# Large Equilateral Sets in Subspaces of $\ell_{\infty}^{n}$ of Small Codimension 

Nóra Frankl ${ }^{1,2}{ }^{(1)}$

Received: 8 May 2020 / Revised: 22 November 2020 / Accepted: 15 December 2020 /
Published online: 22 January 2021
© The Author(s) 2021, corrected publication 2021


#### Abstract

For fixed $k$ we prove exponential lower bounds on the equilateral number of subspaces of $\ell_{\infty}^{n}$ of codimension $k$. In particular, we show that subspaces of codimension 2 of $\ell_{\infty}^{n+2}$ and subspaces of codimension 3 of $\ell_{\infty}^{n+3}$ have an equilateral set of cardinality $n+1$ if $n \geq 7$ and $n \geq 12$ respectively. Moreover, the same is true for every normed space of dimension $n$, whose unit ball is a centrally symmetric polytope with at most $4 n / 3-o(n)$ pairs of facets.


## 1 Introduction

Let $(X,\|\cdot\|)$ be a normed space. A set $S \subseteq X$ is called $c$-equilateral if $\|x-y\|=c$ for all distinct $x, y \in S . S$ is called equilateral if it is $c$-equilateral for some $c>0$. The equilateral number $e(X)$ of $X$ is the cardinality of the largest equilateral set of $X$. Petty [7] made the following conjecture regarding lower bounds on $e(X)$.

Conjecture 1.1 For all normed spaces $X$ of dimension $n, e(X) \geq n+1$.
Petty [7] proved Conjecture 1.1 for $n=3$, and Makeev [6] for $n=4$. For $n \geq 5$ the conjecture is still open, except for some special classes of norms. The best general lower bound is $e(X) \geq \exp \Omega(\sqrt{\log n})$, proved by Swanepoel and Villa [10]. Regarding upper bounds on the equilateral number, a classical result of Petty [7] and Soltan [8] shows that $e(X) \leq 2^{n}$ for any $X$ of dimension $n$, with equality if and only if the unit ball of $X$ is an affine image of the $n$-dimensional cube. For more background on the equilateral number see Sect. 3 of the survey [9].

[^0][^1]The norm $\|\cdot\|_{\infty}$ of $x \in \mathbb{R}^{n}$ is defined as $\|x\|_{\infty}=\max _{1 \leq i \leq n}\left|x_{i}\right|$, and $\ell_{\infty}^{n}$ denotes the normed space $\left(\mathbb{R}^{n},\|\cdot\|_{\infty}\right)$. In [5] Kobos studied subspaces of $\ell_{\infty}^{n}$ of codimension 1, and proved the lower bound $e(X) \geq 2^{\lfloor n / 2\rfloor}$, which in particular implies Conjecture 1.1 for these spaces for $n \geq 6$. In the same paper he proposed as a problem to prove Petty's conjecture for subspaces of $\ell_{\infty}^{n}$ of codimension 2. In Theorem 1.2 we prove exponential lower bounds on the equilateral number of subspaces of $\ell_{\infty}^{n}$ of codimension $k$. This, in particular, solves Kobos' problem if $n \geq 9$.

Theorem 1.2 Let $X$ be an $(n-k)$-dimensional subspace of $\ell_{\infty}^{n}$. Then

$$
\begin{align*}
& e(X) \geq \frac{2^{n-k}}{(n-k)^{k}},  \tag{1}\\
& e(X) \geq 1+\frac{1}{2^{k-1}} \sum_{r=1}^{\ell}\binom{n-k \ell}{r} \text { for every } 1 \leq \ell \leq \frac{n}{k+1}, \quad \text { and }  \tag{2}\\
& e(X) \geq 1+\sum_{r=1}^{\ell}\binom{n-2 k \ell}{r} \text { for every } 1 \leq \ell \leq \frac{n}{2 k+1} . \tag{3}
\end{align*}
$$

Note that none of the three bounds follows from the other two in Theorem 1.2, hence none of them is redundant. Comparing (1) and (3), for fixed $k$ we have

$$
\max _{\ell} \sum_{1 \leq r \leq \ell}\binom{n-2 k \ell}{r}=O\left(2^{c_{k} n}\right)
$$

for some $0<c_{k}<1$, while $2^{n-k} /(n-k)^{k}=2^{n-k-k \log (n-k)}=2^{n-o(n)}$. On the other hand, when we let $k$ vary, it can be as large as $\Omega(n)$ in (3) to still give a non-trivial estimate, while $k$ can only be chosen up to $O(n / \log n)$ for (1) to be non-trivial. Finally, (2) is beaten by (1) or (3) in most cases, however, for $k=2,3$ and for small values of $n$, (2) gives the best bound. Table 1 shows the lower bounds given by Theorem 1.2 for small values of $k$ and $n$, whenever it is at least 3 .

For two $n$-dimensional normed spaces $X, Y$ by $d_{\mathrm{BM}}(X, Y)=\inf _{T}\left\{\|T\| \cdot\left\|T^{-1}\right\|\right\}$ we denote their Banach-Mazur distance, where the infimum is over all linear isomorphisms $T: X \rightarrow Y$. The metric space of isometry classes of normed spaces endowed with the logarithm of the Banach-Mazur distance is the Banach-Mazur compactum. It is not hard to see that $e(X)$ is upper semi-continuous on the Banach-Mazur compactum. This, together with the fact that any convex polytope can be obtained as a section of a cube of sufficiently large dimension (see for example p. 72 of Grünbaum's book [4]) implies that it would be sufficient to prove Conjecture 1.1 for $k$-codimensional subspaces of $\ell_{\infty}^{n}$ for all $1 \leq k \leq n-4$ and $n \geq 5$. (This was also pointed out in [5].) Unfortunately, our bounds are only non-trivial if $n$ is sufficiently large compared to $k$. However, we deduce an interesting corollary.

Table 1 Lower bounds provided by Theorem 1.2 for small values of $k$ and $n$

| Bounds by (1) |  |  |  | Bounds by (2), $\ell$ value in bracket |  |  |  |  | Bounds by (3), $\ell$ value in bracket |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $n$ | $k$ |  |  | $n$ | $k$ |  |  |  | $n$ | $k$ |  |  |  |  |  |
|  | 1 | 2 | 3 |  | 1 | 2 | 3 | 4 |  | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 |  |  |  | 3 | 3 [1] |  |  |  | 3 |  |  |  |  |  |  |
| 4 | 3 |  |  | 4 | 4 [1] |  |  |  | 4 | 3 [1] |  |  |  |  |  |
| 5 | 4 |  |  | 5 | 7 [2] | 3 [1] |  |  | 5 | 4 [1] |  |  |  |  |  |
| 6 | 7 |  |  | 6 | 11 [2] | 3 [1] |  |  | 6 | 5 [1] | 3 [1] |  |  |  |  |
| 7 | 11 |  |  | 7 | 16 [2] | 4 [1] |  |  | 7 | 7 [2] | 4 [1] |  |  |  |  |
| 8 | 19 |  |  | 8 | 26 [3] | 6 [2] | 3 [1] |  | 8 | 11 [2] | 5 [1] | 3 [1] |  |  |  |
| 9 | 32 | 3 |  | 9 | 42 [3] | 9 [2] | 3 [1] |  | 9 | 16 [2] | 6 [1] | 4 [1] |  |  |  |
| 10 | 57 | 4 |  | 10 | 64 [3] | 12 [2] | 4 [2] |  | 10 | 22 [2] | 7 [1] | 5 [1] | 3 [1] |  |  |
| 11 | 103 | 7 |  | 11 | 99 [4] | 15 [2] | 5 [2] |  | 11 | 29 [2] | 8 [1] | 6 [1] | 4 [1] |  |  |
| 12 | 187 | 11 |  | 12 | 163 [4] | 22 [3] | 7 [2] | 3 [2] | 12 | 43 [3] | 11 [2] | 7 [1] | 5 [1] | 3 [1] |  |
| 13 | 342 | 17 |  | 13 | 256 [4] | 33 [3] | 8 [2] | 3 [1] | 13 | 64 [3] | 16 [2] | 8 [1] | 6 [1] | 4 [1] |  |
| 14 | 631 | 29 |  | 14 | 386 [4] | 47 [3] | 10 [2] | 4 [2] | 14 | 93 [3] | 22 [2] | 9 [1] | 7 [1] | 5 [1] | 3 [1] |
| 15 | 1171 | 49 | 3 | 15 | 638 [5] | 66 [3] | 13 [2] | 5 [2] | 15 | 130 [3] | 29 [2] | 10 [1] | 8 [1] | 6 [1] | 4 [1] |

Corollary 1.3 Let $P$ be an origin-symmetric convex polytope in $\mathbb{R}^{d}$ with at most

$$
\frac{4 d}{3}-\frac{1+\sqrt{8 d+9}}{6}=\frac{4 d}{3}-o(d)
$$

opposite pairs of facets. If $X$ is a d-dimensional normed space with $P$ as a unit ball, then $e(X) \geq d+1$.

There have been some extensions of lower bounds obtained on the equilateral number of certain normed spaces to other norms that are close to them according to the BanachMazur distance. These results are based on using Brouwer's fixed point theorem, first applied in this context by Brass [2] and Dekster [3]. Swanepoel and Villa [10] proved that if $d_{\mathrm{BM}}\left(Y, \ell_{\infty}^{n}\right) \leq 3 / 2$, then $e(Y) \geq n+1$. Kobos [5] proved that if $X$ is a subspace of $\ell_{\infty}^{n}$ of dimension $n-1$ and $Y$ is a normed space of dimension $n-1$ such that $d_{\mathrm{BM}}(X, Y) \leq 2$, then $e(Y) \geq\lceil n / 2\rceil$. We prove a similar result for subspaces of $\ell_{\infty}^{n}$ of codimension at least 1 .

Theorem 1.4 For $k \geq 1$ let $X$ be an $(n-k)$-dimensional subspace of $\ell_{\infty}^{n}$, and $Y$ be an $(n-k)$-dimensional normed space such that

$$
d_{\mathrm{BM}}(X, Y) \leq 1+\frac{\ell}{2(n-2 k-\ell k-1)}
$$

for some integer $1 \leq \ell \leq(n-2 k) / k$. Then $e(Y) \geq n-k(2+\ell)$.

## 2 Norms with Polytopal Unit Ball and Small Codimension

We recall the following well-known fact to prove Corollary 1.3. (For a proof, see for example [1].)

Lemma 2.1 Any centrally symmetric convex $d$-polytope with $f \geq d$ opposite pairs of facets is a d-dimensional section of the $f$-dimensional cube.

Proof of Corollary 1.3 By Lemma 2.1, $P$ can be obtained as an $d$-dimensional section of the $(4 d / 3-(1+\sqrt{8 d+9}) / 6)$-dimensional cube. Choose

$$
n=\frac{4 d}{3}-\frac{1+\sqrt{8 d+9}}{6}, \quad \ell=2, \quad k=\frac{d}{3}-\frac{d+\sqrt{8 d+9}}{6},
$$

and apply inequality (3) from Theorem 1.2. This yields $e(X) \geq d+1$.
To confirm Petty's conjecture for subspaces of $\ell_{\infty}^{n}$ of codimension 2 and 3 when $n \geq 9$ and respectively $n \geq 15$, apply inequality (2) from Theorem 1.2 with $\ell=2$.

## 3 Large Equilateral Sets

## Notation

We denote vectors by bold lowercase letters, and the $i$-th coordinate of a vector $\mathbf{a} \in \mathbb{R}^{n}$ by $a^{i}$. We treat vectors by default as column vectors. By subspace we mean linear subspace. We write span $\left(\mathbf{a}_{1}, \ldots, \mathbf{a}_{k}\right)$ for the subspace spanned by $\mathbf{a}_{1}, \ldots, \mathbf{a}_{k} \in \mathbb{R}^{n}$. For a subspace $X \subseteq \mathbb{R}^{n}$ we denote by $X^{\perp}$ the orthogonal complement of $X$. We denote by $[n]$ the set $\{1, \ldots, n\}$ and by $2^{[n]}$ the set of all subsets of $[n] .0$ denotes the vector $(0, \ldots, 0) \in \mathbb{R}^{n}$. For two vectors $\mathbf{a}$ and $\mathbf{b}$, let $\mathbf{a} \cdot \mathbf{b}=\sum_{i=1}^{n} a^{i} b^{i}$ be their scalar product.

## Idea of the Constructions

For two vectors $\mathbf{x}, \mathbf{y} \in X$ we have $\|\mathbf{x}-\mathbf{y}\|_{\infty}=c$ if and only if the following hold:

$$
\begin{align*}
& \text { there is an } 1 \leq i \leq n \text { such that }\left|x^{i}-y^{i}\right|=c, \text { and }  \tag{4}\\
& \left|x^{j}-y^{j}\right| \leq c \text { for all } 1 \leq j \leq n . \tag{5}
\end{align*}
$$

In our constructions of $c$-equilateral sets $S \subseteq X$, we split the index set [ $n$ ] of the coordinates into two parts, $[n]=N_{1} \cup N_{2}$. In the first part $N_{1}$, we choose all the coordinates from the set $\{0,1,-1\}$, so that for each pair from $S$ there will be an index in $N_{1}$ for which (4) holds, and (5) is not violated by any index in $N_{1}$. We use $N_{2}$ to ensure that all of the points we choose are indeed in the subspace $X$. For each vector, this will lead to a system of linear equations. The main difficulty will be to choose the values of the coordinates in $N_{1}$ so that the coordinates in $N_{2}$, obtained as a solution to those systems of linear equations, do not violate (5).

## Proof of Theorem 1.2

For vectors $\mathbf{v}_{1}, \ldots, \mathbf{v}_{k} \in \mathbb{R}^{k}$ let $B\left(\mathbf{v}_{1}, \ldots, \mathbf{v}_{k}\right) \in \mathbb{R}^{k \times k}$ be the matrix whose $i$-th column is $\mathbf{v}_{i}$. For a matrix $B \in \mathbb{R}^{k \times k}$, a vector $\mathbf{v} \in \mathbb{R}^{k}$, and an index $i \in[k]$, we denote by $B(i, \mathbf{v})$ the matrix obtained from $B$ by replacing its $i$-th column by $\mathbf{v}$.

Let $\left\{\mathbf{a}_{i}: 1 \leq i \leq k\right\}$ be a set of $k$ linearly independent vectors in $\mathbb{R}^{n}$ spanning $X^{\perp}$. That is, $\mathbf{x} \in X$ if and only if $\mathbf{a}_{i} \cdot \mathbf{x}=0$ for all $1 \leq i \leq k$. Further, let $A \in \mathbb{R}^{k \times n}$ be the matrix whose $i$-th row is $\mathbf{a}_{i}^{T}$, and let $\mathbf{b}_{j}=\left(a_{1}^{j}, \ldots, a_{k}^{j}\right)$ be the $j$-th column of $A$. For $I \subseteq[n]$ and for $\sigma \in\{ \pm 1\}^{n}$ let $\mathbf{b}_{I}=\sum_{i \in I} \mathbf{b}_{i}$ and $\mathbf{b}_{I, \sigma}=\sum_{i \in I} \sigma^{i} \mathbf{b}_{i}$. For a subset $N \subseteq[n]$ we denote by $\left.A\right|_{N}$ the submatrix of $A$ formed by those columns of $A$ whose index is in $N$. Similarly, for a vector $\mathbf{v} \in \mathbb{R}^{n}$ we denote by $\left.\mathbf{v}\right|_{N}$ the vector that is formed by those coordinates of $\mathbf{v}$ whose index is in $N$ (without changing the order of the indices).

Proof of (1) We construct a 2-equilateral set of size $2^{n-k} /(n-k)^{k}$. Let $B$ be a $k \times k$ submatrix of $A$ for which $|\operatorname{det} B|$ is maximal among all $k \times k$ submatrices. Further, let $N_{2}$ be the set of indices of the columns of $B$ and $N_{1}=[n] \backslash N_{2}$. Note that det $B \neq 0$, since the vectors $\left\{\mathbf{a}_{i}: i \in[k]\right\}$ are linearly independent. First, we find many vectors in $X$ whose coordinates in $N_{1}$ are from $\{1,-1\}$, and then we select a subset of these that form an equilateral set. For every $J \subseteq N_{1}$ we define the vector $\mathbf{w}(J) \in \mathbb{R}^{n}$ as

$$
w(J)^{i}= \begin{cases}1 & \text { if } i \in J \\ -1 & \text { if } i \in N_{1} \backslash J \\ \frac{\operatorname{det} B\left(j, \mathbf{b}_{N_{1} \backslash J}-\mathbf{b}_{J}\right)}{\operatorname{det} B} & \text { if } i \in N_{2} \text { is the } j \text {-th element of } N_{2}\end{cases}
$$

To see that $\mathbf{w}(J)$ is in $X$, we need to check that $A \mathbf{w}(J)=\mathbf{0}$. This is indeed the case, since by Cramer's rule $\left.\mathbf{w}(J)\right|_{N_{2}}$ is the solution of

$$
B \mathbf{x}=\mathbf{b}_{N_{1} \backslash J}-\mathbf{b}_{J}
$$

Further, by the multilinearity of the determinant for every $J$ and $j$ we have

$$
\operatorname{det} B\left(j, \mathbf{b}_{N_{1} \backslash J}-\mathbf{b}_{J}\right)=\sum_{r \in N_{1} \backslash J} \operatorname{det} B\left(j, \mathbf{b}_{r}\right)-\sum_{r \in J} \operatorname{det} B\left(j, \mathbf{b}_{r}\right) .
$$

By the maximality of $|\operatorname{det} B|$ and by the triangle inequality we have

$$
\left|\operatorname{det} B\left(j, \mathbf{b}_{N_{1} \backslash J}-\mathbf{b}_{J}\right)\right| \leq(n-k)|\operatorname{det} B|
$$

This implies that for each $J$ and $i \in N_{2}$ we have $-(n-k) \leq w(J)^{i} \leq n-k$. Consider the set $W=\left\{\mathbf{w}(J): J \in 2^{N_{1}}\right\} . W$ is not necessarily a 2-equilateral set, because for $J_{1}, J_{2} \in 2^{N_{1}}$ and for $i \in N_{2}$ we only have that $\left|w\left(J_{1}\right)^{i}-w\left(J_{2}\right)^{i}\right| \leq 2(n-k)$. However, we can find a 2-equilateral subset of $W$ that has large cardinality, as follows.

For each vector $\mathbf{s} \in\{-(n-k),(-n-k)+2, \ldots, n-k-2\}^{k}=T^{k}$ let $W(\mathbf{s})$ be the set of those vectors $\mathbf{w}(J)$ for which

$$
w(J)^{j} \in\left[s^{i}, s^{i}+2\right] \quad \text { for every } i \in k,
$$

where $j$ is the $i$-th element of $N_{2}$. Since $W \subseteq \bigcup_{\mathbf{s} \in T^{k}} W(\mathbf{s})$, there is an $\mathbf{s}$ for which $|W(\mathbf{s})| \geq 2^{n-k} /(n-k)^{k}$. It is not hard to check that $W(\mathbf{s})$ is 2-equilateral. Indeed, for every $J_{1}, J_{2} \in W(\mathbf{s})$, we have $\left|w^{i}\left(J_{1}\right)-w^{i}\left(J_{2}\right)\right| \leq 2$ for $i \in N_{2}$ by the definition of $W(\mathbf{s})$, and for $i \in N_{1}$ by the definition of $\mathbf{w}(J)$. Further, by the definition of $\mathbf{w}(J)$ there is an index $j \in N_{1}$ for which $\left\{w\left(J_{1}\right)^{j}, w\left(J_{2}\right)^{j}\right\}=\{1,-1\}$ (assuming $J_{1} \neq J_{2}$ ).

Proof of (2) Fix some $1 \leq \ell \leq n /(k+1)$. We will construct a 1-equilateral set of cardinality

$$
\frac{1}{2^{k-1}} \sum_{1 \leq r \leq \ell}\binom{n-k \ell}{r}+1
$$

Let $I_{1}, \ldots, I_{k} \subseteq[n]$ be sets of cardinality at most $\ell$, and $\sigma \in\{ \pm 1\}^{n}$ be a sign vector with the following properties:
(*) The determinant of $B=B\left(\mathbf{b}_{I_{1}, \sigma}, \ldots, \mathbf{b}_{I_{k}, \boldsymbol{\sigma}}\right)$ is maximal among all possible choices of $k$ disjoint sets $I_{1}, \ldots, I_{k} \subseteq[n]$ of cardinality at most $\ell$ and $\sigma \in\{ \pm 1\}^{n}$.
$(* *) \operatorname{det} B\left(k, \sigma^{j} \mathbf{b}_{j}\right) \geq 0$ for every $j \in[n] \backslash\left(I_{1} \cup \cdots \cup I_{k}\right)$.
Note that det $B>0$, since the vectors $\mathbf{a}_{1}, \ldots, \mathbf{a}_{k}$ are linearly independent. Let $N_{2}=$ $\bigcup_{i \in[k]} I_{i}$ and $N_{1}=[n] \backslash N_{2}$. We find a set of vectors whose coordinates in $N_{1}$ are from $\{0,1,-1\}$, and then we find a large subset of them that forms an equilateral set.

For every subset $J \subseteq N_{1}$ of cardinality at most $\ell$ we define the vector $\mathbf{w}(J) \in \mathbb{R}^{n}$ as

$$
w(J)^{i}= \begin{cases}-\sigma^{i} & \text { if } i \in J, \\ 0 & \text { if } i \in N_{1} \backslash J, \\ \frac{\sigma^{i} \operatorname{det} B\left(j, \mathbf{b}_{J, \sigma}\right)}{\operatorname{det} B} & \text { if } i \in I_{j} \text { for some } j \in[k] .\end{cases}
$$

To see that $\mathbf{w}(J)$ is in $X$, we have to check that $A \mathbf{w}(J)=\mathbf{0}$. This follows by showing that $\left.\mathbf{w}(J)\right|_{N_{2}}$ is a solution of

$$
\begin{equation*}
\left.A\right|_{N_{2}} \mathbf{x}=\mathbf{b}_{J, \sigma} . \tag{6}
\end{equation*}
$$

By Cramer's rule $\mathbf{y} \in \mathbb{R}^{k}$ defined as

$$
y^{j}=\frac{\operatorname{det} B\left(j, \mathbf{b}_{J, \sigma}\right)}{\operatorname{det} B}
$$

is a solution of $B \mathbf{y}=\mathbf{b}_{J, \sigma}$. Thus, $\left.\mathbf{w}(J)\right|_{N_{2}}$ is indeed a solution of (6) of a special form where $x^{i}=\sigma^{i} y^{j}$ if $i \in I_{j}$. Note that $B\left(j, \mathbf{b}_{J, \sigma}\right)=B\left(\mathbf{b}_{J_{1}, \sigma}, \ldots, \mathbf{b}_{J_{k}, \sigma}\right)$ for some
disjoint sets $J_{1}, \ldots, J_{k}$, hence by property $(*)$ we have

$$
\begin{equation*}
\left|w(J)^{i}\right| \leq 1 \quad \text { for each } 1 \leq i \leq n \tag{7}
\end{equation*}
$$

Further, the multilinearity of the determinant together with property $(* *)$ implies that for $i \in I_{k}$ we have

$$
\begin{align*}
\sigma^{i} w(J)^{i} & =\frac{\operatorname{det} B\left(k, \mathbf{b}_{J, \sigma}\right)}{\operatorname{det} B}=\frac{\operatorname{det} B\left(k, \sum_{j \in J} \sigma^{j} \mathbf{b}_{j}\right)}{\operatorname{det} B}  \tag{8}\\
& =\frac{\sum_{j \in J} \operatorname{det} B\left(k, \sigma^{j} \mathbf{b}_{j}\right)}{\operatorname{det} B} \geq 0 .
\end{align*}
$$

Consider the set $W=\left\{\mathbf{w}(J): J \subseteq N_{1},|J| \leq \ell\right\} . W$ is not necessarily a 1-equilateral set, because for $J_{1}, J_{2} \subseteq N_{1}$ with $\left|J_{1}\right|,\left|J_{2}\right| \leq \ell$ and for some $i_{1} \in I_{1} \cup \cdots \cup I_{k-1}$ we only have that $w\left(J_{1}\right)^{i}, w\left(J_{2}\right)^{i} \in[-1,1]$, and thus $\left|w\left(J_{1}\right)^{i}-w\left(J_{2}\right)^{i}\right| \leq 2$. However, we can find a 1 -equilateral subset of $W$ of large cardinality.

For each vector $\mathbf{s} \in\{ \pm 1\}^{k-1}$ let $W(\mathbf{s}) \subseteq W$ be the set of those vectors $\mathbf{w}(J) \in W$ for which

$$
s^{j} w(J)^{i} \sigma^{i} \geq 0 \quad \text { for each } i \in I_{1} \cup \cdots \cup I_{k-1},
$$

where $j \in[k-1]$ is such that $i \in I_{j}$. Then $\bigcup_{\mathbf{s} \in\{ \pm 1\}^{k-1}} W(\mathbf{s})$ is a partition of $W$, hence there is an $\mathbf{s}$ for which

$$
|W(\mathbf{s})| \geq \frac{1}{2^{k-1}}|W| \geq \frac{1}{2^{k-1}} \sum_{1 \leq r \leq \ell}\binom{n-k \ell}{r}
$$

$W(\mathbf{s})$ is a 1-equilateral set, because for any two vectors $\mathbf{w}_{1}, \mathbf{w}_{2} \in W(\mathbf{s})$, there is an index $i \in N_{1}$ for which either $\left\{w_{1}^{i}, w_{2}^{i}\right\}=\{0,-1\}$ or $\left\{w_{1}^{i}, w_{2}^{i}\right\}=\{0,1\}$, and for all $i \in[n]$ we have $\left|w_{1}^{i}-w_{2}^{i}\right| \leq 1$ by (7), by the definition of $W$ (s), and by (8). Finally, it is not hard to see that we can add $\mathbf{0}$ to $W(\mathbf{s})$. Thus $W(\mathbf{s}) \cup\{\mathbf{0}\}$ is a 1-equilateral set of the promised cardinality.

In the proof of (3) we will need the following simple lemma.
Lemma 3.1 Let A be a real matrix of size $k \times n$. Then for every $\varepsilon>0$ there exists a real matrix $A^{\prime}$ of size $k \times n$ such that $\left|a_{j}^{i}-a_{j}^{i}\right| \leq \varepsilon$ for all $i, j$, and every $k \times k$ minor of $A^{\prime}$ is non-zero.

Proof Associating matrices with points of $\mathbb{R}^{n \times k}$, the set of those matrices in which a given $k \times k$ minor is 0 is the zero set of a non-zero polynomial, which is nowhere dense. Thus, the set of those matrices for which there is a $k \times k$ minor which is zero is nowhere dense, which implies the statement.

Proof of (3) Fix some $1 \leq \ell \leq n /(2 k+1)$. We will construct a 1 -equilateral set of cardinality

$$
\sum_{1 \leq r \leq \ell}\binom{n-2 k \ell}{r}+1
$$

Pick $2 \ell$ disjoint submatrices $B_{1}, \ldots, B_{2 \ell}$ of $A$ of size $k \times k$, such that for every $1 \leq m \leq k$ and $1 \leq i \leq 2 \ell$ and for any column $\mathbf{b}_{r}$ that is not a column of any $B_{j}$ we have

$$
\begin{equation*}
\left|\operatorname{det} B_{i}\right| \geq\left|\operatorname{det} B_{i}\left(m, \mathbf{b}_{r}\right)\right| . \tag{9}
\end{equation*}
$$

This we may do by choosing the $B_{i}$ 's after each other, always choosing the submatrix with the largest determinant disjoint from the previous submatrices. Using Lemma 3.1 and the upper semi-continuity of the equilateral number, we may further assume that $\left|\operatorname{det} B_{i}\right|>0$ for all $1 \leq i \leq 2 \ell$.

Let $U_{i}$ be the set of indices of the columns of $B_{i}$, let $N_{2}=U_{1} \cup \cdots \cup U_{2 \ell}$ and $N_{1}=[n] \backslash N_{2}$. We will find vectors in $X$ (denoted by $\mathbf{y}(J)$ ) whose coordinates in $N_{1}$ are from the set $\{0,-1\}$, and whose coordinates in $N_{2}$ have absolute value at most $1 / 2$. We do not construct them directly, but as the sum of some other vectors $\mathbf{w}(J, i), \mathbf{z}(J, i) \in X$, whose coordinates in $N_{1}$ are from $\{0,-1 / 2\}$.

For every set $J=\left\{q_{1}, \ldots, q_{|J|}\right\} \subseteq\left[N_{1}\right]$ of cardinality at most $\ell$ with $q_{1}<\cdots<q_{|J|}$, and for every $1 \leq i \leq|J|$ we define $\mathbf{w}(J, i) \in \mathbb{R}^{n}$ and $\mathbf{z}(J, i) \in \mathbb{R}^{n}$ as

$$
\begin{aligned}
& w(J, i)^{j}= \begin{cases}-\frac{1}{2} & \text { if } j=q_{i}, \\
0 & \text { if } j \in[n] \backslash\left(\left\{q_{i}\right\} \cup U_{2 i}\right), \\
\frac{\operatorname{det} B_{2 i}\left(m, \mathbf{b}_{q_{i}} / 2\right)}{\operatorname{det} B_{2 i}} & \text { if } j \text { is the } m \text {-th element of } U_{2 i},\end{cases} \\
& z(J, i)^{j}= \begin{cases}-\frac{1}{2} & \text { if } j=q_{i}, \\
0 & \text { if } j \in[n] \backslash\left(\left\{q_{i}\right\} \cup U_{2 i-1}\right), \\
\frac{\operatorname{det} B_{2 i-1}\left(m, \mathbf{b}_{q_{i}} / 2\right)}{\operatorname{det} B_{2 i-1}} & \text { if } j \text { is the } m \text {-th element of } U_{2 i-1}\end{cases}
\end{aligned}
$$

To see that $\mathbf{w}(J, i)$ and $\mathbf{z}(J, i)$ are in $X$, we need to check that $A \mathbf{w}(J, i)=0$ and $A \mathbf{z}(J, i)=\mathbf{0}$. This is indeed the case, since by Cramer's rule both $\left.\mathbf{w}(J, i)\right|_{U_{2 i}}$ and $\left.\mathbf{z}(J, i)\right|_{U_{2 i-1}}$ are solutions of

$$
\left.A\right|_{J} \mathbf{x}=\frac{\mathbf{b}_{q_{i}}}{2}
$$

Therefore $\mathbf{y}(J)=\sum_{1 \leq i \leq|J|}(\mathbf{w}(J, i)+\mathbf{z}(J, i))$ is also in $X$. Note that by assumption (9) and by the multilinearity of the determinant we have $\left|w(J, i)^{j}\right|,\left|z(J, i)^{j}\right| \leq 1 / 2$ for all $1 \leq j \leq n$. For the coordinates of $\mathbf{y}(J)$ we have

$$
\begin{array}{ll}
y(J)^{j}=-1 & \text { if } j \in J, \\
y(J)^{j}=0 & \text { if } j \in\left[N_{1}\right] \backslash J
\end{array}
$$

$$
\left|y(J)^{j}\right| \leq \frac{1}{2} \quad \text { if } j \in N_{2} .
$$

Thus, for any two distinct sets $J_{1}, J_{2} \subseteq\left[N_{1}\right]$ of cardinality at most $\ell$ there is a coordinate $j \in\left[N_{1}\right]$ for which $\left\{y\left(J_{1}\right)^{j}, y\left(J_{2}\right)^{j}\right\}=\{0,-1\}$, and for all $1 \leq$ $j \leq n$ we have $\left|y\left(J_{1}\right)^{j}-y\left(J_{2}\right)^{j}\right| \leq 1$. This means $\left\|\mathbf{y}\left(J_{1}\right)-\mathbf{y}\left(J_{2}\right)\right\|_{\infty}=1$, and $\left\{\mathbf{y}(J): J \subseteq\left[N_{1}\right],|J| \leq \ell\right\} \cup\{\mathbf{0}\}$ is a 1-equilateral set of cardinality

$$
\sum_{1 \leq r \leq \ell}\binom{N_{1}}{r}+1=\sum_{1 \leq r \leq \ell}\binom{n-2 k \ell}{r}+1
$$

## 4 Equilateral Sets in Normed Spaces Close to Subspaces of $\ell_{\infty}^{n}$

In this section we prove Theorem 1.4. The construction we use is similar to the one from [10]. Fix some $1 \leq \ell \leq(n-2 k) / k$, and let $N=n-k(2+\ell)$ and $c=$ $\ell /(2(N-1))>0$. We assume that the linear structure of $Y$ is identified with the linear structure of $X$ and the norm $\|\cdot\|_{Y}$ of $Y$ satisfies

$$
\|x\|_{Y} \leq\|x\|_{\infty} \leq(1+c)\|x\|_{Y}
$$

for each $x \in X$. Further, let $M=\{(i, j): 1 \leq i<j \leq N\}$. For every $\boldsymbol{\varepsilon}=$ $\left(\varepsilon_{j}^{i}\right)_{(i, j) \in M} \in[0, c]^{M}$ and $j \in[N]$ we will define a vector $\mathbf{p}_{j}(\boldsymbol{\varepsilon}) \in Y$ such that

$$
\begin{array}{rlrl}
p_{j}(\varepsilon)^{i} & =-1 & & \text { if } i=j, \\
p_{j}(\varepsilon)^{i} & =\varepsilon_{j}^{i} & & \text { if } i<j, \\
p_{j}(\varepsilon)^{i} & =0 & & \text { if } i \in[N] \backslash[j], \\
\left|p_{j}(\varepsilon)^{i}\right| \leq \frac{1}{2} & & \text { if } i \in[n] \backslash[N] . \tag{13}
\end{array}
$$

Conditions (10)-(13) imply that $\left\|\mathbf{p}_{s}(\boldsymbol{\varepsilon})-\mathbf{p}_{t}(\boldsymbol{\varepsilon})\right\|_{\infty}=1+\varepsilon_{t}^{s}$ for every $1 \leq s<t \leq N$. Define $\varphi:[0, c]^{M} \rightarrow \mathbb{R}^{M}$ by

$$
\varphi_{j}^{i}(\boldsymbol{\varepsilon})=1+\varepsilon_{j}^{i}-\left\|\mathbf{p}_{i}(\boldsymbol{\varepsilon})-\mathbf{p}_{j}(\boldsymbol{\varepsilon})\right\|_{Y}
$$

for every $1 \leq i<j \leq N$. From

$$
\begin{aligned}
0=1+\varepsilon_{j}^{i}-\left\|\mathbf{p}_{i}(\boldsymbol{\varepsilon})-\mathbf{p}_{j}(\boldsymbol{\varepsilon})\right\|_{\infty} & \leq \varphi_{j}^{i}(\boldsymbol{\varepsilon})=1+\varepsilon_{j}^{i}-\left\|\mathbf{p}_{i}(\boldsymbol{\varepsilon})-\mathbf{p}_{j}(\boldsymbol{\varepsilon})\right\|_{Y} \\
& \leq 1+\varepsilon_{j}^{i}-(1+c)^{-1}\left\|\mathbf{p}_{i}(\boldsymbol{\varepsilon})-\mathbf{p}_{j}(\boldsymbol{\varepsilon})\right\|_{\infty} \leq c
\end{aligned}
$$

it follows that the image of $\varphi$ is contained in $[0, c]^{M}$. Since $\varphi$ is continuous, by Brouwer's fixed point theorem, $\varphi$ has a fixed point $\boldsymbol{\varepsilon}_{0} \in[0, c]^{M}$. Then $\left\{\mathbf{p}_{j}\left(\boldsymbol{\varepsilon}_{0}\right)\right.$ : $j \in[N]\}$ is a 1-equilateral set in $Y$ of cardinality $N=n-k(2+\ell)+1$.

To finish the proof, we only have to find vectors $\mathbf{p}_{j}(\boldsymbol{\varepsilon})$ that satisfy conditions (10)-(13). We construct them in a similar way as the equilateral sets in the proof of Theorem 1.2. Select $2+\ell$ disjoint submatrices $B_{1}, \ldots, B_{2 \ell}$ of $A$ of size $k \times k$, such that for every $1 \leq m \leq k$ and $1 \leq i \leq \ell+2$, and for any column $\mathbf{b}_{r}$ that is not a column of any $B_{j}$, we have

$$
\begin{equation*}
\left|\operatorname{det} B_{i}\right| \geq\left|\operatorname{det} B_{i}\left(m, \mathbf{b}_{r}\right)\right| . \tag{14}
\end{equation*}
$$

This we may do by choosing the $B_{i}$ 's after each other, always choosing the submatrix with the largest determinant disjoint from the previous submatrices. Using Lemma 3.1 and the upper semi-continuity of the equilateral number, we may further assume that $\left|\operatorname{det} B_{i}\right|>0$ for all $1 \leq i \leq \ell+2$. Let $U_{i}$ be the set of indices of the columns of $B_{i}$, and let $N_{2}=U_{1} \cup \cdots \cup U_{\ell+2}$. By permuting the coordinates, we may assume that $N_{2} \cap[N]=\emptyset$. Indeed, permuting the coordinates gives a subspace that is linearly isometric to the initial one, and the equilateral number is the same for isometric normed spaces.

We construct $\mathbf{p}_{j}(\boldsymbol{\varepsilon})$ as a sum of $2+\ell$ other vectors $\mathbf{p}_{j}(\boldsymbol{\varepsilon}, 1), \ldots, \mathbf{p}_{j}(\boldsymbol{\varepsilon}, 2+\ell)$. For $1 \leq j \leq N$ we define $\mathbf{p}_{j}(\boldsymbol{\varepsilon}, q)$ as follows. For $q \in\{1,2\}$ let

$$
p_{j}(\boldsymbol{\varepsilon}, q)^{i}= \begin{cases}-\frac{1}{2} & \text { if } i=j, \\ 0 & \text { if } i \in[n] \backslash\left(\{j\} \cup U_{q}\right), \\ \frac{\operatorname{det} B_{q}\left(m, \mathbf{b}_{j} / 2\right)}{\operatorname{det} B_{q}} & \text { if } i \text { is the } m \text {-th element of } U_{q}\end{cases}
$$

Let $\mathbf{s}(\boldsymbol{\varepsilon}, j)=\sum_{r<j} \varepsilon_{j}^{r} \mathbf{b}_{r} / \ell$ and for $q \in\{3, \ldots, 2+\ell\}$ let

$$
p_{j}(\varepsilon, q)^{i}= \begin{cases}\frac{\varepsilon_{j}^{i}}{\ell} & \text { if } i<j \\ 0 & \text { if } i \in[n] \backslash\left([j-1] \cup U_{q}\right), \\ \frac{\operatorname{det} B_{q}(m,-\mathbf{s}(\boldsymbol{\varepsilon}, j))}{\operatorname{det} B_{q}} & \text { if } i \text { is the } m \text {-th element of } U_{q}\end{cases}
$$

As before, by Cramer's rule we obtain that $\mathbf{p}_{j}(\boldsymbol{\varepsilon}, q)$ is contained in $Y$ for every $q$ and $j$, thus $\mathbf{p}_{j}(\boldsymbol{\varepsilon})=\sum_{q \in[2+\ell]} \mathbf{p}_{j}(\boldsymbol{\varepsilon}, q)$ is also contained in $Y$. It follows immediately that $\mathbf{p}_{j}(\boldsymbol{\varepsilon})$ satisfies conditions (10)-(12). It only remains to check condition (13).

By the multilinearity of the determinant, (14), and the triangle inequality, for every $j$, for $q \in\{1,2\}$ and $i \in U_{q}$ we have

$$
\left|p_{j}(\boldsymbol{\varepsilon}, q)^{i}\right|=\left|\frac{\operatorname{det} B_{q}\left(m, \mathbf{b}_{j} / 2\right)}{\operatorname{det} B_{q}}\right| \leq \frac{1}{2} .
$$

For every $j$, for $q \in\{3, \ldots, 2+\ell\}$ and $i \in U_{q}$ we have

$$
\begin{aligned}
\left|p_{j}(\boldsymbol{\varepsilon}, q)^{i}\right| & =\left|\frac{\operatorname{det} B_{q}(m,-\mathbf{s}(\boldsymbol{\varepsilon}, j))}{\operatorname{det} B_{m}}\right| \leq\left|\sum_{r<j} \frac{\varepsilon_{j}^{r}}{\ell} \cdot \frac{\operatorname{det} B_{q}\left(m,-\mathbf{b}_{r}\right)}{\operatorname{det} B_{q}}\right| \\
& \leq \sum_{r<j} \frac{\varepsilon_{j}^{r}}{\ell} \cdot\left|\frac{\operatorname{det} B_{q}\left(m,-\mathbf{b}_{r}\right)}{\operatorname{det} B_{q}}\right| \leq \sum_{r<j} \frac{\varepsilon_{j}^{r}}{\ell} \leq(N-1) \frac{c}{\ell}=\frac{1}{2} .
\end{aligned}
$$

The above bounds on $\left|p_{j}(\boldsymbol{\varepsilon}, q)^{i}\right|$ imply that condition (13) holds for $\mathbf{p}_{j}(\boldsymbol{\varepsilon})$, finishing the proof.

## 5 Concluding Remarks

In Theorem 1.4 we did not make an attempt to find the most optimal bound on the equilateral number in terms of the Banach-Mazur distance that can be achieved with this approach. Our goal with this theorem was to illustrate how to find non-trivial lower bounds on the equilateral number depending on the codimension and the BanachMazur distance, while keeping the proof short.

One possible immediate improvement is to increase the lower bound by 1 without changing the bound on the Banach-Mazur distance. This can be done by adding a vector to the construction of the form $\mathbf{p}_{N+1}(\varepsilon)=\left(\varepsilon_{N+1}^{1}, \ldots, \varepsilon_{N+1}^{N}, 0, \ldots, 0, \leq\right.$ $1 / 2, \ldots, \leq 1 / 2$ ). As this improvement is negligible for large $k$, we decided not to include it in the proof to keep it simpler.

With this plus one improvement, the lower bound on $e(Y)$ for $k=0$ would match the bound from [10]. However, the bound on the Banach-Mazur distance in the assumption would be weaker. To explain this, note that our proof consists of two parts. In the first part we find possible candidates for points of a large equilateral set. This part is very similar to the proof in [10]. In the second part we have to fit these candidates in a subspace, and with doing this we lose some flexibility with the Banach-Mazur distance. For $k=0$, however, we would not have a second part, thus the proof would be exactly the same as in [10].

Note also that for $k=1, \ell=n / 2+1$ our lower bound in Theorem 1.4 is the same as the bound in [5], but again with a worse bound on the Banach-Mazur distance. The reason for this is that our intention was to present a non-trivial general bound that can be proven in a short way in various settings of the parameters.

Finally, it would be interesting to find some applications of Theorem 1.4 (or of a more optimized form of it). One corollary which was pointed out by an anonymous referee is that if $Y$ is a subspace of $\ell_{p}^{n}$ of codimension $k$ for $p \geq 2$, then $e(Y) \geq$ $\Omega_{k, p}\left(n^{1-1 / p}\right)$, since $d_{\mathrm{BM}}\left(\ell_{\infty}^{n}, \ell_{p}^{n}\right)=n^{1 / p}$.

Acknowledgements We thank Konrad Swanepoel for bringing our attention to Corollary 1.3, for discussions on the topic, and for helpful comments on the manuscript. We also thank the anonymous referees for helpful suggestions that improved the presentation of the paper. Research was supported by an LMS Early Career Fellowship, by the Ministry of Education and Science of the Russian Federation in the framework
of MegaGrant no. 075-15-2019-1926 and by the National Research, Development, and Innovation Office, NKFIH Grant K119670.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

## References

1. Ball, K.: An elementary introduction to modern convex geometry. In: Flavors of Geometry. Math. Sci. Res. Inst. Publ., vol. 31, pp. 1-58. Cambridge Univ. Press, Cambridge (1997)
2. Braß, P.: On equilateral simplices in normed spaces. Beiträge Algebra Geom. 40(2), 303-307 (1999)
3. Dekster, B.V.: Simplexes with prescribed edge lengths in Minkowski and Banach spaces. Acta Math. Hungar. 86(4), 343-358 (2000)
4. Grünbaum, B.: Convex Polytopes. Graduate Texts in Mathematics, vol. 221. Springer, New York (2003)
5. Kobos, T.: Equilateral dimension of certain classes of normed spaces. Numer. Funct. Anal. Optim. 35(10), 1340-1358 (2014)
6. Makeev, V.V.: On equilateral simplices in a four-dimensional normed space. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 329, 88-91 (2005). (in Russian)
7. Petty, C.M.: Equilateral sets in Minkowski spaces. Proc. Am. Math. Soc. 29, 369-374 (1971)
8. Soltan, P.S.: Analogues of regular simplexes in normed spaces. Dokl. Akad. Nauk SSSR 222(6), 1303-1305 (1975). (in Russian)
9. Swanepoel, K.J.: Combinatorial distance geometry in normed spaces. In: New Trends in Intuitive Geometry. Bolyai Soc. Math. Stud., vol. 27, pp. 407-458. János Bolyai Math. Soc., Budapest (2018)
10. Swanepoel, K.J., Villa, R.: A lower bound for the equilateral number of normed spaces. Proc. Am. Math. Soc. 136(1), 127-131 (2008)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.


[^0]:    Editor in Charge: János Pach

[^1]:    Nóra Frankl
    n.frankl@1se.ac.uk

    1 London School of Economics, London, UK
    2 Moscow Institute of Physics and Technology, Moscow, Russia

