Discounted optimal stopping problems in first-passage time models with random thresholds

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We derive closed-form solutions to some discounted optimal stopping problems related to the perpetual American cancellable dividend paying put and call option pricing problems in an extension of the Black-Merton-Scholes model. The cancellation times are assumed to occur when the underlying risky asset price process hits some unobservable random thresholds. The optimal stopping times are shown to be the first times at which the asset price reaches stochastic boundaries depending on the current values of its running maximum and minimum processes. The proof is based on the reduction of the original optimal stopping problems to the associated free-boundary problems and the solution of the latter problems by means of the smooth-fit and modified normal-reflection conditions. We show that the optimal stopping boundaries are characterised as the maximal and minimal solutions of certain first-order nonlinear ordinary differential equations.

1. Introduction

The main aim of this paper is to present closed-form solutions to the discounted optimal stopping problems with the values:

$$V_1^* = \sup_\tau E\Big[e^{-r\tau}\,(K_1 - X_\tau)\,I(\tau < \theta_1) + e^{-r\theta_1}\,(\eta_1 + \varkappa_1\,X_{\theta_1})\,I(\theta_1 \le \tau) + \frac{\nu_1}{r}\,(1 - e^{-r(\tau\wedge\theta_1)})\Big] \quad (1.1)$$

and

$$V_2^* = \sup_\tau E\Big[e^{-r\tau}\,(X_\tau - K_2)\,I(\tau < \theta_2) - e^{-r\theta_2}\,(\eta_2 + \varkappa_2\,X_{\theta_2})\,I(\theta_2 \le \tau) + \frac{\nu_2}{r}\,(1 - e^{-r(\tau\wedge\theta_2)})\Big] \quad (1.2)$$

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Mathematics Subject Classification 2010: Primary 60G40, 60G44, 60J65. Secondary 91B25, 60J60, 35R35.

Key words and phrases: Discounted optimal stopping problem, Brownian motion, first passage times, running maximum and minimum processes, smooth-fit and normal-reflection conditions, perpetual American options, random dividends, free-boundary problem, a change-of-variable formula with local time on surfaces.

Date: September 17, 2020

for some given constants $K_i > 0$, $\eta_i > 0$, $\varkappa_i > 0$, and $\nu_i > 0$, for every $i = 1, 2$, where $I(\cdot)$ denotes the indicator function. Here, for a precise formulation of the problem, we consider a probability space $(\Omega, \mathcal{G}, P)$ with a standard Brownian motion $B = (B_t)_{t \ge 0}$ and a strictly positive integrable random variable ξ which has a strictly increasing continuously differentiable cumulative distribution function $F(x)$ such that $F(0) \equiv 1 - F(\infty) = 0$ and $0 < F(x) < 1$ as well as $F'(x) > 0$, for all $x > 0$ (B and ξ are supposed to be independent under the probability measure P). We assume that the process $X = (X_t)_{t \ge 0}$ is given by:

$$X_t = x \, \exp\Big(\big(\mu - \sigma^2/2 \big) \, t + \sigma \, B_t \Big) \tag{1.3}$$

so that it solves the stochastic differential equation:

$$dX_t = \mu \, X_t \, dt + \sigma \, X_t \, dB_t \quad (X_0 = x) \tag{1.4}$$

where $x > 0$ is fixed, and $r > 0$, $\delta > 0$, and $\sigma > 0$ are some given constants. Suppose that the process X describes the price of a risky asset in a financial market, where r is the riskless interest rate, δ is the dividend rate paid to the asset holders, and σ is the volatility rate. We also define the random times θ_i, $i = 1, 2$, by:

$$\theta_1 = \inf\{t \ge 0 \, | \, X_t \ge \xi\} \quad \text{and} \quad \theta_2 = \inf\{t \ge 0 \, | \, X_t \le \xi\} \tag{1.5}$$

and assume that cancellations of certain dividend paying contingent claims are announced at these times, by the issuers of those products depending on the market price of the underlying risky asset. Here, K_i is the strike price, $\eta_i + \varkappa_i X$ is a (linear) recovery, and ν_i is the promised rate of continuously paid dividends. These properties particularly mean that the holders of such contingent claims impose some prior (Bayesian) distribution on the unknown for them and unobservable from the market cancellation thresholds ξ. Note that European contingent claims with fixed finite-time horizon which have similar payoff and dividend structure were described in Bielecki and Rutkowski [7; Section 2.1] and in the related references therein.

Suppose that the suprema in (1.1) and (1.2) are taken over all stopping times τ with respect to the natural filtration $(\mathcal{F}_t)_{t \ge 0}$ of the process X, and the expectations there are taken with respect to the risk-neutral probability measure P. In this view, the values V_1^* and V_2^* in (1.1) and (1.2) can be interpreted as the rational (or no-arbitrage) ex-dividend prices of the perpetual American cancellable dividend paying put and call options in an extension of the Black-Merton-Scholes model (see, e.g. [40; Chapter VII, Section 3g]). Observe that the structure of the reward functionals in (1.1) and (1.2) allows to describe the associated contracts as standard game (or Israeli) contingent claims introduced by Kifer [23]. Such contacts enable their issuers to exercise their right to withdraw the contracts prematurely, by paying some penalties agreed in advance. Further developments of the Israeli options and the associated zero-sum optimal stopping (Dynkin) games were provided by Kyprianou [25], Kühn and Kyprianou [24], Kallsen and Kühn [22], Baurdoux and Kyprianou [3]-[5], Ekström and Villeneuve [11], and Baurdoux, Kyprianou, and Pardo [6] among others. In contrast to the concept of game contingent claims mentioned above, in the present paper, we study the cancellable American options in which the exogenous terminations of the contracts occur at the first times when the underlying risky asset price processes reach certain random thresholds being unknown and unobservable to the usual investors trading in the market. We assume that these thresholds are independent of the geometric Brownian motion describing the underlying risky asset price. Some extensive overviews of the perpetual American options in diffusion models of financial markets and other related results in the area are provided in Shiryaev [40; Chapter VIII; Section 2a], Peskir and Shiryaev [34; Chapter VII; Section 25], and Detemple [9] among others. Note that other applications of the concept described above include the consideration of perpetual American dividend paying options with credit risk which are defaulted at the times when the underlying risky asset price processes reach random thresholds. Other perpetual American defaultable dividend paying options were considered in [14] in some models with full and partial information.

We further study the problems of (1.1) and (1.2) as the associated optimal stopping problems of (2.5) and (2.6) for the two-dimensional continuous Markov processes having the underlying risky asset price X and its running maximum S or minimum Q as their state space components. The resulting problems turn out to be necessarily two-dimensional in the sense that they cannot be reduced to optimal stopping problems for one-dimensional Markov processes. Note that the reward functionals of the optimal stopping problems in (2.5) and (2.6) contain complicated stochastic integrals with respect to the running maximum and minimum processes. This feature initiates further developments of techniques to determine the structure of the associated continuation and stopping regions as well as appropriate modifications of the normal-reflection conditions in the equivalent free-boundary problems. Discounted optimal stopping problems for the running maxima and minima of the initial continuous (diffusion-type) processes were initiated by Shepp and Shiryaev [37]-[39] and further developed by Pedersen [29], Guo and Shepp [20], Gapeev [12], Guo and Zervos [21], Peskir [32]-[33], Glover, Hulley, and Peskir [18], Gapeev and Rodosthenous [15]-[17], Rodosthenous and Zervos [36], and Gapeev, Kort, and Lavrutich [13] among others. It was shown, by means of the established by Peskir [30] maximality principle for solutions of optimal stopping problems, which is equivalent to the superharmonic characterisation of payoff functions, that the optimal stopping boundaries are given by the appropriate extremal solutions of certain (systems of) first-order nonlinear ordinary differential equations. More complicated optimal stopping problems in models with spectrally negative Lévy processes and their running maxima were studied by Asmussen, Avram, and Pistorius [1], Avram, Kyprianou, and Pistorius [2], Ott [28], and Kyprianou and Ott [26] among others.

The rest of the paper is organised as follows. In Section 2, we embed the original problems of (1.1) and (1.2) into the optimal stopping problems of (2.5) and (2.6) for the two-dimensional continuous Markov processes (X, S) and (X, Q) defined in (1.3) and (2.1). It is shown that the optimal stopping times τ_1^* and τ_2^* are the first times at which the process X reaches some lower or upper boundaries $a^*(S)$ and $b^*(Q)$ depending on the current values of the processes S or Q, respectively. In Section 3, we derive closed-form expressions for the associated value functions $V_1^*(x, s)$ and $V_2^*(x, q)$ as solutions to the equivalent free-boundary problems and apply the modified normal-reflection conditions at the edges of the two-dimensional state spaces for (X, S) or (X, Q) to characterise the optimal stopping boundaries $a^*(S)$ and $b^*(Q)$ as the maximal and minimal solutions to the resulting first-order nonlinear ordinary differential equations. In Section 4, by using the change-of-variable formula with local time on surfaces from Peskir [31], we verify that the solutions of the free-boundary problems provide the solutions of the original optimal stopping problems. The main results of the paper are stated in Theorem 4.1.

2. Preliminaries

In this section, we introduce the setting and notation of the two-dimensional optimal stopping problems which are related to the pricing of perpetual American cancellable dividend paying put and call options and formulate the equivalent free-boundary problems.

2.1 The optimal stopping problems. Let us now define the associated with X *running maximum and minimum* processes $S = (S_t)_{t \ge 0}$ and $Q = (Q_t)_{t \ge 0}$ by:

$$S_t = s \vee \Big(\max_{0 \le u \le t} X_u \Big) \quad \text{and} \quad Q_t = q \wedge \Big(\min_{0 \le u \le t} X_u \Big) \tag{2.1}$$

for some arbitrary $s \ge x \ge q > 0$. Then, the conditional probabilities of the events that cancellation occurs before any time $t \ge 0$ take the form:

$$P(\theta_1 \le t \,|\, \mathcal{F}_t) = P(S_t \ge \xi \,|\, \mathcal{F}_t) = F(S_t) \quad \text{and} \quad P(\theta_2 \le t \,|\, \mathcal{F}_t) = P(Q_t \le \xi \,|\, \mathcal{F}_t) = G(Q_t), \tag{2.2}$$

where $F(x)$ is the cumulative distribution function of ξ, and we set $G(x) = 1 - F(x)$, for all $x > 0$. Thus, by virtue of the assumptions made above, we have $G(0) = 1 - G(\infty) = 1$ and $0 < G(x) < 1$ as well as $G'(x) < 0$, for all $x > 0$. In this case, the values of (1.1) and (1.2) admit the representations:

$$V_1^* = \sup_\tau E\bigg[e^{-r\tau} \, (K_1 - X_\tau) \, G(S_\tau) + \int_0^\tau e^{-rt} \, (\eta_1 + \varkappa_1 \, X_t) \, dF(S_t) + \int_0^\tau e^{-rt} \, \nu_1 \, G(S_t) \, dt \bigg] \tag{2.3}$$

and

$$V_2^* = \sup_\tau E\bigg[e^{-r\tau} \, (X_\tau - K_2) \, F(Q_\tau) - \int_0^\tau e^{-rt} \, (\eta_2 + \varkappa_2 \, X_t) \, dG(Q_t) + \int_0^\tau e^{-rt} \, \nu_2 \, F(Q_t) \, dt \bigg] \tag{2.4}$$

where the suprema are taken over all stopping times of τ with respect to $(\mathcal{F}_t)_{t \ge 0}$. In this case, taking into account the facts that the processes S and Q may change their values only when $X_t = S_t$ and $X_t = Q_t$, for $t \ge 0$, respectively, we see that the problems in (2.3) and (2.4) can be naturally embedded into the optimal stopping problems for the (time-homogeneous strong) Markov processes $(X, S) = (X_t, S_t)_{t \ge 0}$ and $(X, Q) = (X_t, Q_t)_{t \ge 0}$ with the value functions:

$$\begin{aligned} &V_1^*(x, s) \\ &= \sup_\tau E_{x,s}\bigg[e^{-r\tau} \, (K_1 - X_\tau) \, G(S_\tau) + \int_0^\tau e^{-rt} \, (\eta_1 + \varkappa_1 \, S_t) \, F'(S_t) \, dS_t + \int_0^\tau e^{-rt} \, \nu_1 \, G(S_t) \, dt \bigg] \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &V_2^*(x, q) \\ &= \sup_\tau E_{x,q}\bigg[e^{-r\tau} \, (X_\tau - K_2) \, F(Q_\tau) - \int_0^\tau e^{-rt} \, (\eta_2 + \varkappa_2 \, Q_t) \, G'(Q_t) \, dQ_t + \int_0^\tau e^{-rt} \, \nu_2 \, F(Q_t) \, dt \bigg] \end{aligned} \tag{2.6}$$

where $E_{x,s}$ and $E_{x,q}$ denote the expectations with respect to the probability measures $P_{x,s}$ and $P_{x,q}$ under which the two-dimensional Markov processes (X, S) and (X, Q) defined in (1.3) and (2.1) start at $(x, s) \in E_1 = \{(x, s) \in \mathbb{R}^2 \,|\, 0 < x \le s\}$ and $(x, q) \in E_2 = \{(x, q) \in \mathbb{R}^2 \,|\, 0 < q \le x\}$, respectively. We further obtain solutions to the optimal stopping problems in (2.5) and (2.6) and verify in Theorem 4.1 below that the value functions $V_1^*(x, s)$ and $V_2^*(x, q)$ are the solutions of the problems in (2.3) and (2.4), and thus, of the original problems in (1.1) and (1.2) under $s = x$ and $q = x$, respectively.

2.2 The structure of optimal stopping times. By means of standard applications of Itô's formula (see, e.g. [27; Theorem 4.4] or [35; Chapter II, Theorem 3.2]) to the processes $e^{-rt}(K_1 - X_t)G(S_t)$ and $e^{-rt}(X_t - K_2)F(Q_t)$, we obtain the representations:

$$
\begin{aligned}
&e^{-rt}\,(K_1 - X_t)\,G(S_t) = (K_1 - x)\,G(s) + N_t^1 \\
&+ \int_0^t e^{-ru}\,(\delta X_u - rK_1)\,G(S_u)\,du + \int_0^t e^{-ru}\,(K_1 - X_u)\,I(X_u = S_u)\,G'(S_u)\,dS_u
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
&e^{-rt}\,(X_t - K_2)\,F(Q_t) = (x - K_2)\,F(q) + N_t^2 \\
&+ \int_0^t e^{-ru}\,(rK_2 - \delta X_u)\,F(Q_u)\,du + \int_0^t e^{-ru}\,(X_u - K_2)\,I(X_u = Q_u)\,F'(Q_u)\,dQ_u,
\end{aligned}
\tag{2.8}
$$

for all $t \ge 0$. Here, the processes $N^i = (N_t^i)_{t \ge 0}$, $i = 1, 2$, defined by:

$$
N_t^1 = -\int_0^t e^{-ru}\,\sigma\,X_u\,G(S_u)\,dB_u \quad \text{and} \quad N_t^2 = \int_0^t e^{-ru}\,\sigma\,X_u\,F(Q_u)\,dB_u \tag{2.9}
$$

are continuous uniformly integrable martingales under the probability measures $P_{x,s}$ and $P_{x,q}$, for each $(x, s) \in E_1$ and $(x, q) \in E_2$, respectively. Then, by applying Doob's optional sampling theorem (see, e.g. [27; Chapter III, Theorem 3.6] or [35; Chapter II, Theorem 3.2]), we obtain that the expected rewards from (2.5) and (2.6) admit the representations:

$$
\begin{aligned}
&E_{x,s}\bigg[e^{-r\tau}\,(K_1 - X_\tau)\,G(S_\tau) + \int_0^\tau e^{-rt}\,(\eta_1 + \varkappa_1\,S_t)\,F'(S_t)\,dS_t + \int_0^\tau e^{-rt}\,\nu_1\,G(S_t)\,dt\bigg] \\
&= (K_1 - x)\,G(s) + E_{x,s}\bigg[\int_0^\tau e^{-rt}\,(\delta X_t - rK_1 + \nu_1)\,G(S_t)\,dt \\
&\qquad + \int_0^\tau e^{-rt}\,\big(K_1 - \eta_1 - (1 + \varkappa_1)\,S_t\big)\,I(X_t = S_t)\,G'(S_t)\,dS_t\bigg]
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
&E_{x,q}\bigg[e^{-r\tau}\,(X_\tau - K_2)\,F(Q_\tau) - \int_0^\tau e^{-rt}\,(\eta_2 + \varkappa_2\,Q_t)\,G'(Q_t)\,dQ_t + \int_0^\tau e^{-rt}\,\nu_2\,F(Q_t)\,dt\bigg] \\
&= (x - K_2)\,F(q) + E_{x,q}\bigg[\int_0^\tau e^{-rt}\,(rK_2 + \nu_2 - \delta X_t)\,F(Q_t)\,dt \\
&\qquad + \int_0^\tau e^{-rt}\,\big((1 + \varkappa_2)\,Q_t - K_2 + \eta_2\big)\,I(X_t = Q_t)\,F'(Q_t)\,dQ_t\bigg]
\end{aligned}
\tag{2.11}
$$

for $(x, s) \in E_1$ and $(x, q) \in E_2$, for any stopping time τ of the process (X, S) or (X, Q), respectively. Observe from the structure of the integrands and the facts that $0 < G(x) < 1$ and $0 < F(x) < 1$, for all $x > 0$, that the expectations of the integrals in the second lines of the formulas in (2.10) and (2.11) are finite. Moreover, by virtue of the assumed integrability of the random variable ξ, it is seen that the expectations of the integrals in the third lines of the formulas in (2.10) and (2.11) are finite too.

We now recall the assumptions that $0 < F(x) < 1$ and $F'(x) > 0$, so that $0 < G(x) < 1$ and $G'(x) < 0$ holds, for all $x > 0$. Then, according to the properties that $0 < G(S_t) < 1$ and $G'(S_t) < 0$, for any $t \geq 0$, by virtue of the fact that the process S is positive and increasing, it is seen from the structure of the integrands in (2.10) that the optimal stopping time τ_1^* is infinite, whenever $K_1 \leq \nu_1/r$ holds. Furthermore, by virtue of the properties that $0 < G(S_t) < 1$ and $0 < F(Q_t) < 1$, for any $t \geq 0$, it follows from the structure of the first integrands in (2.10) and (2.11) that it is not optimal to exercise the cancellable put option when $\overline{a} \leq X_t < S_t$ with $\overline{a} = (rK_1 - \nu_1)/\delta$ under $K_1 > \nu_1/r$, while it is not optimal to exercise the cancellable call option when $Q_t < X_t \leq \underline{b}$ with $\underline{b} = (rK_2 + \nu_2)/\delta$, for any $t \geq 0$, respectively. In other words, these facts mean that the set $\{(x,s) \in E_1 \,|\, \overline{a} \leq x < s\}$ under $K_1 > \nu_1/r$ belongs to the continuation region C_1^* which has the form:

$$C_1^* = \big\{(x,s) \in E_1 \,\big|\, V_1^*(x,s) > (K_1 - x)\, G(s)\big\}, \tag{2.12}$$

while the set $\{(x,q) \in E_2 \,|\, q < x \leq \underline{b}\}$ belongs to the continuation region C_2^* which is given by:

$$C_2^* = \big\{(x,q) \in E_2 \,\big|\, V_2^*(x,q) > (x - K_2)\, F(q)\big\} \tag{2.13}$$

(see, e.g. [34; Chapter I, Subsection 2.2]).

Note that, by virtue of properties of the running maximum S and minimum Q from (2.1) of the geometric Brownian motion X from (1.3)-(1.4) (see, e.g. [10; Subsection 3.3] for similar arguments applied to the running maxima of the Bessel processes), it is seen that, for any $s' > 0$ and $q' > 0$ fixed and an infinitesimally small deterministic time interval Δ, we have:

$$S_\Delta = s' \vee \max_{0 \leq u \leq \Delta} X_u = s' \vee (s' + \Delta X) + o(\Delta) \quad \text{as } \Delta \downarrow 0 \tag{2.14}$$

and

$$Q_\Delta = q' \wedge \min_{0 \leq u \leq \Delta} X_u = q' \wedge (q' + \Delta X) + o(\Delta) \quad \text{as } \Delta \downarrow 0 \tag{2.15}$$

where we set $\Delta X = X_\Delta - s'$ and $\Delta X = X_\Delta - q'$, respectively. Observe that $\Delta S = o(\Delta)$ when $\Delta X \leq 0$, $\Delta S = \Delta X + o(\Delta)$ when $\Delta X > 0$, $\Delta Q = o(\Delta)$ when $\Delta X \geq 0$, and $\Delta Q = \Delta X + o(\Delta)$ when $\Delta X < 0$, where we set $\Delta S = S_\Delta - s'$ and $\Delta Q = Q_\Delta - q'$, and recall that $o(\Delta)$ denotes a random function satisfying $o(\Delta)/\Delta \to 0$ as $\Delta \downarrow 0$ (P-a.s.). In this case, using the following asymptotic formulas:

$$E_{s',s'}\big[\Delta X \,;\, \Delta X > 0\big] \equiv E_{s',s'}\big[\Delta X\, I(\Delta X > 0)\big] \sim s' \sqrt{\frac{\Delta}{2\pi}} \quad \text{as } \Delta \downarrow 0 \tag{2.16}$$

and

$$E_{q',q'}\big[\Delta X \,;\, \Delta X < 0\big] \equiv E_{q',q'}\big[\Delta X\, I(\Delta X < 0)\big] \sim -q' \sqrt{\frac{\Delta}{2\pi}} \quad \text{as } \Delta \downarrow 0 \tag{2.17}$$

as well as applying the representations in (2.10) and (2.11), we get:

$$\begin{aligned} &E_{s',s'}\Big[e^{-r\Delta}\,(\delta s' - rK_1 + \nu_1)\, G(s')\, \Delta + e^{-r\Delta}\, \big(K_1 - \eta_1 - (1 + \varkappa_1)\, s'\big)\, G'(s')\, \Delta S\Big] \\ &\sim e^{-r\Delta}\,(\delta s' - rK_1 + \nu_1)\, G(s')\, \Delta + e^{-r\Delta}\, \big(K_1 - \eta_1 - (1 + \varkappa_1)\, s'\big)\, G'(s')\, s' \sqrt{\frac{\Delta}{2\pi}} \quad \text{as } \Delta \downarrow 0 \end{aligned} \tag{2.18}$$

and

$$E_{q',q'}\Big[e^{-r\Delta}\,(rK_2+\nu_2-\delta q')\,F(q')\,\Delta+e^{-r\Delta}\,\big((1+\varkappa_2)\,q'-K_2+\eta_2\big)\,F'(q')\,\Delta Q\Big] \tag{2.19}$$
$$\sim e^{-r\Delta}\,(rK_2+\nu_2-\delta q')\,F(q')\,\Delta-e^{-r\Delta}\,\big((1+\varkappa_2)\,q'-K_2+\eta_2\big)\,F'(q')\,q'\sqrt{\frac{\Delta}{2\pi}} \quad \text{as } \Delta\downarrow 0$$

for each $s'>0$ and $q'>0$ fixed. Since we have $G'(s)<0$ and $F'(q)>0$, for all $s>0$ and $q>0$, we see that the resulting coefficients by the terms of order $\sqrt{\Delta}$ in the expressions of (2.18) and (2.19) are strictly positive, when $s'>s^*$ with $s^*=(K_1-\eta_1)/(1+\varkappa_1)$ under $K_1>\eta_1$ (or when $s'>0$ under $K_1\le\eta_1$), as well as $q'<q^*$ with $q^*=(K_2-\eta_2)/(1+\varkappa_2)$ under $K_2>\eta_2$. Hence, taking into account the facts that the process S is positive and increasing and the process Q is positive and decreasing, by virtue of the properties that $G'(S_t)<0$ and $F'(Q_t)>0$, for any $t\ge 0$, we may therefore conclude from the structure of the second integrands in (2.10) and (2.11) as well as the heuristic arguments presented in (2.18) and (2.19) above that it is not optimal to exercise the cancellable put option when $s^*<S_t=X_t$ with $s^*=(K_1-\eta_1)/(1+\varkappa_1)$ under $K_1>\eta_1$ (or when $0<S_t=X_t$ under $K_1\le\eta_1$), while it is not optimal to exercise the cancellable call option when $X_t=Q_t<q^*$ with $q^*=(K_2-\eta_2)/(1+\varkappa_2)$ under $K_2>\eta_2$, for any $t\ge 0$, respectively. In other words, these facts mean that the sets $d_1'=\{(x,s)\in E_1\,|\,x=s>s^*\}$ under $K_1>\eta_1$ (which becomes the whole diagonal $d_1=\{(x,s)\in E_1\,|\,x=s\}$ under $K_1\le\eta_1$) and $d_2'=\{(x,q)\in E_2\,|\,x=q<q^*\}$ under $K_2>\eta_2$ (which becomes an empty set under $K_2\le\eta_2$) surely belong to the continuation regions C_1^* and C_2^* in (2.12) and (2.13) above. For simplicity of presentation, we further assume that the inequalities $K_1>\eta_1\vee(\nu_1/r)$ and $K_2>\eta_2$ hold.

On the other hand, it follows from the definition of the processes (X,S) and (X,Q) in (1.3) and (2.1) and the structure of the rewards in (2.5) and (2.6) with the representations in (2.10) and (2.11) that, for each $s>0$ fixed, there exists a sufficiently small $x>0$ such that the point (x,s) belongs to the stopping region D_1^* which has the form:

$$D_1^*=\big\{(x,s)\in E_1\,\big|\,V_1^*(x,s)=(K_1-x)\,G(s)\big\}, \tag{2.20}$$

while, for each $q>0$ fixed, there exists a sufficiently large $x>0$ such that the point (x,q) belongs to the stopping region D_2^* which is given by:

$$D_2^*=\big\{(x,q)\in E_2\,\big|\,V_2^*(x,q)=(x-K_2)\,F(q)\big\}, \tag{2.21}$$

respectively (see, e.g. [34; Chapter I, Subsection 2.2]). According to arguments similar to the ones applied in [10; Subsection 3.3] and [30; Subsection 3.3], the latter properties can be explained by the facts that the costs of waiting until the process X coming from such a small $x>0$ increases the current value of the running maximum process S, as well as the costs of waiting until the process X coming from such a large $x>0$ decreases the current value of the running minimum process Q, may be too large due to the presence of the discounting factor in the reward functionals of (2.5) and (2.6). It is seen from the results of Theorem 4.1 proved below that the value functions $V_1^*(x,s)$ and $V_2^*(x,q)$ are continuous, so that the sets C_1^* and C_2^* in (2.12) and (2.13) are open, while the sets D_1^* and D_2^* in (2.20) and (2.21) are closed.

Observe that, if we take some $(x,s)\in D_1^*$ from (2.20) and use the fact that the process (X,S) started at some (x',s) such that $x'<x$ passes through the point (x,s) before hitting

the diagonal $d_1 = \{(x,s) \in E_1 \,|\, x = s\}$, then the equalities in (2.5) and (2.10) imply that $V_1^*(x',s) - (K_1 - x')G(s) \le V_1^*(x,s) - (K_1 - x)G(s) = 0$ holds, so that $(x',s) \in D_1^*$. Moreover, if we take some $(x,q) \in D_2^*$ from (2.21) and use the fact that the process (X,Q) started at some (x',q) such that $x' > x$ passes through the point (x,q) before hitting the diagonal $d_2 = \{(x,q) \in E_2 \,|\, x = q\}$, then the equalities in (2.6) and (2.11) imply that $V_2^*(x',q) - (x' - K_2)F(q) \le V_2^*(x,q) - (x - K_2)F(q) = 0$ holds, so that $(x',q) \in D_2^*$. On the other hand, if take some $(x,s) \in C_1^*$ from (2.12) and use the fact that the process (X,S) started at (x,s) passes through some point (x'',s) such that $x'' > x$ before hitting the diagonal d_1, then the equalities in (2.5) and (2.10) yield that $V_1^*(x'',s) - (K_1 - x'')G(s) \ge V_1^*(x,s) - (K_1 - x)G(s) > 0$ holds, so that $(x'',s) \in C_1^*$. Moreover, if we take some $(x,q) \in C_2^*$ from (2.13) and use the fact that the process (X,Q) started at (x,q) passes through some point (x'',q) such that $x'' < x$ before hitting the diagonal d_2, then the equalities in (2.6) and (2.11) yield that $V_2^*(x'',q) - (x'' - K_2)F(q) \ge V_2^*(x,q) - (x - K_2)F(q) > 0$ holds, so that $(x'',q) \in C_2^*$. Hence, combining these arguments together with the comments in [10; Subsection 3.3] and [30; Subsection 3.3] and recalling the facts that the sets $d_1' = \{(x,s) \in E_1 \,|\, x = s > s^*\}$ and $d_2' = \{(x,q) \in E_2 \,|\, x = q < q^*\}$ surely belong to the continuation regions C_1^* and C_2^* in (2.12) and (2.13), respectively, we may conclude that there exist functions $a^*(s)$ and $b^*(q)$ satisfying the inequalities $a^*(s) < s \wedge \overline{a}$ with $\overline{a} = (rK_1 - \nu_1)/\delta$ and $b^*(q) > q \vee \underline{b}$ with $\underline{b} = (rK_2 + \nu_2)/\delta$, for all $s > \underline{s}$ and $q < \overline{q}$ and some $0 \le \underline{s} \le s^* \wedge \overline{a}$ and $\overline{q} \ge q^* \vee \underline{b}$ fixed, as well as the equalities $a^*(s) = s$ and $b^*(q) = q$, for all $s \le \underline{s}$ and $q \ge \overline{q}$, such that the continuation regions C_1^* and C_2^* in (2.12) and (2.13) have the form:

$$C_1^* = \big\{(x,s) \in E_1 \,\big|\, a^*(s) < x \le s\big\} \quad \text{and} \quad C_2^* = \big\{(x,q) \in E_2 \,\big|\, q \le x < b^*(q)\big\}, \tag{2.22}$$

while the stopping regions D_1^* and D_2^* in (2.20)-(2.21) are given by:

$$D_1^* = \big\{(x,s) \in E_1 \,\big|\, x \le a^*(s)\big\} \quad \text{and} \quad D_2^* = \big\{(x,q) \in E_2 \,\big|\, x \ge b^*(q)\big\}, \tag{2.23}$$

under $K_1 > \eta_1 \vee (\nu_1/r)$ and $K_2 > \eta_2$, respectively (see Figures 1 and 2 below for computer drawings of the optimal exercise boundaries $a^*(s)$ and $b^*(q)$).

We summarise the arguments shown above in the following assertion.

Lemma 2.1 *Let the processes (X,S) and (X,Q) be given by (1.3) and (2.1), with some $r > 0$, $\delta > 0$, and $\sigma > 0$ fixed, and the inequalities $K_1 > \eta_1 \vee (\nu_1/r)$ and $K_2 > \eta_2$ hold. Suppose that the random times θ_i, $i = 1, 2$, are defined in (1.5) for a strictly positive continuous integrable random variable ξ with a strictly increasing continuously differentiable cumulative distribution function $F(x) \equiv 1 - G(x)$ such that $F(0) = 1 - F(\infty) = 0$ and $0 < F(x) < 1$ as well as $F'(x) > 0$, for all $x > 0$. Then, the optimal stopping times in the problems of (2.5) and (2.6) have the structure:*

$$\tau_1^* = \inf\{t \ge 0 \,|\, X_t \le a^*(S_t)\} \quad \text{and} \quad \tau_2^* = \inf\{t \ge 0 \,|\, X_t \ge b^*(Q_t)\} \tag{2.24}$$

for some functions $a^*(s)$ and $b^*(q)$ satisfying the inequalities $a^*(s) < s \wedge \overline{a}$ with $\overline{a} = (rK_1 - \nu_1)/\delta$ and $b^*(q) > q \vee \underline{b}$ with $\underline{b} = (rK_2 + \nu_2)/\delta$, for all $s > \underline{s}$ and $q < \overline{q}$ and some $0 \le \underline{s} \le s^* \wedge \overline{a}$ and $\overline{q} \ge q^* \vee \underline{b}$ fixed, as well as the equalities $a^*(s) = s$ and $b^*(q) = q$, for all $s \le \underline{s}$ and $q \ge \overline{q}$, respectively.

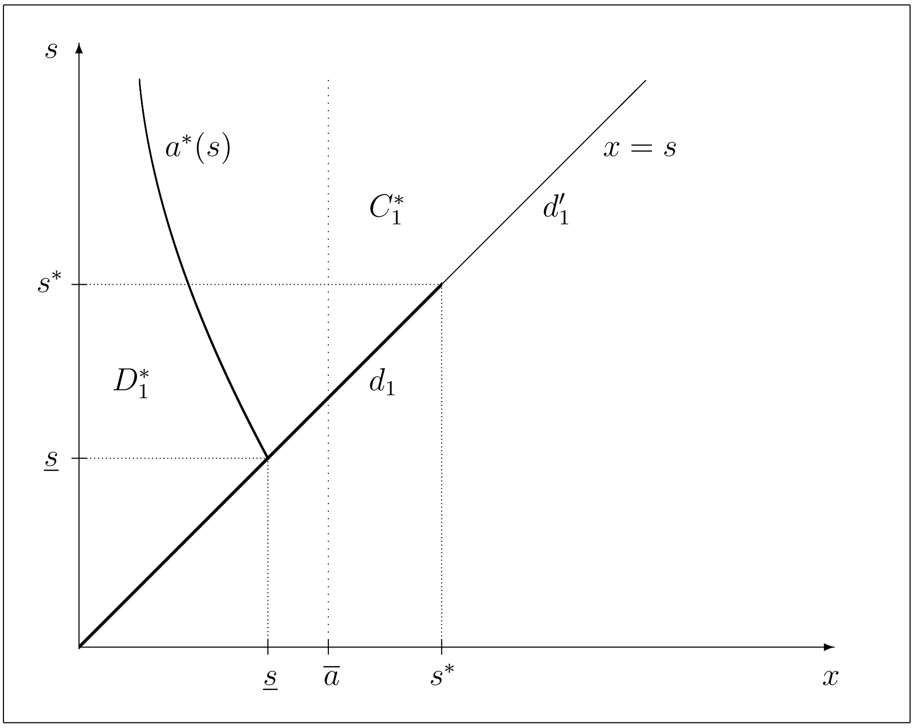

Figure 1. A computer drawing of the optimal exercise boundary $a^*(s)$.

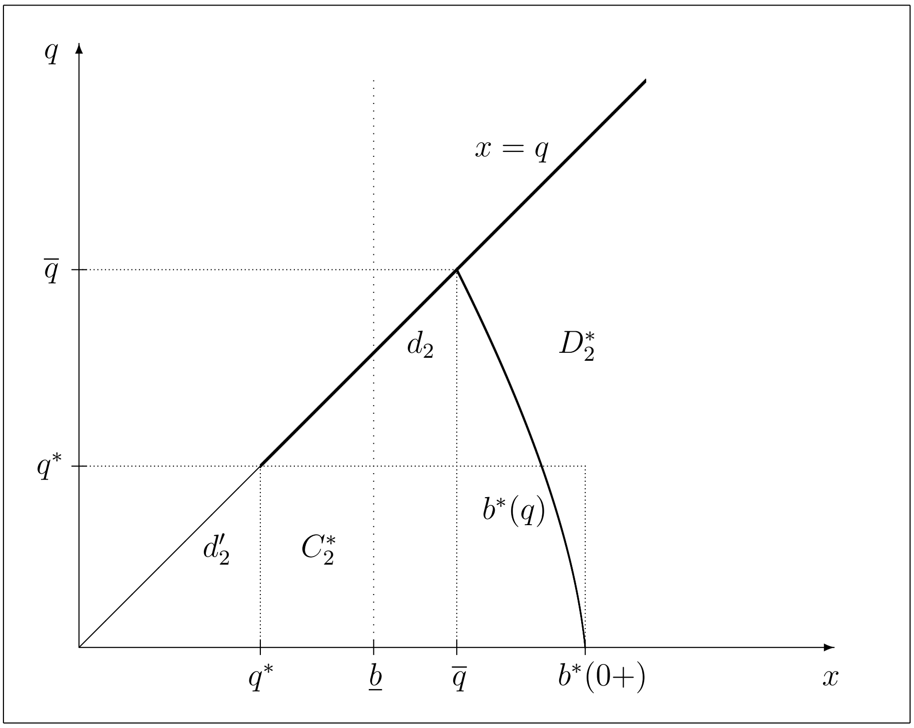

Figure 2. A computer drawing of the optimal exercise boundary $b^*(q)$.

2.3 The free-boundary problems. By means of standard arguments based on the application of Itô's formula, it is shown that the infinitesimal operator $\mathbb{L}$ of the process (X, S) or (X, Q) from (1.4) and (2.1) has the form:

$$\mathbb{L} = (r - \delta)\, x\, \partial_x + \frac{\sigma^2 x^2}{2}\, \partial_{xx} \quad \text{in} \quad 0 < x < s \quad \text{or} \quad 0 < q < x \tag{2.25}$$

$$\partial_s = 0 \quad \text{at} \quad x = s \quad \text{or} \quad \partial_q = 0 \quad \text{at} \quad x = q \tag{2.26}$$

(see, e.g. [30; Subsection 3.1]). In order to find analytic expressions for the unknown value functions $V_1^*(x, s)$ and $V_2^*(x, q)$ in (2.5) and (2.6) and the unknown boundaries $a^*(s)$ and $b^*(q)$ from (2.24), we apply the results of general theory for solving optimal stopping problems for Markov processes presented in [34; Chapter IV, Section 8] among others (see also [34; Chapter V, Sections 15-20] for optimal stopping problems for maxima processes and other related references). More precisely, for the original optimal stopping problems in (2.5) and (2.6), we formulate the associated free-boundary problems (see, e.g. [34; Chapter IV, Section 8]) and then verify in Theorem 4.1 below that the appropriate candidate solutions of the latter problems coincide with the solutions of the original problems. In other words, we reduce the optimal stopping problems of (2.5) and (2.6) to the following equivalent free-boundary problems:

$$(\mathbb{L}V_1 - rV_1)(x, s) = -\nu_1 G(s) \text{ for } (x, s) \in C_1', \quad (\mathbb{L}V_2 - rV_2)(x, q) = -\nu_2 F(q) \text{ for } (x, q) \in C_2' \tag{2.27}$$

$$V_1(x, s)\big|_{x=a(s)+} = (K_1 - a(s))\, G(s), \quad V_2(x, q)\big|_{x=b(q)-} = (b(q) - K_2)\, F(q) \tag{2.28}$$

$$\partial_x V_1(x, s)\big|_{x=a(s)+} = -G(s), \quad \partial_x V_2(x, q)\big|_{x=b(q)-} = F(q) \tag{2.29}$$

$$\partial_s V_1(x, s)\big|_{x=s-} = -(\eta_1 + \varkappa_1\, s)\, F'(s), \quad \partial_q V_2(x, q)\big|_{x=q+} = (\eta_2 + \varkappa_2\, q)\, G'(q) \tag{2.30}$$

$$V_1(x, s) = (K_1 - x)\, G(s) \text{ for } (x, s) \in D_1, \quad V_2(x, q) = (x - K_2)\, F(q) \text{ for } (x, q) \in D_2 \tag{2.31}$$

$$V_1(x, s) > (K_1 - x)\, G(s) \text{ for } (x, s) \in C_1, \quad V_2(x, q) > (x - K_2)\, F(q) \text{ for } (x, q) \in C_2 \tag{2.32}$$

$$(\mathbb{L}V_1 - rV_1)(x, s) < -\nu_1 G(s) \text{ for } (x, s) \in D_1, \quad (\mathbb{L}V_2 - rV_2)(x, q) < -\nu_2 F(q) \text{ for } (x, q) \in D_2 \tag{2.33}$$

where C_i and D_i, $i = 1, 2$, are defined as C_i^* and D_i^*, $i = 1, 2$, in (2.22) and (2.23) with $a(s)$ and $b(q)$ instead of $a^*(s)$ and $b^*(q)$, respectively, and we set $C_1' = C_1 \setminus \{(x, s) \in E_1 \,|\, x = s\}$ and $C_2' = C_2 \setminus \{(x, q) \in E_2 \,|\, x = q\}$. Here, the instantaneous-stopping as well as the smooth-fit and modified normal-reflection conditions of (2.28)-(2.30) are satisfied, for all $s > \underline{s}$ and $q < \overline{q}$, respectively. Observe that the superharmonic characterisation of the value function (see, e.g. [34; Chapter IV, Section 9]) implies that $V_1^*(x, s)$ and $V_2^*(x, q)$ are the smallest functions satisfying (2.27)-(2.28) and (2.31)-(2.32) with the boundaries $a^*(s)$ and $b^*(q)$, respectively. Note that the inequalities in (2.33) follow directly from the assertion of Lemma 2.1 proved in Subsection 2.2 above.

3. Solutions to the free-boundary problems

In this section, we obtain solutions to the free-boundary problems in (2.27)-(2.33) and derive first-order nonlinear ordinary differential equations for the candidate optimal stopping boundaries.

3.1 The candidate value functions. It is shown that the second-order ordinary differential equations in (2.27) have the general solutions:

$$V_1(x,s) = C_{1,1}(s)\, x^{\gamma_1} + C_{1,2}(s)\, x^{\gamma_2} + \nu_1\, G(s)/r \tag{3.1}$$

and

$$V_2(x,q) = C_{2,1}(q)\, x^{\gamma_1} + C_{2,2}(q)\, x^{\gamma_2} + \nu_2\, F(q)/r, \tag{3.2}$$

where $C_{1,j}(s)$ and $C_{2,j}(q)$, $j = 1, 2$, are some arbitrary (continuously differentiable) functions, and γ_j, $j = 1, 2$, are given by:

$$\gamma_j = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - (-1)^j \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \tag{3.3}$$

so that $\gamma_2 < 0 < 1 < \gamma_1$ holds. Then, by applying the conditions of (2.28)-(2.30) to the functions in (3.1), we obtain the equalities:

$$C_{1,1}(s)\, a^{\gamma_1}(s) + C_{1,2}(s)\, a^{\gamma_2}(s) + \nu_1\, G(s)/r = (K_1 - a(s))\, G(s) \tag{3.4}$$
$$\gamma_1\, C_{1,1}(s)\, a^{\gamma_1}(s) + \gamma_2\, C_{1,2}(s)\, a^{\gamma_2}(s) = -a(s)\, G(s) \tag{3.5}$$
$$C'_{1,1}(s)\, s^{\gamma_1} + C'_{1,2}(s)\, s^{\gamma_2} + \nu_1\, G'(s)/r = -(\eta_1 + \varkappa_1\, s)\, F'(s) \tag{3.6}$$

for all $s > \underline{s}$, and

$$C_{2,1}(q)\, b^{\gamma_1}(q) + C_{2,2}(q)\, b^{\gamma_2}(q) + \nu_2\, F(q)/r = (b(q) - K_2)\, F(q) \tag{3.7}$$
$$\gamma_1\, C_{2,1}(q)\, b^{\gamma_1}(q) + \gamma_2\, C_{2,2}(q)\, b^{\gamma_2}(q) = b(q)\, F(q) \tag{3.8}$$
$$C'_{2,1}(q)\, q^{\gamma_1} + C'_{2,2}(q)\, q^{\gamma_2} + \nu_2\, F'(q)/r = (\eta_2 + \varkappa_2\, q)\, G'(q) \tag{3.9}$$

for all $q < \overline{q}$, respectively. Hence, by solving the systems of equations in (3.4)-(3.5) and (3.7)-(3.8), we obtain that the candidate value functions admit the representations:

$$V_1(x,s;a(s)) = C_{1,1}(s;a(s))\, x^{\gamma_1} + C_{1,2}(s;a(s))\, x^{\gamma_2} + \nu_1\, G(s)/r \tag{3.10}$$

for $a(s) < x \le s$, with

$$C_{1,j}(s;a(s)) = \frac{(\gamma_{3-j}(K_1 - \nu_1/r) - (\gamma_{3-j} - 1)a(s))G(s)}{(\gamma_{3-j} - \gamma_j)a^{\gamma_j}(s)} \tag{3.11}$$

for $j = 1, 2$, and

$$V_2(x,q;b(q)) = C_{2,1}(q;b(q))\, x^{\gamma_1} + C_{2,2}(q;b(q))\, x^{\gamma_2} + \nu_2\, F(q)/r \tag{3.12}$$

for $q \le x < b(q)$, with

$$C_{2,j}(q;b(q)) = \frac{((\gamma_{3-j} - 1)b(q) - \gamma_{3-j}(K_2 + \nu_2/r))F(q)}{(\gamma_{3-j} - \gamma_j)b^{\gamma_j}(q)} \tag{3.13}$$

for $j = 1, 2$, respectively.

3.2 The candidate stopping boundaries. By applying the conditions of (3.6) and (3.9) to the functions in (3.11) and (3.13), we conclude that the candidate boundaries satisfy the first-order nonlinear ordinary differential equations:

$$a'(s) = \frac{\Psi_{1,1}(s, a(s)) s^{\gamma_1} + \Psi_{1,2}(s, a(s)) s^{\gamma_2} + (\eta_1 + \varkappa_1 s - \nu_1/r) G'(s)}{\Phi_{1,1}(s, a(s)) s^{\gamma_1} + \Phi_{1,2}(s, a(s)) s^{\gamma_2}} \tag{3.14}$$

for $s > \underline{s}$, and

$$b'(q) = \frac{\Psi_{2,1}(q, b(q)) q^{\gamma_1} + \Psi_{2,2}(q, b(q)) q^{\gamma_2} + (\eta_2 + \varkappa_2 q + \nu_2/r) F'(q)}{\Phi_{2,1}(q, b(q)) q^{\gamma_1} + \Phi_{2,2}(q, b(q)) q^{\gamma_2}} \tag{3.15}$$

for $q < \overline{q}$, respectively. Here, the functions $\Phi_{1,j}(s, a(s))$, $\Psi_{1,j}(s, a(s))$ and $\Phi_{2,j}(q, b(q))$, $\Psi_{2,j}(q, b(q))$ are defined by:

$$\Phi_{1,j}(s, a(s)) = \frac{((\gamma_1 - 1)(\gamma_2 - 1) - \gamma_1 \gamma_2 (K_1 - \nu_1/r)/a(s)) G(s)}{(\gamma_{3-j} - \gamma_j) a^{\gamma_j}(s)} \tag{3.16}$$

$$\Psi_{1,j}(s, a(s)) = \frac{((\gamma_{3-j} - 1) a(s) - \gamma_{3-j}(K_1 - \nu_1/r)) G'(s)}{(\gamma_{3-j} - \gamma_j) a^{\gamma_j}(s)} \tag{3.17}$$

for $s > 0$, and

$$\Phi_{2,j}(q, b(q)) = \frac{((\gamma_1 - 1)(\gamma_2 - 1) - \gamma_1 \gamma_2 (K_2 + \nu_2/r)/b(q)) F(q)}{(\gamma_{3-j} - \gamma_j) b^{\gamma_j}(q)} \tag{3.18}$$

$$\Psi_{2,j}(q, b(q)) = \frac{((\gamma_{3-j} - 1) b(q) - \gamma_{3-j}(K_2 + \nu_2/r)) F'(q)}{(\gamma_{3-j} - \gamma_j) b^{\gamma_j}(q)} \tag{3.19}$$

for $q > 0$, and every $j = 1, 2$. We have also used the obvious facts that $F'(s) = -G'(s)$, for all $s > 0$, and $G'(q) = -F'(q)$, for all $q > 0$, by virtue of the definition of the function $G(x) = 1 - F(x)$, for all $x > 0$.

3.3 The maximal and minimal admissible solutions $a^*(s)$ and $b^*(q)$. We further consider the *maximal and minimal admissible* solutions of first-order nonlinear ordinary differential equations as the largest and smallest possible solutions $a^*(s)$ and $b^*(q)$ of the equations in (3.14) and (3.15) with (3.16)-(3.17) and (3.18)-(3.19) which satisfy the inequalities $a^*(s) < s \wedge \overline{a}$ and $b^*(q) > q \vee \underline{b}$ with $\overline{a} = (rK_1 - \nu_1)/\delta$ and $\underline{b} = (rK_2 + \nu_2)/\delta$, for all $s > \underline{s}$ and $q < \overline{q}$ and some $0 \le \underline{s} \le s^*$ and $\overline{q} \ge q^*$ fixed. By virtue of the classical results on the existence and uniqueness of solutions for first-order nonlinear ordinary differential equations, we may conclude that these equations admit (locally) unique solutions, in view of the facts that the right-hand sides in (3.14) and (3.15) with (3.16)-(3.17) and (3.18)-(3.19) are (locally) continuous in $(s, a(s))$ and $(q, b(q))$ and (locally) Lipschitz in $a(s)$ and $b(q)$, for each $s > \underline{s}$ and $q < \overline{q}$ fixed (see also [30; Subsection 3.9] for similar arguments based on the analysis of other first-order nonlinear ordinary differential equations). Then, it is shown by means of technical arguments based on Picard's method of successive approximations that there exist unique solutions $a(s)$ and $b(q)$ to the equations in (3.14) and (3.15) with (3.16)-(3.17) and (3.18)-(3.19), for $s > \underline{s}$ and $q < \overline{q}$, started at some points (s_0, s_0) and (q_0, q_0) such that $s_0 > \underline{s}$ and $q_0 < \overline{q}$ (see also [19;

Subsection 3.2] and [30; Example 4.4] for similar arguments based on the analysis of other first-order nonlinear ordinary differential equations). Hence, in order to construct the appropriate functions $a^*(s)$ and $b^*(q)$ which satisfy the equations in (3.14) and (3.15) and stays strictly above and below the appropriate diagonal, for $s > \underline{s}$ and $q < \overline{q}$, respectively, we can follow the arguments from [33; Subsection 3.5] (among others) which are based on the construction of sequences of the so-called bad-good solutions which intersect the diagonals. For this purpose, for any sequences $(s_l)_{l \in \mathbb{N}}$ and $(q_l)_{l \in \mathbb{N}}$ such that $s_l > \underline{s}$ and $q_l < \overline{q}$ as well as $s_l \uparrow \infty$ and $q_l \downarrow 0$ as $l \to \infty$, we can construct the sequence of solutions $a_l(s)$ and $b_l(q)$, $l \in \mathbb{N}$, to the equations (3.14) and (3.15), for all $s > \underline{s}$ and $q < \overline{q}$ such that $a_l(s_l) = s_l$ and $b_l(q_l) = q_l$ holds, for each $l \in \mathbb{N}$. It follows from the structure of the equations in (3.14) and (3.15) as well as the functions in (3.16)-(3.17) and (3.18)-(3.19) that the properties $a_l'(s_l) < 1$ and $b_l'(q_l) > 1$ holds, for each $l \in \mathbb{N}$ (see also [29; pages 979-982] for the analysis of solutions of another first-order nonlinear differential equation). Observe that, by virtue of the uniqueness of solutions mentioned above, we know that each two curves $s \mapsto a_l(s)$ and $s \mapsto a_m(s)$ as well as $q \mapsto b_l(q)$ and $q \mapsto b_m(q)$ cannot intersect, for $l, m \in \mathbb{N}$, $l \neq m$, and thus, we see that the sequence $(a_l(s))_{l \in \mathbb{N}}$ is increasing and the sequence $(b_l(q))_{l \in \mathbb{N}}$ is decreasing, so that the limits $a^*(s) = \lim_{l \to \infty} a_l(s)$ and $b^*(q) = \lim_{l \to \infty} b_l(q)$ exist, for each $s > \underline{s}$ and $q > \overline{q}$, respectively. We may therefore conclude that $a^*(s)$ and $b^*(q)$ provides the maximal and minimal solutions to the equations in (3.14) and (3.15) such that $a^*(s) < s \wedge \overline{a}$ and $b^*(q) > q \vee \underline{b}$ holds, for all $s > \underline{s}$ and $q < \overline{q}$. Moreover, since the right-hand sides of the first-order nonlinear ordinary differential equations in (3.14) and (3.15) with (3.16)-(3.17) and (3.18)-(3.19) are (locally) Lipschitz in s and q, respectively, one can deduce by means of Gronwall's inequality that the functions $a_l(s)$ and $b_l(q)$, $l \in \mathbb{N}$, are continuous, so that the functions $a^*(s)$ and $b^*(q)$ are continuous too. The corresponding *maximal admissible* solutions of first-order nonlinear ordinary differential equations and the associated maximality principle for solutions of optimal stopping problems which is equivalent to the superharmonic characterisation of the payoff functions were established in [30] and further developed in [19], [29], [20], [12], [5], [21], [32]-[33], [18], [28], [26], [15]-[17], [36], and [13] among other subsequent papers (see also [34; Chapter I; Chapter V, Section 17] for other references).

4. Main results and proofs

In this section, based on the expressions computed above, we formulate and prove the main results of the paper.

Theorem 4.1 *Suppose that the assumptions of Lemma 2.1 are satisfied. Then, the value functions of the perpetual American cancellable put and call option optimal stopping problems in (2.5) and (2.6) have the expressions:*

$$V_1^*(x,s) = \begin{cases} V_1(x,s;a^*(s)), & \text{if} \quad a^*(s) < x \le s \quad \text{and} \quad s > \underline{s} \\ (K_1 - x)\, G(s), & \text{if} \quad 0 < x \le a^*(s) \quad \text{and} \quad s > \underline{s} \\ (K_1 - x)\, G(s), & \text{if} \quad 0 < x \le s \le \underline{s} \end{cases} \tag{4.1}$$

and

$$V_2^*(x,q) = \begin{cases} V_2(x,q;b^*(q)), & \text{if} \quad q \le x < b^*(q) \quad \text{and} \quad 0 < q < \overline{q} \\ (x-K_2)\,F(q), & \text{if} \quad x \ge b^*(q) \quad \text{and} \quad 0 < q < \overline{q} \\ (x-K_2)\,F(q), & \text{if} \quad x \ge q \ge \overline{q} \end{cases} \tag{4.2}$$

and the optimal exercise times have the form of (2.24). Here, the functions $V_1(x,s;a(s))$ and $V_2(x,q;b(q))$ are given by (3.10) and (3.12) with (3.11) and (3.13), and the optimal exercise boundaries $a^*(s)$ and $b^*(q)$ provide the maximal and minimal solutions of the first-order nonlinear ordinary differential equations in (3.14) and (3.15) with (3.16)-(3.17) and (3.18)-(3.19) satisfying the inequalities $a^*(s) < s \wedge \overline{a}$ with $\overline{a} = (rK_1 - \nu_1)/\delta$ and $b^*(q) > q \vee \underline{b}$ with $\underline{b} = (rK_2 + \nu_2)/\delta$, for all $s > \underline{s}$ and $q < \overline{q}$ and some $0 \le \underline{s} \le s^* \wedge \overline{a}$ and $\overline{q} \ge q^* \vee \underline{b}$ fixed, as well as the equalities $a^*(s) = s$ and $b^*(q) = q$, for all $s \le \underline{s}$ and $q \ge \overline{q}$, respectively.

Since both assertions stated above are proved using similar arguments, we only give a proof for the case of the two-dimensional optimal stopping problem of (2.6) related to the dividend paying perpetual American cancellable call option. Observe that we can put $s = x$ and $q = x$ to obtain the values of the original perpetual American cancellable option pricing problems of (2.3) and (2.4) from the values of the optimal stopping problems of (2.5) and (2.6).

Proof In order to verify the assertion stated above, it remains for us to show that the function defined in (4.2) coincides with the value function in (2.6) and that the stopping time τ_2^* in (2.24) is optimal with the boundary $b^*(q)$ specified above. For this purpose, let $b(q)$ be any solution of the ordinary differential equation in (3.15) satisfying the inequality $b(q) > q \vee \underline{b}$, for all $q < \overline{q}$ and some $\overline{q} \ge q^* \vee \underline{b}$ fixed. Let us also denote by $V_2^b(x,q)$ the right-hand side of the expression in (4.2) associated with $b(q)$. Then, it is shown by means of straightforward calculations from the previous section that the function $V_2^b(x,q)$ solves the system of (2.27) with (2.31)-(2.33) and satisfies the conditions of (2.28)-(2.30). Recall that the function $V_2^b(x,q)$ is $C^{2,1}$ on the closure $\overline{C}_2$ of C_2 and is equal to $(x-K_2)F(q)$ on D_2, which are defined as $\overline{C}_2^*$, C_2^* and D_2^* in (2.22) and (2.23) with $b(q)$ instead of $b^*(q)$, respectively. Hence, taking into account the assumption that the boundary $b(q)$ is continuously differentiable, for all $q < \overline{q}$, by applying the change-of-variable formula from [31; Theorem 3.1] to the process $e^{-rt}V_2^b(X_t,Q_t)$ (see also [34; Chapter II, Section 3.5] for a summary of the related results and further references), we obtain the expression:

$$\begin{aligned} e^{-rt}\,V_2^b(X_t,Q_t) = V_2^b(x,q) &+ \int_0^t e^{-ru}\,(\mathbb{L}V_2^b - rV_2^b)(X_u,Q_u)\,I(X_u \ne b(Q_u), X_u \ne Q_u)\,du \\ &+ \int_0^t e^{-ru}\,\partial_q V_2^b(X_u,Q_u)\,I(X_u = Q_u)\,dQ_u + M_t^2 \end{aligned} \tag{4.3}$$

for all $t \ge 0$. Here, the process $M^2 = (M_t^2)_{t \ge 0}$ defined by:

$$M_t^2 = \int_0^t e^{-ru}\,\partial_x V_2^b(X_u,Q_u)\,I(X_u \ne Q_u)\,\sigma\,X_u\,dB_u \tag{4.4}$$

is a continuous local martingale with respect to the probability measure $P_{x,q}$. Note that, since the time spent by the process (X,Q) at the boundary surface $\partial C_2 = \{(x,q) \in E_2 \,|\, x = b(q)\}$

as well as at the diagonal $d_2 = \{(x,q) \in E_2 \,|\, x = q\}$ is of the Lebesgue measure zero (see, e.g. [8; Chapter II, Section 1]), the indicators in the first line of the formula in (4.3) as well as in the expression of (4.4) can be ignored. Moreover, since the component Q decreases only when the process (X,Q) is located on the diagonal $d_2 = \{(x,q) \in E_2 \,|\, x = q\}$, the indicator in the second line of (4.3) and the one in (4.4) can also be set equal to one. Observe that the integral in the second line of (4.3) will actually be compensated accordingly, due to the fact that the candidate value function $V_2^b(x,q)$ satisfies the modified normal-reflection condition of (2.30) at the diagonal d_2.

It follows from straightforward calculations and the arguments of the previous section that the function $V_2^b(x,q)$ satisfies the second-order ordinary differential equation in (2.27), which together with the conditions of (2.28)-(2.29) and (2.31) as well as the fact that the inequality in (2.33) holds imply that the inequality $(\mathbb{L}V_2^b - rV_2^b)(x,q) \le -\nu_2 F(q)$ is satisfied, for all $0 < q < x$ such that $q < \overline{q}$ and $x \neq b(q)$. Moreover, it is shown by means of standard arguments applied to the expressions in (3.12)-(3.13) that the property in (2.32) also holds, which together with the conditions of (2.28)-(2.29) and (2.31) imply that the inequality $V_2^b(x,q) \ge (x - K_2)F(q)$ is satisfied, for all $(x,q) \in E_2$. Let $(\sigma_n)_{n \in \mathbb{N}}$ be the localising sequence of stopping times for the process M^2 from (4.4) such that $\sigma_n = \inf\{t \ge 0 \,|\, |M_t^2| \ge n\}$, for each $n \in \mathbb{N}$. It therefore follows from the expression in (4.3) that the inequalities:

$$
\begin{aligned}
& e^{-r(\tau\wedge\sigma_n)}\,(X_{\tau\wedge\sigma_n} - K_2)\,F(Q_{\tau\wedge\sigma_n}) \\
& - \int_0^{\tau\wedge\sigma_n} e^{-ru}\,(\eta_2 + \varkappa_2\,Q_u)\,G'(Q_u)\,dQ_u + \int_0^{\tau\wedge\sigma_n} e^{-ru}\,\nu_2\,F(Q_u)\,du \\
& \le e^{-r(\tau\wedge\sigma_n)}\,V_2(X_{\tau\wedge\sigma_n}, Q_{\tau\wedge\sigma_n}) \\
& \quad - \int_0^{\tau\wedge\sigma_n} e^{-ru}\,(\eta_2 + \varkappa_2\,Q_u)\,G'(Q_u)\,dQ_u + \int_0^{\tau\wedge\sigma_n} e^{-ru}\,\nu_2\,F(Q_u)\,du \\
& \le V_2^b(x,q) + M^2_{\tau\wedge\sigma_n}
\end{aligned}
\tag{4.5}
$$

hold, for any stopping time τ of the process X and each $n \in \mathbb{N}$ fixed. Then, taking the expectation with respect to $P_{x,q}$ in (4.5), by means of Doob's optional sampling theorem, we get:

$$
\begin{aligned}
& E_{x,q}\Big[e^{-r(\tau\wedge\sigma_n)}\,(X_{\tau\wedge\sigma_n} - K_2)\,F(Q_{\tau\wedge\sigma_n}) \\
& \qquad - \int_0^{\tau\wedge\sigma_n} e^{-ru}\,(\eta_2 + \varkappa_2\,Q_u)\,G'(Q_u)\,dQ_u + \int_0^{\tau\wedge\sigma_n} e^{-ru}\,\nu_2\,F(Q_u)\,du\Big] \\
& \le E_{x,q}\Big[e^{-r(\tau\wedge\sigma_n)}\,V_2^b(X_{\tau\wedge\sigma_n}, Q_{\tau\wedge\sigma_n}) \\
& \qquad - \int_0^{\tau\wedge\sigma_n} e^{-ru}\,(\eta_2 + \varkappa_2\,Q_u)\,G'(Q_u)\,dQ_u + \int_0^{\tau\wedge\sigma_n} e^{-ru}\,\nu_2\,F(Q_u)\,du\Big] \\
& \le V_2^b(x,q) + E_{x,q}\big[M^2_{\tau\wedge\sigma_n}\big] = V_2^b(x,q)
\end{aligned}
\tag{4.6}
$$

for all $0 < q \le x$ such that $q < \overline{q}$, and each $n \in \mathbb{N}$. Hence, letting n go to infinity and using

Fatou's lemma, we obtain from the expressions in (4.6) that the inequalities:

$$E_{x,q}\Big[e^{-r\tau}\,(X_\tau - K_2)\,F(Q_\tau) - \int_0^\tau e^{-ru}\,(\eta_2 + \varkappa_2\,Q_u)\,G'(Q_u)\,dQ_u + \int_0^\tau e^{-ru}\,\nu_2\,F(Q_u)\,du\Big] \tag{4.7}$$
$$\le E_{x,q}\Big[e^{-r\tau}\,V_2^b(X_\tau, Q_\tau) - \int_0^\tau e^{-ru}\,(\eta_2 + \varkappa_2\,Q_u)\,G'(Q_u)\,dQ_u + \int_0^\tau e^{-ru}\,\nu_2\,F(Q_u)\,du\Big]$$
$$\le V_2^b(x,q)$$

are satisfied, for any stopping time τ, and all $0 < q \le x$ such that $q < \overline{q}$. Thus, taking the supremum over all stopping times τ and then the infimum over all boundaries b in the expressions of (4.7), we may therefore conclude that the inequalities:

$$\sup_\tau E_{x,q}\Big[e^{-r\tau}\,(X_\tau - K_2)\,F(Q_\tau) \tag{4.8}$$
$$- \int_0^\tau e^{-ru}\,(\eta_2 + \varkappa_2\,Q_u)\,G'(Q_u)\,dQ_u + \int_0^\tau e^{-ru}\,\nu_2\,F(Q_u)\,du\Big] \le \inf_b V_2^b(x,q) = V_2^{b^*}(x,q)$$

hold, for all $0 < q \le x$ such that $q < \overline{q}$, where $b^*(q)$ is the minimal solution of the ordinary differential equation in (3.15) as well as satisfying the inequality $b^*(q) > q \vee \underline{b}$, for all $q < \overline{q}$. By using the fact that the function $V_2^b(x,q)$ is (strictly) increasing in the value $b(q)$, for each $q < \overline{q}$ fixed, we see that the infimum in (4.8) is attained over any sequence of solutions $(b_m(q))_{m\in\mathbb{N}}$ to (3.15) satisfying the inequality $b_m(q) > q \vee \underline{b}$, for all $q < \overline{q}$, for each $m \in \mathbb{N}$, and such that $b_m(q) \downarrow b^*(q)$ as $m \to \infty$, for each $q < \overline{q}$ fixed. It follows from the (local) uniqueness of the solutions to the first-order (nonlinear) ordinary differential equation in (3.15) that no distinct solutions intersect, so that the sequence $(b_m(q))_{m\in\mathbb{N}}$ is decreasing and the limit $b^*(q) = \lim_{m\to\infty} b_m(q)$ exists, for each $q < \overline{q}$ fixed. Since the inequalities in (4.7) hold for $b^*(q)$ too, we see that the expression in (4.8) holds, for $b^*(q)$ and $(x,q) \in E_2$, as well. We also note from the inequality in (4.6) that the function $V_2^b(x,q)$ is superharmonic for the Markov process (X,Q) on E_2. Hence, taking into account the facts that $V_2^b(x,q)$ is increasing in $b(q) > q \vee \underline{b}$, for all $q < \overline{q}$, and the inequality $V_2^b(x,q) \ge (x - K_2)F(q)$ holds, for all $(x,q) \in E_2$, we observe that the selection of the minimal solution $b^*(q)$ which stays strictly above the diagonal $d_2 = \{(x,q) \in E_2 \,|\, x = q\}$ and the level $x = \underline{b}$ is equivalent to the implementation of the superharmonic characterisation of the value function as the smallest superharmonic function dominating the payoff function (cf. [30] or [34; Chapter I and Chapter V, Section 17]).

In order to prove the fact that the boundary $b^*(q)$ is optimal, we consider the sequence of stopping times τ_m, $m \in \mathbb{N}$, defined as in the right-hand part of (2.24) with $b_m(q)$ instead of $b^*(q)$, where $b_m(q)$ is a solution to the first-order ordinary differential equation in (3.15), and such that $b_m(q) \downarrow b^*(q)$ as $m \to \infty$, for each $q < \overline{q}$ fixed. Then, by virtue of the fact that the function $V_2^{b_m}(x,q)$ from the right-hand side of the expression in (4.2) associated with the boundary $b_m(q)$ satisfies the conditions of (2.27) and (2.28), and taking into account the structure of τ_2^* in (2.24), it follows from the expression which is equivalent to the one in (4.3) that the equalities:

$$
\begin{aligned}
&e^{-r(\tau_m \wedge \sigma_n)}\,(X_{\tau_m \wedge \sigma_n} - K_2)\,F(Q_{\tau_m \wedge \sigma_n}) \\
&- \int_0^{\tau_m \wedge \sigma_n} e^{-ru}\,(\eta_2 + \varkappa_2\,Q_u)\,G'(Q_u)\,dQ_u + \int_0^{\tau_m \wedge \sigma_n} e^{-ru}\,\nu_2\,F(Q_u)\,du \\
&= e^{-r(\tau_m \wedge \sigma_n)}\,V_2^{b_m}(X_{\tau_m \wedge \sigma_n}, Q_{\tau_m \wedge \sigma_n}) \\
&\quad - \int_0^{\tau_m \wedge \sigma_n} e^{-ru}\,(\eta_2 + \varkappa_2\,Q_u)\,G'(Q_u)\,dQ_u + \int_0^{\tau_m \wedge \sigma_n} e^{-ru}\,\nu_2\,F(Q_u)\,du \\
&= V_2^{b_m}(x,q) + M^2_{\tau_m \wedge \sigma_n}
\end{aligned} \tag{4.9}
$$

hold, for all $0 < q \le x$ such that $q < \overline{q}$ and each $n, m \in \mathbb{N}$. Observe that, by virtue of the arguments from [40; Chapter VIII, Section 2a], the property:

$$
\begin{aligned}
E_{x,q}\bigg[\sup_{t \ge 0}\bigg(&e^{-r(\tau_2^* \wedge t)}\,(X_{\tau_2^* \wedge t} - K_2)\,F(Q_{\tau_2^* \wedge t}) \\
&- \int_0^{\tau_2^* \wedge t} e^{-ru}\,(\eta_2 + \varkappa_2\,Q_u)\,G'(Q_u)\,dQ_u + \int_0^{\tau_2^* \wedge t} e^{-ru}\,\nu_2\,F(Q_u)\,du\bigg)\bigg] < \infty
\end{aligned} \tag{4.10}
$$

holds, for all $(x,q) \in E_2$, as well as the variable $e^{-r\tau_2^*}(X_{\tau_2^*} - K_2)F(Q_{\tau_2^*})$ is equal to zero on the event $\{\tau_2^* = \infty\}$, because the value $b^*(0+)$ is finite. Hence, letting m and n go to infinity and using the condition of (2.28) as well as the property $\tau_m \downarrow \tau_2^*$ ($P_{x,q}$-a.s.) as $m \to \infty$, we can apply the Lebesgue dominated convergence theorem to the appropriate (diagonal) subsequence in the expression of (4.9) to obtain the equality:

$$
\begin{aligned}
E_{x,q}\bigg[&e^{-r\tau_2^*}\,(X_{\tau_2^*} - K_2)\,F(Q_{\tau_2^*}) \\
&- \int_0^{\tau_2^*} e^{-ru}\,(\eta_2 + \varkappa_2\,Q_u)\,G'(Q_u)\,dQ_u + \int_0^{\tau_2^*} e^{-ru}\,\nu_2\,F(Q_u)\,du\bigg] = V_2^{b^*}(x,q)
\end{aligned} \tag{4.11}
$$

for all $0 < x \le q$ such that $q < \overline{q}$, which together with the inequalities in (4.8) directly implies the desired assertion. $\square$

Acknowledgments. The paper was essentially written during the time when the second author was visiting the Department of Mathematics at the London School of Economics and Political Science and she is grateful for the hospitality.

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