The Steinhaus-Weil property:
IV. Other interior-point properties
by

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In memory of Harry I. Miller (1939-2018)

Abstract. In this the final part of the four-part series [BinO3; 4,5,6] on theorems of Steinhaus-Weil type, as a companion piece to Part III, [BinO6], on Weil-type refinement topologies, we study from the perspective of refinement topologies the relation of the composite AB^{-1} to the classical setting of the AA^{-1} Steinhaus interior-point theorem.

Keywords. Steinhaus-Weil property, Cameron-Martin space, Borell's theorem, refinement topology, Haar density topology, quasi-invariance.
Classification: Primary 22A10, 43A05; Secondary 28C10.

1 The Steinhaus property AA^{-1} versus the Steinhaus property AB^{-1}

We clarify below the relation between two versions of the Steinhaus interior points property: the simple (sometimes called 'classical') version concerning sets AA^{-1} and the composite, more embracing one, concerning sets AB^{-1}, for sets from a given family $\mathcal{H}$. The latter is connected to a strong form of metric transitivity: Kominek [Kom] shows, for a general separable Baire topological group G equipped with an inner-regular measure μ defined on some σ-algebra $\mathcal{M}$, that AB^{-1} has non-empty interior for all $A, B \in \mathcal{M}_+(\mu)$, the sets in $\mathcal{M}$ of positive μ-measure, iff for each countable dense set D and each $E \in \mathcal{M}_+(\mu)$ the set $X \backslash DE \in \mathcal{M}_0(\mu)$, the sets in $\mathcal{M}$ of μ-measure zero; this is recalled in Theorem K below. The composite property is thus related to the Smital property, for which see [BarFN]. Care is required when moving to the alternative property for AB, since the family $\mathcal{H}$ need not be preserved under inversion.

In general the simple property does not imply the composite: Matoušková and Zelený [MatZ] show that in any non-locally compact abelian Polish group there are closed non-(left) Haar null sets A, B such that $A + B$ has empty interior. Jabłońska [Jab] has shown that likewise in any non-locally compact abelian Polish group there are closed non-Haar meager sets A, B such that $A + B$ has empty interior; see also [BanGJS]. Bartoszewicz and M. and T. Filipczak [BarFF, Ths. 1, 4] analyze the Bernoulli product measure on $\{0,1\}^{\mathbb{N}}$ with p the probability of the digit 1; see [BinO3, §8.15]. The product space may be regarded as comprising canonical binary digit expansions of the additive reals modulo 1 (in which case the measure is not invariant). Here the (Borel) set A of binary expansions with asymptotic frequency p of the digit 1 has $[0,1)$ as its difference set iff $\frac{1}{4} \leq p \leq \frac{3}{4}$; however $A+A$ has empty interior unless $p = \frac{1}{2}$ (the base 2 simple-normal-numbers case).

Below we identify some conditions on a family of sets A with the simple AA^{-1} property which do imply the AB^{-1} property. What follows is a generalization to a group context of relevant observations from [BinO3] from the classical context of $\mathbb{R}$.

The motivation for the definition below is that its subject, the space H, is a subgroup of a topological group G from which it inherits a (necessarily) translation-invariant (either-sidedly) topology τ. Various notions of 'density at a point' give rise to 'density topologies' [BinO1], which are translation-invariant since they may be obtained via translation to a fixed reference point: early examples, which originate in spirit with Denjoy as interpreted by Haupt and Pauc [HauP], were studied intensively in [GofW], [GofNN], soon followed by [Mar1,2] and [Mue]; more recent examples include [FilW] and others investigated by the Wilczyński school, cf. [Wil].

Proposition 1 below embraces as an immediate corollary the case $H = G$ with G locally compact and σ the Haar density topology (see [BinO2]). Proposition 2 proves that Proposition 1 applies also to the ideal topology (in the sense of [LukMZ]) generated from the ideal of Haar null sets of an abelian Polish group.

We recall that a group H carries a *left semi-topological* structure τ if the topology τ is left invariant [ArhT] ($hU \in \tau$ iff $U \in \tau$); the structure is *semi-topological* if it is also right invariant, i.e. briefly: τ is translation invariant. H is a *quasi-topological group* under τ if τ is both left and right invariant and inversion is τ-continuous.

Definition. For H a group with a translation-invariant topology τ, call a topology $\sigma \supseteq \tau$ a *Steinhaus refinement* if:
i) $\mathrm{int}_\tau(AA^{-1}) \neq \emptyset$ for each non-empty $A \in \sigma$, and
ii) σ is involutive-translation invariant: $hA^{-1} \in \sigma$ for all $A \in \sigma$ and all $h \in H$.

Property (ii) above (called simply 'invariance' in [BarFN]) apparently calls for only left invariance, but in fact, via double inversion, delivers translation invariance, since $Uh = (h^{-1}U^{-1})^{-1}$; then H under σ is a semi-topological group with a continuous inverse, so a *quasi-topological group*. We address the step from the simple property to the composite in

Proposition 1. *If τ is translation-invariant, and $\sigma \supseteq \tau$ is a Steinhaus refinement topology, then $\mathrm{int}_\tau(AB^{-1}) \neq \emptyset$ for non-empty $A, B \in \sigma$. In particular, as σ is preserved under inversion, also $\mathrm{int}_\tau(AB) \neq \emptyset$ for $A, B \in \sigma$.*

Proof. Suppose $A, B \in \sigma$ are non-empty; as $B^{-1} \in \sigma$, choose $a \in A$ and $b \in B$; then by (ii)

$$1_H \in C := a^{-1}A \cap b^{-1}B^{-1} \in \sigma.$$

By (i), for some non-empty $W \in \tau$,

$$W \subseteq CC^{-1} = (a^{-1}A \cap b^{-1}B) \cdot (A^{-1}a \cap B^{-1}b) \subseteq (a^{-1}A) \cdot (B^{-1}b).$$

As τ is translation invariant, $aWb^{-1} \in \tau$ and

$$aWb^{-1} \subseteq AB^{-1},$$

the latter since for each $w \in W$ there are $x \in A, y \in B^{-1}$ with

$$w = a^{-1}x.yb : \qquad awb^{-1} = xy \in AB^{-1}.$$

So $\mathrm{int}_\tau(AB^{-1}) \neq \emptyset$. □

Corollary 1. *In a locally compact group the Haar density topology is a Steinhaus refinement.*

Proof. Property (i) follows from Weil's theorem since density-open sets are non-null measurable; left translation invariance in (ii) follows from left invariance of Haar measure, while involutive invariance holds, as any measurable set of positive Haar measure has non-null inverse ([HewR, 15.14], cf. Part II [BinO5, §2 Lemma H]). □

A weaker version, inspired by metric transitivity, comes from applying the following concept.

Definition. Say that a group H *acts transitively* on a family $\mathcal{H} \subseteq \wp(G)$ if for each $A, B \in \mathcal{H}$ there is $h \in H$ with $A \cap hB \in \mathcal{H}$.

Thus a locally compact topological group acts transitively on its non-null Haar measurable subsets (in fact, either-sidedly); this follows from Fubini's Theorem [Hal, 36C], via the average theorem [Hal, 59.F]:

$$\int_G |g^{-1}A \cap B| dg = |A| \cdot |B^{-1}| \qquad (A, B \in \mathcal{M}),$$

$(g = ab^{-1}$ iff $g^{-1}a = b)$ – cf. [TomW, §11.3 after Th. 11.17].

[MatZ] show that in any non-locally compact abelian Polish group G there exist two non-Haar null sets, $A, B \notin \mathcal{HN}$, such that $A \cap hB \in \mathcal{HN}$ for all h; that is, G there does *not* act transitively on the non-Haar null sets.

Definition (cf. [BarFN]). In a quasi-topological group (H, τ) say that a proper σ-ideal $\mathcal{H}$ has the *Simple Steinhaus Property* AA^{-1} if AA^{-1} has interior points for universally measurable subsets $A \notin \mathcal{H}$.

Proposition 1′ (cf. [Kha, Th. 1])**.** *In a group (H, τ) with τ translation-invariant, if H acts transitively on a family of subsets $\mathcal{H}$ with the simple Steinhaus property, then $\mathcal{H}$ has the (composite) Steinhaus property:*

$$\text{int}_\tau(AB^{-1}) \neq \emptyset \textit{ for } A, B \in \mathcal{H}.$$

Furthermore, if $\mathcal{H}$ is preserved under inversion, then also

$$\text{int}_\tau(AB) \neq \emptyset \textit{ for } A, B \in \mathcal{H}.$$

Proof. For $A, B \in \mathcal{H}$ choose h with $C := A \cap hB \in \mathcal{H}$; then

$$CC^{-1}h = (A \cap hB)(A^{-1} \cap B^{-1}h^{-1}) \subseteq AB^{-1}. \qquad \square$$

Proposition 2. *If (H, τ) is a quasi-topological group (i.e. τ is invariant with continuous inversion) carrying a left invariant σ-ideal $\mathcal{H}$ with the Steinhaus property and $\tau \cap \mathcal{H} = \{\emptyset\}$, then the ideal-topology σ with basis*

$$\mathcal{B} := \{U \backslash N : U \in \tau, N \in \mathcal{H}\}$$

is a Steinhaus refinement of τ*.*

In particular, for (H,τ) *an abelian Polish group, the ideal topology generated by its* σ*-ideal of Haar null subsets is a Steinhaus refinement.*

Proof. If $U,V\in\mathcal{B}$ and $w\in U\cap V$, choose $M,N\in\mathcal{H}$ and $W_M,W_N\in\tau$ such that $x\in(W_M\backslash M)\subseteq U$ and $x\in(W_N\backslash N)\subseteq V$. Then as $M\cup N\in\mathcal{H}$,

$$x\in(W_M\cap W_N)\backslash(M\cup N)\in\mathcal{B}.$$

So $\mathcal{B}$ generates a topology σ refining τ. With the same notation, $hU=hW_M\backslash hM\in\sigma$, as $hM\in H$, and $U^{-1}=W_M^{-1}\backslash M^{-1}$. Finally, UU^{-1} has non-empty τ-interior, as $U\notin\mathcal{H}$ and is non-empty.

As for the final assertion concerned with an *abelian* Polish group context, note that if N is Haar null ($N\in\mathcal{HN}$), then $\mu(hN)=0$ for some $\mu\in\mathcal{P}(G)$ and all $h\in H$, so $hN\in\mathcal{HN}$ for all $h\in H$. Furthermore, if $A\notin\mathcal{HN}$, then $A^{-1}\notin\mathcal{HN}$: for otherwise, $\mu(hA^{-1})=0$ for some $\mu\in\mathcal{P}(G)$ and all $h\in H$; then, taking $\tilde{\mu}(B)=\mu(B^{-1})$ for Borel B, we have $\tilde{\mu}(A)=0$ and $\tilde{\mu}(hA)=\mu(A^{-1}h^{-1})=0$ for all $h\in H$, a contradiction. □

Remark. A left Haar null set need not be right Haar null: for one example see [ShiT], and for more general non-coincidence see Solecki [Sol1, Cor. 6]. So the argument in Prop. 2 does not extend to the family of left Haar null sets $\mathcal{HN}$ of a *non-commutative* Polish group. Indeed, Solecki [Sol2, Th. 1.4] shows in the context of a countable product of countable groups that the simpler Steinhaus property holds for $\mathcal{HN}_{\text{amb}}$ (involving simultaneous left- and right-sided translation – see Part III §1) iff $\mathcal{HN}_{\text{amb}}=\mathcal{HN}$.

Next, we reproduce a result from [Kom]. Recall that μ is *quasi-invariant* if μ-nullity is translation invariant. The transitivity assumption (of co-nullity) is motivated by *Smítal's lemma*, which refers to a countable dense set – see [KucS].

Theorem K ([Kom, Th. 5]**).** *If* $\mu\in\mathcal{P}(G)$ *is quasi-invariant and there exists a countable subset* $H\subseteq G$ *with* HM *co-null for all* $M\in\mathcal{M}_+(\mu)$*, then* $\text{int}(AB^{-1})\neq\emptyset$ *for all* $A,B\in\mathcal{M}_+(\mu)$*.*

Proof. By regularity, we may assume $A,B\in\mathcal{M}_+(\mu)$ are compact, so AB^{-1} is compact. Fix $g\in G$; then by quasi-invariance $\mu(gB)>0$. So by the transitivity assumption, both $G\backslash HgB$ and $G\backslash HA$ are null, and so

$HA \cap HgB \neq \emptyset$. Say $h_1 a = h_2 g b$, for some $a \in A, b \in B, h_1, h_2 \in H$; then $g = h_2^{-1} h_1 a b^{-1}$. As g was arbitrary,

$$G = \bigcup_{h \in H} h_2^{-1} h_1 A B^{-1}.$$

By Baire's Theorem, as H is countable, $\mathrm{int}(AB^{-1}) \neq \emptyset$. □

2 Borell's interior-point property

For completeness of this overview of the Steinhaus-Weil interior-point property, we offer in brief here the context and statement of a (by now) classical Steinhaus-like result in probability theory; this differs in that the Polish group now specializes to an infinite-dimensional topological vector space and the reference measure is Gaussian, so no longer invariant. We refer to the related paper [BinO2] for further details and background literature, and to our generalizations to Polish groups and other reference measures.

For X a locally convex topological vector space, γ a probability measure on the σ-algebra of the cylinder sets generated by the dual space X^* (equivalently, for X separable Fréchet, e.g. separable Banach, the Borel sets), with $X^* \subseteq L^2(\gamma)$: then γ is called *Gaussian* on X ('gamma for Gaussian', following [Bog]) iff $\gamma \circ \ell^{-1}$ defined by

$$\gamma \circ \ell^{-1}(B) = \gamma(\ell^{-1}(B)) \qquad (\text{Borel } B \subseteq \mathbb{R})$$

is Gaussian (normal) on $\mathbb{R}$ for every $\ell \in X^* \subseteq L^2(\gamma)$. For a monograph treatment of Gaussianity in a Hilbert-space setting, see Janson [Jan]. Write $\gamma_h(K) := \gamma(K+h)$ for the translate by h. *Relative quasi-invariance* of γ_h and γ, that for all compact K

$$\gamma_h(K) > 0 \text{ iff } \gamma(K) > 0,$$

holds relative to a set of vectors $h \in X$ (the *admissible directions*) forming a vector subspace known as the *Cameron-Martin space,* $H(\gamma)$. Then γ_h and γ are equivalent, $\gamma \sim \gamma_h$, iff $h \in H(\gamma)$. Indeed, if $\gamma \sim \gamma_h$ fails, then the two measures are mutually singular, $\gamma_h \perp \gamma$ (the Hajek-Feldman Theorem – cf. [Bog, Th. 2.4.5, 2.7.2]).

Continuing with the assumption above on X^*, as $X \subseteq X^{**} \subseteq L^2(\gamma)$, one can equip $H = H(\gamma)$ with a norm derived from that on $L^2(\gamma)$. In brief, this is done with reference to a natural covariance under γ obtained by regarding $f \in X^*$ as a random variable and working with its zero-mean version $f - \gamma(f)$; then, for $h \in H$, δ_h^γ, the (shifted) evaluation map defined by $\delta_h^\gamma(f) := f(h) - \gamma(f)$ for $f \in X^*$, is represented as $\langle f - \gamma(f), \hat{h} \rangle_{L^2(\gamma)}$ for some $\hat{h} \in L^2(\gamma)$. (Here for γ symmetric $\gamma(f) = 0$, so $\delta_h^\gamma = \delta_h$ is the Dirac measure at h.) This is followed by identifying h with $\hat{h}$ (for $h \in H$), and $|h|_H := ||\hat{h}||_{L^2(\gamma)}$ is a norm on H arising from the inner product

$$(h, k)_H := \int_X \hat{h}(x)\hat{k}(x) d\gamma(x).$$

Formally, the construction first requires an extension of the domain of δ_h^γ to X_γ^*, the closed span of $\{x^* - \gamma(x^*) : x^* \in X^*\}$ in $L^2(\gamma)$, a Hilbert subspace in which to apply the Riesz Representation Theorem.

We may now state the Steinhaus-like property due, essentially in this form, to Christer Borell. ([LeP, Prop. 1] offers a weaker, 'one-dimensional-section' form with the origin an interior point of the difference set relative to each line of H passing through it; we may call it the H-radial form by analogy with the $\mathbb{Q}$-radial form [Kuc, §10.1] of Euclidean spaces: the rational points are indeed an additive subgroup. The alternative term 'algebraic interior point' is also in use, e.g. in the literature of functional equations – cf. [Brz].)

Theorem B (Borell's Interior-point Theorem, [Bor, Cor. 4.1] – see [Bog, p. 64]). *For γ a Gaussian measure on a locally convex topological space X with $X^* \subseteq L^2(\gamma)$, and A any non-null γ-measurable subset A of X, the difference set $A - A$ contains a $|.|_H$-open nhd (neighbourhood) of 0 in the Cameron Martin space $H = H(\gamma)$, i.e. $(A - A) \cap H$ contains a H-open nhd of 0.*

This follows from the continuity in h of the density of γ_h wrt γ ([Bog, Cor. 2.4.3]), as given in the *Cameron-Martin-Girsanov formula*:

$$\exp\left(\hat{h}(x) - \frac{1}{2}||\hat{h}||_{L^2(\gamma)}^2\right) \tag{CM}$$

(where $\hat{h}$ 'Riesz-represents' h, i.e. $x^*(h) = \langle x^*, \hat{h} \rangle$, for $x^* \in X^*$, as above). Thus here a modified Steinhaus Theorem holds: the *relative-interior-point theorem*.

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