac{2\kappa}{r}\Big)\psi' + \cancel{\frac{\ddot{\psi}}{V^2}} + \cancel{\frac{1}{r^2 V}\mathring{\Delta}\psi} - \theta\psi\Big) \\
&\qquad + \frac{1}{2}\left(\frac{1}{V^2}\right)'\dot{\psi}^2 + \cancel{\frac{1}{V^2}\dot{\psi}\dot{\psi}'} + \frac{1}{2}\left(\frac{1}{r^2 V}\right)'|\mathring{\nabla}\psi|^2 + \cancel{\frac{1}{r^2 V}\langle\mathring{\nabla}\psi, (\mathring{\nabla}\psi)'\rangle} \\
&\qquad - \frac{\epsilon'}{r^3}\psi^2 + \frac{\epsilon'}{r^2}\psi\psi'\Bigg)\mathrm{d}t\mathrm{d}\Omega \\
&\quad= \int_{\mathbb{R}\times S^{n-1}} \Bigg(-\Big(\frac{V'}{V} + \frac{n-1}{r} - \frac{2\kappa}{r}\Big)\psi'^2 + \frac{1}{2}\left(\frac{1}{V^2}\right)'\dot{\psi}^2
\end{aligned}$$

[6] This will be proved later in the subsection on redshift estimates.

[7] The part of the Sobolev embedding theorem concerning inclusion in Hölder spaces holds for a complete Riemannian manifold with a positive injectivity radius and a bounded sectional curvature; see e.g. [16, §3.3, Thm.3.4] or [4, Ch.2].

$$+\frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2-\frac{\epsilon'}{r^3}\psi^2\Big)\,\mathrm{d}t\mathrm{d}\Omega$$
$$+\Big(\frac{\epsilon'}{r^2}-\theta\Big)\int_{\mathbb{R}\times S^{n-1}}\psi\psi'\mathrm{d}t\mathrm{d}\Omega.$$

Again, for getting rid of the spherical Laplacian, we use the divergence theorem, noting that the sphere S^{n-1} has no boundary. For handling the second time derivative, as before, we note that there is a boundary at infinity (with two connected components), which can be seen to be equal to 0, by Lemma 7.3 from Appendix B (Section 7). Thus,

$$\begin{aligned}\widetilde{\mathcal{E}}'(r)=&\int_{\mathbb{R}\times S^{n-1}}\Big(-\Big(\frac{V'}{V}+\frac{n-1}{r}-\frac{2\kappa}{r}\Big)\psi'^2+\frac{1}{2}\Big(\frac{1}{V^2}\Big)'\dot{\psi}^2+\frac{1}{2}\Big(\frac{1}{r^2V}\Big)'|\mathring{\nabla}\psi|^2\\&-\frac{\epsilon'}{r^3}\psi^2\Big)\,\mathrm{d}t\mathrm{d}\Omega\\&+\Big(\frac{\epsilon'}{r^2}-\theta\Big)\int_{\mathbb{R}\times S^{n-1}}\psi\psi'\mathrm{d}t\mathrm{d}\Omega.\end{aligned}$$

Now, there exists an r_0 large enough such that for all $r\geqslant r_0$, we have:

(i) $\displaystyle\frac{V'}{V}+\frac{n-1}{r}-\frac{2\kappa}{r}\geqslant\frac{2-\epsilon'}{r}+\frac{n-1}{r}-\frac{2\kappa}{r}=\frac{1-\epsilon'+(n-2\kappa)}{r}\geqslant\frac{1-\epsilon'}{r}$, using $n-2\kappa\geqslant 0$.

(ii) $\displaystyle\Big(\frac{1}{V^2}\Big)'\leqslant-\frac{2(2-\epsilon')}{r}\cdot\frac{1}{V^2}$.

(iii) $\displaystyle\Big(\frac{1}{r^2V}\Big)'\leqslant-\frac{1}{r^2V}\Big(\frac{2}{r}+\frac{2-\epsilon'}{r}\Big)$.

Using (i), (ii) and (iii), it can be seen that

$$\begin{aligned}\widetilde{\mathcal{E}}'(r)\leqslant&\int_{\mathbb{R}\times S^{n-1}}\Big(-\frac{(1-\epsilon')}{r}\psi'^2-\frac{1}{2}\frac{2(2-\epsilon')}{rV^2}\dot{\psi}^2-\frac{1}{2}\frac{1}{r^2V}\Big(\frac{2}{r}+\frac{2-\epsilon'}{r}\Big)|\mathring{\nabla}\psi|^2-\frac{\epsilon'}{r^3}\psi^2\Big)\,\mathrm{d}t\mathrm{d}\Omega\\&+\Big(\frac{\epsilon'}{r^2}-\theta\Big)\int_{\mathbb{R}\times S^{n-1}}\psi\psi'\mathrm{d}t\mathrm{d}\Omega\\\leqslant&-\frac{1}{r}\frac{1}{2}\int_{\mathbb{R}\times S^{n-1}}\Big(2(1-\epsilon')\psi'^2+2(2-\epsilon')\frac{1}{V^2}\dot{\psi}^2+(4-\epsilon')\frac{1}{r^2V}|\mathring{\nabla}\psi|^2+\frac{2\epsilon'}{r^2}\psi^2\Big)\,\mathrm{d}t\mathrm{d}\Omega\\&+\Big(\frac{\epsilon'}{r^2}-\theta\Big)\int_{\mathbb{R}\times S^{n-1}}\psi\psi'\mathrm{d}t\mathrm{d}\Omega.\end{aligned}$$

Hence, $\displaystyle\widetilde{\mathcal{E}}'(r)\leqslant-\frac{2(1-\epsilon')}{r}\widetilde{\mathcal{E}}(r)+\Big(\frac{\epsilon'}{r^2}-\theta\Big)\int_{\mathbb{R}\times S^{n-1}}\psi\psi'\mathrm{d}t\mathrm{d}\Omega$. We have

$$\begin{aligned}\theta&=\frac{m^2}{V}+\frac{\kappa}{r}\Big(\frac{1}{r}-\frac{V'}{V}\Big)-\frac{\kappa}{r^2}(n-1-\kappa)\\&=\frac{1}{r^2}\Big(\frac{m^2}{\frac{V}{r^2}}+\kappa\Big(1-\frac{V'}{V}r\Big)-\kappa(n-1-\kappa)\Big).\end{aligned}$$

As $\displaystyle\frac{V}{r^2}\stackrel{r\to\infty}{\longrightarrow}1$ and $\displaystyle\frac{V'}{V}r\stackrel{r\to\infty}{\longrightarrow}2$, it follows that

$$r^2\theta\stackrel{r\to\infty}{\longrightarrow}\frac{m^2}{1}+\kappa(1-2)-\kappa(n-1-\kappa)=m^2-\kappa n+\kappa^2=0.$$

Thus, given $\epsilon' > 0$, there exists an r_0 large enough such that for $r \geqslant r_0$, $|r^2\theta| < \epsilon'$, that is, $|\theta| < \frac{\epsilon'}{r^2}$. So

$$\begin{aligned}\widetilde{\mathcal{E}}'(r) &\leqslant -\frac{2(1-\epsilon')}{r}\widetilde{\mathcal{E}}(r) + \Big(\frac{\epsilon'}{r^2} - \theta\Big)\int_{\mathbb{R}\times S^{n-1}} \psi\psi' \mathrm{d}t\mathrm{d}\Omega \\ &\leqslant -\frac{2(1-\epsilon')}{r}\widetilde{\mathcal{E}}(r) + \Big(\frac{\epsilon'}{r^2} + \frac{\epsilon'}{r^2}\Big)\left|\int_{\mathbb{R}\times S^{n-1}} \psi\psi' \mathrm{d}t\mathrm{d}\Omega\right|.\end{aligned}$$

The Cauchy–Schwarz inequality applied to the last integral gives

$$\begin{aligned}\left|\int_{\mathbb{R}\times S^{n-1}} \psi\psi' \mathrm{d}t\mathrm{d}\Omega\right| &\leqslant \sqrt{\int_{\mathbb{R}\times S^{n-1}} \psi^2 \mathrm{d}t\mathrm{d}\Omega} \cdot \sqrt{\int_{\mathbb{R}\times S^{n-1}} \psi'^2 \mathrm{d}t\mathrm{d}\Omega} \\ &\leqslant \sqrt{\frac{2r^2}{\epsilon'}\widetilde{\mathcal{E}}(r)} \cdot \sqrt{2\,\widetilde{\mathcal{E}}(r)} = \frac{2r}{\sqrt{\epsilon'}}\widetilde{\mathcal{E}}(r).\end{aligned}$$

So we obtain

$$\widetilde{\mathcal{E}}'(r) \leqslant -\frac{2(1-\epsilon')}{r}\widetilde{\mathcal{E}}(r) + \frac{2\epsilon'}{r^2}\frac{2r}{\sqrt{\epsilon'}}\widetilde{\mathcal{E}}(r) = (-2+2\epsilon'+4\sqrt{\epsilon'})\frac{1}{r}\widetilde{\mathcal{E}}(r).$$

Application of Grönwall's inequality yields

$$\widetilde{\mathcal{E}}(r) \leqslant \widetilde{\mathcal{E}}(r_0)e^{\int_{r_0}^{r}(-2+2\epsilon'+4\sqrt{\epsilon'})\frac{1}{r}\mathrm{d}r} = \frac{\widetilde{\mathcal{E}}(r_0)}{r_0^{-2+2\epsilon'+4\sqrt{\epsilon'}}}r^{-2+2\epsilon'+4\sqrt{\epsilon'}}.$$

So

$$\begin{aligned}\int_{\mathbb{R}\times S^{n-1}} \psi^2 \mathrm{d}t\mathrm{d}\Omega &= \frac{2r^2}{\epsilon'}\frac{1}{2}\int_{\mathbb{R}\times S^{n-1}} \frac{\epsilon'}{r^2}\psi^2\mathrm{d}t\mathrm{d}\Omega \leqslant \frac{2r^2}{\epsilon'}\widetilde{\mathcal{E}}(r) \\ &\leqslant \frac{2r^2}{\epsilon'}\frac{\widetilde{\mathcal{E}}(r_0)}{r_0^{-2+2\epsilon'+4\sqrt{\epsilon'}}}r^{-2+2\epsilon'+4\sqrt{\epsilon'}} = \frac{2\widetilde{\mathcal{E}}(r_0)}{\epsilon' r_0^{-2+2\epsilon'+4\sqrt{\epsilon'}}}r^{2\epsilon'+4\sqrt{\epsilon'}}.\end{aligned}$$

Thus, $\|\psi(r,\cdot)\|_{L^2(\mathbb{R}\times S^{n-1})} \leqslant \sqrt{\frac{2\widetilde{\mathcal{E}}(r_0)}{\epsilon'}}\frac{1}{r_0^{-1+\epsilon'+2\sqrt{\epsilon'}}}r^{\epsilon'+2\sqrt{\epsilon'}}$, and so

$$\|\phi(r,\cdot)\|_{L^2(\mathbb{R}\times S^{n-1})} \leqslant \sqrt{\frac{2\widetilde{\mathcal{E}}(r_0)}{\epsilon'}}\frac{1}{r_0^{-1+\epsilon'+2\sqrt{\epsilon'}}}r^{-\kappa+\epsilon'+2\sqrt{\epsilon'}}.$$

Given $\epsilon > 0$, arbitrarily small, we can choose $\epsilon' = \epsilon'(\epsilon) > 0$ small enough so that $\epsilon' + 2\sqrt{\epsilon'} < \epsilon$ at the outset, so that $\|\phi(r,\cdot)\|_{L^2(\mathbb{R}\times S^{n-1})} \lesssim r^{-\kappa+\epsilon}$. Again assuming at the moment that at r_0 we have $\|\phi(r_0,\cdot)\|_{H^k(\{r=r_0\})} < +\infty$, and by commuting with the Killing vector fields $\frac{\partial}{\partial t}$ and L_{ij}, then we also obtain for all $r \geqslant r_0$ that

$$\|\phi(r,\cdot)\|_{H^{k'}(\mathbb{R}\times S^{n-1})} \lesssim r^{-(\frac{n}{2}-\sqrt{\frac{n^2}{4}-m^2})+\epsilon},$$

where $k' = k-2 > \frac{n}{2}$. By the Sobolev inequality, this yields

$$\|\phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \lesssim r^{-(\frac{n}{2}-\sqrt{\frac{n^2}{4}-m^2})+\epsilon}.$$

This completes the proof of Theorem 4.2 in the case when $|m| \leqslant \frac{n}{2}$ (provided we show the finiteness of energy, which will be carried out in the subsection on redshift estimates below).

4.5. Redshift Estimates

The last step is to use redshift estimates to transfer finiteness of the energies along the branches $\mathcal{CH}_1^+$ and $\mathcal{CH}_2^+$ of the cosmological horizon to finiteness at $r = r_0$, justifying the finiteness of the energies $\mathcal{E}(r_0)$ and $\widetilde{\mathcal{E}}(r_0)$ assumed in the previous two subsections. Here, we do not include all the details of the computations, since they are analogous to the corresponding estimates given in [8, §3.6], and the interested reader can also find the details for our case spelt out in the arxiv version of our paper [19].

Define $u := t + \int_{r_*}^{r} \frac{1}{V} \mathrm{d}r$, where $r_* > r_c$ is arbitrary, but fixed. Then,

$$du = dt + \frac{1}{V} dr.$$

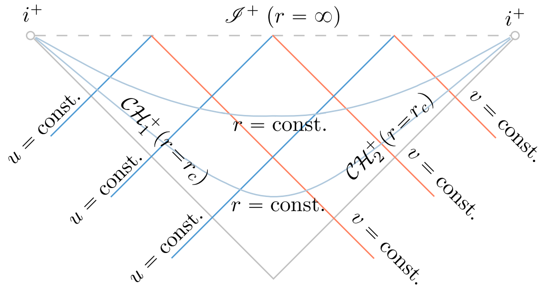
The Reissner–Nordström–de Sitter metric can be rewritten using the coordinates $(u, r, \ldots)$, instead of the old $(t, r, \ldots)$-coordinates, as follows:

$$g = -\frac{1}{V} dr^2 + V dt^2 + r^2 d\Omega^2 = V du^2 - 2du dr + r^2 d\Omega^2.$$

This new coordinate system $(u, r, \ldots)$ extends across the cosmological horizon $r = r_c$ (where $V = 0$). The hypersurfaces of constant u are null and transverse to the cosmological horizon. Thus, only one of the branches of the cosmological horizon, namely $\mathcal{CH}_1^+$, is covered by the $(u, r, \ldots)$-coordinates. (In order to cover the other branch $\mathcal{CH}_2^+$, where $u = -\infty$, we can introduce

$$v := -t + \int_{r_*}^{r} \frac{1}{V} \mathrm{d}r,$$

and use the $(v, r, \ldots)$-coordinate chart.) We will only consider $\mathcal{CH}_1^+$ in the remainder of this subsection, since $\mathcal{CH}_2^+$ can be treated analogously.



The Killing vector field $K = \frac{\partial}{\partial u} = \frac{\partial}{\partial t}$ is well defined across $\mathcal{CH}_1^+$ and is null on the cosmological horizon $\mathcal{CH}_1^+$, even though the t-coordinate is not defined there. Consider the vector field in the $(u, r, \ldots)$-coordinate chart, $Y = (\frac{\partial}{\partial r})_u$. The subscript u means that the integral curves of Y in the $(u, r, \ldots)$-coordinate chart have a constant u-coordinate. Then, $du(Y) = 0$ and $dr(Y) =$

1. In the $(t, r, \ldots)$-coordinate chart, Y can be expressed as $Y = \frac{\partial}{\partial r} - \frac{1}{V}\frac{\partial}{\partial t}$. Let the vector field X be defined by $X = \frac{V^{\frac{1}{2}}}{r^{n-1}} N = \frac{V}{r^{n-1}}\frac{\partial}{\partial r}$ in the old $(t, r, \ldots)$-coordinate chart. To find the expression for X in the $(u, r, \ldots)$-coordinate chart induced basis vectors, we first find $N = -\frac{\text{grad } r}{|\text{grad } r|}$ in the $(u, r, \ldots)$-coordinate chart induced basis vectors. If $\omega := -\frac{1}{\sqrt{V}} dr$, then $N = g^{\mu\nu}\omega_\nu = \frac{1}{\sqrt{V}}\left(\frac{\partial}{\partial u} + V\frac{\partial}{\partial r}\right)$. So $X = \frac{\sqrt{V}}{r^{n-1}} N = \frac{1}{r^{n-1}}(\frac{\partial}{\partial u} + V\frac{\partial}{\partial r})$. The energy

$$E(r) = \int_{\mathbb{R}\times S^{n-1}} T(X, N) \mathrm{d}V_n = \frac{1}{2}\int_{\mathbb{R}\times S^{n-1}} \left(V^2\phi'^2 + \dot{\phi}^2 + \frac{V}{r^2}|\mathring{\nabla}\phi|^2 + m^2 V\phi^2\right) \mathrm{d}t\mathrm{d}\Omega$$
$$\xrightarrow{r \to r_c} \frac{1}{2}\int_{\mathbb{R}\times S^{n-1}\ (\mathcal{CH}_1^+)} (K\cdot\phi)^2 \mathrm{d}u\mathrm{d}\Omega + \frac{1}{2}\int_{\mathbb{R}\times S^{n-1}\ (\mathcal{CH}_2^+)} (K\cdot\phi)^2 \mathrm{d}v\mathrm{d}\Omega$$

(since $V(r_c) = 0$). So $E(r)$ 'loses control' of the transverse and angular derivatives as $r \to r_c$. To remedy this problem, we define a new energy $\widetilde{E}$, by adding Y to X, obtaining

$$\widetilde{E}(r) := E(r) + \int_{\mathbb{R}\times S^{n-1}} T(Y, N)\mathrm{d}V_n.$$

In $(t, r, \ldots)$-coordinates, $N = \sqrt{V}\partial_r$. So $T(Y, N) = \frac{1}{\sqrt{V}}(T(N, N) - T(\partial_t, \partial_r))$. We have $T(\partial_t, \partial_r) = \dot{\phi}\phi'$. So

$$\widetilde{E}(r) = E(r) + \int_{\mathbb{R}\times S^{n-1}} \left(\frac{1}{\sqrt{V}}\frac{1}{2}\left(V\phi'^2 + \frac{\dot{\phi}^2}{V} + \frac{|\mathring{\nabla}\phi|^2}{r^2} + m^2\phi^2\right) - \frac{1}{\sqrt{V}}\dot{\phi}\phi'\right)\sqrt{V} r^{n-1}\mathrm{d}t\mathrm{d}\Omega$$
$$= E(r) + \int_{\mathbb{R}\times S^{n-1}} \frac{1}{2}\left(V(Y\cdot\phi)^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + m^2\phi^2\right) r^{n-1}\mathrm{d}t\mathrm{d}\Omega.$$

We now have

$$\widetilde{E}(r_c) = E(r_c) + \frac{r_c^{n-3}}{2}\int_{\mathbb{R}\times S^{n-1}} |\mathring{\nabla}\phi|^2 \mathrm{d}u\mathrm{d}\Omega + \frac{r_c^{n-3}}{2}\int_{\mathbb{R}\times S^{n-1}} |\mathring{\nabla}\phi|^2 \mathrm{d}v\mathrm{d}\Omega$$
$$+ \frac{m^2 r_c^{n-1}}{2}\int_{\mathbb{R}\times S^{n-1}} \phi^2 \mathrm{d}u\mathrm{d}\Omega + \frac{m^2 r_c^{n-1}}{2}\int_{\mathbb{R}\times S^{n-1}} \phi^2 \mathrm{d}v\mathrm{d}\Omega,$$

so that using $\widetilde{E}$ instead of E allows some control of the angular derivatives as $r \to r_c$. Note that $\widetilde{E}(r_c)$ is equivalent to $\|\phi\|^2_{H^1(\mathcal{CH}_1^+)} + \|\phi\|^2_{H^1(\mathcal{CH}_2^+)}$. We will now compute the deformation tensor Ξ corresponding to the multiplier Y. We have $[-\frac{1}{V}\partial_t, \partial_r] = -\frac{V'}{V}\partial_t$, $\mathcal{L}_{\partial_r} g = \frac{V'}{V}dr^2 + V'dt^2 + 2r d\Omega^2$, and $\mathcal{L}_{-\frac{1}{V}\partial_t} g = 2V dt\ (-\frac{V'}{V^2})dr$. Hence, $\Xi = \frac{1}{2}\mathcal{L}_Y g = \frac{1}{2}V' du^2 + r d\Omega^2$. We have $du = -g(Y, \cdot)$. Also, we recall that $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{g_{\mu\nu}}{2}(\partial_\alpha\phi\partial^\alpha\phi + m^2\phi^2)$,

$$T^{\mu\nu}\Xi_{\mu\nu} = \frac{1}{2}V'(Y\cdot\phi)^2 + \frac{1}{r^3}|\mathring{\nabla}\phi|^2 - \frac{n-1}{2r}\langle d\phi, d\phi\rangle - \frac{n-1}{2r}m^2\phi^2. \quad (12)$$

We have $\langle d\phi, d\phi\rangle = -2(K\cdot\phi)(Y\cdot\phi) - V(Y\cdot\phi)^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2$. Combining (12) and the above, we obtain

$$\begin{aligned} T^{\mu\nu}\Xi_{\mu\nu} &= \Big(\frac{V'}{2}+\frac{n-1}{2r}V\Big)(Y\cdot\phi)^2+\frac{n-1}{r}(K\cdot\phi)(Y\cdot\phi)-\frac{n-3}{2r^3}|\mathring{\nabla}\phi|^2-\frac{n-1}{2r}m^2\phi^2 \\ &= \frac{V'}{2}\Big(Y\cdot\phi+\frac{n-1}{rV'}(K\cdot\phi)\Big)^2-\frac{(n-1)^2}{2r^2V'}(K\cdot\phi)^2-\frac{n-3}{2r^3}|\mathring{\nabla}\phi|^2-\frac{n-1}{2r}m^2\phi^2. \end{aligned}$$

Now as $V'(r) > 0$ for $r \geqslant r_c$ (global redshift), it follows that the first summand in the last expression is nonnegative, and so we obtain the inequality

$$T^{\mu\nu}\Xi_{\mu\nu} \geqslant -\frac{(n-1)^2}{2r^2V'}(K\cdot\phi)^2 - \frac{n-3}{2r^3}|\mathring{\nabla}\phi|^2 - \frac{n-1}{2r}m^2\phi^2. \tag{13}$$

Now suppose that r_0 is fixed. As $r^2V'(r) > 0$ for all $r \in [r_c, r_0]$, we have $\min\limits_{r\in[r_c,r_0]} r^2V'(r) > 0$. Thus, $-\frac{(n-1)^2}{2r^2V'} \geqslant -\frac{(n-1)^2}{\min\limits_{r\in[r_c,r_0]} r^2V'(r)} =: -C_1(r_0)$. Similarly, for $r \in [r_c, r_0]$, $\dfrac{1}{r^3} \leqslant \dfrac{1}{r_c^3}$, and so $-\frac{n-3}{2r^3} \geqslant -\frac{n-3}{2r_c^3} =: -\widetilde{C}_2(r_c)$. Also, for $r \in [r_c, r_0]$, $-\frac{(n-1)}{2r}m^2 \geqslant -\frac{(n-1)}{2r_c}m^2 =: -C_3(r_c)$. If we set $\Pi := \frac{1}{2}\mathcal{L}_X g$, then from Step 1 of the proof of Theorem 5.3 (see in particular the inequality (17) on page 35), we have for $r > r_c$ that

$$T^{\mu\nu}\Pi_{\mu\nu} \geqslant -\left(1+\frac{n}{2}+\frac{e^2}{2r_c^{n-1}}\right)\frac{1}{r^{n+2}}|\mathring{\nabla}\phi|^2 \geqslant -\underbrace{\left(1+\frac{n}{2}+\frac{e^2}{2r_c^{n-1}}\right)\frac{1}{r_c^{n+2}}}_{=:\widetilde{\widetilde{C}}_2(r_c)}|\mathring{\nabla}\phi|^2.$$

Set $C_2(r_c) := \widetilde{C}_2(r_c) + \widetilde{\widetilde{C}}_2(r_c)$. We have

$$T^{\mu\nu}\Pi_{\mu\nu} + T^{\mu\nu}\Xi_{\mu\nu} \geqslant -\underbrace{\max\{C_1(r_0), C_2(r_c), C_3(r_c)\}}_{=:C(r_c,r_0)>0}\cdot\Big((K\cdot\phi)^2 + |\mathring{\nabla}\phi|^2 + \phi^2\Big).$$

For $r_1 \in (r_c, r_0)$, and $T > 0$, define $\mathcal{D} = \{r = r_1\} \cap \{-T \leqslant t \leqslant T\}$. We now apply the divergence theorem, with the current J corresponding to the multiplier $X + Y$, in the region $\mathcal{T} = D^+(\mathcal{D}) \cap \{r \leqslant r_0\}$. Noticing that the flux across the future null boundaries is less than or equal to 0, we obtain, after passing the limit $T \to \infty$, that

$$\widetilde{E}(r_1) - \widetilde{E}(r_0) \geqslant -\int_{r_1}^{r_0}\int_{\mathbb{R}\times S^{n-1}} C(r_0, r_c)\Big((K\cdot\phi)^2 + |\mathring{\nabla}\phi|^2 + \phi^2\Big) r^{n-1}\mathrm{d}t\mathrm{d}\Omega\mathrm{d}r. \tag{14}$$

But

$$\begin{aligned} \widetilde{E}(r) &= \frac{1}{2}\int_{\mathbb{R}\times S^{n-1}}\Big(V^2\phi'^2 + \dot{\phi}^2 + \frac{V}{r^2}|\mathring{\nabla}\phi|^2 + m^2V\phi^2\Big)\mathrm{d}t\mathrm{d}\Omega \\ &\quad + \frac{1}{2}\int_{\mathbb{R}\times S^{n-1}}\Big(V(Y\cdot\phi)^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + m^2\phi^2\Big) r^{n-1}\mathrm{d}t\mathrm{d}\Omega. \end{aligned}$$

We have

$$\int_{r_1}^{r_0}\int_{\mathbb{R}\times S^{n-1}} (K\cdot\phi)^2 r^{n-1}\mathrm{d}t\mathrm{d}\Omega\mathrm{d}r \leqslant \int_{r_1}^{r_0}\int_{\mathbb{R}\times S^{n-1}} \dot{\phi}^2 r_0^{n-1}\mathrm{d}t\mathrm{d}\Omega\mathrm{d}r \leqslant 2r_0^{n-1}\int_{r_1}^{r_0}\widetilde{E}(r)\mathrm{d}r$$
$$\int_{r_1}^{r_0}\int_{\mathbb{R}\times S^{n-1}} |\mathring{\nabla}\phi|^2 r^{n-1}\mathrm{d}t\mathrm{d}\Omega\mathrm{d}r \leqslant \int_{r_1}^{r_0}\int_{\mathbb{R}\times S^{n-1}} \frac{|\mathring{\nabla}\phi|^2}{r^2} r^{n-1}r^2\mathrm{d}t\mathrm{d}\Omega\mathrm{d}r \leqslant 2r_0^2\int_{r_1}^{r_0}\widetilde{E}(r)\mathrm{d}r$$
$$\int_{r_1}^{r_0}\int_{\mathbb{R}\times S^{n-1}} \phi^2 r^{n-1}\mathrm{d}t\mathrm{d}\Omega\mathrm{d}r \leqslant \frac{2}{m^2}\int_{r_1}^{r_0}\widetilde{E}(r)\mathrm{d}r.$$

Using the above three estimates, it follows from (14) that

$$\widetilde{E}(r_1)-\widetilde{E}(r_0) \geqslant= -\int_{r_1}^{r_0} k(r_0,r_c)\widetilde{E}(r)\mathrm{d}r, \tag{15}$$

where $k(r_0,r_c) := C(r_0,r_c)(2r_0^{n-1}+2r_0^2+\frac{2}{m^2})$. Now suppose r_2 is such that $r_c<r_1<r_2<r_0$. If we redo all of the above steps in order to obtain (15), but with r_2 replacing r_0, we obtain

$$\widetilde{E}(r_1)-\widetilde{E}(r_2) \geqslant -\int_{r_1}^{r_2} k(r_2,r_c)\widetilde{E}(r)\mathrm{d}r, \tag{16}$$

where $k(r_2,r_c) = C(r_2,r_c)(2r_2^{n-1}+2r_2^2+\frac{2}{m^2})$. But

$$k(r_2,r_c) \leqslant \max\{C_1(r_2),\, C_2(r_c),\, C_3(r_c)\}\cdot\left(2r_0^{n-1}+2r_0^2+\frac{2}{m^2}\right).$$

As $C_1(r_2)=\frac{(n-1)^2}{\min\limits_{r\in[r_c,r_2]} r^2V'(r)} \leqslant \frac{(n-1)^2}{\min\limits_{r\in[r_c,r_0]} r^2V'(r)} = C_1(r_0)$, $k(r_2,r_c)\leqslant k(r_0,r_c)$. From (16), we get

$$\widetilde{E}(r_1)-\widetilde{E}(r_2) \geqslant -\int_{r_1}^{r_2} k(r_2,r_c)\widetilde{E}(r)\mathrm{d}r \geqslant -\int_{r_1}^{r_2} k(r_0,r_c)\widetilde{E}(r)\mathrm{d}r.$$

Consequently, for all $r_2\in[r_1,r_0)$, $\widetilde{E}(r_2)\leqslant\widetilde{E}(r_1)+\displaystyle\int_{r_1}^{r_2}k(r_0,r_c)\widetilde{E}(r)\mathrm{d}r$. By the integral form of Grönwall's inequality (see e.g. [24, Thm. 1.10]), we obtain for all $r_2\in[r_1,r_0)$ that

$$\widetilde{E}(r_2)\leqslant\widetilde{E}(r_1)e^{\int_{r_1}^{r_2}k(r_0,r_c)\mathrm{d}r} = \widetilde{E}(r_1)e^{k(r_0,r_c)\cdot(r_2-r_1)}.$$

As $r_2\nearrow r_0$, we get $\widetilde{E}(r_0)\leqslant\widetilde{E}(r_1)e^{k(r_0,r_c)\cdot(r_0-r_1)}$. This holds for all $r_1\in(r_c,r_0)$. Passing the limit as $r_1\searrow r_c$, we obtain $\widetilde{E}(r_0)\leqslant\widetilde{E}(r_c)e^{k(r_0,r_c)\cdot(r_0-r_c)}$. So $E(r_0)\leqslant\widetilde{E}(r_0)\leqslant\widetilde{E}(r_c)e^{k(r_0,r_c)\cdot(r_0-r_c)}\lesssim\widetilde{E}(r_c)\lesssim\|\phi\|^2_{H^1(\mathcal{CH}_1^+)}+\|\phi\|^2_{H^1(\mathcal{CH}_2^+)}<\infty$.

Commuting with the Killing vector fields $\frac{\partial}{\partial t}$ and L_{ij}, we see that the hypothesis from Theorem 4.2, namely $\|\phi\|_{H^k(\mathcal{CH}_1^+)}<+\infty$ and $\|\phi\|_{H^k(\mathcal{CH}_2^+)}<+\infty$, for some $k>\frac{n}{2}+2$, yields also $\|\phi\|_{H^k(\{r=r_0\})}\lesssim\|\phi\|_{H^k(\mathcal{CH}_1^+)}+\|\phi\|_{H^k(\mathcal{CH}_2^+)}<+\infty$. We now show that this justifies the assumption used in the previous two subsections. For simplicity, we only consider just one of the energies

$$\mathcal{E}(r)=\frac{1}{2}\int_{\mathbb{R}\times S^{n-1}}\left(\psi'^2+\frac{1}{V^2}\dot{\psi}^2+\frac{1}{r^2V}|\mathring{\nabla}\psi|^2+\theta\psi^2\right)\mathrm{d}t\mathrm{d}\Omega.$$

(The proof of the finiteness of $\widetilde{\mathcal{E}}(r_0)$ is entirely analogous.) As $\psi = r^\kappa \phi$, we obtain finiteness of the last summand, namely

$$\int_{\mathbb{R}\times S^{n-1}} \theta(r_0)\left(\psi(r_0,\cdot)\right)^2 \mathrm{d}t\mathrm{d}\Omega \leqslant \theta(r_0) r_0^{2\kappa} \|\phi(r_0,\cdot)\|^2_{H^1(\{r=r_0\})} < +\infty.$$

We have $\displaystyle\int_{\mathbb{R}\times S^{n-1}} \left(\phi'(r_0,\cdot)\right)^2 \mathrm{d}t\mathrm{d}\Omega \leqslant \frac{2E(r_0)}{(V(r_0))^2} < +\infty.$

Since $\psi'(r_0,\cdot) = \kappa r_0^{\kappa-1}\phi(r_0,\cdot) + r_0^\kappa \phi'(r_0,\cdot)$, and as $\phi(r_0,\cdot) \in H^1(\{r = r_0\})$, we have $\psi(r_0,\cdot) \in L^2(\mathbb{R}\times S^{n-1})$, that is, $\displaystyle\int_{\mathbb{R}\times S^{n-1}} \left(\psi'(r_0,\cdot)\right)^2 \mathrm{d}t\mathrm{d}\Omega < +\infty$.

We also have

$$\int_{\mathbb{R}\times S^{n-1}} \frac{1}{\left(V(r_0)\right)^2}\left(\dot{\psi}(r_0,\cdot)\right)^2 \mathrm{d}t\mathrm{d}\Omega \leqslant \frac{r_0^{2\kappa}}{\left(V(r_0)\right)^2} \|\phi(r_0,\cdot)\|^2_{H^1(\{r=r_0\})} < +\infty.$$

Finally,

$$\int_{\mathbb{R}\times S^{n-1}} \frac{1}{r_0^2 V(r_0)} |\mathring{\nabla}\psi(r_0,\cdot)|^2 \mathrm{d}t\mathrm{d}\Omega \lesssim \|\phi(r_0,\cdot)\|^2_{H^1(\{r=r_0\})} < +\infty.$$

Thus, each summand in the expression for $\mathcal{E}(r_0)$ is finite. This completes the proof of Theorem 4.2.

5. Decay in RNdS When $m = 0$, the Wave Equation

In [8, Theorem 2], the following result was shown:

Theorem 5.1. *Suppose that*

- $\delta > 0$,
- $M > 0$,
- $e \geqslant 0$,
- $n > 2$,
- (M, g) *is the* $(n+1)$*-dimensional subextremal Reissner–Nordström–de Sitter solution given by the metric*

$$g = -\frac{1}{V} dr^2 + V dt^2 + r^2 d\Omega^2,$$

  where

$$V = r^2 + \frac{2M}{r^{n-2}} - \frac{e^2}{r^{n-1}} - 1,$$

  and $d\Omega^2$ *is the metric of the unit* $(n-1)$*-dimensional sphere* S^{n-1},
- $k > \frac{n}{2} + 2$, *and*
- ϕ *is a smooth solution to* $\Box_g \phi = 0$ *such that*

$$\|\phi\|_{H^k(\mathcal{CH}_1^+)} < +\infty \quad \text{and} \quad \|\phi\|_{H^k(\mathcal{CH}_2^+)} < +\infty,$$

  where $\mathcal{CH}_1^+ \simeq \mathcal{CH}_2^+ \simeq \mathbb{R}\times S^{n-1}$ *are the two components of the future cosmological horizon, parameterised by the flow parameter of the global Killing vector field* $\frac{\partial}{\partial t}$.

Then, there exists a r_0 large enough so that for all $r \geqslant r_0$,

$$\|\partial_r \phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \lesssim r^{-3+\delta}.$$

Using a method similar to the one we used to show Rendall's conjecture in Theorem 2.3, we can improve the almost-exact bound of $r^{-3+\delta}$ to r^{-3}.

Remark 5.2. As observed in [8, Remark 1.4], this decay rate bound of r^{-3} for $\partial_r\phi$ is in fact the decay rate one would expect in the light of Rendall's conjecture. Indeed, for freely falling observers in the cosmological region, one has

$$r(\tau) \sim e^{\tau} \sim a(\tau),$$

where τ is the proper time, and $a(\tau)$ is the radius of a comparable de Sitter universe in flat FLRW form, giving

$$\partial_r\phi \sim \frac{\partial_\tau \phi}{\partial_\tau r} \sim \frac{e^{-2\tau}}{e^{\tau}} \sim \frac{1}{r^3}.$$

Thus, our improved version of Theorem 5.1 is the following result.

Theorem 5.3. *Suppose that*

- $M > 0$,
- $e \geqslant 0$,
- $n > 2$,
- (M, g) *is the* $(n+1)$*-dimensional subextremal Reissner–Nordström–de Sitter solution given by the metric*

$$g = -\frac{1}{V}dr^2 + V dt^2 + r^2 d\Omega^2,$$

where

$$V = r^2 + \frac{2M}{r^{n-2}} - \frac{e^2}{r^{n-1}} - 1,$$

and $d\Omega^2$ is the metric of the unit $(n-1)$-dimensional sphere S^{n-1},

- $k > \frac{n}{2} + 2$, *and*
- ϕ *is a smooth solution to $\Box_g \phi = 0$ such that*

$$\|\phi\|_{H^k(\mathcal{CH}_1^+)} < +\infty \quad \text{and} \quad \|\phi\|_{H^k(\mathcal{CH}_2^+)} < +\infty,$$

where $\mathcal{CH}_1^+ \simeq \mathcal{CH}_2^+ \simeq \mathbb{R}\times S^{n-1}$ are the two components of the future cosmological horizon, parameterised by the flow parameter of the global Killing vector field $\frac{\partial}{\partial t}$.

Then, there exists a r_0 large enough so that for all $r \geqslant r_0$,

$$\|\partial_r \phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \lesssim r^{-3}.$$

Proof. **Step 1:** We will first establish the following estimates: there exists an r_0 large enough such that for all $r \geqslant r_0$,

$$\|\ddot{\phi}(r,\cdot)\|_{L^\infty(\mathbb{R},S^{n-1})} \lesssim 1, \qquad \|\mathring{\Delta}\phi(r,\cdot)\|_{L^\infty(\mathbb{R},S^{n-1})} \lesssim 1.$$

We will follow [8, §3.2] in order to obtain the bounds above, which will be needed in Step 2 of our proof below. We repeat this preliminary step here from [8, §3.2] for the sake of completeness and for the convenience of the reader.

Suppose ϕ satisfies the wave equation $\Box_g \phi = 0$. The energy–momentum tensor associated with ϕ is given by $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \partial^\alpha \phi$. Thus,

$$T(N,N) = \Big(\phi'^2 - \frac{1}{2}\frac{(-1)}{V}\Big(\phi'^2(-V) + \frac{\dot\phi^2}{V} + \frac{1}{r^2}|\mathring{\nabla}\phi|^2\Big)\Big)V = \frac{1}{2}\Big(V\phi'^2 + \frac{\dot\phi^2}{V} + \frac{1}{r^2}|\mathring{\nabla}\phi|^2\Big).$$

Define $X := \dfrac{V^{\frac{1}{2}}}{r^{n-1}} N = \dfrac{V^{\frac{1}{2}}}{r^{n-1}} V^{\frac{1}{2}} \dfrac{\partial}{\partial r} = \dfrac{V}{r^{n-1}} \dfrac{\partial}{\partial r}$.

The current J is given by $J_\mu := T_{\mu\nu} X^\nu$. We define the energy

$$E(r) := \int_{\mathbb{R}\times S^{n-1}} T(X,N) \mathrm{d}V_n = \frac{1}{2}\int_{\mathbb{R}\times S^{n-1}} \Big(V^2\phi'^2 + \dot\phi^2 + \frac{V}{r^2}|\mathring{\nabla}\phi|^2\Big)\mathrm{d}t\mathrm{d}\Omega.$$

The deformation tensor Π associated with the multiplier X is given by

$$\Pi = \frac{1}{2}\mathcal{L}_X g = -\frac{1}{V} dr \mathcal{L}_X dr + \frac{V}{2r^{n-1}}\frac{V'}{V^2}dr^2 + \frac{V'V}{2r^{n-1}}dt^2 + \frac{V}{r^{n-1}} r d\Omega^2.$$

It can be shown that $\mathcal{L}_X dr = \Big(\dfrac{V'}{r^{n-1}} - \dfrac{(n-1)V}{r^n}\Big)dr$. Thus,

$$\begin{aligned}
\Pi &= -\frac{1}{V}\Big(\frac{V'}{r^{n-1}} - \frac{(n-1)V}{r^n}\Big)dr^2 + \frac{V'}{2r^{n-1}V}dr^2 + \frac{V'V}{2r^{n-1}}dt^2 + \frac{V}{r^{n-1}} r d\Omega^2 \\
&= \underbrace{\frac{V'}{2r^{n-1}}\Big(-\frac{1}{V}dr^2 + V dt^2\Big)}_{=:\Pi^{(1)}} + \underbrace{\frac{n-1}{r^n}dr^2}_{=:\Pi^{(2)}} + \underbrace{\frac{V}{r^{n-2}}d\Omega^2}_{\Pi^{(3)}}.
\end{aligned}$$

We have

$$\begin{aligned}
T^{\mu\nu}\Pi^{(1)}_{\mu\nu} &= \frac{V'}{2r^{n-1}}\Big(-\frac{1}{2}V\phi'^2 - \frac{1}{2V}\dot\phi^2 - \frac{1}{2r^2}|\mathring{\nabla}\phi|^2 \\
&\quad + \frac{1}{V}\Big(\dot\phi^2 - \frac{1}{2}V\Big(\phi'^2(-V) + \dot\phi^2\frac{1}{V} + \frac{1}{r^2}|\mathring{\nabla}\phi|^2\Big)\Big)\Big) \\
&= \frac{V'}{2r^{n-1}}\Big(-\frac{1}{r^2}|\mathring{\nabla}\phi|^2\Big).
\end{aligned}$$

Also, $T^{\mu\nu}\Pi^{(2)}_{\mu\nu} = \dfrac{(n-1)}{2r^n}\Big(V^2\phi'^2 + \dot\phi^2 + \dfrac{V}{r^2}|\mathring{\nabla}\phi|^2\Big)$. Finally,

$$\begin{aligned}
T^{\mu\nu}\Pi^{(3)}_{\mu\nu} &= \frac{V}{r^{n-2}}\Big(\frac{1}{r^4}|\mathring{\nabla}\phi|^2 - \frac{(n-1)}{2r^2}\Big(\phi'^2(-V) + \dot\phi^2\frac{1}{V} + \frac{1}{r^2}|\mathring{\nabla}\phi|^2\Big)\Big) \\
&= \frac{V}{r^{n+2}}|\mathring{\nabla}\phi|^2 + \frac{(n-1)}{2r^n}\Big(V^2\phi'^2 - \dot\phi^2 - \frac{V}{r^2}|\mathring{\nabla}\phi|^2\Big).
\end{aligned}$$

Consequently, the full bulk term is

$$\nabla_\mu J^\mu = T^{\mu\nu}\Pi_{\mu\nu} = \frac{(n-1)}{r^n}V^2\phi'^2 + |\mathring{\nabla}\phi|^2\Big(\frac{V}{r^{n+2}} - \frac{V'}{2r^{n+1}}\Big).$$

Using the expression for V, we compute

$$\frac{V}{r^{n+2}} - \frac{V'}{2r^{n+1}} = -\frac{1}{r^{n+2}}\Big(1 + \frac{(n+1)e^2}{2r^{n-1}} - \frac{nM}{r^{n-2}}\Big).$$

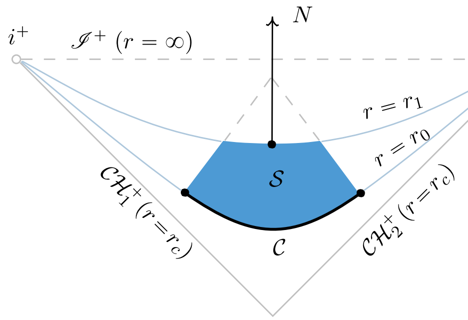
For $r > r_c$, we have $V(r) > 0$, and so

$$\begin{aligned}1+\frac{(n+1)e^2}{2r^{n-1}}-\frac{nM}{r^{n-2}} &= 1-\frac{n}{2}\Big(r^2+\frac{2M}{r^{n-2}}-\frac{e^2}{r^{n-1}}-1\Big)-\frac{n}{2}+\frac{e^2}{2r^{n-1}}\\ &= 1-\frac{n}{2}V(r)-\frac{n}{2}+\frac{e^2}{2r^{n-1}}\\ &\leqslant 1-0-\frac{n}{2}+\frac{e^2}{2r^{n-1}} \leqslant 1+\frac{n}{2}+\frac{e^2}{2r_c^{n-1}} =: C.\end{aligned}$$

Hence,

$$\nabla_\mu J^\mu = T^{\mu\nu}\Pi_{\mu\nu} \geqslant -\frac{C}{r^{n+2}}|\mathring{\nabla}\phi|^2. \tag{17}$$

For each $T < 0$, define the set $\mathcal{C} := \{r = r_0\} \cap \{-T \leqslant t \leqslant T\}$. Also, consider the region $\mathcal{S} := D^+(\mathcal{C}) \cap \{r \leqslant r_1\}$.



We will apply the divergence theorem to the current J on the region $\mathcal{S}$. As the flux across the future null boundaries is nonpositive, we have

$$-\int_{r_0}^{r_1}\int_{\mathbb{R}\times S^{n-1}}\frac{C}{r^{n+2}}|\mathring{\nabla}\phi|^2\mathrm{d}t\mathrm{d}\Omega\mathrm{d}r \leqslant \int_{\mathcal{S}}(\nabla_\mu J^\mu)\epsilon \underbrace{\int_{\substack{\text{null}\\\text{part}}} J\lrcorner\,\epsilon}_{\leqslant 0} + \underbrace{\int_{\{r=r_1\}} J\lrcorner\,\epsilon}_{-E(r_1)} + \underbrace{\int_{\{r=r_0\}} J\lrcorner\,\epsilon}_{E(r_0)}.$$

So $E(r_0) - E(r_1) \geqslant -\displaystyle\int_{r_0}^{r_1}\int_{\mathbb{R}\times S^{n-1}}\frac{C}{r^3}|\mathring{\nabla}\phi|^2\mathrm{d}t\mathrm{d}\Omega\mathrm{d}r$. We have

$$\int_{\mathbb{R}\times S^{n-1}}|\mathring{\nabla}\phi|^2\mathrm{d}t\mathrm{d}\Omega = \int_{\mathbb{R}\times S^{n-1}}\frac{V}{r^2}|\mathring{\nabla}\phi|^2\frac{r^2}{V}\mathrm{d}t\mathrm{d}\Omega \leqslant 2E(r)\cdot\frac{r^2}{V} = \frac{2r^2}{V}E(r).$$

Since

$$\lim_{r\to\infty}\frac{2r^2}{V} = \lim_{r\to\infty}\frac{2}{1+\frac{2M}{r^n}-\frac{e^2}{r^{n+1}}-\frac{1}{r^2}} = \frac{2}{1},$$

there exists an r_0 large enough such that for all $r \geqslant r_0$, $|\frac{2r^2}{V}-2| < 1$, and in particular, $(0 <)\frac{2r^2}{V} < 3$. Hence,

$$E(r_1) \leqslant E(r_0) + \int_{r_0}^{r_1}\int_{\mathbb{R}\times S^{n-1}}\Big(\frac{C}{r^3}|\mathring{\nabla}\phi|^2\Big)\mathrm{d}t\mathrm{d}\Omega\mathrm{d}r \leqslant E(r_0) + \int_{r_0}^{r_1}\frac{3C}{r^3}E(r)\mathrm{d}r.$$

Using Grönwall's inequality (see for e.g. [24, Thm. 1.10]), we obtain

$$E(r_1) \leqslant E(r_0) e^{\int_{r_0}^{r_1} \frac{3C}{r^3} \mathrm{d}r} \lesssim C(r_0) E(r_0),$$

as was also noted in [8, Eq.(71)]. Thus, we have in particular that there exists an r_0 large enough such that for all $r \geqslant r_0$,

$$\int_{\mathbb{R}\times S^{n-1}} \dot{\phi}^2 \mathrm{d}t\mathrm{d}\Omega \lesssim 1, \quad \text{and} \quad \int_{\mathbb{R}\times S^{n-1}} |\mathring{\nabla}\phi|^2 \mathrm{d}t\mathrm{d}\Omega \lesssim 1.$$

Commuting with the Killing vector fields $\frac{\partial}{\partial t}$ and L_{ij}, we obtain (after the transferral of the finiteness of the energies along the branches $\mathcal{CH}_1^+$ and $\mathcal{CH}_2^+$ of the cosmological horizon to finiteness at $r = r_0$, and an application of Sobolev's inequality) that for all $r \geqslant r_0$,

$$\|\ddot{\phi}(r,\cdot)\|_{L^\infty(\mathbb{R},S^{n-1})} \lesssim 1, \quad \text{and} \quad \|\mathring{\Delta}\phi(r,\cdot)\|_{L^\infty(\mathbb{R},S^{n-1})} \lesssim 1.$$

Step 2: In this step, we will write the wave equation in new coordinates which 'equalises' the magnitude of the coefficient weights for the r and t coordinates in the matrix of the metric.

To this end, we define $\rho = \displaystyle\int_{r_0}^{r} \frac{1}{V(r)} \mathrm{d}r$. Then, $\dfrac{\mathrm{d}\rho}{\mathrm{d}r} = \dfrac{1}{V(r)}$ and $V(r)\dfrac{\mathrm{d}}{\mathrm{d}r} = \dfrac{\mathrm{d}}{\mathrm{d}\rho}$.

With a slight abuse of notation, we write $V(\rho) := V(r(\rho))$. We have

$$g = -\frac{1}{V} dr^2 + V dt^2 + r^2 d\Omega^2 = -V d\rho^2 + V dt^2 + (r(\rho))^2 d\Omega^2.$$

The wave equation $\Box_g \phi = 0$ can be rewritten as $\partial_\mu(\sqrt{-g}\,\partial^\mu \phi) = 0$, which becomes $\partial_\mu(V r^{n-1} \partial^\mu \phi) = 0$. Separating the differential operators with respect to the $\rho, t, \ldots$ coordinates, we obtain

$$\partial_\rho(r^{n-1}\partial_\rho \phi) = r^{n-1}\ddot{\phi} + V r^{n-3} \mathring{\Delta}\phi.$$

Integrating from $\rho_0 := \rho(r_0) = 0$ to $\rho = \rho(r)$, we obtain

$$r^{n-1}\partial_\rho \phi - r_0^{n-1} \left.(\partial_\rho \phi)\right|_{\rho=\rho_0} = \int_0^\rho \left(r^{n-1}\ddot{\phi} + V r^{n-3}\mathring{\Delta}\phi\right) \mathrm{d}\rho,$$

and so $r^{n-1} V \partial_r \phi = r_0^{n-1} V(r_0) \left.(\partial_r \phi)\right|_{r=r_0} + \displaystyle\int_0^\rho \left(r^{n-1}\ddot{\phi} + V r^{n-3}\mathring{\Delta}\phi\right) \mathrm{d}\rho$, i.e.,

$$\partial_r \phi = \left(\frac{r_0}{r}\right)^{n-1} \frac{V(r_0)}{V(r)} \left.(\partial_r \phi)\right|_{r=r_0} + \frac{1}{r^{n-1}V} \int_0^\rho \left(r^{n-1}\ddot{\phi} + V r^{n-3}\mathring{\Delta}\phi\right) \mathrm{d}\rho.$$

Hence,

$$\begin{aligned} &\|\partial_r \phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \\ &\quad\leqslant \left(\frac{r_0}{r}\right)^{n-1} \frac{V(r_0)}{V(r)} \|(\partial_r\phi)(r_0,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \\ &\qquad + \frac{1}{r^{n-1}V} \int_{\rho_0}^{\rho} \left(r^{n-1}\|\ddot{\phi}(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} + V r^{n-3}\|\mathring{\Delta}\phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})}\right) \mathrm{d}\rho. \end{aligned}$$

Using the fact that $V \sim r^2$ for $r \geqslant r_0$, with r_0 large enough, and the estimates from Step 1 above, we obtain

$$\begin{aligned}
\|\partial_r\phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} &\lesssim \frac{A}{r^{n+1}} + \frac{B}{r^{n+1}}\int_{\rho_0}^{\rho} (r(\rho))^{n-1}\,\mathrm{d}\rho \\
&\lesssim \frac{A}{r^{n+1}} + \frac{B}{r^{n+1}}\int_{r_0}^{r} r^{n-1}\frac{1}{V(r)}\mathrm{d}r \\
&\lesssim \frac{A}{r^{n+1}} + \frac{B'}{r^{n+1}}\int_{r_0}^{r} r^{n-3}\mathrm{d}r.
\end{aligned}$$

Recalling that $n > 2$, we have

$$\|\partial_r\phi(r,\cdot)\|_{L^\infty(\mathbb{R}\times S^{n-1})} \lesssim \frac{A}{r^{n+1}} + \frac{B'}{r^{n+1}}\frac{1}{(n-2)}(r^{n-2} - r_0^{n-2}) \lesssim \frac{1}{r^3}.$$

This completes the proof of Theorem 5.3. □

Acknowledgements

We thank Pedro Girão for suggesting the idea behind the proofs of Theorems 2.2 and 5.3. JN was partially supported by FCT/Portugal through UIDP/MAT/04459/2020, UID/MAT/04459/2013 and Grant (GPSEinstein) PTDC/MAT-ANA/1275/2014.

6. Appendix A: Fourier Modes (de Sitter in Flat FLRW Form)

In this appendix, we outline the Fourier modal analysis that motivates the specific estimates given in Theorem 3.1, starting with spatially periodic solutions to the Klein–Gordon equation. We only give highlights, since the computations are similar to the ones in [8], and moreover, the details can be found in the arxiv version of our paper [19].

Let $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$. Suppose that the 'spatially periodic' $\phi : \mathbb{R} \times \mathbb{T}^n \to \mathbb{R}$ satisfies the Klein–Gordon equation (5). Writing

$$\phi = \sum_{\mathbf{k}\in\mathbb{Z}^n} c_{\mathbf{k}}(t)e^{i\langle \mathbf{k},\mathbf{x}\rangle},$$

(5) yields $\ddot{c}_{\mathbf{k}} + \frac{n\dot{a}}{a}\dot{c}_{\mathbf{k}} + \frac{k^2}{a^2}c_{\mathbf{k}} + m^2 c_{\mathbf{k}} = 0$, where $k^2 := \langle \mathbf{k}, \mathbf{k}\rangle$. So

$$\frac{\mathrm{d}}{\mathrm{d}t}(a^n \dot{c}_{\mathbf{k}}) + a^{n-2}(k^2 + m^2a^2)c_{\mathbf{k}} = 0. \tag{18}$$

Let $\tau = \int \frac{1}{a(t)}\mathrm{d}t$. Then, $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}\tau}$, and so (18) becomes (with $\frac{\mathrm{d}}{\mathrm{d}\tau} =:'$)

$$(a^{n-1}c'_{\mathbf{k}})' + a^{n-1}(k^2 + m^2a^2)c_{\mathbf{k}} = 0. \tag{19}$$

Defining $d_{\mathbf{k}}$ by $c_{\mathbf{k}} =: a^{-\frac{n-1}{2}}d_{\mathbf{k}}$, we have

$$d''_{\mathbf{k}} + \left(k^2 + m^2a^2 - \frac{(n-1)}{2}\frac{a''}{a} - \frac{(n-1)(n-3)}{4}\left(\frac{a'}{a}\right)^2\right)d_{\mathbf{k}} = 0. \tag{20}$$

Now if $a(t) = e^t$, then we may take $\tau = -e^{-t}$, so that $-t = \log(-\tau)$, that is, $-\tau = e^{-t}$. We remark that relative to our earlier use of conformal coordinates in (4) on page 10, we are taking $t_0 = +\infty$ for simplicity. Then, (20) becomes

$$d''_{\mathbf{k}} + \left(k^2 - \frac{\mu}{\tau^2}\right)d_{\mathbf{k}} = 0, \tag{21}$$

where $\mu := n - 1 + \frac{(n-1)(n-3)}{4} - m^2$. The general solution to this equation is[8] given by

$$C_1\sqrt{\tau}\, J_\nu(|k|\tau) + C_2\sqrt{\tau}\, Y_\nu(|k|\tau), \tag{22}$$

where ν satisfies $\nu^2 = \frac{1}{4} + \mu = \frac{n^2}{4} - m^2$. Here, J_ν denotes the Bessel function of the first kind,

$$J_\nu(z) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+\nu+1)}\left(\frac{z}{2}\right)^{2m+\nu},$$

and Y_ν is the Bessel function of the second kind,

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)},$$

where the right-hand side is replaced by its limiting value if ν is an integer. Without loss of generality, in the solution (22), we may only consider ν such that $\mathrm{Re}(\nu) \geqslant 0$.

We note that as $t \to \infty$, $-\tau = e^{-t} \searrow 0$, and so $\tau \nearrow 0$. We now use the asymptotic expansions of $J_\nu(z)$ and $Y_\nu(z)$ as $z \nearrow 0$ (see e.g. [1, 9.1.7–9]):

1° If $\nu \neq 0$ (that is, $m \neq \pm\frac{n}{2}$), then as $\tau \nearrow 0$, we have

$$J_\nu(|k|\tau) = C(-\tau)^\nu + O(|\tau|),$$
$$Y_\nu(|k|\tau) = A(-\tau)^\nu + B(-\tau)^{-\nu} + C(-\tau)^{2-\nu} + O(|\tau|).$$

[8] See for example [28, p. 95]. For the relevant notation, see also [28, pages 82,100,101].

Using these, as $\tau \nearrow 0$ or $t \to \infty$, $|c_{\mathbf{k}}| = C' e^{-\left(\frac{n}{2}-\mathrm{Re}(\nu)\right)t} + O(e^{-\frac{n+1}{2}t})$. Thus, we expect ϕ to satisfy

$$\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim \begin{cases} a^{-\frac{n}{2}} & \text{if } |m| > \frac{n}{2}, \\ a^{-\frac{n}{2}+\sqrt{\frac{n^2}{4}-m^2}} & \text{if } |m| < \frac{n}{2}. \end{cases}$$

$\underline{2}^\circ$ If $\nu = 0$ (that is, $m = \pm\frac{n}{2}$), then as $\tau \nearrow 0$, we have

$$\begin{aligned} J_\nu(|k|\tau) &= C + O(|\tau|), \\ Y_\nu(|k|\tau) &= C\log(-\tau) + O(|\tau|). \end{aligned}$$

Using these, we get $|c_{\mathbf{k}}| = (A+Bt)e^{-\frac{n}{2}t} + O(e^{-\frac{n+1}{2}t})$ as $t \to +\infty$. Thus, we expect ϕ to satisfy

$$\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim a^{-\frac{n}{2}} \log a \quad \text{if } \ m = \pm\frac{n}{2}.$$

Summarising, ϕ is expected to have the decay

$$\|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \lesssim \begin{cases} a^{-\frac{n}{2}} & \text{if } |m| > \dfrac{n}{2}, \\ a^{-\frac{n}{2}} \log a & \text{if } |m| = \dfrac{n}{2}, \\ a^{-\frac{n}{2}+\sqrt{\frac{n^2}{4}-m^2}} & \text{if } |m| < \dfrac{n}{2}. \end{cases}$$

This motivates the decay estimates in Theorem 3.1.

7. Appendix B

In this section, we prove the technical result we had used in the proof of Theorem 3.1, in Sect. 3.

Lemma 7.1. *If $f, g \in H^1(\mathbb{R}^n)$, then* $\lim_{r\to+\infty} \int_{S_r} fg \,\mathrm{d}\sigma_r = 0.$

Proof. By the Cauchy–Schwarz inequality,

$$\left| \int_{S_r} fg \,\mathrm{d}\sigma_r \right|^2 \leqslant \int_{S_r} |f|^2 \mathrm{d}\sigma_r \cdot \int_{S_r} |g|^2 \mathrm{d}\sigma_r,$$

and so, it is enough to show that

$$\lim_{r\to\infty} \int\limits_{S_r} |f|^2 \mathrm{d}\sigma_r = 0.$$

Suppose this does not hold. Then, there exists an increasing sequence $(r_k)_k$ such that $r_k \overset{k\to\infty}{\longrightarrow} \infty$, and there exists an $\epsilon > 0$ such that for each k,

$$\int_{S_{r_k}} |f|^2 \mathrm{d}\sigma_{r_k} > \epsilon.$$

(The plan is to use the trace theorem to fatten these S_{r_k}-slices to 'annuli' A_k and obtain $\|f\|^2_{H^1(A_k)} > \widetilde{\epsilon} > 0$ for all k, giving the contradiction that

$$\infty > \|f\|^2_{H^1(\mathbb{R}^n)} > \sum_k \|f\|^2_{H^1(A_k)} > \sum_k \tilde{\epsilon} = +\infty.$$

So we will construct a subsequence $(r_{k_m})_m$ of $(r_k)_k$ and a sequence $(\delta_m)_m$ of positive numbers such that $r_{k_1} < r_{k_1} + \delta_1 < r_{k_2} < r_{k_2} + \delta_2 < r_{k_3} < \cdots$, and such that for the 'annuli' $A_m := \{\mathbf{x} : r_{k_m} < |\mathbf{x}| < r_{k_m} + \delta_m\}$, we have $\|f\|^2_{H^1(A_m)} > \widetilde{\epsilon}$. We will need to keep track of the constants in the trace theorems on our annuli A_m, and we will use the following [15, p. 41].) $\square$

Theorem 7.2. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ. Then, for $f \in H^1(\Omega)$ and for all $\epsilon \in (0,1)$,*

$$\|f\|^2_{L^2(\partial\Omega)} \leqslant \frac{\|\boldsymbol{\mu}\|_{C^1(\overline{\Omega})}}{\delta}\left(\epsilon^{1/2}\|\nabla f\|^2_{L^2(\Omega)} + (1+\epsilon^{-1/2})\|f\|^2_{L^2(\Omega)}\right),$$

where $\boldsymbol{\mu} \in C^1(\overline{\Omega}, \mathbb{R}^n)$ is such that $\boldsymbol{\mu}\cdot\mathbf{n} \geqslant \delta$ on $\partial\Omega$, and $\mathbf{n}$ is the outer normal vector.

If Ω is an annulus $A = \{\mathbf{x} : r < \|\mathbf{x}\| < R\}$ (which is clearly bounded, open, and also it has the Lipschitz boundaries which are the two spheres S_r and S_R), then with $\boldsymbol{\mu}(\mathbf{x}) = \mathbf{x}$, we have

$$\boldsymbol{\mu}\cdot\mathbf{n} = \|\mathbf{x}\| = \left\{\begin{matrix} R \text{ on } S_R, \\ r \text{ on } S_r \end{matrix}\right\} \geqslant r =: \delta.$$

Also, if we take $\epsilon = 1/4$, then

$$\|f\|^2_{L^2(S_r)} \leqslant \|f\|^2_{L^2(\partial A)} \leqslant 3\frac{\|\boldsymbol{\mu}\|_{C^1(\overline{A})}}{r}\|f\|^2_{H^1(A)}.$$

As $\|\boldsymbol{\mu}\|_{C^1(\overline{A})} = \max_{\overline{A}}\|\boldsymbol{\mu}\| + \max_{\overline{A}}|\nabla\cdot\boldsymbol{\mu}| = R + n$, we obtain

$$\|f\|^2_{L^2(S_r)} \leqslant 3\frac{R+n}{r}\|f\|^2_{H^1(A)}.$$

Now, we will construct $(r_{k_m})_m$ and $(\delta_m)_m$.

We choose k_1 such that $r_{k_1} > n$. Let δ_1 be such that $0 < \delta_1 < r_{k_1} - n$. Then, for the annulus $A_1 := \{\mathbf{x} : r_{k_1} < \|\mathbf{x}\| < r_{k_1} + \delta_1\}$, we have

$$\|f\|^2_{H^1(A_1)} \geqslant \frac{r_{k_1}/3}{(r_{k_1}+\delta_1)+n}\|f\|^2_{L^2(S_{r_{k_1}})} \geqslant \frac{1/3}{1+\frac{\delta_1+n}{r_{k_1}}}\epsilon > \frac{1/3}{1+1}\epsilon = \frac{\epsilon}{6} =: \widetilde{\epsilon}.$$

Now suppose $r_{k_1}, \ldots, r_{k_m}, \delta_1, \ldots, \delta_m$ possessing the desired properties have been constructed. Choose k_{m+1} such that $r_{k_{m+1}} > r_{k_m} + \delta_m$. Let δ_{m+1} be such that $0 < \delta_{m+1} < r_{k_{m+1}} - n$.

Then, for the annulus $A_{m+1} := \{\mathbf{x} : r_{k_{m+1}} < \|\mathbf{x}\| < r_{k_{m+1}} + \delta_{m+1}\}$, we have

$$\|f\|^2_{H^1(A_{m+1})} > \frac{r_{k_{m+1}}/3}{(r_{k_{m+1}}+\delta_{m+1})+n}\|f\|_{L^2(S_{r_{k_{m+1}}})} \geqslant \frac{1/3}{1+\frac{\delta_{m+1}+n}{r_{k_{m+1}}}}\epsilon > \frac{\epsilon}{6} = \widetilde{\epsilon}.$$

This completes the induction step.

So we have arrived at the contradiction that

$$+\infty > \|f\|^2_{H^1(\mathbb{R}^n)} \geqslant \sum_m \|f\|^2_{H^1(A_m)} \geqslant \sum_m \widetilde{\epsilon} = +\infty.$$

This shows that our original assumption was incorrect, and so

$$\lim_{r\to\infty}\int_{S_r}|f|^2\mathrm{d}\sigma_r=0,$$

completing the proof of our lemma. □

An analogous result also holds for the cylinder $\mathbb{R}\times S^{n-1}$. This was used in the proof of our Theorem 4.2.

Lemma 7.3. *If* $n\geqslant 3$ *and* $f,g\in H^1(\mathbb{R}\times S^{n-1})$*, then*

$$\lim_{t\to+\infty}\int_{S^{n-1}}fg\mathrm{d}\Omega=0=\lim_{t\to-\infty}\int_{S^{n-1}}fg\mathrm{d}\Omega.$$

Proof. (*Sketch*) The proof is based on the same idea as the above, but is somewhat simpler, since the radius of S^{n-1} does not change, and the constants one has in the trace theorem for a 'cylindrical band' of the form $(a,b)\times S^{n-1}$ already work, as opposed to having to keep careful track, via Theorem 7.2, of the constants in the earlier case when the radii of the S_r^{n-1} were changing. Proceeding in the same way as in the previous lemma, we assume that

$$\neg\left(\lim_{t\to+\infty}\int_{S^{n-1}}|f|^2\mathrm{d}\Omega=0\right),$$

and so there exists an $\epsilon>0$ and a sequence $(t_k)_{k\in\mathbb{N}}$ such that $\lim\limits_{k\to\infty}t_k=+\infty$, and

$$\lim_{k\to+\infty}\int_{S^{n-1}}|f(t_k,\cdot)|^2\mathrm{d}\Omega>\epsilon.$$

In order to fatten the 'circle' $\{t_k\}\times S^{n-1}$ to a cylindrical band of the form $I=(t_k,t_k+\delta)\times S^{n-1}$, while keeping the L^2-norm of f on the band uniformly (in k) bigger than a fixed positive quantity, one can use the inequality

$$\|f(t_k,\cdot)\|_{L^2(S^{n-1})}\leqslant C\|f\|_{H^1(I\times S^{n-1})}.$$

This follows from [25, Prop. 4.5, p. 287], by taking $\Omega=[t_k,t_k+\delta]\times S^{n-1}$. The rest of the proof is along the same lines. □

8. Appendix C: Sharpness of Bound When $|m|=\frac{n}{2}$ in Theorem 3.1

In this appendix, we will show the sharpness of the bound from Theorem 3.1 we had obtained for the decay of the solution to the Klein–Gordon equation in the de Sitter universe in flat FLRW form, when $|m|=\frac{n}{2}$. Let us recall this bound:

$$\forall t\geqslant t_0,\ \ \|\phi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}\lesssim a^{-\frac{n}{2}}\log a.$$

If $|m|=\frac{n}{2}$, then with $\psi:=a^{\frac{n}{2}}\phi$, we had seen that

$$\ddot{\psi}-\frac{1}{a^2}\Delta\psi=0.$$

We will now construct a solution ψ that satisfies

$$\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} \sim A + Bt \ \text{ as } t \to \infty,$$

showing that

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{R}^n)} \sim (A+Bt)a^{-\frac{n}{2}} \ \text{ as } t \to \infty,$$

and so the bound

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim (A+Bt)a^{-\frac{n}{2}} \ \text{ for all large } t$$

cannot be improved.

We want

$$\ddot{\psi} - \frac{1}{e^{2t}}\Delta\psi = 0. \tag{23}$$

Taking the Fourier transform with respect to only the (spatial) $\mathbf{x}$-variable, and denoting

$$\widehat{\psi}(t,\boldsymbol{\xi}) := \int_{\mathbb{R}^n} \psi(t,\mathbf{x}) e^{i\langle \boldsymbol{\xi},\mathbf{x}\rangle} d^n\mathbf{x},$$

(23) becomes

$$\frac{\partial^2}{\partial t^2}\widehat{\psi}(t,\boldsymbol{\xi}) + \frac{\|\boldsymbol{\xi}\|^2}{e^{2t}}\widehat{\psi}(t,\boldsymbol{\xi}) = 0, \tag{24}$$

which is a family of ordinary differential equations in t, parameterised by $\boldsymbol{\xi} \in \mathbb{R}^n$. For a fixed $\boldsymbol{\xi} \in \mathbb{R}^n$, the general solution to the ODE (24) is given by

$$\widehat{\psi}(t,\boldsymbol{\xi}) = C_1(\boldsymbol{\xi}) \cdot J_0(\|\boldsymbol{\xi}\|e^{-t}) + C_2(\boldsymbol{\xi}) \cdot Y_0(\|\boldsymbol{\xi}\|e^{-t}),$$

where

- J_0 is the Bessel function of first kind and of order 0, and
- Y_0 is the Bessel function of second kind and of order 0.

In order to construct our ψ, we will make special choices of C_1 and C_2.

We recall [1, (9.1.7–8)] that

$$J_0(z) \sim 1,$$
$$Y_0(z) \sim \frac{2}{\pi}\log z$$

as $z \searrow 0$ ($z \in \mathbb{R}$).

Now as $t \to \infty$, $e^{-t} \searrow 0$, and so from the above limiting behaviour of J_0 and Y_0, we obtain that as $t \to \infty$,

$$\begin{aligned}\widehat{\psi}(t,\boldsymbol{\xi}) &\sim C_1(\boldsymbol{\xi}) \cdot 1 + C_2(\boldsymbol{\xi}) \cdot \left(\frac{2}{\pi}\log(\|\boldsymbol{\xi}\|e^{-t})\right) \\ &= C_1(\boldsymbol{\xi}) + \frac{2}{\pi}C_2(\boldsymbol{\xi})\log\|\boldsymbol{\xi}\| - \frac{2}{\pi}t \cdot C_2(\boldsymbol{\xi}).\end{aligned}$$

By Plancherel's identity (see e.g. [25, Prop. 3.2]),

$$\|\widehat{\psi}(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)}.$$

Since we want the linear behaviour in t of $\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)}$, we keep C_2 nonzero, but may take $C_1 \equiv 0$. Then, as $t \to \infty$,

$$\widehat{\psi}(t,\boldsymbol{\xi}) = C_2(\boldsymbol{\xi}) \cdot Y_0(\|\boldsymbol{\xi}\| e^{-t}).$$

In order to have $\widehat{\psi}(t,\cdot)$ (and so also $\psi(t,\cdot)$) in $L^2(\mathbb{R}^n)$ for all t, we choose C_2 to have a sufficiently fast decay.

We recall [1, §9.2.2] that

$$Y_0(z) = \sqrt{\frac{2}{\pi z}} \left(\sin\left(z - \frac{\pi}{4}\right) + O\Big(\frac{1}{|z|}\Big)\right)$$

as $z \to \infty$ ($z \in \mathbb{R}$). So we have

$$Y_0(\|\boldsymbol{\xi}\| e^{-t}) = \sqrt{\frac{2}{\pi \|\boldsymbol{\xi}\| e^{-t}}} \left(\sin\left(\|\boldsymbol{\xi}\| e^{-t} - \frac{\pi}{4}\right) + O\Big(\frac{1}{\|\boldsymbol{\xi}\| e^{-t}}\Big)\right)$$

as $\|\boldsymbol{\xi}\| \to +\infty$ (and t is kept fixed). So to arrange $\widehat{\psi}(t,\cdot) \in L^2(\mathbb{R}^n)$ for all t, we may take

$$C_2(\boldsymbol{\xi}) := \frac{\|\boldsymbol{\xi}\|}{(\|\boldsymbol{\xi}\|^2+1)^{1+\frac{n}{4}}}.$$

(Also this choice makes

$$\boldsymbol{\xi} \mapsto C_2(\boldsymbol{\xi}) \log \|\boldsymbol{\xi}\| \in L^2(\mathbb{R}^n),$$

which will be needed below.)

Then, $\widehat{\psi}(t,\cdot) \in L^2(\mathbb{R}^n)$ for all t. Also, as $t \to \infty$,

$$\widehat{\psi}(t,\boldsymbol{\xi}) \sim \frac{2}{\pi} \Bigg(\underbrace{\frac{\|\boldsymbol{\xi}\|}{(\|\boldsymbol{\xi}\|^2+1)^{1+\frac{n}{4}}} \log \|\boldsymbol{\xi}\|}_{=:f \in L^2(\mathbb{R}^n)} - t \underbrace{\frac{\|\boldsymbol{\xi}\|}{(\|\boldsymbol{\xi}\|^2+1)^{1+\frac{n}{4}}}}_{=:g \in L^2(\mathbb{R}^n)} \Bigg),$$

and

$$\|\widehat{\psi}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \geqslant \frac{2}{\pi}\Big(t \underbrace{\|g\|_{L^2(\mathbb{R}^n)}}_{\neq 0} - \|f\|_{L^2(\mathbb{R}^n)} \Big) \geqslant 0$$

for large t.

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Communicated by Mihalis Dafermos.

Received: August 12, 2020.

Accepted: November 10, 2021.