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On optimal stopping problems with positive discounting rates and related Laplace transforms of first hitting times in models with geometric Brownian motions

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Abstract: We derive closed-form solutions to some optimal stopping problems for one-dimensional geometric Brownian motions with positive discounting rates. It is assumed that the original processes can be trapped or reflected or sticky at some fixed lower levels and the conditions on the gain functions imply that the the optimal stopping times turn out to be the first times at which the processes hit some upper level which are to be determined. The proof is based on the reduction of the original optimal stopping problems to the to the equivalent free-boundary problems and the solutions of the latter problems by means of the instantaneous-stopping and smooth-fit conditions for the value functions at the optimal stopping boundaries.

We also obtain explicit expressions for the Laplace transforms or moment generating functions (with positive exponents or parameters) of the first hitting times for the geometric Brownian motion of given upper levels under various conditions on the parameters of the model. In particular, we determine the upper bounds for the hitting levels and given positive exponents or parameters of the Laplace transforms for which the resulting expectations are finite under various relations between the parameters of the model. Moreover, we determine the upper bounds for the positive exponents or parameters of the Laplace transforms and given hitting levels for which the resulting expectations are finite under various relations between the parameters of the model.

The main aim of this short article is to derive closed-form solutions to the optimal stopping problem of (2) for the geometric Brownian motion X defined in (1) with a positive exponential discounting rate $\lambda > 0$. We assume that the process X can be trapped or reflected or sticky at some level $a > 0$ and the gain function $G(x)$ is a twice continuously differentiable positive and strictly increasing concave function on $(0, \infty)$. Optimal stopping problems for one-dimensional diffusion processes with *negative* exponential discounting rates have been studied after Dynkin (1963) by many authors in the literature including Fakeev (1970), Mucci (1978), Salminen (1985), Øksendal and Reikvam (1998), Alvarez (2001), Dayanik and Karatzas (2003), and Lamberton and Zervos (2013) among others (we refer to Øksendal (1998, Chapter X), Peskir and Shiryaev (2006) and Gapeev and Lerche (2011) for further references). The consideration of optimal stopping problems for diffusions with *positive* discounting rates was initiated by Shepp and Shiryaev (1996) and then has been continued by other authors in the literature (we refer to Gapeev (2019) and Gapeev (2020) for further references). In this short article, we also present explicit expressions for the Laplace transforms (with positive exponents or parameters) of the first hitting times of given upper levels under various conditions on the parameters of the model (see Borodin and Salminen (2002, Part II) for other computations of the Laplace transforms of first hitting times).

Keywords: *Optimal stopping problem, positive discounting rate, Brownian motion, first hitting time, Laplace transform or moment generating function.*

1 PRELIMINARIES

In this section, we give a formulation of an optimal stopping problem for a geometric Brownian motion with positive exponential discounting rates.

1.1 The optimal stopping problem

For a precise formulation of the problem, let us consider a probability space $(\Omega, \mathcal{F}, P)$ with a standard Brownian motion $B = (B_t)_{t \geq 0}$ and its natural filtration $(\mathcal{F}_t)_{t \geq 0}$. Let us define the process $X = (X_t)_{t \geq 0}$ by:

$$X_t = x \, \exp\Big(\big(\mu - \sigma^2/2 \big) \, t + \sigma \, B_t \Big) \tag{1}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are some given constants, and $x > a > 0$ is fixed. The purpose of the present paper is to study the optimal stopping problem for the value function $V_*(x) = V_*(x; a; \lambda)$ given by:

$$V_*(x) = \inf_\tau E_x \big[e^{\lambda \tau} \, G(X_\tau) \big] \tag{2}$$

where $G(x)$ is a twice continuously differentiable positive and strictly increasing concave function on $(0, \infty)$, and $\lambda > 0$ is given and fixed. Here, E_x denotes the expectation with respect to the probability measure P given that the process X starts at some point $x \geq a$, for some $a > 0$ fixed, under the assumptions that the process X is trapped or reflected or sticky at $x = a$ (see Borodin and Salminen (2002, Part I, Appendix 1) for the definitions of these notions). Observe from the structure of the reward in the expression of (2) that the infimum there should be taken over finite stopping times τ of the process X for which the condition:

$$E_x \big[e^{\lambda \tau} \big] < \infty \tag{3}$$

holds. We will search for the optimal stopping time τ_* for the problem of (2) in the form:

$$\tau_* = \inf\{ t \geq 0 \,|\, X_t \geq b_* \} \tag{4}$$

for some $b_* > a > 0$ to be determined, for which the condition of (3) holds.

1.2 The free-boundary problem

In order to find closed-form expressions for the unknown value function $V_*(x)$ from (2), we may use the results of general theory of optimal stopping problems for continuous time Markov processes (see, e.g. Shiryaev (1978, Chapter III, Section 8) and Peskir and Shiryaev (2006, Chapter IV, Section 8)) and formulate the associated free-boundary-value problem for the infinitesimal operator $\mathbb{L}$ of the process X from (1) of the form:

$$(\mathbb{L}V)(x) \equiv \mu \, x \, F'(x) + ((\sigma^2 \, x^2)/2) \, F''(x) = -\lambda \, V(x) \quad \text{for} \quad a < x < b \tag{5}$$

$$V(b-) = G(b) \quad \text{and} \quad V'(b-) = G'(b) \tag{6}$$

$$\alpha \, V''(a+) = \beta \, V'(a+) - \gamma \, V(a+) \tag{7}$$

$$V(x) = G(x) \quad \text{for} \quad x > b \tag{8}$$

$$V(x) < G(x) \quad \text{for} \quad a < x < b \tag{9}$$

$$(\mathbb{L}V)(x) \equiv \mu \, x \, F'(x) + ((\sigma^2 \, x^2)/2) \, F''(x) > -\lambda \, V(x) \quad \text{for} \quad x > b \tag{10}$$

for some $\alpha, \beta, \gamma > 0$ given and fixed. Here, we have the instantaneous-stopping and smooth-fit conditions of (6), and the condition of (7) at the point $x = a$, which reflects the properties of the infinitesimal operator $\mathbb{L}$.

2 SOLUTIONS TO THE FREE-BOUNDARY PROBLEM

We now look for functions which solve the boundary-value problems. For this purpose, we consider three separate cases based on the different relations between the parameters of the model.

2.1 The general solution to the ordinary differential equation and other conditions

The general solution to the second-order ordinary differential equation in (5) has the form:

$$V(x) = C_1 \, U_1(x) + C_2 \, U_2(x) \tag{11}$$

where C_i, for $i = 1, 2$, are some arbitrary constants. Here, the functions $U_i(x)$, for $i = 1, 2$, represent fundamental solutions (independent particular solutions) of the ordinary differential equation in (5) (see, e.g. Rogers and Williams (1987, Chapter V, Section 50)). Then, by applying the conditions of (6)-(7) to the function in (11), we see that the equations:

$$C_1\, U_1(b) + C_2\, U_2(b) = G(b) \quad \text{and} \quad C_1\, U_1'(b) + C_2\, U_2'(b) = G'(b) \tag{12}$$

$$\alpha \left(C_1\, U_1''(a) + C_2\, U_2''(a)\right) = \beta \left(C_1\, U_1'(a) + C_2\, U_2'(a)\right) - \gamma \left(C_1\, U_1(a) + C_2\, U_2(a)\right) \tag{13}$$

should hold, for $0 < a < b < \infty$. Hence, solving the system of equations in (12)-(13), we obtain:

$$V(x; a, b) = C_1(a, b)\, U_1(x) + C_2(a, b)\, U_2(x) \tag{14}$$

for $a < x < b$, with

$$C_1(a, b) = -\frac{\alpha U_2''(a) - \beta U_2'(a) + \gamma U_2(a)}{(\alpha U_1''(a) - \beta U_1'(a) + \gamma U_1(a))U_2(b) - U_1(b)(\alpha U_2''(a) - \beta U_2'(a) + \gamma U_2(a))} \tag{15}$$

and

$$C_2(a, b) = \frac{\alpha U_1''(a) - \beta U_1'(a) + \gamma U_1(a)}{(\alpha U_1''(a) - \beta U_1'(a) + \gamma U_1(a))U_2(b) - U_1(b)(\alpha U_2''(a) - \beta U_2'(a) + \gamma U_2(a))} \tag{16}$$

as well as

$$-\frac{C_2(a, b)}{C_1(a, b)} \equiv \frac{\alpha U_1''(a) - \beta U_1'(a) + \gamma U_1(a)}{\alpha U_2''(a) - \beta U_2'(a) + \gamma U_2(a)} = \frac{G(b)U_1'(b) - G'(b)U_1(b)}{G(b)U_2'(b) - G'(b)U_2(b)} \tag{17}$$

for $0 < a < b < \infty$.

2.2 The expressions for the candidate value function and optimal stopping boundary

Let us now consider three different cases of fundamental systems $U_i(x)$, $i = 1, 2$, of solutions of the second-order ordinary differential equation in (5) for (11) under certain relations on the parameters of the model.

(i). Let us first assume that $0 < \lambda < (\mu - \sigma^2/2)^2/(2\sigma^2)$ and $\mu > \sigma^2/2$ holds. In this case, we have $U_i(x) = x^{\eta_i}$, for $i = 1, 2$, in (11) and (12)-(13), where η_j, for $j = 1, 2$, are given by:

$$\eta_j = \frac{1}{2} - \frac{\mu}{\sigma^2} - (-1)^j \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 - \frac{2\lambda}{\sigma^2}} \tag{18}$$

so that $\eta_2 < \eta_1 < 0$. Then, by applying the conditions in (12)-(13), we obtain that the candidate value function has the form:

$$V(x; a, b) = \frac{1}{\eta_1 - \eta_2} \left(\big(b\, G'(b) - \eta_2\, G(b)\big) \left(\frac{x}{b}\right)^{\eta_1} + \big(\eta_1\, G(b) - b\, G'(b)\big) \left(\frac{x}{b}\right)^{\eta_2} \right) \tag{19}$$

while the candidate optimal stopping boundary should solve the arithmetic equation:

$$\frac{\eta_1 G(b) - bG'(b)}{\eta_2 G(b) - bG'(b)} = \left(\frac{a}{b}\right)^{\eta_1 - \eta_2} \frac{\alpha\eta_1(\eta_1 - 1) - \beta\eta_1 a + \gamma a^2}{\alpha\eta_2(\eta_2 - 1) - \beta\eta_2 a + \gamma a^2} \tag{20}$$

for some $0 < a < b < \infty$.

(ii). Let us now assume that $\lambda = (\mu - \sigma^2/2)^2/(2\sigma^2)$ and $\mu > \sigma^2/2$ holds. In this case, we have $U_1(x) = x^\nu \ln x$ and $U_2(x) = x^\nu$ in (11) and (12)-(13), where ν is given by:

$$\nu = \frac{1}{2} - \frac{\mu}{\sigma^2} \tag{21}$$

so that $\nu < 0$. Hence, by applying the conditions in (12)-(13), we get that the candidate value function has the form:

$$V(x; a, b) = \big(b\, G'(b) - \nu\, G(b)\big) \left(\frac{x}{b}\right)^{\nu} \ln\left(\frac{x}{b}\right) + G(b) \left(\frac{x}{b}\right)^{\nu} \tag{22}$$

while the candidate optimal stopping boundary should solve the arithmetic equation:

$$\frac{G(b)(\nu \ln b + 1) - G'(b) b \ln b}{\nu G(b) - bG'(b)} = \frac{\alpha(\nu(\nu - 1)\ln a + 2\nu - 1) - \beta a(\nu \ln a + 1) + \gamma a^2}{\alpha\nu(\nu - 1) - \beta\nu a + \gamma a^2} \tag{23}$$

for some $0 < a < b < \infty$.

(iii). Let us finally assume that $\lambda > (\mu - \sigma^2/2)^2/(2\sigma^2)$ holds. In this case, we have $U_1(x) = x^\nu \sin(\theta \ln x)$ and $U_2(x) = x^\nu \cos(\theta \ln x)$ in (11) and (12)-(13), where ν is defined in (21) and θ is given by:

$$\theta = \sqrt{\frac{2\lambda}{\sigma^2} - \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2}. \tag{24}$$

Thus, by applying the conditions in (12)-(13), we obtain that the candidate value function has the form:

$$V(x; a, b) = \big(b\, G'(b) - \nu\, G(b)\big) \left(\frac{x}{b}\right)^\nu \sin\left(\theta \, \ln\left(\frac{x}{b}\right)\right) + \theta\, G(b) \left(\frac{x}{b}\right)^\nu \cos\left(\theta \, \ln\left(\frac{x}{b}\right)\right) \tag{25}$$

while the candidate optimal stopping boundary should solve the arithmetic equation:

$$\begin{aligned} &\frac{(\nu G(b) - bG'(b)) \tan(\theta \ln b) - \theta G(b)}{\nu G(b) - bG'(b) - \theta G(b) \tan(\theta \ln b)} \\ &= \frac{(\alpha(\nu(\nu-1) - \theta^2) - \beta\nu a + \gamma a^2) \tan(\theta \ln a) + \alpha(2\nu - 1)\theta - \beta\theta a}{\alpha(\nu(\nu-1) - \theta^2) - \beta\nu a + \gamma a^2 - (\alpha(2\nu-1)\theta - \beta\theta a)\tan(\theta \ln a)} \end{aligned} \tag{26}$$

for some $0 < a < b < \infty$.

3 MAIN RESULTS (VERIFICATION)

Theorem 1. *Let the process X be geometric Brownian motion given by (1) with $\mu \in \mathbb{R}$ and $\sigma > 0$, which is trapped or reflected or sticky with the rates $\alpha, \beta, \gamma > 0$ at some point $a > 0$ given and fixed. Assume that the payoff function $G(x)$ is a twice continuously differentiable positive and strictly increasing convex function, and the threshold b_* is a minimal solution of the arithmetic equation in (17) which admits its particular cases (20) or (23) or (26) on the interval (a, ∞). Assume that the resulting candidate function:*

$$V(x) = \begin{cases} V(x; a, b_*), & \text{if } a \le x < b_* \\ G(x), & \text{if } x \ge b_* \end{cases} \tag{27}$$

where $V(x; a, b)$ is given by (11) which admits its particular cases (19) or (22) or (25), satisfies the inequalities in (9)-(10). Then, the function $V(x)$ from (27) provides the value function $V_*(x)$ of the optimal stopping problem in (2), while the first hitting time τ_* from (4) is optimal, whenever is satisfies the condition of (3), for $\lambda > 0$ given and fixed.

Proof. In order to verify the assertions stated above, we recall that the function $V(x)$ from (27) solves the ordinary differential equation of (5) and satisfies the conditions of (6)-(8) by construction. Then, applying the local time-space formula from Peskir and Shiryaev (2006, Chapter II, Section 3.5) to the process $e^{\lambda t} V(X_t)$, we get:

$$e^{\lambda t}\, V(X_t) = V(x) + \int_0^t e^{\lambda u}\, (\mathbb{L}V + rV)(X_u)\, I(X_u \neq a, X_u \neq b_*)\, du + M_t \tag{28}$$

where the process $M = (M_t)_{t \ge 0}$ defined by:

$$M_t = \int_0^t e^{\lambda u}\, V'(X_u)\, I(X_u \neq a)\, \sigma(X_u)\, dB_u \tag{29}$$

for all $t \ge 0$, is a continuous local martingale with respect to P_x, which is a probability measure under which the process X starts at $x \ge a$. Note that, since the time spent by the process X at the levels $x = a$ and $x = b_*$ is of Lebesgue measure zero (see, e.g. Borodin and Salminen (2002, Chapter II, Section 1)), the indicators which appear in the formulas of (28) and (29) can be ignored (see also Shiryaev (1999, Chapter VIII, Section 2) and Gapeev (2019) for further arguments).

By using the assumptions of the theorem that the inequalities in (9)-(10) hold with the boundary b_* specified above, we conclude from the conditions in (6) and (8) that the inequality $(\mathbb{L}V + \lambda V)(x) \ge 0$ holds, for any $x > a$ such that $x \neq b_*$, as well as the inequality $V(x) \le G(x)$ holds, for all $x > 0$. It therefore follows from the expression in (28) that the inequalities:

$$e^{\lambda \tau}\, G(X_\tau) \ge e^{\lambda \tau}\, V(X_\tau) \ge V(x) + M_\tau \tag{30}$$

hold for any stopping time τ of the process X started at $x \geq a$.

Now, consider the localising sequence $(\tau_n)_{n\in\mathbb{N}}$ for the local martingale M from (29) such that $\tau_n = \inf\{t \geq 0 \,|\, |M_t| \geq n\}$, for each $n \in \mathbb{N}$. Then, inserting $\tau \wedge \tau_n$ instead of τ in (30) and taking the expectations with respect to the probability measure P_x there, by means of Doob's optional sampling theorem (see, e.g. Liptser and Shiryaev (2001, Chapter III, Theorem 3.6)), we get that the inequalities:

$$E_x\big[e^{\lambda(\tau\wedge\tau_n)}\, G(X_{\tau\wedge\tau_n})\big] \geq E_x\big[e^{\lambda(\tau\wedge\tau_n)}\, V(X_{\tau\wedge\tau_n})\big] \geq V(x) + E_x\big[M_{\tau\wedge\tau_n}\big] = V(x) \tag{31}$$

hold any stopping time τ satisfying the condition of (3) and all $x > a$. Hence, letting n go to infinity and applying the Lebesgue dominated convergence theorem under the condition of (3), we obtain that the inequalities:

$$E_x\big[e^{\lambda\tau}\, G(X_\tau)\big] \geq E_x\big[e^{\lambda\tau}\, V(X_\tau)\big] \geq V(x) \tag{32}$$

hold, for any stopping time τ satisfying the condition of (3) and all $x > a$. By virtue of the structure of the stopping time in (4), it is readily seen that the equalities in (32) hold with τ_* instead of τ when $x \geq b_*$.

It remains to show that the equalities are attained in (32) when τ_* replaces τ for $a \leq x < b_*$. By virtue of the fact that the function $V(x; a, b_*)$ and the boundaries a and b_* satisfy the conditions in (5) and (6), it follows from the expression in (28) and the structure of the stopping time in (4) that the equalities:

$$e^{\lambda(\tau_*\wedge\tau_n)}\, V(X_{\tau_*\wedge\tau_n}; a, b_*) = V(x; a, b_*) + M_{\tau_*\wedge\tau_n} \tag{33}$$

are satisfied for all $a \leq x < b_*$. Hence, taking into account the assumption of the theorem that the stopping time τ_* from (4) satisfies the condition of (3) and letting n go to infinity as well as using the condition of (6), we apply again the Lebesgue dominated convergence theorem to obtain the equalities:

$$E_x\big[e^{\lambda\tau_*}\, G(X_{\tau_*})\big] = E_x\big[e^{\lambda\tau_*}\, V(X_{\tau_*}; a, b_*)\big] = V(x) \tag{34}$$

for all $a \leq x < b_*$. The latter, together with the inequalities in (32), implies the fact that the candidate function $V(x)$ from (27) coincides with the value function $V_*(x)$ from (2). □

4 COMPUTATIONS OF THE LAPLACE TRANSFORMS

We now compute the Laplace transforms (or moment generating functions) under particular relations between the parameters of the model. We therefore assume that $G(x) = 1$ in (2).

(i). Let us first consider the case $\alpha = \beta = 0$ and $\gamma = 1$, so that the geometric Brownian motion X from (1) is trapped at the point a. Such a situation appears in the perpetual American barrier call option problem for selling short (see Gapeev (2019)). Then, under $0 < \lambda \leq \sigma^2\nu^2/2$ and $\nu < 0$, the expression in (14) with (15)-(16) which takes the form of (19) and (22) is finite, for all $a \leq x \leq b < \infty$, where ν is given by (21). On the other hand, under $\lambda > \sigma^2\nu^2/2$, the expression in (14) with (15)-(16) takes the form of (25) which is equivalent to:

$$\begin{aligned} & V_1(x; a, b; \lambda) \\ & = \Big(\frac{x}{b}\Big)^{\nu} \frac{\sin(\theta \ln(x/a))}{\sin(\theta \ln(b/a))} \quad \text{with} \quad \bar{b}(\lambda) = a \exp\bigg(\frac{\pi}{2\theta(\lambda)}\bigg) \quad \text{and} \quad \bar{\lambda} = \frac{\pi\sigma^2}{4\ln(b/a)} + \frac{\sigma^2\nu^2}{2} \end{aligned} \tag{35}$$

for all $a \leq x \leq b < \infty$, where $\theta = \theta(\lambda)$ is given by (24). Hence, the expression in (35) is finite if and only if $-\pi/(2\theta) < \ln(x/a) \leq \ln(b/a) < \pi/(2\theta)$, so that we can determine $\bar{b} = \bar{b}(\lambda, a)$ from the equation $\ln(b/a) = \pi/(2\theta)$ which represents the upper bound for the thresholds $a \leq b < \bar{b}$ such that the expectation in (3) is finite, for $\lambda > \sigma^2\nu^2/2$ and $a > 0$ given and fixed. Thus, we can also determine $\bar{\lambda} = \bar{\lambda}(b, a)$ as the upper bound for the Laplace exponents $\sigma^2\nu^2/2 < \lambda < \bar{\lambda}$ such that the expectation in (3) is finite, for $0 < a < b < \infty$ given and fixed.

(ii). Let us now consider the case $\alpha = \gamma = 0$ and $\beta = 1$, so that the process X is reflected at the point a. Such a situation appears in the contract of the dual Russian call option introduced in Shepp and Shiryaev (1996) (see also Gapeev (2019) and Gapeev (2020)) with $a = 1$. Then, under $0 < \lambda \leq \sigma^2\nu^2/2$ and $\nu < 0$, the expression in (14) with (15)-(16) which takes the form of (19) and (22) is finite, for all $a \leq x \leq b < \infty$, where ν is given by (21). On the other hand, under $\lambda > \sigma^2\nu^2/2$, the expression in (14) with (15)-(16) takes the form of (25) which is equivalent to:

$$V_2(x;a,b;\lambda) = \left(\frac{x}{b}\right)^{\nu} \frac{\nu \sin(\theta \ln(x/a)) - \theta \cos(\theta \ln(x/a))}{\nu \sin(\theta \ln(b/a)) - \theta \cos(\theta \ln(b/a))} = \left(\frac{x}{b}\right)^{\nu} \frac{\sin(\theta \ln(x/c_2))}{\sin(\theta \ln(b/a_2))} \tag{36}$$

for all $a \le x \le b < \infty$, where $\theta = \theta(\lambda)$ is given by (24) and we set:

$$a_2 \equiv a_2(\lambda) = a \, \exp\left(\frac{1}{\theta(\lambda)} \, \arctan\left(\frac{\theta(\lambda)}{\nu}\right)\right) \tag{37}$$

when $\nu \neq 0$. Hence, the expression in (36) is finite if and only if $-\pi/(2\theta) < \ln(x/a_2) \le \ln(b/a_2) < \pi/(2\theta)$, so that we can determine $\bar{b} = \bar{b}(\lambda, a)$ from the equation $\ln(b/a_2) = \pi/(2\theta)$ which represents the upper bound for the thresholds $a \le b < \bar{b}$ such that the expectation in (3) is finite, for $\lambda > \sigma^2\nu^2/2$ and $a > 0$ fixed. Thus, we can also determine $\bar{\lambda} = \bar{\lambda}(b, a)$ as the upper bound for the Laplace exponents $\sigma^2\nu^2/2 < \lambda < \bar{\lambda}$ such that the expectation in (3) is finite, for $0 < a < b < \infty$ given and fixed. In this case, we have:

$$\bar{b}(\lambda) = a_2(\lambda) \, \exp\left(\frac{\pi}{2\theta(\lambda)}\right) \quad \text{and} \quad \bar{\lambda} = \frac{\pi\sigma^2}{4\ln(b/a_2(\bar{\lambda}))} + \frac{\sigma^2\nu^2}{2} \tag{38}$$

where $\bar{\lambda}$ is the appropriate root of the arithmetic equation in (41) and $a_2(\lambda)$ is given by (37).

(iii). Let us now consider the case $\alpha = 0$, $\beta = 1$, and $\gamma > 0$, so that the process X from (1) is trapped or reflected at the point a. Then, under $0 < \lambda \le \sigma^2\nu^2/2$ and $\nu < 0$, the expression in (14) with (15)-(16) which takes the form of (19) and (22) is finite, for all $a \le x \le b < \infty$, where ν is given by (21). On the other hand, under $\lambda > \sigma^2\nu^2/2$, the expression in (14) with (15)-(16) takes the form of (25) which is equivalent to:

$$V_3(x;a,b;\lambda) = \left(\frac{x}{b}\right)^{\nu} \frac{(\nu - \gamma a)\sin(\theta \ln(x/a)) - \theta \cos(\theta \ln(x/a))}{(\nu - \gamma a)\sin(\theta \ln(b/a)) - \theta \cos(\theta \ln(b/a))} = \left(\frac{x}{b}\right)^{\nu} \frac{\sin(\theta \ln(x/a_3))}{\sin(\theta \ln(b/a_3))} \tag{39}$$

for all $a \le x \le b < \infty$, where $\theta = \theta(\lambda)$ is given by (24) and we set:

$$a_3 \equiv a_3(\lambda) = a \, \exp\left(\frac{1}{\theta(\lambda)} \, \arctan\left(\frac{\theta(\lambda)}{\nu - \gamma a}\right)\right) \tag{40}$$

when $\nu \neq \gamma a$. Hence, the expression in (39) is finite if and only if $-\pi/(2\theta) < \ln(x/a_3) \le \ln(b/a_3) < \pi/(2\theta)$, so that we can determine $\bar{b} = \bar{b}(\lambda, a)$ from the equation $\ln(b/a_3) = \pi/(2\theta)$ which represents the upper bound for the thresholds $a \le b < \bar{b}$ such that the expectation in (3) is finite, for $\lambda > 0$ and $a > 0$ fixed. Thus, we can also determine $\bar{\lambda} = \bar{\lambda}(b, a)$ as the upper bound for the Laplace exponents $0 < \lambda < \bar{\lambda}$ such that the expectation in (3) is finite, for $0 < a < b < \infty$ given and fixed. In this case, we have:

$$\bar{b}(\lambda) = a_3(\lambda) \, \exp\left(\frac{\pi}{2\theta(\lambda)}\right) \quad \text{and} \quad \bar{\lambda} = \frac{\pi\sigma^2}{4\ln(b/a_3(\bar{\lambda}))} + \frac{\sigma^2\nu^2}{2} \tag{41}$$

where $\bar{\lambda}$ is the appropriate root of the arithmetic equation in (41) and $a_3(\lambda)$ is given by (40).

(iv) Let us now consider the case $\alpha = 1$ and $\beta = \gamma = 0$, so that the process X is sticky at the point a. Then, under $0 < \lambda \le \sigma^2\nu^2/2$ and $\nu < 0$, the expression in (14) with (15)-(16) which takes the form of (19) and (22) is finite, for all $a \le x \le b < \infty$, where ν is given by (21). On the other hand, under $\lambda > \sigma^2\nu^2/2$, the expression in (14) with (15)-(16) takes the form of (25) which is equivalent to:

$$\begin{aligned} &V_4(x;a,b;\lambda) \\ &= \left(\frac{x}{b}\right)^{\nu} \frac{(\nu(\nu-1) - \theta^2)\sin(\theta \ln(x/a)) - ((2\nu-1)\theta)\cos(\theta \ln(x/a))}{(\nu(\nu-1) - \theta^2)\sin(\theta \ln(b/a)) - ((2\nu-1)\theta)\cos(\theta \ln(b/a))} = \left(\frac{x}{b}\right)^{\nu} \frac{\sin(\theta \ln(x/a_4))}{\sin(\theta \ln(b/a_4))} \end{aligned} \tag{42}$$

for all $a \le x \le b < \infty$, where $\theta = \theta(\lambda)$ is given by (24) and we set:

$$a_4 \equiv a_4(\lambda) = a \, \exp\left(\frac{1}{\theta(\lambda)} \, \arctan\left(\frac{(2\nu-1)\theta(\lambda)}{\nu(\nu-1) - \theta^2(\lambda)}\right)\right) \tag{43}$$

when $\nu(\nu-1) \neq \theta^2$. Hence, the expression in (39) is finite if and only if $-\pi/(2\theta) < \ln(x/a_4) \leq \ln(b/a_4) < \pi/(2\theta)$, so that we can determine $\bar{b} = \bar{b}(\lambda, a)$ from the equation $\ln(b/a_4) = \pi/(2\theta)$ which represents the upper bound for the thresholds $a \leq b < \bar{b}$ such that the expectation in (3) is finite, for $\lambda > \sigma^2\nu^2/2$ and $a > 0$ fixed. Thus, we can also determine $\bar{\lambda} = \bar{\lambda}(b, a)$ as the upper bound for the Laplace exponents $\sigma^2\nu^2/2 < \lambda < \bar{\lambda}$ such that the expectation in (3) is finite, for $0 < a < b < \infty$ given and fixed. In this case, we have:

$$\bar{b}(\lambda) = a_4(\lambda) \, \exp\left(\frac{\pi}{2\theta(\lambda)}\right) \quad \text{and} \quad \bar{\lambda} = \frac{\pi\sigma^2}{4\ln(b/a_4(\bar{\lambda}))} + \frac{\sigma^2\nu^2}{2} \tag{44}$$

where $\bar{\lambda}$ is the appropriate root of the arithmetic equation in (44) and $a_4(\lambda)$ is given by (43).

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