The Steinhaus-Weil property:
III. Weil Topologies
by
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In memory of Harry I. Miller (1939-2018)

Abstract. We study Weil topologies, linking the topological-group structure with the measure-theoretic structure. This paper is a companion piece to Parts I, II, IV [BinO7,8,9] on theorems of Steinhaus-Weil type. (See [BinO6] for the fuller arXiv version combining all four.)

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1 Weil-like topologies: preliminaries

We are concerned with relatives of the *Weil topology* as generators of the Steinhaus-Weil interior-point property [Ste]. For background, we refer to Weil's book [Wei, Ch. VII] and Halmos's book [Hal, Ch. XII] (see also [BinO6, §8.4]). Weil regarded his result as a *Converse Haar Theorem,* in retrieving the topological-group structure from the measure-algebra structure [Fre] as encoded by the Haar-measurable subsets – cf. [Kod]. (Here one may work either, following Weil, to within a dense embedding in a locally compact group, as in the Remark to Theorem 1M below, or, following Mackey, uniquely up to homeomorphism, granted the further assumption of an analytic Borel structure [Mac, Th. 7.1]; for further information see [BinO6, §8.16].) The alternative view below throws light on this result in that the measure structure is already encoded by the density topology $\mathcal{D}$ via the Haar density theorem, for which see [Mue], [Hal, §61(5), p. 268], cf. [BinO1, §7; Th. 6.10], [BinO3]. This view is partially implicit in [Amb]: writing $\mathcal{M}_+(\mu)$ for the μ-measurable sets of positive μ-measure, refinement of one invariant measure μ_1 by another μ_2 holds when sets in $\mathcal{M}_+(\mu_2)$, contain sets in $\mathcal{M}_+(\mu_1)$ (as in the refinement of one topology by another). This falls within the broader aim of retrieving a *topological* group structure from a given (one-sidedly) invariant topology τ on a group G, when τ arises from refinement of a topological group structure (i.e. starting from a *semitopological* group structure (G,τ)). Also relevant here are *Converse Steinhaus-Weil* results, as in Part II Prop. 1 of [BinO6,§3], [BinO8, §2] (see also [BinO6, §8.5]). For background on group-norms see the textbook treatment in [ArhT, §3.3] (who trace this notion back to Markov) or [BinO1], but note their use of 'pre-norm' for what we call (following Pettis [Pet]) a *pseudo-norm*; for quasi-interiors and regular open sets see [BinO6, §8.6]. Thus a norm $||\cdot||: G \to [0,\infty)$ satisfies all the three conditions 1-3 below and generates a right-invariant metric $d(x,y) = ||xy^{-1}||$ and so a topology $\mathcal{T} = \mathcal{T}_d$, just as a right-invariant metric d derives from a separable topology $\mathcal{T}_G$ and generates, via the Birkhoff-Kakutani Theorem ([HewR, Th. 8.3], [Gao, Th. 2.1.1]), the norm $||x|| = d(x, 1_G)$. A pseudo-norm differs in possibly lacking condition 1.i. (so generates a *pseudo*-metric).

1.i (positivity): $||g|| > 0$ for $g \neq 1_G$, and 1.ii: $||1_G|| = 0$;
2 (subadditivity): $||gh|| \leq ||g|| + ||h||$,
3 (symmetry): $||g^{-1}|| = ||g||$.

With $\mathcal{U}(G)$ the universally measurable subsets of G, recall from the Introduction of Part I [BinO6,7] that $\lambda \in \mathcal{M}_{\mathrm{sub}}$ if λ is a set function λ defined on $\mathcal{U}(G)$ and is a *submeasure,* i.e. is monotone and subadditive with $\lambda(\emptyset) = 0$ (Introduction, [Fre, Ch. 39, §392], [Tal]); by analogy with the term *finitely additive measure* (for background see [Bin], [TomW, Ch. 12]; cf. [Pat]), this is a *finitely subadditive outer measure*, similarly as in Maharam [Mah], albeit in the context of Boolean algebras, but without her positivity condition. Recall from Halmos [Hal, Ch. II §10] that a submeasure is an *outer measure* if in addition it is *countably subadditive*. The set function λ is *left invariant* if $\lambda(gE) = \lambda(E)$ for all $g \in G$ and $E \in \mathcal{U}(G)$.

Propositions 1 and 2 below are motivated by [Hal, Ch. XII §62, cf. Ch. II §9 (2-4)], where G is a locally compact group with λ its left Haar measure, but here the context is broader, allowing in *amenable* groups G (cf. [TomW, Ch. 12], [Pat]). The two results enable the introduction in §2 of Weil-like topologies generated from families of left-invariant *pseudo-metrics* derived from invariant submeasures. The latter rely on the natural *measure-metric*, also known as the *Fréchet-Nikodym metric* ([Fre, §323Ad], [Hal, §40 Th. A], [Bog, p. 53, 102-3, 418]); see [Dre1,2] (cf. [Web]) for the related literature of Fréchet-Nikodym topologies and their relation to the Vitali-Hahn-Saks Theorem. Maharam [Mah] studies sequential continuity of the order relation (of inclusion, here in the measure algebra), and requires positivity to obtain a (measure-) *metric*; see Talagrand [Tal] (cf. [Fre, §394] and the literature cited there) for a discussion of pathological submeasures (the only measures they dominates under $\ll$ being trivial), and [ChrH] for corresponding exotic abelian Polish groups.

In the setting of a locally compact group G, these pseudo-metrics are implicit in work of Struble: initially, in 1953 [Str1], he used a ('sampler') family of pre-compact open sets $\{E_t : t > 0\}$ to construct a mean on G, thereby refering to a one-parameter family of pseudo-metrics corresponding to the sets E_t; some twenty years later in 1974 [Str2] (cf. [DieS, Ch. 8]) identifies a left-invariant (proper) metric on G by taking the supremum of pseudo-metrics, each generated from some open set in a countable open base at 1_G. The pseudo-metric makes a very brief appearance in Yamasaki's textbook treatment [Yam, Ch. 1] of Weil's theorem.

Proposition 1 (**Weil pseudo-norm**, cf. [Fre, §392H], [Yam, Ch. 1, Proof of Th. 4.1]). *For G a Polish group, $\lambda \in \mathcal{M}_{\mathrm{sub}}(G)$, a left-invariant submeasure on $\mathcal{U}(G)$, and $E \in \mathcal{U}(G)$ with $\lambda(E) > 0$, put*

$$||g||_E^\lambda := \lambda(gE \triangle E) \qquad (g \in G).$$

Then $||.||_E$ defines a group pseudo-norm with associated right-invariant pseudo-metric

$$d_E^\lambda(g,h) = ||gh^{-1}||_E^\lambda \qquad (g, h \in G).$$

Likewise, for λ right-invariant, a pseudo-norm is defined by

$$||g||_E^\lambda := \lambda(E \triangle Eg) \qquad (g \in G).$$

Proof. Since $\lambda(\emptyset) = 0$, $||1_G||_E^\lambda = 0$. By left invariance under a,

$$||a^{-1}||_E^\lambda = \lambda(a^{-1}E \triangle E) = \lambda(a(a^{-1}E \triangle E)) = \lambda(E \triangle aE) = ||a||_E^\lambda.$$

Also,

$$||ab||_E^\lambda \le ||a||_E^\lambda + ||b||_E^\lambda$$

follows from monotonicity, subadditivity and $\lambda(abE \triangle aE) = \lambda(bE \triangle E)$:

$$\begin{aligned}\lambda(abE\backslash E \cup E\backslash abE) \le & \lambda(abE\backslash aE) \cup (aE\backslash E) \cup (E\backslash aE) \cup (aE\backslash abE)) \\ = & \lambda(abE\backslash aE) \cup (aE\backslash abE) \cup (aE\backslash E) \cup (E\backslash aE)) \\ \le & \lambda(abE \triangle aE) + \lambda(E \triangle aE) = \lambda(bE \triangle E) + \lambda(E \triangle aE).\end{aligned}$$

□

Corollary 1 (Kneser for Haar measure, [Kne, Hilfs. 4]). *For G a Polish group, $\lambda \in \mathcal{M}_{\mathrm{sub}}(G)$, a left-invariant submeasure on $\mathcal{U}(G)$, and $E \in \mathcal{U}(G)$ with $\lambda(E) > 0$, the set*

$$H := \{g \in G : \lambda(gE \triangle E) = 0\}$$

is a subgroup of G closed under the norm $||g||_E^\lambda$.

Proof. Indeed $H = \{g \in G : ||g||_E^\lambda = 0\}$, and so H is a subgroup, since for $g, h \in H$, $||gh^{-1}||_E^\lambda \leq ||g||_E^\lambda + ||h||_E^\lambda = 0$. □

Recall now that a subset A of a Polish group G is *left Haar null* if it is contained in a universally measurable set B such that for some $\mu \in \mathcal{P}(G)$

$$\mu(gB) = 0 \qquad (g \in G).$$

It is *Haar null:* $A \in \mathcal{HN}_{\mathrm{amb}}$ [Sol1] (cf. [HofT, p. 374]), if it is contained in a universally measurable set B such that for some $\mu \in \mathcal{P}(G)$

$$\mu(gBh) = 0 \qquad (g, h \in G).$$

This motivates the following application of Proposition 1 beyond Haar measure. Extending the notation of [BinO6,§3], Part II §1, below $\mathcal{M}_0^L(G)$ (resp. $\mathcal{M}_0(G)$) denotes the family of left-Haar-null (resp. Haar-null) sets of G, and we write

$$\mathcal{U}_+^L(G) := \mathcal{U}(G) \backslash \mathcal{M}_0^L(G), \qquad \mathcal{U}_+(G) := \mathcal{U}(G) \backslash \mathcal{M}_0(G).$$

Prop. 1 may be applied to the following measures; those constructed from μ a normalized counting measure (of finite support) are studied in [Sol1].

Proposition 2. *In a Polish group G, for $\mu \in \mathcal{P}(G)$ put*

$$\begin{aligned} \mu_L^*(E) \ &: \ = \sup\{\mu(gE) : g \in G\} \qquad (E \in \mathcal{U}(G)), \\ \hat{\mu}(E) \ &: \ = \sup\{\mu(gEh) : g, h \in G\} \qquad (E \in \mathcal{U}(G)). \end{aligned}$$

Then μ_L^* (resp. $\hat{\mu}$) is a left invariant (resp. bi-invariant) submeasure on $\mathcal{U}(G)$, which is positive for $E \in \mathcal{U}_+^L(G)$ (resp. for $E \in \mathcal{U}_+(G)$), i.e. for universally measurable, non-left-Haar null (resp. non-Haar-null) sets.

Proof. We consider only $\hat{\mu}$, as the case μ_L^* is similar and simpler (through the omission of h and b below). The set function $\hat{\mu}$ is well defined, with

$$\mu(E) \leq \hat{\mu}(E) \leq 1 \qquad (E \in \mathcal{U}(G)),$$

since μ is a probability measure; it is bi-invariant, since

$$\hat{\mu}(aEb) := \sup\{\mu(gaEbh) : g, h \in G\} = \sup\{\mu(gEh) : g, h \in G\},$$

and G is a group. Furthermore, for $B \in \mathcal{U}(G)$

$$\mu(gBh) \leq \hat{\mu}(B) \leq 1, \qquad (g, h \in G).$$

So, for $\mu \in \mathcal{P}(G)$

$$0 < \hat{\mu}(B) \leq 1 \qquad (B \in \mathcal{U}_+(G)),$$

since there are $g, h \in G$ with $\mu(gBh) > 0$. Countable subadditivity follows (on taking suprema of the leftmost term over g, h) from

$$\mu(g(\bigcup\nolimits_n A_n)h) \leq \sum\nolimits_n \mu(gA_nh) \leq \sum\nolimits_n \hat{\mu}(gA_nh) = \sum\nolimits_n \hat{\mu}(A_n),$$

for any sequence of sets $A_n \in \mathcal{U}(G)$. $\square$

Definition. For $\mu \in \mathcal{P}(G), E \in \mathcal{U}(G)$, put

$$B_\varepsilon^E(\mu) := \{x \in G : ||x||_E^\mu < \varepsilon\}.$$

Our next step uses Prop. 2 to inscribe these balls into EE^{-1} for all small enough $\varepsilon > 0$.

Lemma 1 (**Self-intersection Lemma**). *In a Polish group G for $E \in \mathcal{U}_+(G)$, and respectively for $E \in \mathcal{U}_+^L(G)$, and $\mu \in \mathcal{P}(G)$,*

$$\begin{array}{ll} 1_G \in B_\varepsilon^E(\hat{\mu}) \subseteq EE^{-1} & (0 < \varepsilon < \hat{\mu}(E)), \\ 1_G \in B_\varepsilon^E(\mu_L^*) \subseteq EE^{-1} & (0 < \varepsilon < \mu_L^*(E)). \end{array}$$

Equivalently, for $0 < \varepsilon < \hat{\mu}(E)$, and respectively for $0 < \varepsilon < \mu_L^*(E)$,

$$E \cap xE \neq \emptyset \qquad (x \in B_\varepsilon^E(\hat{\mu})); \qquad E \cap xE \neq \emptyset \qquad (x \in B_\varepsilon^E(\mu_L^*)).$$

Proof. We check only the $\hat{\mu}$ case; the other is similar and simpler (through the omission of h below). For $E \in \mathcal{U}_+(G)$, since $\hat{\mu}(E) > 0$ by Prop. 2, we may pick $g,h\in G$ such that $\varepsilon_E:=\mu(gEh)>0$. Consider x and $\varepsilon>0$ with $||x||_E^{\hat{\mu}}<\varepsilon\leq\varepsilon_E$. If E and xE are disjoint, then

$$\begin{aligned}\varepsilon_E &= \mu(gEh)\leq\mu(g(E\cup xE)h)\leq\hat{\mu}(g(E\cup xE)h)=\hat{\mu}(E\cup xE)\\ &= \hat{\mu}(xE\triangle E)=||x||_E^{\hat{\mu}}<\varepsilon\leq\varepsilon_E,\end{aligned}$$

a contradiction. So E and xE do meet. Now first pick $t\in xE\cap E$ and next $s\in E$ so that $t=xs$; then $x=ts^{-1}\in EE^{-1}$. The argument is valid when $\varepsilon_E=\mu(gEh)$ assumes any value in $(0,\hat{\mu}(E)]$. The converse is clear. $\square$

We need a simple analogue of a result due to Weil ([Wei, Ch. VII, §31], cf. [Hal, Ch. XII §62]). Below τ_1 denotes the τ-open *neighbourhoods of* 1_G. For G locally compact with $\lambda=\eta_G$, the identity

$$2\eta(E)-2\eta(E\cap xE)=\eta(E\triangle xE)=1-2\int 1_E(t)1_{E^{-1}}(t^{-1}x)d\eta(t)\qquad(\dagger)$$

connects the continuity of the (pseudo-) norm to $\mathcal{T}_d$-continuity of translation in the topological group structure $(G,\mathcal{T}_d)$ of the locally compact group, and to continuity of the convolution function here (for E of finite η-measure) – see [HewR, Th. 20.16]; see also [HewR, Th. 20.17] for the well-known connection between the Steinhaus-Weil Theorem and convolution. Such continuity guarantees that $B_\varepsilon^E(\eta)$ contains points other than 1_G.

Lemma 2 (**Fragmentation Lemma**; cf. [Hal, Ch. XII §62 Th. A]). *For $\lambda\in\mathcal{M}_{\rm sub}(G)$ a left-invariant submeasure on $\mathcal{U}(G)$ in a Polish group G equipped with a finer right-invariant topology τ with 1_G-open-nhd family $\tau_1\subseteq\mathcal{U}_+^L(G)$:*

if the map

$$x\mapsto||x||_E^\lambda$$

is continuous under τ at $x=1_G$ for each $E\in\mathcal{U}_+^L(G)$
– then, for each $\emptyset\neq E,F\in\tau$ and $\varepsilon>0$ with $\varepsilon<\lambda(E)$, there exists $H\in\tau_1$ with $HH^{-1}\subseteq FF^{-1}$ and

$$||h'h^{-1}||_E^\lambda<\varepsilon\qquad(h,h'\in H):\qquad HH^{-1}\subseteq B_\varepsilon^E,$$

so that $\mathrm{diam}_E^\lambda(H)\leq\varepsilon$.

Proof. Pick any $f\in F$, and $D\in\tau_1$ satisfying $||x||_E^\lambda<\varepsilon/2$ for all $x\in D$. As τ is right-invariant and $1_G\in D\cap Ff^{-1}\in\tau$, pick $H\in\tau_1$ with $H\subseteq D\cap Ff^{-1}$; then

$$HH^{-1}=Hff^{-1}H^{-1}\subseteq FF^{-1}.$$

For $h,h' \in H$, as $h,h' \in D$

$$||h'f(hf)^{-1}||_E^\lambda = ||h'h^{-1}||_E^\lambda \leq ||h'||_E^\lambda + ||h^{-1}||_E^\lambda = ||h'||_E^\lambda + ||h||_E^\lambda < \varepsilon. \quad \square$$

In the presence of a refinement topology τ on the group G, the lemma motivates further notation: write $\mathcal{P}_{\mathrm{cont}}(G,\tau)$, or just

$$\mathcal{P}(\tau) := \{\mu \in \mathcal{P}(G, \mathcal{T}_d) : g \mapsto ||g||_E^{\hat{\mu}} := \hat{\mu}(gE \triangle E) \text{ is } \tau\text{-continuous at } 1_G\}.$$

Of necessity attention here focuses on continuity. The characterization question as to which topologies τ yield a non-empty $\mathcal{P}(\tau)$ is in part answered by Theorem 1M below. Indeed, for Haar measure η in the locally compact case,

$$\mu \in \mathcal{P}(\tau) \qquad (\mu \ll \eta, \tau \supseteq \mathcal{T}_d),$$

by (†) in the presence of $d\mu/d\eta$ as a kernel:

$$||x||_E^\mu = 1 - 2\int 1_E(t) 1_{E^{-1}}(t^{-1}x)\frac{d\mu}{d\eta} d\eta(t). \tag{††}$$

However, $\mathcal{P}(G)$ will contain measures μ singular with respect to η : for such μ, by the Simmons-Mospan Theorem [BinO6,8, Th. SM] there will be Borel subsets B of positive μ-measure such that BB^{-1} has void $\mathcal{T}_d$-interior.

2 Weil-like topologies: theorems

Prop. 2 now yields the following result, which embraces known Hashimoto topologies [BinO3] in both the Polish abelian setting, where the left Haar null sets form a σ-ideal (Christensen [Chr]), and likewise in (the not necessarily abelian) Polish groups that are *amenable at* 1 (Solecki [Sol1,2]); this includes, as additive groups, F- (hence also Banach) spaces – cf. [BinO3,4], where use is made of Hashimoto topologies.

Theorem 1. *Let G be a Polish group and τ both a left- and a right-invariant refinement topology with 1_G-open-nhd family $\tau_1 \subseteq \mathcal{U}_+(G)$.*
Then both the families $\{AA^{-1} : A \in \tau_1\}$ and $\{B_\varepsilon^E(\hat{\mu}) : \emptyset \neq E \in \tau, \mu \in \mathcal{P}(\tau)$ and $0 < \varepsilon \leq \hat{\mu}(E)\}$ generate neighbourhoods of the identity under which G is a topological group. Moreover, the pseudo-norms

$$\{||.||_E^{\hat{\mu}} : \emptyset \neq E \in \tau, \mu \in \mathcal{P}(\tau)\}$$

are downward directed by refinement as follows: for $\emptyset \neq E, F \in \tau_1$, $\lambda, \mu \in \mathcal{P}(\tau)$ *and* $\varepsilon < \min\{\hat{\lambda}(E), \hat{\mu}(F)\}$, *there is* $H \in \tau_1$ *such that for* $0 < \delta < \min\{\tilde{\lambda}(H), \hat{\mu}(H)\}$

$$B_\delta^H(\lambda) \cap B_\delta^H(\mu) \subseteq B_\varepsilon^E(\lambda) \cap B_\varepsilon^F(\mu).$$

Proof. The proof is similar to but simpler than that of [Hal, Ch. XII §62 Th. A]. With the notation of Prop. 2 for $\lambda, \mu \in \mathcal{P}(\tau)$, given two (non-left-Haar-null) sets $E, F \in \tau_1$ and $\varepsilon < \min\{\hat{\lambda}(E), \hat{\mu}(F)\}$, by the Fragmentation Lemma (Lemma 2 of §1) applied separately to $\hat{\lambda}$ and to $\hat{\mu}$, there are $A, B \in \tau_1$ with

$$AA^{-1} \subseteq B_\varepsilon^E(\hat{\lambda}), \quad BB^{-1} \subseteq B_\varepsilon^F(\hat{\mu}).$$

Take any $H \in \tau_1$ with $H \subseteq A \cap B$; then

$$HH^{-1} \subseteq AA^{-1} \cap BB^{-1}.$$

Since $H \in \mathcal{U}_+(G)$ (as $\tau_1 \subseteq \mathcal{U}_+(G)$), take δ with $0 < \delta < \min\{\hat{\lambda}(H), \hat{\mu}(H)\}$; then by $(*)$ of I, Lemma 1,

$$B_\delta^H(\hat{\lambda}) \cap B_\delta^H(\hat{\mu}) \subseteq HH^{-1} \subseteq AA^{-1} \cap BB^{-1} \subseteq B_\varepsilon^E(\hat{\lambda}) \cap B_\varepsilon^F(\hat{\mu}).$$

(So 'mutual refinement' holds between the sets of the form AA^{-1} and those of the form B_ε^E.) As $|| \cdot ||_E^{\hat{\mu}}$ is a pre-norm,

$$B_{\varepsilon/2}^E(\hat{\mu}) B_{\varepsilon/2}^E(\hat{\mu})^{-1} = B_{\varepsilon/2}^E(\hat{\mu}) B_{\varepsilon/2}^E(\hat{\mu}) \subseteq B_\varepsilon^E(\hat{\mu}).$$

By the Fragmentation Lemma again, given any $x \in G$ and $\varepsilon > 0$, choose $H \in \tau_1$ with $HH^{-1} \subseteq B_\varepsilon^E(\tilde{\mu})$. Then with $F := xH \in \tau$,

$$B_\varepsilon^F(\hat{\mu}) = \{z : ||z||_F^{\hat{\mu}} < \varepsilon\} \subseteq (xH)(xH)^{-1} = xHH^{-1}x^{-1} \subseteq xB_\varepsilon^E(\hat{\mu})x^{-1}.$$

Finally, for any x_0 with $||x_0||_E^{\hat{\mu}} < \varepsilon$, put $\delta := \varepsilon - ||x_0||_E^{\hat{\mu}}$. Then for $||y||_E^{\hat{\mu}} < \delta$,

$$||x_0 \cdot y||_E^{\hat{\mu}} \leq ||x_0||_E^{\hat{\mu}} + ||y||_E^{\hat{\mu}} < ||x_0||_E^{\hat{\mu}} + \varepsilon - ||x_0||_E^{\hat{\mu}} < \varepsilon,$$

i.e.

$$x_0 B_\delta^E(\hat{\mu}) \subseteq B_\varepsilon^E(\hat{\mu}). \qquad \square$$

Specializing to locally compact groups yields as a corollary, on writing $B_\varepsilon^E := B_\varepsilon^E(\eta)$:

Theorem 1M. *For G a locally compact group with left Haar measure η, if:*
(i) *τ is both a left- and a right-invariant refinement topology with $\tau_1 \subseteq \mathcal{M}_+$,*
(ii) *for every non-empty $E \in \tau$, the pseudo-norm*

$$g \mapsto ||g||_E := \eta(gE \triangle E) \qquad (g \in G)$$

is continuous under τ at $g = 1_G$
– then both the families $\{AA^{-1} : A \in \tau_1\}$ and $\{B_\varepsilon^E : \emptyset \neq E \in \tau$ and $0 < \varepsilon \leq 2\eta(E)\}$ generate neighbourhoods of the identity under which G is a topological group. Moreover, the pseudo-norms

$$\{||.||_E : \emptyset \neq E \in \tau\}$$

are downward directed by refinement; indeed, for $\emptyset \neq E, F \in \tau$ and $\varepsilon < 2\min\{\eta(E), \eta(F)\}$, there is $H \in \tau_1$ such that for $0 < \delta < \eta(H)$

$$B_\delta^H \subseteq B_\varepsilon^E \cap B_\varepsilon^F.$$

Proof. It is enough to replace $\mathcal{P}(G)$ by $\{\eta\}$ (so that λ and μ both refer to η), and to note that if xE and E are disjoint, then $\eta(xE \triangle E) = 2\eta(E)$, so that in Lemma 1 the bound $\eta^*(E)$ in the restriction governing inclusion may be replaced by $2\eta(E)$. $\square$

Remark. As in [Hal, Ch. XII §62 Th. F], but by the Fragmentation Lemma (and by the countable additivity of η), the Weil-like topology on a locally compact G in Theorem 1M is locally bounded (norm-totally-bounded in some ball). Then G with the Weil-like topology may be densely embedded in its completion $\hat{G}$, which is in turn locally compact, being locally complete and (totally) bounded. However, the corresponding argument in the case of the preceeding more general Theorem 1 fails, since $\hat{\mu}$ there is not necessarily countably additive.

Finally, we give a category version of Theorem 1M, as an easy corollary; indeed, our main task is merely to define what is meant by 'mutatis mutandis' in the present context. Denote by $\mathcal{B}_+(\tau)$ the non-meagre *Baire* sets (= with the Baire property, [Oxt2]) of a topology τ. Given the assumption $\tau_1 \subseteq \mathcal{B}_+$ below, we are entitled to refer to the usual quasi-interior of any $E \in \mathcal{B}_+$, denoted below by $\tilde{E}$, as in Part I Cor. 2′ [BinO6, Cor. 2′]; we also write $\tilde{B}_\varepsilon^E$ for $B_\varepsilon^{\tilde{E}}(\eta)$.

Theorem 1B. *For G a locally compact group with left Haar measure η, if:*
(i) *τ is both a left- and a right-invariant refinement topology with $\tau_1 \subseteq \mathcal{B}_+$ and with the left Nikodym property (preservation of category under left shifts),*
(ii) *for every non-empty $E \in \tau$ the pseudo-norm*

$$g \mapsto ||g||_{\tilde{E}} := \eta(g\tilde{E} \triangle \tilde{E}) \qquad (g \in G)$$

is continuous under τ at $g = 1_G$
– then both the families $\{AA^{-1} : A \in \tau_1\}$ and $\{\tilde{B}^E_\varepsilon : \emptyset \neq E \in \tau$ and $0 < \varepsilon \leq 2\eta(\tilde{E})\}$ generate neighbourhoods of the identity under which G is a topological group. Moreover, the pseudo-norms

$$\{||.||_{\tilde{E}} : \emptyset \neq E \in \tau\}$$

are downward directed by refinement; indeed, for $\emptyset \neq E, F \in \tau$ and $\varepsilon < 2\min\{\eta(\tilde{E}), \eta(\tilde{F})\}$, there is $H \in \tau_1$ such that for $0 < \delta < 2\eta(\tilde{H})$

$$\tilde{B}^H_\delta \subseteq \tilde{B}^E_\varepsilon \cap \tilde{B}^F_\varepsilon.$$

Proof. In place of the inclusion of Lemma 1 we note a result stronger than that valid for $\tilde{E}$ (i.e. inclusion only in $\tilde{E}\tilde{E}^{-1}$): since meagreness is translation-invariant (the 'Nikodym property' of [BinO3]), $(xE)\tilde{} = x\tilde{E}$ for non-meagre Baire E, so $x\tilde{E} \cap \tilde{E} \neq \emptyset$ implies $xE \cap E \neq \emptyset$, and so again

$$\tilde{B}^E_\varepsilon = B^{\tilde{E}}_\varepsilon \subseteq EE^{-1};$$

here again in Lemma 1 the bound $\eta^*(E)$ in the restriction governing inclusion may be replaced by $2\eta(E)$. The proof of Theorem 1 may now be followed verbatim, but for the replacement of $\mathcal{P}(G)$ by $\{\eta\}$, using the stronger inclusion just observed, and of $B^{\cdot}_\varepsilon(\eta)$ by $\tilde{B}^{\cdot}_\varepsilon$. □

Remark. The last result follows more directly from Th. 1M in a context where there exists on G a *Marczewski measure* (see [TomW, Ch. 13, cf. Ch. 11]), i.e. a finitely additive invariant measure on $\mathcal{B}$ vanishing on bounded members of $\mathcal{B}_0$; this includes $\mathbb{R}, \mathbb{R}^2, \mathbb{S}^1$, albeit under AC [TomW, Cor. 13.3]; cf. [Myc], but not $\mathbb{R}^d$ for $d \geq 3$ [DouF].

With the groundwork of Part I [BinO6,7] on translation-continuity for *compacts* completed, we close by establishing the promised dichotomy associated with the map

$$x \mapsto ||x||^\mu_E = \mu(xE \triangle E),$$

for measurable E : the Fubini Null Theorem [BinO6,7, Th. FN (Part I §1)] creates a duality between the vanishing of the F-based pseudo-norm and a *dichotomy* for x-translates of E^{-1} in relation to F according as $x \in E$ or $x \notin E$, which are thus unable in each case to distinguish between the points of F. Below we write $\forall^{\mu}$ for the generalized quantifier "for μ-a.a.' (cf. [Kec, 8.J]).

Theorem 2 (Almost Inclusion-Exclusion). *For G a Polish group $\mu \in \mathcal{P}(G)$ and non-null μ-measurable E, F, the vanishing μ-a.e. on F of the E-norm under μ :*

$$||x||_F^{\mu} = \mu(xE \triangle E) = 0 \qquad (x \in F),$$

is equivalent to the following Almost Inclusion-Exclusion for translates of E^{-1}:
(i) Inclusion: *F is μ-almost covered by μ-almost every translate $xE^{-1}$for $x \in E$:*

$$\mu(F \backslash xE^{-1}) = 0 \qquad (\forall^{\mu} x \in E),$$

(ii) Exclusion: *F is μ-almost disjoint from μ-almost every translate $xE^{-1}$for $x \notin E$:*

$$\mu(F \cap xE^{-1}) = 0 \qquad (\forall^{\mu} x \notin E).$$

Proof. By the Fubini Null Theorem [BinO6,7, Th. FN (Part I §1)], applied to the set H of Part I Prop. 3 [BinO6, Prop. 3], i.e.

$$H := \bigcup\nolimits_{x \in F} \{x\} \times (xE \triangle E),$$

H has vertical sections H_x almost all μ-null iff μ-almost all of its horizontal sections H^y are μ-null. But, since $y \in xE$ iff $x \in yE^{-1}$, $H^y = F \backslash yE^{-1}$ for $y \in E$ and $H^y := F \cap yE^{-1}$ for $y \in G \backslash E$. □

Remark. If the inclusion side of the dichotomy of Th. 8 holds for all $x \in E$, then $F \subseteq EE^{-1}$. The converse direction may fail: consider $E = (1, 2) \subseteq \mathbb{R}$ and $F = (-1, 1)$, so that $E - E = F$, but no translate of $-E$ may cover F.

3 Complements

1. *Inclusion-Exclusion dichotomy.* Above we focus on inclusions amongst sets of the form EE^{-1}, for $E \in \mathcal{U}(G)$, the exception being the Inclusion-Exclusion of a set $F \in \mathcal{U}(G)$ by an E-, or non-E, x-translate of E^{-1} in

Theorem 2 (a dichotomy as between E and its complement). This places most of our study on one side of a related inclusion-exclusion dichotomy – for subsets $H,B\in\mathcal{U}(G)$ in a group G one has either inclusion , or 'near-disjointness':

$$HH^{-1}\subseteq BB^{-1},\qquad\text{or}\qquad HH^{-1}\cap BB^{-1}=\{1_G\}.$$

Inclusion may be equivalently re-phrased to the meeting of distinct pairs of H^{-1}-translates of B :

$$kB\cap k'B\neq\emptyset\qquad(k,k'\in H^{-1}),\tag{In}$$

whereas exclusion to their disjointness:

$$kB\cap k'B=\emptyset\qquad(\text{distinct }k,k'\in H^{-1}).\tag{Ex}$$

The duality of the relation of (Ex) to the results in Th. 2 is clarified by observing that $\mu(F\cap xE^{-1})=0$, for a.a. $x\in C$, is equivalent to $\mu(C\cap yE)=0$, for a.a. $y\in F$. Indeed,

$$0=\int\int 1_C(x)1_F(y)1_{xE^{-1}}(y)d(\mu\times\mu)=\int\int 1_F(y)1_C(x)1_{yE}(x)d(\mu\times\mu).$$

The condition (Ex) gives rise to $\mathcal{I}_0$, the σ-ideal introduced in Balcerzak et al. [BalRS], generated by Borel sets B having perfectly many disjoint translates, as in (Ex) above with H^{-1} a perfect compact set (i.e. compact and dense-in-itself); continuum-many disjoint translates of a compactum also emerge in a theorem of Ulam concerning a non-locally compact Polish group: see [Oxt1, Th. 1]. Such *perfect exclusions* offer a combinatorial tool, akin to *shift-compactness* (as in Part I Th. 3 or [BinO6, Th. 3], the latter requiring a subsequence embedding under translation of any null sequence into a non-negligible set – cf. [BinO1,2] [MilO], [BanJ]), and play a key role in the context of groups with *ample generics*; see for instance the small-index property of [HodHLS].

Solecki [Sol3] proves a 'Fubini for negligibles'-type theorem (cf. Theorem FN in Part I §1 or [BinO6, §1]): the non-negligible vertical sections (relative to a uniformly Steinhaus ideal) of a planar $\mathcal{I}_0$-negligible set form a horizontal $\mathcal{I}_0$-negligible set. The ideal $\mathcal{I}_0$ is of particular interest, as it violates the countable (anti)-chain condition, [BalRS].

2. *Regular open sets.* Recall that, in a topological space X, U is *regular open* if U =int(cl$U$), and that int(cl$U$) is itself regular open; for background see e.g. [GivH, Ch. 10]. For $\mathcal{D} = \mathcal{D}_{\mathcal{B}}$ the Baire-density topology of a normed topological group, let $\mathcal{D}_{\mathcal{B}}^{RO}$ denote the regular open sets. For $D \in \mathcal{D}_{\mathcal{B}}^{RO}$, put

$$N_D := \{t \in G : tD \cap D \neq \emptyset\} = DD^{-1}, \qquad \mathcal{N}_1 := \{N_D : 1_G \in D \in \mathcal{D}_{RO}\};$$

then $\mathcal{N}_1$ is a base at 1_G (since $1_G \in C \in \mathcal{D}_{RO}$ and $1_G \in D \in \mathcal{D}_{RO}$ yield $1_G \in C \cap D \in \mathcal{D}_{RO}$) comprising $\mathcal{T}$-neighbourhoods that are $\mathcal{D}_{\mathcal{B}}$-open (since $DD^{-1} = \bigcup\{Dd^{-1} : d \in D\}$). We raise the (metrizability) question, by analogy with the Weil topology of a measurable group (see §1 and §3.1 above): with $\mathcal{D}_{\mathcal{B}}$ above replaced by a general density topology $\mathcal{D}$ on a group G, when is the topology generated by $\mathcal{N}_1$ on G a norm topology? Some indications of an answer may be found in [ArhT, §3.3]. We note the following answer in the context of Theorem 1B; compare Struble's Theorem [Str2], or [DieS, Ch. 8]. If there exists a separating sequence D_n, i.e. such that for each $g \neq 1_G$ there is n with $||g||_{D_n} = 1$, then

$$||g|| := \sum\nolimits_n 2^{-n}||g||_{D_n}$$

is a norm, since it is separating and, by the Nikodym property, $(D \cap g^{-1}D) = g^{-1}(gD \cap D) \in \mathcal{B}_0$.
3. *The Effros Theorem* asserts that a transitive continuous action of a Polish group G on a space X of second category in itself is necessarily 'open', or more accurately is *microtransitive* (the (continuous) evaluation map $e_x : g \mapsto g(x)$ takes open neighbourhoods E of 1_G to open neighbourhoods that are the orbit sets $E(x)$ of x). It emerges that this assertion is very close to the shift-compactness property: see [Ost]. The Effros Theorem reduces to the Open Mapping Theorem when G, X are Banach spaces regarded as additive groups, and G acts on X by a linear surjection $L : G \to X$ via $g(x) = L(g) + x$. Indeed, here $e_0(E) = L(E)$. For a neat proof, choose an open neighbourhood U of 0 in G with $E \supseteq U - U$; then $L(U)$ is Baire (being analytic) and non-meagre (since $\{L(nU) : n \in \mathbb{N}\}$ covers X), and so $L(U) - L(U) \subseteq L(E)$ is an open neighbourhood of 0 in X.
4. *Beyond local compactness: Haar category-measure duality.* In the absence of Haar measure, the definition of left Haar null subsets of a topological group G requires $\mathcal{U}(G)$, the universally measurable sets – by dint of the role of the totality of (probability) measures on G. The natural dual of $\mathcal{U}(G)$ is the class $\mathcal{U}_{\mathcal{B}}(G)$ of *universally Baire sets*, defined for G with a Baire topology as those

sets B whose preimages $f^{-1}(B)$ are Baire in any compact Hausdorff space K for any continuous $f : K \to G$. Initially considered in [FenMW] for $G = \mathbb{R}$, these have attracted continued attention for their role in the investigation of axioms of determinacy and large cardinals – see especially [Woo], cf. [MarS] – and is a key notion in [BanJ].

Analogously to the left Haar null sets, define a *left Haar meagre* set as any set M coverable by a universally Baire set B for which there are a compact Hausdorff space K and a continuous $f : K \to G$ with $f^{-1}(gB)$ meagre in K for all $g \in G$. Here, as recently noted in [BanGJS, Prop. 5.1], K may be replaced by the Cantor space $2^{\mathbb{N}}$. These were introduced, in the abelian Polish group setting with K metrizable, by Darji [Dar], cf. [Jab], and shown there to form a σ-ideal of meagre sets (co-extensive with the meagre sets for G locally compact).

5. *Metrizability and Christensen's Theorem.* An analytic topological group is metrizable; so if also it is a Baire space, then it is a Polish group – [HofT, Th. 2.3.6].

6. *Metrizability of refinements.* Underlying the Disaggregation Theorem (Part II Th. 1) which refines the topology $\mathcal{T}_d$ of G there are refining metrics:

$$d_K(x,y) := d(x,y) + |\mu(Kx) - \mu(Ky)|$$

(for a family of sets $K \in \mathcal{K}_+(\mu)$ – cf. the Struble sampler of §1 above), reminiscent of Theorem 1 above.

7. *Quasi-invariance and the Mackey topology of analytic Borel groups.* We comment on the force of full quasi-invariance of a measure in connection with a *Steinhaus triple* (H,G,μ) [BinO5] with H and G completely metrizable. Both groups, being absolutely Borel, are analytic spaces. So both carry a 'standard' Borel structures with H a Borel substructure of G. Mackey [Mac] investigates such Borel groups, defining also a (Borel) measure μ to be 'standard' if it has a Borel support. It emerges that every σ-finite Borel measure in an analytic Borel space is standard [Mac, Th. 6.1]. Of interest to us is Mackey's notion of a 'measure class' C_μ, comprising all Borel measures ν with the same null sets as $\mu : \mathcal{M}_0(\nu) = \mathcal{M}_0(\mu)$. Such a measure class may be closed under translation, and may be right or left invariant; then their mutually common null sets are themselves invariant, and so may be viewed as witnessing quasi-invariance of the measure μ. Mackey shows that a Borel group with a one-sided invariant measure class has a both-sidedly invariant measure class [Mac, Lemma 7.2]; furthermore, if the class is countably generated, then the class contains a left-invariant and a right-invariant measure

[Mac, Lemma 7.3]. This enables Mackey to improve on Weil's theorem in showing that an analytic Borel group G with a one-sidedly invariant measure class, in particular one generated by a quasi-invariant measure, has a unique locally compact topology on G both yielding a topological group structure and generating the given Borel structure.

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