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Discounted optimal stopping zero-sum games in diffusion-type models with maxima and minima

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We present a closed-form solution to a discounted optimal stopping zero-sum game in a model based on a generalised geometric Brownian motion with coefficients depending on its running maximum and minimum processes. The optimal stopping times forming a Nash equilibrium are shown to be the first times at which the original process hits certain boundaries depending on the running values of the associated maximum and minimum processes. The proof is based on the reduction of the original game to the equivalent coupled free-boundary problem and the solution of the latter problem by means of the smooth-fit and normal-reflection conditions. We show that the optimal stopping boundaries are partially determined as either unique solutions of the appropriate system of arithmetic equations or unique solutions of the appropriate first-order nonlinear ordinary differential equations. The obtained results are related to the valuation of the perpetual lookback game options with floating strikes in the appropriate diffusion-type extension of the Black-Merton-Scholes model.

1. Formulation of the problem

For a precise formulation of the problem, let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a standard Brownian motion $B = (B_t)_{t \geq 0}$. Suppose that the process $X = (X_t)_{t \geq 0}$ is given by:

$$X_t = x \, \exp\left(\int_0^t \left(r - \delta(S_u, Q_u) - \frac{\sigma^2(S_u, Q_u)}{2} \right) du + \int_0^t \sigma(S_u, Q_u) \, dB_u \right) \tag{1.1}$$

so that it solves the stochastic differential equation:

$$dX_t = \big(r - \delta(S_t, Q_t) \big) \, X_t \, dt + \sigma(S_t, Q_t) \, X_t \, dB_t \quad (X_0 = x) \tag{1.2}$$

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where $r > 0$ is a given constant, $\delta(s,q) > 0$ and $\sigma(s,q) > 0$ are continuously differentiable bounded functions on $[0,\infty]^2$, and $x > 0$ is fixed. We further assume that the function $\delta(s,q)$ is increasing in the both variables s and q on $[0,\infty]^2$. Here, $S = (S_t)_{t \ge 0}$ and $Q = (Q_t)_{t \ge 0}$ are the associated with X *running maximum* and *minimum* processes defined by:

$$S_t = s \vee \max_{0 \le u \le t} X_u \quad \text{and} \quad Q_t = q \wedge \min_{0 \le u \le t} X_u \tag{1.3}$$

for arbitrary $0 < q \le x \le s$, respectively. Since the functions $\delta(s,q)$ and $\sigma(s,q)$ are assumed to be bounded on $[0,\infty]^2$, it follows from [48; Chapter IV, Theorem 4.8] that there exists a (pathwise) unique strong solution to the stochastic differential equation in (1.2). It can be assumed that the process X describes the price of a risky asset on a financial market, where r is the riskless interest rate, $\delta(s,q)$ is the dividend rate paid to the asset holders, and $\sigma(s,q)$ is the volatility rate.

The main aim of this paper is to present closed-form solutions to a discounted optimal stopping zero-sum game with the value:

$$V_* = \sup_{\tau} \inf_{\zeta} \mathbb{E}\big[e^{-r\tau}\, F(X_\tau, S_\tau)\, I(\tau < \zeta) + e^{-r\zeta}\, G(X_\zeta, S_\zeta, Q_\zeta)\, I(\zeta \le \tau)\big] \tag{1.4}$$

where we set:

$$F(x,s) = s - K\,x \quad \text{and} \quad G(x,s,q) = (s - K\,x) \vee (q - L\,x) \equiv \max\{s - K\,x, q - L\,x\} \tag{1.5}$$

for all $0 < q \le x \le s$ and some given constants $0 < L < K < L + 1$, while $I(\cdot)$ denotes the indicator function. Suppose that the supremum and infimum in (1.4) are taken over all stopping times τ and ζ of the process X, and the expectation there is taken with respect to the risk-neutral probability measure $\mathbb{P}$. In that case, the value of (1.4) can therefore be interpreted as the rational (or no-arbitrage) price of a perpetual lookback game (or Israeli) option with the floating strikes KX and LX in the diffusion-type extension of the Black-Merton-Scholes model considered in Gapeev and Rodosthenous [30]-[32]. Such game-type contingent claims were introduced by Kifer [41] and further studied by Kyprianou [45], Kühn and Kyprianou [44], Kallsen and Kühn [39], Baurdoux and Kyprianou [3]-[5], Gapeev and Kühn [25], Ekström and Villeneuve [15], Ekström and Peskir [14], Peskir [55]-[56], and Baurdoux, Kyprianou, and Pardo [6] among others. We also refer to Shiryaev [68; Chapter VIII; Section 2a], Peskir and Shiryaev [59; Chapter VII; Section 25], and Detemple [12] for extensive overviews of the solutions to the American option pricing problems as well as other related results on optimal stopping problems in financial mathematics.

Discounted optimal stopping problems for certain reward functionals depending on the running maxima and minima of continuous Markov (diffusion-type) processes were initiated by Shepp and Shiryaev [64] and further developed by Pedersen [51], Guo and Shepp [36], Peskir [53], Gapeev [19]-[20], Guo and Zervos [37], Peskir [57]-[58], Glover, Hulley, and Peskir [33], Gapeev and Rodosthenous [30]-[32], Kitapbayev [42], Rodosthenous and Zervos [63], Gapeev, Kort, and Lavrutich [23], Gapeev and Li [27]-[28], Gapeev and Al Motairi [29], Gapeev et al. [24], and Gapeev [22] among others. The main feature in the analysis of such optimal stopping problems was that the normal-reflection conditions hold for the value functions at the diagonal planes of the state spaces of the multi-dimensional continuous Markov processes having the initial processes and the running extrema as their components. It was shown, by using the established by Peskir [52] maximality principle for solutions of optimal stopping problems for maxima of the original diffusion processes, which is equivalent to the superharmonic characterisation of the value functions, that the optimal stopping boundaries for the original processes represent functions of the running values of the associated maxima processes and are characterised by the appropriate extremal solutions of certain first-order nonlinear ordinary differential equations. In this paper, we continue these developments and study the problem of (1.4) related to the pricing of the floating-strike lookback game options as the associated optimal stopping zero-sum game of (2.3) for a three-dimensional (continuous) Markov diffusion-type process which has the underlying risky asset price X as well as its running maximum S and minimum Q as their state space components.

Note that the resulting problems turn out to be necessarily three-dimensional in the sense that they cannot be reduced to optimal stopping problems for Markov processes of lower dimensions. It is shown that the optimal exercise times forming a Nash equilibrium are the first times at which the original process exits certain two-sided regions restricted by stochastic boundaries depending on the running values of the associated maximum and minimum processes. We apply the smooth-fit and normal-reflection conditions for the value functions to determine the optimal stopping boundaries as either unique solutions of the appropriate system of arithmetic equations or unique solutions of the appropriate first-order nonlinear ordinary differential equations. Optimal stopping problems with the one-sided continuation regions in similar models based on the original diffusion-type processes with coefficients depending on the running maximum and the running maximum drawdown were considered in Gapeev and Rodosthenous [30]-[32]. Other optimal stopping problems in models with spectrally negative Lévy processes and their running maxima were studied by Asmussen, Avram, and Pistorius [1], Avram, Kyprianou, and Pistorius [2], Ott [50], and Kyprianou and Ott [46] among others.

The dependence of the local drift and diffusion coefficients on the past dynamics of observable diffusion-type processes through certain processes playing the role of sufficient statistics is often used in financial practice and well studied in the related literature. For instance, an increase of the running maximum or decrease of the running minimum of a risky asset price normally causes a structural change in the local drift representing its expected return and dividend policy. It also triggers changes in the diffusion coefficient representing the volatility rate of an asset price with a higher impact under either a maximum increase or a minimum decrease rather than either a minimum increase or maximum decrease, respectively. Such sufficient statistics transparently exhibit the risk levels of the assets and therefore usually influence the decisions taken by the market participants. The demand for option pricing in models with stochastic interest rates and volatility initiated the development and subsequent calibration of these models, based on diffusion-type processes with tractable path-dependent coefficients, which were realised by Henry-Labordère [38] and Ren, Madan, and Qian [61] among others (see also [31] for further discussions on diffusion-type models for prices of financial assets with coefficients depending on the running maxima and minima as well as the maxima drawdowns and maxima drawups).

The paper is organised as follows. In Section 2, we formulate the optimal stopping zero-sum game for a necessarily three-dimensional continuous Markov process, which has the underlying asset price and the running values of its maximum and minimum as the state space components. The resulting optimal stopping game is reduced to the equivalent coupled free-boundary problem for the value function which satisfies the smooth-fit conditions at the stopping boundaries and the normal-reflection conditions at the edges of the state space of the three-dimensional process. In Section 3, we obtain closed-form expressions for the candidate value functions as well as derive the appropriate arithmetic equations and first-order nonlinear ordinary differential equations for the candidate stopping boundaries as solutions of the associated free-boundary problems. We specify the starting conditions for the solutions of the first-order nonlinear ordinary differential equations and provide a recursive algorithm to determine the value functions and the optimal stopping boundaries along with their intersection lines with the edges of the three-dimensional state space. In Section 4, by applying the change-of-variable formula with local time on surfaces from Peskir [54], it is verified that the resulting solution of the free-boundary problem provides the expressions for the value function and the optimal stopping boundaries for the underlying asset price process in the original problem. In Section 5, we give closed-form solutions to some auxiliary optimal stopping problems in the same model, which give the appropriate bounds for the value functions and optimal stopping boundaries for the original game. We apply the maximality principle from Peskir [52] to the framework of the considered three-dimensional optimal stopping problem to show that the optimal stopping boundaries provide the extremal solutions of the associated first-order nonlinear ordinary differential equations (see also [58] and [30]-[32] for other optimal stopping problems in other related three-dimensional models). The main results of the paper are stated in Theorems 4.1 and 5.1.

2. The optimal stopping game and free-boundary problem

In this section, we introduce the setting and notation of the three-dimensional optimal stopping zero-sum game associated with the value of (1.4), which is related to the pricing of the perpetual floating-strike lookback game options. We specify the structure of the optimal stopping times forming a Nash equilibrium and formulate the equivalent free-boundary problem.

2.1 The three-dimensional optimal stopping zero-sum game. Suppose that an investor writes a perpetual lookback game option and sells the contract to another investor at time 0. Then, the holder of the option can exercise the contract at some random time τ which they can choose by collecting the amount of the running maximum S and paying the floating strike KX to the writer, for some $K > 0$. At the same time, the writer of the option can either recall the contract at some random time ζ which they choose by paying the amount of the running minimum Q to the holder and collecting the floating strike LX, for some $L > 0$, when $Q - LX > S - KX$ holds, or agree with the holder on the payment of S and the collection of KX, otherwise. In this respect, the holder of the option looks for the exercise time τ_* maximising the expected total payoff received from the writer, while, at the same time, the writer of the contract looks for the recall time ζ_* minimising the expected total payoff sent to the holder. In other words, the perpetual lookback game option pricing problem seeks to determine the couple of stopping times τ_* and ζ_* of the process X being a *saddle point* for the total expected reward functional given by:

$$\mathbb{J}(\tau, \zeta) = \mathbb{E}\big[e^{-r\tau} F(X_\tau, S_\tau) \, I(\tau < \zeta) + e^{-r\zeta} G(X_\zeta, S_\zeta, Q_\zeta) \, I(\zeta \le \tau)\big] \tag{2.1}$$

with the functions $F(x,s)$ and $G(x,s,q)$ defined in (1.5), for some $0<L<K<L+1$ fixed, which means that the inequalities:

$$\mathbb{J}(\tau,\zeta_*) \le \mathbb{J}(\tau_*,\zeta_*) \le \mathbb{J}(\tau_*,\zeta) \tag{2.2}$$

should hold, for any exercise and recall times τ and ζ. Such a couple τ_* and ζ_* satisfying the inequalities of (2.2) with the functional in (2.1) defined above is called the *Nash equilibrium* in the optimal stopping zero-sum game of (1.4) (see, e.g. [8], [14] and [55] for a precise definition of this notion).

It thus follows from the results of [41] and [39] that the rational (or no-arbitrage) price of the game contingent claim described above coincides with the value function $V_*(x,s,q)$ of the optimal stopping zero-sum game for the (time-homogeneous strong) Markov process $(X,S,Q)=(X_t,S_t,Q_t)_{t\ge0}$ of the form:

$$V_*(x,s,q) = \sup_\tau \inf_\zeta \mathbb{E}_{x,s,q}\big[e^{-r\tau}\,F(X_\tau,S_\tau)\,I(\tau<\zeta) + e^{-r\zeta}\,G(X_\zeta,S_\zeta,Q_\zeta)\,I(\zeta\le\tau)\big] \tag{2.3}$$

where the supremum and infimum are taken over all stopping times τ and ζ with respect to the natural filtration $(\mathcal{F}_t)_{t\ge0}$ of the process X, and the functions $F(x,s)$ and $G(x,s,q)$ are given by (1.5), for some $0<L<K<L+1$ fixed. Here, we denote by $\mathbb{E}_{x,s,q}$ the expectation with respect to the probability measure $\mathbb{P}$ under the assumption that the three-dimensional (strong Markov) process (X,S,Q) defined in (1.1)-(1.2) and (1.3) starts at $(x,s,q)\in E$, and by $E=\{(x,s,q)\in\mathbb{R}^3 \,|\, 0<q\le x\le s\}$ the state space of the process (X,S,Q). It therefore follows from the results of [11; Theorem 4.1] based on the solutions of the associated (doubly) reflected backward stochastic differential equations that the game-type optimal stopping problem of (2.3) has a value. The existence of the associated *Stackelberg equilibria* in various optimal stopping games is proved in the results of [69]-[70], [47], and [55]-[56] among others. We further establish the existence and describe the structure of the stopping times τ_* and ζ_* forming a *Nash equilibrium* of the optimal stopping zero-sum game of (2.3).

2.2 The structure of the optimal stopping times. Let us first determine the structure of the stopping times forming a Nash equilibrium in the optimal stopping game of (2.3).

(i) By means of the results of general optimal stopping theory for Markov processes (see, e.g. [59; Chapter I, Section 2.2]) and the results of general theory of optimal stopping games (see, e.g. [7]-[8], [16]-[17], [43], [69], [47], and [11] among others), we obtain from the structure of the reward functional that the stopping times forming a Nash equilibrium in the optimal stopping game of (2.3) exist and are given by:

$$\tau_* = \inf\big\{t\ge0 \;\big|\; V_*(X_t,S_t,Q_t)=F(X_t,S_t)\big\} \tag{2.4}$$

and

$$\zeta_* = \inf\big\{t\ge0 \;\big|\; V_*(X_t,S_t,Q_t)=G(X_t,S_t,Q_t)\big\} \tag{2.5}$$

so that the associated continuation and stopping regions have the form:

$$C_* = \big\{(x,s,q)\in E \;\big|\; F(x,s)<V_*(x,s,q)<G(x,s,q)\big\} \tag{2.6}$$

and

$$D_* = \big\{ (x,s,q) \in E \;\big|\; \text{either} \;\; V_*(x,s,q) = F(x,s) \;\; \text{or} \;\; V_*(x,s,q) = G(x,s,q) \big\} \tag{2.7}$$

respectively, where the functions $F(x,s)$ and $G(x,s,q)$ are given by (1.5), for some $0 < L < K < L+1$ fixed. It is seen from the results of Theorem 4.1 below that the value function $V_*(x,s,q)$ is continuous, so that the set C_* is open and the set D_* is closed.

It follows from the structure of the payoff functions $F(x,s)$ and $G(x,s,q)$ given by (1.5) that the inequality $F(X_t,S_t) < G(X_t,S_t,Q_t)$ holds, when $X_t > (S_t - Q_t)/(K-L)$, while the equality $F(X_t,S_t) = G(X_t,S_t,Q_t)$ holds, when $X_t \le (S_t - Q_t)/(K-L)$, for any $t \ge 0$. Moreover, by virtue of the structure of the processes S and Q in (1.3), we may conclude that the inequalities $S_t - KX_t \ge (1-K)X_t \ge Q_t - LX_t$ hold, when $X_t \ge Q_t/(L-K+1)$, because of the assumption $0 < L < K < L+1$, so that the equality $F(X_t,S_t) = G(X_t,S_t,Q_t)$ is also satisfied in that case, for any $t \ge 0$. Observe that the latter condition particularly holds in the case $0 < L < K \le 1$, which is more restrictive but allows to keep the both payoffs $F(X_t,S_t)$ and $G(X_t,S_t,Q_t)$ positive, for all $t \ge 0$, that also represents an important feature from the point of view of callable financial contracts. Note that, in the case $L \ge K$, the inequality $S_t - KX_t < Q_t - LX_t$ cannot be satisfied, for any $t \ge 0$, which implies that the solution of the optimal stopping zero-sum game in (2.3) is trivial in that case, for each $0 < q < s$ fixed.

Summarising the arguments above, we realise that we consider the three-dimensional continuous strong Markov process (X,S,Q) defined in (1.1)-(1.2) and (1.3) within the discounted optimal stopping zero-sum game of (2.3) with the continuous payoff functions $F(x,s)$ and $G(x,s,q)$ from (1.5) such that the equality $F(x,s) = G(x,s,q)$ holds, when $x \le (s-q)/(K-L)$ as well as $x \ge q/(L-K+1)$, for all $(x,s,q) \in E$, under the assumption $0 < L < K < L+1$. Observe that the arguments of the proofs from [14; Theorem 2.1] and [55; Theorem 2.1] (also applicable for continuous strong Markov time-space processes with constant killing rates) can be naturally extended to the case of the discounted optimal stopping game of (2.3) for the process (X,S,Q) with the second and third components changing only at the diagonals $d_1 = \{(x,s,q) \in \mathbb{R}^3 \,|\, 0 < q \le x = s\}$ and $d_2 = \{(x,s,q) \in \mathbb{R}^3 \,|\, 0 < q = x \le s\}$ of the state space E, respectively (see also [56; Theorem 3.1] for the corresponding result in a model based on a standard Brownian motion in the interval $[0,1]$ absorbed at either 0 or 1). Note that the natural analogues of the conditions of [14; Formula (2.1)] and [55; Formulae (2.9)+(2.12)] are clearly satisfied for the discounted payoffs $e^{-rt}F(X_t,S_t)$ and $e^{-rt}G(X_t,S_t,Q_t)$ from (1.5) (see, e.g. [64; Formula (2.16)]). Hence, by applying the resulting extensions mentioned above, we may therefore conclude that the continuation region C_* in (2.6) should belong to the set:

$$E' = \big\{ (x,s,q) \in \mathbb{R}^3 \;\big|\; 0 < a'(s,q) \vee q \le x \le s \wedge b'(s,q) \big\} \tag{2.8}$$

with

$$a'(s,q) = (s-q)/(K-L) \quad \text{and} \quad b'(s,q) = q/(L-K+1) \tag{2.9}$$

for $0 < q < s$, representing all points (x,s,q) from the state space E of the process (X,S,Q) for which the solution of the original problem of (2.3) may be nontrivial, while the complement $E \setminus E'$ surely belongs to the stopping region D_* in (2.7), under the assumption $0 < L < K < L+1$. In this respect, the property $\tau_* \wedge \zeta_* \le \theta$ ($\mathbb{P}_{x,s,q}$-a.s.) holds, for τ_* and ζ_* from (2.4) and (2.5) and for any point $(x,s,q) \in E'$, under the assumption $0 < L < K < L+1$, where we put:

$$\theta = \inf \big\{ t \ge 0 \;\big|\; \text{either} \;\; X_t \le a'(S_t,Q_t) \;\; \text{or} \;\; X_t \ge b'(S_t,Q_t) \big\} \tag{2.10}$$

which is a stopping time of the process (X,S,Q).

(ii) We now describe the structure of the continuation and stopping regions C_* and D_* from (2.6)-(2.7). For this purpose, by means of standard applications of Itô's formula (see, e.g. [48; Theorem 4.4] or [62; Chapter IV, Theorem 3.3]) to the processes $e^{-rt}(S_t - KX_t)$ and $e^{-rt}(Q_t - LX_t)$, we obtain the representations:

$$e^{-rt}\,(S_t - K\,X_t) = s - K\,x + \int_0^t e^{-ru}\,\big(K\,\delta(S_u,Q_u)\,X_u - r\,S_u\big)\,du + \int_0^t e^{-ru}\,dS_u + N_t^1 \tag{2.11}$$

and

$$e^{-rt}\,(Q_t - L\,X_t) = q - L\,x + \int_0^t e^{-ru}\,\big(L\,\delta(S_u,Q_u)\,X_u - r\,Q_u\big)\,du + \int_0^t e^{-ru}\,dQ_u + N_t^2 \tag{2.12}$$

for all $t \ge 0$. Here, the processes $N^i = (N^i_t)_{t\ge0}$, for $i=1,2$, defined by:

$$N_t^1 = -K \int_0^t e^{-ru}\,\sigma(S_u,Q_u)\,X_u\,dB_u \quad \text{and} \quad N_t^2 = -L \int_0^t e^{-ru}\,\sigma(S_u,Q_u)\,X_u\,dB_u \tag{2.13}$$

are continuous uniformly integrable martingales under the probability measure $\mathbb{P}_{x,s,q}$. Then, inserting $\tau\wedge\zeta$ in place of t and applying Doob's optional sampling theorem (see, e.g. [48; Chapter III, Theorem 3.6] and [62; Chapter II, Theorem 3.2]) to the expressions in (2.11) and (2.12), we get that the equalities:

$$\begin{aligned}
&\mathbb{E}_{x,s,q}\big[e^{-r\tau}\,(S_\tau - K\,X_\tau)\,I(\tau<\zeta) + e^{-r\zeta}\,(Q_\zeta - L\,X_\zeta)\,I(\zeta\le\tau)\big] \\
&= \mathbb{E}_{x,s,q}\big[e^{-r(\tau\wedge\zeta)}\,(S_{\tau\wedge\zeta} - K\,X_{\tau\wedge\zeta}) - e^{-r\zeta}\,(S_\zeta - K\,X_\zeta - Q_\zeta + L\,X_\zeta)\,I(\zeta\le\tau)\big] \\
&= s - K\,x - \mathbb{E}_{x,s,q}\big[e^{-r\zeta}\,(S_\zeta - K\,X_\zeta - Q_\zeta + L\,X_\zeta)\,I(\zeta\le\tau)\big] \\
&\quad + \mathbb{E}_{x,s,q}\bigg[\int_0^{\tau\wedge\zeta} e^{-ru}\,\big(K\,\delta(S_u,Q_u)\,X_u - r\,S_u\big)\,du + \int_0^{\tau\wedge\zeta} e^{-ru}\,dS_u\bigg]
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
&\mathbb{E}_{x,s,q}\big[e^{-r\tau}\,(S_\tau - K\,X_\tau)\,I(\tau<\zeta) + e^{-r\zeta}\,(Q_\zeta - L\,X_\zeta)\,I(\zeta\le\tau)\big] \\
&= \mathbb{E}_{x,s,q}\big[e^{-r(\tau\wedge\zeta)}\,(Q_{\tau\wedge\zeta} - L\,X_{\tau\wedge\zeta}) + e^{-r\tau}\,(S_\tau - K\,X_\tau - Q_\tau + L\,X_\tau)\,I(\tau<\zeta)\big] \\
&= q - L\,x + \mathbb{E}_{x,s,q}\big[e^{-r\tau}\,(S_\tau - K\,X_\tau - Q_\tau + L\,X_\tau)\,I(\tau<\zeta)\big] \\
&\quad + \mathbb{E}_{x,s,q}\bigg[\int_0^{\tau\wedge\zeta} e^{-ru}\,\big(L\,\delta(S_u,Q_u)\,X_u - r\,Q_u\big)\,du + \int_0^{\tau\wedge\zeta} e^{-ru}\,dQ_u\bigg]
\end{aligned} \tag{2.15}$$

hold, for any stopping times τ and ζ such that $\tau\wedge\zeta\le\theta$ ($\mathbb{P}_{x,s,q}$-a.s.) holds with θ defined in (2.10), and for any starting point $(x,s,q)\in E'$ of (X,S,Q). Hence, it follows from the expressions in (2.14) and (2.15) and the structure of the optimal stopping times in (2.4) and (2.5) that the value function of the optimal stopping game in (2.3) admits the representations:

$$\begin{aligned}
V_*(x,s,q) &= s - K\,x - \mathbb{E}_{x,s,q}\big[e^{-r\zeta_*}\,(S_{\zeta_*} - K\,X_{\zeta_*} - Q_{\zeta_*} + L\,X_{\zeta_*})\,I(\zeta_*\le\tau_*)\big] \\
&\quad + \mathbb{E}_{x,s,q}\bigg[\int_0^{\tau_*\wedge\zeta_*} e^{-ru}\,\big(K\,\delta(S_u,Q_u)\,X_u - r\,S_u\big)\,du + \int_0^{\tau_*\wedge\zeta_*} e^{-ru}\,dS_u\bigg]
\end{aligned} \tag{2.16}$$

and

$$V_*(x,s,q) = q - L\,x + \mathbb{E}_{x,s,q}\big[e^{-r\tau_*}\,(S_{\tau_*} - K\,X_{\tau_*} - Q_{\tau_*} + L\,X_{\tau_*})\,I(\tau_* < \zeta_*)\big] \tag{2.17}$$
$$+\,\mathbb{E}_{x,s,q}\bigg[\int_0^{\tau_*\wedge\zeta_*} e^{-ru}\,\big(L\,\delta(S_u,Q_u)\,X_u - r\,Q_u\big)\,du + \int_0^{\tau_*\wedge\zeta_*} e^{-ru}\,dQ_u\bigg]$$

for the optimal stopping times τ_* and ζ_* forming a Nash equilibrium in (2.3), because the property $\tau_* \wedge \zeta_* \le \theta$ ($\mathbb{P}_{x,s,q}$-a.s.) holds with θ defined in (2.10), for any $(x,s,q) \in E'$.

Here and after, we denote by $\tau_* = \tau_*(x,s,q)$ and $\zeta_* = \zeta_*(x,s,q)$ the optimal stopping times forming a Nash equilibrium in (2.3) for the starting point $(x,s,q) \in E'$ of the process (X,S,Q), where E' is defined in (2.8) above. Thus, on the one hand, it follows from the structure of the integrand in the first integral of (2.16) and the fact that the second integral there increases whenever the process (X,S,Q) is located at the diagonal $d_1 = \{(x,s,q) \in \mathbb{R}^3 \,|\, 0 < q \le x = s\}$ that it should not be optimal for the option holder (maximiser of the expected reward) to exercise the contract earlier than the option writer recalls it, when the inequalities $a'(S_t,Q_t) \vee rS_t/(K\delta(S_t,Q_t)) < X_t \le S_t \wedge b'(S_t,Q_t)$ hold, for any $t \ge 0$. Moreover, it follows from the structure of the integrand in first integral of (2.17) and the fact that the second integral there decreases whenever the process (X,S,Q) is located at the diagonal $d_2 = \{(x,s,q) \in \mathbb{R}^3 \,|\, 0 < q = x \le s\}$ that it should not be optimal for the option writer (minimiser of the expected reward) to recall the contract earlier than the option holder exercises it, when the inequalities $a'(S_t,Q_t) \vee Q_t \le X_t < rQ_t/(L\delta(S_t,Q_t)) \wedge b'(S_t,Q_t)$ hold, for any $t \ge 0$. Since the both participants of the contract are acting simultaneously, these facts yield that the set:

$$C' = \big\{(x,s,q) \in E' \;\big|\; a'(s,q) \vee q \vee \overline{a}(s,q) < x < \underline{b}(s,q) \wedge s \wedge b'(s,q)\big\} \tag{2.18}$$

with

$$\overline{a}(s,q) = rs/(K\delta(s,q)) \quad \text{and} \quad \underline{b}(s,q) = rq/(L\delta(s,q)) \tag{2.19}$$

for $0 < q < s$, which can be nonempty because of the assumption that $0 < L < K < L+1$ holds and represents a part of the continuation region C_* in (2.6).

(iii) Let us finally prove the connectivity of the left-hand and right-hand parts of the stopping region D_* from (2.7) which are located outside the region C' from (2.18). For this purpose, we first recall from the arguments of Part (i) above that all the points (x',s,q) and (x'',s,q) of the set $E \setminus E'$ with E' defined in (2.8) such that $0 < x' < a'(s,q)$ and $x'' > b'(s,q)$, respectively, belong to the stopping region D_*, for each $0 < q < s$ fixed, under the assumption $0 < L < K < L+1$. We also remind that the process X admits the explicit expression of (1.1) as well as provides a (pathwise) unique strong solution of the stochastic differential equation in (1.2) with the processes S and Q given by (1.3), so that the solutions starting from the different points $x > 0$ do not intersect each other over the whole infinite time interval. For the ease of presentation, in the rest of this section, we indicate by $(X^{(x)}, S^{(s,x)}, Q^{(q,x)})$ the dependence of the process (X,S,Q) defined in (1.1)-(1.2) and (1.3) from its starting point $(x,s,q) \in E'$.

Observe that, if we take some $(x,s,q) \in D_*$ such that $a'(s,q) < x < \overline{a}(s,q) \wedge s \wedge b'(s,q)$ holds, for each $0 < q < s$ fixed, then the arguments of Part (ii) above would imply that it is not optimal for the option writer to recall the contract earlier than the holder exercises it, so that the value in (2.3) would admit the representation:

$$V_*(x', s, q) - (s - K\,x') \tag{2.20}$$
$$= \mathbb{E}\bigg[\int_0^{\tau_*'} e^{-ru} \big(K\,\delta\big(S_u^{(s,x')}, Q_u^{(q,x')}\big)\, X_u^{(x')} - r\, S_u^{(s,x')} \big)\, du + \int_0^{\tau_*'} e^{-ru}\, dS_u^{(s,x')} \bigg]$$

where $\tau_*' = \tau_*(x', s, q)$ denotes the optimal exercise time of the holder in the problem of (2.3) under the assumption that the process (X, S, Q) starts at (x', s, q), for all $a'(s, q) \le x' < x$ and each $0 < q < s$ fixed. Hence, because of the assumption that the function $\delta(s, q)$ is increasing in the both variables s and q on $[0, \infty]^2$ and the fact that the process (X, S, Q) started at (x', s, q) reaches the point (x, s, q') for some $0 < q' \le q$ before hitting the upper diagonal $d_1 = \{(x, s, q) \in \mathbb{R}^3 \,|\, 0 < q \le x = s\}$, we see that the integrand in the first integral in the right-hand side of (2.20) (so that the resulting total expected reward functional there) increases in x' on $a'(s, q) \le x' < x$, for each $0 < q < s$ fixed. Thus, we may conclude that the inequalities:

$$V_*(x', s, q) - (s - K\,x') \le V_*(x, s, q) - (s - K\,x) = 0 \tag{2.21}$$

hold, so that the point (x', s, q) such that $a'(s, q) \le x' < x < \overline{a}(s, q) \wedge s \wedge b'(s, q)$ also belongs to the left-hand part of the stopping region D_* in (2.7).

Similarly, if we take some $(x, s, q) \in D_*$ such that $a'(s, q) \vee q \vee \underline{b}(s, q) < x < b'(s, q)$ holds, for each $0 < q < s$ fixed, then the arguments of Part (ii) above would imply that it is not optimal for the option holder to exercise the contract earlier than the writer recalls it, so that the value in (2.3) would admit the representation:

$$V_*(x'', s, q) - (q - L\,x'') \tag{2.22}$$
$$= \mathbb{E}\bigg[\int_0^{\zeta_*'} e^{-ru} \big(L\,\delta\big(S_u^{(s,x'')}, Q_u^{(q,x'')}\big)\, X_u^{(x'')} - r\, Q_u^{(q,x'')} \big)\, du + \int_0^{\zeta_*'} e^{-ru}\, dQ_u^{(q,x'')} \bigg]$$

where $\zeta_*' = \zeta_*(x'', s, q)$ denotes the optimal exercise time of the writer in the problem of (2.3) under the assumption that the process (X, S, Q) starts at (x'', s, q), for all $x < x'' \le b'(s, q)$ and each $0 < q < s$ fixed. Hence, due to the assumption that the function $\delta(s, q)$ is increasing in the both variables s and q on $[0, \infty]^2$ and the fact that the process (X, S, Q) started at (x'', s, q) reaches the point (x, s', q) for some $0 < s \le s'$ before hitting the lower diagonal $d_2 = \{(x, s, q) \in \mathbb{R}^3 \,|\, 0 < q = x \le s\}$, we see that the integrand in the first integral in the right-hand side of (2.22) (so that the resulting total expected reward functional there) increases in x'' on $x < x'' \le b'(s, q)$, for each $0 < q < s$ fixed. Thus, we may conclude that the inequalities:

$$V_*(x'', s, q) - (q - L\,x'') \ge V_*(x, s, q) - (q - L\,x) = 0 \tag{2.23}$$

hold, so that the point (x'', s, q) such that $a'(s, q) \vee q \vee \underline{b}(s, q) < x < x'' \le b'(s, q)$ also belongs to the right-hand part of the stopping region D_* in (2.7).

In this view, combining these arguments together with the facts deduced in Parts (i)-(ii) above and checking with the comments in [13; Subsection 3.3] and [52; Subsection 3.3], we may therefore conclude that there exist functions $a_*(s, q)$ and $b_*(s, q)$ satisfying the inequalities

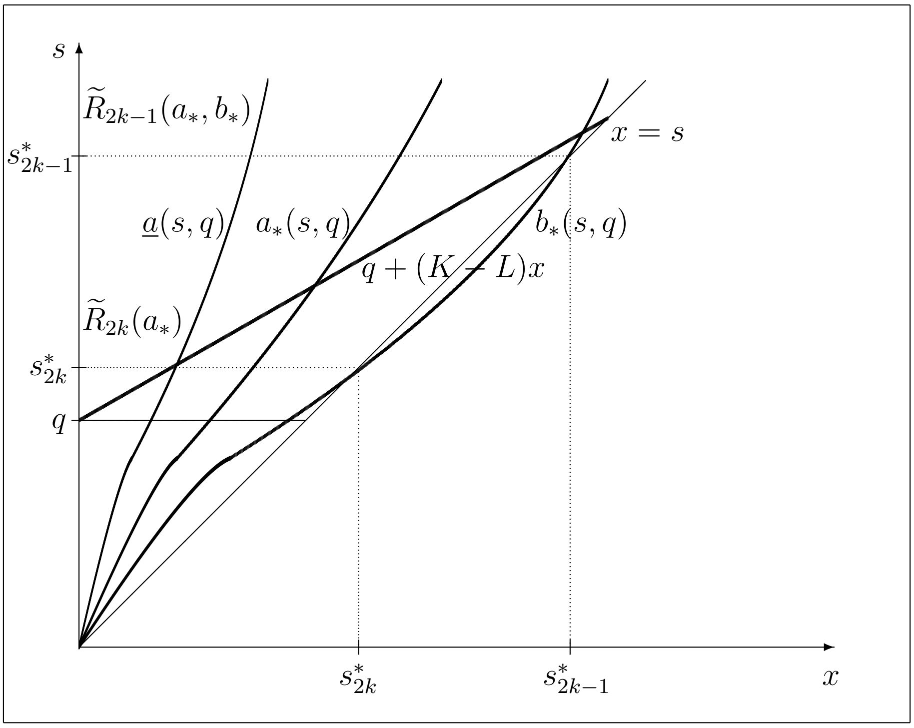

Figure 1. A computer drawing of the optimal exercise boundaries $a_*(s,q)$, $b_*(s,q)$ and $\underline{a}(s,q)$, for each $q > 0$ fixed.

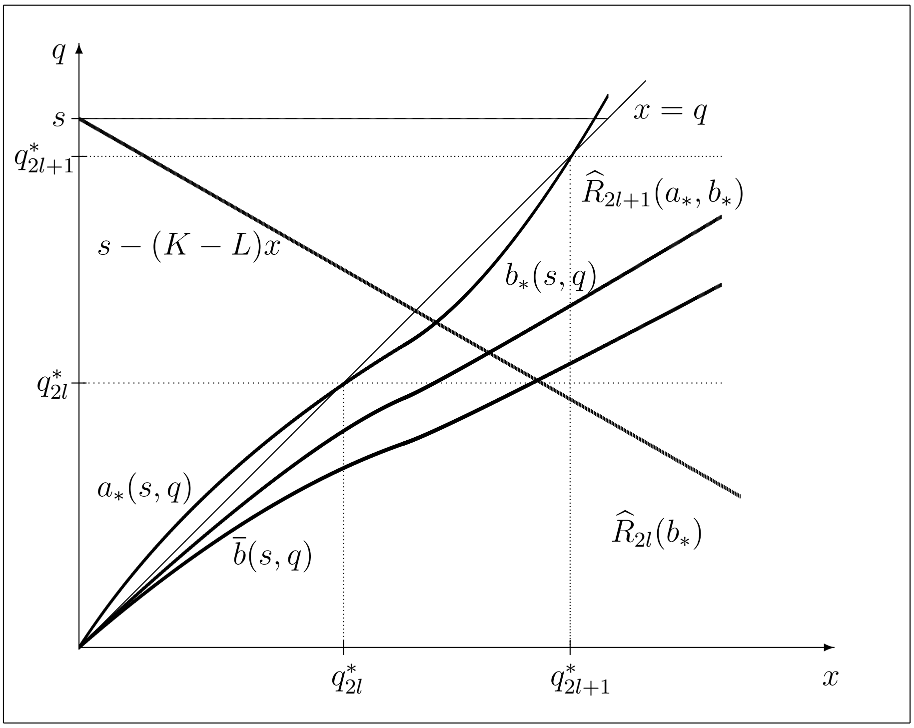

Figure 2. A computer drawing of the optimal exercise boundaries $a_*(s,q)$, $b_*(s,q)$ and $\overline{b}(s,q)$, for each $s > 0$ fixed.

$a'(s,q) \le a_*(s,q) \le \overline{a}(s,q) \wedge s \wedge b'(s,q)$ and $a'(s,q) \vee q \vee \underline{b}(s,q) \le b_*(s,q) \le b'(s,q)$, for $0 < q < s$, such that the continuation region C_* in (2.6) has the form:

$$C_* = \big\{ (x,s,q) \in E' \; \big| \; a_*(s,q) < x < b_*(s,q) \big\} \tag{2.24}$$

while the stopping region D_* in (2.7) is given by:

$$D_* = \big\{ (x,s,q) \in E \; \big| \; \text{either} \;\; x \le a_*(s,q) \;\; \text{or} \;\; x \ge b_*(s,q) \big\}. \tag{2.25}$$

(iv) Finally, in order to determine some upper and lower bounds for the value function in (2.3) and optimal stopping boundaries in (2.24) and (2.25), we consider the optimal stopping problems with the value functions $\overline{V}(x,s,q)$ and $\underline{V}(x,s,q)$ from (5.1). It is shown in Section 5 below that the functions $\overline{V}(x,s,q)$ and $\underline{V}(x,s,q)$ admit the explicit expressions in (5.11) and (5.12), while the associated optimal stopping times $\overline{\tau}$ and $\underline{\zeta}$ are given by (5.2), where the boundaries $\underline{a}(s,q)$ and $\overline{b}(s,q)$ are determined as the maximal and minimal solutions of the first-order nonlinear ordinary differential equations in (3.24) and (3.27) staying below or above the diagonals $d_1 = \{(x,s,q) \in \mathbb{R}^3 \,|\, 0 < q \le x = s\}$ and $d_2 = \{(x,s,q) \in \mathbb{R}^3 \,|\, 0 < q = x \le s\}$, respectively. If we suppose that either the inequality $a_*(s,q) < \underline{a}(s,q)$ or $b_*(s,q) > \overline{b}(s,q)$ holds, then, for each $x > 0$ such that either $x \in (a_*(s,q), \underline{a}(s,q))$ or $x \in (\overline{b}(s,q), b_*(s,q))$ given and fixed, we would have either $V_*(x,s,q) > s - Kx = \overline{V}(x,s,q)$ or $V_*(x,s,q) < q - Lx = \underline{V}(x,s,q)$, respectively, contradicting the obvious facts that the inequalities $V_*(x,s,q) \le \overline{V}(x,s,q)$ and $V_*(x,s,q) \ge \underline{V}(x,s,q)$ hold, for all $(x,s,q) \in E'$. Thus, we may conclude that the inequalities $a_*(s,q) \ge \underline{a}(s,q)$ and $b_*(s,q) \le \overline{b}(s,q)$ should be satisfied, for all $0 < q < s$ (see Figures 1 and 2 above for computer drawings of the optimal stopping boundaries $a_*(s,q)$ and $b_*(s,q)$ as well as their estimates $\underline{a}(s,q)$ and $\overline{b}(s,q)$).

2.3 The three-dimensional coupled free-boundary problem. By means of standard arguments based on an application of Itô's formula, it is shown that the infinitesimal operator $\mathbb{L}$ of the process (X,S,Q) has the form:

$$\mathbb{L} = \big(r - \delta(s,q)\big)\, x\, \partial_x + \frac{\sigma^2(s,q) x^2}{2}\, \partial_{xx} \quad \text{in} \quad 0 < q < x < s \tag{2.26}$$

$$\partial_s = 0 \quad \text{at} \quad 0 < q \le x = s \quad \text{and} \quad \partial_q = 0 \quad \text{at} \quad 0 < q = x \le s \tag{2.27}$$

(see, e.g. [52; Subsection 3.1]). In order to find analytic expressions for the unknown value function $V_*(x,s,q)$ from (2.3) and the unknown boundaries $a_*(s,q)$ and $b_*(s,q)$ from (2.24)-(2.25), let us build on the results of general theory of optimal stopping problems for Markov processes (see, e.g. [59; Chapter IV, Section 8]). We can reduce the optimal stopping game of (2.3) to the equivalent coupled free-boundary problem for $V(x,s,q)$ with $a(s,q)$ and $b(s,q)$ given by:

$$(\mathbb{L}V - rV)(x,s,q) = 0 \quad \text{for} \quad q \vee a(s,q) < x < b(s,q) \wedge s \tag{2.28}$$

$$V(x,s,q)\big|_{x=a(s,q)+} = s - K\,a(s,q), \quad V(x,s,q)\big|_{x=b(s,q)-} = q - L\,b(s,q) \tag{2.29}$$

$$V(x,s,q) = s - K\,x \quad \text{for} \quad x < a(s,q), \quad V(x,s,q) = q - L\,x \quad \text{for} \quad x > b(s,q) \tag{2.30}$$

$$s - K\,x < V(x,s,q) < q - L\,x \quad \text{for} \quad a(s,q) < x < b(s,q) \tag{2.31}$$

$$(\mathbb{L}V - rV)(x,s,q) < 0 \quad \text{for} \quad x < a(s,q), \quad (\mathbb{L}V - rV)(x,s,q) > 0 \quad \text{for} \quad x > b(s,q) \tag{2.32}$$

where the *instantaneous-stopping* conditions in (2.29) are satisfied, when either $a(s,q) \ge a'(s,q) \vee q$ or $b(s,q) \le s \wedge b'(s,q)$ holds, for each $0 < q < s$, respectively. Moreover, we further assume that the *smooth-fit* conditions:

$$\partial_x V(x,s,q)\big|_{x=a(s,q)+} = -K, \quad \partial_x V(x,s,q)\big|_{x=b(s,q)-} = -L \tag{2.33}$$

are satisfied, when either $a(s,q) > a'(s,q) \vee q$ or $b(s,q) < s \wedge b'(s,q)$ holds, while the *normal-reflection* conditions:

$$\partial_s V(x,s,q)\big|_{x=s-} = 0, \quad \partial_q V(x,s,q)\big|_{x=q+} = 0 \tag{2.34}$$

are satisfied, when either $b'(s,q) \ge b(s,q) > s$ or $a'(s,q) \le a(s,q) < q$ holds, for each $0 < q < s$, respectively. On the one hand, when either the inequality $a(s,q) > a'(s,q) \vee q$ or $b(s,q) < s \wedge b'(s,q)$ holds, for some $0 < q < s$, the continuous process X can cross the left-hand boundary $a(S,Q)$ before hitting the lower diagonal d_2 or cross the right-hand boundary $b(S,Q)$ before hitting the upper diagonal d_1, so that we can assume that the left-hand or the right-hand smooth-fit conditions of (2.33) are satisfied for the candidate value function $V(x,s,q)$ at $a(s,q)$ or $b(s,q)$, respectively. On the other hand, when either the inequalities $b'(s,q) \ge b(s,q) > s$ or $a'(s,q) \le a(s,q) < q$ holds, for some $0 < q < s$, the process X can hit the upper diagonal $d_1 = \{(x,s,q) \in \mathbb{R}^3 \,|\, 0 < q \le x = s\}$ before crossing the right-hand boundary $b(S,Q)$ or hit the lower diagonal $d_2 = \{(x,s,q) \in \mathbb{R}^3 \,|\, 0 < q = x \le s\}$ before crossing the left-hand boundary $a(S,Q)$, so that we can assume that either the left-hand or the right-hand normal-reflection conditions of (2.34) are satisfied for $V(x,s,q)$ at d_1 and d_2, respectively. These properties are verified in the proof of Theorem 4.1 below, while the inequalities in (2.32) follow directly from the arguments of Parts (i)-(ii) of Subsection 2.2 above.

3. Solutions to the coupled free-boundary problem

In this section, we obtain closed-form expressions for the value function $V_*(x,s,q)$ in (2.3) associated with the perpetual floating-strike lookback game option and derive first-order nonlinear ordinary differential equations for the optimal stopping boundaries $a_*(s,q)$ and $b_*(s,q)$ from (2.24)-(2.25) forming a solution to the free-boundary problem in (2.28)-(2.32) with (2.33) and (2.34).

3.1 The candidate value function. We first observe that the general solution of the second-order ordinary differential equation in (2.28) has the form:

$$V(x,s,q) = C_1(s,q)\, x^{\gamma_1(s,q)} + C_2(s,q)\, x^{\gamma_2(s,q)} \tag{3.1}$$

where $C_i(s,q)$, for $i = 1,2$, are some continuously differentiable functions and $\gamma_i(s,q)$, for $i = 1,2$, are given by:

$$\gamma_i(s,q) = \frac{1}{2} - \frac{r - \delta(s,q)}{\sigma^2(s,q)} - (-1)^i \sqrt{\left(\frac{1}{2} - \frac{r-\delta(s,q)}{\sigma^2(s,q)}\right)^2 + \frac{2r}{\sigma^2(s,q)}} \tag{3.2}$$

so that $\gamma_2(s,q) < 0 < 1 < \gamma_1(s,q)$ holds, for all $0 < q < s$. Then, by applying the instantaneous-stopping conditions from (2.29) to the function in (3.1), we get that the equalities:

$$C_1(s,q)\, a^{\gamma_1(s,q)}(s,q) + C_2(s,q)\, a^{\gamma_2(s,q)}(s,q) = s - K\, a(s,q) \tag{3.3}$$
$$C_1(s,q)\, b^{\gamma_1(s,q)}(s,q) + C_2(s,q)\, b^{\gamma_2(s,q)}(s,q) = q - L\, b(s,q) \tag{3.4}$$

are satisfied, when $a(s,q) \geq a'(s,q) \vee q$ and $b(s,q) \leq s \wedge b'(s,q)$ holds, for each $0 < q < s$, respectively. Hence, by using the smooth-fit conditions from (2.33), we obtain that the equalities:

$$C_1(s,q)\, \gamma_1(s,q)\, a^{\gamma_1(s,q)}(s,q) + C_2(s,q)\, \gamma_2(s,q)\, a^{\gamma_2(s,q)}(s,q) = -K\, a(s,q) \tag{3.5}$$
$$C_1(s,q)\, \gamma_1(s,q)\, b^{\gamma_1(s,q)}(s,q) + C_2(s,q)\, \gamma_2(s,q)\, b^{\gamma_2(s,q)}(s,q) = -L\, b(s,q) \tag{3.6}$$

are satisfied, when $a(s,q) > a'(s,q) \vee q$ and $b(s,q) < s \wedge b'(s,q)$ holds, for each $0 < q < s$, respectively. Thus, by applying the normal-reflection conditions from (2.34) to the function in (3.1), we obtain that the equalities:

$$\sum_{i=1}^{2} \Big(\partial_s C_i(s,q)\, s^{\gamma_i(s,q)} + C_i(s,q)\, \partial_s \gamma_i(s,q)\, s^{\gamma_i(s,q)} \ln s \Big) = 0 \tag{3.7}$$
$$\sum_{i=1}^{2} \Big(\partial_q C_i(s,q)\, q^{\gamma_i(s,q)} + C_i(s,q)\, \partial_q \gamma_i(s,q)\, q^{\gamma_i(s,q)} \ln q \Big) = 0 \tag{3.8}$$

are satisfied, when $b'(s,q) \geq b(s,q) > s$ and $a'(s,q) \leq a(s,q) < q$ holds, for each $0 < q < s$, respectively. Here, the partial derivatives $\partial_s \gamma_i(s,q)$ and $\partial_q \gamma_i(s,q)$ take the form:

$$\partial_s \gamma_i(s,q) = \varphi(s,q) - (-1)^i \frac{\varphi(s,q)(\gamma_1(s,q)+\gamma_2(s,q))\sigma^3(s,q) - 4r\partial_s\sigma(s,q)}{\sigma^2(s,q)\sqrt{(\gamma_1(s,q)+\gamma_2(s,q))^2\sigma^2(s,q)+8r}} \tag{3.9}$$
$$\partial_q \gamma_i(s,q) = \psi(s,q) - (-1)^i \frac{\psi(s,q)(\gamma_1(s,q)+\gamma_2(s,q))\sigma^3(s,q) - 4r\partial_q\sigma(s,q)}{\sigma^2(s,q)\sqrt{(\gamma_1(s,q)+\gamma_2(s,q))^2\sigma^2(s,q)+8r}} \tag{3.10}$$

for $i = 1, 2$, and the functions $\varphi(s,q)$ and $\psi(s,q)$ are defined by:

$$\varphi(s,q) = \frac{\sigma(s,q)\partial_s\delta(s,q) + 2(r-\delta(s,q))\partial_s\sigma(s,q)}{\sigma^3(s,q)} \tag{3.11}$$
$$\psi(s,q) = \frac{\sigma(s,q)\partial_q\delta(s,q) + 2(r-\delta(s,q))\partial_q\sigma(s,q)}{\sigma^3(s,q)} \tag{3.12}$$

for $0 < q < s$.

Now, by solving the system of equations in (3.3)+(3.4), we obtain that the function in (3.1) admits the representation:

$$V(x,s,q;a(s,q),b(s,q)) = \sum_{i=1}^{2} C_i(s,q;a(s,q),b(s,q))\, x^{\gamma_i(s,q)} \tag{3.13}$$

for $a'(s,q) \vee q \leq a(s,q) < x < b(s,q) \leq s \wedge b'(s,q)$, where

$$C_i(s,q;a(s,q),b(s,q)) = \frac{(s - Ka(s,q))b^{\gamma_{3-i}(s,q)}(s,q) - (q - Lb(s,q))a^{\gamma_{3-i}(s,q)}(s,q)}{a^{\gamma_i(s,q)}(s,q)b^{\gamma_{3-i}(s,q)}(s,q) - b^{\gamma_i(s,q)}(s,q)a^{\gamma_{3-i}(s,q)}(s,q)} \tag{3.14}$$

when $a'(s,q) \vee q \leq a(s,q) < b(s,q) \leq s \wedge b'(s,q)$ holds, for every $i = 1, 2$. Then, by solving the system of equations in (3.3)+(3.5), we obtain that the function in (3.1) admits the representation:

$$V(x,s,q;a(s,q)) = C_1(s,q;a(s,q))\, x^{\gamma_1(s,q)} + C_2(s,q;a(s,q))\, x^{\gamma_2(s,q)} \tag{3.15}$$

for $a'(s,q) \vee q \leq a(s,q) < x \leq s < b(s,q) \leq b'(s,q)$, where

$$C_i(s,q;a(s,q)) = \frac{\gamma_{3-i}(s,q)(s - Ka(s,q)) + Ka(s,q)}{(\gamma_{3-i}(s,q) - \gamma_i(s,q))a^{\gamma_i(s,q)}(s,q)} \tag{3.16}$$

when $a'(s,q) \vee q \leq a(s,q) < s < b(s,q) \leq b'(s,q)$ holds, for every $i = 1, 2$. Also, by solving the system of equations in (3.4)+(3.6), we obtain that the function in (3.1) admits the representation:

$$V(x,s,q;b(s,q)) = C_1(s,q;b(s,q))\, x^{\gamma_1(s,q)} + C_2(s,q;b(s,q))\, x^{\gamma_2(s,q)} \tag{3.17}$$

for $a'(s,q) \leq a(s,q) < q \leq x < b(s,q) \leq s \wedge b'(s,q)$, where

$$C_i(s,q;b(s,q)) = \frac{\gamma_{3-i}(s,q)(q - Lb(s,q)) + Lb(s,q)}{(\gamma_{3-i}(s,q) - \gamma_i(s,q))b^{\gamma_i(s,q)}(s,q)} \tag{3.18}$$

when $a'(s,q) \leq a(s,q) < q < b(s,q) \leq s \wedge b'(s,q)$ holds, for every $i = 1, 2$.

Finally, by means of straightforward computations, it can be deduced from the expression in (3.13) that the first-order and second-order partial derivatives $\partial_x V(x,s,q;a(s,q),b(s,q))$ and $\partial_{xx} V(x,s,q;a(s,q),b(s,q))$ of the function $V(x,s,q;a(s,q),b(s,q))$ take the form:

$$\partial_x V(x,s,q;a(s,q),b(s,q)) = \sum_{i=1}^{2} C_i(s,q;a(s,q),b(s,q))\, \gamma_i(s,q)\, x^{\gamma_i(s,q)-1} \tag{3.19}$$

and

$$\partial_{xx} V(x,s,q;a(s,q),b(s,q)) = \sum_{i=1}^{2} C_i(s,q;a(s,q),b(s,q))\, \gamma_i(s,q)(\gamma_i(s,q) - 1)\, x^{\gamma_i(s,q)-2} \tag{3.20}$$

for $a'(s,q) \vee q \leq a(s,q) < x < b(s,q) \leq s \wedge b'(s,q)$. Note that the same first-order and second-order partial derivatives of the functions $V(x,s,q;a(s,q))$, for $a'(s,q) \vee q \leq a(s,q) < s < b(s,q) \leq b'(s,q)$, and $V(x,s,q;b(s,q))$, for $a'(s,q) \leq a(s,q) < q < b(s,q) \leq s \wedge b'(s,q)$, from (3.15) with (3.16) and (3.17) with (3.18) are computed similarly.

3.2 The candidate stopping boundaries. We now apply the conditions of (3.5)-(3.6) to the functions $C_i(s,q;a(s,q),b(s,q))$, for $i=1,2$, in (3.14) to obtain the equalities:

$$\frac{\gamma_i(s,q)(s-Ka(s,q))+Ka(s,q)}{\gamma_i(s,q)(q-Lb(s,q))+Lb(s,q)}=\left(\frac{a(s,q)}{b(s,q)}\right)^{\gamma_{3-i}(s,q)} \tag{3.21}$$

for $a'(s,q)\vee q\le a(s,q)<b(s,q)\le s\wedge b'(s,q)$ and $i=1,2$. Then, we set $b(s,q)=b'(s,q)$ and apply the condition of (3.5) to the same functions to obtain the equality:

$$\sum_{i=1}^{2}\frac{(s-Ka(s,q))b'^{\gamma_{3-i}(s,q)}(s,q)-(q-Lb'(s,q))a^{\gamma_{3-i}(s,q)}(s,q)}{a^{\gamma_i(s,q)}(s,q)b'^{\gamma_{3-i}(s,q)}(s,q)-b'^{\gamma_i(s,q)}(s,q)a^{\gamma_{3-i}(s,q)}(s,q)}\,\gamma_i(s,q)\,a^{\gamma_i(s,q)}(s,q) \tag{3.22}$$
$$=-K\,a(s,q)$$

for $a'(s,q)\vee q\le a(s,q)<s\wedge b'(s,q)<b(s,q)$. Also, we set $a(s,q)=a'(s,q)$ and apply the condition of (3.6) to the same functions to obtain the equality:

$$\sum_{i=1}^{2}\frac{(s-Ka'(s,q))b^{\gamma_{3-i}(s,q)}(s,q)-(q-Lb(s,q))a'^{\gamma_{3-i}(s,q)}(s,q)}{a'^{\gamma_i(s,q)}(s,q)b^{\gamma_{3-i}(s,q)}(s,q)-b^{\gamma_i(s,q)}(s,q)a'^{\gamma_{3-i}(s,q)}(s,q)}\,\gamma_i(s,q)\,b^{\gamma_i(s,q)}(s,q) \tag{3.23}$$
$$=-L\,b(s,q)$$

for $a(s,q)<a'(s,q)\vee q<b(s,q)\le s\wedge b'(s,q)$.

The existence and uniqueness of solutions of the system of arithmetic equations in (3.21) as well as of the equations in (3.22) and (3.23) on the admissible intervals follow from the arguments of Subsection 3.4 below. Observe that the system of arithmetic equations in (3.21) as well as the equation in (3.23) satisfy the conditions of the classical (two-dimensional) implicit function theorem, so that the resulting solutions $a_*(s,q)$ and $b_*(s,q)$ turn out to be continuously differentiable. Furthermore, assuming that the candidate boundary functions $a(s,q)$ and $b(s,q)$ are continuously differentiable, we apply the condition of (3.7) to the functions $C_i(s,q;a(s,q))$, for $i=1,2$, in (3.16) to conclude that the candidate boundary $a(s,q)$ satisfies the ordinary differential equation:

$$\partial_s a(s,q)=\sum_{j=1}^{2}\frac{C_i(s,q;a(s,q))\partial_s\gamma_i(s,q)s^{\gamma_i(s,q)}\ln s+\Psi_{1,i}(a(s,q),s,q)(s/a(s,q))^{\gamma_i(s,q)}}{\Phi_1(a(s,q),s,q)((s/a(s,q))^{\gamma_1(s,q)}-(s/a(s,q))^{\gamma_2(s,q)})} \tag{3.24}$$

for $a'(s,q)\vee q\le a(s,q)<s<b(s,q)\le b'(s,q)$, where we set:

$$\Phi_1(x,s,q)=\frac{(\gamma_1(s,q)+\gamma_2(s,q))K+\gamma_1(s,q)\gamma_2(s,q)(s-Kx)/x}{\gamma_2(s,q)-\gamma_1(s,q)} \tag{3.25}$$

and

$$\Psi_{1,i}(x,s,q)=\frac{\partial_s\gamma_{3-i}(s,q)(s-Kx)+\gamma_{3-i}(s,q)}{\gamma_{3-i}(s,q)-\gamma_i(s,q)} \tag{3.26}$$
$$-\frac{(Kx+\gamma_{3-i}(s,q)(s-Kx))(\partial_s\gamma_i(s,q)\,\ln x+\partial_s\ln(\gamma_{3-i}(s,q)-\gamma_i(s,q)))}{\gamma_{3-i}(s,q)-\gamma_i(s,q)}$$

for all $0 < q \le x \le s$ and every $i = 1, 2$. We also apply the condition of (3.8) to the functions $C_i(s, q; b(s, q))$, for $i = 1, 2$, in (3.18) to conclude that the candidate boundary $b(s, q)$ satisfies the ordinary differential equation:

$$\partial_q b(s,q) = \sum_{i=1}^{2} \frac{C_i(s,q;b(s,q))\partial_q \gamma_j(s,q) q^{\gamma_i(s,q)} \ln q + \Psi_{2,i}(b(s,q),s,q)(q/b(s,q))^{\gamma_i(s,q)}}{\Phi_2(b(s,q),s,q)((q/b(s,q))^{\gamma_1(s,q)} - (q/b(s,q))^{\gamma_2(s,q)})} \tag{3.27}$$

for $a'(s, q) \le a(s, q) < q < b(s, q) \le s \wedge b'(s, q)$, where we set:

$$\Phi_2(x,s,q) = \frac{(\gamma_1(s,q) + \gamma_2(s,q))L + \gamma_1(s,q)\gamma_2(s,q)(q - Lx)/x}{\gamma_2(s,q) - \gamma_1(s,q)} \tag{3.28}$$

and

$$\Psi_{2,i}(x,s,q) = \frac{\partial_q \gamma_{3-i}(s,q)(q - Lx) + \gamma_{3-i}(s,q)}{\gamma_{3-i}(s,q) - \gamma_i(s,q)} - \frac{(Lx + \gamma_{3-i}(s,q)(q - Lx))(\partial_q \gamma_i(s,q) \ln x + \partial_q \ln(\gamma_{3-i}(s,q) - \gamma_i(s,q)))}{\gamma_{3-i}(s,q) - \gamma_i(s,q)} \tag{3.29}$$

for all $0 < q \le x \le s$ and every $i = 1, 2$. Note that, by virtue of the assumptions on the coefficients $\delta(s, q) > 0$ and $\sigma(s, q) > 0$ of the diffusion-type process X from (1.1)-(1.2) and (1.3), the right-hand sides of the expressions in (3.24) with (3.25)-(3.26) and (3.27) with (3.28)-(3.29) are (locally) continuous in $(s, q, a(s, q))$ and $(s, q, b(s, q))$ and (locally) Lipschitz in $a(s, q)$ and $b(s, q)$, for each $0 < q < s$ fixed. Thus, by means of the classical results on the existence and uniqueness of solutions for first-order nonlinear ordinary differential equations, the equations in (3.24) and (3.27) admit (locally) unique solutions, which can be constructed by means of Picard's method of successive approximations (see Subsection 5.3 below for further constructions and references).

3.3 The structure of the continuation region. In order to specify the optimal exercise boundaries for the floating-strike lookback game options, let us consider the functions $a_*(s, q)$ and $b_*(s, q)$, which either provide a unique solution to the system of arithmetic equations in (3.21) such that $a'(s, q) \le a_*(s, q) < b_*(s, q) \le b'(s, q)$, or the function $a_*(s, q) < b_*(s, q) = b'(s, q)$ represents the largest root of the equation in (3.22), or the function $b_*(s, q) > a_*(s, q) = a'(s, q)$ represents the largest root of the equation in (3.23), or otherwise, the properties $a_*(s, q) = a'(s, q)$ and $b_*(s, q) = b'(s, q)$ hold, for each $0 < q < s$ fixed.

On the one hand, we can set $s_0^*(q) = \infty$ and define the functions $s_{2k-1}^*(q) = \sup\{s < s_{2k-2}^*(q) \,|\, b_*(s, q) > s\}$ and $s_{2k}^*(q) = \sup\{s < s_{2k-1}^*(q) \,|\, b_*(s, q) < s\}$, whenever they exist, and put $s_{2k-1}^*(q) = s_{2k}^*(q) = 0$ otherwise, so that the inequalities $0 \le s_{2k-1}^*(q) \le s_{2k-2}^*(q) \le \infty$ hold, for all $k \in \mathbb{N}$, and each $q > 0$ fixed. In other words, the boundary $b_*(s, q)$ exits the region E from the side of the diagonal $d_1 = \{(x, s, q) \in \mathbb{R}^3 \,|\, 0 < q \le x = s\}$ passing through the points $(s_{2k-1}^*(q), s_{2k-1}^*(q), q)$ and comes back to E from the side of d_1 passing through the points $(s_{2k}^*(q), s_{2k}^*(q), q)$, for $k \in \mathbb{N}$, for each $q > 0$ fixed. Hence, the candidate value function $V(x, s, q; a_*(s, q), b_*(s, q))$ admits the representation of (3.13) with (3.14) and the candidate stopping boundaries $a_*(s, q)$ and $b_*(s, q)$ solve the system of arithmetic equations in (3.21) in the regions:

$$\widetilde{R}_{2k-1}(a_*, b_*) = \big\{(x, s, q) \in E' \,\big|\, s_{2k-1}^*(q) \le s < s_{2k-2}^*(q)\big\} \tag{3.30}$$

while the candidate value function $V(x,s,q;a_*(s,q))$ admits the representation of (3.15) with (3.16) and the candidate stopping boundary $a_*(s,q)$ either solves the first-order nonlinear ordinary differential equation in the regions:

$$\widetilde{R}_{2k}(a_*) = \big\{ (x,s,q) \in E' \, \big| \, s^*_{2k}(q) \le s < s^*_{2k-1}(q) \big\} \tag{3.31}$$

both representing subsets of the continuation region C_* in (2.24), or coincides with $a'(s,q)$, for each $q>0$ fixed, and every $k \in \mathbb{N}$. Furthermore, we observe that the process (X,S,Q) can enter the region $\widetilde{R}_{2k-1}(a_*,b_*)$ in (3.31) from the region $\widetilde{R}_{2k}(a_*)$ in (3.30) only through the point $(s^*_{2k-1}(q), s^*_{2k-1}(q), q)$, for any $k \in \mathbb{N}$, by hitting the plane $d_1 = \{(x,s,q) \in \mathbb{R}^3 \, | \, 0 < q \le x = s\}$, so that by increasing its second component S. In this respect, the candidate value function should satisfy the instantaneous-stopping and smooth-fit conditions at the points $(s^*_{2k-1}(q), s^*_{2k-1}(q), q)$, that is expressed by the equalities:

$$\sum_{i=1}^{2} C_i\big(s^*_{2k-1}(q)-, q; a_*(s^*_{2k-1}(q)-, q)\big) \, (s^*_{2k-1}(q))^{\gamma_i(s^*_{2k-1}(q),q)} \tag{3.32}$$
$$= V\big(s^*_{2k-1}(q), s^*_{2k-1}(q), q; a_*(s^*_{2k-1}(q), q), b_*(s^*_{2k-1}(q), q)\big)$$
$$\sum_{i=1}^{2} C_i\big(s^*_{2k-1}(q)-, q; a_*(s^*_{2k-1}(q)-, q)\big) \, \gamma_i(s^*_{2k-1}(q)-, q) \, (s^*_{2k-1}(q))^{\gamma_i(s^*_{2k-1}(q),q)} \tag{3.33}$$
$$= s^*_{2k-1}(q) \, \partial_x V\big(s^*_{2k-1}(q), s^*_{2k-1}(q), q; a_*(s^*_{2k-1}(q), q), b_*(s^*_{2k-1}(q), q)\big)$$

where the functions $C_i(s,q;a_*(s,q))$, for $i=1,2$, are given by (3.16) and the function $V(x,s,q;a_*(s,q),b_*(s,q))$ has the form of (3.13) with (3.14), while the boundary $a_*(s,q)$ provides a unique solution of the first-order nonlinear ordinary differential equation in (3.24) in the region $\widetilde{R}_{2k}(a_*)$ from (3.31) satisfying the (starting) conditions of (3.32)-(3.33) above, for each $q>0$ fixed, and every $k \in \mathbb{N}$, respectively.

On the other hand, we can set $q^*_0(s) = 0$ and define the functions $q^*_{2l-1}(s) = \inf\{q > q^*_{2l-2}(s) \, | \, a_*(s,q) < q\}$ and $q^*_{2l}(s) = \inf\{q > q^*_{2l-1}(s) \, | \, a_*(s,q) > q\}$, whenever they exist, and put $q^*_{2l-1}(s) = q^*_{2l}(s) = \infty$ otherwise, so that the inequalities $0 \le q^*_{2l-2}(s) \le q^*_{2l-1}(s) \le \infty$ hold, for all $l \in \mathbb{N}$, and each $s>0$ fixed. In other words, the boundary $a_*(s,q)$ exits the region E from the side of the diagonal $d_2 = \{(x,s,q) \in \mathbb{R}^3 \, | \, 0 < q = x \le s\}$ passing through the points $(q^*_{2l-1}(s), s, q^*_{2l-1}(q))$ and comes back to E from the side of the diagonal d_2 passing through the points $(q^*_{2l-1}(s), s, q^*_{2l-1}(s))$, for $k \in \mathbb{N}$, for each $s>0$ fixed. Hence, the candidate value function $V(x,s,q;a_*(s,q),b_*(s,q))$ admits the representation of (3.13) with (3.14) and the candidate stopping boundaries $a_*(s,q)$ and $b_*(s,q)$ solve the system of arithmetic equations in (3.21) in the regions:

$$\widehat{R}_{2l-1}(a_*,b_*) = \big\{ (x,s,q) \in E' \, \big| \, q^*_{2l-2}(s) < q \le q^*_{2l-1}(s) \big\} \tag{3.34}$$

while the candidate value function $V(x,s,q;b_*(s,q))$ admits the representation of (3.17) with (3.18) and the candidate stopping boundary $b_*(s,q)$ solves the first-order nonlinear ordinary differential equation in the regions:

$$\widehat{R}_{2l}(b_*) = \big\{ (x,s,q) \in E' \, \big| \, q^*_{2l-1}(s) < q \le q^*_{2l}(s) \big\} \tag{3.35}$$

both representing subsets of the continuation region C_* in (2.24), for each $s>0$ fixed, and every $l \in \mathbb{N}$. Furthermore, we observe that the process (X,S,Q) can enter the region $\widehat{R}_{2l-1}(a_*,b_*)$ in (3.35) from the region $\widehat{R}_{2l}(b_*)$ in (3.34) only through the point $(q^*_{2l-1}(s), s, q^*_{2l-1}(s))$, for any $l \in \mathbb{N}$, by hitting the plane $d_2 = \{(x,s,q) \in \mathbb{R}^3 \,|\, 0<q=x\le s\}$, so that by decreasing its third component Q. In this respect, the candidate value function should satisfy the instantaneous-stopping and smooth-fit conditions at the points $(q^*_{2l-1}(s), s, q^*_{2l-1}(s))$, that is expressed by the equalities:

$$\sum_{i=1}^{2} C_i\big(s, q^*_{2l-1}(s)+; b_*(s, q^*_{2l-1}(s)+)\big)\,(q^*_{2l-1}(s))^{\gamma_j(s,q^*_{2l-1}(s))} \tag{3.36}$$
$$= V\big(q^*_{2l-1}(s), s, q^*_{2l-1}(s); a_*(s, q^*_{2l-1}(s)), b_*(s, q^*_{2l-1}(s))\big)$$
$$\sum_{i=1}^{2} C_i\big(s, q^*_{2l-1}(s)+; b_*(s, q^*_{2l-1}(s)+)\big)\,\gamma_i(s, q^*_{2l-1}(s))\,(q^*_{2l-1}(s))^{\gamma_i(s,q^*_{2l-1}(s))} \tag{3.37}$$
$$= q^*_{2l-1}(s)\,\partial_x V\big(q^*_{2l-1}(s), s, q^*_{2l-1}(s); a_*(s, q^*_{2l-1}(s)), b_*(s, q^*_{2l-1}(s))\big)$$

where the functions $C_i(s,q;b_*(s,q))$, for $i=1,2$, are given by (3.18) and the function $V(x,s,q;a_*(s,q),b_*(s,q))$ has the form of (3.13) with (3.14), while the boundary $b_*(s,q)$ provides a unique solution of the first-order nonlinear ordinary differential equation in (3.27) in the region $\widehat{R}_{2l}(b_*)$ from (3.35) satisfying the (starting) conditions of (3.36)-(3.37) above, for each $s>0$ fixed, and every $k \in \mathbb{N}$, respectively. Note that the process (X,S,Q) cannot come from the region $\widetilde{R}_{2k}(a_*)$ in (3.31) directly to the region $\widehat{R}_{2l}(b_*)$ in (3.35) and vice versa without crossing the regions $\widetilde{R}_{2k}(a_*,b_*)$ in (3.30) or $\widehat{R}_{2l}(a_*,b_*)$ in (3.34), for every $k,l \in \mathbb{N}$, respectively.

Finally, we observe that, if we have $\gamma_i(s,q)=\gamma_i(q)$, for $i=1,2$, in (3.2), then the appropriate left-hand exercise boundary for the floating-strike lookback game option in (2.24)-(2.25) takes the form $a_*(s,q) = a'(s,q) \vee g_*(q)s$ with some function $0<g_*(q)<1$, while, if we have $\gamma_i(s,q)=\gamma_i(s)$, for $i=1,2$, in (3.2), then the appropriate right-hand exercise boundary for that contract there takes the form $b_*(s,q)=b'(s,q)\wedge h_*(s)q$ with some function $h_*(s)>1$, for all $0<q<s$. In these cases, we have the sole regions $\widetilde{R}_1(a_*,b_*)$, $\widetilde{R}_2(a_*)$ in (3.30)-(3.31) with $k=1$ and $\widehat{R}_1(a_*,b_*)$, $\widehat{R}_2(b_*)$ in (3.34)-(3.35) with $l=1$, respectively. Moreover, if we have $\gamma_i(s,q)=\gamma_i$, for $i=1,2$, in (3.2), then we have $a_*(s,q)=a'(s,q)\vee a_*(s)=a'(s,q)\vee g_*s$ and $b_*(s,q)=b'(s,q)\wedge b_*(q)=b'(s,q)\wedge h_*q$ with some constants $0<g_*<1$ and $h_*>1$, for all $0<q<s$. The latter property can be explained by the fact that the original problem of (1.4), and thus, the three-dimensional problem of (2.3) can be reduced to an optimal stopping zero-sum game for the process $(S/X, Q/X) = (S_t/X_t, Q_t/X_t)_{t\ge0}$ representing a two-dimensional Markov diffusion process with reflection, by means of the change-of-measure arguments from [65] (see also [21]).

3.4 The system of arithmetic equations for the boundaries. Let us now extend the arguments from [26; Example 4.2] (see also [18; Section 3] and [60; Theorem 1]) to show that the system of arithmetic power equations in (3.21) admits a unique solution. For this purpose, by virtue of straightforward calculations, we first observe that the system of equations in (3.21) is equivalent to the following one:

$$\Xi_i(a(s,q);s,q) = \Upsilon_i(b(s,q);s,q) \tag{3.38}$$

for $a'(s,q) \vee q \le a(s,q) < b(s,q) \le s \wedge b'(s,q)$, where we set:

$$\Xi_i(a;s,q) = \frac{\gamma_i(s,q)(s-Ka)+Ka}{a^{\gamma_{3-i}(s,q)}} \quad \text{and} \quad \Upsilon_i(b;s,q) = \frac{\gamma_i(s,q)(q-Lb)+Lb}{b^{\gamma_{3-i}(s,q)}} \tag{3.39}$$

for all $0<q<a<b<s$ and every $i=1,2$.

In order to show the existence and uniqueness of a solution of the system of arithmetic power equations in (3.38) with (3.39), we develop the idea of proof of the existence and uniqueness of solutions applied to the systems of arithmetic power equations in [26; Example 4.2] (see also the systems (4.73)-(4.74) in [67; Chapter IV, Section 2], the system (3.16)-(3.17) in [18; Section 3], and [60; Theorem 1]). For this purpose, we observe that, for the derivatives of the functions $\Xi_i(a)$ and $\Upsilon_i(b)$, for $i=1,2$, defined in (3.38), the expressions:

$$\Xi_i'(a;s,q) = \frac{(\gamma_1(s,q)-1)(\gamma_2(s,q)-1)K(a-\overline{a}(s,q))}{a^{\gamma_{3-i}(s,q)+1}} \tag{3.40}$$

and

$$\Upsilon_i'(b;s,q) = \frac{(\gamma_1(s,q)-1)(\gamma_2(s,q)-1)L(b-\underline{b}(s,q))}{b^{\gamma_{3-i}(s,q)+1}} \tag{3.41}$$

hold, so that the following inequalities:

$$\Xi_i'(a;s,q)>0 \quad \text{for} \quad a<\overline{a}(s,q) \quad \text{and} \quad \Upsilon_i'(b;s,q)<0 \quad \text{for} \quad b>\underline{b}(s,q) \tag{3.42}$$

are satisfied, for $a'(s,q) \vee q < a < b < s \wedge b'(s,q)$ and every $i=1,2$, where the functions $a'(s,q)$ and $b'(s,q)$ are given by (2.9). Hence, we may conclude that the following properties hold, for each $0<q<s$ fixed. Namely, the function $\Xi_1(a;s,q)$ increases on the interval $(0,\overline{a}(s,q))$ with $\Xi_1(0+;s,q)=0$ and $\Xi_1(\overline{a}(s,q);s,q)>0$, so that the range of its values is given by the interval $(0,\Xi_1(\overline{a}(s,q);s,q))$. The function $\Upsilon_1(b;s,q)$ decreases on the interval $(\underline{b}(s,q),\infty)$ with $\Upsilon_1(\underline{b}(s,q);s,q)>0$ and $\Upsilon_1(\infty;s,q)=-\infty$, so that the range of its values is given by the interval $(-\infty,\Upsilon_1(\underline{b}(s,q);s,q))$. The function $\Xi_2(a;s,q)$ increases on the interval $(0,\overline{a}(s,q))$ with $\Xi_2(0+;s,q)=-\infty$ and $\Xi_2(\overline{a}(s,q);s,q)>0$, so that the range of its values is given by the interval $(-\infty,\Xi_2(\overline{a}(s,q);s,q))$. The function $\Upsilon_2(b;s,q)$ decreases on the interval $(\underline{b}(s,q),\infty)$ with $\Upsilon_2(\underline{b}(s,q);s,q)>0$ and $\Upsilon_2(\infty;s,q)=0$, so that the range of its values is given by the interval $(0,\Upsilon_2(\underline{b}(s,q);s,q))$.

We now observe that, when $\Xi_i(\overline{a}(s,q);s,q) \le \Upsilon_i(\underline{b}(s,q);s,q)$ holds, one can determine some $\widehat{b}_i(s,q) \ge \underline{b}(s,q)$ from the equation $\Xi_i(\overline{a}(s,q);s,q) = \Upsilon_i(\widehat{b}_i(s,q);s,q)$, while, when $\Xi_i(\overline{a}(s,q);s,q) \ge \Upsilon_i(\underline{b}(s,q);s,q)$ holds, one can determine some $\widehat{a}_i(s,q) \le \overline{a}(s,q)$ from the equation $\Xi_i(\widehat{a}_i(s,q);s,q) = \Upsilon_i(\underline{b}(s,q);s,q)$, for each $0<q<s$ fixed, and every $i=1,2$. Hence, it follows from the equations in (3.38) with (3.39) that, for each $a \in (\underline{a}(s,q), \widehat{a}_1(s,q) \wedge \overline{a}(s,q))$, there exists a unique number $b \in (\underline{b}(s,q) \vee \widehat{b}_1(s,q), \overline{b}(s,q))$, while, for each $a \in (\underline{a}(s,q), \widehat{a}_2(s,q) \wedge \overline{a}(s,q))$, there exists a unique number $b \in (\underline{b}(s,q) \vee \widehat{b}_2(s,q), \overline{b}(s,q))$, for each $0<q<s$ fixed. (Recall that the values $\overline{a}(s,q)$ and $\underline{b}(s,q)$ are specified in Part (ii) of Subsection 2.2 above and given by the expressions in (2.19), while the values $\underline{a}(s,q)$ and $\overline{b}(s,q)$ are determined in Theorem 5.1 below, for each $0<q<s$ fixed.) In other words, we may conclude that the first and second equations in (3.38) uniquely determine the function $b_1(a)$ on $(\underline{a}(s,q), \widehat{a}_1(s,q) \wedge \overline{a}(s,q))$ with the range $(\underline{b}(s,q) \vee \widehat{b}_1(s,q), \overline{b}(s,q))$ and the function $b_2(a)$ on $(\underline{a}(s,q), \widehat{a}_2(s,q) \wedge \overline{a}(s,q))$

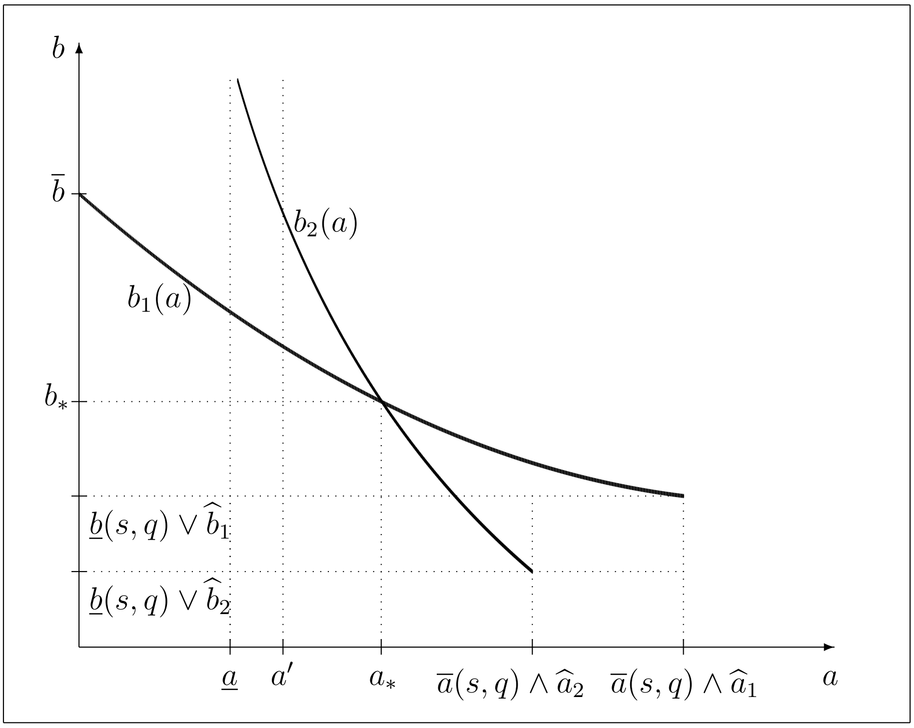

Figure 3. A computer drawing of the boundary functions $b_1(a)$ and $b_2(a)$ in the case $a'(s,q) \le a_*(s,q) < b_*(s,q) \le b'(s,q)$, for each $0 < q < s$ fixed.

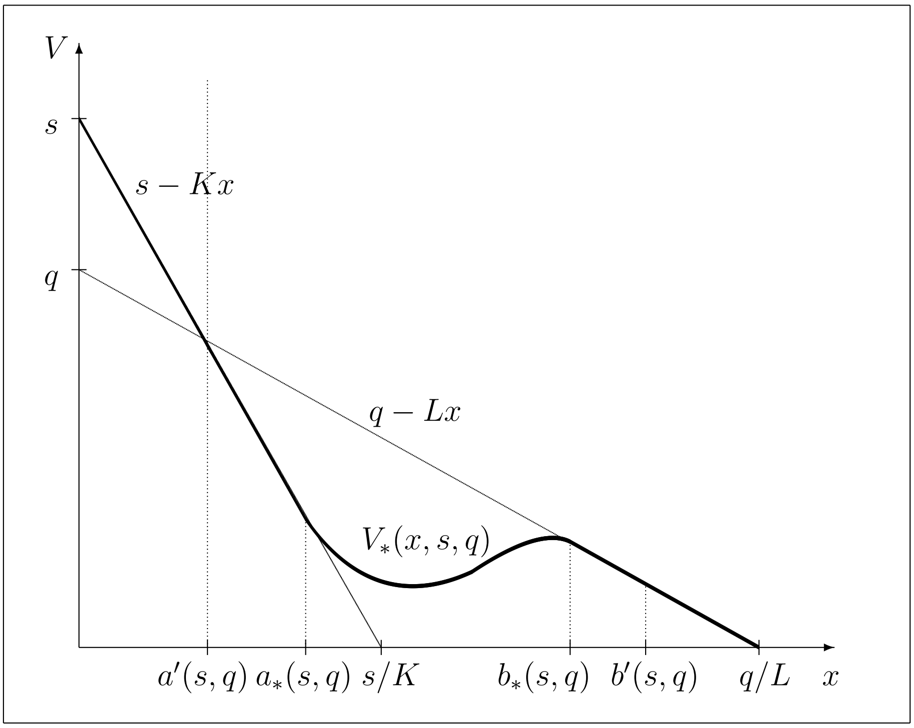

Figure 4. A computer drawing of the value function $V_*(x,s,q)$ and optimal exercise boundaries $a'(s,q) < a_*(s,q) < b_*(s,q) < b'(s,q)$, for each $0 < q < s$ fixed.

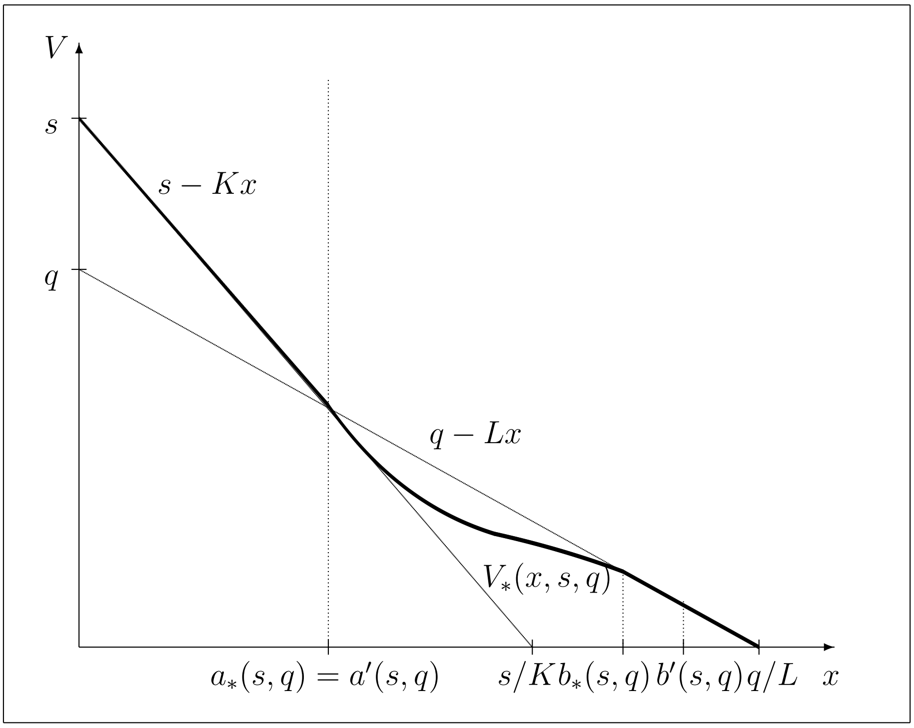

Figure 5. A computer drawing of the value function $V_*(x,s,q)$ and optimal exercise boundaries $a'(s,q) = a_*(s,q) < b_*(s,q) < b'(s,q)$, for each $0 < q < s$ fixed.

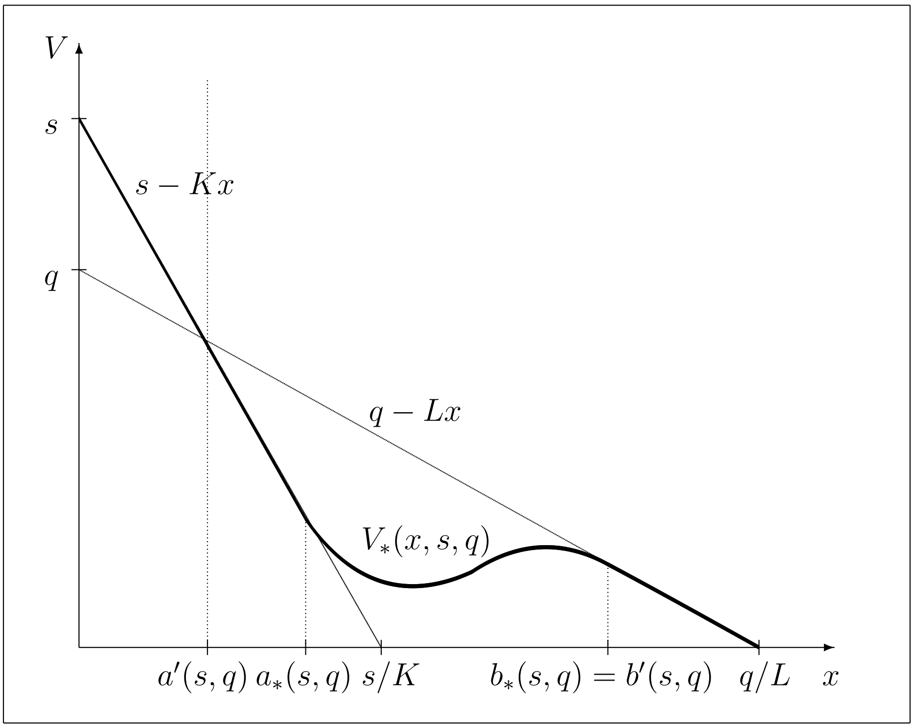

Figure 6. A computer drawing of the value function $V_*(x,s,q)$ and optimal exercise boundaries $a'(s,q) < a_*(s,q) < b_*(s,q) = b'(s,q)$, for each $0 < q < s$ fixed.

with the range $(\underline{b}(s,q)\vee\widehat{b}_2(s,q), \overline{b}(s,q))$, for each $0<q<s$ fixed, respectively. These arguments also imply that the expression $\underline{b}(s,q)\vee\widehat{b}_1(s,q) < b_1(\underline{a}(s,q)) < \overline{b}(s,q) < b_2(\underline{a}(s,q))$ holds too. Moreover, the same arguments and assumptions directly yield that there exists exactly one intersection point with the coordinates $a_*(s,q)$ and $b_*(s,q)$ of the curves associated with the functions $b_1(a)$ and $b_2(a)$ on the interval $a\in(\underline{a}(s,q), \widehat{a}_2(s,q)\wedge\overline{a}(s,q))$ such that $\underline{b}(s,q)\vee\widehat{b}_1(s,q) < b_1(a_*(s,q)) \equiv b_*(s,q) \equiv b_2(a_*(s,q)) < \overline{b}(s,q)$ holds, for each $0<q<s$ fixed (see Figure 3 above).

More precisely, let us assume that there exists at least two intersection points $(a_*(s,q), b_*(s,q))$ and $(\widetilde{a}(s,q), \widetilde{b}(s,q))$ of the curves $b_1(a)$ and $b_2(a)$ such that $\underline{a}(s,q) < \widetilde{a}(s,q) < a_*(s,q) \le \overline{a}(s,q)\wedge\widehat{a}_2(s,q)$ and $\underline{b}(s,q)\vee\widehat{b}_1(s,q) \le \widetilde{b}(s,q) < b_*(s,q) < \overline{b}(s,q)$ (or $\underline{a}(s,q) < a_*(s,q) < \widetilde{a}(s,q) \le \overline{a}(s,q)\wedge\widehat{a}_2(s,q)$ and $\underline{b}(s,q)\vee\widehat{b}_1(s,q) \le b_*(s,q) < \widetilde{b}(s,q) < \overline{b}(s,q)$) as well as $b_2(a) > b_1(a)$, for $a\in(\widetilde{a}(s,q), a_*(s,q))$, and any $0<q<s$ fixed. Observe that, by virtue of assumptions made above and according to the implicit function theorem, it follows from the representations in (3.40)-(3.41) that the expressions:

$$b_i'(a) = \frac{\Xi_i'(a)}{\Upsilon_i'(b)} = \frac{K(a-\overline{a}(s,q))}{L(b-\underline{b}(s,q))}\left(\frac{b}{a}\right)^{\gamma_{3-i}(s,q)+1} < 0 \tag{3.43}$$

hold, for every $i=1,2$, for all $a\in(\widetilde{a}(s,q), a_*(s,q))$ and $b\in(\widetilde{b}(s,q), b_*(s,q))$, from where it directly follows that the inequality:

$$\frac{b_2'(a)}{b_1'(a)} = \left(\frac{b}{a}\right)^{\gamma_1(s,q)-\gamma_2(s,q)} > 1 \tag{3.44}$$

is satisfied, for all $a\in(\widetilde{a}(s,q), a_*(s,q))$. Since the derivatives $b_i'(a)$, for $i=1,2$, from (3.43) are continuous functions on $(\widetilde{a}(s,q), a_*(s,q))$, we may conclude that there exists an open interval $(\widetilde{a}(s,q)-\varepsilon, \widetilde{a}(s,q)+\varepsilon)$, for some relatively small $\varepsilon>0$, such that the inequality $b_2'(a) > b_1'(a)$ holds, so that the inequality $b_2(a) > b_1(a)$ should hold, for $a\in(\widetilde{a}(s,q)-\varepsilon, \widetilde{a}(s,q)+\varepsilon)$, too. However, the latter fact contradicts the assumption that the equality $b_1(\widetilde{a}(s,q)) = b_2(\widetilde{a}(s,q))$ holds, which means that the curves $b_1(a)$ and $b_2(a)$ may have only one intersection point, and thus, it completes the proof of the claim.

Furthermore, we recall that the functions $a_*(s,q)$ and $b_*(s,q)$ determined by means of the arguments above provide the candidate optimal stopping boundaries, whenever the inequalities $a'(s,q) \le a_*(s,q) < b_*(s,q) \le b'(s,q)$ hold for the solution of the system in (3.38) with (3.39), with $a'(s,q)$ and $b'(s,q)$ from (2.9), for each $0<q<s$ fixed. Otherwise, on the one hand, it is shown by means of arguments similar to the ones used above that, if the inequalities $a'(s,q) \le a_*(s,q) < b'(s,q) < b_*(s,q)$ hold for the solution of the system in (3.38) with (3.39), then the right-hand candidate stopping boundary $b_*(s,q)$ should coincide with $b'(s,q)$, while the the left-hand candidate stopping boundary $a_*(s,q) < b'(s,q)$ represents the smallest root (or the minimal solution) of the arithmetic equation in (3.22), which takes the form:

$$\begin{aligned}&\sum_{i=1}^{2}(-1)^i\left(\frac{\gamma_{3-i}(s,q)s}{b'(s,q)} + (1-\gamma_{3-i}(s,q))\,K\,\frac{a(s,q)}{b'(s,q)}\right)\left(\frac{a(s,q)}{b'(s,q)}\right)^{-\gamma_i(s,q)} \\ &= (\gamma_1(s,q)-\gamma_2(s,q))\,(q/b'(s,q)-L)\end{aligned} \tag{3.45}$$

for $0 < q < s$. On the other hand, it is shown by means of arguments similar to the ones used above that, if the inequalities $a_*(s,q) < a'(s,q) < b_*(s,q) \le b'(s,q)$ hold for the solution of the system in (3.38) with (3.39), then the left-hand candidate stopping boundary $a_*(s,q)$ should coincide with $a'(s,q)$, while the the right-hand candidate stopping boundary $b_*(s,q) > a'(s,q)$ represents the largest root (or the maximal solution) of the arithmetic equation in (3.23), which takes the form:

$$\sum_{i=1}^{2}(-1)^i \left(\frac{\gamma_{3-i}(s,q)q}{a'(s,q)} + (1-\gamma_{3-i}(s,q))\, L\, \frac{b(s,q)}{a'(s,q)}\right)\left(\frac{b(s,q)}{a'(s,q)}\right)^{-\gamma_i(s,q)} \\ = (\gamma_1(s,q)-\gamma_2(s,q))\,(s/a'(s,q)-K) \tag{3.46}$$

for $0 < q < s$. We finally note that, in the case in which neither the system of arithmetic equations in (3.38) with (3.39) nor the equations in (3.45) and (3.46) admit solutions, for which either the inequalities $a_*(s,q) \ge a'(s,q)$ or $b_*(s,q) \le b'(s,q)$ hold, respectively, we may conclude that the both candidate stopping boundaries $a_*(s,q)$ and $b_*(s,q)$ should coincide with the boundaries $a'(s,q)$ and $b'(s,q)$, for $0 < q < s$, respectively.

4. Main results and proofs

In this section, being based on the facts proved above, we formulate and prove the main result of the paper concerning the three-dimensional optimal stopping zero-sum game of (2.3) in the model from (1.1)-(1.2) and (1.3).

Theorem 4.1 *Let the process (X,S,Q) be defined in (1.1)-(1.2) and (1.3), where $r > 0$ is a constant, and $\delta(s,q) > 0$ and $\sigma(s,q) > 0$ are continuously differentiable bounded functions on $[0,\infty]^2$. Assume that the function $\delta(s,q)$ is increasing in the both variables s and q on $[0,\infty]^2$. Then, the value function in (2.3) of the perpetual floating-strike lookback game option takes the form:*

$$V_*(x,s,q) = \begin{cases} V(x,s,q;a_*(s,q),b_*(s,q)), & \text{if} \quad 0 < q \le a_*(s,q) < x < b_*(s,q) \le s \\ V(x,s,q;a_*(s,q)), & \text{if} \quad 0 < q \le a_*(s,q) < x \le s < b_*(s,q) \\ V(x,s,q;b_*(s,q)), & \text{if} \quad 0 < a_*(s,q) < q \le x < b_*(s,q) \le s \\ F(x,s), & \text{if} \quad 0 < q \le x \le a_*(s,q) < s \\ G(x,s,q), & \text{if} \quad 0 < q < b_*(s,q) \le x \le s \end{cases} \tag{4.1}$$

where the functions $F(x,s)$ and $G(x,s,q)$ are defined in (1.5), for some $0 < L < K < L+1$ given and fixed, and the optimal exercise times forming a Nash equilibrium in the game are given by:

$$\tau_* = \inf\big\{t \ge 0 \,\big|\, X_t \le a_*(S_t,Q_t)\big\} \quad \text{and} \quad \zeta_* = \inf\big\{t \ge 0 \,\big|\, X_t \ge b_*(S_t,Q_t)\big\} \tag{4.2}$$

where the stopping boundaries satisfy the inequalities $a'(s,q) \vee \underline{a}(s,q) \le a_*(s,q) \le \overline{a}(s,q) \wedge s \wedge b'(s,q)$ and $a'(s,q) \vee q \vee \underline{b}(s,q) \le b_*(s,q) \le \overline{b}(s,q) \wedge b'(s,q)$ with $a'(s,q)$ and $b'(s,q)$ given by (2.9) as well as $\overline{a}(s,q)$ and $\underline{b}(s,q)$ given by (2.19), while $\underline{a}(s,q)$ and $\overline{b}(s,q)$ determined *in Theorem 5.1 below, and together with the candidate value functions are further specified as follows:*

(i) the function $V(x,s,q;a_*(s,q),b_*(s,q))$ is given by (3.13)-(3.14), while either the boundaries $a_*(s,q)$ and $b_*(s,q)$ provide a unique solution of the system of arithmetic equations in (3.21), whenever $a'(s,q) \le a_*(s,q) < b_*(s,q) \le b'(s,q)$ (see Figure 4 above), or $b_*(s,q) = b'(s,q)$ and $a_*(s,q)$ provides the smallest root of the arithmetic equation in (3.22), whenever $a'(s,q) \le a_*(s,q) < b'(s,q)$ (see Figure 5 above), or $a_*(s,q) = a'(s,q)$ and $b_*(s,q)$ provides the largest root of the arithmetic equation in (3.23), whenever $b'(s,q) \ge b_*(s,q) > a'(s,q)$ (see Figure 6 above), or $a_*(s,q) = a'(s,q)$ and $b_*(s,q) = b'(s,q)$, otherwise, in the regions $\widetilde{R}_{2k-1}(a_*,b_*)$ and $\widehat{R}_{2l-1}(a_*,b_*)$ from (3.30) and (3.34), for $k,l \in \mathbb{N}$, for $0<q<s$;

(ii) the function $V(x,s,q;a_*(s,q))$ is given by (3.15)-(3.16), while the boundary $a_*(s,q)$ provides either a unique solution of the first-order nonlinear ordinary differential equation in (3.24) started at $(s^*_{2k-1}(q), s^*_{2k-1}(q), q)$, whenever $a_*(s,q) \ge a'(s,q)$, or coincides with $a'(s,q)$ otherwise, in the regions $\widetilde{R}_{2k}(a_*)$ from (3.31), for $k \in \mathbb{N}$, for $0<q<s$;

(iii) the function $V(x,s,q;b_*(s,q))$ is given by (3.17)-(3.18), while the boundary $b_*(s,q)$ provides a unique solution of the first-order nonlinear ordinary differential equation in (3.27) started at $(q^*_{2l-1}(s), s, q^*_{2l-1}(s))$, whenever $b_*(s,q) \le b'(s,q)$, or coincides with $b'(s,q)$ otherwise, in the regions $\widehat{R}_{2l}(b^*_i)$ from (3.35), for $l \in \mathbb{N}$, for $0<q<s$.

Observe that we can put $s=q=x$ to obtain the value of the original perpetual floating-strike lookback game option pricing problem of (1.4) from the value function of the optimal stopping zero-sum game of (2.3).

Proof In order to verify the assertion stated above, it remains for us to show that the function defined in (4.1) coincides with the value function in (2.3) and the stopping times τ_* and ζ_* in (4.2) form a Nash equilibrium with the boundaries $a_*(s,q)$ and $b_*(s,q)$ specified in the previous section. For this purpose, let us denote by $V(x,s,q)$ the right-hand side of the expression in (4.1) associated with these boundaries $a_*(s,q)$ and $b_*(s,q)$. Then, it follows from the straightforward calculations presented in the previous section that the function $V(x,s,q)$ solves the system of (2.28)-(2.30), while the smooth-fit and normal-reflection conditions of (2.33)-(2.34) are satisfied in the appropriate regions $\widetilde{R}_{2k-1}(a_*,b_*)$, $\widetilde{R}_{2k}(a_*)$ from (3.30)-(3.31), for $k \in \mathbb{N}$, and $\widehat{R}_{2l-1}(a_*,b_*)$, $\widehat{R}_{2l}(b_*)$ from (3.34)-(3.35), for $l \in \mathbb{N}$, respectively. We also observe that the function $V(x,s,q)$ is $C^{2,1,1}$ on the closure $\overline{C}_*$ of C_* from (2.24) and D_* from (2.25), by construction. Hence, taking into account the fact that the boundaries $a_*(s,q)$ and $b_*(s,q)$ are assumed to be continuously differentiable, for $0<q<s$, by applying the change-of-variable formula from [54; Theorem 3.1] (see also [59; Chapter II, Section 3.5] for a summary of the related results and further references) to the process $e^{-r(t\wedge\theta)}V(X_{t\wedge\theta}, S_{t\wedge\theta}, Q_{t\wedge\theta})$, we obtain:

$$e^{-r(t\wedge\theta)}\, V(X_{t\wedge\theta}, S_{t\wedge\theta}, Q_{t\wedge\theta}) = V(x,s,q) + M_{t\wedge\theta} \tag{4.3}$$
$$+ \int_0^{t\wedge\theta} e^{-ru}\, (\mathbb{L}V - rV)(X_u,S_u,Q_u)\, I\big(X_u \neq S_u, X_u \neq Q_u, X_u \neq a_*(S_u,Q_u), X_u \neq b_*(S_u,Q_u)\big)\, du$$
$$+ \int_0^{t\wedge\theta} e^{-ru}\, \partial_s V(X_u,S_u,Q_u)\, I(X_u = S_u)\, dS_u + \int_0^{t\wedge\theta} e^{-ru}\, \partial_q V(X_u,S_u,Q_u)\, I(X_u = Q_u)\, dQ_u$$

for all $t \geq 0$, holds with θ defined in (2.10). Here, the process $(M_{t\wedge\theta})_{t\geq 0}$ defined by:

$$M_{t\wedge\theta} = \int_0^{t\wedge\theta} e^{-ru}\, \partial_x V(X_u, S_u, Q_u)\, I\big(X_u \neq S_u, X_u \neq Q_u\big)\, \sigma(S_u, Q_u)\, X_u\, dB_u \tag{4.4}$$

is a continuous local martingale under the probability measure $\mathbb{P}_{x,s,q}$. Note that, since the time spent by the process (X, S, Q) at the boundary surfaces $\{(x, s, q) \in E' \,|\, x = a_*(s, q)\}$ and $\{(x, s, q) \in E' \,|\, x = b_*(s, q)\}$ as well as at the planes $d_1 = \{(x, s, q) \in \mathbb{R}^3 \,|\, 0 < q \leq x = s\}$ and $d_2 = \{(x, s, q) \in \mathbb{R}^3 \,|\, 0 < q = x \leq s\}$ is of the Lebesgue measure zero (see, e.g. [10; Chapter II, Section 1]), the indicators in the second line of the formula of (4.3) as well as in the formula of (4.4) can be ignored. Moreover, since the process S increases only when the process (X, S, Q) is located on the plane d_1, while the process Q decreases only when the process (X, S, Q) is located on the plane d_2, the indicators in the third line of (4.3) can be set equal to one. Finally, taking into account the fact that the candidate value function $V(x, s, q)$ satisfies the normal reflection conditions of (2.34) at the diagonals d_1 and d_2 in the regions $\widetilde{R}_{2k}(a_*)$, for $k \in \mathbb{N}$, in (3.35) and $\widehat{R}_{2l}(b_*)$, for $l \in \mathbb{N}$, we may conclude that the integrals in the third line of (4.3) are actually equal to zero.

By using straightforward calculations and the arguments from the previous section, it is verified that the inequality $(\mathbb{L}V - rV)(x, s, q) \leq 0$ holds, for all $a'(s, q) < x < b_*(s, q)$ such that $x \neq a_*(s, q)$, and the inequality $(\mathbb{L}V - rV)(x, s, q) \geq 0$ holds, for all $x > a_*(s, q)$ such that $x \neq b_*(s, q)$, as well as $x \neq s$ and $x \neq q$. Moreover, we observe directly from the expressions in (3.13), (3.15), and (3.17) that the function $V(x, s, q) - (s - Kx)$ increases in the variable x on the interval $q \vee a_*(s, q) < x < b_*(s, q) \wedge s$ from 0 (when $a_*(s, q) \geq q$) to the value $q - s + (K - L)b_*(s, q)$ (when $b_*(s, q) \leq s$), because the expression $\partial_x V(x, s, q) + K$ for the first-order partial derivative in (3.19) is positive there, for $(x, s, q) \in E'$. We also note from the expressions in (3.13), (3.15), and (3.17) that the function $V(x, s, q)$ is convex in the variable x in a left-hand neighborhood of $q \vee a_*(s, q)$ and concave in a right-hand neighborhood of $b_*(s, q) \wedge s$, because its second-order partial derivative $\partial_{xx} V(x, s, q)$ is positive in a left-hand neighborhood of $q \vee a_*(s, q)$ and negative in a right-hand neighborhood of $b_*(s, q) \wedge s$, for $(x, s, q) \in E'$. Thus, we may conclude that the inequalities in (2.31) hold, which together with the conditions of (2.29)-(2.30) and (2.33) imply that the inequalities $s - Kx \leq V(x, s, q) \leq q - Lx$ are satisfied, for all $(x, s, q) \in E'$, under $0 < L < K < L + 1$ given and fixed. It therefore follows from the expression in (4.3) that the inequalities:

$$e^{-r(\tau\wedge\zeta_*)}\, F(X_{\tau\wedge\zeta_*}, S_{\tau\wedge\zeta_*}) \leq e^{-r(\tau\wedge\zeta_*)}\, V(X_{\tau\wedge\zeta_*}, S_{\tau\wedge\zeta_*}, Q_{\tau\wedge\zeta_*}) \leq V(x, s, q) + M_{\tau\wedge\zeta_*} \tag{4.5}$$

and

$$e^{-r(\tau_*\wedge\zeta)}\, G(X_{\tau_*\wedge\zeta}, S_{\tau_*\wedge\zeta}, Q_{\tau_*\wedge\zeta}) \geq e^{-r(\tau_*\wedge\zeta)}\, V(X_{\tau_*\wedge\zeta}, S_{\tau_*\wedge\zeta}, Q_{\tau_*\wedge\zeta}) \geq V(x, s, q) + M_{\tau_*\wedge\zeta} \tag{4.6}$$

hold, for any stopping times τ and ζ of the process (X, S, Q), because $\tau_* \wedge \zeta_* \leq \theta$ ($\mathbb{P}_{x,s,q}$-a.s.) holds with θ defined in (2.10).

Now, consider the localising sequence $(\varkappa_n)_{n\in\mathbb{N}}$ for the local martingale $(M_{t\wedge\theta})_{t\geq 0}$ from (4.4) such that $\varkappa_n = \inf\{t \geq 0 \,|\, |M_{t\wedge\theta}| \geq n\}$, for each $n \in \mathbb{N}$. Then, inserting $\tau \wedge \varkappa_n$ and $\zeta \wedge \varkappa_n$ instead of τ and ζ in (4.5) and (4.6) and taking the expectations with respect to the probability measure $\mathbb{P}_{x,s,q}$ in (4.5) and (4.6), by means of Doob's optional sampling theorem, we get:

$$\begin{aligned}&\mathbb{E}_{x,s,q}\big[e^{-r(\tau\wedge\zeta_*\wedge\varkappa_n)}\,\big(F(X_{\tau\wedge\varkappa_n},S_{\tau\wedge\varkappa_n})\,I(\tau\wedge\varkappa_n<\zeta_*)+G(X_{\zeta_*},S_{\zeta_*},Q_{\zeta_*})\,I(\zeta_*\le\tau\wedge\varkappa_n)\big)\big] \\ &\le \mathbb{E}_{x,s,q}\big[e^{-r(\tau\wedge\zeta_*\wedge\varkappa_n)}\,V(X_{\tau\wedge\zeta_*\wedge\varkappa_n},S_{\tau\wedge\zeta_*\wedge\varkappa_n},Q_{\tau\wedge\zeta_*\wedge\varkappa_n})\big] \\ &\le V(x,s,q)+\mathbb{E}_{x,s,q}\big[M_{\tau\wedge\zeta_*\wedge\varkappa_n}\big]=V(x,s,q)\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}&\mathbb{E}_{x,s,q}\big[e^{-r(\tau_*\wedge\zeta\wedge\varkappa_n)}\,\big(F(X_{\tau_*},S_{\tau_*})\,I(\tau_*<\zeta\wedge\varkappa_n)+G(X_{\zeta\wedge\varkappa_n},S_{\zeta\wedge\varkappa_n},Q_{\zeta\wedge\varkappa_n})\,I(\zeta\wedge\varkappa_n\le\tau_*)\big)\big] \\ &\ge \mathbb{E}_{x,s,q}\big[e^{-r(\tau_*\wedge\zeta\wedge\varkappa_n)}\,V(X_{\tau_*\wedge\zeta\wedge\varkappa_n},S_{\tau_*\wedge\zeta\wedge\varkappa_n},Q_{\tau_*\wedge\zeta\wedge\varkappa_n})\big] \\ &\ge V(x,s,q)+\mathbb{E}_{x,s,q}\big[M_{\tau_*\wedge\zeta\wedge\varkappa_n}\big]=V(x,s,q)\end{aligned} \tag{4.8}$$

for all $(x,s,q)\in E'$ and each $n\in\mathbb{N}$. Observe that, taking into account the arguments from [64; pages 635-636], it follows from the structure of the stopping times τ_* and ζ_* in (4.2) that the property:

$$\mathbb{E}_{x,s,q}\Big[\sup_{t\ge0}e^{-r((\tau_*\wedge\zeta_*)\wedge t)}\,S_{(\tau_*\wedge\zeta_*)\wedge t}\Big]=\mathbb{E}_{x,s,q}\Big[\sup_{t\ge0}e^{-r((\tau_*\wedge\zeta_*)\wedge t)}\,\big(X_{(\tau_*\wedge\zeta_*)\wedge t}\vee s\big)\Big]<\infty \tag{4.9}$$

holds, and the variables $e^{-r(\tau_*\wedge\zeta_*)}(S_{\tau_*\wedge\zeta_*}-KX_{\tau_*\wedge\zeta_*})$ and $e^{-r(\tau_*\wedge\zeta_*)}(Q_{\tau_*\wedge\zeta_*}-LX_{\tau_*\wedge\zeta_*})$ are finite on the set $\{\tau_*\wedge\zeta_*=\infty\}$ ($\mathbb{P}_{x,s,q}$-a.s.) as well as $\mathbb{P}_{x,s,q}(\tau_*\wedge\zeta_*<\infty)=1$, for all $(x,s,q)\in E'$. Hence, letting n go to infinity and using Fatou's lemma, we obtain that the inequalities:

$$\mathbb{E}_{x,s,q}\big[e^{-r(\tau\wedge\zeta_*)}\,\big(F(X_\tau,S_\tau)\,I(\tau<\zeta_*)+G(X_{\zeta_*},S_{\zeta_*},Q_{\zeta_*})\,I(\zeta_*\le\tau)\big)\big]\le V(x,s,q) \tag{4.10}$$

and

$$\mathbb{E}_{x,s,q}\big[e^{-r(\tau_*\wedge\zeta)}\,\big(F(X_{\tau_*},S_{\tau_*})\,I(\tau_*<\zeta)+G(X_\zeta,S_\zeta,Q_\zeta)\,I(\zeta\le\tau_*)\big)\big]\ge V(x,s,q) \tag{4.11}$$

hold, for any stopping times τ and ζ such that $\tau\wedge\zeta\le\theta$ ($\mathbb{P}_{x,s,q}$-a.s.) holds, and all $(x,s,q)\in E'$. Therefore, using the fact that the function $V(x,s,q)$ and the continuously differentiable boundaries $a_*(s,q)$ and $b_*(s,q)$ solve the second-order ordinary differential equation in (2.28) and satisfy the conditions of (2.29)-(2.30) and (2.33)-(2.34), inserting τ_* in place of τ and ζ_* in place of ζ into (4.10) and (4.11), we obtain that the equality:

$$\mathbb{E}_{x,s,q}\big[e^{-r(\tau_*\wedge\zeta_*)}\,\big(F(X_{\tau_*},S_{\tau_*})\,I(\tau_*<\zeta_*)+G(X_{\zeta_*},S_{\zeta_*},Q_{\zeta_*})\,I(\zeta_*\le\tau_*)\big)\big]=V(x,s,q) \tag{4.12}$$

holds, so that the candidate function $V(x,s,q)$ coincides with the value function $V_*(x,s,q)$ of the optimal stopping game in (2.3), for all $(x,s,q)\in E'$, and the optimal stopping times τ_* and ζ_* form a Nash equilibrium of the zero-sum game. We finally recall from the results of Part (ii) of Subsection 2.2 above implied by standard comparison arguments applied to the value functions of the appropriate optimal stopping problems that the inequalities $a'(s,q)\vee\underline{a}(s,q)\le a_*(s,q)<\overline{a}(s,q)\wedge s\wedge b'(s,q)$ and $a'(s,q)\vee q\vee\underline{b}(s,q)<b_*(s,q)\le\overline{b}(s,q)\wedge b'(s,q)$ should hold, for $0<q<s$, that completes the verification. □

5 Appendix

In this section, we derive closed-form expressions for the value functions and optimal stopping boundaries of some auxiliary optimal stopping problems, which provide the upper and lower bounds for the value function and optimal stopping boundaries of the original optimal stopping zero-sum game of (2.3).

5.1 The optimal stopping and free-boundary problems. In order to provide the upper and lower bounds for the value functions and optimal stopping boundaries in the optimal stopping game of (2.3) above, let us introduce the value functions $\overline{V}(x,s,q)$ and $\underline{V}(x,s,q)$ of the optimal stopping problems:

$$\overline{V}(x,s,q) = \sup_{\tau} \mathbb{E}_{x,s,q}\big[e^{-r\tau}\,(S_\tau - K\,X_\tau)\big] \quad \text{and} \quad \underline{V}(x,s,q) = \inf_{\zeta} \mathbb{E}_{x,s,q}\big[e^{-r\zeta}\,(Q_\zeta - L\,X_\zeta)\big] \tag{5.1}$$

for some given constants $K, L > 0$, respectively, where the supremum and infimum are taken with respect to all stopping times τ and ζ of the process X. It is shown by means of arguments similar to the ones used in Part (ii) of Subsection 2.2 based on the representations (2.14) and (2.15) for the reward functional, under the assumptions $\zeta = \infty$ and $\tau = \infty$, respectively, or by using the easily shown convexity and concavity of the value functions $\overline{V}(x,s,q)$ and $\underline{V}(x,s,q)$, that the optimal stopping times in the problems of (5.1) have the form:

$$\overline{\tau} = \inf\big\{t \geq 0 \;\big|\; X_t \leq \underline{a}(S_t, Q_t)\big\} \quad \text{and} \quad \underline{\zeta} = \inf\big\{t \geq 0 \;\big|\; X_t \geq \overline{b}(S_t, Q_t)\big\} \tag{5.2}$$

with some functions $\underline{a}(s,q)$ and $\overline{b}(s,q)$, for $0 < q < s$, to be determined (see [51] and [36] as well as [23; Section 3] and [21]). By means of arguments similar to the ones applied in [13; Subsection 3.2] and [52; Proposition 2.1], the existence of such boundaries $\underline{a}(s,q)$ and $\overline{b}(s,q)$ can be explained by the facts that the costs for the holder (maximiser of the expected discounted payoff) of waiting until the process X from (1.1) coming from such a small $x > 0$ increases to the current value of the running maximum process S and the costs for the writer (minimiser of the expected discounted payoff) of waiting until the process X coming from such a large $x > 0$ decreases to the current value of the running minimum process Q may be too large due to the structure of the integrands in the reward functionals of (2.16) and (2.17), respectively.

Following the arguments from [51] and [36] (see also [23; Section 3] and [21]) extended to the considered three-dimensional model, we may conclude that the unknown value functions $\overline{V}(x,s,q)$ and $\underline{V}(x,s,q)$ from (5.1) and the unknown boundaries $\underline{a}(s,q)$ and $\overline{b}(s,q)$ from (5.2) should solve the equivalent free-boundary problems:

$$(\mathbb{L}V - rV)(x,s,q) = 0 \quad \text{for} \quad q \vee a(s,q) < x < s \quad \text{or} \tag{5.3}$$
$$(\mathbb{L}V - rV)(x,s,q) = 0 \quad \text{for} \quad q < x < b(s,q) \wedge s \tag{5.4}$$
$$V(x,s,q)\big|_{x=a(s,q)+} = s - K\,a(s,q), \quad V(x,s,q)\big|_{x=b(s,q)-} = q - L\,b(s,q) \tag{5.5}$$
$$\partial_x V(x,s,q)\big|_{x=a(s,q)+} = -K, \quad \partial_x V(x,s,q)\big|_{x=b(s,q)-} = -L \tag{5.6}$$
$$\partial_s V(x,s,q)\big|_{x=s-} = 0, \quad \partial_q V(x,s,q)\big|_{x=q+} = 0 \tag{5.7}$$
$$V(x,s,q) = s - K\,x \quad \text{for} \quad x < a(s,q), \quad V(x,s,q) = q - L\,x \quad \text{for} \quad x > b(s,q) \tag{5.8}$$
$$V(x,s,q) > s - K\,x \quad \text{for} \quad a(s,q) < x \leq s, \quad V(x,s,q) < q - L\,x \quad \text{for} \quad q \leq x < b(s,q) \tag{5.9}$$
$$(\mathbb{L}V - rV)(x,s,q) < 0 \quad \text{for} \quad x < a(s,q), \quad (\mathbb{L}V - rV)(x,s,q) > 0 \quad \text{for} \quad x > b(s,q) \tag{5.10}$$

where the conditions in (5.5)-(5.7) are satisfied, for each $0 < q < s$, respectively.

5.2 Solutions to the free-boundary problems. It follows from the arguments of [51] and [36] (see also [23; Section 3] and [21]) that the solution to the left-hand system in (5.3)+(5.5)-(5.10) has the form of (3.15) with (3.16), for $0 < q \leq a(s,q) < x \leq s$, while the boundary

$a(s,q)$ solves the first-order nonlinear ordinary differential equation in (3.24), for any $q > 0$ fixed. We also see that the solution to the right-hand system in (5.4)+(5.5)-(5.10) has the form of (3.17) with (3.18), $0 < q \le x < b(s,q) \le s$, while the boundary $b(s,q)$ solves the first-order nonlinear ordinary differential equation in (3.27), for any $s > 0$ fixed.

We further consider the *maximal and minimal admissible* solutions of first-order nonlinear ordinary differential equations as the largest and smallest possible solutions $\underline{a}(s,q)$ and $\overline{b}(s,q)$ of the equations in (3.24) and (3.27) which satisfy the inequalities $\underline{a}(s,q) < s$ and $\overline{b}(s,q) > q$, for all $0 < q < s$. By virtue of the classical results on the existence and uniqueness of solutions for first-order nonlinear ordinary differential equations, we may conclude that these equations admit (locally) unique solutions, in view of the facts that the right-hand sides in (3.24) and (3.27) are (locally) continuous in $(s,q,a(s,q))$ and $(s,q,b(s,q))$ and (locally) Lipschitz in $a(s,q)$ and $b(s,q)$, for each (s,q) fixed (see also [52; Subsection 3.9] for similar arguments based on the analysis of other first-order nonlinear ordinary differential equations). Then, it is shown by means of technical arguments based on Picard's method of successive approximations that there exist unique solutions $a(s,q)$ and $b(s,q)$ to the equations in (3.24) and (3.27), started at some points (s_0,s_0,q) and (s,q_0,q_0) such that $s_0 > 0$ and $q_0 > 0$, for each $0 < q < s$ fixed (see also [34; Subsection 3.2] and [52; Example 4.4] for similar arguments based on the analysis of other first-order nonlinear ordinary differential equations).

Hence, in order to construct the appropriate functions $\underline{a}(s,q)$ and $\overline{b}(s,q)$ which satisfy the equations in (3.24) and (3.27) and stays strictly above or below the appropriate diagonal, for $0 < q < s$, respectively, we can follow the arguments from [58; Subsection 3.5] (among others) which are based on the construction of sequences of the so-called bad-good solutions which intersect the diagonals. For this purpose, for any positive sequences $(s_k,q_k)_{k\in\mathbb{N}}$ and $(s_l,q_l)_{l\in\mathbb{N}}$ such that $s_k \uparrow \infty$ as $k \to \infty$ and $q_l \downarrow 0$ as $l \to \infty$, we can construct the sequence of solutions $a_k(s,q)$, for $k \in \mathbb{N}$, and $b_l(s,q)$, for $l \in \mathbb{N}$, to the equations (3.24) and (3.27) such that $a_k(s_k,q_k) = s_k$ and $b_l(s_l,q_l) = q_l$ holds, for each $k,l \in \mathbb{N}$. It follows from the structure of the equations in (3.24) and (3.27) that the properties $\partial_s a_k(s_k,q_k) < 1$ and $\partial_q b_l(s_l,q_l) > 1$ hold, for each $k,l \in \mathbb{N}$ (see also [51; pages 979-982] for the analysis of solutions of the non-parametrised version of the first-order nonlinear differential equation of (3.24)). Observe that, by virtue of the uniqueness of solutions mentioned above, we know that each two curves $s \mapsto a_k(s,q)$ and $s \mapsto a_m(s,q)$ as well as $q \mapsto b_l(s,q)$ and $q \mapsto b_n(s,q)$ cannot intersect, for $l,k,m,n \in \mathbb{N}$ such that $k \neq m$ and $l \neq n$, and thus, we see that the sequence $(a_k(s,q))_{k\in\mathbb{N}}$ is increasing and the sequence $(b_l(s,q))_{l\in\mathbb{N}}$ is decreasing, so that the limits $\underline{a}(s,q) = \lim_{k\to\infty} a_k(s,q)$ and $\overline{b}(s,q) = \lim_{l\to\infty} b_l(s,q)$ exist, for each $0 < q < s$, respectively. We may therefore conclude that $\underline{a}(s,q)$ and $\overline{b}(s,q)$ provides the maximal and minimal solutions to the equations in (3.24) and (3.27) such that $\underline{a}_k(s,q) < s$ holds, for each $k \in \mathbb{N}$, and $\overline{b}_l(s,q) > q$ holds, for each $l \in \mathbb{N}$, for all $0 < q < s$.

Moreover, since the right-hand sides of the first-order nonlinear ordinary differential equations in (3.24) and (3.27) are (locally) Lipschitz in s and q, for each $0 < q < s$, respectively, one can deduce by means of Gronwall's inequality that the functions $a_k(s,q)$ and $b_l(s,q)$, for each $k,l \in \mathbb{N}$, are continuous, so that the functions $\underline{a}(s,q)$ and $\overline{b}(s,q)$ are continuous too. The appropriate *maximal admissible* solutions of first-order nonlinear ordinary differential equations and the associated maximality principle for solutions of optimal stopping problems which is equivalent to the superharmonic characterisation of the payoff functions were established in [52] and further developed in [34], [51], [36], [20], [5], [37], [57]-[58], [33], [50], [46], [30]-[32],

[63], and [23] among other subsequent papers (see also [59; Chapter I; Chapter V, Section 17] for other references).

5.3 The results. Summarising the facts shown above, we state the following result which can be proved by means of the same arguments as Theorem 4.1 above in combinations with the arguments from [21] (see [64]-[66] for the original derivation and [20]-[23] for the related comparison arguments).

Theorem 5.1 *Let the process* (X, S, Q) *be defined in (1.1) and (1.3), where* $r > 0$ *is a constant, and* $\delta(s, q) > 0$ *and* $\sigma(s, q) > 0$ *are continuously differentiable bounded functions on* $[0, \infty]^2$. *Assume that the function* $\delta(s, q)$ *is increasing in the both variables* s *and* q *on* $[0, \infty]^2$. *Then, the following assertions hold:*

(i) the value function $\overline{V}(x, s, q)$ *of the left-hand optimal stopping problem in (5.1) with* $K > 0$ *takes the form:*

$$\overline{V}(x,s,q) = \begin{cases} V(x,s,q;\underline{a}(s,q)), & \text{if } \; 0 < \underline{a}(s,q) < x \le s \\ s - K\,x, & \text{if } \; 0 < x \le \underline{a}(s,q) \end{cases} \tag{5.11}$$

and the optimal stopping time $\overline{\tau}$ *has the form of (5.2), where the function* $V(x, s, q; a(s, q))$ *is given by (3.15)-(3.16), while the optimal stopping boundary* $\underline{a}(s, q)[\le rs/(K\delta(s, q))]$ *provides the maximal solution of the first-order nonlinear ordinary differential equation in (3.24) staying below the diagonal* $d_1 = \{(x, s, q) \in \mathbb{R}^3 \,|\, 0 < q \le x = s\}$, *for* $0 < q < s$;

(ii) the value function $\underline{V}(x, s, q)$ *of the right-hand optimal stopping problem in (5.1) with* $L > 0$ *takes the form:*

$$\underline{V}(x,s,q) = \begin{cases} V(x,s,q;\overline{b}(s,q)), & \text{if } \; 0 < q \le x < \overline{b}(s,q) \\ q - L\,x, & \text{if } \; x \ge \overline{b}(s,q) \end{cases} \tag{5.12}$$

and the optimal stopping time $\underline{\zeta}$ *has the form of (5.2), where the function* $V(x, s, q; b(s, q))$ *is given by (3.17)-(3.18), while the optimal stopping boundary* $\overline{b}(s, q)[\ge rq/(L\delta(s, q))]$ *provides the minimal solution of the first-order nonlinear ordinary differential equation in (3.27) staying above the diagonal* $d_2 = \{(x, s, q) \in \mathbb{R}^3 \,|\, 0 < q = x \le s\}$, *for* $0 < q < s$.

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