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Multi-asset Noisy Rational Expectations Equilibrium with Contingent Claims

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We study a noisy rational expectations equilibrium in a multi-asset economy populated by informed and uninformed investors and noise traders. The assets can include state contingent claims such as Arrow–Debreu securities, assets with only positive payoffs, options or other derivative securities. The probabilities of states depend on an aggregate shock, which is observed only by the informed investor. We derive a three-factor CAPM with asymmetric information, establish conditions under which asset prices reveal information about the shock, and show that information asymmetry amplifies the effects of payoff skewness on asset returns. We also find that volatility derivatives make incomplete markets effectively complete, and their prices quantify market illiquidity and shadow value of information to uninformed investors.

Key words: Asymmetric information, Learning from prices, Multi-asset economy.

JEL Codes: D82, G12, G14

1. INTRODUCTION

Informed investors in financial markets often use their private information to extract gains from trading in a multitude of financial assets, such as stocks, corporate bonds, and derivative securities. This information can be partially learned from asset prices by uninformed investors for their own advantage. Learning from asset prices in realistic multi-asset economies is complicated by the joint effects of non-normality of asset payoff distributions, complex payoff structures of derivative securities, and interdependence of asset prices arising because each price may contain information about the payoffs of other assets. Therefore, despite the fact that asymmetry of information across investors is a prominent feature of financial markets, its impact on equilibrium asset prices in multi-asset economies remains relatively unexplored.

In this article, we develop a new tractable approach for studying asset prices in multi-asset noisy rational expectations equilibrium (REE) economies with realistic asset payoffs and

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asymmetric information. Our approach provides a bridge between risk-neutral asset pricing and models with asymmetric information, and sheds new light on the effects of information asymmetry on asset prices. In particular, we derive a three-factor extension of the standard Capital Asset Pricing Model (CAPM) to economies with asymmetric information, derive conditions under which prices of assets (such as debt, equity, and derivatives) reveal information about hidden economic shocks, and show that information asymmetry amplifies the effects of higher order moments of asset payoffs on asset returns. We also uncover a unique economic role of volatility derivatives. We show that these derivatives make incomplete markets effectively complete, so that investors achieve Pareto optimal asset allocations. Moreover, their prices quantify both market illiquidity and the shadow value of information for uninformed investors. The latter results complement the widely-used interpretation of market volatility indices such as VIX (i.e. the risk-neutral volatility of S&P500 stock index) as “fear gauges” of financial markets.

We consider a multi-asset economy with two dates and a finite but arbitrary number of random states. The probabilities of states are functions of a state-dependent aggregate risk factor multiplied by an aggregate shock. This risk factor is a key fundamental variable, such as aggregate output or any other source of systematic risk, that determines asset prices and expected payoffs. The assets can be state contingent claims such as Arrow–Debreu (AD) securities, assets with only positive payoffs, and derivatives. The economy is populated by two types of price-taking rational investors, informed and uninformed, with constant absolute risk aversion (CARA) preferences over terminal wealth, and noise traders with exogenous jointly normally distributed asset demands. The informed investor observes the aggregate shock, whereas the uninformed investor uses asset prices to extract the information about the shock, which is obfuscated by noise traders. The investors trade assets at the initial date, and asset prices are determined from the market clearing conditions.

We start by studying markets where the number of states equals the number of assets, henceforth complete markets, and extend the methods of risk-neutral valuation and portfolio choice to these complete markets with asymmetric information. We show that the risk-neutral probability measure is unique and the asset prices are given in closed form by expected discounted payoffs with respect to risk-neutral probabilities. The tractability of our analysis is further facilitated by the assumption that the probabilities have the same structure as in logit models widely employed in econometrics so that their log-ratios are linear functions of the risk factor. Consequently, because investors have exponential CARA preferences, the risk factor and wealth affect the expected utility of the informed investor in a similar way, which makes their portfolios linear in the aggregate shock.

We then derive an approximate linear three-factor CAPM with asymmetric information. This three-factor model helps us isolate the effects of asymmetric information on asset returns. The first factor in the model is the payoff of the market portfolio, as in the standard CAPM. The second and third factors are the aggregate risk factor and its square, which we interpret as the factor’s realized volatility.

Next, we extend our analysis to a subset of realistic incomplete-market economies where the number of states exceeds the number of assets. These economies include economies with one risky asset as in the previous literature (e.g. Grossman and Stiglitz, 1980; Breon-Drish, 2015), the latter economies with added derivatives, and economies with risky corporate debt and equity. Asset prices in these economies solve a system of non-linear equations and are less tractable than in economies with complete markets.

We provide several further applications of our model that shed light on the effects of information asymmetry on asset prices. First, we provide an answer to a fundamental question: which assets help mitigate the information asymmetry in the economy? We show that the information about the aggregate shock is impounded into asset prices only via the informed investor's demand for those assets that make up a replicating portfolio for the aggregate risk factor. Investing in this replicating portfolio allows the informed investors to trade on their information and transfer more wealth to more likely states. The amount of wealth invested in this portfolio depends on the aggregate shock. The information about the shock then becomes incorporated into asset prices via the market clearing conditions. The assets which are not included in the replicating portfolio are informationally irrelevant in the sense that their presence in the economy does not contribute towards revealing more information. We also demonstrate an informational analogue of the irrelevance of capital structure according to which, absent additional frictions, the information jointly revealed by the prices of risky corporate debt and equity does not depend on the face value of debt.

We also identify realistic situations when securities are not informationally irrelevant. In particular, we consider an economy with costly defaults on corporate debt that destroy a fraction of the firm's value, and allow investors to trade AD securities on default states, similar to credit default swaps. We find that the informativeness of prices (measured by the posterior variance of the aggregate shock) is a decreasing function of the face value of debt and the cost of default. The intuition is that default becomes more likely as the face value of debt increases, and hence, more AD securities are needed for the replication of cash flows and more information is revealed through their prices. Other realistic situations where derivatives are not informationally redundant include the cases in which noise trader demands are correlated across markets or derivatives help hedge the exposure to the aggregate risk factor.

The multi-asset nature of our setup also allows us to uncover the important economic role of volatility derivatives. These derivatives are widely traded in financial markets, and adding them to our model makes it both more realistic and tractable. The analysis of these derivatives when the payoff of the underlying asset is not normally distributed is unique to our model. Specifically, we study a security with a payoff equal to the squared payoff of the underlying, which we interpret as a volatility derivative. We show that introducing this security in an incomplete market with one risky underlying asset makes this market effectively complete, that is, allows informed and uninformed investors to achieve Pareto optimal asset allocations. Moreover, the asset prices in the resulting effectively complete market are available in closed form.

Furthermore, we show that the prices of volatility derivatives, given by risk-neutral variances of asset payoffs, determine the shadow value of information and illiquidity in financial markets. We define the shadow value of information as the amount of wealth that an uninformed investor is willing to give up to become informed. When the market is complete, the value of information is proportional to the risk-neutral variance of the aggregate risk factor, which can be interpreted as the price of a volatility derivative.

We define market illiquidity as the matrix of asset price sensitivities to noise trader demands and show that illiquidity is proportional to the risk-neutral variance–covariance matrix of asset payoffs. Such a relationship between illiquidity and risk-neutral variance is in line with empirical findings in Chung and Chuwonganant (2014) who show that liquidity (defined as the inverse of illiquidity) is negatively related to VIX.

We build on the role of volatility derivatives discussed above to develop a new method for studying the effects of asymmetric information on asset prices. We consider an incomplete market with a single risky asset and then make it effectively complete by adding a volatility derivative. To facilitate the comparison of incomplete and effectively complete markets, we exclude noise traders from trading the derivative; even then we show that the equilibrium is not fully revealing because the derivative is informationally irrelevant (in this particular case). For several broad classes of asset payoff distributions, we find asset prices in the effectively complete market in closed-form. Using our new method, we also show that asset risk premia are decreasing functions of payoff skewness and the asymmetry of information amplifies the sensitivity of risk premia to skewness.

Finally, we demonstrate the generality and robustness of our results by extending our analysis in several directions. First, we show that when the market is complete the model can be solved for general probabilities of states and distributions of shocks and noise trader demands. Second, we extend the analysis to economies where noise traders are replaced by noisy endowments of investors and show that the equilibrium prices have the same structure and the quadratic derivatives play the same role as in the baseline analysis. Third, we study economies with multiple heterogeneously informed investors. We show that quadratic derivatives make the market effectively complete, prove the existence and uniqueness of a symmetric equilibrium, and derive prices in closed form. Then, we extend the analysis to the case of multi-dimensional economic shocks. As in other extensions, the prices are in closed form and have the same structure as in the baseline analysis. Moreover, the security that effectively completes the market has payoffs given by quadratic forms of the aggregate risk factor. Our final extension considers economies in which informed investors receive a signal about the aggregate shock instead of observing it directly.

1.1. **Related literature.**

Our article is related to a large literature on noisy REE models, which was pioneered by Grossman (1976), Grossman and Stiglitz (1980), and Hellwig (1980). These early works typically consider economies with CARA investors and one risky asset with normally distributed payoffs. Admati (1985) extends these models to the case of multiple securities and Wang (1993) develops a dynamic model. Pálvölgyi and Venter (2014) prove the uniqueness of equilibrium in Grossman and Stiglitz (1980) in the class of continuous prices and find multiple equilibria with discontinuous prices. Diamond and Verrecchia (1981), Vives (2008), García and Urošević (2013), and Kurlat and Veldkamp (2015) discuss further extensions and applications of CARA-normal models. Davila and Parlatore (2021) study the effect of trading costs on the acquisition and aggregation of information. Cespa and Foucault (2014) study an economy in which investors learn information about an asset also from the information about other assets and show that such learning may lead to liquidity crashes. In contrast to this literature, we allow for multiple assets with more general payoff distributions.

Breon-Drish (2015) studies an economy with CARA investors and one risky asset without normality. He obtains a general characterization of prices in terms of inverse functions and proves the existence and uniqueness of equilibrium when the asset payoff conditional on the informed investor's signal has a distribution from the exponential family. He also obtains closed-form exponential-linear asset prices when payoffs have a binomial distribution, and uses conjugate prior for the economic shock similar to ours. Relative to the latter paper, we study a muti-asset economy with multinomial distributions. We show that (effective) market completeness offers significant additional tractability and closed-form prices. We provide new economic results on the informational irrelevance of assets, the role of quadratic derivatives, effective completeness, value of information, and liquidity.

Malamud (2015) studies an REE with AD securities in a continuous-space complete-market economy and shows that the equilibria with non-CARA preferences are fully revealing. Albagli, Hellwig, and Tsyvinski (2021) consider a noisy REE model where the information is dispersed across all investors, preferences are general, and investors face position limits. Our model differs from their setup in that our investors are either informed or uninformed and their trades are unconstrained. Han (2018) studies numerically dynamic information acquisition with ex ante identical investors but without derivative securities, and shows that the marginal cost of information precision is proportional to risk-neutral variance. In contrast to Han (2018), we derive the value of information and show that this value is reduced by market incompleteness. Although our model is one-period, investors are not ex-ante identical and solutions are in closed form.

Bernardo and Judd (2000) solve models with general distributions and preferences numerically and demonstrate that the REE in Grossman and Stiglitz (1980) is not robust to parametric assumptions. Breugem and Buss (2019) study portfolio and information choice of investors concerned about their performance relative to an index. Glebkin, Malamud, and Teguia (2020) study asset prices in a model with non-normally distributed payoffs and market power, but without asymmetric information. Similar to our model, they find that illiquidity is proportional to the risk-neutral variance of asset returns. Banerjee, Marinovic, and Smith (2021) study a disclosure problem with asymmetric information where managers disclose the asset payoff when its value exceeds an endogenous threshold. Other related models that do not rely on the normality of asset payoffs include Barlevy and Veronesi (2000), Peress (2004), and Yuan (2005) among others.

Our model is also related to works on the informational role of derivatives. Brennan and Cao (1996) consider a CARA-normal model with one risky underlying asset and a quadratic derivative written on it. They show that the derivative effectively completes the market and does not reveal any information which is not already in the price of the underlying asset. We show that the latter result extends to economies without payoff normality. Moreover, the quadratic derivative in our model plays a unique role by allowing us to obtain closed-form prices whereas in their work introducing this derivative does not affect the price of the underlying asset. Other related models in this literature include Back (1993), Biais and Hillion (1994), Vanden (2008), and Huang (2014) among others.

2. MODEL

2.1. *Securities markets and information structure*

We consider an economy with two dates $t=0$ and $t=T$, and N states $\omega_1,\dots,\omega_N$ at the terminal date, where $N\geq 2$. The probabilities $\pi_n(\varepsilon)$ of states ω_n are normalized exponential functions of the *aggregate shock* $\varepsilon\in\mathbb{R}$ adopted from logit models in econometrics:

$$\pi_n(\varepsilon)=\frac{e^{a_n+b_n\varepsilon}}{\sum_{j=1}^{N}e^{a_j+b_j\varepsilon}},\quad n=1,\dots,N, \tag{1}$$

where a_n and b_n are state-dependent variables. Variable b_n determines the effect of shocks on probabilities and can be interpreted as an *aggregate risk factor* of fundamental importance for the economy, such as aggregate output or other source of systematic risk. With such interpretation, probabilities (1) intuitively imply that, for example, bad shock $\varepsilon<0$ makes states with higher output b_n less likely than states with lower output. We also interpret variables b_n as the *sensitivities* of probabilities $\pi_n(\varepsilon)$ to shock ε. Below, we discuss economic settings where probabilities (1) arise endogenously and factor b emerges as a source of information asymmetry. By $b(\omega)$ we denote a random variable with realizations b_n in states ω_n. We assume that $b(\omega)$ is an injective mapping, so that its values are different for different states, and hence, factor realizations b_n are analogous to states.[1]

There are $M\geq 2$ assets traded in the economy: a riskless bond in perfectly elastic supply paying \$1 at date T and $M-1$ state-contingent risky assets in zero net supply with terminal payoff $C_m(\omega)$ in state ω, where $m=1,\dots,M-1$. The risky assets may include AD securities, assets with only positive payoffs, options or other derivatives. All assets are non-redundant in the sense that each asset's terminal payoff cannot be replicated by trading in other assets. We denote

1. Assuming that function $b(\omega)$ is injective is without loss of generality because the states such that $b(\omega_k)=b(\omega_m)$ can be lumped together into one state.

the vector of $M-1$ risky asset payoffs in state ω by $C(\omega)=\Big(C_1(\omega),\dots,\ C_{M-1}(\omega)\Big)^\top$, and use C_m and C as shorthand notation for $C_m(\omega)$ and $C(\omega)$. The bond price is set to $p_0=e^{-rT}$, where r is an exogenous risk-free interest rate. We denote the vector of observed date-0 prices of the risky assets by $p=(p_1,\dots,p_{M-1})^\top$. These prices are determined in equilibrium, defined below.

The economy is populated by three types of investors, informed and uninformed, labeled I and U, and noise traders. There is a continuum of identical investors of each type that act as price-takers. Fraction $h/2$ of all investors are informed and fraction $(1-h)/2$ are uninformed, where $h\in(0,1)$, and fraction $1/2$ are noise traders. Investors I and U have CARA preferences over terminal wealth with risk aversions γ_I and γ_U, respectively. Noise traders submit exogenous normally distributed demands $\nu=(\nu_1,\dots,\nu_{M-1})^\top\sim\mathcal{N}(0,\Sigma_\nu)$ for risky assets, where $\Sigma_\nu\in\mathbb{R}^{(M-1)\times(M-1)}$ is a symmetric positive-definite matrix.

Both investors I and U know all asset payoffs in all states, $C_m(\omega_n)$. Before the markets open, investor I observes shock ε. Investor U observes only asset prices p at date $t=0$ and knows how the equilibrium prices, given by some function $P(\varepsilon,\nu)\in\mathbb{R}^{M-1}$, depend on shock ε and noises ν. Investor U has the following conjugate prior probability density function (PDF) over ε:

$$\varphi_\varepsilon(x)=\frac{\Big(\sum_{j=1}^N e^{a_j+b_jx}\Big)e^{-0.5(x-\mu_0)^2/\sigma_0^2}}{\int_{-\infty}^{\infty}\Big(\sum_{j=1}^N e^{a_j+b_jx}\Big)e^{-0.5(x-\mu_0)^2/\sigma_0^2}dx}. \tag{2}$$

Below, we discuss economies where PDF $\varphi_\varepsilon(x)$ arises endogenously.

For fixed parameters a_n and b_n, we choose μ_0 and σ_0^2 such that ε has any desired mean μ_ε and variance σ_ε^2. We refer to the distribution of ε as *generalized normal* $\widehat{\mathcal{N}}(\mu_\varepsilon,\sigma_\varepsilon^2)$ with mean $\mathbb{E}[\varepsilon]=\mu_\varepsilon$ and variance $\mathrm{var}[\varepsilon]=\sigma_\varepsilon^2$, and note that it can be rewritten as a weighted average of PDFs of normally distributed random variables. The relationship between (μ_0,σ_0^2) and $(\mu_\varepsilon,\sigma_\varepsilon^2)$ is given by equations (A.1) and (A.2) in the Appendix.

Our economy also has an equivalent interpretation as an economy where the aggregate risk factor b has an arbitrary prior distribution $\widetilde{\pi}_n$ and the informed investor receives a conditionally normal signal $\varepsilon=b\sigma_0^2+u$, where $u\sim\mathcal{N}(\mu_0,\sigma_0^2)$. Then, Bayes' law and the formula for total probability give rise to probabilities (1) and distribution (2), where $a_n=\ln(\widetilde{\pi}_n)-0.5\sigma_0^2b_n^2$. According to this interpretation, learning via ε about factor b is the source of the asymmetry of information, which justifies the interpretation of b as an aggregate risk factor.

Allowing shocks ε to be conditionally normal and noise trader demands ν normal variables captures in a tractable way the fact that these variables can be either positive or negative, which makes their normality less restrictive than the normality of asset payoffs in the related literature. Moreover, ν and ε in reality are likely to be sums of smaller zero-mean shocks, in which case they are approximately normal by the central limit theorem.

2.2. *Investors' optimization and definition of equilibrium*

Each investor $i=I,U$ is endowed with initial wealth $W_{i,0}$, and allocates it to buy α_i units of the riskless asset and $\theta_{i,m}$ units of the risky asset m. By $\theta_i=(\theta_{i,1},\dots,\theta_{i,M-1})^\top$ we denote the vector of units of risky assets purchased by investor i. The value of riskless assets held by investor i is given by $W_{i,0}-p^\top\theta_i$ but does not play any role in our analysis. After observing prices p, the uninformed investor updates the prior PDF (2) conditioning on the information that shock ε and noise ν satisfy equation $P(\varepsilon,\nu)=p$. Investors I and U maximize their expected utilities over terminal wealth:

$$\max_{\theta_I}\mathbb{E}\Big[-e^{-\gamma_I W_{I,T}}\Big|\varepsilon,p\Big], \tag{3}$$

$$\max_{\theta_U} \mathbb{E}\Big[-e^{-\gamma_U W_{U,T}}\Big| P(\varepsilon,\nu)=p\Big], \tag{4}$$

respectively, subject to their self-financing budget constraints

$$W_{i,T}=W_{i,0}e^{rT}+(C-e^{rT}p)^\top\theta_i, \quad i=I,U. \tag{5}$$

Definition of equilibrium. *A competitive noisy rational expectations equilibrium is a vector of risky asset prices $P(\varepsilon,\nu)$ and investors' portfolios of risky assets $\theta_I^*(p;\varepsilon)$ and $\theta_U^*(p)$ that solve optimization problems* (3) *and* (4) *subject to self-financing budget constraints* (5), *taking asset prices as given, and satisfy the market clearing condition:*

$$h\theta_I^*(P(\varepsilon,\nu);\varepsilon)+(1-h)\theta_U^*(P(\varepsilon,\nu))+\nu=0. \tag{6}$$

Remark 1 (Particular cases). Our model incorporates the economies with one risky asset in which asset payoffs are normally distributed as in Grossman and Stiglitz (1980) or drawn from more general distributions as in Breon-Drish (2015). Lemma A.2 in the Appendix demonstrates that the latter economies can be obtained in the limit $N\to\infty$ of our N-state economy with asset payoffs $C_1(\omega_n)=\underline{C}_N+(\overline{C}_N-\underline{C}_N)(n-1)/(N-1)$ and distribution parameters

$$a_n=-\frac{C_1(\omega_n)^2}{2\sigma_0^2}-\frac{\mu_0 C_1(\omega_n)}{\sigma_0^2}+\ln[\varphi_C(C_1(\omega_n))], \quad b_n=\frac{C_1(\omega_n)}{\sigma_0^2}, \tag{7}$$

where $\underline{C}_N$ and $\overline{C}_N$ converge to the lower and upper limits of payoff C_1 with PDF $\varphi_C(x)$.

3. CHARACTERIZATION OF EQUILIBRIUM

In this section, we first consider our baseline economy where the number of assets equals the number of states, $M=N$, and find the equilibrium in closed form. Then, we consider a more general economy with $M\leq N$ assets and derive the equilibrium in terms of inverse functions. We use the latter economy to establish conditions for the effective completeness of financial markets and to study the impact of market incompleteness on asset prices.

3.1. *Economy with $M=N$ securities*

We start with an economy where the number of assets equals the number of states, that is, $M=N$, which we label as a *complete-market economy*.[2] Our methodological contribution is to show that market completeness substantially simplifies the derivation of equilibrium, so that asset prices and investors' portfolios are available in closed form.

Due to market completeness, we look for equilibrium prices p in the following form:

$$p=\Big[\pi_1^{\text{RN}}C(\omega_1)+\pi_2^{\text{RN}}C(\omega_2)+\cdots+\pi_N^{\text{RN}}C(\omega_N)\Big]e^{-rT}, \tag{8}$$

where π_n^{RN} are the *risk-neutral* probabilities of states ω_n. The risk-neutral probabilities exist and are unique in equilibrium because investors I and U are unconstrained and can eliminate any arising arbitrage opportunities (e.g. Duffie, 2001, p. 4).

2. The realizations of shock ε can be interpreted as a continuum of states of the economy in addition to states ω_n. However, N non-redundant assets still suffice to replicate any contingent claim in our economy because the payoffs of such claims do not vary across ε for a fixed state ω_n. In other words, ε-states can be clumped together so that only ω_n states matter for replication. Furthermore, noise trader demands do not contribute to market incompleteness because they affect date-0 prices but do not affect future payoffs.

Informed and uninformed investors agree on the risk-neutral probabilities because these probabilities are uniquely determined from equation (8) as functions of prices p. However, investors assign different real probabilities to states ω_n. In particular, investor U's real *posterior probabilities* are given by $\pi_n^U(p)=\mathbb{E}\Big[\pi_n(\varepsilon)|P(\varepsilon,\nu)=p\Big]$. The expression for $\pi_n^U(p)$ can be obtained by rewriting investor U's expected utility (4) as follows:

$$\begin{aligned}\mathbb{E}\Big[-e^{-\gamma_U W_{U,T}}|P(\varepsilon,\nu)=p\Big] &= -\sum_{n=1}^N\Big(\mathbb{E}\Big[\pi_n(\varepsilon)|P(\varepsilon,\nu)=p\Big]e^{-\gamma_U W_{U,T,n}}\Big)\\ &= -\sum_{n=1}^N\pi_n^U(p)e^{-\gamma_U W_{U,T,n}}.\end{aligned} \tag{9}$$

Taking probabilities $\pi_n(\varepsilon)$ and $\pi_n^U(p)$ as given, investors' optimizations can be solved using the methods of complete-market portfolio choice. The informed and uninformed investors have different state price densities (SPDs), which are given by discounted ratios of risk-neutral and their real probabilities (e.g. Duffie, 2001, p. 11): $\pi_n^{\text{RN}}e^{-rT}/\pi_n(\varepsilon)$ and $\pi_n^{\text{RN}}e^{-rT}/\pi_n^U(p)$, respectively. The first order conditions (FOCs) of investors equate their marginal utilities and SPDs (e.g. Duffie, 2001, p. 5) and are given by:

$$\gamma_I e^{-\gamma_I W_{I,T,n}}=\ell_I\frac{\pi_n^{\text{RN}}e^{-rT}}{\pi_n(\varepsilon)},\quad \gamma_U e^{-\gamma_U W_{U,T,n}}=\ell_U\frac{\pi_n^{\text{RN}}e^{-rT}}{\pi_n^U(p)}, \tag{10}$$

where ℓ_i are Lagrange multipliers for investors' budget constraints. From equation (10), we find optimal wealths $W_{i,T}$. Then, we use these wealths to recover optimal portfolios from budget constraints (5). Lemma 1 below reports investor I's portfolio.

Lemma 1 (Investor I's optimal portfolio). *Investor I's optimal portfolio of risky assets is given by:*

$$\theta_I^*(p;\varepsilon)=\frac{\lambda\varepsilon}{\gamma_I}-\frac{\Omega^{-1}(\widetilde{v}(p)-\widetilde{a})}{\gamma_I}, \tag{11}$$

where $\Omega\in\mathbb{R}^{(N-1)\times(N-1)}$ is a matrix of excess payoffs with elements $\Omega_{n,k}=C_k(\omega_n)-C_k(\omega_N)$, $\lambda=\Omega^{-1}(b_1-b_N,\dots,b_{N-1}-b_N)^\top\in\mathbb{R}^{N-1}$, $\widetilde{a}=(a_1-a_N,\dots,a_{N-1}-a_N)^\top\in\mathbb{R}^{N-1}$, and $\widetilde{v}(p)\in\mathbb{R}^{N-1}$ is the vector of log ratios of risk-neutral probabilities, given by:

$$\widetilde{v}(p)=\left(\ln\left(\frac{\pi_1^{\text{RN}}}{\pi_N^{\text{RN}}}\right),\dots,\ln\left(\frac{\pi_{N-1}^{\text{RN}}}{\pi_N^{\text{RN}}}\right)\right)^\top. \tag{12}$$

Equation (11) decomposes portfolio of risky assets $\theta_I^*(p;\varepsilon)$ into two terms, which we label as *information-sensitive* and *information-insensitive* demands, respectively. The information-sensitive demand is linear in shock ε, and the portfolio is separable in ε and prices p, as in the related literature (e.g. Grossman and Stiglitz, 1980; Breon-Drish, 2015). The information-insensitive demand depends on risk-neutral probabilities, which can be found analytically as functions of prices p by solving risk-neutral pricing equations (8). Hence, portfolio (11) is in closed form in terms of shock ε and price p, which are observable by investor I.

We also observe that portfolio (11) features variables a_n-a_N and b_n-b_N, normalized relative to state ω_N. The normalization arises because adding a constant to variables a_n and b_n that determine the probabilities of states (1) does not affect these probabilities, and hence, should not affect optimal portfolios and equilibrium asset prices.

Vector λ in portfolio (11) has the following interpretation. The definition of λ in Lemma 1 implies that λ satisfies equations $b_n=\lambda_0+C(\omega_n)^\top\lambda$, where λ_0 is a constant. Therefore, λ is a replicating portfolio for the aggregate risk factor b, up to a constant λ_0. In Section 3.3 below, we provide detailed discussion of portfolio λ and separability of portfolio (11) in shock ε and price p in a more general incomplete-market economy.

The linearity of portfolio θ_I^* in the aggregate shock ε and its separability in ε and prices p are the key sources of tractability of our model. The latter properties of portfolio θ_I^* along with the market clearing condition allow us to characterize asset prices in terms of a sufficient statistic that is linear and separable in the aggregate shock ε and noise trader demands ν. In turn, the linearity of the sufficient statistic and joint normality of noise trader demands greatly simplify learning from prices by uninformed investors, which ultimately allows us to obtain asset prices in closed form.

Specifically, substituting $\theta_I^*(p;\varepsilon)$ and $\theta_U^*(p)$ into the market clearing condition (6), we find that

$$\frac{h\lambda\varepsilon}{\gamma_I}+\nu+H(p)=0, \tag{13}$$

where $H(p)$ is a function of asset prices p, discussed below, which is given by:

$$H(p)=(1-h)\theta_U^*(p)-\frac{h}{\gamma_I}\Omega^{-1}\Big(\widetilde{v}(p)-\widetilde{a}\Big), \tag{14}$$

and $\widetilde{v}(p)$, $\widetilde{a}$ are as in Lemma 1. Substituting prices p into equation (13), investor U learns the *sufficient statistic* $s=h\lambda\varepsilon/\gamma_I+\nu$. This sufficient statistic is equal to function $-H(p)$, which we label as *the informational content of prices*. Next, investor U finds posterior probabilities of states by computing conditional expectations $\pi_n^U(p)=\mathbb{E}\Big[\pi_n(\varepsilon)|h\lambda\varepsilon/\gamma_I+\nu=-H(p)\Big]$. Lemma 2 reports these probabilities.

Lemma 2 (Posterior probabilities of states). *The posterior probabilities of states ω_n conditional on observing the sufficient statistic $s=h\lambda\varepsilon/\gamma_I+\nu$ are given by:*

$$\pi_n^U(p)=\frac{1}{G(p)}\exp\left\{a_n+\frac{1}{2}\frac{b_n^2+2b_n\Big(\mu_0/\sigma_0^2-h\lambda^\top\Sigma_\nu^{-1}H(p)/\gamma_I\Big)}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}\right\}, \tag{15}$$

where function $H(p)$ is given by equation (14), and $G(p)$ is a normalizing function.

We observe that probabilities (15) are proportional to an exponent of a quadratic function of sensitivities b_n, which is similar to a moment generating function $\mathbb{E}[e^{b_n z}]$ of a normally distributed variable z. This similarity emerges because, as discussed in Section 2.1, noise ν and shock ε are (conditionally) normally distributed, and state probability $\pi_n(\varepsilon)$ has an exponential-linear form. Consequently, Bayesian updating of the latter probability gives exponential-quadratic posterior probability (15).

We then find the equilibrium as follows. From the FOC (10) for investor U we find investor U's optimal portfolio $\theta_U^*(p)$ in terms of probabilities (15), similar to finding investor I's portfolio (11). We observe, that probabilities (15) themselves depend on portfolio $\theta_U^*(p)$ via the informational content of prices $H(p)$. Hence, portfolio $\theta_U^*(p)$ solves a fixed-point problem [equation (A.15) in the Appendix], which we do not show here for brevity. The fixed-point problem turns out to be linear, and hence, we find portfolio $\theta_U^*(p)$ in closed form. Then, we find the equilibrium risk-neutral probabilities and asset prices from the market clearing condition (13). In this article, we focus on equilibria in which asset prices are continuous functions of the sufficient statistic s, and the sufficient statistic is the only information revealed by prices.[3] Proposition 1 reports the equilibrium.

Proposition 1 (Equilibrium with $M=N$ assets). *(i) There exists unique equilibrium in which prices only reveal the sufficient statistic $s=h\lambda\varepsilon/\gamma_I+\nu$. In this equilibrium, the vector of risky asset prices $P(\varepsilon,\nu)$ and risk-neutral probabilities π_n^{RN} are given by:*

$$P(\varepsilon,\nu)=\Big[\pi_1^{\text{RN}}C(\omega_1)+\pi_2^{\text{RN}}C(\omega_2)+\cdots+\pi_N^{\text{RN}}C(\omega_N)\Big]e^{-rT}, \tag{16}$$

$$\pi_n^{\text{RN}}=\frac{e^{\upsilon_n}}{\sum_{j=1}^{N}e^{\upsilon_j}}, \tag{17}$$

where probability parameters υ_n are given in closed form by:

$$\upsilon_n=a_n+\frac{1}{2}\frac{(1-h)/\gamma_U}{h/\gamma_I+(1-h)/\gamma_U}\frac{b_n^2+2(\mu_0/\sigma_0^2)b_n}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}+\frac{C(\omega_n)^\top\Big(E+Q\Big)s}{h/\gamma_I+(1-h)/\gamma_U}, \tag{18}$$

$$Q=\frac{h(1-h)}{\gamma_I\gamma_U}\frac{\lambda\lambda^\top\Sigma_\nu^{-1}}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}, \tag{19}$$

where E is an identity matrix, $Q,E\in\mathbb{R}^{(N-1)\times(N-1)}$, and $\lambda=\Omega^{-1}(b_1-b_N,\ldots,b_{N-1}-b_N)^\top$.
(ii) Portfolio $\theta_I^*(p;\varepsilon)$ is given by equation (11) and portfolio $\theta_U^*(p)$ is given by

$$\theta_U^*(p)=\Big(E+Q\Big)^{-1}\left(\frac{hQ\Omega^{-1}(\widetilde{\upsilon}(p)-\widetilde{a})}{(1-h)\gamma_I}-\frac{\Omega^{-1}(\widetilde{\upsilon}(p)-\widehat{a})}{\gamma_U}+\frac{\mu_0/(\gamma_U\sigma_0^2)\lambda}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}\right), \tag{20}$$

where $\widetilde{\upsilon}(p)\in\mathbb{R}^{N-1}$ is given by (12) and has elements $\upsilon_n-\upsilon_N$ in equilibrium, and $\widetilde{a},\widehat{a}\in\mathbb{R}^{N-1}$ have elements a_n-a_N and $a_n-a_N+0.5(b_n^2-b_N^2)/(h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2)$, respectively.

Proposition 1 extends the no-arbitrage valuation approach to economies with asymmetric information and provides asset prices in closed form in terms of expected discounted payoffs under risk-neutral probabilities, familiar from the asset-pricing literature. These risk-neutral probabilities are functions of the sufficient statistic s such that the log-likelihood ratios of any two states, given by $\ln(\pi_n^{\text{RN}}/\pi_m^{\text{RN}})$, are linear functions of s. Consequently, the sufficient statistic emerges as a state variable that affects optimal portfolios, asset prices, and asset returns, similar to the related literature (e.g. Breon-Drish, 2015; Albagli *et al.*, 2021). The resulting equilibrium prices are non-linear functions of the sufficient statistic, in contrast to CARA-normal noisy REE models (e.g. Grossman and Stiglitz 1980; Admati 1985, among others].

The tractability of equilibrium prices and portfolios allows us to study the sensitivities of optimal portfolios to prices p, which we report in Proposition 2 below.

3. Pálvölgyi and Venter (2015) use our characterization of equilibrium asset prices to study the uniqueness of equilibrium in Proposition 1. They show that our complete market equilibrium is unique among all equilibria with continuous prices. Pálvölgyi and Venter (2014) demonstrate the existence of multiple discontinuous equilibria in a Grossman and Stiglitz (1980) economy, which may also exist in our model. Finding such equilibria in multi-asset economies is a challenging task and is beyond the scope of our work.

Proposition 2 (Slopes of asset demands). *The sensitivities of investors' portfolios of risky assets to prices p are given by:*

$$\frac{\partial\theta_I^*(p;\varepsilon)}{\partial p} = -\frac{\left(\text{var}^{\text{RN}}[C]\right)^{-1}e^{rT}}{\gamma_I}, \tag{21}$$

$$\frac{\partial\theta_U^*(p)}{\partial p} = -\left(E - \frac{(h/\gamma_I+(1-h)/\gamma_U)h\lambda\lambda^\top\Sigma_\nu^{-1}}{(h/\gamma_I+(1-h)/\gamma_U)h\lambda^\top\Sigma_\nu^{-1}\lambda+\gamma_I/\sigma_0^2}\right)\frac{\left(\text{var}^{\text{RN}}[C]\right)^{-1}e^{rT}}{\gamma_U}, \tag{22}$$

where $\text{var}^{\text{RN}}[\cdot]$ *is the risk-neutral variance-covariance matrix and* $\partial\theta_i^*/\partial p$ *for* $i=I,U$ *are matrices with elements* $\partial\theta_{i,n}^*/\partial p_m$ *in rows n and columns m. Furthermore, the informed investor's demand for risky asset m is a downward-sloping function of that asset's price* p_m*, holding other prices fixed.*

The portfolio sensitivities in equations (21) and (22) can be also interpreted as matrices of demand slopes. The sensitivities of portfolio $\theta_I^*(p;\varepsilon)$ to prices p in equation (21) are determined by the inverse risk-neutral variance–covariance matrix. Consequently, the informed investor's demand for risky asset m is a downward sloping function of that asset's own price p_m because the elements on the main diagonal of a positive-definite matrix $(\text{var}^{\text{RN}}[C])^{-1}$ are all positive [see Proof of Proposition 2].

The sensitivities of portfolio $\theta_U^*(p)$ to prices p in equation (22) are given by the negative of the product of two matrices that capture the information and substitution effects, respectively. Investor U's demand for asset m can be an upward sloping function of p_m because the product of these matrices in (22), in general, is not positive-definite and may have negative elements on the main diagonal. Intuitively, high asset prices can be interpreted as positive information about shock ε, in which case the uninformed demand for asset m may increase through this information effect.[4] Admati (1985) finds a similar result in a multi-asset CARA-normal model. Our analysis extends the finding of Admati (1985) to economies without normality and reveals the important roles of portfolio λ and the risk-neutral variance of asset payoffs in generating these effects.

Similar to Admati (1985), the equilibrium is unique despite possibly upward sloping uninformed demand. The intuition is that for the uninformed demand to be upward sloping there should be a sufficiently large fraction of informed investors in the economy so that the information effect is sufficiently strong. However, as shown in Proposition 2, the informed investors have downward sloping demands. As a result, the aggregate demand is downward sloping because the information effect is subdued by the substitution effect after aggregation. At a more technical level, the equilibrium is unique because uninformed portfolios and log-ratios of risk-neutral probabilities solve a linear system of equations that has a unique solution [see the Proof of Proposition 1].

4. A sufficient condition for investor U's demands to be downward sloping is that the first matrix (in parentheses) on the right-hand side of (22), which captures the information effects, is sufficiently close to the identity matrix E. The latter condition is satisfied when, for example, it is difficult to learn from prices because there are few informed traders ($h\approx 0$) or noise trader demands are very volatile ($\Sigma_\nu^{-1}\approx 0$). Then, the slopes are approximately given by $-\text{var}^{\text{RN}}[C]^{-1}e^{rT}/\gamma_U$, which implies downward sloping demands, as shown in the previous paragraph. Section IA6.1 of the Supplementary Appendix provides an example of a market with upward sloping asset demands. In that example, investors have identical risk aversions, and hence, differences in preferences are not necessary for generating upward sloping demands.

3.2. *Capital asset pricing model with asymmetric information*

In this section, we derive a capital asset pricing model (CAPM) with asymmetric information and study its economic implications. Our CAPM provides a bridge between risk-based models of asset pricing in the no-arbitrage tradition and models with asymmetric information, and also highlights how information aggregation alters asset returns. The existence of a state price density allows us to derive this CAPM as a simple corollary of the results in Section 6.1 of Cochrane (2005), which we report below.

Corollary 1 (Asymmetric information CAPM). *Let $R_i = C_i/P_i$ be the gross return of asset i. In the complete market the asset risk premia are given by*

$$\mathbb{E}^U[R_i]-e^{rT}=-\mathrm{cov}^U\left(R_i,\frac{\pi^{\mathrm{RN}}}{\pi^U}\right), \tag{23}$$

where π^{RN}/π^U is the ratio of risk-neutral and investor U's probabilities of states, and the covariance is computed under investor U's probability measure π^U. Furthermore, let R^* be the projection of the rescaled state price density $(\pi^{\mathrm{RN}}/\pi^U)/\mathbb{E}^{\mathrm{RN}}[\pi^{\mathrm{RN}}/\pi^U]$ on the space of tradable assets in the Euclidean space with scalar product $\mathbb{E}^U[XY]$. Then

$$\mathbb{E}^U[R_i]-e^{rT}=\frac{\mathrm{cov}^U\left(R_i,R^*\right)}{\mathrm{var}^U[R^*]}(\mathbb{E}^U[R^*]-e^{rT}). \tag{24}$$

Corollary 1 presents the CAPM from the point of view of an uninformed investor or an econometrician, and demonstrates that the asset risk premia are determined by the covariance of returns with the state price density and admit a beta representation (24), similar to standard risk-based models discussed in Cochrane (2005, pp. 101–102). Following Cochrane (2005, p. 109), we observe that return R^* in (24) can be interpreted as a return of a factor-mimicking portfolio for the state price density. We also note that the risk premia in our analysis are contingent on the sufficient statistic s because the posterior probabilities π_n^U given by equation (15) are functions of s.

The risk premia in (24) non-linearly depend on the aggregate risk factor b, which complicates the analysis of asset returns. To isolate the effects of asymmetric information on the risk premia, following Cochrane (2005, p. 161), we derive an approximate linear three-factor model by linearizing the SPD. Corollary 2 reports our results.

Corollary 2 (Conditional three-factor CAPM). *The SPD has linearization*

$$\frac{\pi_n^{\mathrm{RN}}}{\pi_n^U}e^{-rT}=g_0(s)\left(1+\frac{C(\omega_n)^\top\mathbb{E}[-\nu|s]}{(1-h)/\gamma_U+h/\gamma_I}-\frac{1}{2}\frac{h/\gamma_I}{(1-h)/\gamma_U+h/\gamma_I}\frac{b_n^2-2b_n\mathbb{E}^U[b]}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}\right)$$
$$+O\left(\frac{1}{((1-h)/\gamma_U+h/\gamma_I)^2}\right), \tag{25}$$

where $g_0(s)$ does not depend on state ω_n. Consequently, when either γ_I or γ_U is small, the asset risk premia are given by:

$$\mathbb{E}^U[R_i]-e^{rT}\approx\beta_{1i}\Lambda_1+\beta_{2i}\Lambda_2+\beta_{3i}\Lambda_3, \tag{26}$$

where β_{1i}, β_{2i}, and β_{3i} are the multiple regression coefficients of R_i on factors $C(\omega)^\top\mathbb{E}[-\nu|s]$, b, b^2, and a constant, and the factor risk premia are given by $\Lambda_1=(\mathbb{E}^U[C(\omega_n)]-pe^{rT})^\top\mathbb{E}[-\nu|s]$, $\Lambda_2=\mathbb{E}^U[b]-\mathbb{E}^{\mathrm{RN}}[b]$, $\Lambda_3=\mathbb{E}^U[b^2]-\mathbb{E}^{\mathrm{RN}}[b^2]$.

The three-factor model (26) is the main result of this section. The betas in (26) can be interpreted as time-series regression coefficients of asset returns on factors in a setting where returns each period are determined from our one-period model, as in the mean-variance CAPM. We also note that model (26) is conditional on the investors' ability to observe the sufficient statistic s. Hence, betas should be estimated as functions (e.g. polynomials) of s, similar to Cochrane (2005, p. 144).

Next, we discuss the factors. The first factor $C(\omega)^\top \mathbb{E}[-\nu|s]$ is analogous to the payoff of the market portfolio in our model. In particular, from the market clearing condition (6), the exogenous supply of risky assets in the economy is given by $-\nu$. The uninformed investor does not observe $-\nu$, and hence, evaluates the payoff of the market portfolio by using the best estimate $\mathbb{E}[-\nu|s]$. The other two factors, the aggregate risk factor b and the volatility factor b^2, capture the effects of asymmetric information on the SPD.

The risk factor b can be any macroeconomic or financial variable that determines payoffs of other assets in the economy and about which some investors have better information (e.g. a signal $\varepsilon = b\sigma_0^2 + u$, where $u \sim \mathcal{N}(\mu_0, \sigma_0^2)$) than the others. The examples include factors with economy-wide impact, such as aggregate output, stock indices, and oil prices. Other examples include future returns or prices of commodities, currencies, or financial assets on which derivative contracts are traded, in which case the three-factor CAPM (26) describes returns in specialized markets restricted to derivatives written on those commodities and assets.[5] For example, if b is the price of oil in one year from now, and some investors receive a signal about b so that they can predict it better than the others, then equation (26) describes returns of derivatives written on b.

The three-factor model (26) reduces to the standard mean-variance CAPM in three special cases of our model in which 1) there are no informed traders ($h=0$), 2) exogenous asset supply is deterministic ($\Sigma_\nu = 0$), and 3) shock ε is known to everyone ($\sigma_0 = 0$). This is because factors b and b^2 cancel out in the SPD (25) in these three special cases when $h=0$, $\lambda^\top \Sigma_\nu^{-1} \lambda \to \infty$, or $\sigma_0 \to 0$, respectively. Hence, these factors would not be priced.

Finally, we evaluate the contribution of the information-specific factors to asset returns. For tractability, we focus on the returns of AD securities. We also assume that the informed and uninformed investors have the same risk aversion so that the effects of information asymmetry are not confounded by other sources of investor heterogeneity. Lemma 3 presents the expected returns and comparative statics.

Lemma 3 (Expected returns of AD securities). *Suppose, both investors have identical risk aversions, $\gamma_I = \gamma_U = \gamma$. Then, the expected return of the AD security with payoff $1_{\{\omega=\omega_n\}}$ is given by:*

$$\mathbb{E}^U[R_n] = \frac{\pi_n^U}{\pi_n^{\text{RN}}} e^{rT} = \frac{\exp\{-\gamma C(\omega_n)^\top \mathbb{E}[-\nu|s] + 0.5t(b_n - \mathbb{E}^U[b])^2 + rT\}}{\mathbb{E}^{\text{RN}}\Big[\exp\{-\gamma C(\omega)^\top \mathbb{E}[-\nu|s] + 0.5t(b - \mathbb{E}^U[b])^2\}\Big]}, \tag{27}$$

where $t = h/(h^2 \lambda^\top \Sigma_\nu^{-1} \lambda/\gamma^2 + 1/\sigma_0^2)$ quantifies the asymmetry of information. Moreover, $\partial \mathbb{E}^U[R_n]/\partial t > 0$ if and only if $(b_n - \mathbb{E}^U[b])^2 > \text{var}^U[b]$.

5. Dew-Becker, Giglio, and Kelly (2021) study the returns on straddles with payoffs $\max(b-K,0)+\max(K-b,0)$ and strangles with payoffs $\max(b-K_1,0)+\max(b-K_2,0)$ written on 14 commodities and 5 financial assets futures (e.g. oil, gold, silver, British pound, and Swiss franc) using factor models that include the returns and squared returns (interpreted as realized volatility) on the futures prices of the underlying assets, and find that these factors help explain the returns. The latter factors are similar to our factors b and b^2. Ang, Hodrick, Xing, and Zhang (2006) empirically demonstrate a conditional relationship between asset returns and the market volatility, measured by VIX. Factors based on output and oil prices have been studied in Chen, Roll, and Ross (1986) and Ferson and Harvey (1994), albeit without volatility factors. However, these papers do not discuss possible role of asymmetric information in generating asset returns.

Equation (27) disentangles the effects of risk attitudes and information asymmetry on expected returns, which are captured by terms $\gamma C(\omega_n)^\top \mathbb{E}[-\nu|s]$ and $t(b_n - \mathbb{E}^U[b])^2$, respectively. Parameter t quantifies information asymmetry. Larger t corresponds to more information asymmetry and $t=0$ to no asymmetry, in which case either there are no informed investors ($h=0$), or asset supply is deterministic ($\Sigma_\nu=0$), or shock ε is known to everyone ($\sigma_0=0$). When $t=0$, expected returns are only determined by risk aversion γ and market portfolio payoffs $C(\omega_n)^\top(-\nu)$. Equation (27) allows for simple comparative statics showing that if two states have the same market portfolio payoffs then, in the presence of information asymmetry, the AD security of the state with higher squared deviation $(b-\mathbb{E}^U[b])^2$ has higher return, and the effect is stronger with larger parameter t.

The risk premia of AD securities are given by $\mathbb{E}^U[R_n]-e^{rT}=(\pi_n^U/\pi_n^{\text{RN}}-1)e^{rT}$. There are always AD securities with positive and negative risk premia because assuming that, e.g., all risk premia are positive, $\pi_n^U/\pi_n^{\text{RN}}-1>0$, leads to a contradiction $1=\sum_{i=1}^N \pi_n^U > \sum_{i=1}^N \pi_n^{\text{RN}}=1$. Next, we find out which AD securities have positive risk premia. From equation (27) we observe that $\pi_n^U > \pi_n^{\text{RN}}$ for sufficiently large $|b_n|$ because the ratio $\pi_n^U/\pi_n^{\text{RN}}$ is mainly determined by the quadratic term $(b_n-\mathbb{E}^U[b])^2$. Therefore, posterior probabilities π_n^U are higher than risk-neutral ones for tail risks (i.e. when $|b_n|$ is large), and hence, corresponding AD securities have positive risk premia. The latter effects are stronger with larger asymmetry parameter t.

3.3. *General economy with $M \le N$ securities*

In this section, we study an economy with M securities, where $M \le N$, which subsumes complete and incomplete market economies as special cases. For tractability, we focus on a subset of economies which satisfy the following condition.

Informational spanning condition: *There exists a replicating portfolio for the risk factor b. That is, there exist constant λ_0 and a vector of asset units $\lambda=(\lambda_1,\dots,\lambda_{M-1})^\top$ such that*

$$b_n = \lambda_0 + C(\omega_n)^\top \lambda. \tag{28}$$

Condition (28) implies the existence of a portfolio of assets λ that replicates the risk factor b. This portfolio is a tradable analogue of the risk factor b, similar to factor mimicking portfolios in the asset pricing literature. Next, we provide several plausible economies that satisfy condition (28). First example is a complete-market economy, where $M=N$, and hence, there always exist constant λ_0 and vector λ satisfying equation (28). Second example is an incomplete-market economy with only one risky asset with payoff $C_1=b/\lambda_1$, where λ_1 is a constant. Lemma A.2 in the Appendix shows that single risky asset economies in Grossman and Stiglitz (1980) and Breon-Drish (2015) satisfy condition $C_1=b/\lambda_1$. Third example is an economy with an asset paying $C_1=b/\lambda_1$, and call options on C_1 with payoffs $C_2=\max(C_1-K_2,0),\dots, C_{M-1}=\max(C_1-K_{M-1},0)$, in which case $\lambda_0=0$, $\lambda=(\lambda_1,0,\dots,0)^\top$. Fourth example is an economy with a firm that has cash flow b and issues risky debt and equity with payoffs $\min(b,K)$ and $\max(b-K,0)$, where K is the face value of debt. It is easy to verify that in this last example $\lambda_0=0$ and $\lambda=(1,1)^\top$ because $b=\min(b,K)+\max(b-K,0)$.

We begin with the derivation of the informed investor's portfolio in Lemma 4 below.

Lemma 4 (Investor I's optimal portfolio). *If condition (28) is satisfied, then investor I's optimal portfolio of risky assets is linear in ε, and given by*

$$\theta_I^*(p;\varepsilon) = \frac{\lambda\varepsilon}{\gamma_I} - \frac{\widehat{\theta}_I^*(p)}{\gamma_I}, \tag{29}$$

where vector λ *is such that condition (28) is satisfied, and* $\widehat{\theta}_I^*(p)$ *is a function of* p.

Lemma 4 shows that condition (28) gives rise to optimal portfolio (29) that is separable in shock ε and price p and linear in ε, as in the complete market economy of Section 3.1. Consequently, learning from prices and the derivation of equilibrium follow the same steps as in Section 3.1. We label the first and second terms of (29) as information-sensitive and information-insensitive demands, respectively. Next, we discuss the economic intuition for condition (28) and portfolio (29), and the derivation of equilibrium.

First, we provide intuition for the informational spanning condition (28). This spanning condition implies the existence of an information-independent portfolio λ whose payoffs replicate the realizations of the aggregate risk factor b. Because this portfolio is perfectly correlated with the risk factor, it is the natural portfolio to use when trading on private information. In particular, investing in information-sensitive portfolio $\lambda\varepsilon/\gamma_I$, as part of the optimal portfolio (29), captures the informational motive for trading and allows the informed investors to have higher payoffs in more likely states. The latter investment strategy is similar to market timing in the real world when those investors who can predict market booms and recessions tilt their portfolios towards securities that do better in those economic conditions. The remaining information-insensitive demand $-\widehat{\theta}_I^*(p)/\gamma_I$ in portfolio (29) captures non-informational, risk-sharing motives.

Next, we discuss the separability of portfolio (29) in shock ε and price p and its linearity in ε. Equation (1) for real probabilities and condition (28) imply that the PDF of the payoff vector $C(\omega)$ conditional on shock ε is a distribution from a multivariate exponential family, given by $\varphi_{C|\varepsilon}(x) = H(\varepsilon)\exp(A(x)+B(x)\varepsilon)$ (Casella and Berger, 2002, p. 114) with a linear function $B(x) = \lambda^\top x$. Breon-Drish (2015) shows that the latter exponential family distribution with a linear function $B(x)$ is necessary and sufficient for the separability of portfolio θ_I^* in the signal and prices in a single risky asset economy. The separability in the multi-asset case can be demonstrated along the same lines as in Breon-Drish (2015).[6]

Specifically, because the distribution $\pi_n(\varepsilon)$ given by equation (1) is from the exponential family, it is conjugate to the exponential CARA utility so that the risk factor $b\varepsilon$ and payoffs $-\gamma_I\theta_I^\top C$ are substitutes in the informed investors' expected utility. The latter observation and the informational spanning condition (28) then imply that the risk factor $b\varepsilon$ affects portfolio choice in the same way as endowing the investor with a portfolio $-\lambda\varepsilon/\gamma_I$ in the economy with zero shock.[7] Consequently, the offsetting portfolio $\lambda\varepsilon/\gamma_I$ that allocates more wealth to states with lower endowment (equivalently, higher probabilities) is linear in ε. Furthermore, because the replicating portfolio λ is independent of states and prices and there are no wealth effects, the investors can separably meet their informational and non-informational demands, and hence, portfolio (29) is separable in shock ε and price p.

Finally, we turn to the characterization of asset prices. Substituting portfolio (29) into the market clearing condition, similar to the complete-market economy, we obtain $h\lambda\varepsilon/\gamma_I + \nu + \widehat{H}(p) = 0$, where $\widehat{H}(p) \equiv (1-h)\theta_U^*(p) - h\widehat{\theta}_I^*(p)/\gamma_I$ is the informational content of prices. Investor U finds the posterior probabilities following the same steps as in the complete-market case. Given these probabilities, we derive the portfolios of investors, and then the asset prices from the market clearing condition. Proposition 3 reports the equilibrium.

6. In Section IA2 of the Supplementary Appendix, we show that the informational spanning condition (28), and hence the linearity of function $B(x)$, can be relaxed in a setting where noise trader demands are replaced with random endowments, and show the robustness of our results in that setting.

7. High probability of a given state has the same effect on portfolio choice as having low endowment in that state because both cases imply higher marginal utility of an extra unit of wealth in that state.

Proposition 3 (Equilibrium with $M \le N$ assets). *(i) Let the informational spanning condition (28) be satisfied. Then, there exists unique price vector $P(\varepsilon,\nu)$ that is a continuous, differentiable, and invertible on its range function of sufficient statistic $s = h\lambda\varepsilon/\gamma_I + \nu$. Price vector $P(\varepsilon,\nu)$ is the unique solution of equation*

$$\frac{hf_I^{-1}\Big(e^{rT}P(\varepsilon,\nu)\Big)}{\gamma_I}+\frac{(1-h)f_U^{-1}\Big(e^{rT}P(\varepsilon,\nu)\Big)}{\gamma_U}=\Big(E+Q\Big)s+\frac{(1-h)\mu_0/(\gamma_U\sigma_0^2)\lambda}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}, \tag{30}$$

where E is the identity matrix, Q is a matrix given by equation (19), $E,Q \in \mathbb{R}^{(M-1)\times(M-1)}$, and functions $f_I, f_U : \mathbb{R}^{M-1} \to \mathbb{R}^{M-1}$ are invertible on their ranges and given by

$$f_I(x)=\frac{\sum_{j=1}^N C(\omega_j)\exp\{a_j+C(\omega_j)^\top x\}}{\sum_{j=1}^N \exp\{a_j+C(\omega_j)^\top x\}}, \tag{31}$$

$$f_U(x)=\frac{\sum_{j=1}^N C(\omega_j)\exp\{a_j+\frac{1}{2}\frac{b_j^2}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}+C(\omega_j)^\top x\}}{\sum_{j=1}^N \exp\{a_j+\frac{1}{2}\frac{b_j^2}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}+C(\omega_j)^\top x\}}. \tag{32}$$

(ii) The informed and uninformed investors' optimal portfolios of risky assets are given by

$$\theta_I^*(p;\varepsilon)=\frac{\lambda\varepsilon}{\gamma_I}-\frac{f_I^{-1}\Big(e^{rT}p\Big)}{\gamma_I}, \tag{33}$$

$$\theta_U^*(p)=\Big(E+Q\Big)^{-1}\left(\frac{hQf_I^{-1}\Big(e^{rT}p\Big)}{(1-h)\gamma_I}-\frac{f_U^{-1}\Big(e^{rT}p\Big)}{\gamma_U}+\frac{\mu_0/(\gamma_U\sigma_0^2)\lambda}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}\right). \tag{34}$$

Portfolios (33) and (34) have similar structure as their complete-market counterparts. Furthermore, both in incomplete and complete markets prices are non-linear functions of the sufficient statistic $h\lambda\varepsilon/\gamma_I+\nu$. However, in incomplete markets, in general, the prices are not available in closed form and satisfy non-linear equations (30). Solving and analysing these equations, in general, is a challenging task when the number of unknowns is large. The intractability of the incomplete-market equilibrium is in stark contrast with our complete-market equilibrium where prices are available in closed form.

Next, we introduce a *measure of effective market incompleteness* which we use in our analysis below. This measure sheds light on the risk-sharing motives in the economy and quantifies the deviation from Pareto optimality. Risk-neutral probabilities in incomplete markets (i.e. $M<N$), in general, are not unique. In particular, consider the following two probability measures:

$$\pi_n^{\text{RN},I}=\frac{\pi_n(\varepsilon)\exp\{-\gamma_I W_{I,T,n}\}}{\sum_{j=1}^N \pi_j(\varepsilon)\exp\{-\gamma_I W_{I,T,j}\}},\quad \pi_n^{\text{RN},U}=\frac{\pi_n^U(p)\exp\{-\gamma_U W_{U,T,n}\}}{\sum_{j=1}^N \pi_j^U(p)\exp\{-\gamma_U W_{U,T,j}\}}. \tag{35}$$

The first order conditions for investor optimizations (3) and (4) imply that $p=e^{-rT}\mathbb{E}^{\text{RN},I}[C]$ and $p=e^{-rT}\mathbb{E}^{\text{RN},U}[C]$, respectively, where the expectations are under the measures (35). Hence, probabilities (35) can be interpreted as risk-neutral probabilities.

We define the measure of effective incompleteness as the symmetric Kullback-Leibler (KL) divergence of the two risk-neutral measures (35), given by

$$\widehat{\kappa}=\mathbb{E}^{\mathrm{RN},U}\left[\ln\left(\frac{\pi^{\mathrm{RN},U}}{\pi^{\mathrm{RN},I}}\right)\right]+\mathbb{E}^{\mathrm{RN},I}\left[\ln\left(\frac{\pi^{\mathrm{RN},I}}{\pi^{\mathrm{RN},U}}\right)\right]. \tag{36}$$

The KL divergence (36) is always non-negative, and takes zero value if and only if the probability measures (35) coincide (e.g. MacKay, 2017).[8] We also observe that the probability measures (35) coincide if and only if the probability-weighted ratios of marginal utilities are state-independent, so that

$$\frac{\pi_n(\varepsilon)\exp\{-\gamma_I W_{I,T,n}\}}{\pi_n^U(p)\exp\{-\gamma_U W_{U,T,n}\}}=\exp\{\bar{\ell}\}, \tag{37}$$

where $\bar{\ell}$ is state-independent. Therefore, investors achieve Pareto-optimal allocations in markets in which $\widehat{\kappa}=0$. Following the literature, we call such markets *effectively complete* (e.g. Amershi, 1985; Ingersoll, 1987, p. 192; Brennan and Cao, 1996).

We also note that complete markets are always effectively complete but the converse is not necessarily true because the number of available securities required to achieve Pareto optimal allocations might be smaller than the number of states of the economy, as we demonstrate in Section 4.2. An important distinction between completeness and effective completeness of financial markets is that the latter requires specification of investor preferences whereas the former is preference free and only requires that the available assets span all possible contingent claims.

Next, we derive the measure of incompleteness $\widehat{\kappa}$ in Lemma 5 and study its properties.

Lemma 5 (Measure of effective market incompleteness). *The measure of effective market incompleteness defined as the symmetric KL divergence (36) of the informed and uninformed investors' risk-neutral probabilities is equal to*

$$\widehat{\kappa}=\frac{\mathrm{var}^{\mathrm{RN},U}[b]-\mathrm{var}^{\mathrm{RN},I}[b]}{h^2\lambda^\top\Sigma_v^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}, \tag{38}$$

where the variances are under the risk-neutral probabilities (35), respectively.

Lemma 5 has several economic implications. The first implication is that the asymmetry of information is in itself an important source of market incompleteness. It is known that without the asymmetry of information the market is effectively complete even when investors have different risk aversions and $M<N$ (see Rubinstein, 1974). Second, equation (38) shows that the extent of the risk-sharing imperfection is characterized by the wedge between the risk-neutral variances of the risk factor b under the risk-neutral measures of investors U and I. Third, Lemma 5 implies that $\mathrm{var}^{\mathrm{RN},U}[b]\geq\mathrm{var}^{\mathrm{RN},I}[b]$ because $\widehat{\kappa}\geq 0$, and hence, investor U faces higher uncertainty. Finally, Lemma 5 implies that the risk-neutral measures (35) coincide if and only if $\mathrm{var}^{\mathrm{RN},U}[b]=\mathrm{var}^{\mathrm{RN},I}[b]$ because then $\widehat{\kappa}=0$.

Lemma 5 also pins down the source of risk-sharing imperfections in the economy. First, we note that the informational spanning condition (28) and the risk-neutral measures (35)

8. Lemma A.9 in the Appendix shows that the probability measures (35) and divergence $\widehat{\kappa}$ are functions of the sufficient statistic s, and hence, are observed by both investors I and U. Section IA6.2 of the Supplementary Appendix derives measure $\widehat{\kappa}$ in a CARA-normal economy and discusses its comparative statics.

imply that $\mathbb{E}^{\mathrm{RN},U}[b]=\mathbb{E}^{\mathrm{RN},I}[b]\equiv\widetilde{p}$, and hence, investors U and I agree on the valuation of the replicating portfolio for the risk factor b. Next, we observe that the risk-neutral variances $\mathrm{var}^{\mathrm{RN},U}[b]$ and $\mathrm{var}^{\mathrm{RN},I}[b]$ in (38) can be interpreted as the informed and uninformed investors' subjective valuations of a non-tradable quadratic derivative with payoff $(b-\widetilde{p})^2$, respectively. Therefore, because $\mathrm{var}^{\mathrm{RN},U}[b]\geq\mathrm{var}^{\mathrm{RN},I}[b]$, as discussed above, there are potential gains from trade if investor I could sell this derivative to investor U. Equation (38) then implies that market incompleteness arises due to the different subjective valuations of this derivative and inability to trade it. The intuition is that buying the derivative at a price lower than $\mathrm{var}^{\mathrm{RN},U}[b]$ compensates investor U for the disutility of higher uncertainty due to being uninformed. Consistent with this intuition, in Section 4.2 below, we show that allowing investors to trade quadratic derivatives makes the market effectively complete and it is the informed investor who sells the quadratic derivative to the uninformed.

4. ECONOMIC APPLICATIONS

In this section, we provide several applications of our model to market microstructure, corporate finance, and asset pricing. In Section 4.1, we study the informativeness of derivative prices, introduce a new concept of informationally irrelevant securities, and explore the informational content of corporate debt and equity. Then, in Section 4.2, we study volatility derivatives and show that they make financial markets effectively complete and their prices quantify the value of information. In Section 4.3, we derive new closed-form asset prices in effectively complete markets and study the effect of payoff skewness on prices and returns. In Section 4.4, we study the effect of financial innovation on market liquidity.

4.1. *Information revelation in asset markets*

In this section, we study how learning about the aggregate shock depends on the characteristics of traded securities. In Proposition 4, we derive conditions under which certain assets do not contribute to learning about shock ε.

Proposition 4 (Informational irrelevance). *Consider an asset m such that two conditions are satisfied. First, the replicating portfolio λ for the risk factor b, defined in equation (28), does not invest in this asset, that is, $\lambda_m=0$. Second, the noise trader demand ν_m for this asset is such that $\mathrm{cov}(\nu_m,\nu_k)=0$ for all $k\neq m$. Then, removing this asset from the market does not affect the posterior distribution of shock ε conditional on observing asset prices.*

Proposition 4 demonstrates that assets with corresponding position $\lambda_m=0$ in the replicating portfolio λ are *informationally irrelevant* in the sense that removing such assets from the economy does not affect the information that can be extracted from asset prices when noise ν_m is uncorrelated with noises ν_k in other markets. The informed investor's portfolio given by equation (29) sheds light on informational irrelevance. This equation implies that the informed investor's demand for asset m depends on shock ε if and only if $\lambda_m\neq0$. Otherwise, if $\lambda_m=0$, the informed investor's demand for asset m does not contain any additional information about ε. Intuitively, asset m is not part of the replicating portfolio for the risk factor, and hence, is not used for the informational trade.

The informativeness of prices also depends on the covariances of noise trader demands across markets. Consider an example with two risky assets and perfectly correlated noises $\nu_1=\nu_2$. Taking the difference of the market clearing conditions (13) for the two markets we find that $(\lambda_1-\lambda_2)\varepsilon/\gamma_I+(\nu_1-\nu_2)+(1,-1)^\top H(p)=0$. Shock ε can then be perfectly learned from prices if $\lambda_1 \neq \lambda_2$ and is given by $\varepsilon = \gamma_I(-1,1)^\top H(p)/(\lambda_1 - \lambda_2)$. The effects of the informed investor's demand and noise correlations on the learning from prices are captured by vector λ and matrix Σ_ν in equation (15) for posterior probabilities, respectively.

The informational irrelevance of derivatives is a generic property of economies with one underlying asset with payoff $C_1 = b/\lambda_1$ and $M-1$ derivative securities written on this asset.[9] In such economies, the informational spanning condition (28) is satisfied with the replicating portfolio $\lambda = (\lambda_1, 0, \ldots, 0)^\top$. Prices then reveal the sufficient statistic $\lambda\varepsilon/\gamma_I + \nu = (\lambda_1\varepsilon/\gamma_I + \nu_1, \nu_2, \ldots, \nu_{M-1})^\top$. Assuming that noises ν_m are i.i.d., only the first element $\lambda_1\varepsilon/\gamma_I + \nu_1$ of the sufficient statistic provides information about ε. Hence, the number of assets does not affect the posterior distribution of ε and posterior probabilities $\pi_n^U(p)$. The derivative securities are thus informationally irrelevant (in this particular case).[10] Informational irrelevance also implies that derivatives prices are not fully revealing when noise traders do not trade in derivatives markets (i.e. $\nu_i = 0$ for $i \geq 2$).

As shown in Section 3, assets with $\lambda_m \neq 0$ are used by investor I for replicating the aggregate risk factor $b\varepsilon$. From equations (33) and (34) we observe that assets with $\lambda_m = 0$ are still held by investors for reasons unrelated to information, such as trading against noise traders. Moreover, we note that portfolio λ does not depend on payoff distributions.[11]

We now apply the results of this section to study the informational content of risky debt and equity. Consider a firm with cash flow b_n in state ω_n. The real probabilities of states $\pi_n(\varepsilon)$ and PDF $\varphi_\varepsilon(x)$ of shock ε are given by equations (1) and (2), respectively. Suppose, investors trade the firm's debt and equity with payoffs $\min(b, K)$ and $\max(b-K, 0)$, respectively, where $K > 0$ is the face value of debt. The informational spanning condition is satisfied for debt and equity with the replicating portfolio $\lambda = (1,1)^\top$ because $b = \min(b, K) + \max(b-K, 0)$. The prices of debt and equity can be obtained by solving equation (30). Portfolio $\lambda = (1,1)^\top$ does not depend on K, and hence, the sufficient statistic, the posterior distribution of ε, and the posterior probabilities of states also do not depend on K. As a result, the face value of debt K is irrelevant for the amount of information jointly revealed by the prices of debt and equity.

The informational irrelevance emerges because the number of units of debt and equity that the informed investor buys to satisfy the informational demand and replicate the risk factor b does not depend on the face value of debt K. The informational irrelevance is not a consequence of the capital structure irrelevance theorem for firm values in Modigliani and Miller (1958) because the latter result relies on arbitraging away value differences across firms with different capital structures, which the informational irrelevance does not require. Consequently, the irrelevance does not hold in our setting for firm values in incomplete markets even absent informational asymmetry.

Similar to the Modigliani–Miller theorem, the economic value of the irrelevance conditions in Proposition 4 is that they help identify plausible situations when the capital structure is relevant for

9. As demonstrated in Lemma A.2 in the Appendix, condition $C_1 = b/\lambda_1$ is satisfied in the single risky asset economies with payoffs drawn from normal distributions as in Grossman and Stiglitz (1980) or from distributions that belong to the exponential family as in Breon-Drish (2015). Therefore, adding derivatives to the latter economies does not reveal any additional information.

10. Our model also allows for plausible situations when derivatives are not informationally irrelevant. The latter situations may arise when noise trader demands are correlated or derivatives help replicate the risk factor b as, for example, in the economy with risky debt and costly defaults that we study below.

11. Consider, for example, a firm with cash flows b and three tradable securities with payoffs $C_1 = b - g_1(b)$, $C_2 = g_1(b) - g_2(b)$, $C_3 = g_2(b)$, where $g_1(b)$ and $g_2(b)$ are arbitrary continuous functions. The replicating portfolio for cash flow b is given by $\lambda = (1,1,1)^\top$, because $b = C_1 + C_2 + C_3$, and does not depend on the distribution of cash flow b or functions $g_1(b)$ and $g_2(b)$. The latter result is similar to how replicating portfolios for options in a binomial model do not depend on real probabilities of states of the economy.

the informativeness of asset prices. For example, our analysis of debt and equity relies on costless default on debt whereas in reality firms may face considerable bankruptcy costs (e.g. Altman, 1984; Andrade and Kaplan, 1998). Hence, the irrelevance result breaks down in the presence of costly defaults, as we show below.

Consider a firm with cash flow b_n in state ω_n such that $b_1 < b_2 < ... < b_N$. The firm loses fraction χ of its value in the case of default so that the final payoffs of debt and equity are given by $\min(b,K) - \chi b 1_{\{b<K\}}$ and $\max(b-K,0)$, respectively. The cash flow b can no longer be replicated by trading only debt and equity. In addition to debt and equity, we allow the investors to trade AD securities with payoffs $1_{\{\omega=\omega_n\}}$, where $n \le N-2$. These securities help achieve protection in default states, and in this respect are similar to credit default swaps. The firm cash flow can be replicated as follows:

$$b = \chi \sum_{n:b_n<K} b_n 1_{\{\omega=\omega_n\}} + \underbrace{(\min(b,K) - \chi b 1_{\{b<K\}})}_{\text{debt with costly default}} + \underbrace{\max(b-K,0)}_{\text{equity}},$$

and hence, the replicating portfolio is given by $\lambda = (\chi b_1, \ldots, \chi b_{\overline{n}}, 0, \ldots, 0, 1, 1)^\top$, where $\overline{n}$ is the largest integer such that $b_n < K$. From equation (16) for asset prices we then observe that these prices depend on the face value of debt via portfolio λ, and hence, debt and equity are no longer informationally irrelevant.

In Section IA6.3 of the Supplementary Appendix, we evaluate the informativeness of prices using posterior variance $\mathrm{var}[\varepsilon|s]$ and its approximation when σ_0^2 is small and/or b_n has low variability. Our analysis reveals that the informativeness of prices is a decreasing function of the face value of debt K and the cost of default χ. The intuition is that default becomes more likely as the face value of debt K increases, and hence, more AD securities are needed for the replication of cash flows and more information is revealed through their prices.

4.2. *Effective completeness and volatility derivatives*

We now apply our model to study the economic role of volatility derivatives. These derivatives are widely traded in financial markets, and hence, adding them to our analysis makes it more realistic. Our analysis has three main economic implications. First, we show that introducing volatility derivatives to incomplete markets facilitates risk sharing to the extent that investors achieve Pareto optimal asset allocations. Second, the prices of volatility derivatives quantify the shadow value of information to uninformed investors defined as the amount of wealth that these investors are willing to give up to become informed. Finally, we establish the direction of volatility trading by showing that the informed investors sell volatility derivatives to uninformed investors. Proposition 5 below provides necessary and sufficient conditions for effective completeness and sheds light on the economic role of volatility derivatives.

Proposition 5 (Conditions for effective completeness).
(i) Suppose, the market is incomplete, that is, $M < N$, and the informational spanning condition (28) is satisfied. Then, the market is effectively complete if and only if there exists a replicating portfolio for squared risk factor b_n^2. That is, there exist constant $\widetilde{\lambda}_0$ and vector $\widetilde{\lambda} \in \mathbb{R}^{M-1}$ such that $b_n^2 = \widetilde{\lambda}_0 + C(\omega_n)^\top \widetilde{\lambda}$, for all $n = 1, \ldots, N$.
(ii) If the quadratic derivative with payoffs b_n^2 is tradable, then investor I holds a short position and investor U hold a long position in this derivative.
(iii) If the market is effectively complete, the prices of risky assets are given by equation (16), as in a complete market, with the only difference that vectors $\lambda, \nu \in \mathbb{R}^{M-1}$, and matrices $E, Q \in \mathbb{R}^{(M-1)\times(M-1)}$ are of lower dimensions. Moreover, the CAPMs in Corollaries 1 and 2 also hold.

Proposition 5 confirms the risk sharing analysis in Section 3.3. As discussed in Section 3.3, the measure of market incompleteness (38) implies that investor U has higher subjective valuation of non-traded assets with quadratic payoffs b^2 than investor I when the market is incomplete. Therefore, allowing investor I to sell quadratic derivatives to investor U generates gains from trade and improves risk sharing so that the market becomes effectively complete.

We demonstrate the implications of Proposition 5 in a simple economy with one asset that has payoff $C_1 = b/\lambda_1$. The proposition implies that the market can be effectively completed by introducing a derivative security with a quadratic payoff such as C_1^2 or $(C_1 - \mathbb{E}^{\text{RN}}[C_1])^2$. The last security is similar to a simple variance swap (SVIX) introduced in Martin (2013). The prices of this security and the underlying asset are then given in closed form by $\text{var}^{\text{RN}}[C_1]e^{-rT}$ and $\mathbb{E}^{\text{RN}}[C_1]e^{-rT}$, respectively. Therefore, we interpret assets with quadratic payoffs as volatility derivatives.

The intuition for why quadratic derivatives achieve effective completeness is that these securities have the functional form that allows investors to reach Pareto optimal allocations. To derive the latter functional form, we take logs on both sides of equation (37) for the marginal rates of substitution and, after simple algebra, find that the market is effectively complete if and only if $\gamma_U W_{U,T,n} - \gamma_I W_{I,T,n} - \overline{\ell} = \ln\big(\pi_n^U(p)/\pi_n(\varepsilon)\big)$. The expression $\gamma_U W_{U,T,n} - \gamma_I W_{I,T,n} - \overline{\ell}$ can be interpreted as the payoff of a portfolio that invests in bonds and $\gamma_U \theta_U^* - \gamma_I \theta_I^*$ units of the risky assets. Hence, effective completeness is feasible if only if the log-ratio of probabilities $\ln\big(\pi_n^U(p)/\pi_n(\varepsilon)\big)$ can be replicated by a portfolio of tradable assets [see Lemma A.8 in the Appendix]. Moreover, the log-probability $\ln(\pi_n^U(p))$ is a quadratic function of b_n [see equations (15) and (A.32)], and hence, Pareto allocations of wealth are quadratic functions of b_n. Therefore, the existence of a replicating portfolio for b_n^2 is essential for Pareto optimality.

Finally, we apply our results to study the valuation of information. Conditional on observing the sufficient statistic s, we define the shadow value of information as the amount of wealth $\widehat{W}$ that an atomistic uninformed investor is willing to give up to become informed. Proposition 6 reports value $\widehat{W}$ in effectively complete markets.

Proposition 6 (Shadow value of information). *Suppose, the market is effectively complete. Then, the amount of wealth that an atomistic uninformed investor, who does not affect asset prices, is willing to give up to become informed is given by*

$$\widehat{W} = \frac{1}{2\gamma_U} \frac{\text{var}^{\text{RN}}[b]}{h^2 \lambda^\top \Sigma_v^{-1} \lambda / \gamma_I^2 + 1/\sigma_0^2}, \tag{39}$$

where the variance is computed under the unique risk-neutral measure.

Proposition 6 sheds further light on the economic role of volatility derivatives. In particular, consider an effectively complete market with two securities, a security with payoff $C_1 = b/\lambda_1$ and a volatility derivative with payoff $(C_1 - \mathbb{E}^{\text{RN}}[C_1])^2$. Then, according to Proposition 6, the price of volatility derivative quantifies the value of information and also shows by how much investor I is better off than investor U. The latter result complements the widespread use of the volatility index VIX as the "fear gauge" in financial markets, which increases during times of uncertainty when information is more valuable.[12]

12. Equation (A.67) in the Appendix generalizes the value of information for the case of incomplete markets where the volatility derivative is not traded. The information value (A.67) features the risk-neutral variance of risk factor b computed under the risk-neutral measure of investor U. Moreover, the value of information is reduced by market incompleteness because incompleteness constrains the use of information to fewer assets.

4.3. *Closed form asset prices and payoff skewness*

In this section, for a broad class of payoff distributions, we derive new analytic expressions for asset prices in effectively complete markets and study the effects of payoff skewness on asset prices and returns. Our benchmark is the economy with one risky asset with payoff C_1 that has a continuous unconditional PDF $\varphi_C(x)$. The informed investor observes a signal $\varepsilon = C_1 + u$, where $u \sim \mathcal{N}(\mu_0, \sigma_0^2)$. The uninformed investor learns about ε from prices, and noise traders submit exogenous demand ν_1. This economy is a special case of our economy in Section 2 with distribution parameters a_n and b_n given by equations (7) when $N \to \infty$, as noted in Remark 1. From equation (7), we observe that the informational spanning condition (28) is satisfied because $b_n = C_1(\omega_n)/\sigma_0^2$.

Following Section 4.2, we make the market effectively complete by adding a volatility derivative with payoff C_1^2. Asset prices then can be obtained by taking $N \to \infty$ limit in equation (16). We assume that the noise traders do not trade in the derivative. As a result, the prices in the economies with and without the derivative depend on the same scalar sufficient statistic $s = h\varepsilon/(\gamma_I \sigma_0^2) + \nu_1$. The equilibrium in the effectively complete market is not fully revealing because the derivative is informationally irrelevant[13] (as defined in Section 4.1), that is, $b = C_1/\sigma_0^2 + 0 \cdot C_1^2$.

We illustrate the tractability of our model by providing analytic price $P_{com}(s)$ of the asset with payoff C_1 in effectively complete markets for the following distributions of C_1:

$$\varphi_C(x) = \sum_{l=1}^{L} w_l \frac{1}{\sqrt{2\pi}\widehat{\sigma}_{C,l}} \exp\left(-\frac{(x-\widehat{\mu}_{C,l})^2}{2\widehat{\sigma}_{C,l}^2}\right), \; x \in \mathbb{R}, \quad \text{(Mixture of Normals)}, \tag{40}$$

$$\varphi_C(x) = \frac{1}{\Lambda} x^{k-1} \exp\left(-\frac{x^2}{2\widehat{\sigma}_C^2} - \delta x\right), \; x \geq 0, \quad \text{(Generalized Gamma)}, \tag{41}$$

$$\varphi_C(x) = \frac{2}{\sqrt{2\pi}\widehat{\sigma}_C} \exp\left(-\frac{(x-\widehat{\mu}_C)^2}{2\widehat{\sigma}_C^2}\right) \Phi\left(\alpha \frac{x-\widehat{\mu}_C}{\widehat{\sigma}_C}\right), \; x \in \mathbb{R}, \quad \text{(Skew-normal)}, \tag{42}$$

respectively, where $w_1 + \cdots + w_L = 1$, $w_l \geq 0$, Λ is a normalizing constant, $\widehat{\sigma}_{C,l} > 0$ and $\widehat{\sigma}_C > 0$ are scale parameters, $\widehat{\mu}_{C,l}$, δ and $\widehat{\mu}_C$ are shift parameters, and $k \geq 1$ is an integer power, α is a skewness parameter, and $\Phi(x)$ is the standard normal cumulative distribution function (CDF). We note that: mixture of normals (40) is widely employed for non-parametric estimation of general PDFs (Greene, 2008, p. 416); generalized gamma (41) has positive support and incorporates exponential ($k=1$, $\widehat{\sigma}_C \to \infty$), gamma ($\widehat{\sigma}_C \to \infty$), Rayleigh ($k=2$), and truncated normal ($k=1$) distributions; and the skew-normal distribution (42) extends the normal distribution to allow for skewness [Azzalini (1985)]. Proposition 7 reports asset prices for distributions (40)–(42).

Proposition 7 (Analytic prices). *Asset prices for distributions (40)–(42) in effectively complete markets are given by* $P_{com}(s) = P_C(\mu_{com}, \sigma_{com})$*, where* $\mu_{com} = \gamma_I \sigma_0^2 s/h - \mu_0$*,*

$$\sigma_{com} = \frac{\sigma_0}{\sqrt{1 - \frac{(1-h)/\gamma_U}{h/\gamma_I + (1-h)/\gamma_U} \frac{1}{1 + h^2/\left(\gamma_I^2 \sigma_\nu^2 \sigma_0^2\right)}}}, \tag{43}$$

13. The volatility derivative is made informationally irrelevant only to have a one-dimensional sufficient statistic s. In general, this derivative is not informationally irrelevant as, e.g., when $b = C_1/\sigma_0^2 + C_1^2$.

and function $P_C(\mu,\sigma)$ is given for each distribution by equations (44)–(46), respectively.
(i) When $\varphi_C(x)$ is a mixture of normals (40), the pricing function $P_C(\mu,\sigma)$ is given by:

$$P_C(\mu,\sigma)=\frac{\sum_{l=1}^{L} w_l \frac{\mu\widehat{\sigma}_{C,l}^2+\widehat{\mu}_{C,l}\sigma^2}{\left(\widehat{\sigma}_{C,l}^2+\sigma^2\right)^{3/2}}\exp\left(-\frac{1}{2}\frac{(\mu-\widehat{\mu}_{C,l})^2}{\widehat{\sigma}_{C,l}^2+\sigma^2}\right)}{\sum_{l=1}^{L} w_l \frac{1}{\sqrt{\widehat{\sigma}_{C,l}^2+\sigma^2}}\exp\left(-\frac{1}{2}\frac{(\mu-\widehat{\mu}_{C,l})^2}{\widehat{\sigma}_{C,l}^2+\sigma^2}\right)}. \tag{44}$$

(ii) When $\varphi_C(x)$ is a generalized gamma PDF (41) with an integer power k the pricing function $P_C(\mu,\sigma;k)$ is given in terms of $P_C(\mu,\sigma;k-1)$ by the following recursive formula:

$$P_C(\mu,\sigma;k)=\begin{cases}\frac{\widehat{\sigma}_C^2\sigma^2}{\widehat{\sigma}_C^2+\sigma^2}\left(\frac{\mu}{\sigma^2}-\delta+\frac{k-1}{P_C(\mu,\sigma;k-1)}\right), \text{ if } k>1,\\[2ex] \frac{\widehat{\sigma}_C^2\sigma^2}{\widehat{\sigma}_C^2+\sigma^2}\left(\frac{\mu}{\sigma^2}-\delta\right)+\frac{\widehat{\sigma}_C\sigma}{\sqrt{\widehat{\sigma}_C^2+\sigma^2}}\widehat{\Phi}\left(\frac{\widehat{\sigma}_C\sigma}{\sqrt{\widehat{\sigma}_C^2+\sigma^2}}\left(\frac{\mu}{\sigma^2}-\delta\right)\right), \text{ if } k=1,\end{cases} \tag{45}$$

where $\widehat{\Phi}(x)=\exp(-0.5x^2)/\left(\sqrt{2\pi}\Phi(x)\right)$, and $\Phi(x)$ is the standard normal CDF.
(iii) When $\varphi_C(x)$ is a skew-normal PDF (42) the pricing function $P_C(\mu,\sigma)$ is given by:

$$P_C(\mu,\sigma)=\frac{\mu\widehat{\sigma}_C^2+\widehat{\mu}_C\sigma^2}{\widehat{\sigma}_C^2+\sigma^2}+\frac{\widehat{\sigma}_C^2\sigma^2}{\widehat{\sigma}_C^2+\sigma^2}\frac{\operatorname{sgn}(\alpha)}{\sqrt{\frac{\widehat{\sigma}_C^2}{\alpha^2}+\frac{\widehat{\sigma}_C^2\sigma^2}{\widehat{\sigma}_C^2+\sigma^2}}}\widehat{\Phi}\left(\frac{\widehat{\sigma}_C^2}{\widehat{\sigma}_C^2+\sigma^2}\frac{\operatorname{sgn}(\alpha)(\mu-\widehat{\mu}_C)}{\sqrt{\frac{\widehat{\sigma}_C^2}{\alpha^2}+\frac{\widehat{\sigma}_C^2\sigma^2}{\widehat{\sigma}_C^2+\sigma^2}}}\right), \tag{46}$$

where $\widehat{\Phi}(x)=\exp(-0.5x^2)/\left(\sqrt{2\pi}\Phi(x)\right)$, $\Phi(x)$ is the standard normal CDF, and $\operatorname{sgn}(x)=x/|x|$ when $x\neq 0$ and $\operatorname{sgn}(0)=0$.

Proposition 7 expresses asset prices in terms of elementary functions and inverse Mills ratio $\widehat{\Phi}(x)=\exp(-0.5x^2)/\left(\sqrt{2\pi}\Phi(x)\right)$, widely employed in statistics. Panels (a), (b), and (c) of Figure 1 plot prices $P_{com}(s)$ when asset payoff C_1 is drawn from distributions (40)–(42), respectively. These panels show that non-normality of payoff C_1 makes asset prices non-linear functions of the sufficient statistic $s=h\varepsilon/(\gamma_I\sigma_0^2)+\nu_1$. In particular, Figure 1(a) demonstrates that even a small change in s can lead to large price changes (see also Breon-Drish, 2010). Therefore, more general distributions give rise to effects that are not captured by CARA-normal models, where prices are linear functions of s.

Figure 1(c) plots price $P_{com}(s)$ for the standard normal distribution with zero skewness and the skew-normal distribution for which the distribution parameters are chosen in such a way that payoff C_1 has mean $\mu_C=0$ and variance $\sigma_C^2=1$, as for the standard normal, but has skewness of 0.75.[14] Equation (46) decomposes the price for the case of a skew-normal distribution (42) into

14. The mean, variance, and skewness of a skew-normal random variable are given by $\mu_C=\widehat{\mu}_C+\widehat{\sigma}_C\sqrt{2/\pi}\widehat{\alpha}$, $\sigma_C^2=\widehat{\sigma}_C^2\left(1-2\widehat{\alpha}^2/\pi\right)$, $\text{skew}=0.5(4-\pi)\left(\widehat{\alpha}\sqrt{2/\pi}\right)^3/\left(1-\widehat{\alpha}^2 2/\pi\right)^{3/2}$, respectively, where $\widehat{\alpha}=\alpha/\sqrt{1+\alpha^2}$ (e.g. Azzalini, 1985). We calibrate the parameters so that $\mu_C=0$, $\sigma_C=1$, and $\text{skew}=0.75$.

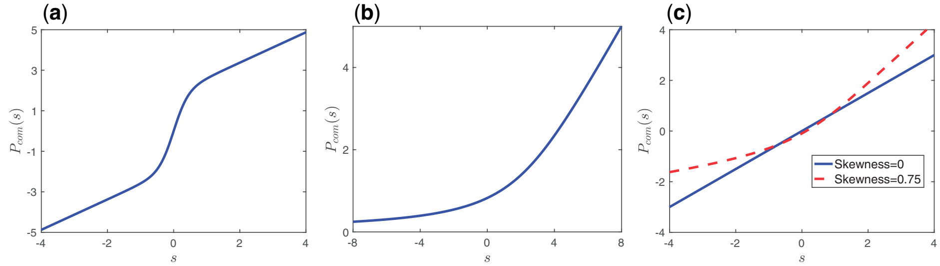

FIGURE 1

Asset prices in effectively complete markets.

Notes: Panels (a), (b), and (c) show the effectively complete market price $P_{com}(s)$ when payoff PDF $\varphi_C(x)$ is a mixture of normals (40), a generalized gamma (41), and a skew-normal (42), respectively. Panel (c) shows price $P_{com}(s)$ for the case of zero skewness (solid blue line) and positive skewness 0.75 (dashed red line). For panel (a) $\widehat{\sigma}_{C,1}=\widehat{\sigma}_{C,2}=1$, $\widehat{\mu}_{C,1}=-\widehat{\mu}_{C,2}=3$, $w_1=w_2=0.5$; for panel (b) $\widehat{\sigma}_C=1$, $\delta=2$, $k=3$; for panel (c) $\mu_C=0$, $\sigma_C^2=1$. The remaining parameters are: $\mu_0=0$, $\sigma_0=1$, $\sigma_\nu=1$, $h=0.5$, $\gamma_I=\gamma_U=1$, $r=0$, $T=1$.

two terms, where the first term is the price when the payoff is normally distributed and the second term isolates the effect of skewness and shows that skewness is priced. The comparison of prices on Figure 1(c) shows that skewness introduces non-linearity and convexity in asset prices and gives rise to higher prices than in the case of normally distributed payoffs when $|s|$ is large.

Next, we study the impact of payoff skewness on asset risk premia for a general payoff distribution. Specifically, we derive the unconditional risk premium $\overline{\mathbb{E}^U[C_1]-P_{com}(s)e^{rT}}$, averaged over different values of s, which captures the effects of skewness that do not depend on particular realizations of s. To focus on the effects of the asymmetry of information, we assume that the informed and uninformed investors have the same risk aversion γ. First, using the tractability of effectively complete markets, we obtain an expansion of asset prices in terms of all higher order moments of general payoff distributions [Lemma A.12 in the Appendix]. Then, assuming that payoff volatility σ_C is small, in the latter expansion we keep only the terms associated with skewness and obtain a relationship between the risk premium and skewness, which we report in Proposition 8.

Proposition 8 *Suppose, $\gamma_I=\gamma_U=\gamma$. Then, the unconditional risk premium of the risky asset, averaged over the realizations of s, is given by:*

$$\overline{\mathbb{E}^U[C_1]-P_{com}(s)e^{rT}}=-\frac{m_3\sigma_C^3}{2}\left(\gamma^2\sigma_\nu^2+\frac{h(1-h)}{\sigma_0^2+h^2/(\gamma^2\sigma_\nu^2)}\right)+o(\sigma_C^3), \tag{47}$$

where m_3 is the skewness of payoff C_1 under the prior cash flow distribution $\varphi_C(x)$. Consequently, the unconditional risk premium is negatively related to the skewness of the asset payoff. Moreover, the effect of skewness on the risk premium is amplified by the asymmetry of information.

Equation (47) reveals that the risk premium is a decreasing function of skewness and that assets with negative (positive) skewness on average have a positive (negative) risk premium. The two components in brackets on the right-hand side of the equation decompose the skewness effect into two terms, the effect of the random asset supply and the effect of asymmetric information, respectively. The first term implies that the relationship between the risk premium and skewness holds due to noisy asset supply even without the asymmetry of information. However, the second term demonstrates that the asymmetry of information amplifies the sensitivity of the risk premium to skewness.

The relationship between the risk premium and skewness arises because positive skewness makes low payoffs less likely, and hence, makes holding the security less risky and decreases the risk premium. Similarly, negative skewness increases the risk premium. The latter effect is amplified by learning from prices. For example, in the case of positive skewness, the uninformed investors would attribute a large negative sufficient statistic s to noises ν_1 and u rather than to low payoff C_1, and hence, would require lower risk premium than in the absence of learning.

Albagli *et al.* (2021) find similar results in a model with risk-neutral investors, position limits, and heterogeneously informed investors. They also discuss large empirical literature documenting a negative relation between the risk premium and skewness. Breon-Drish (2015) considers a single-asset binomial model with more general distributions for ε and supply shocks and derives conditions under which uninformed demand curves bend backwards, in which case asset prices react more strongly than in a setting in which all traders are informed. Our Proposition 8 complements the latter results by showing that the relationship between the risk premium and skewness also holds in a no-arbitrage setting with risk averse and asymmetrically informed investors and for general payoff distributions.

4.4. *Liquidity and financial innovation*

In this section, we study the effects of asymmetric information and financial innovation on market liquidity. We first define market illiquidity in terms of sensitivities of asset prices to noise trader demands, similar to the literature (e.g. Kyle, 1985; Xiong, 2001; Vayanos and Wang, 2012; Cespa and Foucault, 2014).[15] Specifically, illiquidity is a matrix $\partial P/\partial \nu$ with price sensitivities $\partial P_n/\partial \nu_m$ as elements in rows n and columns m. Proposition 9 presents illiquidities in the effectively complete and incomplete markets.

Proposition 9 (Price impacts). *In effectively complete markets the illiquidity is proportional to the risk neutral variance–covariance matrix of asset payoffs. In incomplete markets the illiquidity is proportional to the harmonic average of variance–covariance matrices evaluated under subjective risk neutral probabilities of informed and uninformed investors. The illiquidities in the effectively complete and incomplete markets are given by:*

$$\frac{\partial P_{com}}{\partial \nu} = \frac{e^{-rT}\,\mathrm{var}^{\mathrm{RN}}[C](E+Q)}{h/\gamma_I+(1-h)/\gamma_U}, \tag{48}$$

$$\frac{\partial P_{inc}}{\partial \nu} = e^{-rT}\left(\frac{h}{\gamma_I}\mathrm{var}^{\mathrm{RN},I}[C]^{-1}+\frac{(1-h)}{\gamma_U}\mathrm{var}^{\mathrm{RN},U}[C]^{-1}\right)^{-1}(E+Q), \tag{49}$$

where matrix Q is given by equation (19). Moreover, if information is symmetric and/or asset payoffs C are normally distributed then effectively completing an incomplete market by adding a quadratic derivative with zero noise trader demand does not affect the illiquidity of the underlying asset.

Equation (48) demonstrates that the price impact of noise traders is proportional to the risk-neutral variance–covariance matrix $\mathrm{var}^{\mathrm{RN}}[C]$ when the market is effectively complete. For

15. In contrast to Kyle (1985) the investors in our model are non-strategic. Deriving illiquidity when investors are strategic is not tractable in our model. Glebkin *et al.* (2020) derive illiquidity in a setting with strategic traders and non-normal payoffs but with symmetric information. Similar to our results in Proposition 9, they find that illiquidity is proportional to the risk-neutral variance of asset payoffs.

example, in a simple market with only two tradable assets C_1 and C_1^2 and no noise traders in the market for the quadratic derivative the first asset's illiquidity is proportional to $\text{var}^{\text{RN}}[C_1]$. For this asset, we define its liquidity as the reciprocal of illiquidity, given by $1/\text{var}^{\text{RN}}[C_1]$. The novelty of our analysis is that it links liquidity directly to the risk-neutral rather than physical variance as in the related literature (e.g. Vayanos and Wang, 2012). The inverse relationship between liquidity and risk-neutral volatility is in line with empirical findings in Chung and Chuwonganant (2014).

The price impact of noise traders can be decomposed into substitution and learning effects, which correspond to terms $\text{var}^{\text{RN}}[C]$ and $\text{var}^{\text{RN}}[C]Q$ in equation (48), respectively. The first term shows that a demand shock ν_l to asset l affects prices of all assets correlated with asset l. This is because the change in the price of asset l leads to portfolio rebalancing across all assets. For example, if shock ν_l is positive, the price of asset l increases, and hence, investors partially substitute asset l with asset m positively correlated with asset l. This substitution then increases the price of asset m.

The second term, $\text{var}^{\text{RN}}[C]Q$, arises due to the difficulty of disentangling the effects of noise ν and shock ε. If noisy demand ν_l increases the prices of some assets, such increases may be partially attributed to a positive shock ε. Hence, noisy demand ν_l affects the posterior distribution of ε, and through this distribution affects the prices of assets other than l. Thus, learning from prices generates contagions by spreading demand shocks across assets. The latter effect of learning is similar to the mechanism in Cespa and Foucault (2014) where investors learn information about an asset from the information about other assets, which may lead to liquidity crashes. Finally, we observe that matrix Q given by equation (19) has a hump-shaped coefficient proportional to $h(1-h)$. Consequently, the learning effect vanishes when investors have the same information (i.e. $h=0$ or $h=1$).

Proposition 9 also shows that illiquidity is unaffected by introducing quadratic derivatives when investors have symmetric information or asset payoffs are normally distributed. The latter result holds because absent information asymmetries the market is effectively complete even without quadratic derivatives (see Rubinstein, 1974) and the prices of assets with normally distributed payoffs are the same in incomplete and effectively complete markets [see Proposition IA6.3 in the Supplementary Appendix] provided that noise traders do not trade in the derivatives market.

Next, we study the effect of completing the market on illiquidity in the economy of Section 4.3 where investors trade a single risky asset with payoff C_1 and the informed investor receives a signal $\varepsilon = C_1 + u$, $u \sim N(\mu_0, \sigma_0^2)$. Investors can also complete the market by introducing a security with quadratic payoff C_1^2. Figure 2(a) shows the illiquidity of the underlying asset with payoff C_1 in the effectively complete and incomplete markets when the payoff is drawn from a mixture of normal distributions (40).

From Figure 2(a), we observe that the effect of introducing a quadratic derivative has an ambiguous effect on (il)liquidity, which goes up or down depending on the sufficient statistic s. To isolate the economic determinants of liquidity, we differentiate the market clearing condition (6) for the underlying asset with respect to noise ν and obtain the following expression for liquidity, defined as the reciprocal of illiquidity:

$$\frac{1}{\partial P/\partial \nu} = -\frac{\partial \theta_U^*}{\partial p}(1-h) - \frac{\partial \theta_I^*}{\partial p} h, \tag{50}$$

where θ_U^* and θ_I^* are the units of the underlying asset with payoff C_1 held by uninformed and informed investors, respectively. Equation (50) decomposes the liquidity into a weighted sum of investors' demand slopes, given by $-\partial\theta_U^*/\partial p$ and $-\partial\theta_I^*/\partial p$, respectively. Figures 2(b,c) plot investors' demands as functions of the price of the underlying asset in the incomplete

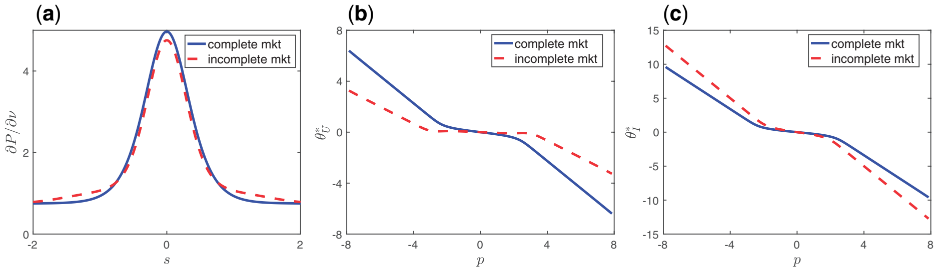

FIGURE 2

Illiquidity and trading strategies of uninformed and informed investors when payoff C_1 is drawn from a mixture of normals distribution.

Notes: (a) The illiquidities for the asset with payoff C_1 in the effectively complete and incomplete markets as functions of the sufficient statistic s. The illiquidity is measured as this asset's price sensitivity to noise trader demand ν. (b,c) The trading strategies of uninformed and informed investors (for $\varepsilon=0$), respectively, in the underlying asset in incomplete and complete markets as functions of the price of the underlying asset. The PDF of payoff C_1 is a mixture of normals (40) with parameters $\widehat{\sigma}_{C,1}=\widehat{\sigma}_{C,2}=1$, $\widehat{\mu}_{C,1}=-\widehat{\mu}_{C,2}=3$, $w_1=w_2=0.5$. The remaining parameters are: $\mu_0=0$, $\sigma_0=1$, $\sigma_\nu=1$, $h=0.5$, $\gamma_I=\gamma_U=1$, $r=0$, $T=1$.

and complete markets. The informed investor's strategy is shown for $\varepsilon=0$. We observe that completing the market has opposite effects on the demand slopes of informed and uninformed investors. Consequently, the overall effect of completing the market on liquidity (and illiquidity) is ambiguous.

The latter ambiguity is due to two forces pulling liquidity in opposite directions. As discussed in Section 4.2, the informed investor sells quadratic derivatives to the uninformed investor, and by doing so compensates the uninformed investor for the uncertainty in the market. Consequently, on the one hand, the uninformed investor trades more aggressively in the risky asset so that the trading strategy becomes more sensitive to asset prices. On the other hand, the informed investor, who takes more risk by selling the quadratic derivative, trades less aggressively, which decreases liquidity.

5. EXTENSIONS AND ROBUSTNESS

5.1. *Complete markets with general distributions.*

In Section IA1 of the Supplementary Appendix, we extend our no-arbitrage complete-market methodology to an economy with general probabilities of states $\pi_n(\varepsilon)$, distribution $\varphi_\varepsilon(x)$ of the aggregate shock, and distribution $\varphi_\nu(x)$ of noise trader demands. We obtain closed-form solutions in the general case. We also consider a special case where noise trader demand is drawn from a mixture of normal distributions (40). Using analytical expressions, we show that in the latter economy the asset price is a non-monotone function of the sufficient statistic and also identify a security which makes an incomplete market effectively complete.

5.2. *Noisy endowments and financial innovation.*

In Section IA2 of the Supplementary Appendix, we extend our analysis to economies in which noise traders are replaced with noisy endowments at the final date. These endowments prevent prices from fully revealing the shock ε that determines state probabilities $\pi_n(\varepsilon)$. The main advantage of this new setting is that it allows us to analyse the welfare effects of a financial innovation and to relax the informational spanning condition (28).

We assume that the state probabilities $\pi_n(\varepsilon)$ are given by equation (1) and that there are no noise traders in the economy. The informed investor receives state-dependent noisy income at date T, which is generated by claims to non-tradable assets, and is given by

$$e_{I,n} = \widetilde{W}_{0,I}(\alpha + \beta b_n + \eta), \tag{51}$$

where α is a constant, $\beta \sim \mathcal{N}(0, \sigma_\beta^2)$, $\eta \sim \mathcal{N}(0, \sigma_\eta^2)$, where variables β and η are uncorrelated with each other and with shock ε. The structure of endowments (51) is motivated by the CAPM and captures the endowment's exposure to the aggregate risk factor b. The exposure coefficient β (analogous to CAPM beta) is known to the informed investors but not to the uninformed investors. The endowment may represent a net asset or liability. Therefore, β can have either sign, and we model it as a normally distributed variable.

First, we show that a security with payoff b^2 effectively completes the market and the asset prices, derived in Proposition IA2.1 of the Supplementary Appendix, are analogous to those in our main analysis. Hence, our results on the effect of asymmetric information and the role of quadratic derivatives are robust to introducing noisy endowments. Second, in Proposition IA2.2 we derive equilibrium in an incomplete market and replace the informational spanning condition with a weaker condition $C = g(b)$, where $g(x)$ is a monotone function. We also study the welfare effects of financial innovation.

5.3. *Information aggregation with multiple types of investors.*

In Section IA3 of the Supplementary Appendix, we study an economy with K types of investors that have different risk aversions and each type receives a signal $e_i = \varepsilon + x_i$, where $x_i \sim \mathcal{N}(0, 1/\tau_i)$, and τ_i is the signal precision. The investors maximize their expected CARA utility conditional on observing asset prices p and the private signal e_i.

We conjecture and verify that investors' optimal portfolios are separable in prices p and signals e_i, and can be decomposed as $\theta_i^*(p; e_i) = \theta_{1,i}(p) + \lambda \kappa_i e_i$. We show that asset prices in this setting are functions of the sufficient statistic given by $\widehat{s} = \lambda \sum_{i=1}^{K} k_i e_i + \nu$, where constants k_i solve a system of non-linear equations. We show that this system of equations has at least one solution, and exactly one solution in the symmetric equilibrium where investors' risk aversions γ_i and signal precisions τ_i are exactly the same.

Similar to our baseline setting, we show that the quadratic derivative makes the financial market effectively complete. Moreover, asset prices in the latter market are available in closed form and have similar structure to asset prices in our baseline setting. When the market is incomplete, asset prices can be characterized in terms of inverse functions, similar to Proposition 3 in our baseline analysis.

5.4. *Multidimensional shock ε.*

In Section IA4 of the Supplementary Appendix, we generalize our model to incorporate multidimensional shocks $\varepsilon \in \mathbb{R}^K$. We consider an economy where the probabilities of states $\pi_n(\varepsilon)$ and the prior PDF of shock ε are given by

$$\pi_n(\varepsilon) = \frac{e^{a_n + b_n^\top \varepsilon}}{\sum_{j=1}^{N} e^{a_j + b_j^\top \varepsilon}}, \quad n = 1, \ldots, N,$$

$$\varphi_\varepsilon(x) = \frac{\left(\sum_{j=1}^{N} e^{a_j + b_j^\top x}\right) e^{-0.5(x-\mu_0)^\top \Sigma_0^{-1}(x-\mu_0)}}{\int_{\mathbb{R}^K} \left(\sum_{j=1}^{N} e^{a_j + b_j^\top x}\right) e^{-0.5(x-\mu_0)^\top \Sigma_0^{-1}(x-\mu_0)} dx},$$

where $\varepsilon, b_n, \mu_0 \in \mathbb{R}^K$, and $\Sigma_0 \in \mathbb{R}^{K\times K}$ is a positive-definite matrix. When the market is complete, we show that the risk-neutral probabilities, asset prices, and investor portfolios have the same structure as in our baseline analysis in Section 3, with the main difference that the replicating portfolio λ is replaced by a matrix Λ.

Similar to the baseline setting, we impose the informational spanning condition when the market is incomplete. This condition requires the existence of K tradable portfolios $\Lambda \in \mathbb{R}^{(M-1)\times K}$ that replicate the risk factor b (up to a constant vector $\Lambda_0 \in \mathbb{R}^K$), so that

$$b_n^\top = \Lambda_0 + C(\omega_n)^\top \Lambda. \tag{52}$$

We show that only one security is needed to effectively complete the market. The payoffs of this security are quadratic forms of the risk factor realizations $b_n \in \mathbb{R}^K$ and are given by $b_n^\top (h^2 \Lambda^\top \Sigma_\nu^{-1} \Lambda / \gamma_I^2 + \Sigma_0^{-1})^{-1} b_n$, where Λ is the matrix of replicating portfolios such that condition (52) is satisfied. In the economy with a scalar shock ($K=1$) the latter payoff reduces to a quadratic derivative $b_n^2/(h^2 \lambda^\top \Sigma_\nu^{-1} \lambda + 1/\sigma_0^2)$, as in Section 4.2.

5.5. *Noisy signals of shock ε.*

Our baseline model in Section 2.1 assumes that the informed investors observe the realization of economic shock ε, whereas in reality they may observe a signal given by $\widehat{\varepsilon} = \varepsilon + z$, where $z \sim \mathcal{N}(0, \sigma_z^2)$. Section IA5 of the Supplementary Appendix shows that the latter economy with a signal is equivalent to our baseline economy where the probabilities of states and the distribution of shock are given by

$$\pi_n(\widehat{\varepsilon}) = \frac{e^{\widehat{a}_n + \widehat{b}_n \widehat{\varepsilon}}}{\sum_{j=1}^N e^{\widehat{a}_j + \widehat{b}_j \widehat{\varepsilon}}}, \qquad \varphi_{\widehat{\varepsilon}}(x) = \frac{\left(\sum_{j=1}^N e^{\widehat{a}_j + \widehat{b}_j x}\right) e^{-0.5(x-\mu_0)^2/(\sigma_0^2+\sigma_z^2)}}{\int_{-\infty}^{\infty} \left(\sum_{j=1}^N e^{\widehat{a}_j + \widehat{b}_j x}\right) e^{-0.5(x-\mu_0)^2/(\sigma_0^2+\sigma_z^2)} dx}, \tag{53}$$

where $\widehat{a}_n = a_n + 0.5(b_n + \mu_0/\sigma_0^2)^2/(1/\sigma_z^2 + 1/\sigma_0^2)$ and $\widehat{b}_n = b_n/(1+\sigma_z^2/\sigma_0^2)$. The probabilities and the shock distribution in (53) have the same structure as in the baseline analysis, and the risk factor $\widehat{b}$ is proportional to b. Hence, our results remain valid in this extension.

6. CONCLUSION

We develop a tractable REE model with multiple risky assets with realistic payoff structures. Our model provides a tractable setting for studying the effects of information aggregation on asset prices where prices and portfolios are available in closed form. We derive a three-factor CAPM with asymmetric information, which provides a bridge between standard risk-based models of asset pricing and models with asymmetric information.

Our results yield necessary and sufficient conditions under which the informed demands for derivative securities reveal information about the underlying asset. Absent any additional frictions other than the asymmetry of information, these conditions imply the irrelevance of the face value of debt for the total amount of information revealed by the debt and equity of a firm. The conditions for the irrelevance of the capital structure also help identify situations in which the capital structure is not informationally redundant. In particular, we show that the irrelevance of capital structure breaks down in the presence of bankruptcy costs. Our analysis also uncovers an important role of volatility derivatives. These derivatives make incomplete markets effectively complete, and their prices quantify market illiquidity and the shadow value of information for uninformed investors.

Data Availability Statement

The code underlying this research is available on Zenodo at https://doi.org/10.5281/zenodo.5579265.

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Supplementary Data

Supplementary data are available at *Review of Economic Studies* online. And the replication packages are available at https://dx.doi.org/10.5281/zenodo.5579265.

A. APPENDIX: PROOFS

Lemma A.1 (Prior mean and prior variance of ε and prior probabilities). *Let ε have PDF (2). Then, its mean μ_ε and variance σ_ε^2 in terms of (μ_0,σ_0^2) are given by:*

$$\mu_\varepsilon = \frac{\sum_{j=1}^N \exp\left(a_j+\frac{\mu_j^2}{2\sigma_0^2}\right)\mu_j}{\sum_{j=1}^N \exp\left(a_j+\frac{\mu_j^2}{2\sigma_0^2}\right)}, \tag{A.1}$$

$$\sigma_\varepsilon^2 = \sigma_0^2+\left[\frac{\sum_{j=1}^N \exp\left(a_j+\frac{\mu_j^2}{2\sigma_0^2}\right)\mu_j^2}{\sum_{j=1}^N \exp\left(a_j+\frac{\mu_j^2}{2\sigma_0^2}\right)}-\frac{\left(\sum_{j=1}^N \exp\left(a_j+\frac{\mu_j^2}{2\sigma_0^2}\right)\mu_j\right)^2}{\left(\sum_{j=1}^N \exp\left(a_j+\frac{\mu_j^2}{2\sigma_0^2}\right)\right)^2}\right], \tag{A.2}$$

where $\mu_j=b_j\sigma_0^2+\mu_0$.

Proof of Lemma A.1. We compute $\mu_\varepsilon=\mathbb{E}[\varepsilon]$ and $\sigma_\varepsilon^2=\mathrm{var}[\varepsilon]$ with PDF $\varphi_\varepsilon(x)$ given by (2), and after straightforward integration, we obtain equations (A.1) and (A.2). ■

Lemma A.2 (Special cases). *Let payoff C_1 have general unconditional continuous PDF $\varphi_C(x)$. Suppose, the informed investor receives signal $\varepsilon=C_1+u$, where $u\sim\mathcal{N}(\mu_0,\sigma_0^2)$. Then, the latter economy is a limiting case (when $N\to\infty$) of our N-state economy with $C_1(\omega_n)=\underline{C}_N+(\overline{C}_N-\underline{C}_N)(n-1)/(N-1)$ and distribution parameters*

$$a_n=-\frac{C_1(\omega_n)^2}{2\sigma_0^2}-\frac{\mu_0 C_1(\omega_n)}{\sigma_0^2}+\ln[\varphi_C(C_1(\omega_n))], \quad b_n=\frac{C_1(\omega_n)}{\sigma_0^2}, \tag{A.3}$$

when $N\to\infty$ and $\underline{C}_N$ and $\overline{C}_N$ converge to lower and upper limits of payoff C_1.

Proof of Lemma A.2. Consider a discretized economy with one risky asset with payoff $C_1(\omega_n)=\underline{C}_N+(\overline{C}_N-\underline{C}_N)(n-1)/(N-1)$, where $n=1,\dots,N$, and let the informed investor receive a signal $\varepsilon=C_1(\omega)+u$, where $u\sim\mathcal{N}(\mu_0,\sigma_0^2)$. The unconditional probabilities of $C_1(\omega_n)$ are given by $\mathrm{Prob}(C_1(\omega_n))=\varphi_C(C_1(\omega_n))/\left(\sum_{n=1}^N\varphi_C(C_1(\omega_n))\right)$. The original continuous-space economy is a limiting case of the latter economy because as $N\to\infty$, the distributions $\text{Prob}(C_1(\omega)\le x)$ and $\text{Prob}(C_1(\omega)\le x|\varepsilon)$ converge pointwise to the respective continuous-space distributions. We show that the latter discretized economy is a special case of ours when parameters a_n and b_n are given by equations (A.3) by verifying that in our economy $\text{Prob}(C_1(\omega_n))=\varphi_C(C_1(\omega_n))/\Big(\sum_{n=1}^N\varphi_C(C_1(\omega_n))\Big)$ and $\varepsilon=C_1(\omega)+u$.

Consider the unconditional probability $\text{Prob}(C_1(\omega_n))$ in the model of Section 2:

$$\text{Prob}(C_1(\omega_n))=\int_{-\infty}^{\infty}\pi_n(x)\varphi_\varepsilon(x)dx=\frac{1}{\Lambda}\int_{-\infty}^{\infty}e^{a_n+b_nx-0.5(x-\mu_0)^2/\sigma_0^2}dx=\frac{1}{\widetilde{\Lambda}}e^{a_n+0.5(\mu_0+b_n\sigma_0^2)^2/\sigma_0^2},$$

where Λ and $\widetilde{\Lambda}$ are constants. Substituting a_n and b_n from equations (A.3) into the above equation, after some algebra, we verify that $\text{Prob}(C_1(\omega_n))=\varphi_C(C_1(\omega_n))/\sum_{n=1}^N\varphi_C(C_1(\omega_n))$.

Next, we verify that $\varepsilon=C_1(\omega)+u$, where $u\sim\mathcal{N}(\mu_0,\sigma_0^2)$. Substituting a_n and b_n from (A.3) into equation (2) for PDF $\varphi_\varepsilon(x)$, after some algebra, we obtain:

$$\varphi_\varepsilon(x)=\frac{\sum_{n=1}^N e^{-0.5(x-C_1(\omega_n)-\mu_0)^2/\sigma_0^2}\varphi_C(C_1(\omega_n))}{\int_{-\infty}^{\infty}\Big(\sum_{n=1}^N e^{-0.5(x-C_1(\omega_n)-\mu_0)^2/\sigma_0^2}\varphi_C(C_1(\omega_n))\Big)dx}.$$

The above PDF is the convolution of the unconditional distributions of $C_1(\omega)$ and a normal distribution $\mathcal{N}(\mu_0,\sigma_0^2)$, and hence is the PDF of $\varepsilon=C_1(\omega)+u$, where $u\sim\mathcal{N}(\mu_0,\sigma_0^2)$. ■

Proof of Lemma 1. Taking log on both sides of investor I's FOC (10), and substituting wealth $W_{I,T,n}$ from the budget constraint (5), we obtain:

$$(\theta_I^*)^\top(C(\omega_n)-e^{rT}p)=\frac{1}{\gamma_I}\Big(\ln\Big(\pi_n(\varepsilon)\Big)-\ln\Big(\pi_n^{\text{RN}}\Big)\Big)+const,\quad n=1,\dots,N,\tag{A.4}$$

where *const* is a constant. Subtracting equation (A.4) for $n=N$ from the other equations in (A.4), we obtain a system of $N-1$ equations with $N-1$ unknown components of θ_I^*:

$$(\theta_I^*)^\top(C(\omega_n)-C(\omega_N))=\frac{1}{\gamma_I}\left(\ln\left(\frac{\pi_n(\varepsilon)}{\pi_N(\varepsilon)}\right)-\ln\left(\frac{\pi_n^{\text{RN}}}{\pi_N^{\text{RN}}}\right)\right),\quad n=1,\dots,N-1,\tag{A.5}$$

Solving the system of equations (A.5), we obtain investor I's optimal portfolio

$$\theta_I^*(p;\varepsilon)=\frac{\Omega^{-1}}{\gamma_I}\left\{\left(\ln\left(\frac{\pi_1(\varepsilon)}{\pi_N(\varepsilon)}\right),\dots,\ln\left(\frac{\pi_{N-1}(\varepsilon)}{\pi_N(\varepsilon)}\right)\right)-\left(\ln\left(\frac{\pi_1^{\text{RN}}}{\pi_N^{\text{RN}}}\right),\dots,\ln\left(\frac{\pi_{N-1}^{\text{RN}}}{\pi_N^{\text{RN}}}\right)\right)\right\}^\top.\tag{A.6}$$

Finally, substituting $\pi_n(\varepsilon)$ from (1) into the above equation, we obtain portfolio (11). ■

Proof of Lemma 2. Let $s\equiv h\lambda\varepsilon/\gamma_I+\nu$ denote the sufficient statistic. From Bayes rule, the PDF of ε conditional on s is given by:

$$\varphi_{\varepsilon|s}(x|y)=\frac{\varphi_{s|\varepsilon}(y|x)\varphi_\varepsilon(x)}{\int_{-\infty}^{\infty}\varphi_{s|\varepsilon}(y|x)\varphi_\varepsilon(x)dx}.\tag{A.7}$$

Because $\nu\sim\mathcal{N}(0,\Sigma_\nu)$, $s=h\lambda\varepsilon/\gamma_I+\nu$ conditional on ε has a multivariate normal distribution $\mathcal{N}(h\lambda\varepsilon/\gamma_I,\Sigma_\nu)$. Hence, substituting $\varphi_{s|\varepsilon}(y|x)$ into equation (A.7), we have

$$\varphi_{\varepsilon|s}(x|y)\;=\;\frac{\exp\Big\{-0.5\Big(y-h\lambda x/\gamma_I\Big)^\top\Sigma_\nu^{-1}\Big(y-h\lambda x/\gamma_I\Big)\Big\}\varphi_\varepsilon(x)}{G_1(y)},\tag{A.8}$$

where $G_1(y)$ normalizes the density. Next, to find probability $\pi_n^U(p)$, from the market clearing condition (13), we note that in equilibrium $s=-H(p)$. We focus on equilibrium where prices only reveal the sufficient statistic s. Hence, $\pi_n^U(p)=\mathbb{E}\Big[\pi_n(\varepsilon)|P(\varepsilon,\nu)=p\Big]=\mathbb{E}[\pi_n(\varepsilon)|s=-H(p)]$. Calculating the last conditional expectation, we obtain:

$$\begin{aligned}\pi_n^U(p)\;&=\;\mathbb{E}[\pi_n(\varepsilon)|s=-H(p)]\\&=\;\int_{-\infty}^{\infty}\frac{e^{a_n+b_nx}}{\sum_{j=1}^N e^{a_j+b_jx}}\varphi_{\varepsilon|s}\Big(x|-H(p)\Big)dx=\frac{1}{G_1(y)}\int_{-\infty}^{\infty}e^{d_n(x)}dx,\end{aligned}\tag{A.9}$$

where $d_n(x)$ is a quadratic function of x given by:

$$\begin{aligned}d_n(x)\;&=\;a_n+b_nx-0.5\Big(h\lambda x/\gamma_I+H(p)\Big)^\top\Sigma_\nu^{-1}\Big(h\lambda x/\gamma_I+H(p)\Big)-0.5(x-\mu_0)^2/\sigma_0^2\\&=\;-\frac{h^2(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2)+1/\sigma_0^2}{2}\left(x-\frac{\mu_0/\sigma_0^2+b_n-h\lambda^\top\Sigma_\nu^{-1}H(p)/\gamma_I}{h^2(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2)+1/\sigma_0^2}\right)^2\\&\quad+a_n+\frac{1}{2}\frac{b_n^2+2b_n\Big(\mu_0/\sigma_0^2-h\lambda^\top\Sigma_\nu^{-1}H(p)/\gamma_I\Big)}{h^2(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2)+1/\sigma_0^2}+g(p),\end{aligned}\tag{A.10}$$

where $g(p)$ is a normalizing function. Substituting $d_n(x)$ from equation (A.10) into integral (A.9) and integrating, we obtain equation (15) for $\pi_n^U(p)$. ■

Proof of Proposition 1. First, we find the optimal portfolios of investors, and then recover the equilibrium prices from the market clearing condition. The portfolio of investor I is given by equation (11) in Lemma 1. To find investor U's portfolio $\theta_U^*(p)$, we follow similar steps as in Lemma 1: 1) take the log of both sides of investor U's FOC (10); 2) subtract the Nth equation from the rest; 3) solve the $N-1$ equations for the $N-1$ positions of U's portfolio. This gives $\theta_U^*(p)$ in terms of investor U's probabilities $\pi_n^U(p)$

$$\theta_U^*(p) = \frac{1}{\gamma_U}\Omega^{-1}\left\{\left(\ln\left(\frac{\pi_1^U(p)}{\pi_N^U(p)}\right),\dots,\ln\left(\frac{\pi_{N-1}^U(p)}{\pi_N^U(p)}\right)\right)^{\top} - \widetilde{v}(p)\right\}, \tag{A.11}$$

where $\widetilde{v}$ is given by equation (12). Substituting $\pi_n^U(p)$ from (15) into (A.11) we obtain:

$$\theta_U^*(p) = \frac{1}{\gamma_U}\Omega^{-1}\left\{\widetilde{a} + \frac{1}{2}\frac{\widetilde{b}^{(2)} - 2\widetilde{b}\left(h\lambda^{\top}\Sigma_\nu^{-1}H(p)/\gamma_I - \mu_0/\sigma_0^2\right)}{h^2\lambda^{\top}\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \widetilde{v}(p)\right\}, \tag{A.12}$$

where vectors $\widetilde{a}, \widetilde{b}, \widetilde{b}^{(2)} \in \mathbb{R}^{N-1}$ are given by:

$$\widetilde{a} = (a_1 - a_N, \dots, a_{N-1} - a_N), \quad \widetilde{b} = (b_1 - b_N, \dots, b_{N-1} - b_N), \tag{A.13}$$

$$\widetilde{b}^{(2)} = (b_1^2 - b_N^2, \dots, b_{N-1}^2 - b_N^2). \tag{A.14}$$

Substituting $\lambda = \Omega^{-1}\widetilde{b}$ and $H(p) = (1-h)\theta_U^* - h\Omega^{-1}\left(\widetilde{v} - \widetilde{a}\right)/\gamma_I$ from (14) into (A.12), we obtain:

$$\begin{aligned} \theta_U^*(p) &= \frac{\Omega^{-1}\left(\widehat{a} - \widetilde{v}(p)\right)}{\gamma_U} + \frac{\mu_0/(\gamma_U\sigma_0^2)\lambda}{h^2\lambda^{\top}\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - \frac{QH(p)}{1-h}, \\ &= \frac{hQ\Omega^{-1}\left(\widetilde{v}(p) - \widetilde{a}\right)}{(1-h)\gamma_I} - \frac{\Omega^{-1}(\widetilde{v}(p) - \widehat{a})}{\gamma_U} + \frac{\mu_0/(\gamma_U\sigma_0^2)\lambda}{h^2\lambda^{\top}\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} - Q\theta_U^*(p), \end{aligned} \tag{A.15}$$

where $\widehat{a}$ and matrix Q are given by:

$$\widehat{a} = \widetilde{a} + \frac{0.5\widetilde{b}^{(2)}}{h^2\lambda^{\top}\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}, \quad Q = \frac{h(1-h)}{\gamma_U\gamma_I}\frac{\lambda\lambda^{\top}\Sigma_\nu^{-1}}{h^2\lambda^{\top}\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}. \tag{A.16}$$

Solving linear equation (A.15) for portfolio $\theta_U^*(p)$, we obtain portfolio $\theta_U^*(p)$ in (20).

Next, we find the equilibrium prices. Substituting optimal portfolios $\theta_I^*(p;\varepsilon)$ and $\theta_U^*(p)$ from equations (11) and (20) into the market clearing condition $h\theta_I^*(p;\varepsilon) + (1-h)\theta_U^*(p) + \nu = 0$, after rearranging terms, we obtain the following equation for vector $\widetilde{v}(p)$:

$$\begin{aligned} &\left(E + Q\right)^{-1}\left(\frac{hQ\Omega^{-1}(\widetilde{v}(p) - \widetilde{a})}{\gamma_I} - \frac{(1-h)\Omega^{-1}(\widetilde{v}(p) - \widehat{a})}{\gamma_U} + \frac{(1-h)\mu_0/(\gamma_U\sigma_0^2)\lambda}{h^2\lambda^{\top}\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2}\right) \\ &\qquad - \frac{h\Omega^{-1}\left(\widetilde{v}(p) - \widetilde{a}\right)}{\gamma_I} + \frac{h\lambda\varepsilon}{\gamma_I} + \nu = 0. \end{aligned} \tag{A.17}$$

The above equation can be further simplified by noting that

$$\begin{aligned} (E+Q)^{-1}\frac{1}{\gamma_I}Q\Omega^{-1}(\widetilde{v}(p) - \widetilde{a}) &= (E+Q)^{-1}(E+Q-E)\frac{1}{\gamma_I}\Omega^{-1}(\widetilde{v}(p) - \widetilde{a}) \\ &= \frac{1}{\gamma_I}\Omega^{-1}(\widetilde{v}(p) - \widetilde{a}) - (E+Q)^{-1}\frac{1}{\gamma_I}\Omega^{-1}(\widetilde{v}(p) - \widetilde{a}). \end{aligned}$$

Substituting the latter expression into equation (A.17), cancelling like terms, substituting $\widehat{a}$ from equation (A.16) into equation (A.17), and solving it for $\widetilde{v}(p) - \widetilde{a}$ we obtain

$$\widetilde{v}(p) = \widetilde{a} + \frac{1}{2}\frac{(1-h)/\gamma_U}{h/\gamma_I + (1-h)/\gamma_U}\frac{\widetilde{b}^{(2)} + 2(\mu_0/\sigma_0^2)\widetilde{b}}{h^2\lambda^{\top}\Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} + \frac{\Omega\left(E+Q\right)s}{h/\gamma_I + (1-h)/\gamma_U}, \tag{A.18}$$

where $\tilde{a}$, $\tilde{b}$, and $\tilde{b}^{(2)}$ are given by (A.13) and (A.14), or element-wise for $n=1,\dots,N-1$:

$$\widetilde{v}_n = a_n - a_N + \frac{1}{2}\frac{(1-h)/\gamma_U}{h/\gamma_I+(1-h)/\gamma_U}\frac{(b_n^2-b_N^2)+2(\mu_0/\sigma_0^2)(b_n-b_N)}{h^2\lambda^\top\Sigma_v^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}+\frac{(C(\omega_n)-C(\omega_N))^\top\Big(E+Q\Big)s}{h/\gamma_I+(1-h)/\gamma_U}.$$

Let v_n be given by equation (18). Then, vector $\widetilde{v}(p)\in\mathbb{R}^{N-1}$ has elements v_n-v_N. From the definition of vector $\widetilde{v}$ in equation (12), we find that $\pi_n^{\text{RN}} = e^{v_n-v_N}/\Big(\sum_{j=1}^N e^{v_j-v_N}\Big)$ for $n=1,\dots,N$. Cancelling e^{-v_N}, we obtain probabilities (17).

Finally, we show that $P(\varepsilon,\nu)$ is an invertible on its range function of $s=h\lambda\varepsilon/\gamma_I+\nu$, and hence, observing prices reveals unique s. First, because the risk-neutral probabilities are unique, there is a one-to-one mapping between these probabilities and prices. From equation (12) we observe that there is an one-to-one mapping between $\widetilde{v}$ and the vector of risk-neutral probabilities. Finally, from equation (A.18), there is an one-to-one mapping between $\widetilde{v}$ and s, which, by transitivity, completes the proof. ■

Proof of Proposition 2. We now find the derivatives of portfolios θ_i^* with respect to prices p. Substituting $\pi_N^{\text{RN}} = 1-\pi_1^{\text{RN}}-\cdots-\pi_{N-1}^{\text{RN}}$ into the risk-neutral valuation equation (8), we obtain:

$$p_m = \Big[\pi_1^{\text{RN}}\Big(C_m(\omega_1)-C_m(\omega_N)\Big)+\cdots+\pi_{N-1}^{\text{RN}}\Big(C_m(\omega_{N-1})-C_m(\omega_N)\Big)+C_m(\omega_N)\Big]e^{-rT}, \quad \text{(A.19)}$$

where $m=1,\dots,N-1$. From the definition of vector $\widetilde{v}$ in (12) we observe that $\pi_n^{\text{RN}} = e^{\widetilde{v}_n}/(1+\sum_{n=1}^{N-1}e^{\widetilde{v}_n})$. First, we need to compute $\partial\widetilde{v}/\partial p$. To do this, we find the Jacobian $J_p=\partial p/\partial\widetilde{v}$ and then by the inverse function theorem we have $\partial\widetilde{v}/\partial p = J_p^{-1}$. Let J_π be the Jacobian of vector $(\pi_1^{\text{RN}},\dots,\pi_{N-1}^{\text{RN}})^\top$, that is, a matrix with (n,k) element given by $\partial\pi_n^{\text{RN}}/\partial\widetilde{v}_k$. Differentiating equation (A.19) we find that $J_p=\Omega^\top J_\pi e^{-rT}$, and hence

$$J_p\Omega e^{rT} = \Omega^\top J_\pi\Omega. \quad \text{(A.20)}$$

To find J_π, we first calculate $\partial\pi_n^{\text{RN}}/\partial\widetilde{v}_k$, where π_n^{RN} is given by equation (17):

$$\frac{\partial\pi_n^{\text{RN}}}{\partial\widetilde{v}_k} = \frac{\partial\pi_n^{\text{RN}}}{\partial v_k} = \begin{cases} -\pi_n^{\text{RN}}\pi_k^{\text{RN}}, & \text{if } n\neq k,\\ \pi_n^{\text{RN}}-(\pi_n^{\text{RN}})^2, & \text{if } n=k.\end{cases} \quad \text{(A.21)}$$

for $n,k=1,\dots,N-1$, where the first equality follows from $\widetilde{v}_k=v_k-v_N$. From equation (A.21) we find $J_\pi = \text{diag}\{\pi_1^{\text{RN}},\dots,\pi_{N-1}^{\text{RN}}\}-(\pi_1^{\text{RN}},\dots,\pi_{N-1}^{\text{RN}})^\top(\pi_1^{\text{RN}},\dots,\pi_{N-1}^{\text{RN}})$, where $\text{diag}\{\dots\}$ is a diagonal matrix. Substituting J_π into equation (A.20) we obtain:

$$J_p\Omega e^{rT} = \Omega^\top\Big(\text{diag}\{\pi_1^{\text{RN}},\dots,\pi_{N-1}^{\text{RN}}\}-(\pi_1^{\text{RN}},\dots,\pi_{N-1}^{\text{RN}})^\top(\pi_1^{\text{RN}},\dots,\pi_{N-1}^{\text{RN}})\Big)\Omega. \quad \text{(A.22)}$$

Recalling that Ω is a matrix with rows $(C(\omega_n)-C(\omega_N))^\top$, and denoting $\widetilde{C}_n = \Big(C_n(\omega_1)-C_n(\omega_N),\dots,C_n(\omega_{N-1})-C_n(\omega_N)\Big)^\top$, we find that the (n,k) element of matrix $J_p\Omega e^{rT}$ is:

$$\begin{aligned}\{J_p\Omega e^{rT}\}_{n,k} &= \widetilde{C}_n^\top\text{diag}\{\pi_1^{\text{RN}},\dots,\pi_{N-1}^{\text{RN}}\}\widetilde{C}_k-\widetilde{C}_n^\top(\pi_1^{\text{RN}},\dots,\pi_{N-1}^{\text{RN}})^\top(\pi_1^{\text{RN}},\dots,\pi_{N-1}^{\text{RN}})\widetilde{C}_k\\ &= \textstyle\sum_{i=1}^N\Big(C_n(\omega_i)-C_n(\omega_N)\Big)\Big(C_k(\omega_i)-C_k(\omega_N)\Big)\pi_i^{\text{RN}}\\ &\quad -\Big(\textstyle\sum_{i=1}^N\Big(C_n(\omega_i)-C_n(\omega_N)\Big)\pi_i^{\text{RN}}\Big)\Big(\textstyle\sum_{i=1}^N\Big(C_k(\omega_i)-C_k(\omega_N)\Big)\pi_i^{\text{RN}}\Big)\\ &= \text{cov}^{\text{RN}}(C_n,C_k),\end{aligned}$$

where to derive the second equality we added zero terms $\Big(C_n(\omega_N)-C_n(\omega_N)\Big)\Big(C_k(\omega_N)-C_k(\omega_N)\Big)\pi_N^{\text{RN}}$, $\Big(C_n(\omega_N)-C_n(\omega_N)\Big)\pi_N^{\text{RN}}$ and $\Big(C_k(\omega_N)-C_k(\omega_N)\Big)\pi_N^{\text{RN}}$ to summations, and then removed constants $C_n(\omega_N)$ and $C_k(\omega_N)$, because they do not affect covariances.

Therefore, we conclude that $J_p\Omega e^{rT}=\text{var}^{\text{RN}}[C]$. Then, by the inverse function theorem, we now find that $\Omega^{-1}\partial\widetilde{v}/\partial p = \Big(\text{var}^{\text{RN}}[C]\Big)^{-1}e^{rT}$. Using the latter equality and differentiating optimal portfolios (11) and (20) with respect to p we obtain that the first of these two partial derivatives is given by equation (21) and the second is given by:

$$\frac{\partial\theta_U^*(p)}{\partial p} = \frac{1}{1-h}\Big(\frac{h}{\gamma_I}E-\Big(\frac{h}{\gamma_I}+\frac{1-h}{\gamma_U}\Big)(E+Q)^{-1}\Big)\Big(\text{var}^{\text{RN}}[C]\Big)^{-1}e^{rT}. \quad \text{(A.23)}$$

We note the following equation for the inverse matrix $(E+Q)^{-1}$:

$$(E+Q)^{-1} = E-\frac{(1-h)h\lambda\lambda^\top\Sigma_v^{-1}}{\gamma_U\gamma_I\Big(h^2\lambda^\top\Sigma_v^{-1}\lambda/\gamma_I^2+1/\sigma_0^2\Big)+h(1-h)\lambda^\top\Sigma_v^{-1}\lambda},$$

which can be verified by multiplying both sides of the latter equation by $E+Q$. Substituting $(E+Q)^{-1}$ above into equation (A.23), we obtain equation (22) for $\partial\theta_U^*(p)/\partial p^\top$.

Finally, we demonstrate that $\theta^*_{I,m}(p;\varepsilon)$ is downward sloping in p_m. This result follows from the fact that matrix $(\text{var}^{\text{RN}}[C])^{-1}$ is positive-definite (as the inverse of a positive-definite matrix), and its element m of the diagonal is given by $e_m^\top(\text{var}^{\text{RN}}[C])^{-1}e_m>0$, where $e_m=(0,0,\ldots,1,\ldots,0)^\top$ is a vector with mth element equal to 1 and other elements equal to zero. Then, from equation (21) it follows that $\partial\theta^*_{I,m}(p;\varepsilon)/\partial p_m<0$. ■

Proof of Corollary 1. Let $m=\pi^{\text{RN}}/\pi^U e^{-rT}$. Then, equation (23) follows from equation (1.12) in Cochrane (2005, p. 14), and equation (24) is derived analogously to equation (6.4) in Cochrane (2005, p. 102) in the proof of a theorem in Cochrane (2005, p. 101). ■

Proof of Corollary 2. From equations (17) and (15) for the probabilities π^{RN} and π^U, we obtain

$$\frac{\pi_n^{\text{RN}}}{\pi_n^U}=G(s)\exp\left\{-\frac{1}{2}\frac{h/\gamma_I}{(1-h)/\gamma_U+h/\gamma_I}\frac{b_n^2+2b_n(\mu_0/\sigma_0+h\lambda^\top\Sigma_v^{-1}s/\gamma_I)}{h^2\lambda^\top\Sigma_v^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}+\frac{C(\omega_n)^\top s}{(1-h)/\gamma_U+h/\gamma_I}\right\}, \tag{A.24}$$

where $G(s)$ does not depend on state ω. First, we find $\mathbb{E}[\varepsilon|s]$ as follows:

$$\mathbb{E}[\varepsilon|s]=\int_{-\infty}^{\infty}x\varphi_{\varepsilon|s}(x|s)dx=\frac{1}{G_1(s)}\sum_{n=1}^{N}\int_{-\infty}^{\infty}xe^{d_n(x)}dx, \tag{A.25}$$

where conditional PDF $\varphi_{\varepsilon|s}(x|s)$ is given by (A.8), $G_1(s)$ is a normalization function, and $d_n(x)$ is given by equation (A.10). Evaluating the integral (A.25), we obtain:

$$\mathbb{E}[\varepsilon|s]=\sum_{n=1}^{N}\frac{\mu_0/\sigma_0^2+b_n+h\lambda^\top\Sigma_v^{-1}s/\gamma_I}{h^2(\lambda^\top\Sigma_v^{-1}\lambda/\gamma_I^2)+1/\sigma_0^2}\pi_n^U(p)=\frac{\mu_0/\sigma_0^2+E^U[b_n]+h\lambda^\top\Sigma_v^{-1}s/\gamma_I}{h^2(\lambda^\top\Sigma_v^{-1}\lambda/\gamma_I^2)+1/\sigma_0^2}. \tag{A.26}$$

Next, we express s in terms of $\mathbb{E}[v|s]$. We note that $\mathbb{E}[v|s]=\mathbb{E}[s-h\lambda\varepsilon/\gamma_I|s]$. Then, using equation (A.26) for $\mathbb{E}[\varepsilon|s]$, we obtain

$$s=\mathbb{E}[v|s]+\frac{h\lambda}{\gamma_I}\frac{\mu_0/\sigma_0^2+\mathbb{E}^U[b_n]+h\lambda^\top\Sigma_v^{-1}s/\gamma_I}{h^2(\lambda^\top\Sigma_v^{-1}\lambda/\gamma_I^2)+1/\sigma_0^2}. \tag{A.27}$$

Substituting s from the above equation into equation (A.24) and then using the informational spanning condition $b_n=\lambda_0+C(\omega_n)^\top\lambda$, we obtain:

$$\frac{\pi_n^{\text{RN}}}{\pi_n^U}=g_0(s)\exp\left\{-\frac{1}{2}\frac{h/\gamma_I}{(1-h)/\gamma_U+h/\gamma_I}\frac{b_n^2-2b_n\mathbb{E}^U[b]}{h^2\lambda^\top\Sigma_v^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}+\frac{C(\omega_n)^\top\mathbb{E}[v|s]}{(1-h)/\gamma_U+h/\gamma_I}\right\}. \tag{A.28}$$

Assuming that either γ_I or γ_U is small, and using approximation $e^x\approx 1+x$ for small x, from the above equation we obtain equation (25). Three-factor model (26) and the interpretations of β_{ki} follow from the theorem in Cochrane (2005, p. 107). The expressions for the risk premia Λ_i are given in Cochrane (2005, p. 108). ■

Proof of Lemma 3. The price of an AD security is given by $\mathbb{E}^{\text{RN}}[1_{\{\omega=\omega_n\}}]e^{-rT}=\pi_n^{\text{RN}}e^{-rT}$. Hence, the return is given by $\mathbb{E}^U[1_{\{\omega=\omega_n\}}]/(\pi_n^{\text{RN}}e^{-rT})=\pi_n^U/\pi_n^{\text{RN}}e^{rT}$. Equation (27) follows, after some algebra, from equation (A.28). Differentiating (27) w.r.t. t, we obtain:

$$\begin{aligned}\frac{\partial\mathbb{E}^U[R_n]}{\partial t} &= (b_n-\mathbb{E}^U[b])^2\mathbb{E}^U[R_n]-\mathbb{E}^U[R_n]\mathbb{E}^{\text{RN}}\Big[(b-\mathbb{E}^U[b])^2\frac{\pi^U}{\pi^{\text{RN}}}\Big]\\ &= (b_n-\mathbb{E}^U[b])^2\mathbb{E}^U[R_n]-\mathbb{E}^U[R_n]\mathbb{E}^U\Big[(b-\mathbb{E}^U[b])^2\Big]\\ &= \Big[(b_n-\mathbb{E}^U[b])^2-\text{var}^U[b]\Big]\mathbb{E}^U[R_n]. \ ■\end{aligned}$$

Proof of Lemma 4. Substituting probabilities $\pi_n(\varepsilon)$, given by equation (1), into the objective function (3) of investor I and dismissing terms that do not affect the optimal portfolio, we observe that the portfolio choice of the informed investor is equivalent to:

$$\max_{\theta_I}-\sum_{j=1}^{N}\exp\{a_j+b_j\varepsilon-\gamma_I\theta_I^\top(C(\omega_j)-e^{rT}p)\}. \tag{A.29}$$

Substituting $\theta_I=\lambda\varepsilon/\gamma_I-\widehat{\theta}_I(p)/\gamma_I$ into (A.29) and using condition (28), we find that $b_j\varepsilon-\gamma_I\theta_I^\top(C(\omega_j)-e^{rT}p)=(\lambda_0+\lambda^\top pe^{rT})\varepsilon+\widehat{\theta}(p)^\top(C(\omega_j)-e^{rT}p)$. Hence, $b\varepsilon$ and the component $\lambda\varepsilon/\gamma_I$ of portfolio θ_I cancel out, and $\widehat{\theta}_I^*$ solves an optimization problem that does not depend on ε:

$$\max_{\widehat{\theta}_I}e^{rT}p^\top\widehat{\theta}_I-g_I(\widehat{\theta}_I), \tag{A.30}$$

where $g_I(\widehat{\theta}_I)=\ln\Big(\sum_{i=1}^N\exp\{a_i+C(\omega_i)^\top\widehat{\theta}_I\}\Big)$. Hence, $\widehat{\theta}_I^*$ only depends on prices p. ■

Proof of Proposition 3. Step 1 (Portfolio of investor I). Investor I's optimization problem (A.30) yields the FOC for the optimal $\widehat{\theta}_I^* = \lambda\varepsilon - \gamma_I \theta_I^*$:

$$f_I\big(\widehat{\theta}_I^*\big) = e^{rT} p, \tag{A.31}$$

where $f_I(x) \equiv g_I'(x)$ is given by (31). Assuming that $f_I(\cdot)$ is invertible (as verified below), gives $\widehat{\theta}_I^* = f_I^{-1}(e^{rT} p)$. Then, using $\widehat{\theta}_I^* = \lambda\varepsilon - \gamma_I \theta_I^*$, we find portfolio $\theta_I^*(p;\varepsilon)$ in (33).

Step 2 (Posterior probabilities and portfolio of investor U). Substituting investor I's portfolio (29) into the market clearing condition, we obtain: $h\lambda\varepsilon/\gamma_I + \nu + \widehat{H}(p) = 0$, where $\widehat{H}(p) = -hf_I^{-1}(e^{rT}p)/\gamma_I + (1-h)\theta_U^*(p)$, analogous to condition (13) for complete markets. Hence, the posterior probabilities can be found similar to Lemma 2, and are given by equation (15) in which $H(p)$ is replaced by $\widehat{H}(p)$. Substituting $b_n = \lambda_0 + C(\omega_n)^\top \lambda$ from Condition (28) into equation (15) with $\widehat{H}(p)$ instead of $H(p)$, we obtain:

$$\pi_n^U(p) = \frac{1}{G_3(p)} \exp\left(a_n + \frac{1}{2} \frac{b_n^2 + 2C(\omega_n)^\top (\lambda\mu_0/\sigma_0^2 - h\lambda\lambda^\top \Sigma_\nu^{-1} \widehat{H}(p)/\gamma_I)}{h^2(\lambda^\top \Sigma_\nu^{-1}\lambda/\gamma_I^2) + 1/\sigma_0^2} \right), \tag{A.32}$$

where $G_3(p)$ is a normalizing function. Substituting probabilities $\pi_n^U(p)$ into investor U's objective function (9), after some algebra, we obtain:

$$\begin{aligned} & -\textstyle\sum_{n=1}^N \pi_n^U(p) \exp\left\{ -\gamma_U \left(W_{U,0} e^{rT} + \left(C(\omega_n) - e^{rT} p \right)^\top \theta_U \right) \right\} \\ & = -\frac{1}{G_4(p)} \exp\left(\gamma_U e^{rT} p^\top \theta_U + g_U \left(\frac{\lambda\mu_0/\sigma_0^2 - h\lambda\lambda^\top \Sigma_\nu^{-1} \widehat{H}(p)/\gamma_I}{h^2(\lambda^\top \Sigma_\nu^{-1}\lambda/\gamma_I^2) + 1/\sigma_0^2} - \gamma_U \theta_U \right) \right), \end{aligned} \tag{A.33}$$

where $G_4(p)$ is some function of prices and $g_U : \mathbb{R}^{M-1} \to \mathbb{R}$ is a function given by:

$$g_U(x) = \ln\left(\sum_{j=1}^N \exp\left\{ a_j + \frac{1}{2} \frac{b_j^2}{h^2(\lambda^\top \Sigma_\nu^{-1}\lambda/\gamma_I^2) + 1/\sigma_0^2} + C(\omega_j)^\top x \right\} \right).$$

From equation (A.33), we find that investor U's optimization problem becomes

$$\min_{\theta_U} \gamma_U e^{rT} p^\top \theta_U + g_U \left(\frac{\lambda\mu_0/\sigma_0^2 - h\lambda\lambda^\top \Sigma_\nu^{-1} \widehat{H}(p)/\gamma_I}{h^2(\lambda^\top \Sigma_\nu^{-1}\lambda/\gamma_I^2) + 1/\sigma_0^2} - \gamma_U \theta_U \right).$$

Let $f_U(x) \equiv g_U'(x)$, then the FOC for investor U's optimal portfolio θ_U^* is,

$$f_U \left(\frac{\lambda\mu_0/\sigma_0^2 - h\lambda\lambda^\top \Sigma_\nu^{-1} \widehat{H}(p)/\gamma_I}{h^2(\lambda^\top \Sigma_\nu^{-1}\lambda/\gamma_I^2) + 1/\sigma_0^2} - \gamma_U \theta_U^* \right) = e^{rT} p.$$

Assuming that f_U is invertible (as verified below), and $e^{rT}p$ belongs to its range, we obtain

$$\frac{\lambda\mu_0/\sigma_0^2 - h\lambda\lambda^\top \Sigma_\nu^{-1} \widehat{H}(p)/\gamma_I}{h^2(\lambda^\top \Sigma_\nu^{-1}\lambda/\gamma_I^2) + 1/\sigma_0^2} - \gamma_U \theta_U^* = f_U^{-1}\left(e^{rT} p\right). \tag{A.34}$$

Substituting for $\widehat{H}(p) = -hf_I^{-1}\left(e^{rT}p\right)/\gamma_I + (1-h)\theta_U^*(p)$ and factoring out $\gamma_U \theta_U^*(p)$ we have

$$\frac{\lambda\mu_0/\sigma_0^2 + h^2\lambda\lambda^\top \Sigma_\nu^{-1} f_I^{-1}\left(e^{rT}p\right)/\gamma_I^2}{h^2(\lambda^\top \Sigma_\nu^{-1}\lambda/\gamma_I^2) + 1/\sigma_0^2} - \gamma_U \theta_U^*(p)(E+Q) = f_U^{-1}\left(e^{rT}p\right),$$

where E is the identity matrix and matrix Q is given in (A.16). Solving for $\theta_U^*(p)$ yields

$$\begin{aligned} \theta_U^*(p) &= \frac{1}{\gamma_U} (E+Q)^{-1} \left(\frac{\lambda\mu_0/\sigma_0^2 + h^2\lambda\lambda^\top \Sigma_\nu^{-1} f_I^{-1}\left(e^{rT}p\right)/\gamma_I^2}{h^2(\lambda^\top \Sigma_\nu^{-1}\lambda/\gamma_I^2) + 1/\sigma_0^2} - f_U^{-1}\left(e^{rT}p\right) \right) \\ &= (E+Q)^{-1} \left(\frac{hQf_I^{-1}\left(e^{rT}p\right)}{(1-h)\gamma_I} - \frac{f_U^{-1}\left(e^{rT}p\right)}{\gamma_U} + \frac{\mu_0/(\gamma_U\sigma_0^2)\lambda}{h^2\lambda^\top \Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2} \right). \end{aligned} \tag{A.35}$$

Step 3 (Invertibility of $f_I(x)$ and $f_U(x)$). Functions $f_I(x)$ and $f_U(x)$ are special cases of function $f(x;t)$ in equation (A.39) below for $t = 0$ and $t = 0.5/(h^2\lambda^\top \Sigma_\nu^{-1}\lambda/\gamma_I^2 + 1/\sigma_0^2)$. Function $f(x;t)$ has positive-definite and invertible Jacobian by Lemma A.3 below. Hence, by Lemma A.4 below, $f_I(x)$ and $f_U(x)$ are invertible on their ranges.

Step 4 (Equation for asset prices). Substituting θ_I^* and θ_U^* from equations (33) and (34) into the market clearing condition $h\theta_I^*(p;\varepsilon)+(1-h)\theta_U^*(p)+\nu=0$ yields, after some algebra, equation (30) for price vector $P(\varepsilon,\nu)$.

Step 5 (Existence of equilibrium). Finally, we show that there exists unique vector of prices satisfying equation (30). Denote $x_I=f_I^{-1}\left(e^{rT}P(\varepsilon,\nu)\right)$ and $x_U=f_U^{-1}\left(e^{rT}P(\varepsilon,\nu)\right)$. Hence, $f_I(x_I)=f_U(x_U)=e^{rT}P(\varepsilon,\nu)$. From the latter equation and equation (30) for $P(\varepsilon,\nu)$, we obtain the following system of equations for x_I and x_U:

$$\frac{hx_I}{\gamma_I}+\frac{(1-h)x_U}{\gamma_U} = \left(E+Q\right)s+\frac{(1-h)\mu_0/(\gamma_U\sigma_0^2)\lambda}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}, \tag{A.36}$$

$$f_I(x_I) = f_U(x_U). \tag{A.37}$$

From equations (A.31) and (A.34), we note that x_I and x_U are related to portfolios:

$$x_I=\lambda\varepsilon-\gamma_I\theta_I^*,\quad x_U=\frac{\lambda\mu_0/\sigma_0^2-h\lambda\lambda^\top\Sigma_\nu^{-1}\widehat{H}(p)/\gamma_I}{h^2(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2)+1/\sigma_0^2}-\gamma_U\theta_U^*. \tag{A.38}$$

From (A.36), we find that $x_U=\bar{x}-\eta x_I$, where $\eta=\gamma_U h/(\gamma_I(1-h))$ and $\bar{x}$ is given by

$$\bar{x}=\frac{\gamma_U\left(E+Q\right)s}{1-h}+\frac{(\mu_0/\sigma_0^2)\lambda}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}.$$

Substituting $x_U=\bar{x}-\eta x_I$ into equation (A.37), we find that x_I solves $f_I(x_I)=f_U(\bar{x}-\eta x_I)$. The latter equation is a special case of equation $f(x;0)=f(\bar{x}-\eta x;t)$ in (A.42) below with $t=0.5/(h^2(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2)+1/\sigma_0^2)$, because $f(x;0)=f_I(x)$, $f(x;t)=f_U(x)$. Hence, by Lemma A.5, equation $f_I(x_I)=f_U(\bar{x}-\eta x_I)$ has a unique, continuous, and differentiable solution x_I. Because $P(\varepsilon,\nu)=e^{-rT}f_I(x_I)$, price exists, is unique, continuous, differentiable, and invertible on its range function of $\bar{x}$, and hence, also of the sufficient statistic s. ■

Lemma A.3 *(i) Consider function* $f(x;t):\mathbb{R}^{M-1}\times\mathbb{R}\to\mathbb{R}^{M-1}$ *given by:*

$$f(x;t)=\frac{\sum_{j=1}^N C(\omega_j)\exp\{a_j+tb_j^2+C(\omega_j)^\top x\}}{\sum_{j=1}^N \exp\{a_j+tb_j^2+C(\omega_j)^\top x\}}. \tag{A.39}$$

Then, for all x and t, $f(x;t)$ has an invertible positive-definite Jacobian given by:

$$\frac{\partial f(x;t)}{\partial x}=\text{var}^\pi[C], \tag{A.40}$$

where $\text{var}^\pi[C]$ *is a variance–covariance matrix under a certain probability measure* $\pi(x;t)$.
(ii) Consider function $\widehat{f}(x;t)=f(x;0)-f(\bar{x}-\eta x;t)$ *for a fixed* $\bar{x}$ *and* $\eta>0$. *Then, this function also has a positive-definite and invertible Jacobian* $\partial\widehat{f}(x;t)/\partial x$ *for all x and t.*

Proof of Lemma A.3. (i) Differentiating function $f(x;t)$ with respect to x, we obtain:

$$\begin{aligned}\frac{\partial f(x;t)}{\partial x}&=\frac{\sum_{j=1}^N C(\omega_j)C(\omega_j)^\top\exp\{a_j+tb_j^2+C(\omega_j)^\top x\}}{\sum_{j=1}^N\exp\{a_j+tb_j^2+C(\omega_j)^\top x\}}\\&\quad-\frac{\sum_{j=1}^N C(\omega_j)\exp\{a_j+tb_j^2+C(\omega_j)^\top x\}}{\sum_{j=1}^N\exp\{a_j+tb_j^2+C(\omega_j)^\top x\}}\frac{\sum_{j=1}^N C(\omega_j)^\top\exp\{a_j+tb_j^2+C(\omega_j)^\top x\}}{\sum_{j=1}^N\exp\{a_j+tb_j^2+C(\omega_j)^\top x\}}\\&=\text{var}^\pi[C],\end{aligned} \tag{A.41}$$

where the variance $\text{var}^\pi[C]$ is computed under a probability measure given by $\pi_j(x;t)=\exp\{a_j+tb_j^2+C(\omega_j)^\top x\}/\left(\sum_{j=1}^N\exp\{a_j+tb_j^2+C(\omega_j)^\top x\}\right)$. Matrix $\text{var}^\pi[C]$ has elements $\text{cov}^\pi(C_i,C_j)$. It is positive-definite and invertible because these risky assets are non-redundant and do not span a riskless asset.
(ii) Function $\widehat{f}(x;t)$ has Jacobian $\partial f(x;0)/\partial x+\eta\,\partial f(\bar{x}-\eta x;t)/\partial x$, which is the sum of positive-definite and invertible matrices, and hence is positive-definite and invertible. ■

Lemma A.4 **(Gale and Nikaidô).** *Let* $f(x):\mathbb{R}^{M-1}\to\mathbb{R}^{M-1}$ *be a continuous differentiable function with a positive-definite Jacobian. Then, function $f(x)$ is injective, that is, invertible on its range, so that* $\forall x_1,x_2\in\mathbb{R}^{M-1}$ *such that* $f(x_1)=f(x_2)$ *we have* $x_1=x_2$.

Proof of Lemma A.4. See the proof of Theorem 6 in Gale and Nikaidô (1965). ■

Lemma A.5 *Consider function $f(x;t):\mathbb{R}^{M-1}\times\mathbb{R}\to\mathbb{R}^{M-1}$ given by equation (A.39). Then, for all fixed $\bar{x}\in\mathbb{R}^{M-1}$, $\eta>0$ and $t\in\mathbb{R}$ there exists unique x which solves equation*

$$f(x;0)=f(\bar{x}-\eta x;t). \tag{A.42}$$

Moreover, solution $x(t;\bar{x})$ is continuous and differentiable in t and $\bar{x}$.

Proof of Lemma A.5. The proof proceeds in three steps.

Step 1. Let us fix $\bar{x}$ and show that the solution of equation (A.42) exists for all t. For $t=0$ equation (A.42) has solution $x_0=\bar{x}/(1+\eta)$. Function $\widehat{f}(x;t)\equiv f(x;0)-f(\bar{x}-\eta x;t)$ is continuously differentiable and has an invertible Jacobian with respect to x by Lemma A.3. Hence, by the implicit function theorem (Theorem A.1 below), there exists a unique continuously differentiable function $x(t)$ that solves (A.42) in some interval $t\in(-t_-,t_+)$, where $t_\pm>0$. Next, we show that $t_+=+\infty$, and the proof that $t_-=+\infty$ is analogous.

Suppose, t_+ is finite. Let $(-t_-,t_+)$ be the largest open interval in which a unique solution exists. We show in Steps 2 and 3 below that there exists a unique solution of equation $\widehat{f}(x;t_+)=0$. Because $\widehat{f}(x;t)$ has a positive definite and invertible Jacobian [see Lemma A.3], by the implicit function theorem, the solution can be extended to some $t>t_+$, which contradicts the fact that $(-t_-,t_+)$ is the largest interval in which a unique solution exists. Therefore, this leads to a contradiction, and hence, $t_+=+\infty$.

Step 2. We show that $\widehat{f}(x;t_+)=0$ has a unique solution, which implies that $t_+=+\infty$, as shown above. Consider a sequence $t_k\uparrow t_+$ and solutions x_k such that

$$f(x_k;0)=f(\bar{x}-\eta x_k;t_k). \tag{A.43}$$

Suppose, x_k are bounded by some constant A, i.e., $|x_k|<A$. Then, by Weierstrass Theorem (e.g. Rudin, 1976, Theorem 2.42), there exists a convergent subsequence such that $x_{k_n}\to x^*$ as $n\to+\infty$. Taking limit $k_n\to\infty$ in equation (A.43), by the continuity of $f(x;t)$ we find that $\widehat{f}(x^*;t_+)=0$. This solution is unique by Lemma A.4 because $f(x;0)-f(\bar{x}-\eta x;t)$ has positive-definite Jacobian by Lemma A.3. Hence, $t_+=+\infty$.

Step 3. It remains to prove that x_k is indeed bounded. Suppose, x_k is unbounded, i.e., there exist indices k_n such that $|x_{k_n}|\to\infty$, as $k_n\to\infty$. We renumber elements k_n by k, and hence, assume that $|x_k|\to\infty$. Let $j(k)=\arg\max_j C(\omega_j)^\top x_k$. Because $j(k)$ takes only finite number of values from 1 to N, there exists index j^* such that $j^*=j(k_n)$ for an infinite sequence of $k_n\to\infty$. Without loss of generality, we assume that $j^*=1$ (otherwise, we relabel states ω_n accordingly) and also focus on subsequence k_n and relabel its elements by k. Hence, $C(\omega_1)^\top x_k\geq C(\omega_j)^\top x_k$ for all $j=1,\dots,N$. Similarly, we find a subsequence x_k such that $C(\omega_1)^\top x_k\geq C(\omega_2)^\top x_k\geq C(\omega_j)^\top x_k$ for all $j=2,\dots,N$. Similarly, there exists x_k such that

$$C(\omega_1)^\top x_k\geq\dots\geq C(\omega_m)^\top x_k>C(\omega_{m+1})^\top x_k\geq\dots\geq C(\omega_N)^\top x_k, \tag{A.44}$$

for all k, where m is the first index for which $C(\omega_m)^\top x_k-C(\omega_{m+1})^\top x_k\to+\infty$ as $k\to\infty$. The existence of such an index m is guaranteed by Lemma A.7 below.

Next, we take the limit $k\to\infty$ in equation (A.43). Ordering (A.44) simplifies the computation of this limit. Consider the following probability measure $\pi_j(x;t)$:

$$\pi_j(x;t)=\frac{\exp\{a_j+tb_j^2+C(\omega_j)^\top x\}}{\sum_{j=1}^N\exp\{a_j+tb_j^2+C(\omega_j)^\top x\}}. \tag{A.45}$$

Because $0\leq\pi_j(x_k;t)\leq1$, by Weierstrass theorem there exists a subsequence x_k such that $\pi_j(x_k;0)\to\pi_j^+$ and $\pi_j(\bar{x}-\eta x_k;t_k)\to\pi_j^-$ for all $j=1,\dots,N$, where $\sum_{j=1}^N\pi_j^+=\sum_{j=1}^N\pi_j^-=1$, $0\leq\pi_j^+\leq1$ and $0\leq\pi_j^-\leq1$. Next, we demonstrate that

$$\pi_j^+ = 0,\ \text{for } j=m+1,\dots,N, \tag{A.46}$$

$$\pi_j^- = 0,\ \text{for } j=1,\dots,m. \tag{A.47}$$

To derive equalities (A.46) and (A.47), we use inequalities (A.44) and the fact that $C(\omega_m)^\top x_k-C(\omega_{m+1})^\top x_k\to+\infty$ as $k\to\infty$, to obtain for all $j>m$:

$$\pi_j^+=\lim_{k\to+\infty}\pi_j(x_k;0)\leq\lim_{k\to+\infty}\frac{\exp\{a_j+C(\omega_j)^\top x_k\}}{\exp\{a_m+C(\omega_m)^\top x_k\}}\leq\lim_{k\to+\infty}\frac{\exp\{a_j+C(\omega_{m+1})^\top x_k\}}{\exp\{a_m+C(\omega_m)^\top x_k\}}=0.$$

Similarly, for all $j \leq m$, we obtain:

$$\pi_j^- = \lim_{k\to+\infty} \pi_j(\bar{x}-\eta x_k;t_k) \leq \lim_{k\to+\infty} \frac{\exp\{a_j+t_k b_j^2+C(\omega_j)^\top \bar{x}-\eta C(\omega_j)^\top x_k\}}{\exp\{a_{m+1}+t_k b_{m+1}^2+C(\omega_{m+1})^\top \bar{x}-\eta C(\omega_{m+1})^\top x_k\}}$$

$$\leq \lim_{k\to+\infty} \frac{\exp\{a_j+t_k b_j^2+C(\omega_j)^\top \bar{x}-\eta C(\omega_m)^\top x_k\}}{\exp\{a_m+t_k b_m^2+C(\omega_{m+1})^\top \bar{x}-\eta C(\omega_{m+1})^\top x_k\}}=0.$$

Using equations (A.46) and (A.47) and taking the limit $k\to+\infty$ in (A.43), we obtain:

$$\pi_{m+1}^+ C(\omega_{m+1})+\cdots+\pi_N^+ C(\omega_N)=\pi_1^- C(\omega_1)+\cdots+\pi_m^- C(\omega_m).$$

Transposing both sides of the above equation and multiplying by x_k, we obtain:

$$\pi_{m+1}^+ C(\omega_{m+1})^\top x_k+\cdots+\pi_N^+ C(\omega_N)^\top x_k=\pi_1^- C(\omega_1)^\top x_k+\cdots+\pi_m^- C(\omega_m)^\top x_k. \tag{A.48}$$

From the fact that $\sum_{j=1}^N \pi_j^+ = \sum_{j=1}^N \pi_j^- = 1$, demonstrated above, equations (A.46)–(A.48), and inequality (A.44), we obtain:

$$\begin{aligned} \pi_{m+1}^+ C(\omega_{m+1})^\top x_k+\cdots+\pi_N^+ C(\omega_N)^\top x_k &\leq C(\omega_{m+1})^\top x_k < C(\omega_m)^\top x_k \\ &\leq \pi_1^- C(\omega_1)^\top x_k+\cdots+\pi_m^- C(\omega_m)^\top x_k. \end{aligned} \tag{A.49}$$

Inequalities (A.49) contradict equation (A.48). Consequently, x_k is bounded. Hence, as shown in Step 2 above, $t_+=+\infty$, which proves the global existence of $x(t;\bar{x})$. The continuity and differentiability of $x(t;\bar{x})$ follows from the implicit function theorem. ■

Theorem A.1 (Implicit function theorem). *Consider a continuously differentiable function* $\widehat{f}(x;t):\mathbb{R}^{M-1}\times\mathbb{R}\to\mathbb{R}^{M-1}$. *Suppose,* $\widehat{f}(x_0;t_0)=0$ *and the Jacobian* $\partial\widehat{f}(x_0;t_0)/\partial x$ *is invertible. Then, there exist open sets* U *and* V *such that* $x_0\in U$, $t_0\in V$, *and a unique continuously differentiable function* $x(t):V\to U$ *such that* $\widehat{f}(x(t);t)=0$.

Proof of Theorem A.1. This is a special case of Theorem 9.28 in Rudin (1976). ■

Lemma A.6 *Consider a sequence* x_k *such that* $|x_k|\to\infty$ *as* $k\to\infty$. *Then, there exists index* m *such that sequence* $|C(\omega_m)^\top x_k|$ *is unbounded.*

Proof of Lemma A.6. Suppose, on the contrary, there exists constant A such that $|C(\omega_m)^\top x_k|<A$ for all m and k. Because all securities are non-redundant, the matrix with columns $C(\omega_n)$, $n=1,\dots,N$ has rank $M-1$ and vectors $C(\omega_n)$ span $\mathbb{R}^{M-1}$. Without loss of generality, assume that the $M-1$ vectors $C(\omega_1),\dots, C(\omega_{M-1})$ form a basis in $\mathbb{R}^{M-1}$.

Consider vector $e_l=(0,\dots,0,1,0,\dots,0)^\top\in\mathbb{R}^{M-1}$ with lth element equal to 1 and all other elements equal to 0. Then, there exist constants $\alpha_{m,l}$ such that $e_l=\alpha_{1,l}C(\omega_1)+\cdots+\alpha_{M-1,l}C(\omega_N)$. It can be easily observed that x_k is bounded, because for all l

$$|e_l^\top x_k|\leq|\alpha_{1,l}||C(\omega_1)^\top x_k|+\cdots+|\alpha_{M-1,l}||C(\omega_{M-1})^\top x_k|\leq A(M-1)\max_{m,l}|\alpha_{m,l}|,$$

which contradicts $|x_k|\to\infty$. Hence, $|C(\omega_m)^\top x_k|$ is unbounded for some m. ■

Lemma A.7 *Consider a sequence* x_k *such that* $|x_k|\to\infty$ *as* $k\to\infty$. *Then, there exists index* m *such that sequence* $|C(\omega_m)^\top x_k - C(\omega_{m+1})^\top x_k|$ *is unbounded.*

Proof of Lemma A.7. Suppose, on the contrary, sequence $|C(\omega_m)^\top x_k - C(\omega_{m+1})^\top x_k|$ is bounded for all m. The latter easily implies that $|C(\omega_i)^\top x_k - C(\omega_j)^\top x_k|<A$ for all i and j and some constant A. Because all assets are non-redundant, vectors $(C(\omega_n)^\top,1)^\top\in\mathbb{R}^M$, where $n=1,\dots,N$, span $\mathbb{R}^M$. Without loss of generality, assume that the first M vectors $(C(\omega_n)^\top,1)^\top$ form a basis in $\mathbb{R}^M$. Hence, there exist unique $(\alpha_1,\dots,\alpha_M)^\top$ such that:

$$\begin{aligned} \alpha_1 C(\omega_1)+\cdots+\alpha_M C(\omega_M) &= 0, \\ \alpha_1+\cdots+\alpha_M &= 1. \end{aligned} \tag{A.50}$$

We solve equations (A.50) and for an arbitrary index m we obtain:

$$\begin{aligned} |C(\omega_m)^\top x_k| &= |(\alpha_1+\cdots+\alpha_M)C(\omega_m)^\top x_k-(\alpha_1 C(\omega_1)^\top x_k+\cdots+\alpha_M C(\omega_M)^\top x_k)| \\ &\leq |\alpha_1||C(\omega_m)^\top x_k - C(\omega_1)^\top x_k|+\cdots+|\alpha_M||C(\omega_m)^\top x_k - C(\omega_M)^\top x_k| \\ &\leq A\max_l|\alpha_l|, \end{aligned}$$

contradicting the result of Lemma A.6 that $|C(\omega_m)^\top x_k|$ is unbounded for some m. ■

Proof of Lemma 5. Substituting the risk-neutral probabilities (35) into (36), and noting that due to the properties of risk-neutral distributions $\mathbb{E}^{\mathrm{RN},i}[W_{i,T}]=W_{i,0}e^{rT}$, we obtain:

$$\widehat{\kappa}=\mathbb{E}^{\mathrm{RN},U}\Big[\ln\Big(\frac{\pi^U(p)}{\pi(\varepsilon)}\Big)\Big]+\mathbb{E}^{\mathrm{RN},I}\Big[\ln\Big(\frac{\pi(\varepsilon)}{\pi^U(p)}\Big)\Big].$$

Substituting $\pi(\varepsilon)$ from (1) and $\pi^U(p)$ from (A.32) into the above equation, we find:

$$\widehat{\kappa}=0.5\mathbb{E}^{\mathrm{RN},U}\Big[\frac{b^2+2b(\mu_0/\sigma_0^2+h\lambda^\top\Sigma_\nu^{-1}s/\gamma_I)}{h^2(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2)+1/\sigma_0^2}-b\varepsilon\Big]$$

$$-0.5\mathbb{E}^{\mathrm{RN},I}\Big[\frac{b^2+2b(\mu_0/\sigma_0^2+h\lambda^\top\Sigma_\nu^{-1}s/\gamma_I)}{h^2(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2)+1/\sigma_0^2}-b\varepsilon\Big].$$

Using the fact that under the risk-neutral probabilities the price of a portfolio with payoff b is given by $\mathbb{E}^{\mathrm{RN},U}[b]=\mathbb{E}^{\mathrm{RN},I}[b]$, we simplify the above expression for $\widehat{\kappa}$ as follows:

$$\widehat{\kappa}=\frac{1}{2}\frac{\mathbb{E}^{\mathrm{RN},U}[b^2]-\mathbb{E}^{\mathrm{RN},I}[b^2]}{h^2(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2)+1/\sigma_0^2}=\frac{1}{2}\frac{\mathrm{var}^{\mathrm{RN},U}[b]-\mathrm{var}^{\mathrm{RN},I}[b]}{h^2(\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2)+1/\sigma_0^2}.\ \blacksquare$$

Proof of Proposition 4. Consider an informationally irrelevant asset m such that $\lambda_m=0$ and $\mathrm{cov}(\nu_m,\nu_k)=0$ for all $k\neq m$. The sufficient statistic is then given by $s=(\lambda_1\varepsilon/\gamma_I+\nu_1,\dots,\lambda_{m-1}\varepsilon/\gamma_I+\nu_{m-1},\nu_m,\lambda_{m+1}\varepsilon/\gamma_I+\nu_{m+1},\dots)^\top$. Due to normality, ν_m is independent of other noise trader demands, and hence, the m^{th} component of the sufficient statistic does not provide any information about ε. Consequently, the distribution of ε conditional on s is the same as in the economies with and without asset m. ■

Lemma A.8 *i) The market is effectively complete if and only if there exists a portfolio that replicates* $\ln\big(\pi_n(\varepsilon)/\pi_n^U(p)\big)$, *that is, there exist ω-independent $\widehat{\lambda}_0\in\mathbb{R}$ and $\hat{\lambda}\in\mathbb{R}^{M-1}$ such that*

$$\ln\left(\frac{\pi_n(\varepsilon)}{\pi_n^U(p)}\right)=\hat{\lambda}_0+C(\omega_n)^\top\hat{\lambda}. \tag{A.51}$$

ii) The equilibrium Pareto efficient portfolios are given by:

$$\theta_I^* = \frac{1}{\gamma_I}\frac{\widehat{\lambda}(1-h)/\gamma_U-\nu}{h/\gamma_I+(1-h)/\gamma_U}, \tag{A.52}$$

$$\theta_U^* = -\frac{1}{\gamma_U}\frac{\widehat{\lambda}h/\gamma_I+\nu}{h/\gamma_I+(1-h)/\gamma_U}. \tag{A.53}$$

Proof of Lemma A.8. (i) Suppose, the market is effectively complete. Taking logs on both sides of (37) and rearranging terms, we find that $\gamma_U W_{U,T,n}-\gamma_I W_{I,T,n}-\overline{\ell}=\ln\big(\pi_n^U(p)/\pi_n(\varepsilon)\big)$. Hence, the log-ratio of probabilities can be replicated by a portfolio of $\gamma_U\theta_U^*-\gamma_I\theta_I^*$ units of risky assets and $\gamma_U(W_{U,0}-p^\top\theta_U^*)e^{rT}-\gamma_I(W_{I,0}-p^\top\theta_I^*)e^{rT}-\overline{\ell}$ units of bond.

Suppose, there exist $\hat{\lambda}_0\in\mathbb{R}$ and $\hat{\lambda}\in\mathbb{R}^{M-1}$ such that (A.51) holds. Hence, $\pi_n(\varepsilon)=\pi_n^U(p)\exp(\hat{\lambda}_0+C(\omega_n)^\top\hat{\lambda})$. Substituting $\pi_n(\varepsilon)$ into investor I's optimization (3), we obtain:

$$\max_{\theta_I}\mathbb{E}\Big[-e^{-\gamma_I W_{I,T}}\Big|\varepsilon,p\Big]=\max_{\theta_I}\Big[-\sum_{n=1}^N\pi_n^U(p)e^{\hat{\lambda}_0+C(\omega_n)^\top\hat{\lambda}-\gamma_I W_{I,T,n}}\Big]. \tag{A.54}$$

Substituting wealth $W_{I,T,n}$ from the budget constraint (5) into optimization (A.54), and rearranging terms, we observe that this optimization is equivalent to maximizing

$$\max_{\theta_I}\Big[-\sum_{n=1}^N\pi_n^U(p)e^{-\gamma_I(C(\omega_n)-e^{rT}p)^\top(\theta_I-\hat{\lambda}/\gamma_I)}\Big]=\max_{\hat{\theta}}\Big[-\sum_{n=1}^N\pi_n^U(p)e^{-\gamma_U(C(\omega_n)-e^{rT}p)^\top\hat{\theta}}\Big], \tag{A.55}$$

where, by a change of variable, $\hat{\theta}=(\theta_I-\hat{\lambda}/\gamma_I)(\gamma_I/\gamma_U)$. The second optimization in (A.55) is the same as that of investor U. Hence, $\theta_U^*=\hat{\theta}^*=(\theta_I^*-\hat{\lambda}/\gamma_I)(\gamma_I/\gamma_U)$, or, equivalently:

$$\gamma_I\theta_I^*-\gamma_U\theta_U^*=\hat{\lambda}. \tag{A.56}$$

Multiplying (A.56) by $(C(\omega_n)-e^{rT}p)$, we obtain $\gamma_I(C(\omega_n)-e^{rT}p)^\top\theta_I^*-\gamma_U(C(\omega_n)-e^{rT}p)^\top\theta_U^*=(C(\omega_n)-e^{rT}p)^\top\hat{\lambda}$. The latter equation and budget constraints (5) then imply $\gamma_U W_{U,T,n}-\gamma_I W_{I,T,n}-\overline{\ell}=\ln\big(\pi_n^U(p)/\pi_n(\varepsilon)\big)$, where $\overline{\ell}$ does not depend on ω_n, which is equivalent to the Pareto efficiency condition (37).

(ii) The equilibrium Pareto efficient portfolios satisfy the market clearing condition (6) and efficiency condition (A.56). Solving the latter two equations with two unknowns, we obtain portfolios (A.52) and (A.53). ■

Proof of Proposition 5. (i) Substituting $\pi_n(\varepsilon)$ from (1) and $\pi_n^U(p)$ from (A.32) into $\ln(\pi_n(\varepsilon)/\pi_n^U)$, and using market clearing $\widehat{H}(p)=-s$, where $s=(h\lambda\varepsilon/\gamma_I+\nu)$, we find:

$$\ln\left(\frac{\pi_n(\varepsilon)}{\pi_n^U(p)}\right)=b_n\varepsilon-\frac{1}{2}\frac{b_n^2+2b_n\left(\mu_0/\sigma_0^2+h\lambda^\top\Sigma_\nu^{-1}(h\lambda\varepsilon/\gamma_I+\nu)/\gamma_I\right)}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}+const, \tag{A.57}$$

where *const* does not depend on state ω_n. The log-ratio of probabilities in (A.57) is a quadratic function of b_n. Hence, by Lemma A.8, the optimal portfolios are Pareto efficient if and only if there exists a portfolio that replicates b_n^2.
(ii) Substituting $b_n=\lambda_0+C(\omega_n)^\top\lambda$ from (28) and $b_n^2=\widetilde{\lambda}_0+C(\omega_n)^\top\widetilde{\lambda}$ from the conditions of Proposition 5 into (A.57), we find that the replicating portfolio $\widehat{\lambda}$ in (A.51) for the ratio $\ln(\pi_n(\varepsilon)/\pi_n^U)$ is given by:

$$\widehat{\lambda}=\lambda\left(\varepsilon-\frac{\mu_0/\sigma_0^2+h\lambda^\top\Sigma_\nu^{-1}s/\gamma_I}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}\right)-\frac{0.5\widetilde{\lambda}}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}. \tag{A.58}$$

Substituting (A.58) into the investors' portfolios (A.52) and (A.53), we observe that investor I shorts and investor U buys portfolio $\widetilde{\lambda}$, which replicates the quadratic derivative.
(iii) From the FOC of investor I (A.31), we find that $P(\varepsilon,\nu)=f_I(\lambda\varepsilon-\gamma_I\theta_I^*)$. Using the expression for $f_I(x)$ in (31), we obtain:

$$P(\varepsilon,\nu)=\frac{\sum_{j=1}^N C(\omega_j)\exp\{a_j+C(\omega_j)^\top(\lambda\varepsilon-\gamma_I\theta_I^*)\}}{\sum_{j=1}^N\exp\{a_j+C(\omega_j)^\top(\lambda\varepsilon-\gamma_I\theta_I^*)\}}. \tag{A.59}$$

Using θ_I^* from (A.52), and the spanning condition $b_n=\lambda_0+C(\omega_n)^\top\lambda$, we obtain:

$$C(\omega_n)^\top(\lambda\varepsilon-\gamma_I\theta_I^*)=(b_n-\lambda_0)\varepsilon-\frac{(1-h)/\gamma_U C(\omega_n)^\top\hat{\lambda}}{h/\gamma_I+(1-h)/\gamma_U}+\frac{1}{h/\gamma_I+(1-h)/\gamma_U}C(\omega_n)^\top\nu, \tag{A.60}$$

where $\hat{\lambda}$ is such that $\ln(\pi_n(\varepsilon)/\pi_n^U(p))=\hat{\lambda}_0+C(\omega_n)^\top\hat{\lambda}$. Hence, $C(\omega_n)^\top\hat{\lambda}=\ln(\pi_n(\varepsilon)/\pi_n^U(p))-\hat{\lambda}_0$. Substituting $C(\omega_n)^\top\hat{\lambda}$ and $\ln(\pi_n(\varepsilon)/\pi_n^U(p))$ from (A.57) into (A.60), after some algebra, we find that $a_n+C(\omega_n)^\top(\lambda\varepsilon-\gamma_I\theta_I^*)=\upsilon_n+const$, where υ_n is given by (18) in which $\lambda,\nu\in\mathbb{R}^{M-1}$ and $E,Q\in\mathbb{R}^{(M-1)\times(M-1)}$, and *const* does not depend on ω_n. Substituting $a_n+C(\omega_n)^\top(\lambda\varepsilon-\gamma_I\theta_I^*)$ into (A.59), we find the price. ■

Proof of Proposition 6. We derive the shadow value of information for a more general case of incomplete markets and then obtain the value of information for effectively complete markets, given by equation (39) in Proposition 6, as a special case. Because the market is incomplete and not necessarily Pareto efficient, the ratios of the probability-weighted marginal utilities are state-dependent:

$$\frac{\pi_n(\varepsilon)\exp\{-\gamma_I W_{I,T,n}\}}{\pi_n^U(p)\exp\{-\gamma_U W_{U,T,n}\}}=\exp\{\overline{\ell}+\ell_n\}. \tag{A.61}$$

We introduce the following new measure of market incompleteness:

$$\kappa^U=\mathbb{E}^{\text{RN},U}\left[\ln\left(\frac{\pi^{\text{RN},U}}{\pi^{\text{RN},I}}\right)\right], \tag{A.62}$$

which is equal to zero if and only if the risk-neutral probabilities of investors coincide, and has similar intuition as the measure of incompleteness $\widehat{\kappa}$ given by (36). It can be easily verified that measure κ^U can be expressed as

$$\kappa^U=\ln(\mathbb{E}^{\text{RN},U}[\exp\{\ell(\omega)\}])-\mathbb{E}^{\text{RN},U}[\ell(\omega)], \tag{A.63}$$

where $\ell(\omega_n)\equiv\ell_n$.

Consider an atomistic uninformed investor that becomes informed and has risk aversion γ_U and wealth $\widetilde{W}_I$. The prices are not affected when this investor becoming informed. From equation (33) for the trading strategy of the informed investor, we see that $\gamma_I\theta_I^*$ does not depend on γ_I. Hence, $\gamma_I W_I$ is the same for all informed investors, up to a constant. Therefore, equation (A.61) for the marginal rate of substitution also holds (albeit with a different constant $\overline{\ell}$) for the informed investor with risk aversion γ_U. Taking conditional expectation $\mathbb{E}[\cdot|s]$ on both sides of the latter equation, we obtain:

$$\frac{\mathbb{E}[\pi_n(\varepsilon)\exp\{-\gamma_U\widetilde{W}_{I,T,n}\}|s]}{\pi_n^U(p)\exp\{-\gamma_U W_{U,T,n}\}}=e^{\tilde{\ell}+\ell_n}, \tag{A.64}$$

where $\tilde{\ell}$ does not depend on ω_n. We note that ℓ_n is not affected by taking conditional expectations on both sides of (A.61) since it does not depend on ε [see Lemma A.9 below].

Let $J_I(W_0)$ be the expected utility of the atomistic uninformed investor who becomes informed, under probabilities $\pi_n(\varepsilon)$, and $J_U(W_0)$ be the expected utility of the uniformed investor. From the uninformed investor's point of view, the expected utility of the informed investor is $\mathbb{E}[J_I|s]$. Using equations (A.63) and (A.64), we obtain:

$$\begin{aligned}\mathbb{E}[J_I|s] &= -\mathbb{E}[\textstyle\sum_{n=1}^N \pi_n(\varepsilon)\exp\{-\gamma_U \widetilde{W}_{I,T,n}\}|s] = -\sum_{n=1}^N \pi_n^U(p)\exp\{-\gamma_U W_{U,T,n}+\tilde{\ell}+\ell_n\}\\ &= -\textstyle\sum_{n=1}^N \pi_n^{\mathrm{RN},U}(p)\exp\{\tilde{\ell}+\ell_n\}\sum_{n=1}^N \pi_n^U(p)\exp\{-\gamma_U W_{U,T,n}\}\\ &= \exp\{\tilde{\ell}\}\mathbb{E}^{\mathrm{RN},U}[\exp\{\ell(\omega)\}]J_U = \exp\{\hat{\ell}+\kappa^U\}J_U,\end{aligned} \tag{A.65}$$

where $\hat{\ell}=\tilde{\ell}+\mathbb{E}^{\mathrm{RN},U}[\ell_n]$, J_U is investor U's expected utility, and $\mathbb{E}^{\mathrm{RN},U}[\cdot]$ is under the risk-neutral measure of investor U in (35). The second line of (A.65) is derived by dividing and multiplying back the preceding equation in the first line by $\sum_{n=1}^N \pi_n^U(p)\exp\{-\gamma_U W_{U,T,n}\}$ and then using equation (35) for the risk-neutral measure of investor U.

Let $\widehat{W}$ be how much investor U is willing to give up to become informed, so that $\mathbb{E}[J_I(W_0-\widehat{W})|s]=J_U(W_0)$. We note that $\mathbb{E}[J_I(W_0-\widehat{W})|s]=\mathbb{E}[J_I(W_0)|s]\exp\{\gamma_U\widehat{W}\}$. Therefore, $\mathbb{E}[J_I(W_0-\widehat{W})|s]=\exp\{\gamma_U\widehat{W}+\hat{\ell}+\kappa^U\}J_U(W_0)$, and hence, $\widehat{W}=-(\hat{\ell}+\kappa^U)/\gamma_U$. It remains now to find $\hat{\ell}$.

Define function $F(z)=\mathbb{E}[\exp\{z\varepsilon\}/\sum_{j=1}^N \exp\{a_j+b_j\varepsilon\}|s]$. Substituting $\pi_n(\varepsilon)$, $\pi_n^U(p)=E[\pi_n(\varepsilon)|s]$, and θ_I^* from (29) into (A.64), we obtain:

$$\exp\{(\widehat{\theta}_I^*(p)+\gamma_U\theta_U^*)^\top(C(\omega_n)-e^{rT}p)\}\frac{F(\lambda_0+e^{rT}\lambda^\top p)}{F(b_n)}=\exp\{\tilde{\ell}+\ell_n\}. \tag{A.66}$$

Using the informational spanning condition (28), we find that $\lambda_0+e^{rT}\lambda^\top p=\lambda_0+\lambda^\top\mathbb{E}^{\mathrm{RN},U}[C(\omega_n)]=\mathbb{E}^{\mathrm{RN},U}[b_n]$. Using the latter fact, equation (A.68) for function $F(z)$ in Lemma A.10 below, and taking logs on both sides of equation (A.66), we obtain:

$$\begin{aligned}&(\widehat{\theta}_I^*(p)+\gamma_U\theta_U^*)^\top(C(\omega_n)-e^{rT}p)+\frac{(h\lambda^\top\Sigma_\nu^{-1}s/\gamma_I+\mu_0/\sigma_0^2+\mathbb{E}^{\mathrm{RN},U}[b_n])^2}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}\\ &-\frac{(h\lambda^\top\Sigma_\nu^{-1}s/\gamma_I+\mu_0/\sigma_0^2+b_n)^2}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}=\hat{\ell}+(\ell_n-\mathbb{E}^{\mathrm{RN},U}[\ell(\omega)]).\end{aligned}$$

Next, we take expectation $\mathbb{E}^{\mathrm{RN},U}[\cdot]$ on both sides of the above equation. We notice that the first term on the left-hand side cancels out because $\mathbb{E}^{\mathrm{RN},U}[C(\omega_n)]=e^{rT}p$ and the second term on the right-hand side trivially cancels out. Then, after some algebra, we obtain

$$\hat{\ell}=-\frac{1}{2}\frac{\mathrm{var}^{\mathrm{RN},U}[b_n]}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}.$$

As shown above, $\widehat{W}=-(\hat{\ell}+\kappa^U)/\gamma_U$, and hence, we obtain the following equation:

$$\widehat{W}=\frac{1}{2\gamma_U}\frac{\mathrm{var}^{\mathrm{RN},U}[b]}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}-\frac{\kappa^U}{\gamma_U}, \tag{A.67}$$

where the measure of market incompleteness κ^U is given by (A.62). When the market is effectively complete, the risk-neutral measure is unique, and hence, $\mathrm{var}^{\mathrm{RN},U}[b]=\mathrm{var}^{\mathrm{RN}}[b]$, and $\kappa^U=0$. Substituting the latter expressions for $\mathrm{var}^{\mathrm{RN},U}[b]$ and κ^U into equation (A.67), we obtain the value of information in effectively complete markets (39). ■

Lemma A.9 *The measures of market incompleteness $\widehat{\kappa}$ given by (36) and κ^U given by (A.62), and the state-dependent parameter ℓ_n of the probability-weighted ratio of marginal utilities (A.61) are functions of the sufficient statistic s, and depend on ε only via s.*

Proof of Lemma A.9. Substituting θ_I^* from (29) into wealth W_I, we find that the probability $\pi_n^{\mathrm{RN},I}$ in (35) is given by:

$$\pi_n^{\mathrm{RN},I}=\frac{\exp\Big\{a_n+b_n\varepsilon-(\lambda\varepsilon-\widehat{\theta}_I^*(p))^\top C(\omega_n)\Big\}}{\sum_{j=1}^N\exp\Big\{a_j+b_j\varepsilon-(\lambda\varepsilon-\widehat{\theta}_I^*(p))^\top C(\omega_j)\Big\}}=\frac{\exp\Big\{a_n+\widehat{\theta}_I^*(p)^\top C(\omega_n)\Big\}}{\sum_{j=1}^N\exp\Big\{a_j+\widehat{\theta}_I^*(p)^\top C(\omega_j)\Big\}},$$

where the second equality uses $\lambda^\top C(\omega_n)=b_n-\lambda_0$ from the informational spanning condition (28). Hence, $\pi_n^{\mathrm{RN},I}$ only depends on s via the asset prices p. Probability $\pi_n^{\mathrm{RN},U}$ in (35) depends only on s because investor U does not observe ε. Hence, $\widehat{\kappa}$ in (36) and κ^U in (A.62) do not depend on ε. Finally, from equations (35) for the risk-neutral measures and the ratio of probability-weighted marginal utilities (A.61), we observe we can choose $\exp\{\ell_n\}=\pi_n^{\mathrm{RN},I}/\pi_n^{\mathrm{RN},U}$, which does not depend on ε as shown above. ■

Lemma A.10 *Let* $F(z)=\mathbb{E}[\exp\{z\varepsilon\}/\sum_{j=1}^{N}\exp\{a_j+b_j\varepsilon\}|s]$. *Then,* $F(z)$ *is given by*

$$F(z)=\exp\left\{\frac{(h\lambda^\top\Sigma_\nu^{-1}s/\gamma_I+\mu_0/\sigma_0^2+z)^2}{h^2\lambda^\top\Sigma_\nu^{-1}\lambda/\gamma_I^2+1/\sigma_0^2}\right\}\frac{1}{\widehat{G}(s)},\tag{A.68}$$

where $\widehat{G}(s)$ *does not depend on* z.

Proof of Lemma A.10. Using conditional PDF (A.8), we obtain:

$$F(z)=\int_{-\infty}^{\infty}\exp\{z\varepsilon-0.5(s-h\lambda\varepsilon/\gamma_I)^\top\Sigma_\nu^{-1}(s-h\lambda\varepsilon/\gamma_I)-0.5(\varepsilon-\mu_0)^2/\sigma_0^2)\}\frac{d\varepsilon}{G(s)},$$

where $G(s)$ does not depend on z. Finding $F(z)$ is similar to finding integral (A.9) in Lemma 2. Integrating, we obtain equation (A.68). ■

Lemma A.11 **(Equilibrium prices).** *The prices of the asset with payoff* C_1 *that has PDF* $\varphi_C(x)$ *in the effectively complete and incomplete markets are given by*

$$P_{com}(s)=P_C\left(\gamma_I\sigma_0^2 s/h-\mu_0,\sigma_{com}\right)e^{-rT},\quad P_{inc}(s)=P_C\left(\gamma_I\sigma_0^2\widehat{s}(s)/h-\mu_0,\sigma_0\right)e^{-rT},\tag{A.69}$$

respectively, where the pricing function $P_C(\mu,\sigma)$ *is given by equation*

$$P_C(\mu,\sigma)=\frac{\mathbb{E}\left[C_1\exp\left(-\frac{(C_1-\mu)^2}{2\sigma^2}\right)\right]}{\mathbb{E}\left[\exp\left(-\frac{(C_1-\mu)^2}{2\sigma^2}\right)\right]},\tag{A.70}$$

$\mathbb{E}[\cdot]$ *is the expectation with respect to PDF* $\varphi_C(x)$, $s=h\varepsilon/(\gamma_I\sigma_0^2)+\nu_1$, $\widehat{s}(s)$ *in the equation for* $P_{inc}(s)$ *is an increasing implicit function of* s *satisfying equation*

$$P_C\left(\gamma_I\sigma_0^2\widehat{s}(s)/h-\mu_0,\sigma_0\right)=P_C\left(\gamma_I\sigma_0^2 s/h-\mu_0+\gamma_U\sigma_{inc}^2(s-\widehat{s}(s))/(1-h),\sigma_{inc}\right),\tag{A.71}$$

and the volatility parameters σ_{com} *and* σ_{inc} *are given by:*

$$\sigma_{com}=\frac{\sigma_0}{\sqrt{1-\dfrac{(1-h)/\gamma_U}{h/\gamma_I+(1-h)/\gamma_U}\dfrac{1}{1+h^2/\left(\gamma_I^2\sigma_\nu^2\sigma_0^2\right)}}},\quad \sigma_{inc}=\frac{\sigma_0}{\sqrt{1-\dfrac{1}{1+h^2/\left(\gamma_I^2\sigma_\nu^2\sigma_0^2\right)}}}.\tag{A.72}$$

Moreover, both prices $P_{com}(s)$ *and* $P_{inc}(s)$ *are increasing functions of* s.

Proof of Lemma A.11. By Lemma A.2, the model is a limiting case of a discrete model with parameters a_n and b_n given by Equations (7) when $N\to\infty$. Therefore, we derive prices for the discrete state-space case and then take the limit $N\to\infty$. Section IA6.5 of the Supplementary Appendix provides the full proof. ■

Lemma A.12 **(Expansion in terms of moments).** *Let* $m_n=\mathbb{E}\left[\left(\frac{C_1-\mu_C}{\sigma_C}\right)^n\right]$ *be the standardized moments under the PDF* $\varphi_C(x)$. *The pricing function (A.70) is then given by*

$$P_C(\mu,\sigma)=\mu_C+\sigma_C\frac{\sum_{n=0}^{\infty}H_n\left(\frac{\mu-\mu_C}{\sigma}\right)\left(\frac{\sigma_C}{\sigma}\right)^n\frac{m_{n+1}}{n!}}{\sum_{n=0}^{\infty}H_n\left(\frac{\mu-\mu_C}{\sigma}\right)\left(\frac{\sigma_C}{\sigma}\right)^n\frac{m_n}{n!}},\tag{A.73}$$

where $H_n(x)$ *are Hermite polynomials satisfying a recursive equation* $H_{n+1}(x)=xH_n(x)-nH_{n-1}(x)$ *with the initial conditions* $H_1(x)=x$ *and* $H_0(x)=1$.

Proof of Lemma A.12. From formula 8.957 in Gradshteyn and Ryzhik (2007)

$$e^{xt-0.5t^2}=\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!},\tag{A.74}$$

where $H_{n+1}(x)=xH_n(x)-nH_{n-1}(x)$ (formula 8.952 in Gradshteyn and Ryzhik (2007)). Then, we note that

$$\exp\left(-\frac{(C_1-\mu)^2}{2\sigma^2}\right)=\exp\left(-\frac{(C_1-\mu_C)^2}{2\sigma_C^2}\frac{\sigma_C^2}{\sigma^2}+\frac{C_1-\mu_C}{\sigma_C}\frac{\sigma_C}{\sigma}\frac{\mu-\mu_C}{\sigma}-\frac{(\mu-\mu_C)^2}{2\sigma^2}\right).\tag{A.75}$$

Next, we expand (A.75) using (A.74), and then substituting the result into equation (A.70), we obtain the expansion (A.73). ■

Proof of Proposition 7. i) We derive equation (44) by substituting PDF (40) into equation (A.70), and then finding the expectations in equation (A.70) in closed form.
ii) Rewriting the expectations under the generalized gamma distribution (41) in the pricing function (A.70) as integrals, after simple algebra, we obtain:

$$P_C(\mu,\sigma;k)=\frac{\int_0^{+\infty} C_1^k \exp\left\{-\frac{1}{2}\left(\frac{1}{\widehat{\sigma}_C^2}+\frac{1}{\sigma^2}\right)C_1^2+\left(\frac{\mu}{\sigma^2}-\delta\right)C_1\right\}dC_1}{\int_0^{+\infty} C_1^{k-1} \exp\left\{-\frac{1}{2}\left(\frac{1}{\widehat{\sigma}_C^2}+\frac{1}{\sigma^2}\right)C_1^2+\left(\frac{\mu}{\sigma^2}-\delta\right)C_1\right\}dC_1}. \tag{A.76}$$

For the case $k=1$ the above pricing function can be easily computed in closed form, and is given in equation (45) for $k=1$. For general integer $k>1$, denote the integrals in the numerator and the denominator of (A.76) by I_k and I_{k-1}, respectively. Using integration by parts, we obtain the following recursive equation for I_k:

$$\begin{aligned} I_k &= \frac{\widehat{\sigma}_C^2\sigma^2}{\widehat{\sigma}_C^2+\sigma^2}\left[\left(\frac{\mu}{\sigma^2}-\delta\right)I_{k-1}-\int_0^{+\infty} C_1^{k-1}\left(\exp\left\{-\frac{1}{2}\left(\frac{1}{\widehat{\sigma}_C^2}+\frac{1}{\sigma^2}\right)C_1^2+\left(\frac{\mu}{\sigma^2}-\delta\right)C_1\right\}\right)' dC_1\right] \\ &= \frac{\widehat{\sigma}_C^2\sigma^2}{\widehat{\sigma}_C^2+\sigma^2}\left(\left(\frac{\mu}{\sigma^2}-\delta\right)I_{k-1}+(k-1)I_{k-2}\right). \end{aligned}$$

Dividing both sides by I_{k-1} and using $P_C(\mu,\sigma;k)=I_k/I_{k-1}$, we obtain equation (45).
iii) Under the skew-normal distribution (42) the pricing function (A.70) becomes:

$$P_C(\mu,\sigma)=\frac{\int_{-\infty}^{+\infty} C_1 \exp\left\{-\frac{(C_1-\mu)^2}{2\sigma^2}-\frac{(C_1-\widehat{\mu}_C)^2}{2\widehat{\sigma}_C^2}\right\}\Phi\left(\alpha\frac{C_1-\widehat{\mu}_C}{\widehat{\sigma}_C}\right)dC_1}{\int_{-\infty}^{+\infty} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}-\frac{(C_1-\widehat{\mu}_C)^2}{2\widehat{\sigma}_C^2}\right\}\Phi\left(\alpha\frac{C_1-\widehat{\mu}_C}{\widehat{\sigma}_C}\right)dC_1}. \tag{A.77}$$

Denote by J_1 and J_0 the integrals in the numerator and the denominator, respectively. After some algebra and integration by parts, we obtain:

$$\begin{aligned} J_1 &= \exp\left\{-\frac{1}{2}\frac{(\mu-\widehat{\mu}_C)^2}{\widehat{\sigma}_C^2+\sigma^2}-\frac{1}{2}\frac{(\widehat{\mu}-\mu)^2}{\widehat{\sigma}_C^2/\alpha^2+\widehat{\sigma}^2}\right\}\frac{\mathrm{sgn}(\alpha)\widehat{\sigma}^3}{\sqrt{\widehat{\sigma}_C^2/\alpha^2+\widehat{\sigma}^2}}+\frac{\widehat{\sigma}_C^2\mu+\sigma^2\widehat{\mu}_C}{\widehat{\sigma}_C^2+\sigma^2}J_0, \\ J_0 &= \exp\left\{-\frac{1}{2}\frac{(\mu-\widehat{\mu}_C)^2}{\widehat{\sigma}_C^2+\sigma^2}\right\}\int_{-\infty}^{\infty}\exp\left\{-\frac{(x-\widehat{\mu})^2}{2\widehat{\sigma}^2}\right\}\Phi\left(\alpha\frac{x-\widehat{\mu}_C}{\widehat{\sigma}_C}\right)dx \\ &= \sqrt{2\pi}\widehat{\sigma}\exp\left\{-\frac{1}{2}\frac{(\mu-\widehat{\mu}_C)^2}{\widehat{\sigma}_C^2+\sigma^2}\right\}\Phi\left(\frac{\mathrm{sgn}(\alpha)(\widehat{\mu}-\widehat{\mu}_C)}{\sqrt{\widehat{\sigma}_C^2/\alpha^2+\widehat{\sigma}^2}}\right), \end{aligned}$$

where $\widehat{\sigma}=\widehat{\sigma}_C^2\sigma^2/(\widehat{\sigma}_C^2+\sigma^2)$, $\widehat{\mu}=(\widehat{\sigma}_C^2\mu+\sigma^2\widehat{\mu}_C)/(\widehat{\sigma}_C^2+\sigma^2)$, and the computation of J_0 uses the following integral 8.259.1 in Gradshteyn and Ryzhik (2007): $\int_{-\infty}^{\infty}\exp(-px^2)\mathrm{erf}(a+bx)dx=(\sqrt{\pi/p})\mathrm{erf}(a\sqrt{p}/\sqrt{b^2+p})$, where the error function is given by $\mathrm{erf}(x)\equiv 2\Phi(x\sqrt{2})-1$. After some algebra, we obtain $P_C(\mu,\sigma)=J_1/J_0$ given by equation (46). ■

Proof of Proposition 8. Similar to the derivation of the prices in incomplete and complete markets in Lemma A.11, we find that $\mathbb{E}^U[C_1]=P_C(\gamma_I\sigma_0^2 s/h-\mu_0,\sigma_{inc})$, where σ_{inc} is given in equation (A.72). Next, we consider the expansion (A.73) in Lemma A.12 and retain only terms that have the order of magnitude σ_C^3. Hence, from the expansion (A.73) we obtain the following approximation:

$$P_C(\mu,\sigma)=\mu_C+\frac{\sigma_C^2}{\sigma^2}(\mu-\mu_C)+\frac{m_3\sigma_C^3}{2\sigma^2}\left[\left(\frac{\mu-\mu_C}{\sigma}\right)^2-1\right]+o(\sigma_C^3). \tag{A.78}$$

Using equation (A.78), we obtain the following expression for the risk premium:

$$\begin{aligned} \mathbb{E}^U[C]-P_{com}(s)e^{rT} &= P_C(\gamma_I\sigma_0^2 s/h-\mu_0,\sigma_{inc})-P_C(\gamma_I\sigma_0^2 s/h-\mu_0,\sigma_{com}) \\ &\approx \frac{\sigma_C^2}{\sigma_{inc}^2}(\gamma_I\sigma_0^2 s/h-\mu_0-\mu_C)+\frac{m_3\sigma_C^3}{2\sigma_{inc}^2}\left[\left(\frac{\gamma_I\sigma_0^2 s/h-\mu_0-\mu_C}{\sigma_{inc}}\right)^2-1\right] \\ &\quad -\frac{\sigma_C^2}{\sigma_{com}^2}(\gamma_I\sigma_0^2 s/h-\mu_0-\mu_C)-\frac{m_3\sigma_C^3}{2\sigma_{com}^2}\left[\left(\frac{\gamma_I\sigma_0^2 s/h-\mu_0-\mu_C}{\sigma_{com}}\right)^2-1\right], \end{aligned}$$

where σ_{inc} and σ_{com} are given in (A.72). The above expression depends on a random sufficient statistic s. Next, we take the expectation with regard to s, and taking into account that $\mathbb{E}[\gamma_I\sigma_0^2 s/h-\mu_0-\mu_C]=0$ and $\text{var}[s]=h^2\sigma_\varepsilon^2/(\gamma_I^2\sigma_0^4)+\sigma_\nu^2$, we obtain:

$$\overline{\mathbb{E}^U[C]-P_{com}(s)e^{rT}}\approx-\frac{m_3\sigma_C^3}{2}\frac{h/\gamma_I}{h/\gamma_I+(1-h)/\gamma_U}\left(\frac{\sigma_C^2}{\sigma_0^2}\left(\frac{1}{\sigma_{inc}^2}+\frac{1}{\sigma_{com}^2}\right)\frac{1}{1+h^2/(\gamma_I^2\sigma_\nu^2\sigma_0^2)}+\frac{\gamma_I^2\sigma_0^2\sigma_\nu^2/h^2}{\sigma_{com}^2}\right).$$

The above expression shows that $\overline{\mathbb{E}^U[C]-P_{com}(s)e^{rT}}\approx-m_3\sigma_C^3A(h)$, where $A(h)>0$. To further simplify the expression for the risk premium, we ignore terms of the order σ_C^5, and assume that $\gamma_U=\gamma_I=\gamma$. Then, after some algebra, we obtain:

$$\begin{aligned}\overline{\mathbb{E}^U[C]-P_{com}(s)e^{rT}} &= -\frac{m_3\sigma_C^3}{2}\frac{h/\gamma}{h/\gamma+(1-h)/\gamma}\left(\frac{\gamma^2\sigma_0^2\sigma_\nu^2/h^2}{\sigma_{com}^2}\right)+o(\sigma_C^3)\\ &= -\frac{m_3\sigma_C^3}{2}\left(\gamma^2\sigma_\nu^2+\frac{h(1-h)}{\sigma_0^2+h^2/(\gamma^2\sigma_\nu^2)}\right)+o(\sigma_C^3).\quad\blacksquare\end{aligned}$$

Proof of Proposition 9. Differentiating equation (30) for the incomplete market price, we obtain

$$\frac{\partial}{\partial\nu}\left(\frac{hf_I^{-1}(e^{rT}P)}{\gamma_I}+\frac{(1-h)f_U^{-1}(e^{rT}P)}{\gamma_I}\right)=E+Q. \tag{A.79}$$

Next, we use the fact that $\partial f^{-1}(y)/\partial y=f'(x)^{-1}$. We note that $x_I=f_I^{-1}(e^{rT}P)$ and $x_U=f_U^{-1}(e^{rT}P)$ are given by equations (A.38). Then, we differentiate functions $f_I(x)$ and $f_U(x)$ using equation (A.40) in Lemma A.3 and find that $f_I'(x_I)=\text{var}^{\text{RN},I}[C]$ and $f_U'(x_U)=\text{var}^{\text{RN},U}[C]$. Substituting the latter derivatives into equation (A.79), after some algebra, we obtain the expression for $\partial P/\partial\nu$ for the incomplete market in equation (49). Equation (48) for the complete market is a special case of (49) when $\text{var}^{\text{RN},I}[C]=\text{var}^{\text{RN},U}[C]=\text{var}^{\text{RN}}[C]$, because the risk-neutral measure in the complete market is unique.

Next, suppose, the investors have symmetric information. That is, the investors have common probability measure Q. Then, the first order conditions and the market clearing imply the following equation for asset prices:

$$p=\frac{\mathbb{E}^Q[C\exp\{C^\top\nu/(h/\gamma_I+(1-h)/\gamma_U)\}]}{\mathbb{E}^Q[\exp\{C^\top\nu/(h/\gamma_I+(1-h)/\gamma_U)\}]}. \tag{A.80}$$

From equation (A.80) we observe that introducing a new asset m with $\nu_m=0$ does not affect the prices of other assets. Proposition IA6.3 in the Supplementary Appendix shows that asset prices are the same in incomplete and effectively complete markets when C is normally distributed. Hence, the illiquidity is not affected by introducing derivatives in the latter two economies, provided that noise traders do not trade these derivatives. $\blacksquare$

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