# First-to-default and second-to-default options in models with various information flows*

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We continue to study a credit risk model of a financial market introduced recently by the authors, in which the dynamics of intensity rates of two default times are described by linear combinations of three independent geometric Brownian motions. The dynamics of two default-free risky asset prices are modeled by two geometric Brownian motions that are not independent of the ones describing the default intensity rates. We obtain closed form expressions for the no-arbitrage prices of some first-to-default and second-to-default European style contingent claims given the reference filtration initially and progressively enlarged by the two successive default times. The accessible default-free reference filtration is generated by the standard Brownian motions driving the model.

## 1 Introduction

In this paper, we derive closed form expressions for the (no-arbitrage) prices of first-to-default and second-to-default European style contingent claims in a model of a financial market introduced in [19] given the flows of information which are expressed by the reference filtration progressively and initially enlarged by means of the *successive* default times. It is assumed that the option payoffs depend on the default times and the current prices of the underlying default-free risky assets taken at the times of defaults. The dynamics of market prices of the two risky assets are described by geometric Brownian motions driven by constantly correlated standard Brownian motions. The default times are given by the first times at which linear combinations of three integral processes of independent geometric Brownian motions hit certain random thresholds which are independent of each other and of the standard Brownian motions driving

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the model. The dependence between the default times is then expressed by means of the dynamics of their intensity rates given by linear combinations of the three independent geometric Brownian motions which are driven by standard Brownian motions constantly correlated with the ones related to the risky asset prices. The default-free reference filtration accessible from the market is generated by the standard Brownian motions driving the model. The prices of the resulting defaultable European style contingent claims are explicitly expressed through the transition densities of the marginal distributions of the geometric Brownian motions and their integral processes describing the model.

The credit risk models in which the default times are defined as the first times at which the associated cumulative intensity processes reach certain random thresholds were initiated by Lando [22]. The computations of conditional distributions of the default times given the observable filtrations in such a first passage intensity model with independent default intensities and correlated thresholds were presented in Schönbucher [24; Chapter X, Proposition 10.9]. Brigo and Chourdakis [7] studied the problem of pricing of credit default swaps (CDSs) in such a model with counterparty risk in which the intensities of the default times are independent of each other, but the associated random thresholds are correlated. Brigo, Capponi and Pallavicini [6] developed the pricing framework for bilateral counterparty credit risk models and specified the credit and debit valuation adjustments (CVAs and DVAs) in the cases in which the default intensity rates are expressed by means of the (strictly positive) Feller's square root diffusion processes, and the associated thresholds are correlated through a Gaussian copula. Bielecki et al. [3] provided the analytic basis for the quantitative methodology of dynamic hedging of the counterparty risk and developed the main theoretical issues of dynamic hedging of credit valuation adjustments. Assefa et al. [1] derived a model-free general counterparty risk representation formula for the credit valuation adjustment of a netted and collateralised portfolio. Some related discussions on modelling and computational aspects regarding managing of exposure to counterparty risk are provided in the monographs by Gregory [21], Cesari, Aquilina and Charpillon [10], Brigo, Morini and Pallavicini [8], and Crépey, Bielecki and Brigo [11].

El Karoui, Jeanblanc and Jiao [15]-[16] emphasised the roles of conditional distributions of several default times in the intensity credit risk models given the appropriate filtrations and presented general expressions for the prices of various defaultable European style contingent claims. In this paper, we consider a model in which the default intensity rates are explicitly given as linear combinations of three independent geometric Brownian motions which are not independent of the ones describing the dynamics of the risky asset price processes. We then use the Markov property of the resulting multi-dimensional process describing the model and apply the explicit formula from Yor [26] for the joint marginal density of a geometric Brownian motion and its integral process to derive closed form expressions for the prices of first-to-default and second-to-default options given the reference filtration progressively and initially enlarged by means of the default times. We also note that the model proposed in the paper keeps its Markovian feature in the filtrations which are obtained by means of the progressive and initial enlargements of the initial Brownian reference filtration. The results of this paper can naturally be extended to the case of credit risk models with more than two default times and more than two underlying risky assets of a similar dependence structure. The prices of first-to-default and second-to-default options and other European style defaultable contingent claims can then be expressed through the transition densities of the marginal distributions of the resulting multi-dimensional continuous Markov process describing the model. The prices of other defaultable contingent claims in some switching models with partial information were recently computed in [18].

The paper is organised as follows. In Section 2, we reproduce the credit risk model of a financial market introduced in [19; Section 2] with the dependence structure of the dynamics of prices of two risky assets and two default intensity rates described above. In Section 3, we derive explicit expressions for the conditional distributions of the two *successive* default times given the accessible default-free reference filtration and the observable filtrations. In Section 4, we compute closed form expressions for the prices of first-to-default and second-to-default options in the model with two underlying risky assets given the reference filtration progressively and initially enlarged by the ordered default times. In Section 5, we recall explicit expressions from [19; Sections 3 and 4] for the conditional distributions of the two *non-successive* default times given the accessible default-free reference filtration and the observable filtrations, these results being used in the previous sections. The main results of the paper are stated in Propositions 4.1-4.3.

# 2 The model

In this section, we reproduce the model of a financial market with two defaultable risky assets introduced in [19; Section 2]. We also define the accessible default-free reference filtration as well as the observable filtrations and refer some known results and distribution laws.

## 2.1 The dynamics of default intensities and firm values

Let $(\Omega, \mathcal{G}, P)$ be a probability space supporting independent standard Brownian motions $W^l = (W^l_t)_{t \geq 0}$ and $B^l = (B^l_t)_{t \geq 0}$, $l = 0, 1, 2$, as well as the random variables $U_i$, $i = 1, 2$, which are uniformly distributed on $(0, 1)$. Suppose that the variables $U_i$, $i = 1, 2$, are independent of each other and of the processes $W^l$ and $B^l$, $l = 0, 1, 2$. We define the random times $\tau_i$, $i = 1, 2$, by

$$\tau_i = \inf \left\{ t \geq 0 \, \middle| \, \delta_i A^0_t + \lambda_i A^i_t \geq - \ln U_i \right\} \tag{2.1}$$

where the processes $A^l = (A^l_t)_{t \geq 0}$, $l = 0, 1, 2$, are given by

$$A^l_t = \int_0^t Y^l_s \, ds \tag{2.2}$$

for all $t \geq 0$, and some $\delta_i, \lambda_i \geq 0$, $i = 1, 2$, fixed, so that the processes $(\delta_i A^0_t + \lambda_i A^i_t)_{t \geq 0}$, $i = 1, 2$, form the *cumulative intensities*, and the processes $(\delta_i Y^0_t + \lambda_i Y^i_t)_{t \geq 0}$, $i = 1, 2$, are the *intensity rates* of the random times $\tau_i$, $i = 1, 2$. These notions mean that the processes $(I(\tau_i \leq t) - \delta_i A^0_{t \wedge \tau_i} - \lambda_i A^i_{t \wedge \tau_i})_{t \geq 0}$, $i = 1, 2$, are martingales in their natural filtrations. Assume that the processes $Y^l = (Y^l_t)_{t \geq 0}$, $j = 0, 1, 2$, admit the representations

$$Y^l_t = \exp\left( \left( \beta_l - \frac{\gamma_l^2}{2} \right) t + \gamma_l \, W^l_t \right) \tag{2.3}$$

for all $t \geq 0$, and some constants $\beta_l \in \mathbb{R}$ and $\gamma_l > 0$, $l = 0, 1, 2$. Note that the random times $\tau_i$, $i = 1, 2$, defined in (2.1) with (2.2) and (2.3) can occur simultaneously only with probability zero, and thus, the property $P(\tau_1 = \tau_2) = 0$ holds, by construction, that we take into account in the sequel, that is, noting that the events $\{\tau_i < \tau_{3-i}\}$ and $\{\tau_i \le \tau_{3-i}\}$ are equal ($P$-a.s.), for every $i = 1, 2$.

Suppose that the random times $\tau_i$, $i = 1, 2$, represent the default times of two firms (reference credits) with the value dynamics described by the processes $X^i = (X^i_t)_{t \ge 0}$, $i = 1, 2$, given by $X^i_t = (Y^i_t)^{\alpha_i} (Z^0_t)^{\zeta_i} Z^i_t$, for some $\alpha_i$ and $\zeta_i \in \mathbb{R}$, $i = 1, 2$, fixed. Here, the processes $Z^l = (Z^l_t)_{t \ge 0}$, $l = 0, 1, 2$, are defined by

$$Z^l_t = \exp\left( \left( \eta_l - \frac{\theta_l^2}{2} \right) t + \theta_l \, B^l_t \right) \tag{2.4}$$

for all $t \ge 0$, and some constants $\eta_l \in \mathbb{R}$ and $\theta_l > 0$, $l = 0, 1, 2$. We further assume that the discounted firm value processes $(e^{-rt} X^i_t)_{t \ge 0}$, $i = 1, 2$, are martingales with respect to the pricing measure $P$ under which the processes $Y^l$ and $Z^l$, $l = 0, 1, 2$, admit the representations in (2.3) and (2.4), where $r \ge 0$ is the interest rate of a riskless bank account. Thus, taking into account the independence of the driving processes $W^l$ and $B^l$, $l = 0, 1, 2$, we may conclude that the equality

$$\beta_i \alpha_i + \frac{\gamma_i^2}{2} \, \alpha_i(\alpha_i - 1) + \eta_0 \zeta_i + \frac{\theta_0^2}{2} \, \zeta_i(\zeta_i - 1) + \eta_i = r \tag{2.5}$$

should hold, for every $i = 1, 2$.

## 2.2 Some filtrations and distribution laws

Let us denote by $(\mathcal{F}_t)_{t \ge 0}$ the natural filtration of the processes $Y^l$ and $Z^l$, $l = 0, 1, 2$, defined by $\mathcal{F}_t = \sigma(Y^l_t, Z^l_t \,|\, 0 \le s \le t, l = 0, 1, 2)$, for all $t \ge 0$, which coincides with the one of the driving standard Brownian motions $W^l$ and $B^l$, $l = 0, 1, 2$, given by $\sigma(W^l_t, B^l_t \,|\, 0 \le s \le t, l = 0, 1, 2)$, for all $t \ge 0$. We define the progressively enlarged filtrations $(\mathcal{G}^i_t)_{t \ge 0}$, $i = 1, 2$, by $\mathcal{G}^i_t = \mathcal{F}_t \vee \sigma(\tau_i \wedge t)$, and $(\mathcal{G}_t)_{t \ge 0}$ by $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau_1 \wedge t) \vee \sigma(\tau_2 \wedge t)$, for all $t \ge 0$. Let us also introduce the initially enlarged filtrations $(\mathcal{F}^i_t)_{t \ge 0}$, $i = 1, 2$, by $\mathcal{F}^i_t = \mathcal{F}_t \vee \sigma(\tau_i)$, for all $t \ge 0$. We actually consider the smallest right-continuous completed filtrations that contain the appropriate filtrations defined above. The default-free reference filtration $(\mathcal{F}_t)_{t \ge 0}$ reflects the information flow which is accessible for the investors trading in the market, while the filtrations $(\mathcal{G}^i_t)_{t \ge 0}$, $i = 1, 2$, and $(\mathcal{G}_t)_{t \ge 0}$ reflect the accessible information including the one about the appearance of the default times. Note that, by virtue of the independence of the random variables $U_i$, $i = 1, 2$, and the filtration $(\mathcal{F}_t)_{t \ge 0}$, it follows that $(\mathcal{F}_t)_{t \ge 0}$ is immersed in the filtrations $(\mathcal{G}^i_t)_{t \ge 0}$, $i = 1, 2$, and $(\mathcal{G}_t)_{t \ge 0}$ (see, e.g., [5] and [17]). Similarly, we also have that $(\mathcal{G}^i_t)_{t \ge 0}$ is immersed in the filtration $(\mathcal{G}^i_t \vee \sigma(U_{3-i}))_{t \ge 0}$, and hence, in $(\mathcal{G}_t)_{t \ge 0}$, for every $i = 1, 2$. We recall that the immersion of a filtration in a larger filtration, also known as the $(H)$-hypothesis for the two nested filtrations, means that any martingale for the smaller filtration is a martingale for the larger one (see, e.g., [5], [23; Chapter V, Section 4], [4; Chapter VIII, Section 3], or [2; Chapter III]). Note that the immersion of $(\mathcal{F}_t)_{t \ge 0}$ in $(\mathcal{G}^i_t)_{t \ge 0}$ is equivalent to the conditional independence of $\mathcal{G}^i_t$ and $\mathcal{F}_\infty$ with respect to $\mathcal{F}_t$, for all $t \ge 0$, for $i = 1, 2$, while the immersion of $(\mathcal{F}_t)_{t \ge 0}$ in $(\mathcal{G}_t)_{t \ge 0}$ is equivalent to the conditional independence of $\mathcal{G}_t$ and $\mathcal{F}_\infty$ with respect to $\mathcal{F}_t$, for all $t \ge 0$ (see, e.g., [13]).

We now define the random times $\varkappa_1 = \tau_1 \wedge \tau_2$ and $\varkappa_2 = \tau_1 \vee \tau_2$. Along with the filtrations introduced above, let us define the progressively enlarged filtrations $(\mathcal{H}^i_t)_{t \ge 0}$, for $i = 1, 2$, by

$\mathcal{H}_t^i = \mathcal{F}_t \vee \sigma(\varkappa_i \wedge t)$, for all $t \geq 0$. We introduce $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(\varkappa_1 \wedge t) \vee \sigma(\varkappa_2 \wedge t)$, so that $\mathcal{H}_t \subsetneq \mathcal{G}_t$, for all $t \geq 0$, since the filtration $(\mathcal{H}_t)_{t\geq 0}$ does not contain information which default time $\tau_i$, $i = 1, 2$, occurred the first and which occurred the second, except in the trivial case in which the default times $\tau_i$, $i = 1, 2$, are ordered. We also consider the initially enlarged filtrations $\mathcal{H}_t^i \vee \sigma(\varkappa_{3-i})$, for $i = 1, 2$. By virtue of the same arguments as before, we conclude that $(\mathcal{F}_t)_{t\geq 0}$ is immersed in $(\mathcal{H}_t^i)_{t\geq 0}$, $i = 1, 2$, and in $(\mathcal{H}_t)_{t\geq 0}$.

## 2.3 Some implications of the key lemma

Let us now consider a filtration $(\mathcal{K}_t)_{t\geq 0}$ larger than the filtration $(\mathcal{F}_t)_{t\geq 0}$, that is, $\mathcal{F}_t \subseteq \mathcal{K}_t$, for all $t \geq 0$. Then, if $\mathcal{K}_t$ coincides with $\mathcal{F}_t$ on the event $J_t \in \mathcal{K}_t$ such that $P(J_t) > 0$, that is, if for any $K_t \in \mathcal{K}_t$, there exists an event $F_t \in \mathcal{F}_t$ such that $J_t \cap K_t = J_t \cap F_t$, then, on the event $J_t$, the conditional expectation $E[V \,|\, \mathcal{K}_t]$ of an integrable random variable $V$ is equal to an $\mathcal{F}_t$-measurable random variable. Hence, denoting by $I(J)$ the indicator function of the set $J$, according to the results in [12; page 122] and [4; Section 5.1], this fact leads to the equality

$$I(J_t)\, E\big[V \,\big|\, \mathcal{K}_t\big]\, P(J_t \,|\, \mathcal{F}_t) = I(J_t)\, E\big[V\, I(J_t) \,\big|\, \mathcal{F}_t\big] \tag{2.6}$$

and thus, taking into account the fact that $P(J_t \,|\, \mathcal{F}_t) > 0$ on the event $J_t$, we have

$$I(J_t)\, E\big[V \,\big|\, \mathcal{K}_t\big] = I(J_t)\, \frac{E[V I(J_t) \,|\, \mathcal{F}_t]}{P(J_t \,|\, \mathcal{F}_t)} \tag{2.7}$$

for any integrable random variable $V$ and all $t \geq 0$. We further refer to the result in (2.6)-(2.7) as to the *generalised key lemma* for the filtrations $(\mathcal{K}_t)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$. Observe that $\mathcal{G}_t^i$ coincides with $\mathcal{F}_t$ on the event $\{\tau_i > t\}$, and $\mathcal{G}_t$ coincides with $\mathcal{F}_t$ on the event $\{\tau_i \wedge \tau_{3-i} > t\}$, while $\mathcal{G}_t^i \vee \sigma(\tau_{3-i})$ coincides with $\mathcal{F}_t^{3-i} \equiv \mathcal{F}_t \vee \sigma(\tau_{3-i})$ on the event $\{\tau_i > t\}$, for all $t \geq 0$ and every $i = 1, 2$. In these cases, the expressions in (2.6)-(2.7), together with the tower property for conditional expectations, imply that, for each $\mathcal{F}_T$-measurable integrable random variable $V_T^i$, the equality

$$I(\tau_i > t)\, E\big[V_T^i \,\big|\, \mathcal{G}_t^i\big] = I(\tau_i > t)\, \frac{E[V_T^i P(\tau_i > t \,|\, \mathcal{F}_T) \,|\, \mathcal{F}_t]}{P(\tau_i > t \,|\, \mathcal{F}_t)} \tag{2.8}$$

holds, for all $t \geq 0$ and every $i = 1, 2$ (see, e.g., [2; Lemma 2.9]). Moreover, it follows that, for each $(\mathcal{F}_t)_{t\geq 0}$-progressively measurable process $V^i = (V_t^i)_{t\geq 0}$, the equality

$$E\big[V_{\tau_i}^i\, I(\tau_i > t) \,|\, \mathcal{G}_t^i\big] = I(\tau_i > t)\, E\bigg[\int_t^\infty \frac{V_u^i P(\tau_i \in du \,|\, \mathcal{F}_u)}{P(\tau_i > t \,|\, \mathcal{F}_t)} \,\bigg|\, \mathcal{F}_t\bigg] \tag{2.9}$$

holds, for all $t \geq 0$ and every $i = 1, 2$ (see, e.g. [2; Corollary 2.10]). We further refer to the results in (2.8) and (2.9) as to the *first* and the *second part of the key lemma* for the filtrations $(\mathcal{G}_t^i)_{t\geq 0}$ and $(\mathcal{F}_t)_{t\geq 0}$, for every $i = 1, 2$.

For any Borelian bounded function $\psi_i$, let us now compute the conditional expectation $E[\psi_i(\tau_i) \,|\, \mathcal{F}_t \vee \sigma(\tau_{3-i})]$, for all $t \geq 0$ and every $i = 1, 2$. For this purpose, we apply the result of [9; Proposition 2.7] to conclude that any $(\mathcal{F}_t \vee \sigma(\tau_{3-i}))_{t\geq 0}$-progressively measurable process can be written as $\Phi_t^i(\tau_{3-i})$, where $\Phi^i(v) = (\Phi_t^i(v))_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-progressively measurable, for any $v \geq 0$ fixed, while the function $v \mapsto \Phi_t^i(v)$ is Borel measurable, for all $t \geq 0$ and every $i = 1, 2$. In particular, there exists $\Psi^i$ with the above measurability properties such that

$$E\big[\psi_i(\tau_i) \,|\, \mathcal{F}_t \vee \sigma(\tau_{3-i})\big] = \Psi_t^i(\tau_{3-i}) \tag{2.10}$$

for all $t \geq 0$ and every $i = 1, 2$. Then, we observe that, by definition of conditional expectations, for any event $F_t \in \mathcal{F}_t$, and any Borelian bounded function $\varphi$, the equality

$$E\bigg[\Psi_t^i(\tau_{3-i}) \, I(F_t) \, \varphi(\tau_{3-i})\bigg] = E\bigg[I(F_t) \, \psi_i(\tau_i) \, \varphi(\tau_{3-i})\bigg] \tag{2.11}$$

holds, and thus, we have

$$\begin{aligned} &E\bigg[\int_{v=0}^{\infty} \Psi_t^i(v) \, I(F_t) \, \varphi(v) \, P(\tau_{3-i} \in dv \,|\, \mathcal{F}_t)\bigg] \\ &= E\bigg[I(F_t) \int_{u=0}^{\infty} \int_{v=0}^{\infty} \psi_i(u) \, \varphi(v) \, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_t)\bigg] \end{aligned} \tag{2.12}$$

for all $t \geq 0$ and every $i = 1, 2$. Hence, the equality in (2.12) being valid for any Borelian bounded function $\varphi$ and the conditional law of $\tau_{3-i}$ being absolutely continuous with respect to Lebesque's measure imply that the equality

$$\Psi_t^i(v) = \int_{u=0}^{\infty} \frac{\psi_i(u) P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_t)}{P(\tau_{3-i} \in dv \,|\, \mathcal{F}_t)} \tag{2.13}$$

is satisfied, for all $t, v \geq 0$, and every $i = 1, 2$.

Similarly, for any Borelian bounded functions $\widetilde{\psi}_i$ and $\widetilde{\xi}_i$, let us now compute the conditional expectations $E[\widetilde{\psi}_i(\tau_i) \, I(\tau_i > \tau_{3-i}) \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1)]$ and $E[\widetilde{\xi}_i(\tau_i) \, I(\tau_i < \tau_{3-i}) \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1)]$, for all $t \geq 0$ and every $i = 1, 2$. We apply again the result of [9; Proposition 2.7] to conclude that any $(\mathcal{F}_t \vee \sigma(\varkappa_1))_{t \geq 0}$-progressively measurable process can be written as $\widetilde{\Phi}_t^i(\varkappa_1)$, where $\widetilde{\Phi}^i(u) = (\widetilde{\Phi}_t^i(u))_{t \geq 0}$ is $(\mathcal{F}_t)_{t \geq 0}$-progressively measurable, for any $u \geq 0$ fixed, while the function $u \mapsto \widetilde{\Phi}_t^i(u)$ is Borel measurable, for all $t \geq 0$ and every $i = 1, 2$. In particular, there exist $\widetilde{\Psi}^i$ and $\widetilde{\Xi}^i$ such that

$$E\big[\widetilde{\psi}_i(\tau_i) \, I(\tau_i > \tau_{3-i}) \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1)\big] = \widetilde{\Psi}_t^i(\varkappa_1) \tag{2.14}$$

and

$$E\big[\widetilde{\xi}_i(\tau_i) \, I(\tau_i < \tau_{3-i}) \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1)\big] = \widetilde{\Xi}_t^i(\varkappa_1) \tag{2.15}$$

for all $t \geq 0$ and every $i = 1, 2$. Then, we observe that, by definition of conditional expectations, for any event $F_t \in \mathcal{F}_t$, and any (positive measurable) bounded function $\widetilde{\varphi}$, the equalities

$$\begin{aligned} &E\bigg[\int_{u=0}^{\infty} \int_{v=0}^{\infty} \widetilde{\psi}_i(v) \, I(F_t) \, \widetilde{\varphi}(u) \, I(u < v) \, P(\tau_{3-i} \in du, \tau_i \in dv \,|\, \mathcal{F}_t)\bigg] \\ &= E\bigg[I(F_t) \int_{u=0}^{\infty} \int_{v=0}^{\infty} \widetilde{\Psi}_t^i(u) \, \widetilde{\varphi}(u) \, I(u < v) \, \big(P(\tau_1 \in du, \tau_2 \in dv \,|\, \mathcal{F}_t) + P(\tau_2 \in du, \tau_1 \in dv \,|\, \mathcal{F}_t)\big)\bigg] \\ &= E\bigg[I(F_t) \int_{u=0}^{\infty} \widetilde{\Psi}_t^i(u) \, \widetilde{\varphi}(u) \, \big(P(\tau_1 > u, \tau_2 \in du \,|\, \mathcal{F}_t) + P(\tau_2 > u, \tau_1 \in du \,|\, \mathcal{F}_t)\big)\bigg] \end{aligned} \tag{2.16}$$

and

$$E\Big[\int_{u=0}^{\infty}\int_{v=0}^{\infty}\widetilde{\xi}_i(u)\,I(F_t)\,\widetilde{\varphi}(u)\,I(u<v)\,P(\tau_i\in du,\tau_{3-i}\in dv\,|\,\mathcal{F}_t)\Big] \tag{2.17}$$
$$= E\bigg[I(F_t)\int_{u=0}^{\infty}\int_{v=0}^{\infty}\widetilde{\Xi}_t^i(u)\,\widetilde{\varphi}(u)\,I(u<v)\,\big(P(\tau_1\in du,\tau_2\in dv\,|\,\mathcal{F}_t)+P(\tau_2\in du,\tau_1\in dv\,|\,\mathcal{F}_t)\big)\bigg]$$
$$= E\bigg[I(F_t)\int_{u=0}^{\infty}\widetilde{\Xi}_t^i(u)\,\widetilde{\varphi}(u)\,\big(P(\tau_1>u,\tau_2\in du\,|\,\mathcal{F}_t)+P(\tau_2>u,\tau_1\in du\,|\,\mathcal{F}_t)\big)\bigg]$$

hold, for all $t\geq 0$ and every $i=1,2$. Hence, the equality in (2.16) being valid for any Borel function $\widetilde{\varphi}$, and the conditional law $P(\tau_i>du,\tau_{3-i}>u\,|\,\mathcal{F}_t)$ being absolutely continuous with respect to the Lebesque measure (see Subsection 5.2 below), the equalities

$$\widetilde{\Psi}_t^i(u)=\int_{v=0}^{\infty}\frac{\widetilde{\psi}_i(v)I(u<v)P(\tau_{3-i}\in du,\tau_i\in dv\,|\,\mathcal{F}_t)}{P(\tau_1>u,\tau_2\in du\,|\,\mathcal{F}_t)+P(\tau_2>u,\tau_1\in du\,|\,\mathcal{F}_t)} \tag{2.18}$$

and

$$\widetilde{\Xi}_t^i(u)=\frac{\widetilde{\xi}_i(u)P(\tau_{3-i}>u,\tau_i\in du\,|\,\mathcal{F}_t)}{P(\tau_1>u,\tau_2\in du\,|\,\mathcal{F}_t)+P(\tau_2>u,\tau_1\in du\,|\,\mathcal{F}_t)} \tag{2.19}$$

are satisfied, for all $t,u\geq 0$, and every $i=1,2$. Here, we note that the equality $P(\tau_1>u,\tau_2\in du\,|\,\mathcal{F}_t)+P(\tau_2>u,\tau_1\in du\,|\,\mathcal{F}_t)=P(\varkappa_1\in du\,|\,\mathcal{F}_t)$ holds, for all $t,u\geq 0$, which explain the meaning of the denominator.

Finally, for any Borelian bounded function $\widehat{\psi}_i$ and $\widehat{\xi}_i$, let us now compute the conditional expectations $E[\widehat{\psi}_i(\tau_i)\,I(\tau_i<\tau_{3-i})\,|\,\mathcal{F}_t\vee\sigma(\varkappa_2)]$ and $E[\widehat{\xi}_i(\tau_i)\,I(\tau_i>\tau_{3-i})\,|\,\mathcal{F}_t\vee\sigma(\varkappa_2)]$, for all $t\geq 0$ and every $i=1,2$. We apply again the result of [9; Proposition 2.7] to conclude that any $(\mathcal{F}_t\vee\sigma(\varkappa_2))_{t\geq 0}$-progressively measurable process can be written as $\widehat{\Phi}_t^i(\varkappa_2)$, where $\widehat{\Phi}^i(v)=(\widehat{\Phi}_t^i(v))_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-progressively measurable, for any $v\geq 0$ fixed, while the function $v\mapsto\widehat{\Phi}_t^i(v)$ is Borel measurable, for all $t\geq 0$ and every $i=1,2$. In particular, there exist $\widehat{\Psi}^i$ and $\widehat{\Xi}^i$ such that

$$E\big[\widehat{\psi}_i(\tau_i)\,I(\tau_i<\tau_{3-i})\,|\,\mathcal{F}_t\vee\sigma(\varkappa_2)\big]=\widehat{\Psi}_t^i(\varkappa_2) \tag{2.20}$$

and

$$E\big[\widehat{\xi}_i(\tau_i)\,I(\tau_i>\tau_{3-i})\,|\,\mathcal{F}_t\vee\sigma(\varkappa_2)\big]=\widehat{\Xi}_t^i(\varkappa_2) \tag{2.21}$$

for all $t\geq 0$ and every $i=1,2$. Then, we observe that, by definition of conditional expectations, for any event $F_t\in\mathcal{F}_t$, and any Borelian bounded function $\widehat{\varphi}$, the equalities

$$E\Big[\int_{u=0}^{\infty}\int_{v=0}^{\infty}\widehat{\psi}_i(u)\,I(F_t)\,\widehat{\varphi}(v)\,I(u<v)\,P(\tau_i\in du,\tau_{3-i}\in dv\,|\,\mathcal{F}_t)\Big] \tag{2.22}$$
$$= E\bigg[I(F_t)\int_{u=0}^{\infty}\int_{v=0}^{\infty}\widehat{\Psi}_t^i(v)\,\widehat{\varphi}(v)\,I(u<v)\,\big(P(\tau_1\in du,\tau_2\in dv\,|\,\mathcal{F}_t)+P(\tau_2\in du,\tau_1\in dv\,|\,\mathcal{F}_t)\big)\bigg]$$
$$= E\bigg[I(F_t)\int_{v=0}^{\infty}\widehat{\Psi}_t^i(v)\,\widehat{\varphi}(v)\,\big(P(\tau_1\leq v,\tau_2\in dv\,|\,\mathcal{F}_t)+P(\tau_2\leq v,\tau_1\in dv\,|\,\mathcal{F}_t)\big)\bigg]$$

and

$$E\Big[\int_{u=0}^{\infty}\int_{v=0}^{\infty}\widehat{\xi}_i(v)\,I(F_t)\,\widehat{\varphi}(v)\,I(u<v)\,P(\tau_{3-i}\in du,\tau_i\in dv\,|\,\mathcal{F}_t)\Big] \tag{2.23}$$
$$= E\Big[I(F_t)\int_{u=0}^{\infty}\int_{v=0}^{\infty}\widehat{\Xi}_t^i(v)\,\widehat{\varphi}(v)\,I(u<v)\,\big(P(\tau_1\in du,\tau_2\in dv\,|\,\mathcal{F}_t)+P(\tau_2\in du,\tau_1\in dv\,|\,\mathcal{F}_t)\big)\Big]$$
$$= E\Big[I(F_t)\int_{v=0}^{\infty}\widehat{\Xi}_t^i(v)\,\widehat{\varphi}(v)\,\big(P(\tau_1\le v,\tau_2\in dv\,|\,\mathcal{F}_t)+P(\tau_2\le v,\tau_1\in dv\,|\,\mathcal{F}_t)\big)\Big]$$

hold, for all $t\ge 0$ and every $i=1,2$. Hence, the equality in (2.22) being valid for any Borelian bounded function $\widehat{\varphi}$, and the conditional law $P(\tau_i\le v,\tau_{3-i}\in dv\,|\,\mathcal{F}_t)$ being absolutely continuous with respect to the Lebesque measure (see Subsection 5.2 below), the equalities

$$\widehat{\Psi}_t^i(v)=\int_{u=0}^{\infty}\frac{\widehat{\psi}_i(u)I(u<v)P(\tau_i\in du,\tau_{3-i}\in dv\,|\,\mathcal{F}_t)}{P(\tau_1\le v,\tau_2\in dv\,|\,\mathcal{F}_t)+P(\tau_2\le v,\tau_1\in dv\,|\,\mathcal{F}_t)} \tag{2.24}$$

and

$$\widehat{\Xi}_t^i(v)=\frac{\widehat{\xi}_i(v)P(\tau_{3-i}\le v,\tau_i\in dv\,|\,\mathcal{F}_t)}{P(\tau_1\le v,\tau_2\in dv\,|\,\mathcal{F}_t)+P(\tau_2\le v,\tau_1\in dv\,|\,\mathcal{F}_t)} \tag{2.25}$$

are satisfied, for all $t,v\ge 0$, and every $i=1,2$. Here, we note that the equality $P(\tau_1\le v,\tau_2\in dv\,|\,\mathcal{F}_t)+P(\tau_2\le v,\tau_1\in dv\,|\,\mathcal{F}_t)=P(\varkappa_2\in dv\,|\,\mathcal{F}_t)$ holds, for all $t,v\ge 0$, which explain the meaning of the denominator.

# 3 Conditional distributions of the default times

In this section, we derive explicit expressions for the conditional distributions of two *successive* default times given the accessible filtration generated by the market prices of the risky assets as well as given the observable filtrations.

## 3.1 Conditional distributions of $\varkappa_j$, $j=1,2$, under $(\mathcal{F}_t)_{t\ge 0}$

Let us now compute the conditional distributions $P(\varkappa_1>u,\varkappa_2>v\,|\,\mathcal{F}_t)$ of the successive default times $\varkappa_j$, $j=1,2$, given the reference filtration $(\mathcal{F}_t)_{t\ge 0}$, for all $t,u,v\ge 0$. We first observe that the equalities

$$P(\varkappa_1>u,\varkappa_2>v\,|\,\mathcal{F}_t) \tag{3.1}$$
$$=\int_u^{\infty}\int_v^{\infty}I(u'<v')\,\big(P(\tau_1\in du',\tau_2\in dv'\,|\,\mathcal{F}_t)+P(\tau_2\in du',\tau_1\in dv'\,|\,\mathcal{F}_t)\big)$$

hold, for all $t,u,v\ge 0$, where the conditional probabilities $P(\tau_i\in du',\tau_{3-i}\in dv'\,|\,\mathcal{F}_t)$, for all $t,u',v'\ge 0$ every $i=1,2$, are given in Subsection 5.2 below in the expressions of (5.17), (5.19) and (5.22), according to the positions of $u',v'$ with respect to $t$. Moreover, it follows from the expressions in (5.17) that the equalities

$$
\begin{aligned}
&P(\varkappa_1 \in du, \varkappa_2 \in dv \,|\, \mathcal{F}_\infty) \qquad (3.2)\\
&= \big(e^{-\delta_1 A_u^0 - \lambda_1 A_u^1 - \delta_2 A_v^0 - \lambda_2 A_v^2} \, (\delta_1 Y_u^0 + \lambda_1 Y_u^1) \, (\delta_2 Y_v^0 + \lambda_2 Y_v^2) \\
&\quad + e^{-\delta_2 A_u^0 - \lambda_2 A_u^2 - \delta_1 A_v^0 - \lambda_1 A_v^1} \, (\delta_2 Y_u^0 + \lambda_2 Y_u^2) \, (\delta_1 Y_v^0 + \lambda_1 Y_v^1)\big) \, du dv \\
&= P(\varkappa_1 \in du, \varkappa_2 \in dv \,|\, \mathcal{F}_t) \quad \text{for} \quad 0 \le u < v \le t
\end{aligned}
$$

are satisfied. Furthermore, according to the tower property for conditional expectations, it follows from the representation in (3.2) that the equalities

$$
\begin{aligned}
&P(\varkappa_1 \in du, \varkappa_2 \in dv \,|\, \mathcal{F}_t) = E\big[P(\varkappa_1 \in du, \varkappa_2 \in dv \,|\, \mathcal{F}_v) \,\big|\, \mathcal{F}_t\big] \qquad (3.3)\\
&= E\big[\big(e^{-\delta_1 A_u^0 - \lambda_1 A_u^1 - \delta_2 A_v^0 - \lambda_2 A_v^2} \, (\delta_1 Y_u^0 + \lambda_1 Y_u^1) \, (\delta_2 Y_v^0 + \lambda_2 Y_v^2) \\
&\quad + e^{-\delta_2 A_u^0 - \lambda_2 A_u^2 - \delta_1 A_v^0 - \lambda_1 A_v^1} \, (\delta_2 Y_u^0 + \lambda_2 Y_u^2) \, (\delta_1 Y_v^0 + \lambda_1 Y_v^1)\big) \,\big|\, \mathcal{F}_t\big] \, du dv \\
&= \big(e^{-\delta_1 A_u^0 - \lambda_1 A_u^1 - \delta_2 A_t^0 - \lambda_2 A_t^2} \, (\delta_1 Y_u^0 + \lambda_1 Y_u^1) \, D_{v-t}^2(Y_t^0, Y_t^2) \\
&\quad + e^{-\delta_2 A_u^0 - \lambda_2 A_u^2 - \delta_1 A_t^0 - \lambda_1 A_t^1} \, (\delta_2 Y_u^0 + \lambda_2 Y_u^2) \, D_{v-t}^1(Y_t^0, Y_t^1)\big) \, du \, dv \\
&\text{for} \quad 0 \le u \le t < v
\end{aligned}
$$

hold, where $D_{v-t}^1(y_0, y_1)$ and $D_{v-t}^2(y_0, y_2)$, are given as in (5.15) below. Finally, taking into account the representation in (3.3), according to the tower property for conditional expectations, we obtain that the equalities

$$
\begin{aligned}
&P(\varkappa_1 \in du, \varkappa_2 \in dv \,|\, \mathcal{F}_t) = E\big[E\big[P(\varkappa_1 \in du, \varkappa_2 \in dv \,|\, \mathcal{F}_v) \,\big|\, \mathcal{F}_u\big] \,\big|\, \mathcal{F}_t\big] \qquad (3.4)\\
&= E\big[e^{-(\delta_1+\delta_2) A_u^0 - \lambda_1 A_u^1 - \lambda_2 A_u^2} \\
&\qquad \big((\delta_1 Y_u^0 + \lambda_1 Y_u^1) \, D_{v-u}^2(Y_u^0, Y_u^2) + (\delta_2 Y_u^0 + \lambda_2 Y_u^2) \, D_{v-u}^1(Y_u^0, Y_u^1)\big) \,\big|\, \mathcal{F}_t\big] \, du dv \\
&= e^{-(\delta_1+\delta_2) A_t^0 - \lambda_1 A_t^1 - \lambda_2 A_t^2} \, E\big[e^{-(\delta_1+\delta_2) Y_t^0 (A_u^0 - A_t^0)/Y_t^0 - \lambda_1 Y_t^1 (A_u^1 - A_t^1)/Y_t^1 - \lambda_2 Y_t^2 (A_u^2 - A_t^2)/Y_t^2} \\
&\quad \times \big(\big(\delta_1 Y_t^0 (Y_u^0/Y_t^0) + \lambda_1 Y_t^1 (Y_u^1/Y_t^1)\big) \, D_{v-u}^2(Y_t^0 (Y_u^0/Y_t^0), Y_t^2 (Y_u^2/Y_t^2)) \\
&\qquad + \big(\delta_2 Y_t^0 (Y_u^0/Y_t^0) + \lambda_2 Y_t^2 (Y_u^2/Y_t^2)\big) \, D_{v-u}^1(Y_t^0 (Y_u^0/Y_t^0), Y_t^1 (Y_u^1/Y_t^1))\big) \,\big|\, \mathcal{F}_t\big] \, du dv \\
&= e^{-(\delta_1+\delta_2) A_t^0 - \lambda_1 A_t^1 - \lambda_2 A_t^2} \, \big(\overline{D}_{u-t,v-u}^2(Y_t^0, Y_t^2, Y_t^1) + \overline{D}_{u-t,v-u}^1(Y_t^0, Y_t^1, Y_t^2)\big) \, du dv \\
&\text{for} \quad 0 \le t \le u < v
\end{aligned}
$$

are satisfied, where $\overline{D}_{u-t,v-u}^1(y_0, y_1, y_2)$ and $\overline{D}_{u-t,v-u}^2(y_0, y_2, y_1)$, are given as in (5.23) below.

### 3.2 Conditional distributions of $\varkappa_j$, $j = 1, 2$, under $(\mathcal{H}_t^k)_{t \ge 0}$, $k = 1, 2$, and $(\mathcal{H}_t)_{t \ge 0}$

Let finally compute the conditional distributions $P(\varkappa_j > u \,|\, \mathcal{H}_t^k)$ of the successive default times $\varkappa_j$, $j = 1, 2$, given the filtration $(\mathcal{H}_t^k)_{t \ge 0}$, $k = 1, 2$, for all $t, u \ge 0$. In this case, we apply the first part of the key lemma in (2.8) for the filtrations $(\mathcal{H}_t^1)_{t \ge 0}$ and $(\mathcal{F}_t)_{t \ge 0}$, where $\mathcal{H}_t^1$ coincides with $\mathcal{F}_t \vee \sigma(\varkappa_1)$ on the event $\{\varkappa_1 \le t\}$ and with $\mathcal{F}_t$ on $\{\varkappa_1 > t\}$, for all $t \ge 0$, to get

$$
P(\varkappa_1 > u \,|\, \mathcal{H}_t^1) = I(u < \varkappa_1 \le t) + I(\varkappa_1 > t) \, \frac{P(\varkappa_1 > u \vee t \,|\, \mathcal{F}_t)}{P(\varkappa_1 > t \,|\, \mathcal{F}_t)} \quad \text{for} \quad t, u \ge 0 \qquad (3.5)
$$

where the conditional probability $P(\varkappa_1 > u \vee t \,|\, \mathcal{F}_t)$ is computed as in (3.1) above, while

$$P(\varkappa_2 > v \,|\, \mathcal{H}^1_t) = I(\varkappa_1 \le t)\, P(\varkappa_2 > v \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1)) + I(\varkappa_1 > t)\, \frac{P(\varkappa_1 > t, \varkappa_2 > v \,|\, \mathcal{F}_t)}{P(\varkappa_1 > t \,|\, \mathcal{F}_t)} \quad \text{for} \quad t, v \ge 0 \tag{3.6}$$

where the conditional probability $P(\varkappa_1 > t, \varkappa_2 > v \,|\, \mathcal{F}_t)$ is computed as in (3.1) above and, by means of the arguments applied for derivation of equalities in (2.14)-(2.19), we have

$$\begin{aligned} &P(\varkappa_2 > v \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1))[= P(\varkappa_2 > v \vee \varkappa_1 \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1))]] \\ &= P(\tau_1 > v, \tau_1 > \tau_2 \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1)) + P(\tau_2 > v, \tau_2 > \tau_1 \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1)) \\ &= \frac{P(\tau_1 > u \vee v, \tau_2 \in du \,|\, \mathcal{F}_t) + P(\tau_2 > u \vee v, \tau_1 \in du \,|\, \mathcal{F}_t)}{P(\tau_1 > u, \tau_2 \in du \,|\, \mathcal{F}_t) + P(\tau_2 > u, \tau_1 \in du \,|\, \mathcal{F}_t)} \Bigg|_{u=\varkappa_1} \\ &\quad \text{for} \quad t, v \ge 0 \end{aligned} \tag{3.7}$$

where the conditional densities $P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_t)$, for every $i = 1, 2$, are given in the expressions of (5.17), (5.19), (5.22) below.

Now, we apply the first part of the key lemma in (2.8) for the filtrations $(\mathcal{H}^2_t)_{t \ge 0}$ and $(\mathcal{F}_t)_{t \ge 0}$, where $\mathcal{H}^2_t$ coincides with $\mathcal{F}_t \vee \sigma(\varkappa_2)$ on the event $\{\varkappa_2 \le t\}$ and with $\mathcal{F}_t$ on $\{\varkappa_2 > t\}$, for all $t \ge 0$, to get

$$P(\varkappa_1 > u \,|\, \mathcal{H}^2_t) = I(\varkappa_2 \le t)\, P(\varkappa_1 > u \,|\, \mathcal{F}_t \vee \sigma(\varkappa_2)) + I(\varkappa_2 > t)\, \frac{P(\varkappa_1 > u, \varkappa_2 > t \,|\, \mathcal{F}_t)}{P(\varkappa_2 > t \,|\, \mathcal{F}_t)} \quad \text{for} \quad t, u \ge 0 \tag{3.8}$$

where the conditional probability $P(\varkappa_1 > u, \varkappa_2 > t \,|\, \mathcal{F}_t)$ is computed as in (3.1) above and, by means of the arguments applied for derivation of equalities in (2.20)-(2.25), we have

$$\begin{aligned} &P(\varkappa_1 > u \,|\, \mathcal{F}_t \vee \sigma(\varkappa_2))[= P(\varkappa_2 > \varkappa_1 > u \,|\, \mathcal{F}_t \vee \sigma(\varkappa_2))]] \\ &= P(\tau_1 > u, \tau_1 < \tau_2 \,|\, \mathcal{F}_t \vee \sigma(\varkappa_2)) + P(\tau_2 > u, \tau_2 < \tau_1 \,|\, \mathcal{F}_t \vee \sigma(\varkappa_2)) \\ &= \frac{P(u \wedge v < \tau_1 \le v, \tau_2 \in dv \,|\, \mathcal{F}_t) + P(u \wedge v < \tau_2 \le v, \tau_1 \in dv \,|\, \mathcal{F}_t)}{P(\tau_1 \le v, \tau_2 \in dv \,|\, \mathcal{F}_t) + P(\tau_2 \le v, \tau_1 \in dv \,|\, \mathcal{F}_t)} \Bigg|_{v=\varkappa_2} \\ &\quad \text{for} \quad t, u \ge 0 \end{aligned} \tag{3.9}$$

and the conditional densities $P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_t)$, for every $i = 1, 2$, are given in the expressions of (5.17), (5.19), (5.22) below, while

$$P(\varkappa_2 > v \,|\, \mathcal{H}^2_t) = I(v < \varkappa_2 \le t) + I(\varkappa_2 > t)\, \frac{P(\varkappa_2 > v \vee t \,|\, \mathcal{F}_t)}{P(\varkappa_2 > t \,|\, \mathcal{F}_t)} \quad \text{for} \quad t, v \ge 0 \tag{3.10}$$

where the conditional probability $P(\varkappa_2 > v \vee t \,|\, \mathcal{F}_t)$ is computed as in (3.1) above.

Finally, we apply the first part of the key lemma in (2.8) for the filtrations $(\mathcal{H}_t)_{t \ge 0}$ and $(\mathcal{F}_t)_{t \ge 0}$, where $\mathcal{H}_t$ coincides with $\mathcal{F}_t \vee \sigma(\varkappa_1) \vee \sigma(\varkappa_2)$ on the event $\{\varkappa_1 < \varkappa_2 \le t\}$, with $\mathcal{F}_t \vee \sigma(\varkappa_1)$

on $\{\varkappa_1 \le t < \varkappa_2\}$, and with $\mathcal{F}_t$ on $\{\varkappa_2 > \varkappa_1 > t\}$, for all $t \ge 0$, to get

$$P(\varkappa_1 > u, \varkappa_2 > v \,|\, \mathcal{H}_t) = I(u < \varkappa_1 < \varkappa_2 \le t, \varkappa_2 > v) \\ + I(u < \varkappa_1 \le t < \varkappa_2) \, \frac{P(\varkappa_2 > v \vee t \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1))}{P(\varkappa_2 > t \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1))} \\ + I(\varkappa_2 > \varkappa_1 > t) \, \frac{P(\varkappa_1 > u \vee t, \varkappa_2 > v \vee t \,|\, \mathcal{F}_t)}{P(\varkappa_2 > \varkappa_1 > t \,|\, \mathcal{F}_t)} \quad \text{for} \quad t, u, v \ge 0 \tag{3.11}$$

where the conditional probabilities $P(\varkappa_1 > u \vee t, \varkappa_2 > v \vee t \,|\, \mathcal{F}_t)$ and $P(\varkappa_2 > v \vee t \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1))$ are computed as in the expressions of (3.1) and (3.7) above, respectively.

# 4 The prices of first and second-to-default claims (Main results)

In this section, we derive explicit expressions for the prices of first and second-to-default options in the model defined above with some (non-negative measurable) deterministic recovery payoff functions $R_t(x_1, x_2)$, for all $0 \le t \le T$. In order to simplify the notations, without loss of generality, we further assume that the payoffs are already discounted by the dynamics of the bank account, that is equivalent to letting the interest rate $r$ equal to zero. We compute the prices for the option holders in various particular cases of available information contained in the filtrations $(\mathcal{H}_t^k)_{t \ge 0}$, or $(\mathcal{H}_t)_{t \ge 0}$, or $(\mathcal{H}_t^k \vee \sigma(\varkappa_{3-k}))_{t \ge 0}$ defined above, for every $k = 1, 2$.

In those cases, the option holders can observe only the default time $\varkappa_k$, or observe the both default times $\varkappa_k$, $k = 1, 2$, or observe the default time $\varkappa_k$ but know the default time $\varkappa_{3-k}$, for every $k = 1, 2$, from the beginning of observations, respectively.

Recall that the conditional probabilities $P(\varkappa_1 > u \vee t, \varkappa_2 > v \vee t \,|\, \mathcal{F}_t)$, for all $t, u, v \ge 0$, were computed in (3.1) above.

## 4.1 The case of filtrations $(\mathcal{H}_t^k)_{t \ge 0}$, $k = 1, 2$

Let us begin by computing the price $P^{j,k} = (P_t^{j,k})_{t \ge 0}$ for the holder of a first- and second-to-default option in the model with the filtration $(\mathcal{H}_t^k)_{t \ge 0}$ given by

$$P_t^{j,k} = E\big[R_{\varkappa_j}(X_{\varkappa_j}^1, X_{\varkappa_j}^2) \, I(t < \varkappa_j \le T) \,\big|\, \mathcal{H}_t^k\big] \tag{4.1}$$

for all $0 \le t \le T \wedge \varkappa_j$ and every $j, k = 1, 2$.

In order to compute closed-form expressions for $P^{1,k}$ in (4.1), we provide the decomposition

$$\begin{aligned} P_t^{1,k} &= E\big[R_{\varkappa_1}(X_{\varkappa_1}^1, X_{\varkappa_1}^2) \, I(t < \varkappa_1 \le T) \,\big|\, \mathcal{H}_t^k\big] \\ &= E\big[R_{\tau_i}(X_{\tau_i}^1, X_{\tau_i}^2) \, I(\tau_i < \tau_{3-i}, t < \tau_i \le T) \,\big|\, \mathcal{H}_t^k\big] \\ &\quad + E\big[R_{\tau_{3-i}}(X_{\tau_{3-i}}^1, X_{\tau_{3-i}}^2) \, I(\tau_{3-i} < \tau_i, t < \tau_{3-i} \le T) \,\big|\, \mathcal{H}_t^k\big] \\ &= E\big[R_{\tau_i}(X_{\tau_i}^1, X_{\tau_i}^2) \, I(t < \tau_i < \tau_{3-i} \wedge T) \,\big|\, \mathcal{H}_t^k\big] \\ &\quad + E\big[R_{\tau_{3-i}}(X_{\tau_{3-i}}^1, X_{\tau_{3-i}}^2) \, I(t < \tau_{3-i} < \tau_i \wedge T) \,\big|\, \mathcal{H}_t^k\big] \end{aligned} \tag{4.2}$$

for all $0 \le t \le T$ and every $i, k = 1, 2$. Then, we can apply the second part of the key lemma in (2.9) for the filtrations $(\mathcal{H}_t^k)_{t \ge 0}$ and $(\mathcal{F}_t)_{t \ge 0}$, where $\mathcal{H}_t^k$ coincides with $\mathcal{F}_t$ on $\{\varkappa_k > t\}$, for all $t \ge 0$, and use Fubini's theorem for interchanging the order of conditional expectation and integration along with the tower property for conditional expectations, and using the fact that, on the set $\{t > \tau_k\}$, the quantity $I(t < \tau_i < \tau_{3-i} \wedge T)$ is equal to zero, to get the expression

$$\begin{aligned} &E\big[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(t < \tau_i < \tau_{3-i} \wedge T) \,\big|\, \mathcal{H}_t^k\big] \qquad (4.3) \\ &= I(\varkappa_k > t)\, \frac{E[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(t < \tau_i < \tau_{3-i} \wedge T) \,|\, \mathcal{F}_t]}{P(\varkappa_k > t \,|\, \mathcal{F}_t)} \\ &= I(\varkappa_k > t)\, E\bigg[\int_t^T \int_t^\infty I(u < v)\, \frac{R_u(X^1_u, X^2_u)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v)}{P(\varkappa_k > t \,|\, \mathcal{F}_t)} \,\bigg|\, \mathcal{F}_t\bigg] \\ &= I(\varkappa_k > t) \int_t^T \int_t^\infty I(u < v)\, \frac{E[R_u(X^1_u, X^2_u)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,|\, \mathcal{F}_t]}{P(\varkappa_k > t \,|\, \mathcal{F}_t)} \end{aligned}$$

for all $0 \le t \le T$ and every $i, k = 1, 2$. Thus, taking into account the expressions in (5.19), according to the tower property for conditional expectations, we obtain that

$$\begin{aligned} &E\big[R_u(X^1_u, X^2_u)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_u) \,\big|\, \mathcal{F}_t\big] \qquad (4.4) \\ &= E\big[R_u(X^1_u, X^2_u)\, e^{-(\delta_i + \delta_{3-i}) A^0_u - \lambda_i A^i_u - \lambda_{3-i} A^{3-i}_u}\, (\delta_i\, Y^0_u + \lambda_i\, Y^i_u)\, D^{3-i}_{v-u}(Y^0_u, Y^{3-i}_u) \,\big|\, \mathcal{F}_t\big]\, du dv \\ &= e^{-(\delta_i + \delta_{3-i}) A^0_t - \lambda_i A^i_t - \lambda_{3-i} A^{3-i}_t} \\ &\quad \times E\big[R_u\big(X^1_t (Y^i_u / Y^i_t)^{\alpha_i} (Z^0_u / Z^0_t)^{\zeta_i} (Z^i_u / Z^i_t), X^2_t (Y^{3-i}_u / Y^{3-i}_t)^{\alpha_{3-i}} (Z^0_u / Z^0_t)^{\zeta_{3-i}} (Z^{3-i}_u / Z^{3-i}_t)\big) \\ &\quad \times e^{-(\delta_i + \delta_{3-i}) Y^0_t (A^0_u - A^0_t)/Y^0_t - \lambda_i Y^i_t (A^i_u - A^i_t)/Y^i_t - \lambda_{3-i} Y^{3-i}_t (A^{3-i}_u - A^{3-i}_t)/Y^{3-i}_t} \\ &\quad \times \big(\delta_i Y^0_t (Y^0_u / Y^0_t) + \lambda_i Y^i_t (Y^i_u / Y^i_t)\big)\, D^{3-i}_{v-u}(Y^0_t (Y^0_u / Y^0_t), Y^{3-i}_t (Y^{3-i}_u / Y^{3-i}_t)) \,\big|\, \mathcal{F}_t\big]\, du dv \\ &= e^{-(\delta_i + \delta_{3-i}) A^0_t - \lambda_i A^i_t - \lambda_{3-i} A^{3-i}_t}\, \overline{Q}^{1,3-i}_{t,u-t,v-u}(X^1_t, X^2_t, Y^0_t, Y^{3-i}_t, Y^i_t)\, du dv \end{aligned}$$

holds, for each $0 \le t < u < v \le T$, for every $i = 1, 2$. Here, by virtue of the Markov property of the processes $(Y^l, A^l)$ and $Z^l$, $l = 0, 1, 2$, and the fact that the random variables $Y^l_u / Y^l_t$ and $Z^l_u / Z^l_t$ have the same laws as $Y^l_{u-t}$ and $Z^l_{u-t}$, $l = 0, 1, 2$, for each $0 \le t < u$, respectively, we have

$$\begin{aligned} &\overline{Q}^{1,3-i}_{t,u-t,v-u}(x_1, x_2, y_0, y_{3-i}, y_i) \qquad (4.5) \\ &= E\big[R_u\big(x_1 (Y^i_{u-t})^{\alpha_i} (Z^0_{u-t})^{\zeta_i} Z^i_{u-t}, x_2 (Y^{3-i}_{u-t})^{\alpha_{3-i}} (Z^0_{u-t})^{\zeta_{3-i}} Z^{3-i}_{u-t}\big) \\ &\quad \times e^{-(\delta_i + \delta_{3-i}) y_0 A^0_{u-t} - \lambda_i y_i A^i_{u-t} - \lambda_{3-i} y_{3-i} A^{3-i}_{u-t}}\, (\delta_i y_0 Y^0_{u-t} + \lambda_i y_i Y^i_{u-t})\, D^{3-i}_{v-u}(y_0 Y^0_{u-t}, y_{3-i} Y^{3-i}_{u-t})\big] \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty R_u\big(x_1 (y'_i)^{\alpha_i} (z'_0)^{\zeta_i} z'_i, x_2 (y'_{3-i})^{\alpha_{3-i}} (z'_0)^{\zeta_{3-i}} z'_{3-i}\big) \\ &\quad \times e^{-(\delta_i + \delta_{3-i}) y_0 a_0 - \lambda_i y_i a_i - \lambda_{3-i} y_{3-i} a_{3-i}}\, (\delta_i y_0 y'_0 + \lambda_i y_i y'_i)\, D^{3-i}_{v-u}(y_0 y'_0, y_{3-i} y'_{3-i})\, g^0_{u-t}(y'_0, a_0) \\ &\quad \times g^i_{u-t}(y'_i, a_i)\, g^{3-i}_{u-t}(y'_i, a_i)\, h^0_{u-t}(z'_0)\, h^i_{u-t}(z'_i)\, h^{3-i}_{u-t}(z'_{3-i})\, dy'_0 da_0 dy'_i da_i dy'_{3-i} da_{3-i} dz'_0 dz'_i dz'_{3-i} \end{aligned}$$

for all $0 \le t < u < v \le T$ and every $i = 1, 2$, where the functions $g^l$, for $l = 0, 1, 2$, stand for $g$ defined in (5.7) and the functions $h^l$, for $l = 0, 1, 2$, stand for $h$ defined in (5.9). There, $\beta$ and $\gamma$ stand for $\beta_l$ and $\gamma_l$, for $l = 0, 1, 2$, defined in (2.3), while $\eta$ and $\theta$ stand for $\eta_l$ and $\theta_l$, for $l = 0, 1, 2$, defined in (2.4).

In order to compute closed-form expressions for $P^{2,k}$ in (4.1), we provide the decomposition

$$\begin{aligned} P_t^{2,k} &= E\big[R_{\varkappa_2}(X^1_{\varkappa_2}, X^2_{\varkappa_2})\, I(t < \varkappa_2 \le T) \,\big|\, \mathcal{H}_t^k\big] \\ &= E\big[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(\tau_i > \tau_{3-i}, t < \tau_i \le T) \,\big|\, \mathcal{H}_t^k\big] \\ &\quad + E\big[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(\tau_{3-i} > \tau_i, t < \tau_{3-i} \le T) \,\big|\, \mathcal{H}_t^k\big] \\ &= E\big[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(t \vee \tau_{3-i} < \tau_i \le T) \,\big|\, \mathcal{H}_t^k\big] \\ &\quad + E\big[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(t \vee \tau_i < \tau_{3-i} \le T) \,\big|\, \mathcal{H}_t^k\big] \end{aligned} \tag{4.6}$$

for all $0 \le t \le T$ and every $i, k = 1, 2$. Then, we can apply the second part of the key lemma in (2.9) for the filtrations $(\mathcal{H}_t^1)_{t \ge 0}$ and $(\mathcal{F}_t)_{t \ge 0}$, where $\mathcal{H}_t^1$ coincides with $\mathcal{F}_t \vee \sigma(\varkappa_1)$ on the event $\{\varkappa_1 \le t\}$ and with $\mathcal{F}_t$ on $\{\varkappa_1 > t\}$, for all $t \ge 0$, as well as the arguments applied for derivation of equalities in (2.14)-(2.19), and use Fubini's theorem for interchanging the order of conditional expectation and integration along with the tower property for conditional expectations to obtain that

$$\begin{aligned} & E\big[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(t \vee \tau_i < \tau_{3-i} \le T) \,\big|\, \mathcal{H}_t^1\big] \\ &= I(\varkappa_1 \le t)\, E[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(t \vee \tau_i < \tau_{3-i} \le T) \,\big|\, \mathcal{F}_t \vee \sigma(\varkappa_1)] \\ &\quad + I(\varkappa_1 > t)\, \frac{E[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(t < \tau_i < \tau_{3-i} \le T) \,|\, \mathcal{F}_t]}{P(\varkappa_1 > t \,|\, \mathcal{F}_t)} \\ &= I(\varkappa_1 \le t) \int_{v = t \vee u}^T \frac{E[R_v(X^1_v, X^2_v)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,|\, \mathcal{F}_t]}{P(\tau_1 > u, \tau_2 \in du \,|\, \mathcal{F}_t) + P(\tau_2 > u, \tau_1 \in du \,|\, \mathcal{F}_t)} \bigg|_{u = \varkappa_1} \\ &\quad + I(\varkappa_1 > t) \int_t^T \int_t^T I(u < v)\, \frac{E[R_v(X^1_v, X^2_v)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,|\, \mathcal{F}_t]}{P(\varkappa_1 > t \,|\, \mathcal{F}_t)} \end{aligned} \tag{4.7}$$

holds, for all $0 \le t \le T$ and every $i = 1, 2$. Hence, we can apply the second part of the key lemma in (2.9) for the filtrations $(\mathcal{H}_t^2)_{t \ge 0}$ and $(\mathcal{F}_t)_{t \ge 0}$, where $\mathcal{H}_t^2$ coincides with $\mathcal{F}_t$ on $\{\varkappa_1 > t\}$, for all $t \ge 0$, and use Fubini's theorem for interchanging the order of conditional expectation and integration along with the tower property for conditional expectations to get the expression

$$\begin{aligned} & E\big[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(t \vee \tau_i < \tau_{3-i} \le T) \,\big|\, \mathcal{H}_t^2\big] \\ &= I(\varkappa_2 > t)\, \frac{E[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(t \vee \tau_i < \tau_{3-i} \le T) \,|\, \mathcal{F}_t]}{P(\varkappa_2 > t \,|\, \mathcal{F}_t)} \\ &= I(\varkappa_2 > t)\, E\bigg[\int_t^T \int_t^\infty I(u < v)\, \frac{R_v(X^1_v, X^2_v)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v)}{P(\varkappa_2 > t \,|\, \mathcal{F}_t)} \,\bigg|\, \mathcal{F}_t\bigg] \\ &= I(\varkappa_2 > t) \int_t^T \int_t^\infty I(u < v)\, \frac{E[R_v(X^1_v, X^2_v)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,|\, \mathcal{F}_t]}{P(\varkappa_2 > t \,|\, \mathcal{F}_t)} \end{aligned} \tag{4.8}$$

since it is obvious that the left-hand side of (4.8) is equal to zero on the event $\{\varkappa_2 \le t\}$, for all $0 \le t \le T$ and every $i = 1, 2$. Thus, taking into account the expressions in (5.17), according to the tower property for conditional expectations, we obtain that

$$
\begin{aligned}
&E\big[R_v(X^1_v, X^2_v)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,\big|\, \mathcal{F}_u\big] \qquad (4.9)\\
&= e^{-\delta_i A^0_u - \lambda_i A^i_u} \, (\delta_i \, Y^0_u + \lambda_i \, Y^i_u) \, E\big[R_v(X^1_v, X^2_v) \, e^{-\delta_{3-i} A^0_v - \lambda_{3-i} A^{3-i}_v} \, (\delta_{3-i} \, Y^0_v + \lambda_{3-i} \, Y^{3-i}_v) \,\big|\, \mathcal{F}_u\big] \, dudv\\
&= e^{-(\delta_i + \delta_{3-i}) A^0_u - \lambda_i A^i_u - \lambda_{3-i} A^{3-i}_u} \, (\delta_i \, Y^0_u + \lambda_i \, Y^i_u)\\
&\times E\big[R_v\big(X^1_u (Y^i_v / Y^i_u)^{\alpha_i} (Z^0_v / Z^0_u)^{\zeta_i} (Z^i_v / Z^i_u), X^2_u (Y^{3-i}_v / Y^{3-i}_u)^{\alpha_{3-i}} (Z^0_v / Z^0_u)^{\zeta_{3-i}} (Z^{3-i}_v / Z^{3-i}_u)\big)\\
&\times e^{-\delta_{3-i} Y^0_u (A^0_v - A^0_u) / Y^0_u - \lambda_{3-i} Y^{3-i}_u (A^{3-i}_v - A^{3-i}_u) / Y^{3-i}_u} \, \big(\delta_{3-i} \, Y^0_u (Y^0_v / Y^0_u) + \lambda_{3-i} \, Y^{3-i}_u (Y^{3-i}_v / Y^{3-i}_u)\big) \,\big|\, \mathcal{F}_u\big] \, dudv\\
&= e^{-(\delta_i + \delta_{3-i}) A^0_u - \lambda_i A^i_u - \lambda_{3-i} A^{3-i}_u} \, (\delta_i \, Y^0_u + \lambda_i \, Y^i_u) \, Q^{2,3-i}_{u,v-u}(X^1_u, X^2_u, Y^0_u, Y^{3-i}_u, Y^i_u) \, dudv
\end{aligned}
$$

holds, for each $0 \le t < u < v \le T$ and every $i = 1, 2$. Here, by virtue of the Markov property of the processes $(Y^l, A^l)$ and $Z^l$, $l = 0, 1, 2$, and the fact that the random variables $Y^l_v / Y^l_u$ and $Z^l_v / Z^l_u$ have the same laws as $Y^l_{v-u}$ and $Z^l_{v-u}$, $l = 0, 1, 2$, for each $0 \le u < v$, respectively, we have

$$
\begin{aligned}
&Q^{2,3-i}_{u,v-u}(x_1, x_2, y_0, y_{3-i}, y_i) \qquad (4.10)\\
&= E\big[R_v\big(x_1 (Y^i_{v-u})^{\alpha_i} (Z^0_{v-u})^{\zeta_i} Z^i_{v-u}, x_2 (Y^{3-i}_{v-u})^{\alpha_{3-i}} (Z^0_{v-u})^{\zeta_{3-i}} Z^{3-i}_{v-u}\big)\\
&\quad \times e^{-\delta_{3-i} y_0 A^0_{v-u} - \lambda_{3-i} y_{3-i} A^{3-i}_{v-u}} \, (\delta_{3-i} y_0 Y^0_{v-u} + \lambda_{3-i} y_{3-i} Y^{3-i}_{v-u})\big]\\
&= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty R_v\big(x_1 (y'_i)^{\alpha_i} (z'_0)^{\zeta_i} z'_i, x_2 (y'_{3-i})^{\alpha_{3-i}} (z'_0)^{\zeta_{3-i}} z'_{3-i}\big)\\
&\quad \times e^{-\delta_{3-i} y_0 a_0 - \lambda_{3-i} y_{3-i} a_{3-i}} \, (\delta_{3-i} y_0 y'_0 + \lambda_{3-i} y_{3-i} y'_{3-i}) \, g^0_{v-u}(y'_0, a_0) \, g^i_{v-u}(y'_i, a_i) \, g^{3-i}_{v-u}(y'_{3-i}, a_{3-i})\\
&\quad \times h^0_{v-u}(z'_0) \, h^i_{v-u}(z'_i) \, h^{3-i}_{v-u}(z'_{3-i}) \, dy'_0 da_0 dy'_i da_i dy'_{3-i} da_{3-i} dz'_0 dz'_i dz'_{3-i}
\end{aligned}
$$

for all $0 \le t < u < v \le T$ and every $i = 1, 2$, while the functions $g^l$ and $h^l$, $l = 0, 1, 2$, are given in (5.7) and (5.9) below, with an adequate choice of the parameters. Hence, taking into account the expressions in (4.9) and applying again the tower property for conditional expectations, we obtain that

$$
\begin{aligned}
&E\big[E\big[R_v(X^1_v, X^2_v) \, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,\big|\, \mathcal{F}_u\big] \,\big|\, \mathcal{F}_t\big] \qquad (4.11)\\
&= E\big[e^{-(\delta_i + \delta_{3-i}) A^0_u - \lambda_i A^i_u - \lambda_{3-i} A^{3-i}_u} \, (\delta_i \, Y^0_u + \lambda_i \, Y^i_u) \, Q^{2,3-i}_{u,v-u}(X^1_u, X^2_u, Y^0_u, Y^{3-i}_u, Y^i_u) \,\big|\, \mathcal{F}_t\big] \, dudv\\
&= e^{-(\delta_i + \delta_{3-i}) A^0_t - \lambda_i A^i_t - \lambda_{3-i} A^{3-i}_t} \, E\big[e^{-(\delta_i + \delta_{3-i}) Y^0_t (A^0_v - A^0_t) / Y^0_t - \lambda_i Y^i_t (A^i_v - A^i_t) / Y^i_t - \lambda_{3-i} Y^{3-i}_t (A^{3-i}_v - A^{3-i}_t) / Y^{3-i}_t}\\
&\times \big(\delta_i \, Y^0_t (Y^0_u / Y^0_t) + \lambda_i \, Y^i_t (Y^i_u / Y^i_t)\big) \, Q^{2,3-i}_{u,v-u}\big(X^1_t (Y^i_u / Y^i_t)^{\alpha_i} (Z^0_u / Z^0_t)^{\zeta_i} (Z^i_u / Z^i_t), X^2_t (Y^{3-i}_u / Y^{3-i}_t)^{\alpha_{3-i}}\\
&\times (Z^0_u / Z^0_t)^{\zeta_{3-i}} (Z^{3-i}_u / Z^{3-i}_t), Y^0_t (Y^0_u / Y^0_t), Y^{3-i}_t (Y^{3-i}_u / Y^{3-i}_t, Y^i_t (Y^i_u / Y^i_t)\big) \,\big|\, \mathcal{F}_t\big] \, dudv\\
&= e^{-(\delta_i + \delta_{3-i}) A^0_t - \lambda_i A^i_t - \lambda_{3-i} A^{3-i}_t} \, \widehat{Q}^{2,3-i}_{t,u-t,v-u}(X^1_t, X^2_t, Y^0_t, Y^{3-i}_t, Y^i_t) \, dudv
\end{aligned}
$$

is satisfied, for each $0 \le t < u < v$ and every $i = 1, 2$. Here, by virtue of the Markov property of the processes $(Y^l, A^l)$ and $Z^l$, $l = 0, 1, 2$, and the fact that the random variables $Y^l_u / Y^l_t$ and $Z^l_u / Z^l_t$ have the same laws as $Y^l_{u-t}$ and $Z^l_{u-t}$, $l = 0, 1, 2$, for each $0 \le t < u$, respectively, we have

$$
\begin{aligned}
&\widehat{Q}^{2,3-i}_{t,u-t,v-u}(x_1, x_2, y_0, y_{3-i}, y_i) \qquad (4.12)\\
&= E\big[e^{-(\delta_i+\delta_{3-i})y_0 A^0_{u-t} - \lambda_i y_i A^i_{u-t} - \lambda_{3-i} y_{3-i} A^{3-i}_{u-t}} \, (\delta_i y_0 Y^0_{u-t} + \lambda_i y_i Y^i_{u-t})\\
&\quad \times Q^{2,3-i}_{u,v-u}\big(x_1 (Y^i_{u-t})^{\alpha_i} (Z^0_{u-t})^{\zeta_i} Z^i_{u-t}, x_2 (Y^{3-i}_{u-t})^{\alpha_{3-i}} (Z^0_{u-t})^{\zeta_{3-i}} Z^{3-i}_{u-t}, y_0 Y^0_{u-t}, y_{3-i} Y^{3-i}_{u-t}, y_i Y^i_{u-t}\big)\big]\\
&= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-(\delta_i+\delta_{3-i})y_0 a_0 - \lambda_i y_i a_i - \lambda_{3-i} y_{3-i} a_{3-i}} \, (\delta_{3-i} y_0 y'_0 + \lambda_{3-i} y_{3-i} y'_{3-i})\\
&\quad \times Q^{2,3-i}_{u,v-u}\big(x_1 (y'_i)^{\alpha_i} (z'_0)^{\zeta_i} z'_i, x_2 (y'_{3-i})^{\alpha_{3-i}} (z'_0)^{\zeta_{3-i}} z'_{3-i}, y_0 y'_0, y_{3-i} y'_{3-i} y_i y'_i\big) \, g^0_{u-t}(y'_0, a_0)\\
&\quad \times g^i_{u-t}(y'_i, a_i) \, g^{3-i}_{u-t}(y'_{3-i}, a_{3-i}) \, h^0_{u-t}(z'_0) \, h^i_{u-t}(z'_i) \, h^{3-i}_{u-t}(z'_{3-i}) \, dy'_0 da_0 dy'_i da_i dy'_{3-i} da_{3-i} dz'_0 dz'_i dz'_{3-i}
\end{aligned}
$$

for all $0 \le t < u < v \le T$ and every $i = 1, 2$, while the functions $g^l$ and $h^l$, $l = 0, 1, 2$, are given in (5.7) and (5.9) below, with an adequate choice of the parameters, as before.

Therefore, summarising the facts proved above, we now formulate the following assertion.

**Proposition 4.1.** *Suppose that $r = 0$. The no-arbitrage price for the holders of the first- or second-to-default options in (4.1) are given by the sum of the expressions in (4.2) or (4.6) with (4.3) or (4.7)-(4.8), respectively. The latter terms are computed by means of the expressions in (4.4) or (4.9), (4.11) with (4.5) or (4.10), (4.12), respectively.*

## 4.2 The case of filtration $(\mathcal{H}_t)_{t \ge 0}$

Let us now continue by computing the price $\widehat{P}^j = (\widehat{P}^j_t)_{t \ge 0}$ for the holder of a first- or second-to-default option in the model with the filtration $(\mathcal{H}_t)_{t \ge 0}$ given by

$$
\widehat{P}^j_t = E\big[R_{\varkappa_j}(X^1_{\varkappa_j}, X^2_{\varkappa_j}) \, I(t < \varkappa_j \le T) \, \big| \, \mathcal{H}_t\big] \qquad (4.13)
$$

for all $0 \le t \le T \wedge \varkappa_j$ and $j = 1, 2$.

In order to compute closed-form expressions for $\widehat{P}^1$ in (4.13), we provide the decomposition

$$
\begin{aligned}
\widehat{P}^1_t &= E\big[R_{\varkappa_1}(X^1_{\varkappa_1}, X^2_{\varkappa_1}) \, I(t < \varkappa_1 \le T) \, \big| \, \mathcal{H}_t\big] \qquad (4.14)\\
&= E\big[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i}) \, I(\tau_i < \tau_{3-i}, t < \tau_i \le T) \, \big| \, \mathcal{H}_t\big]\\
&\quad + E\big[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}}) \, I(\tau_{3-i} < \tau_i, t < \tau_{3-i} \le T) \, \big| \, \mathcal{H}_t\big]\\
&= E\big[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i}) \, I(t < \tau_i < \tau_{3-i} \wedge T) \, \big| \, \mathcal{H}_t\big]\\
&\quad + E\big[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}}) \, I(t < \tau_{3-i} < \tau_i \wedge T) \, \big| \, \mathcal{H}_t\big]
\end{aligned}
$$

for all $0 \le t \le T$ and every $i = 1, 2$. Then, we can apply the second part of the key lemma in (2.9) for the filtrations $(\mathcal{H}_t)_{t \ge 0}$ and $(\mathcal{F}_t)_{t \ge 0}$, where $\mathcal{H}_t$ coincides with $\mathcal{F}_t$ on the event $\{\varkappa_2 > \varkappa_1 > t\}$, for all $t \ge 0$, and use Fubini's theorem for interchanging the order of conditional expectation and integration along with the tower property for conditional expectations to get the expression

$$\begin{aligned}
&E\big[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(t < \tau_i < \tau_{3-i} \wedge T) \,\big|\, \mathcal{H}_t\big] \qquad (4.15)\\
&= I(\varkappa_2 > \varkappa_1 > t)\, \frac{E[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(t < \tau_i < \tau_{3-i} \wedge T) \,|\, \mathcal{F}_t]}{P(\varkappa_2 > \varkappa_1 > t \,|\, \mathcal{F}_t)}\\
&= I(\varkappa_2 > \varkappa_1 > t)\, E\bigg[\int_t^T \int_t^\infty I(u < v)\, \frac{R_u(X^1_u, X^2_u)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v)}{P(\varkappa_2 > \varkappa_1 > t \,|\, \mathcal{F}_t)} \,\bigg|\, \mathcal{F}_t\bigg]\\
&= I(\varkappa_2 > \varkappa_1 > t) \int_t^T \int_t^\infty I(u < v)\, \frac{E[R_u(X^1_u, X^2_u)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,|\, \mathcal{F}_t]}{P(\varkappa_2 > \varkappa_1 > t \,|\, \mathcal{F}_t)}
\end{aligned}$$

taking into account that the left-hand side of (4.15) is equal to zero on the event $\{\varkappa_k \le t\}$, for all $0 \le t \le T$ and every $i, k = 1, 2$. Here, the conditional expectations of interest are computed in (4.4) with (4.5) above.

In order to compute closed-form expressions for $\widehat{P}^2$ in (4.13), we provide the decomposition

$$\begin{aligned}
\widehat{P}^2_t &= E\big[R_{\varkappa_2}(X^1_{\varkappa_2}, X^2_{\varkappa_2})\, I(t < \varkappa_2 \le T) \,\big|\, \mathcal{H}_t\big] \qquad (4.16)\\
&= E\big[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(\tau_i > \tau_{3-i}, t < \tau_i \le T) \,\big|\, \mathcal{H}_t\big]\\
&\quad + E\big[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(\tau_{3-i} > \tau_i, t < \tau_{3-i} \le T) \,\big|\, \mathcal{H}_t\big]\\
&= E\big[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(t \vee \tau_{3-i} < \tau_i \le T) \,\big|\, \mathcal{H}_t\big]\\
&\quad + E\big[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(t \vee \tau_i < \tau_{3-i} \le T) \,\big|\, \mathcal{H}_t\big]
\end{aligned}$$

for all $0 \le t \le T$ and every $i = 1, 2$. Then, we can apply the second part of the key lemma in (2.9) for the filtrations $(\mathcal{H}_t)_{t \ge 0}$ and $(\mathcal{F}_t)_{t \ge 0}$, where $\mathcal{H}_t$ coincides with $\mathcal{F}_t \vee \sigma(\varkappa_1)$ on the event $\{\varkappa_1 \le t < \varkappa_2\}$ and with $\mathcal{F}_t$ on $\{\varkappa_2 > \varkappa_1 > t\}$, for all $t \ge 0$, as well as the arguments applied for derivation of equalities in (2.14)-(2.19), and use Fubini's theorem for interchanging the order of conditional expectation and integration along with the tower property for conditional expectations to get the expression

$$\begin{aligned}
&E\big[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(t \vee \tau_{3-i} < \tau_i \le T) \,\big|\, \mathcal{H}_t\big] \qquad (4.17)\\
&= I(\varkappa_1 \le t < \varkappa_2)\, \frac{E[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(t \vee \tau_{3-i} < \tau_i \le T) \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1)]}{P(\varkappa_2 > t \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1))}\\
&\quad + I(\varkappa_2 > \varkappa_1 > t)\, \frac{E[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(t < \tau_{3-i} < \tau_i \le T) \,|\, \mathcal{F}_t]}{P(\varkappa_2 > \varkappa_1 > t \,|\, \mathcal{F}_t)}\\
&= I(\varkappa_1 \le t < \varkappa_2) \int_{v = t \vee u}^T \frac{E[R_v(X^1_v, X^2_v)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,|\, \mathcal{F}_t]}{P(\tau_1 > t \vee u, \tau_2 \in du \,|\, \mathcal{F}_t) + P(\tau_2 > t \vee u, \tau_1 \in du \,|\, \mathcal{F}_t)}\bigg|_{u = \varkappa_1}\\
&\quad + I(\varkappa_2 > \varkappa_1 > t) \int_t^T \int_t^T I(u < v)\, \frac{E[R_v(X^1_v, X^2_v)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,|\, \mathcal{F}_t]}{P(\varkappa_2 > \varkappa_1 > t \,|\, \mathcal{F}_t)}
\end{aligned}$$

taking into account that the left-hand side of (4.17) is equal to zero on the event $\{\varkappa_2 \le t\}$, for all $0 \le t \le T$ and every $i = 1, 2$. Here, the conditional expectations of interest are computed in (4.9), (4.11) with (4.10), (4.12) above.

Therefore, summarising the facts proved above, we now formulate the following assertion.

**Proposition 4.2.** *Suppose that $r = 0$. The no-arbitrage price for the holders of the first- or second-to-default options in (4.13) are given by the sum of the expressions in (4.14) or (4.16) with (4.15) or (4.17), respectively. The latter terms are computed by means of the expressions in (4.4) or (4.9), (4.11) with (4.5) or (4.10), (4.12), respectively.*

## 4.3 The case of filtrations $(\mathcal{H}_t^k \vee \sigma(\varkappa_{3-k})))_{t \geq 0}$, $k = 1, 2$

Let us finally compute the price $\widetilde{P}^{j,k}(\varkappa_{3-k}) = (\widetilde{P}^{j,k}(\varkappa_{3-k}))_{t \geq 0}$ for the holder of a first- and second-to-default options in the model with the filtration $(\mathcal{H}_t^k \vee \sigma(\varkappa_{3-k}))_{t \geq 0}$ given by

$$\widetilde{P}_t^{j,k}(\varkappa_{3-k}) = E\big[R_{\varkappa_j}(X_{\varkappa_j}^1, X_{\varkappa_j}^2)\, I(t < \varkappa_j \leq T) \,\big|\, \mathcal{H}_t^k \vee \sigma(\varkappa_{3-k})\big] \tag{4.18}$$

for all $0 \leq t \leq T \wedge \varkappa_j$ and $j, k = 1, 2$.

In order to compute closed-form expressions for $\widetilde{P}^{1,k}(\varkappa_{3-k})$ in (4.18), we provide the decomposition

$$\begin{aligned} \widetilde{P}_t^{1,k}(\varkappa_{3-k}) &= E\big[R_{\varkappa_1}(X_{\varkappa_1}^1, X_{\varkappa_1}^2)\, I(t < \varkappa_1 \leq T) \,\big|\, \mathcal{H}_t^k \vee \sigma(\varkappa_{3-k})\big] \\ &= E\big[R_{\tau_i}(X_{\tau_i}^1, X_{\tau_i}^2)\, I(\tau_i < \tau_{3-i}, t < \tau_i \leq T) \,\big|\, \mathcal{H}_t^k \vee \sigma(\varkappa_{3-k})\big] \\ &\quad + E\big[R_{\tau_{3-i}}(X_{\tau_{3-i}}^1, X_{\tau_{3-i}}^2)\, I(\tau_{3-i} < \tau_i, t < \tau_{3-i} \leq T) \,\big|\, \mathcal{H}_t^k \vee \sigma(\varkappa_{3-k})\big] \\ &= E\big[R_{\tau_i}(X_{\tau_i}^1, X_{\tau_i}^2)\, I(t < \tau_i < \tau_{3-i} \wedge T) \,\big|\, \mathcal{H}_t^k \vee \sigma(\varkappa_{3-k})\big] \\ &\quad + E\big[R_{\tau_{3-i}}(X_{\tau_{3-i}}^1, X_{\tau_{3-i}}^2)\, I(t < \tau_{3-i} < \tau_i \wedge T) \,\big|\, \mathcal{H}_t^k \vee \sigma(\varkappa_{3-k})\big] \end{aligned} \tag{4.19}$$

for all $0 \leq t \leq T$ and every $i, k = 1, 2$. Then, we can apply the second part of the key lemma in (2.9) for the filtrations $(\mathcal{H}_t^1 \vee \sigma(\varkappa_2))_{t \geq 0}$ and $(\mathcal{F}_t \vee \sigma(\varkappa_2))_{t \geq 0}$, where $\mathcal{H}_t^1 \vee \sigma(\varkappa_2)$ coincides with $\mathcal{F}_t \vee \sigma(\varkappa_2)$ on the event $\{\varkappa_1 > t\}$, for all $t \geq 0$, as well as the arguments applied for derivation of equalities in (2.20)-(2.25), and use Fubini's theorem for interchanging the order of conditional expectation and integration along with the tower property for conditional expectations to get the expression

$$\begin{aligned} &E\big[R_{\tau_i}(X_{\tau_i}^1, X_{\tau_i}^2)\, I(t < \tau_i < \tau_{3-i} \wedge T) \,\big|\, \mathcal{H}_t^1 \vee \sigma(\varkappa_2)\big] \\ &= I(\varkappa_1 > t)\, \frac{E[R_{\tau_i}(X_{\tau_i}^1, X_{\tau_i}^2)\, I(t < \tau_i < \tau_{3-i} \wedge T) \,|\, \mathcal{F}_t \vee \sigma(\varkappa_2)]}{P(\varkappa_1 > t \,|\, \mathcal{F}_t \vee \sigma(\varkappa_2))} \\ &= I(\varkappa_2 > \varkappa_1 > t)\, E\bigg[\int_{u=t}^{\infty} \frac{R_u(X_u^1, X_u^2)\, I(t < u < v \wedge T)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v)}{P(t < \tau_1 \leq v, \tau_2 \in dv \,|\, \mathcal{F}_t) + P(t < \tau_2 \leq v, \tau_1 \in dv \,|\, \mathcal{F}_t)} \,\bigg|\, \mathcal{F}_t\bigg]\bigg|_{v = \varkappa_2} \\ &= I(\varkappa_2 > \varkappa_1 > t) \int_{u=t}^{\infty} \frac{E[R_u(X_u^1, X_u^2)\, I(t < u < v \wedge T)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,|\, \mathcal{F}_t]}{P(t < \tau_1 \leq v, \tau_2 \in dv \,|\, \mathcal{F}_t) + P(t < \tau_2 \leq v, \tau_1 \in dv \,|\, \mathcal{F}_t)}\bigg|_{v = \varkappa_2} \end{aligned} \tag{4.20}$$

for all $0 \leq t \leq T$ and every $i, k = 1, 2$.

Now, we can apply the second part of the key lemma in (2.9) for the filtrations $(\mathcal{H}_t^2 \vee \sigma(\varkappa_1))_{t \geq 0}$ and $(\mathcal{F}_t \vee \sigma(\varkappa_1))_{t \geq 0}$, where $\mathcal{H}_t^2 \vee \sigma(\varkappa_1)$ coincides with $\mathcal{F}_t \vee \sigma(\varkappa_1)$ on the event $\{\varkappa_2 > t\}$, for all $t \geq 0$, as well as the arguments applied for derivation of equalities in (2.14)-(2.19), and use Fubini's theorem for interchanging the order of conditional expectation and integration along with the tower property for conditional expectations to get the expression

$$
\begin{aligned}
&E\big[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(t < \tau_i < \tau_{3-i} \wedge T) \,\big|\, \mathcal{H}^2_t \vee \sigma(\varkappa_1)\big] \qquad (4.21)\\
&= I(\varkappa_2 > t)\, \frac{E[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(t < \tau_i < \tau_{3-i} \wedge T) \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1)]}{P(\varkappa_2 > t \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1))}\\
&= I(\varkappa_2 > \varkappa_1 > t)\, E\bigg[\int_{v=u}^{\infty} \frac{R_u(X^1_u, X^2_u)\, I(t < u < v \wedge T)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v)}{P(\tau_1 > u \vee t, \tau_2 \in du \,|\, \mathcal{F}_t) + P(\tau_2 > u \vee t, \tau_1 \in du \,|\, \mathcal{F}_t)} \,\bigg|\, \mathcal{F}_t\bigg]\bigg|_{u=\varkappa_1}\\
&= I(\varkappa_2 > \varkappa_1 > t) \int_{v=u}^{\infty} \frac{E[R_u(X^1_u, X^2_u)\, I(t < u < v \wedge T)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,|\, \mathcal{F}_t]}{P(\tau_1 > u \vee t, \tau_2 \in du \,|\, \mathcal{F}_t) + P(\tau_2 > u \vee t, \tau_1 \in du \,|\, \mathcal{F}_t)}\bigg|_{u=\varkappa_1}
\end{aligned}
$$

on the event $\{\varkappa_1 \le T\}$, for all $0 \le t \le T$ and every $i, k = 1, 2$. Here, the conditional expectations of interest are computed in (4.4) with (4.5) above.

In order to compute closed-form expressions for $\widetilde{P}^{2,k}(\varkappa_{3-k})$ in (4.18), we provide the decomposition

$$
\begin{aligned}
\widetilde{P}^{2,k}_t(\varkappa_{3-k}) &= E\big[R_{\varkappa_2}(X^1_{\varkappa_2}, X^2_{\varkappa_2})\, I(t < \varkappa_2 \le T) \,\big|\, \mathcal{H}^k_t \vee \sigma(\varkappa_{3-k})\big] \qquad (4.22)\\
&= E\big[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(\tau_i > \tau_{3-i}, t < \tau_i \le T) \,\big|\, \mathcal{H}^k_t \vee \sigma(\varkappa_{3-k})\big]\\
&\quad + E\big[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(\tau_{3-i} > \tau_i, t < \tau_{3-i} \le T) \,\big|\, \mathcal{H}^k_t \vee \sigma(\varkappa_{3-k})\big]\\
&= E\big[R_{\tau_i}(X^1_{\tau_i}, X^2_{\tau_i})\, I(t \vee \tau_{3-i} < \tau_i \le T) \,\big|\, \mathcal{H}^k_t \vee \sigma(\varkappa_{3-k})\big]\\
&\quad + E\big[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(t \vee \tau_i < \tau_{3-i} \le T) \,\big|\, \mathcal{H}^k_t \vee \sigma(\varkappa_{3-k})\big]
\end{aligned}
$$

for all $0 \le t \le T$ and every $i, k = 1, 2$. Then, we can apply the second part of the key lemma in (2.9) for the filtrations $(\mathcal{H}^1_t \vee \sigma(\varkappa_2))_{t \ge 0}$ and $(\mathcal{F}_t \vee \sigma(\varkappa_2))_{t \ge 0}$, where $\mathcal{H}^1_t \vee \sigma(\varkappa_2)$ coincides with $\mathcal{F}_t \vee \sigma(\varkappa_1) \vee \sigma(\varkappa_2)$ on the event $\{\varkappa_1 \le t\}$ and with $\mathcal{F}_t \vee \sigma(\varkappa_2)$ on $\{\varkappa_1 > t\}$, for all $t \ge 0$, as well as the arguments applied for derivation of equalities in (2.20)-(2.25), and use Fubini's theorem for interchanging the order of conditional expectation and integration along with the tower property for conditional expectations to get

$$
\begin{aligned}
&E\big[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(t \vee \tau_i < \tau_{3-i} \le T) \,\big|\, \mathcal{H}^1_t \vee \sigma(\varkappa_2)\big] \qquad (4.23)\\
&= I(\varkappa_1 \le t)\, E[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(t \vee \tau_i < \tau_{3-i} \le T) \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1) \vee \sigma(\varkappa_2)]\\
&\quad + I(\varkappa_1 > t)\, \frac{E[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(t < \tau_i < \tau_{3-i} \le T) \,|\, \mathcal{F}_t \vee \sigma(\varkappa_2)]}{P(\varkappa_1 > t \,|\, \mathcal{F}_t \vee \sigma(\varkappa_2))}\\
&= I(\varkappa_1 \le t < \varkappa_2)\, E\big[R_v(X^1_v, X^2_v)\, I(t \vee u < v \le T) \,\big|\, \mathcal{F}_t\big]\big|_{u=\varkappa_1, v=\varkappa_2}\\
&\quad + I(\varkappa_2 > \varkappa_1 > t) \int_{u=t}^{\infty} \frac{E[R_v(X^1_v, X^2_v)\, I(t < u < v \le T)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,|\, \mathcal{F}_t]}{P(t < \tau_1 \le v, \tau_2 \in dv \,|\, \mathcal{F}_t) + P(t < \tau_2 \le v, \tau_1 \in dv \,|\, \mathcal{F}_t)}\bigg|_{v=\varkappa_2}
\end{aligned}
$$

on the event $\{\varkappa_2 \le T\}$, for all $0 \le t \le T$ and every $i = 1, 2$. Thus, by virtue of the Markov property of the processes $(Y^l, A^l)$ and $Z^l$, $l = 0, 1, 2$, and the fact that the random variables $Y^l_v / Y^l_t$ and $Z^l_v / Z^l_t$ have the same laws as $Y^l_{v-t}$ and $Z^l_{v-t}$, $l = 0, 1, 2$, for each $0 \le t < v$, respectively, we have

$$
\begin{aligned}
&E\big[R_v(X^1_v, X^2_v) \,\big|\, \mathcal{F}_t\big] = E\big[R_v\big(x_1 (Y^i_{v-t})^{\alpha_i} (Z^0_{v-t})^{\zeta_i} Z^i_{v-t}, x_2 (Y^{3-i}_{v-t})^{\alpha_{3-i}} (Z^0_{v-t})^{\zeta_{3-i}} Z^{3-i}_{v-t}\big)\big] \qquad (4.24)\\
&= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty R_v\big(x_1 (y'_i)^{\alpha_i} (z'_0)^{\zeta_i} z'_i, x_2 (y'_{3-i})^{\alpha_{3-i}} (z'_0)^{\zeta_{3-i}} z'_{3-i}, y_i y'_i, y_{3-i} y'_{3-i}\big)\\
&\quad \times g^i_{v-t}(y'_i, a_i)\, g^{3-i}_{v-t}(y'_{3-i}, a_{3-i})\, h^0_{v-t}(z'_0)\, h^i_{v-t}(z'_i)\, h^{3-i}_{v-t}(z'_{3-i})\, dy'_i da_i dy'_{3-i} da_{3-i} dz'_0 dz'_i dz'_{3-i}
\end{aligned}
$$

for all $0 \le t < v \le T$ and every $i = 1, 2$, while the functions $g^l$ and $h^l$, $l = 0, 1, 2$, are given in (5.7) and (5.9) above. Here and after, the other conditional expectations of interest are computed in (4.9), (4.11) with (4.10), (4.12) above.

Finally, we can apply the second part of the key lemma in (2.9) for the filtrations $(\mathcal{H}_t^2 \vee \sigma(\varkappa_1))_{t \ge 0}$ and $(\mathcal{F}_t \vee \sigma(\varkappa_1))_{t \ge 0}$, where $\mathcal{H}_t^2 \vee \sigma(\varkappa_1)$ coincides with $\mathcal{F}_t \vee \sigma(\varkappa_1)$ on the event $\{\varkappa_2 \le t\}$ and with $\mathcal{F}_t \vee \sigma(\varkappa_1)$ on $\{\varkappa_2 > t\}$, for all $t \ge 0$, as well as the arguments applied for derivation of equalities in (2.14)-(2.19), and use Fubini's theorem for interchanging the order of conditional expectation and integration along with the tower property for conditional expectations to get

$$
\begin{aligned}
& E\big[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(t \vee \tau_i < \tau_{3-i} \le T) \,\big|\, \mathcal{H}_t^2 \vee \sigma(\varkappa_1)\big] \qquad (4.25) \\
&= I(\varkappa_2 \le t)\, E[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(t \vee \tau_i < \tau_{3-i} \le T) \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1) \vee \sigma(\varkappa_2)] \\
&\quad + I(\varkappa_2 > t)\, \frac{E[R_{\tau_{3-i}}(X^1_{\tau_{3-i}}, X^2_{\tau_{3-i}})\, I(t \vee \tau_i < \tau_{3-i} \le T) \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1)]}{P(\varkappa_2 > t \,|\, \mathcal{F}_t \vee \sigma(\varkappa_1))} \\
&= I(\varkappa_1 < \varkappa_2 \le t)\, E\big[R_v(X^1_v, X^2_v)\, I(t \vee u < v \le T) \,\big|\, \mathcal{F}_t\big]\big|_{u=\varkappa_1, v=\varkappa_2} \\
&\quad + I(\varkappa_2 > t \vee \varkappa_1) \int_{v=u}^{\infty} \frac{E[R_v(X^1_v, X^2_v)\, I(t \vee u < v \le T)\, P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_v) \,|\, \mathcal{F}_t]}{P(\tau_1 > u \vee t, \tau_2 \in du \,|\, \mathcal{F}_t) + P(\tau_2 > u \vee t, \tau_1 \in du \,|\, \mathcal{F}_t)}\bigg|_{u=\varkappa_1}
\end{aligned}
$$

on the event $\{\varkappa_1 \le T\}$, for all $0 \le t \le T$ and every $i = 1, 2$.

Therefore, summarising the facts proved above, we now formulate the following assertion.

**Proposition 4.3.** *Suppose that $r = 0$. The no-arbitrage price for the holders of the first- or second-to-default options in (4.18) are given by the sum of the expressions in (4.19) or (4.22) with (4.20), (4.21) or (4.23), (4.25), respectively. The latter terms are computed by means of the expressions in (4.4) or (4.9), (4.11) with (4.5) or (4.10), (4.12), and (4.24), respectively.*

**Remark 4.4.** This paper continues the development of credit risk in models with various information flows studied in [18] and [19]. The prospective research in this direction include the generalisations of the results to the case of several (three and more) risky assets and several (three and more) defaults given various filtrations representing different information flows. Other potential extensions include elaborations of the models with an explicit form of the joint transition densities of the dependent risky asset prices and default intensities.

# 5 Appendix

In this appendix, we reproduce explicit expressions from [19; Sections 3 and 4] for the conditional distributions of two *non-successive* default times given the accessible filtration generated by the market prices of the risky assets as well as given the observable filtrations.

## 5.1 Transition densities

Let us give the expressions for the transition density functions of the processes $(Y^l, A^l)$, $l = 0, 1, 2$, defined in (2.2)-(2.3) above. For this purpose, deleting the index $l$ in the notation for simplicity, we recall from [26; page 527] that, for a standard Brownian motion $W$, the random variable $A_t^{(\nu)} = \int_0^t e^{2(W_s+\nu s)} ds$ has the conditional distribution

$$P\Big(A_t^{(\nu)} \in da \,\Big|\, W_t + \nu t = x\Big) = p(t,x,a)\, da \tag{5.1}$$

where the density function $p(t,x,a)$ is given by

$$\begin{aligned} p(t,x,a) = \; & \frac{1}{\pi a^2} \exp\left(\frac{x^2+\pi^2}{2t} + x - \frac{1+e^{2x}}{2a}\right) \\ & \times \int_0^\infty \exp\left(-\frac{w^2}{2t} - \frac{e^x}{a}\cosh(w)\right) \sinh(w) \sin\left(\frac{\pi w}{t}\right) dw \end{aligned} \tag{5.2}$$

with $t, a > 0$ and $x \in \mathbb{R}$, and $\nu \in \mathbb{R}$ given and fixed. This fact yields that the random vector $(2(W_t + \nu t), A_t^{(\nu)})$ has the distribution:

$$P\Big(2(W_t + \nu t) \in dx, A_t^{(\nu)} \in da\Big) = q(t,x,a)\, dxda \tag{5.3}$$

where the density function $q(t,x,a)$ is given by

$$\begin{aligned} q(t,x,a) &= p\Big(t, \frac{x}{2}, a\Big) \frac{1}{2\sqrt{t}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\Big(\frac{x-2\nu t}{2\sqrt{t}}\Big)^2\right) \\ &= \frac{1}{(2\pi)^{3/2} a^2 \sqrt{t}} \exp\left(\frac{\pi^2}{2t} + \Big(\frac{\nu+1}{2}\Big) x - \frac{\nu^2}{2} t - \frac{1+e^x}{2a}\right) \\ &\quad \times \int_0^\infty \exp\left(-\frac{w^2}{2t} - \frac{e^{x/2}}{a}\cosh(w)\right) \sinh(w) \sin\left(\frac{\pi w}{t}\right) dw \end{aligned} \tag{5.4}$$

with $t, a > 0$ and $x \in \mathbb{R}$ (see also [14] and [25] for related expressions in terms of Hermite functions). Therefore, defining the Markov process $(Y, A) = (Y_t, A_t)_{t \ge 0}$ by

$$Y_t = \exp\left(\Big(\beta - \frac{\gamma^2}{2}\Big) t + \gamma W_t\right) \quad \text{and} \quad A_t = \int_0^t Y_s \, ds \tag{5.5}$$

for all $t \ge 0$, one obtains that the random vector $(Y_T/Y_t, (A_T - A_t)/Y_t)$ has the distribution

$$P\big(Y_T/Y_t \in dy, (A_T - A_t)/Y_t \in da\big) = P\big(Y_{T-t} \in dy, A_{T-t} \in da\big) = g_{T-t}(y,a)\, dyda \tag{5.6}$$

where the density function $g_{T-t}(y,a)$ is given by

$$\begin{aligned} g_{T-t}(y,a) &= \frac{\gamma^2}{4y} q\left(\frac{\gamma^2}{4}(T-t), \ln(y), \frac{\gamma^2 a}{4}\right) \\ &= \frac{2\sqrt{2}}{\pi^{3/2}\gamma^3} \frac{1}{a^2 y \sqrt{T-t}} \exp\left(\frac{2\pi^2}{\gamma^2(T-t)} + \frac{\beta}{\gamma^2}\ln(y) - \Big(\frac{\beta}{\gamma} - \frac{\gamma}{2}\Big)^2 \frac{(T-t)}{2} - \frac{2(1+y)}{\gamma^2 a}\right) \\ &\quad \times \int_0^\infty \exp\left(-\frac{2w^2}{\gamma^2(T-t)} - \frac{4\sqrt{y}}{\gamma^2 a}\cosh(w)\right) \sinh(w) \sin\left(\frac{4\pi w}{\gamma^2(T-t)}\right) dw \end{aligned} \tag{5.7}$$

for all $T - t, y, a > 0$. Note that the formulas above were also used in [20; Section 4] for the computation of the marginal density of the posterior probability process in the one-dimensional quickest change-point detection problem.

We also recall the transition density functions of the geometric Brownian motions $Z$, defined as in (2.4) above. It follows that the random variable $Z_T/Z_t$ has the distribution

$$P\big(Z_T/Z_t \in dz\big) = P\big(Z_{T-t} \in dz\big) = h_{T-t}(z)\, dz \tag{5.8}$$

where the density function $h_{T-t}(z)$ is given by

$$h_{T-t}(z) = \frac{1}{\theta z \sqrt{2\pi(T-t)}} \exp\bigg( - \frac{(\ln(z) - (\eta - \theta^2/2)(T-t))^2}{2\theta^2(T-t)} \bigg) \tag{5.9}$$

for all $T-t, z > 0$.

## 5.2 Conditional distributions of $\tau_i$, $i = 1, 2$, under $(\mathcal{F}_t)_{t \geq 0}$

We first recall the computations of the conditional distributions $P(\tau_i > u \,|\, \mathcal{F}_t)$ of the default times $\tau_i$, $i = 1, 2$, given the reference filtration $(\mathcal{F}_t)_{t \geq 0}$, for all $t, u \geq 0$, from [19; Section 3]. The equalities

$$P(\tau_i > u \,|\, \mathcal{F}_\infty) = e^{-\delta_i A^0_u - \lambda_i A^i_u} = P(\tau_i > u \,|\, \mathcal{F}_t) \quad \text{for} \quad 0 \leq u \leq t \tag{5.10}$$

hold, so that the equality

$$P(\tau_i \in du \,|\, \mathcal{F}_\infty) = e^{-\delta_i A^0_u - \lambda_i A^i_u} \, (\delta_i Y^0_u + \lambda_i Y^i_u) \, dt = P(\tau_i \in du \,|\, \mathcal{F}_t) \quad \text{for} \quad 0 \leq u \leq t \tag{5.11}$$

is satisfied, for every $i = 1, 2$ (see [19; Formulae (3.1)-(3.3)]). Furthermore, the equality

$$P(\tau_i > u \,|\, \mathcal{F}_t) = e^{-\delta_i A^0_t - \lambda_i A^i_t} \, C^i_{u-t}(Y^0_t, Y^i_t) \quad \text{for} \quad 0 \leq t < u \tag{5.12}$$

holds, where

$$C^i_{u-t}(y_0, y_i) = \int_0^\infty \int_0^\infty e^{-\delta_i y_0 a_0} \, g^0_{u-t}(y'_0, a_0) \, dy'_0 da_0 \int_0^\infty \int_0^\infty e^{-\lambda_i y_i a_i} \, g^i_{u-t}(y'_i, a_i) \, dy'_i da_i \tag{5.13}$$

for all $0 \leq t < u \leq T$ and every $i = 1, 2$, where the functions $g^l$, $l = 0, 1, 2$, stand for $g$ (5.7), while $\beta$ and $\gamma$ stand for $\beta_l$ and $\gamma_l$, for $l = 0, 1, 2$, defined in (2.3) (see [19; Formulae (3.4)-(3.5)]). Moreover, the equality

$$P(\tau_i \in du \,|\, \mathcal{F}_t) = e^{-\delta_i A^0_t - \lambda_i A^i_t} \, D^i_{u-t}(Y^0_t, Y^i_t) \, du \quad \text{for} \quad 0 \leq t < u \tag{5.14}$$

is satisfied, for every $i = 1, 2$, where

$$\begin{aligned} &D^i_{u-t}(y_0, y_i) \\ &= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-\delta_i y_0 a_0 - \lambda_i y_i a_i} \, (\delta_i y_0 y'_0 + \lambda_i y_i y'_i) \, g^0_{u-t}(y'_0, a_0) \, g^i_{u-t}(y'_i, a_i) \, dy'_0 da_0 dy'_i da_i \end{aligned} \tag{5.15}$$

for all $0 \leq t < u \leq T$ and every $i = 1, 2$ (see [19; Formulae (3.6)-(3.7)]).

We now continue with the results of [19; Sections 3 and 4], which tell that the equalities

$$\begin{aligned} &P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_\infty) = e^{-\delta_i A^0_u - \lambda_i A^i_u - \delta_{3-i} A^0_v - \lambda_{3-i} A^{3-i}_v} = P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_t) \\ &\quad \text{for} \quad 0 \leq u, v \leq t \end{aligned} \tag{5.16}$$

hold, so that the equalities

$$P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_\infty) \tag{5.17}$$
$$= e^{-\delta_i A_u^0 - \lambda_i A_u^i - \delta_{3-i} A_v^0 - \lambda_{3-i} A_v^{3-i}} \, (\delta_i Y_u^0 + \lambda_i Y_u^i) \, (\delta_{3-i} Y_v^0 + \lambda_{3-i} Y_v^{3-i}) \, du dv$$
$$= P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_t) \quad \text{for} \quad 0 \le u, v \le t$$

are satisfied, for every $i = 1, 2$ (see [19; Formulae (3.8)-(3.10)]). Furthermore, the equality

$$P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_t) = e^{-\delta_i A_u^0 - \lambda_i A_u^i - \delta_{3-i} A_t^0 - \lambda_{3-i} A_t^{3-i}} \, C_{v-t}^{3-i}(Y_t^0, Y_t^{3-i}) \quad \text{for} \quad 0 \le u \le t < v \tag{5.18}$$

holds, where $C_{v-t}^{3-i}(y_0, y_{3-i})$ is given as in (5.13) above, for every $i = 1, 2$ (see [19; Formulae (3.11)-(3.12)]). Then, the equality

$$P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_t) = e^{-\delta_i A_t^0 - \lambda_i A_t^i - \delta_i A_u^0 - \lambda_i A_u^i} \, (\delta_i Y_u^0 + \lambda_i Y_u^{3-i}) \, D_{v-t}^{3-i}(Y_t^0, Y_t^{3-i}) \, du dv \quad \text{for} \quad 0 \le u \le t < v \tag{5.19}$$

is satisfied, where $D_{v-t}^{3-i}(y_0, y_{3-i})$ is given in (5.15) above, for every $i = 1, 2$ (see [19; Formula (3.13)]).

Finally, we have that the equality

$$P(\tau_i > u, \tau_{3-i} > v \,|\, \mathcal{F}_t) = e^{-(\delta_i + \delta_{3-i}) A_t^0 - \lambda_i A_t^i - \lambda_{3-i} A_t^{3-i}} \, \overline{C}_{u-t, v-u}^{3-i}(Y_t^0, Y_t^{3-i}, Y_t^i) \quad \text{for} \quad 0 \le t < u < v \tag{5.20}$$

holds, where

$$\overline{C}_{u-t,v-u}^{3-i}(y_0, y_{3-i}, y_i) \tag{5.21}$$
$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-(\delta_i + \delta_{3-i}) y_0 a_0 - \lambda_i y_i a_i - \lambda_{3-i} y_{3-i} a_{3-i}} \, C_{v-u}^{3-i}(y_0 y_0', y_{3-i} y_{3-i}')$$
$$\times \, g_{u-t}^0(y_0', a_0) \, g_{u-t}^i(y_i', a_i) \, g_{u-t}^{3-i}(y_{3-i}', a_{3-i}) \, dy_0' da_0 dy_i' da_i dy_{3-i}' da_{3-i}$$

for all $0 \le t < u < v \le T$ and every $i = 1, 2$ (see [19; Formulae (3.16)-(3.18)]). Hence, the equality

$$P(\tau_i \in du, \tau_{3-i} \in dv \,|\, \mathcal{F}_t) = e^{-(\delta_i + \delta_{3-i}) A_t^0 - \lambda_i A_t^i - \lambda_{3-i} A_t^{3-i}} \, \overline{D}_{u-t, v-u}^{3-i}(Y_t^0, Y_t^{3-i}, Y_t^i) \, du dv \quad \text{for} \quad 0 \le t \le u < v \tag{5.22}$$

is satisfied, where

$$\overline{D}_{u-t,v-u}^{3-i}(y_0, y_{3-i}, y_i) \tag{5.23}$$
$$= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-(\delta_i + \delta_{3-i}) y_0 a_0 - \lambda_i y_i a_i - \lambda_{3-i} y_{3-i} a_{3-i}} \, (\delta_i y_0 y_0' + \lambda_i y_i y_i')$$
$$\times \, D_{v-u}^{3-i}(y_0 y_0', y_{3-i} y_{3-i}') \, g_{u-t}^0(y_0', a_0) \, g_{u-t}^i(y_i', a_i) \, g_{u-t}^{3-i}(y_{3-i}', a_{3-i}) \, dy_0' da_0 dy_i' da_i dy_{3-i}' da_{3-i}$$

for all $0 \le t < u < v \le T$ and every $i = 1, 2$ (see [19; Formulae (4.12)-(4.14)]).

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# References


[1] Assefa, S., Bielecki, T. R., Crépey, S. and Jeanblanc, M. (2011). CVA computation for counterparty risk assessment in credit portfolios. In the Volume *Credit Risk Frontiers: Subprime Crisis, Pricing and Hedging, CVA, MBS, Ratings, and Liquidity.* Bielecki, T. R., Brigo, D., and Patras, F. (eds.) Wiley.

[2] Aksamit, A. and Jeanblanc, M. (2017). *Enlargements of Filtrations with Finance in View.* Springer.

[3] Bielecki, T. R., Crépey, S., Jeanblanc, M. and Zargari, B. (2012). Valuation and hedging of CDS counterparty exposure in a Markov copula model. *International Journal of Theoretical and Applied Finance* **15** (1–39).

[4] Bielecki, T. R. and Rutkowski, M. (2004). *Credit Risk: Modeling, Valuation and Hedging.* (Second Edition) Springer, Berlin.

[5] Brémaud, P. and Yor, M. (1978). Changes of filtrations and of probability measures. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **45** (269-–295).

[6] Brigo, D., Capponi, A. and Pallavicini, A. (2014). Arbitrage-free bilateral counterparty risk valuation under collateralization and application to credit default swaps. *Mathematical Finance* **24** (125–146).

[7] Brigo, D. and Chourdakis, K. (2009). Counterparty risk for credit default swaps: impact of spread volatility and default correlation. *International Journal of Theoretical and Applied Finance* **12** (1007–1026).

[8] Brigo, D., Morini, M. and Pallavicini, A. (2013). *Counterparty Credit Risk, Collateral and Funding: with Pricing Cases for all Asset Classes.* Wiley.

[9] Callegaro, G., Jeanblanc, M. and Zargari, B. (2013). Carthagian enlargement of filtrations. *ESAIM: Probability and Statistics* **17** (550–566).

[10] Cesari, G., Aquilina, J. and Charpillon, N. (2010). *Modelling, Pricing, and Hedging Counterparty Credit Exposure.* Springer.

[11] Crépey, S., Bielecki, T. R. and Brigo, D. (2014). *Counterparty Risk and Funding. A Tale of Two Puzzles.* Taylor and Francis.

[12] Dellacherie, C. (1972). *Capacités et Processus Stochastiques.* Springer.

[13] Dellacherie, C. and Meyer, P.-A. (1975). *Probabilités et Potentiel, Chapitres I-IV.* Hermann, Paris. (English translation, 1978: *Probabilities and Potential, Chapters I-IV.* North-Holland.)

[14] Dufresne, D. (2001). The integral of geometric Brownian motion. *Advances in Applied Probability* **33** (223–241).

[15] El Karoui, N., Jeanblanc, M. and Jiao, Y. (2010). What happens after a default: the conditional density approach. *Stochastic Processes and their Applications* **120**(7) (1011–1032).

[16] El Karoui, N., Jeanblanc, M. and Jiao, Y. (2012). Density approach in modeling successive defaults. *SIAM Journal on Financial Mathematics* **6** (1–21).

[17] Elliott, R. J., Jeanblanc, M. and Yor, M. (2000). On models of default risk. *Mathematical Finance* **10** (179-–195).

[18] Gapeev, P. V. and Jeanblanc, M. (2019). Defaultable claims in switching models with partial information. *International Journal of Theoretical and Applied Finance* **22**(1950006) (16 pp).

[19] Gapeev, P. V. and Jeanblanc, M. (2020). Credit default swaps in two-dimensional models with various information flows. *International Journal of Theoretical and Applied Finance* **23**(2050010) (28 pp).

[20] Gapeev, P. V. and Peskir, G. (2006). The Wiener disorder problem with finite horizon. *Stochastic Processes and their Applications* **116**(12) (1770–1791).

[21] Gregory, J. (2009). *Counterparty Credit Risk: The New Challenge for Global Financial Markets.* Wiley.

[22] Lando, D. (1998). On Cox processes and credit risky securities. *Review of Derivatives Research* **2** (99–120).

[23] Mansuy, R. and Yor, M. (2004). *Random Times and Enlargements of Filtrations in a Brownian Setting.* Lecture Notes in Mathematics **1873**. Springer, Berlin

[24] Schönbucher, Ph. (2003). *Credit Derivatives Pricing Models: Models, Pricing and Implementation.* Wiley.

[25] Schröder, M. (2003). On the integral of geometric Brownian motion. *Advances in Applied Probability* **35** (159–183).

[26] Yor, M. (1992). On some exponential functionals of Brownian motion. *Advances in Applied Probability* **24** (509–531).