Optimal stopping games in models with various information flows

Pavel V. Gapeev* Neofytos Rodosthenous†

We study zero-sum optimal stopping games associated with perpetual convertible bonds in an extension of the Black-Merton-Scholes model with random dividends under various information flows. In this type of contracts, the writers have the right to withdraw the bonds, before the holders convert them into assets. We derive closed-form expressions for the associated value function and optimal exercise boundaries in the model with an accessible dividend rate policy which is described by a continuous-time Markov chain with two states. We further consider the optimal stopping game in the model with inaccessible dividend rate policy and prove that the optimal exercise times are the first times at which the asset price process hits monotone boundaries depending on the running state of the filtering dividend rate estimate. We finally present the value of the optimal stopping game for the model in which the dividend rate policy is accessible to the writers but remains inaccessible to the holders of the bonds.

1 Introduction

Stochastic game-theoretic problems in which both participants can select random (stopping) times, at which certain payments are made from one participant to the other, have attracted considerable attention in the literature on optimal stochastic control. The study of such game-theoretic problems was initiated by Dynkin [16]. The purely probabilistic approach for the analysis of such games, based on the application of martingale theory, was developed in Neveu [48], Krylov [41], Bismut [10], Stettner [59], and Lepeltier and Mainguenau [45] among others. The analytical theory of stochastic differential games with stopping times was developed in Bensoussan and Friedman [8]-[9] in Markovian diffusion models. The latter approach, dealing with the analysis of the value functions and saddle points of such games, was based on using the theory of variational inequalities and free-boundary problems for partial differential equations. Cvitanić and Karatzas [11] established a connection between the values of optimal stopping games and the solutions of (doubly) reflected backward stochastic differential equations with general (random) coefficients and provided a pathwise approach to these games. Karatzas and Wang [39] studied such games in a more general non-Markovian setting and connected them with bounded-variation optimal control problems. More

*(Corresponding author) London School of Economics, Department of Mathematics, Houghton Street, London WC2A 2AE, United Kingdom; e-mail: p.v.gapeev@lse.ac.uk

†Queen Mary University of London, School of Mathematical Sciences, Mile End Road, London E1 4NS, United Kingdom; e-mail: n.rodosthenous@qmul.ac.uk

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recently, Ekström and Peskir [18] and Peskir [51]-[52] proved that the value function of a general zero-sum optimal stopping game for a right-continuous (strong) Markov process is measurable, and found necessary and sufficient conditions for the existence of Stackelberg and Nash equilibria in such a game. Bayraktar and Sîrbu [7] applied the stochastic Perron's method and verification without smoothness using viscosity comparisons for solving obstacle problems and Dynkin games.

The related concept of the so-called game-type (or Israeli) contingent claims for models of financial markets was introduced by Kifer [40], who generalised the case of American-type claims, by allowing the writers to withdraw the contracts prematurely, at the costs of some penalties. It was shown that the problems of pricing and hedging of such options can be reduced to solving the associated optimal stopping games. Kyprianou [43] obtained explicit expressions for the rational value functions in two classes of perpetual game options. Kühn and Kyprianou [42] characterised the value functions of finite expiry versions of those options by means of the mixtures of other exotic options and martingale arguments, and produced the same analysis for a more general class of finite expiry game options by means of pathwise pricing formulae. Kallsen and Kühn [38] applied the neutral valuation approach for the American and game options in incomplete markets and introduced a mathematically rigorous dynamic concept of no-arbitrage prices for game contingent claims. The convertible bond optimal stopping games were studied by Sîrbu, Pikovsky, and Shreve [57] and Sîrbu and Shreve [58] within a structural diffusion model in which the dynamics of the underlying risky asset price are given *endogenously*, under an infinite and finite time horizon, respectively. It was particularly shown in [57] and [58] that the original optimal stopping game can be solved as two separate optimal stopping problems for the risky asset price process, when the structural model assumes a constant coupon rate. Further calculations of rational prices of perpetual game options and convertible bonds in certain reduced-form jump-diffusion models were provided by Baurdoux and Kyprianou [3], [4] and [5], Ekström and Villeneuve [19], and Baurdoux, Kyprianou and Pardo [6] among others.

In this paper, we study the (zero-sum) optimal stopping game associated with the perpetual convertible (and callable) bond pricing problem in a reduced-form model in which the dynamics of the underlying risky asset price are given *exogenously*. In particular, we consider an extension of the Black-Merton-Scholes model for the underlying risky asset with the dividend rate dynamics described by means of a two-state continuous-time Markov chain which is observable by both the writer and holder of the bond (*full information*). We further extend the model by allowing the rate of coupon payments, from the writer to the holder of the bond until an exercise time, to be given by a *linear function* of the running underlying risky asset price. Such a feature reflects the current trend, observed in the modern literature on financial markets, of using derivative securities with floating coupon rates, dividend rates, and strike prices rather than fixed ones. In this setting, we obtain closed-form expressions for the value function and optimal exercise boundaries as solutions to the associated free-boundary problem involving a coupled system of two ordinary differential equations. Note that closed-form solutions of the optimal stopping problems related to the pricing of perpetual American lookback and (standard) put options in such a model were obtained by Guo [33] and Guo and Zhang [34], respectively. Similar optimal stopping problems in other models with regime-switching parameters were studied by Jobert and Rogers [36] in an extension of the exponential diffusion-type model to the case with several states for the Markov chain, by Dalang and Hongler [12] in the associated model with a two-state continuous-time Markov chain and no diffusion part, and by Jiang and Pistorius [35] in the framework of an exponential jump-diffusion model (see also [23] for a recently considered regime-switching optimal stopping game with constant rewards used to obtain the solution of a bounded-variation control problem).

In the present paper, we also consider the perpetual convertible bond pricing problem in the version of the model described above in which the dividend rate dynamics is unobservable by both the writer and the holder of the convertible bond (*partial information*). In this setting, the original problem is equivalent to an optimal stopping game for a two-dimensional (Markovian) diffusion process having the underlying risky asset price and the filtering estimate of the dividend rate (Wonham filter) as the state space components. Note that such a two-state hidden Markov model was proposed by Shiryaev [56; Chapter III, Section 4a] for the description of interest rate dynamics, and then applied by Elliott and Wilson [21] for the computation of zero-coupon bond prices and other quantities in the interest rate framework (see also [27]-[28] for the derivation of perpetual American and real option prices in this model). We conclude the paper with the perpetual convertible bond pricing problem in the version of the model in which the dividend rate dynamics is observable by the writer but remains unobservable by the holder of the convertible bond (*asymmetric information*). Some other (zero-sum) optimal stopping games in which the infima and suprema are taken over stopping times with respect to different filtrations have been studied in the very recent literature. For instance, Grün [32] applied the viscosity solution approach for studying such a game for an underlying diffusion process in which only one of two players is informed about the precise structure of the payoffs. Gensbittel and Grün [31] considered a simpler version of such a game in a model in which the dynamics of the underlying process are modelled by continuous-time Markov chains. The asymmetry of information in other models for such optimal stopping games was described in Lempa and Matomäki [44] by a random time horizon which is independent of the underlying process, in Ekström, Glover, and Leniec [17] by heterogeneous beliefs about the drift of the underlying diffusion process, in Esmaeeli, Imkeller, and Nzengang [22] by a random variable which is not necessarily independent of the underlying process, and in De Angelis, Ekström, and Glover [13] by a Bernoulli random variable affecting the drift of the underlying process only at the initial time (see also De Angelis, Gensbittel, and Villeneuve [14] for a similar problem where both players have partial information). In our model, the asymmetry of information is described by a continuous-time Markov chain which is independent of the standard Brownian motion driving the underlying process. We combine the solutions of the perpetual convertible bond pricing problem in models with *full* and *partial information* and use the linear structure of the reward functionals to show that the optimal stopping game does not have any nontrivial solution in the model with *asymmetric information*. In this respect, we actually find optimal stopping times forming a Nash equilibrium in the considered optimal stopping game with asymmetric information in the classes of classical (non-randomised) stopping times with respect to the appropriate different filtrations.

The rest of the paper is organised as follows. In Section 2, we formulate the associated optimal stopping game for the underlying risky asset price and the dividend rate as well as its filtering estimate, in order to consider the various models of information. It is shown that the resulting game can be decomposed into two separate optimal stopping problems for the two-dimensional (strong Markov) diffusion process (see similar results in [57] and [58] for a model of different type). We prove that the optimal exercise times for the writer and the holder of the convertible bond under *partial information* are expressed as the first times at which the asset price process hits stochastic boundaries, which are represented by monotone functions of the running estimate of the dividend rate process (Lemma 2.1). In Section 3, we derive a closed-form solution to the coupled ordinary free-boundary problem associated with the optimal stopping game for the underlying risky asset price and the observable dividend rate which constitute the model of *full information* (Theorem 3.1). In Section 4, we verify that, under the assumption of regularity of the points of the optimal exercise boundaries for the stopping regions relative to the underlying risky asset price and the filtering estimate process, the solution of the associated parabolic-type free-boundary problem provides the solution of the optimal stopping game (Lemma 4.1). We then state the main results concerning the perpetual convertible bond optimal stopping game in the model with *partial information* (Theorem 4.2). We also give a closed-form solution to the optimal stopping game in terms of Gauss' hypergeometric functions under certain relations on the parameters of the model (Corollary 4.3). In Section 5, we conclude with the optimal stopping game in the associated model of *asymmetric information* in which the dividend rate is observable by the writer but remains unobservable by the holder of the bond (Corollary 5.1).

2 Preliminaries

In this section, we introduce the setting and notation of the optimal stopping game which is related to the pricing of perpetual convertible bonds in the various models of information considered in the paper.

2.1 The model. For a precise formulation of the problem, let us consider a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with a standard Brownian motion $B = (B_t)_{t \ge 0}$ and a continuous-time Markov chain $\Theta = (\Theta_t)_{t \ge 0}$ with two states, 0 and 1. Assume that Θ has the initial distribution $\{1-\pi, \pi\}$, for $\pi \in [0,1]$, the transition probability matrix $\{e^{-\lambda t}, 1-e^{-\lambda t}; 1-e^{-\lambda t}, e^{-\lambda t}\}$, for $t \ge 0$, and thus, the intensity matrix $\{-\lambda, \lambda; \lambda, -\lambda\}$, for some $\lambda \ge 0$ fixed. Moreover, suppose that the processes B and Θ are independent. In other words, the Markov chain Θ changes its state from i to $1-i$ at exponentially distributed times of intensity λ, which are independent of the dynamics of the Brownian motion B. Such a process Θ is called a *telegraphic signal* in the literature (see, e.g. [47; Chapter IX, Section 4] or [20; Chapter VIII]). We define the process $S = (S_t)_{t \ge 0}$ by

$$S_t = s \, \exp\bigg(\int_0^t \Big(r - \frac{\sigma^2}{2} - \delta_0 - (\delta_1 - \delta_0)\, \Theta_u \Big)\, du + \sigma \, B_t \bigg) \tag{2.1}$$

which solves the stochastic differential equation

$$dS_t = \big(r - \delta_0 - (\delta_1 - \delta_0)\, \Theta_t \big)\, S_t \, dt + \sigma \, S_t \, dB_t \quad (S_0 = s) \tag{2.2}$$

where $s > 0$ is fixed, and $r > 0$, $\delta_i > 0$, $i = 0, 1$, and $\sigma > 0$ are some given constants. Assume that S describes the risk-neutral dynamics of the market price of a dividend paying risky asset under a martingale (risk-neutral) probability measure $\mathbb{P}$, where r is the interest rate of a riskless bank account and σ is the volatility coefficient. Suppose that Θ reflects the behaviour of the economic state of the firm issuing the asset, where the firm is in the so-called *good* state at $\Theta = 0$, or in the so-called *bad* state at $\Theta = 1$, and the asset pays dividends at the rate $\delta_0(1-\Theta) + \delta_1 \Theta$. We further assume that the process Θ has the same distribution with respect to the martingale measure $\mathbb{P}$ as with respect to the initial or physical probability measure. This property allows us to specify the pricing measure $\mathbb{P}$ from the set of all martingale measures in the incomplete market model defined in (2.1)-(2.2).

It is shown by means of standard arguments (see, e.g. [47; Chapter IX] or [20; Chapter VIII]) that the asset price process S from (2.1)-(2.2) admits the representation

$$dS_t = \big(r - \delta_0 - (\delta_1 - \delta_0)\, \Pi_t \big)\, S_t \, dt + \sigma \, S_t \, d\overline{B}_t \quad (S_0 = s) \tag{2.3}$$

on its natural filtration $(\mathcal{F}_t)_{t \ge 0}$, and the filtering estimate $\Pi = (\Pi_t)_{t \ge 0}$ defined by $\Pi_t = \mathbb{E}[\Theta_t \,|\, \mathcal{F}_t] \equiv \mathbb{P}(\Theta_t = 1 \,|\, \mathcal{F}_t)$ solves the stochastic differential equation

$$d\Pi_t = \lambda \, (1 - 2\Pi_t)\, dt - \frac{\delta_1 - \delta_0}{\sigma} \, \Pi_t (1 - \Pi_t)\, d\overline{B}_t \quad (\Pi_0 = \pi) \tag{2.4}$$

for some $(s, \pi) \in (0, \infty) \times [0,1]$ fixed. Here, the innovation process $\overline{B} = (\overline{B}_t)_{t \ge 0}$ defined by

$$\overline{B}_t = \int_0^t \frac{dS_u}{\sigma S_u} - \frac{1}{\sigma} \int_0^t \big(r - \delta_0 - (\delta_1 - \delta_0)\, \Pi_u \big)\, du \tag{2.5}$$

is a standard Brownian motion, according to P. Lévy's characterisation theorem (see, e.g. [47; Theorem 4.1]). It can be verified that (S, Π) is a (time-homogeneous strong) Markov process, under $\mathbb{P}$ with respect to its natural filtration $(\mathcal{F}_t)_{t \ge 0}$, as a unique strong solution of the system of stochastic differential equations in (2.3)-(2.4) (see, e.g. [49; Theorem 7.2.4]). Note that the model presented above which contains the observable process S and the estimate process Π of the continuous-time Markov chain Θ given running observations is known as the Wonham filter in the literature (see [60] for the original formulation of the model and derivation of the stochastic differential equations).

2.2 The optimal stopping game. Suppose that an investor writes a convertible bond on the underlying risky asset with market price S and sells it to another investor at time zero. Then, the holder of the bond can decide whether to continue holding it and collect the coupon payments at the rate $c + \nu S$, with some $c > 0$ and $\nu > 0$ fixed, or to terminate the contract by converting it into α units of the asset, thus receiving the amount αS, for some $\alpha > 0$ fixed. At the same time, the writer can recall the bond at some (penalty) strike $K > 0$, while offering the opportunity to the holder to convert the bond instantly, who thus receives the amount $K \vee (\alpha S) \equiv \max\{K, \alpha S\}$. In these cases, the total (discounted) amount paid by the writer to the holder is equal to

$$Y_t = \int_0^t e^{-ru} \, (c + \nu S_u) \, du + e^{-rt} \, \alpha S_t \quad \text{or} \quad Z_t = \int_0^t e^{-ru} \, (c + \nu S_u) \, du + e^{-rt} \, \big(K \vee (\alpha S_t) \big) \tag{2.6}$$

at any converting or recalling time $t \geq 0$, respectively. The holder looks for a converting time maximising the expected discounted amount paid by the writer, while the latter looks for a recalling time minimising the equivalent quantity. Taking this into account, it follows from the results of Kifer [40] and Kallsen and Kühn [38] (see also [29] or [6]) that such a contract can be expressed as a standard game contingent claim. More precisely, a rational (or no-arbitrage) value of such a claim is given by the value of the optimal stopping game

$$V_*(s, \pi) = \inf_\zeta \sup_\tau \mathbb{E}_{s,\pi} \big[Y_\tau \, I(\tau < \zeta) + Z_\zeta \, I(\zeta \leq \tau) \big] = \sup_\tau \inf_\zeta \mathbb{E}_{s,\pi} \big[Y_\tau \, I(\tau < \zeta) + Z_\zeta \, I(\zeta \leq \tau) \big] \tag{2.7}$$

where $\mathbb{E}_{s,\pi}$ denotes the expectation with respect to the probability measure $\mathbb{P}_{s,\pi}$ under which the two-dimensional (time-homogeneous strong) Markov process (S, Π) starts at some $(s, \pi) \in (0, \infty) \times [0, 1]$, and $I(\cdot)$ is the indicator function. We first assume that the infimum and the supremum in (2.7) are taken over all stopping times ζ and τ with respect to the filtration generated by the process (S, Π), which corresponds to the case of the model with *partial information*. It follows from the results of Cvitanić and Karatzas [11; Theorem 4.1] based on the solutions of the associated (doubly) reflected backward stochastic differential equations that the game-type optimal stopping problem of (2.7) has a value, for each $(s, \pi) \in (0, \infty) \times [0, 1]$ fixed. In this paper, we aim at studying (2.7) in the model of (S, Π) with *partial information* given by (2.3)-(2.5), deriving closed-form expressions for the value function of (3.1) below in the associated model of (S, Θ) with *full information* given by (2.1)-(2.2), as well as obtaining the value function of (5.1) below in the model with *asymmetric information*. The existence of the values or the associated Stackelberg equilibria in optimal stopping games is proved in the results of Stettner [59], Lepeltier and Mainguenau [45], and Peskir [51]-[52] among others.

By means of the results of general theory of optimal stopping games (see, e.g. [16], [8]-[9], [25]-[26], [41], [59], [45], and [11] among others), we obtain from the structure of the value function that the stopping times forming a Nash equilibrium in the optimal stopping game of (2.7) at which the writer and the holder of the bond should exercise the contract are given by

$$\zeta_* = \inf \big\{ t \geq 0 \; \big| \; V_*(S_t, \Pi_t) = K \vee (\alpha S_t) \big\} \quad \text{and} \quad \tau_* = \inf \big\{ t \geq 0 \; \big| \; V_*(S_t, \Pi_t) = \alpha S_t \big\} \tag{2.8}$$

so that the continuation region has the form

$$C_* = \big\{ (s, \pi) \in (0, \infty) \times [0, 1] \; \big| \; \alpha s < V_*(s, \pi) < K \vee (\alpha s) \big\} \tag{2.9}$$

while the stopping region is the set

$$D_* = \big\{ (s, \pi) \in (0, \infty) \times [0, 1] \; \big| \; V_*(s, \pi) = K \vee (\alpha s) \;\; \text{or} \;\; V_*(s, \pi) = \alpha s \big\}. \tag{2.10}$$

We prove in part (i) of Subsection 4.1 below that $V_*(s, \pi)$ is a continuous function, and thus, the optimal stopping times have the structure of (2.8), while C_* in (2.9) is an open set and D_* in (2.10) is a closed set. It follows from the structure of the reward processes Y and Z in (2.6) that the inequality $Z_t < Y_t$ holds when $S_t < K/\alpha$, and the equality $Z_t = Y_t$ holds when $S_t \geq K/\alpha$, for any $t \geq 0$. Hence, either the event $\{\zeta_* < \tau_*\}$ or $\{\tau_* < \zeta_*\}$ can occur when $s < K/\alpha$, while the event $\{\tau_* = \zeta_*\}$ occurs when $s \geq K/\alpha$. In this respect, we may conclude that the property $\zeta_* \vee \tau_* = \inf\{t \geq 0 \,|\, S_t \geq K/\alpha\}$ holds, so that the continuation region C_* in (2.9) belongs to the set $\{(s, \pi) \in (0, K/\alpha) \times [0, 1]\}$, and thus, we further need to consider only the case in which the inequalities $0 < s \leq K/\alpha$ are satisfied for the starting risky asset price. Note that the same property of separation of the optimal stopping game into two optimal stopping problems was earlier observed in the studies of games related to convertible bonds, such as in [57]-[58], [29], and [6].

2.3 The structure of optimal stopping times. Let us now prove the existence of the optimal stopping times ζ_* and τ_* in (2.8) and clarify the structure of the associated continuation and stopping regions C_* and D_* in (2.9) and (2.10). Due to the strong explicit dependence of the optimal stopping boundaries on the values of the model parameters, in order to convey the main ideas in a transparent way and simplify the subsequent presentation, we further assume that the inequalities $0 < \delta_1 < \delta_0 < r$ and $0 < \nu < \alpha\delta_1 < \alpha\delta_0$ hold. The cases related to other possible relations between the parameters of the model can be considered in an essentially similar way.

(i) Let us first observe that, by applying Tanaka's formula (see, e.g. [55; Chapter VI, Theorem 1.2]) to the processes Y and Z from (2.6) and Doob's optional sampling theorem (see, e.g. [47; Chapter III, Theorem 3.6]) to the corresponding stochastic integral processes which represent uniformly integrable martingales, we get that the expressions

$$\begin{aligned}
&\mathbb{E}_{s,\pi}\big[Y_\tau\, I(\tau < \zeta) + Z_\zeta\, I(\zeta \leq \tau)\big] \qquad (2.11)\\
&= \mathbb{E}_{s,\pi}\bigg[\int_0^{\zeta\wedge\tau} e^{-ru}\,(c + \nu S_u)\, du + e^{-r(\zeta\wedge\tau)}\,\big(K \vee (\alpha S_{\zeta\wedge\tau})\big) - e^{-r\tau}\,(K - \alpha S_\tau)^+\, I(\tau < \zeta)\bigg]\\
&= K \vee (\alpha s) + \mathbb{E}_{s,\pi}\bigg[\int_0^{\zeta\wedge\tau} e^{-ru}\, G(S_u, \Pi_u)\, du + \int_0^{\zeta\wedge\tau} e^{-ru}\, d\ell_u^{K/\alpha}(S) - e^{-r\tau}\,(K - \alpha S_\tau)^+\, I(\tau < \zeta)\bigg]
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_{s,\pi}\big[Y_\tau\, I(\tau < \zeta) + Z_\zeta\, I(\zeta \leq \tau)\big] \qquad (2.12)\\
&= \mathbb{E}_{s,\pi}\bigg[\int_0^{\zeta\wedge\tau} e^{-ru}\,(c + \nu S_u)\, du + e^{-r(\zeta\wedge\tau)}\,\alpha S_{\zeta\wedge\tau} + e^{-r\zeta}\,(K - \alpha S_\zeta)^+\, I(\zeta \leq \tau)\bigg]\\
&= \alpha s + \mathbb{E}_{s,\pi}\bigg[\int_0^{\zeta\wedge\tau} e^{-ru}\, H(S_u, \Pi_u)\, du - e^{-r\zeta}\,(K - \alpha S_\zeta)^+\, I(\zeta \leq \tau)\bigg]
\end{aligned}$$

hold, for any stopping times ζ and τ. Here, we set $G(s, \pi) = c + \nu s - rKI(s \leq K/\alpha) - (\delta_0 + (\delta_1 - \delta_0)\pi)I(s > K/\alpha)$ and $H(s, \pi) = c + (\nu - \alpha\delta_0 - \alpha(\delta_1 - \delta_0)\pi)s$, for all $(s, \pi) \in (0, \infty) \times [0, 1]$, and the process $\ell^{K/\alpha}(S) = (\ell_t^{K/\alpha}(S))_{t \geq 0}$ defined as the limit in probability by

$$\ell_t^{K/\alpha}(S) = \mathbb{P}_{s,\pi} - \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t I\big(-\varepsilon < S_u - K/\alpha < \varepsilon\big)\, \sigma^2 S_u^2\, du \qquad (2.13)$$

is the local time of the process S at the plane K/α. Then, it follows from the expressions in (2.11) and (2.12) and the structure of the optimal stopping times in (2.8) that the value function of the optimal stopping game in (2.7) admits the representations

$$V_*(s, \pi) = K + \mathbb{E}_{s,\pi}\bigg[\int_0^{\zeta_*\wedge\tau_*} e^{-ru}\, G(S_u, \Pi_u)\, du - e^{-r\tau_*}\,(K - \alpha S_{\tau_*})^+\, I(\tau_* < \zeta_*)\bigg] \qquad (2.14)$$

and

$$V_*(s,\pi) = \alpha s + \mathbb{E}_{s,\pi}\bigg[\int_0^{\zeta_* \wedge \tau_*} e^{-ru} \, H(S_u, \Pi_u) \, du - e^{-r\zeta_*} \, (K - \alpha S_{\zeta_*})^+ \, I(\zeta_* \le \tau_*) \bigg] \tag{2.15}$$

for all $(s,\pi) \in (0, K/\alpha) \times [0,1]$. Hence, it is seen from the structure of the integrands in the expressions of (2.14) and (2.15), that it is not optimal for the writer to recall the bond when $G(S_t, \Pi_t) < 0$, while it is not optimal for the holder to convert the bond when $H(S_t, \Pi_t) > 0$, for $0 \le t < \zeta_* \vee \tau_*$. On one hand, these facts mean that the points $(s,\pi) \in (0, K/\alpha) \times [0,1]$, for which both inequalities $G(s,\pi) < 0$ and $H(s,\pi) > 0$ hold simultaneously, belong to the continuation region C_* in (2.9), whenever such points exist. On the other hand, these properties also imply that, if the inequality $c \le (r - \nu/\alpha)K$ is satisfied, meaning that $G(s,\pi) \le 0$, for all $(s,\pi) \in (0, K/\alpha] \times [0,1]$, then $\mathbb{P}_{s,\pi}(\tau_* \le \zeta_*) = 1$ holds. Moreover, if the inequality $c \ge (\delta_0 - \nu/\alpha)K$ is satisfied, meaning that $H(s,\pi) \ge 0$, for all $(s,\pi) \in (0, K/\alpha] \times [0,1]$, then $\mathbb{P}_{s,\pi}(\zeta_* \le \tau_*) = 1$ holds. In particular, we see in the latter case that, if $c \ge rK$ holds, then we have $H(s,\pi) \ge 0$ as well as $G(s,\pi) \ge 0$, for all $(s,\pi) \in (0, K/\alpha] \times [0,1]$, which means that it is optimal for the writer to recall the bond instantly, so that $\zeta_* = 0$ should hold in that case. It finally follows from the arguments above that if $(\delta_0 - \nu/\alpha)K \le c \le (r - \nu/\alpha)K$ holds, then $\mathbb{P}_{s,\pi}(\zeta_* = \tau_*) = 1$, for all $(s,\pi) \in (0, K/\alpha] \times [0,1]$, so that $\zeta_* = \tau_* = \inf\{t \ge 0 \,|\, S_t \ge K/\alpha\}$ should hold in that case.

(ii) Let us now assume that the property $(r - \nu/\alpha)K < c < rK$ holds, which does not depend on the starting point (s,π) of the process (S,Π), so that it implies that $\zeta_*(s,\pi) \le \tau_*(s',\pi')$ ($\mathbb{P}$-a.s.), for all (s,π) and (s',π') from the set $(0, K/\alpha] \times [0,1]$. Here, we denote by $\zeta_*(s,\pi)$ and $\tau_*(s',\pi')$ the optimal exercise times of the writer and the holder of the convertible bond forming a Nash equilibrium in the problem of (2.7) given that the process (S,Π) starts at the points (s,π) and (s',π') from $(0, K/\alpha] \times [0,1]$, respectively. Then, by means of the results of general optimal stopping theory for Markov processes (see, e.g. [54; Chapter I, Section 2.2]), we conclude from the structure of the continuation region C_* in (2.9) and the form of the stopping times in (2.8) as well as from the expression in (2.14) that

$$V_*(s,\pi) - K = \mathbb{E}\bigg[\int_0^{\zeta_*} e^{-ru} \left(c + \nu S_u^{(s,\pi)} - rK \right) du \bigg] < 0 \tag{2.16}$$

holds, for any $(s,\pi) \in C_*$ with $\zeta_* = \zeta_*(s,\pi)$. Here and in the rest of this section, we indicate by $(S^{(s,\pi)}, \Pi^{(\pi)})$ the dependence of the process (S,Π) in (2.3) and (2.4) on the starting point $(s,\pi) \in (0, K/\alpha] \times [0,1]$. Hence, taking any s' such that $s' < s < K/\alpha$ and using the explicit expressions for the process S in (2.1)-(2.3), we obtain from the expression in (2.14) that the inequalities

$$V_*(s',\pi) - K \le \mathbb{E}\bigg[\int_0^{\zeta_*} e^{-ru} \left(c + \nu S_u^{(s',\pi)} - rK \right) du \bigg] \le \mathbb{E}\bigg[\int_0^{\zeta_*} e^{-ru} \left(c + \nu S_u^{(s,\pi)} - rK \right) du \bigg] \tag{2.17}$$

are satisfied. Therefore, by virtue of the inequality in (2.16), we see that $(s',\pi) \in C_*$. Moreover, we may conclude from the expression in (2.17) that all the points (s,π) such that $0 < s < \underline{a} \wedge (K/\alpha)$ with $\underline{a} = (rK - c)/\nu$, belong to the continuation region C_* in (2.9) in this case. Note that the inequalities $0 < \underline{a} < K/\alpha$ are satisfied, whenever $(r - \nu/\alpha)K < c < rK$ holds. On the other hand, if we assume that $(s,\pi) \in D_*$ and use arguments similar to the ones above, we see from the equality in (2.16) that $V_*(s',\pi) - K \ge V_*(s,\pi) - K = 0$ holds, for all $s < s' \le K/\alpha$, so that $(s',\pi) \in D_*$. Getting these arguments together, we may conclude that there exists a function $a_*(\pi)$ such that $\underline{a} \le a_*(\pi) \le K/\alpha$ holds under $(r - \nu/\alpha)K < c < rK$, and all the points (s,π) such that $0 < s < a_*(\pi)$ and $\pi \in [0,1]$ belong to the continuation region C_* in (2.9), while all the points (s,π) such that $a_*(\pi) \le s \le K/\alpha$ and $\pi \in [0,1]$ belong to the stopping region D_* in (2.10).

Let us finally assume that $(s,\pi) \in C_*$ under $(r - \nu/\alpha)K < c < rK$ and fix some π' such that $0 < \pi' < \pi < 1$. Then, taking into account the comparison results for strong solutions of stochastic differential equations (see, e.g. [24; Theorem 1]), we obtain from the explicit expressions in (2.1)-(2.2) and (2.3)-(2.4) that the inequality $S_t^{(s,\pi')} \le S_t^{(s,\pi)}$ holds, for all $t \ge 0$. Hence, using the facts that (S, Π) is a time-homogeneous Markov process and $\zeta_* = \zeta_*(s,\pi)$ does not depend on π', we obtain that the inequalities

$$V_*(s,\pi') \le \mathbb{E}\bigg[\int_0^{\zeta_*} e^{-ru} \left(c + \nu S_u^{(s,\pi')}\right) du + e^{-r\zeta_*} K \bigg] \tag{2.18}$$
$$\le \mathbb{E}\bigg[\int_0^{\zeta_*} e^{-ru} \left(c + \nu S_u^{(s,\pi)}\right) du + e^{-r\zeta_*} K \bigg] < K$$

hold. By virtue of the inequality in (2.16), we may conclude that $(s,\pi') \in C_*$, so that the boundary $a_*(\pi)$ is decreasing on $[0,1]$.

(iii) Let us now assume that the property $c < (\delta_0 - \nu/\alpha)K$ holds, which does not depend on the initial point (s,π) of the process (S,Π), so that implies that $\tau_*(s,\pi) \le \zeta_*(s',\pi')$ ($\mathbb{P}$-a.s.), for all (s,π) and (s',π') from the set $(0, K/\alpha] \times [0,1]$. Here, we denote by $\tau_*(s,\pi)$ and $\zeta_*(s',\pi')$ the optimal exercise times of the writer and the holder of the convertible bond forming a Nash equilibrium in the problem of (2.7) given that the process (S,Π) starts at the points (s,π) and (s',π') from $(0,K/\alpha] \times [0,1]$, respectively. Then, by means of the results of general optimal stopping theory for Markov processes, we conclude from the structure of the continuation region C_* in (2.9) and the form of the stopping times in (2.8) as well as from the expression in (2.15) that

$$V_*(s,\pi) - \alpha s = \mathbb{E}\bigg[\int_0^{\tau_*} e^{-ru} \, H(S_u^{(s,\pi)}, \Pi_u^{(\pi)}) \, du \bigg] > 0 \tag{2.19}$$

holds, for any $(s,\pi) \in C_*$ with $\tau_* = \tau_*(s,\pi)$. Hence, taking any s' such that $s' < s < K/\alpha$ and using the definition of the process S in (2.1)-(2.3), we obtain from the expression in (2.15) that the inequalities

$$V_*(s',\pi) - \alpha s' \ge \mathbb{E}\bigg[\int_0^{\tau_*} e^{-ru} \, H(S_u^{(s',\pi)}, \Pi_u^{(\pi)}) \, du \bigg] \ge \mathbb{E}\bigg[\int_0^{\tau_*} e^{-ru} \, H(S_u^{(s,\pi)}, \Pi_u^{(\pi)}) \, du \bigg] > 0 \tag{2.20}$$

are satisfied. Therefore, by virtue of the inequality in (2.15), we see that $(s',\pi) \in C_*$. Moreover, we may conclude that all the points (s,π) such that $0 < s < \underline{b}(\pi) \wedge (K/\alpha)$ with $\underline{b}(\pi) = c/(\alpha\delta_0 + \alpha(\delta_1 - \delta_0)\pi - \nu)$, for $\pi \in [0,1]$, belong to the continuation region C_* in (2.9) in this case. Observe that the inequalities $0 < \underline{b}(\pi) < K/\alpha$ are satisfied, whenever $c < (\delta_0 + (\delta_1 - \delta_0)\pi - \nu/\alpha)K$ holds, for $\pi \in [0,1]$, and the latter inequality is satisfied, for all $0 \le \pi < \varkappa$ with $\varkappa = (\delta_0 - \nu/\alpha - c/K)/(\delta_0 - \delta_1)$, whenever $c < (\delta_0 - \nu/\alpha)K$ holds. On the other hand, if we assume that $(s,\pi) \in D_*$ and use arguments similar to the ones above, we see from the equality in (2.15) that $V_*(s',\pi) - \alpha s' \le V_*(s,\pi) - \alpha s = 0$ holds, for all $s < s' \le K/\alpha$, so that $(s',\pi) \in D_*$. Getting these arguments together, we may conclude that there exists a function $b_*(\pi)$ such that $\underline{b}(\pi) \le b_*(\pi) \le K/\alpha$ holds under $c < (\delta_0 - \nu/\alpha)K$, and all the points (s,π) such that $0 < s < b_*(\pi)$ and $\pi \in [0,1]$ belong to the continuation region C_* in (2.9), while all the points (s,π) such that $b_*(\pi) \le s \le K/\alpha$ and $\pi \in [0,1]$ belong to the stopping region D_* in (2.10). It also follows from the arguments above that we have $b_*(\pi) = K/\alpha$, for $\varkappa \le \pi \le 1$, whenever $(\delta_1 - \nu/\alpha)K \le c < (\delta_0 - \nu/\alpha)K$ holds.

Let us finally assume that $(s,\pi) \in C_*$ under $c < (\delta_0 - \nu/\alpha)K$ and fix some π' such that $0 < \pi < \pi' < 1$. Then, taking into account the comparison results for strong solutions of stochastic differential differential equations implying that the inequality $S_t^{(s,\pi')} \le S_t^{(s,\pi)}$ holds, for all $t \ge 0$, as well as using the facts that (S,Π) is a time-homogeneous Markov process and $\tau_* = \tau_*(s,\pi)$ does not depend on π', we obtain from the expression in (2.15) that the inequalities

$$V_*(s,\pi') \geq \mathbb{E}\bigg[\int_0^{\tau_*} e^{-ru}\,\big(c+\nu S_u^{(s,\pi')}\big)\,du + e^{-r\tau_*}\,\alpha S_{\tau_*}^{(s,\pi')}\bigg] \tag{2.21}$$
$$\geq \mathbb{E}\bigg[\int_0^{\tau_*} e^{-ru}\,\big(c+\nu S_u^{(s,\pi)}\big)\,du + e^{-r\tau_*}\,\alpha S_{\tau_*}^{(s,\pi)}\bigg] > \alpha s$$

hold. By virtue of the inequality in (2.19), we may conclude that $(s,\pi') \in C_*$, so that the boundary $b_*(\pi)$ is increasing on $[0,1]$ under $\delta_0 > \delta_1$.

Summarising the arguments shown above, let us formulate the following assertion.

Lemma 2.1 *Let the processes S and Π be defined by (2.3)-(2.5). Assume that the inequalities $0 < \delta_1 < \delta_0 < r$ and $0 < \nu < \alpha\delta_1 < \alpha\delta_0$ hold. In this case, if $c \geq rK$ holds, then the writer recalls the bond instantly, so that $\zeta_* = 0$ in the optimal stopping game of (2.7). Otherwise, if $c < rK$ holds, then the optimal stopping times from (2.8) have the structure*

$$\zeta_* = \inf\big\{t \geq 0 \,\big|\, S_t \geq a_*(\Pi_t)\big\} \quad \text{and} \quad \tau_* = \inf\big\{t \geq 0 \,\big|\, S_t \geq b_*(\Pi_t)\big\} \tag{2.22}$$

so that the continuation and stopping regions in (2.9) and (2.10) take either of the forms

$$C_* = \big\{(s,\pi) \in (0,K/\alpha)\times[0,1] \,\big|\, s < a_*(\pi)\big\} \quad \text{or} \quad C_* = \big\{(s,\pi) \in (0,K/\alpha)\times[0,1] \,\big|\, s < b_*(\pi)\big\} \tag{2.23}$$

and

$$D_* = \big\{(s,\pi) \in (0,K/\alpha]\times[0,1] \,\big|\, s \geq a_*(\pi)\big\} \quad \text{or} \quad D_* = \big\{(s,\pi) \in (0,K/\alpha]\times[0,1] \,\big|\, s \geq b_*(\pi)\big\} \tag{2.24}$$

respectively. Here, $a_*(\pi)$ and $b_*(\pi)$ are some functions which are specified as follows:

(i) If $(r-\nu/\alpha)K < c < rK$ holds, then

$$a_*(\pi) : [0,1] \to (0,K/\alpha] \ \text{ is decreasing and } \ \underline{a} \leq a_*(\pi) \leq K/\alpha \ \text{ holds, for } \ \pi \in [0,1], \tag{2.25}$$

with $\underline{a} = (rK-c)/\nu$ and $b_*(\pi) \equiv K/\alpha$.

(ii) If $c < (\delta_0 - \nu/\alpha)K$ holds, then

$$b_*(\pi) : [0,1] \to (0,K/\alpha] \ \text{ is increasing and } \ \underline{b}(\pi) \leq b_*(\pi) \leq K/\alpha \ \text{ holds, for } \ \pi \in [0,1], \tag{2.26}$$

with $\underline{b}(\pi) = c/(\alpha\delta_0 + \alpha(\delta_1-\delta_0)\pi - \nu)$ and $a_*(\pi) \equiv K/\alpha$, where we have $b_*(\pi) = K/\alpha$, for $\varkappa \leq \pi \leq 1$ with $\varkappa = (\delta_0 - \nu/\alpha - c/K)/(\delta_0-\delta_1)$ under $(\delta_1 - \nu/\alpha)K \leq c < (\delta_0 - \nu/\alpha)K$.

(iii) If $(\delta_0 - \nu/\alpha)K \leq c \leq (r-\nu/\alpha)K$ holds, then $a_*(\pi) \equiv b_*(\pi) \equiv K/\alpha$.

It is seen from the assertion of Lemma 2.1 that the writer should recall the bond instantly when the inequality $c \geq rK$ holds. This feature intuitively makes it unfavourable for the writer to issue such a bond, since recalling it immediately turns out to be less expensive irrespective of the underlying asset price. Taking these observations into account, we assume that the inequality $c < rK$ holds for the rest of the section. It also follows from the results of Lemma 2.1 that, when the writer (issuer) pays a relatively large value of fixed coupon rate c compared to the recalling strike price K, then the writer has the incentive to recall the bond before the holder might convert it. On the other hand, when the coupon rate c is relatively small (compared to K), it is not favourable for the writer to recall the bond soon enough, resulting in the holder converting the bond prior to the writer recalling it. Finally, when the fixed coupon rate c takes some intermediate values (compared to K), the writer and the holder should act simultaneously, by waiting until the underlying asset price increases to the critical value K/α. We also observe that the optimal strategy of the holder changes qualitatively, between converting the bond first and waiting to exercise the contact simultaneously with the writer, according to the running estimate Π of the dividend policy Θ, based on their observations of the risky asset price S.

3 The case of full information

In this section, we present a solution to the optimal stopping game in the associated model with *full information*, when both the writer and the holder of the convertible bond have access to the dividend policy of the firm issuing the asset.

3.1 The optimal stopping game. According to the appropriate arguments from Subsection 2.2 above, we may conclude that a rational value of the convertible bond in the model with *full information* is given by the value of the optimal stopping game

$$U_*(s,i) = \inf_{\zeta'} \sup_{\tau'} \mathbb{E}_{s,i}\big[Y_{\tau'} I(\tau' < \zeta') + Z_{\zeta'} I(\zeta' \le \tau')\big] = \sup_{\tau'} \inf_{\zeta'} \mathbb{E}_{s,i}\big[Y_{\tau'} I(\tau' < \zeta') + Z_{\zeta'} I(\zeta' \le \tau')\big] \tag{3.1}$$

where $\mathbb{E}_{s,i}$ denotes the expectation under the assumption that process (S,Θ) starts at some $(s,i) \in (0, K/\alpha] \times \{0,1\}$. The infimum and supremum in (3.1) are taken over all stopping times $\zeta' = \zeta'(S,\Theta)$ and $\tau' = \tau'(S,\Theta)$ with respect to the natural filtration of the process (S,Θ) defined in (2.1)-(2.2). Since the continuous time Markov chain Θ is observable in this formulation, it can be deduced by using arguments similar to the ones in the proof of Lemma 2.1 that the optimal stopping times forming a Nash equilibrium in the problem of (3.1) should be of the form

$$\zeta'_* = \inf\big\{t \ge 0 \,\big|\, S_t \ge g_*(\Theta_t)\big\} \quad \text{and} \quad \tau'_* = \inf\big\{t \ge 0 \,\big|\, S_t \ge h_*(\Theta_t)\big\} \tag{3.2}$$

for some numbers $g_*(i)$ and $h_*(i)$, for $i = 0,1$, which are determined from the analysis of the associated free-boundary problem formulated below. We further assume that the inequalities $0 < \delta_1 < \delta_0 < r$ and $0 < \nu < \alpha\delta_1 < \alpha\delta_0$ as well as $c < rK$ hold. In this case, it also follows from arguments similar to the ones used in the proof of Lemma 2.1 that the inequality $U_*(s,0) \le U_*(s,1)$ holds for the value functions, for $0 < s \le K/\alpha$, as well as $g_*(1) \le g_*(0) \le K/\alpha$ and $h_*(0) \le h_*(1) \le K/\alpha$ are satisfied by the optimal exercise boundaries.

3.2 The free-boundary problem. By means of standard arguments based on the application of Itô's formula, it is shown that the infinitesimal operator $\mathbb{L}_{(S,\Theta)}$ of the process (S,Θ) from (2.1)-(2.2) acts on an arbitrary function $F(\cdot,i)$ from the class C^2 on the interval $(0, K/\alpha)$, for every $i \in \{0,1\}$, according to the rule

$$(\mathbb{L}_{(S,\Theta)}F)(s,i) = (r - \delta_0 - (\delta_1 - \delta_0)\, i)\, s\, F_s(s,i) + \frac{\sigma^2 s^2}{2} F_{ss}(s,i) + \lambda\Big(F(s,1-i) - F(s,i)\Big) \tag{3.3}$$

for all $(s,i) \in (0, K/\alpha) \times \{0,1\}$. We now formulate the following associated free-boundary problem for the function $U_*(s,i)$ and the boundaries $g_*(i)$ and $h_*(i)$ given by

$$(\mathbb{L}_{(S,\Theta)}U - rU)(s,i) = -(c + \nu s) \;\text{ for }\; 0 < s < g(i) \le K/\alpha \quad \text{or} \quad 0 < s < h(i) \le K/\alpha \tag{3.4}$$

$$U(s,i)\big|_{s=g(i)} = K \;\text{ if }\; g(i) \le K/\alpha, \quad \text{or} \quad U(s,i)\big|_{s=h(i)} = \alpha\, h(i) \;\text{ if }\; h(i) \le K/\alpha \tag{3.5}$$

$$U_s(s,i)\big|_{s=g(i)} = 0 \;\text{ if }\; g(i) < K/\alpha, \quad \text{or} \quad U_s(s,i)\big|_{s=h(i)} = \alpha \;\text{ if }\; h(i) < K/\alpha \tag{3.6}$$

$$U(s,i)\big|_{s=0+} \;\text{ is finite} \tag{3.7}$$

$$U(s,i) = K \;\text{ for } s > g(i) \text{ if } g(i) \le K/\alpha, \quad \text{or} \quad U(s,i) = \alpha s \;\text{ for } s > h(i) \text{ if } h(i) \le K/\alpha \tag{3.8}$$

$$\alpha s < U(s,i) < K \;\text{ for }\; 0 < s < g(i) \le K/\alpha \quad \text{or} \quad 0 < s < h(i) \le K/\alpha \tag{3.9}$$

$$(\mathbb{L}_{(S,\Theta)}U - rU)(s,i) > -(c + \nu s) \;\text{ for }\; g(i) < s < K/\alpha \;\text{ if }\; g(i) \le K/\alpha, \quad \text{or} \tag{3.10}$$

$$(\mathbb{L}_{(S,\Theta)}U - rU)(s,i) < -(c + \nu s) \;\text{ for }\; h(i) < s < K/\alpha \;\text{ if }\; h(i) \le K/\alpha \tag{3.11}$$

with some $0 < g(i) \le K/\alpha$ or $0 < h(i) \le K/\alpha$, for $i = 0,1$.

3.3 Solution to the free-boundary problem. By means of straightforward computations, we obtain that the general solution to the two-dimensional system of second-order ordinary differential equations in (3.3)+(3.4) is given by

$$U(s,i) = \sum_{j=1}^{4} C_j(i)\, s^{\beta_j} + A_i(s) \quad \text{with} \quad A_i(s) = \frac{(2\lambda + \delta_1 - (\delta_1 - \delta_0)i)\nu}{(\delta_0 + \lambda)(\delta_1 + \lambda) - \lambda^2}\, s + \frac{c}{r} \tag{3.12}$$

for either $0 < s < g(1)$ or $0 < s < h(0)$, and every $i = 0, 1$, as well as

$$U(s,i) = \sum_{j=1}^{2} D_j(i)\, s^{\gamma_{i,j}} + B_i(s) \quad \text{with} \quad B_i(s) = \frac{\nu + \alpha\lambda i}{\delta_i + \lambda}\, s + \frac{c + \lambda K(1-i)}{r + \lambda} \tag{3.13}$$

for either $g(1) < s < g(0) \leq K/\alpha$ or $h(0) < s < h(1) \leq K/\alpha$, respectively, whenever the appropriate intervals for s exist. Here, $C_j(i)$, $j = 1, 2, 3, 4$, and $D_k(i)$, $k = 1, 2$, are some arbitrary constants, the numbers $\beta_4 < \beta_3 < 0 < \beta_2 < \beta_1$ are the roots of the corresponding characteristic equation

$$R_0(\beta) R_1(\beta) = \lambda^2 \quad \text{with} \quad R_i(\beta) = r + \lambda - \beta(r - \delta_i) - \frac{\sigma^2}{2}\beta(\beta - 1) \tag{3.14}$$

and $\gamma_{i,2} < 0 < 1 < \gamma_{i,1}$ are explicitly given by

$$\gamma_{i,k} = \frac{1}{2} - \frac{r - \delta_i}{\sigma^2} - (-1)^k \sqrt{\left(\frac{1}{2} - \frac{r - \delta_i}{\sigma^2}\right)^2 + \frac{2(r + \lambda)}{\sigma^2}} \tag{3.15}$$

for every $i = 0, 1$ and $k = 1, 2$. Observe that we should have $C_j(i) = 0$, $j = 3, 4$, in the expression of (3.12), since otherwise $U(s,i) \to \pm\infty$ as $s \downarrow 0$, that must be excluded by virtue of the obvious fact that the value function in (3.1) is bounded under $s \downarrow 0$. The latter fact also follows from the property that the point 0 cannot be reached by the process S in a finite time, that is expressed by the condition of (3.7). We now derive closed-form solutions to the free-boundary problem of (3.3)+(3.4)-(3.11) under the following five possible ordered combinations for the optimal exercise boundaries $g_*(i)$ and $h_*(i)$, for $i = 0, 1$.

(i) Suppose that it is optimal for the holder of the bond to convert first, so that the combination $g(1) = g(0) = K/\alpha$ and $h(0) \leq h(1) \leq K/\alpha$ is realised. Then, applying the conditions of (3.5) and (3.6) to the functions in (3.12), under the assumption that $C_j(i) = 0$, $j = 3, 4$, and to the function in (3.13) for $i = 1$, we obtain that the equalities

$$C_j(0) R_0(\beta_j) = C_j(1)\lambda \quad \text{for} \quad j = 1, 2 \tag{3.16}$$

as well as

$$\sum_{j=1}^{2} C_j(0)\, h^{\beta_j}(0) + A_0(h(0)) = \alpha\, h(0), \quad \sum_{j=1}^{2} C_j(0)\, \beta_j\, h^{\beta_j}(0) + h(0)\, A_0'(h(0)) = \alpha\, h(0) \tag{3.17}$$

and

$$\sum_{j=1}^{2} D_j(1)\, h^{\gamma_{1,j}}(1) + B_1(h(1)) = \alpha\, h(1), \quad \sum_{j=1}^{2} D_j(1)\, \gamma_{1,j}\, h^{\gamma_{1,j}}(1) + h(1)\, B_1'(h(1)) = \alpha\, h(1) \tag{3.18}$$

hold. Observe that, since the inequality $h(0) \leq h(1)$ holds, the function in (3.12)-(3.13) for $i = 1$, when the process Θ is in the state 1, should be continuously differentiable, and thus, the equalities

$$\sum_{j=1}^{2} C_j(1)\, h^{\beta_j}(0) + A_1(h(0)) = \sum_{j=1}^{2} D_j(1)\, h^{\gamma_{1,j}}(0) + B_1(h(0)) \tag{3.19}$$

and

$$\sum_{j=1}^{2} C_j(1)\,\beta_j\,h^{\beta_j}(0) + h(0)\,A_1'(h(0)) = \sum_{j=1}^{2} D_j(1)\,\gamma_{1,j}\,h^{\gamma_{1,j}}(0) + h(0)\,B_1'(h(0)) \tag{3.20}$$

are satisfied, for $0 < h(0) < K/\alpha$. Hence, solving the system in (3.17)-(3.20), we obtain that the solution to the free-boundary problem in (3.3)+(3.4)-(3.7) is given by

$$U(s,0;h_*(0)) = C_1(0;h_*(0))\,s^{\beta_1} + C_2(0;h_*(0))\,s^{\beta_2} + A_0(s) \tag{3.21}$$

and

$$U(s,1;h_*(0),h_*(1)) = C_1(1;h_*(0),h_*(1))\,s^{\beta_1} + C_2(1;h_*(0),h_*(1))\,s^{\beta_2} + A_1(s) \tag{3.22}$$

for $0 < s < h_*(0)$, as well as

$$U(s,1;h_*(1)) = D_1(1;h_*(1))\,s^{\gamma_{1,1}} + D_2(1;h_*(1))\,s^{\gamma_{1,2}} + B_1(s) \tag{3.23}$$

for $h_*(0) \le s < h_*(1)$, where

$$C_j(0;h_*(0)) = \frac{(1-\beta_{3-j})r(A_0(h_*(0)) - \alpha\,h_*(0)) - c}{(\beta_{3-j}-\beta_j)rh_*^{\beta_j}(0)} \tag{3.24}$$

$$\begin{aligned} &C_j(1;h_*(0),h_*(1)) \\ &= \sum_{k=1}^{2} \frac{(\beta_{3-j}-\gamma_{1,k})D_k(1;h_*(1))h_*^{\gamma_{1,k}}(0)}{(\beta_{3-j}-\beta_j)h_*^{\beta_j}(0)} + \frac{(\beta_{3-j}-1)(r+\lambda)r(B_1(h_*(0)) - A_1(h_*(0))) - c\lambda}{(\beta_{3-j}-\beta_j)(r+\lambda)rh_*^{\beta_j}(0)} \end{aligned} \tag{3.25}$$

and

$$D_j(1;h_*(1)) = \frac{(1-\gamma_{1,3-j})(r+\lambda)(B_1(h_*(1)) - \alpha\,h_*(1)) - c}{(r+\lambda)(\gamma_{1,3-j}-\gamma_{1,j})h_*^{\gamma_{1,j}}(1)} \tag{3.26}$$

for $j = 1, 2$, and the functions $A_i(s)$, $i = 1, 2$, and $B_1(s)$ are given in (3.12)-(3.13). Here, the couple $h_*(0)$ and $h_*(1)$ is determined as the unique solution to the system of equations in (3.16), whenever it exists, having the form

$$C_j(0;h(0))\,R_0(\beta_j) = \lambda\,C_j(1;h(0),h(1)) \tag{3.27}$$

for $j = 1, 2$, where $R_0(\beta_j)$ is given by (3.14). Assume that the inequalities $c/(\alpha\delta_1 - \nu) < h(1) \le K/\alpha$ and $c/(\alpha\delta_0 - \nu) < H_*(h(1)) < h(0) \le h(1) \le K/\alpha$ hold, so that the inequalities in (3.11), for $i = 0, 1$, are satisfied. Here $H_*(h(1))$ denotes the unique solution to the equation $\lambda(U(H,1;h(1)) - \alpha\,H) = (\alpha\delta_0 - \nu)H - c$ with $U(s,1;h(1))$ given by (3.23), for every $h(1)$ fixed. Then, the latter assumption implies that the case $g(1) = g(0) = K/\alpha$ and $h(0) \le h(1) \le K/\alpha$ can only be realised when $c < (\delta_1 - \nu/\alpha)K$ holds, which also guarantees that $H_*(h(1)) < K/\alpha$ holds. The omitted analysis of this case is given in part (i) of Appendix below. Moreover, taking into account the uniqueness of solutions of systems of second-order ordinary differential equations and applying standard comparison arguments for solutions of the system of equations in (3.4) together with the instantaneous-stopping and smooth-fit conditions on the right-hand sides of (3.5)-(3.6) for $i = 0, 1$, or verifying directly, we obtain that the functions $U(s,0;h_*(0))$, $U(s,1;h_*(0),h_*(1))$ and $U(s,1;h_*(1))$ in (3.21)-(3.23) satisfy the inequalities in (3.9) as well.

(ii) Suppose that it is optimal for the holder of the bond to convert first, only under dividend rate δ_0, when process Θ is in the state 0, so that the combination $g(1) = g(0) = K/\alpha$ and

$h(0) \leq h(1) = K/\alpha$ is realised. Then, applying the conditions of (3.5) and (3.6) to the function in (3.12), under the assumption that $C_j(i) = 0, \ j = 3, 4$, we obtain that the equalities in (3.16)-(3.17) hold, while applying the conditions of (3.5) to the function (3.13) for $i = 1$, when the process S hits the level K/α, we obtain that the equality

$$D_1(1)\,(K/\alpha)^{\gamma_{1,1}} + D_2(1)\,(K/\alpha)^{\gamma_{1,2}} + B_1(K/\alpha) = K \tag{3.28}$$

holds as well. Observe that, since the inequality $h(0) \leq K/\alpha$ holds, the function in (3.12)-(3.13) for $i = 1$, when the process Θ is in the state 1, should be continuously differentiable, and thus, the equalities in (3.19)-(3.20) hold. Hence, solving the system in (3.17), (3.28) and (3.19)-(3.20), we obtain that the solution to the free-boundary problem in (3.3)+(3.4)-(3.7) for $i = 0$, is given by $U(s, 0; h_*(0))$ in (3.21) and

$$U(s, 1; h_*(0), K/\alpha) = C_1(1; h_*(0), f(1))\, s^{\beta_1} + C_2(1; h_*(0), f(1))\, s^{\beta_2} + A_1(s) \tag{3.29}$$

for $0 < s < h_*(0)$, as well as

$$U(s, 1; K/\alpha) = f(1)\, s^{\gamma_{1,1}} + (K - B_1(K/\alpha) - f(1)\,(K/\alpha)^{\gamma_{1,1}})\,(\alpha s/K)^{\gamma_{1,2}} + B_1(s) \tag{3.30}$$

for $h_*(0) \leq s < K/\alpha$, where $C_j(1; h_*(0), f(1))$, $j = 1, 2$, admits the representation of (3.25) with $D_1(1; h_*(1)) = f(1)$ and $D_2(1; h_*(1)) = (K/\alpha)^{-\gamma_{1,2}}(K - B_1(K/\alpha) - f(1)(K/\alpha)^{\gamma_{1,1}})$, for an arbitrary variable $f(1)$, and the functions $A_1(s)$ and $B_1(s)$ are given by (3.12)-(3.13). Here, the couple $f_*(1)$ and $h_*(0)$ is determined as the unique solution to the system of equations in (3.16), whenever it exists, having the form

$$C_j(0; h(0))\, R_0(\beta_j) = \lambda\, C_j(1; h(0), f(1)) \tag{3.31}$$

where $R_0(\beta_j)$ is given by (3.14), for $j = 1, 2$. Following the appropriate analysis presented in part (i) above, assume that the inequalities $c/(\alpha\delta_0 - \nu) < H_*(f(1)) < h(0) \leq K/\alpha$ hold, so that the inequality in (3.11), for $i = 0$, as well as the equality in (3.4), for $i = 1$, are satisfied. Here $H_*(f(1))$ denotes the unique solution to the equation $\lambda(U(H, 1; K/\alpha) - \alpha\, H) = (\alpha\, \delta_0 - \nu)H - c$ with $U(s, 1; K/\alpha)$ given by (3.30), for every $f(1)$ fixed. Then, the latter assumption implies that the case $g(1) = g(0) = K/\alpha$ and $h(0) \leq K/\alpha = h(1)$ can be realised when $c < (\delta_0 - \nu/\alpha)K$ holds. In particular, this case is the only possible combination for the boundaries, when $(\delta_1 - \nu/\alpha)K \leq c < (\delta_0 - \nu/\alpha)K$ holds, while it can also occur when $c < (\delta_1 - \nu/\alpha)K$ holds and the system of equations in (3.27) does not have a solution. The omitted analysis of this case is given in part (ii) of Appendix below. Moreover, taking into account the uniqueness of solutions of systems of second-order ordinary differential equations and applying standard comparison arguments for solutions of the system of equations in (3.4) together with the instantaneous-stopping and smooth-fit conditions on the right-hand sides of (3.5)-(3.6) for $i = 0$, or verifying directly, we obtain that the functions $U(s, 0; h_*(0))$, $U(s, 1; h_*(0), K/\alpha)$ and $U(s, 1; K/\alpha)$ in (3.21) and (3.29)-(3.30) satisfy the inequalities in (3.9) as well.

(iii) Suppose that it is optimal for the writer of the bond to recall first, so that the combination $g(1) \leq g(0) \leq K/\alpha$ and $h(0) = h(1) = K/\alpha$ is realised. Then, applying the conditions of (3.5) and (3.6) to the function in (3.12), under the assumption that $C_j(i) = 0, \ j = 3, 4$, and to the function (3.13) for $i = 0$, we obtain that the equalities

$$\sum_{j=1}^{2} C_j(1)\, g^{\beta_j}(1) + A_1(g(1)) = K, \quad \sum_{j=1}^{2} C_j(1)\, \beta_j\, g^{\beta_j}(1) + g(1)\, A_1'(g(1)) = 0 \tag{3.32}$$

and

$$\sum_{j=1}^{2} D_j(0)\, g^{\gamma_{0,j}}(0) + B_0(g(0)) = K, \quad \sum_{j=1}^{2} D_j(0)\, \gamma_{0,j}\, g^{\gamma_{0,j}}(0) + g(0)\, B_0'(g(0)) = 0 \tag{3.33}$$

hold. Observe that, since the inequality $g(1) \le g(0)$ holds, the function in (3.12)-(3.13) for $i=0$, when the process Θ is in the state 0, should be continuously differentiable, and thus, the equalities

$$\sum_{j=1}^{2} C_j(0)\, g^{\beta_j}(1) + A_0(g(1)) = \sum_{j=1}^{2} D_j(0)\, g^{\gamma_{0,j}}(1) + B_0(g(1)) \tag{3.34}$$

and

$$\sum_{j=1}^{2} C_j(0)\, \beta_j\, g^{\beta_j}(1) + g(1)\, A_0'(g(1)) = \sum_{j=1}^{2} D_j(0)\, \gamma_{0,j}\, g^{\gamma_{0,j}}(1) + g(1)\, B_0'(g(1)) \tag{3.35}$$

are satisfied, for some $0 < g(1) < K/\alpha$. Hence, solving the system in (3.32)-(3.35), we obtain that the solution to the free-boundary problem in (3.3)+(3.4)-(3.7) is given by

$$U(s,1;g_*(1)) = C_1(1;g_*(1))\, s^{\beta_1} + C_2(1;g_*(1))\, s^{\beta_2} + A_1(s) \tag{3.36}$$

and

$$U(s,0;g_*(1),g_*(0)) = C_1(0;g_*(1),g_*(0))\, s^{\beta_1} + C_2(0;g_*(1),g_*(0))\, s^{\beta_2} + A_0(s) \tag{3.37}$$

for $0 < s < g_*(1)$, as well as

$$U(s,0;g_*(0)) = D_1(0;g_*(0))\, s^{\gamma_{0,1}} + D_2(0;g_*(0))\, s^{\gamma_{0,2}} + B_0(s) \tag{3.38}$$

for $g_*(1) \le s < g_*(0)$, where

$$C_j(1;g_*(1)) = \frac{r(\beta_{3-j}-1)A_1(g_*(1)) + c - r\beta_{3-j}K}{r(\beta_j - \beta_{3-j})g_*^{\beta_j}(1)} \tag{3.39}$$

$$C_j(0;g_*(1),g_*(0)) = \tag{3.40}$$

$$\sum_{k=1}^{2} \frac{(\gamma_{0,k}-\beta_{3-i})D_k(0;g_*(0))g_*^{\gamma_{0,k}}(1)}{(\beta_j-\beta_{3-j})g_*^{\beta_j}(1)} + \frac{(\beta_{3-j}-1)(r+\lambda)r(A_0(g_*(1))-B_0(g_*(1))) - \lambda(rK-c)}{(r+\lambda)r(\beta_j-\beta_{3-j})g_*^{\beta_j}(1)}$$

and

$$D_j(0;g_*(0)) = \frac{((\gamma_{0,3-j}-1)\lambda + \gamma_{0,3-j}r)(B_0(g_*(0))-K) - rB_0(g_*(0)) + c}{(r+\lambda)(\gamma_{0,j}-\gamma_{0,3-j})g_*^{\gamma_{0,j}}(0)} \tag{3.41}$$

for every $j=1,2$, and the functions $A_i(s)$, $i=1,2$, and $B_0(s)$ are given by (3.12)-(3.13). Here, the couple $g_*(0)$ and $g_*(1)$ is determined as the unique solution to the system of equations in (3.16), whenever it exists, having the form

$$C_j(0;g(1),g(0))\, R_0(\beta_j) = \lambda\, C_j(1;g(1)) \tag{3.42}$$

where $R_0(\beta_j)$ is given by (3.14), for $j=1,2$. Assume that the inequalities $(rK-c)/\nu < G_*(g(0)) \le g(1) \le g(0) \le K/\alpha$ hold, so that the inequalities in (3.10), for $i=0,1$, are satisfied. Here $G_*(g(0))$ denotes the unique solution to the equation $\lambda(U(G,0;g(0))-K) = rK - \nu G - c$ with $U(s,0;g(0))$ given by (3.38), for every $g(0)$ fixed. Then, the latter assumption implies that the case $g(1) \le g(0) \le K/\alpha$ and $h(0)=h(1)=K/\alpha$ can only be realised when $(r-\nu/\alpha)K < c < rK$ holds. The omitted analysis of this case is given in part (iii) of Appendix below. Moreover, taking into account the uniqueness of solutions of systems of second-order ordinary differential equations and applying standard comparison arguments for solutions of the system of equations in (3.4) together with the instantaneous-stopping and smooth-fit conditions on the left-hand sides of (3.5)-(3.6) for $i = 0, 1$, or verifying directly, we obtain that the functions $U(s, 1; g_*(1))$, $U(s, 0; g_*(1), g_*(0))$ and $U(s, 0; g_*(0))$ in (3.36)-(3.38) satisfy the inequalities in (3.9) as well.

(iv) Suppose that it is optimal for the writer of the bond to recall first, only under dividend rate δ_1, when process Θ is in the state 1, so that the combination $g(1) \leq g(0) = K/\alpha$ and $h(0) = h(1) = K/\alpha$ is realised. Then, applying the conditions of (3.5) and (3.6) to the function in (3.12), under the assumption that $C_j(i) = 0$, $j = 3, 4$, we obtain that the equalities (3.16) and (3.32) hold, while applying the condition of (3.5) to the function (3.13) for $i = 0$, when the process S hits the level K/α, we obtain that the equality

$$D_1(0)\,(K/\alpha)^{\gamma_{0,1}} + D_2(0)\,(K/\alpha)^{\gamma_{0,2}} + B_0(K/\alpha) = K \tag{3.43}$$

holds as well. Observe that, since the inequality $g(1) \leq K/\alpha$ holds, the function in (3.12)-(3.13) for $i = 0$, when the process Θ is in the state 0, should be continuously differentiable, and thus, the equalities in (3.34)-(3.35) hold. Hence, solving the system in (3.32), (3.43) and (3.34)-(3.35), we obtain that the solution to the free-boundary problem in (3.3)+(3.4)-(3.7) for $i = 1$, is given by the function $U(s, 1; g_*(1))$ in (3.36) and

$$U(s, 0; g_*(1), K/\alpha) = C_1(0; g_*(1), f(0))\, s^{\beta_1} + C_2(0; g_*(1), f(0))\, s^{\beta_2} + A_0(s) \tag{3.44}$$

for $0 < s < g_*(1)$, as well as

$$U(s, 0; K/\alpha) = f(0)\, s^{\gamma_{0,1}} + (K - B_0(K/\alpha) - f(0)\,(K/\alpha)^{\gamma_{0,1}})\,(\alpha s/K)^{\gamma_{0,2}} + B_0(s) \tag{3.45}$$

for $g_*(1) \leq s < K/\alpha$, where $C_j(0; g_*(1), f(0))$, $j = 1, 2$, admits the representation of the equation in (3.40) with $D_1(0; g_*(0)) = f(0)$ and $D_2(0; g_*(0)) = (K/\alpha)^{-\gamma_{0,2}}(K - B_0(K/\alpha) - f(0)(K/\alpha)^{\gamma_{0,1}})$ for an arbitrary variable $f(0)$, and the functions $A_0(s)$ and $B_0(s)$ are given by (3.12)-(3.13). Here, the couple $f_*(0)$ and $g_*(1)$ is determined as a unique solution to the system of equations in (3.16), whenever it exists, given by

$$C_j(0; g(1), f(0))\, R_0(\beta_j) = \lambda\, C_j(1; g(1)) \tag{3.46}$$

where $R_0(\beta_j)$ is given by (3.14), for $j = 1, 2$. Following the appropriate analysis presented in part (iii) above, assume that the inequalities $(rK - c)/\nu < G_*(f(0)) < g(1) \leq K/\alpha$ hold, so that the inequality in (3.10), for $i = 1$, as well as the equality in (3.4), for $i = 0$, are satisfied. Here $G_*(f(0))$ denotes the unique solution to the equation $\lambda(U(G, 0; K) - K) = rK - \nu G - c$ with $U(s, 0; K/\alpha)$ given by (3.45), for every $f(0)$ fixed. Then, the latter assumption implies that the case $g(1) \leq g(0) = K/\alpha$ and $h(0) = h(1) = K/\alpha$ can be realised when $(r - \nu/\alpha)K < c < rK$ holds and the system of (3.42) does not have a solution. The omitted analysis of this case is given in part (iv) of Appendix below. Moreover, taking into account the uniqueness of solutions of systems of second-order ordinary differential equations and applying standard comparison arguments for solutions of the system of equations in (3.4) together with the instantaneous-stopping and smooth-fit conditions on the left-hand sides of (3.5)-(3.6) for $i = 1$, or verifying directly, we obtain that the functions $U(s, 1; g_*(1))$, $U(s, 0; g_*(1), K/\alpha)$ and $U(s, 0; K/\alpha)$ in (3.36) and (3.44)-(3.45) satisfy the inequalities in (3.9) as well.

(v) Finally, suppose that it is optimal for both the holder and writer of the bond to convert and recall at the same time, so that the combination $g(1) = g(0) = K/\alpha$ and $h(0) = h(1) = K/\alpha$ is realised. Then, applying the condition of (3.5) to the function in (3.12) under the assumption that $C_j(i) = 0$, for $j = 3, 4$, we obtain that the equality (3.16) as well as

$$C_1(i)\,(K/\alpha)^{\beta_1} + C_2(i)\,(K/\alpha)^{\beta_2} + A_i(K/\alpha) = K \tag{3.47}$$

holds, for $i = 0, 1$. Hence, solving the system in (3.16) and (3.47), we obtain that the solution to the boundary problem in (3.3)+(3.4)-(3.7) is given by

$$U(s, i; K/\alpha) = C_1(i)\, s^{\beta_1} + C_2(i)\, s^{\beta_2} + A_i(s) \tag{3.48}$$

for $0 < s < K/\alpha$ and $i = 0, 1$, where

$$C_j(i) = \frac{R_i(\beta_{3-j}) A_i(K) - \lambda A_{1-i}(K) - (R_i(\beta_{3-j}) - \lambda) K}{(R_i(\beta_j) - R_i(\beta_{3-j})) K^{\beta_i}} \tag{3.49}$$

for $j = 1, 2$ and $i = 0, 1$, and the functions $A_i(s)$, $i = 0, 1$, are defined in (3.12). It is shown by means of straightforward calculations that the case $g(1) = g(0) = K/\alpha$ and $h(0) = h(1) = K/\alpha$ is the only possible combination for the boundaries, when $(\delta_0 - \nu/\alpha)K \le c \le (r - \nu/\alpha)K$ holds, while it can also occur when either $c < (\delta_0 - \nu/\alpha)K$ holds and the systems in (3.27) and (3.31) do not have solutions, or $(r - \nu/\alpha)K < c < rK$ holds and the systems in (3.42) and (3.46) do not have solutions. In this case, the equalities in (3.4), for $i = 0, 1$, are satisfied. Then, taking into account the uniqueness of solutions of systems of second-order ordinary differential equations and applying standard comparison arguments for solutions of the system of equations in (3.4) or verifying directly, we obtain that the functions $U(s, i; K/\alpha)$, $i = 0, 1$, in (3.48) satisfy the inequalities in (3.9).

3.4 The main result. Summarising the computations and arguments above, we can formulate the main result of this section concerning the solution to the convertible bond pricing problem under *full information*. This assertion can be proved by means of arguments similar to the ones used in the proof of Theorem 4.2 above.

Theorem 3.1 *Let the process S be given by (2.1)-(2.2) and Θ be the continuous-time Markov chain with two states defined above. Assume that the inequalities $0 < \delta_1 < \delta_0 < r$ and $0 < \nu < \alpha\delta_1 < \alpha\delta_0$ as well as $c < rK$ hold. Then, the value function $U_*(s, i)$ of the optimal stopping game in (3.1) admits the representation*

$$U_*(s, i) = \begin{cases} U(s, 0; g_*(1), g_*(0)), & \text{if } 0 < s < g_*(1) < g_*(0) \le K/\alpha \\ U(s, 1; h_*(0), h_*(1)), & \text{if } 0 < s < h_*(0) < h_*(1) \le K/\alpha \\ U(s, i; g_*(i)), & \text{if either } g_*(1-i) \le s < g_*(i) \le K/\alpha \\ & \text{or } 0 < s < g_*(i) \le g_*(1-i) \le K/\alpha \\ U(s, i; h_*(i)), & \text{if either } h_*(1-i) \le s < h_*(i) \le K/\alpha \\ & \text{or } 0 < s < h_*(i) \le h_*(1-i) \le K/\alpha \\ K \vee (\alpha s), & \text{if } s \ge g_*(i) \text{ and } g_*(i) \le K/\alpha \\ \alpha s, & \text{if } s \ge h_*(i) \text{ and } h_*(i) \le K/\alpha \end{cases} \tag{3.50}$$

and the stopping times ζ_*' and τ_*' from (3.2) form a Nash equilibrium, where the functions $U(s, 0; g_*(1), g_*(0)$ and $U(s, 1; h_*(0), h_*(1))$, or $U(s, i; g_*(i))$ and $U(s, i; h_*(i))$, as well as the boundaries $g_*(i)$ and $h_*(i)$, for $i = 0, 1$, are specified as follows:

(i) If $(r - \nu/\alpha)K < c < rK$ holds, then $U(s, 1; g_*(1))$ is given by (3.36), while $U(s, 0; g_*(1), g_*(0))$ and $U(s, 0; g_*(0))$ are given by the expressions in (3.37)-(3.38), when the system in (3.42) admits a unique solution with $(rK - c)/\nu \le g_*(i) \le K/\alpha$, for $i = 0, 1$, otherwise, by the expressions in (3.44)-(3.45) with $g_*(0) = K/\alpha$, when the system in (3.46) admits a unique solution with $(rK - c)/\nu \le g_*(1) \le K/\alpha$, and otherwise, by the expression in (3.48) with $g_*(0) = g_*(1) = K/\alpha$.

(ii) If $c < (\delta_0 - \nu/\alpha)K$ holds, then $U(s, 0; h_*(0))$ is given by the expression in (3.21), while $U(s, 1; h_*(0), h_*(1))$ and $U(s, 1; h_*(1))$ are given by the expressions in (3.22)-(3.23), when the system in (3.27) admits a unique solution with $c/(\alpha\delta_i - \nu) \le h_*(i) \le K/\alpha$, for $i = 0, 1$, otherwise, by the

expressions in (3.29)-(3.30) with $h_*(1) = K/\alpha$, when the system in (3.31) admits a unique solution with $c/(\alpha\delta_0 - \nu) \le h_*(0) \le K/\alpha$, and otherwise, by the expression in (3.48) with $h_*(0) = h_*(1) = K/\alpha$.

(iii) If $(\delta_0 - \nu/\alpha)K \le c \le (r - \nu/\alpha)K$ holds, then we have $g_*(i) = h_*(i) = K/\alpha$, and the function $U(s, i; K/\alpha)$ is explicitly given by the expression in (3.48), for $i = 0, 1$.

4 The case of partial information

In this section, we present a solution to the optimal stopping game in the associated model with *partial information*, when neither the writer nor the holder of the convertible bond have access to the dividend policy of the firm issuing the asset.

4.1 Regularity of the value function. In this subsection, we prove some regularity properties of the value function $V_*(s, \pi)$ of the optimal stopping game in (2.7) under the assumption $c < (\delta_0 - \nu/\alpha)K$ implying that $\tau_*(s, \pi) \le \zeta_*(s', \pi')$ ($\mathbb{P}$-a.s.) holds, for all (s, π) and (s', π') from the set $(0, K/\alpha] \times [0, 1]$. Note that the proof of the following properties for the value function $V_*(s, \pi)$ under the assumption $(r - \nu/\alpha)K < c < rK$ implying that $\zeta_*(s, \pi) \le \tau_*(s', \pi')$ ($\mathbb{P}$-a.s.) holds, for all (s, π) and (s', π') from $(0, K/\alpha] \times [0, 1]$, follow by means of arguments similar to the ones presented below.

(i) *The value function $V_*(s, \pi)$ is continuous at $(s, \pi) \in (0, K/\alpha] \times [0, 1]$.* To show this, it is enough to prove that

$$
\begin{aligned}
&s \mapsto V_*(s, \pi) \quad \text{is continuous at} \quad s' \quad \text{uniformly over} \quad \pi \in [\pi' - \varepsilon, \pi' + \varepsilon] &\text{(4.1)}\\
&\pi \mapsto V_*(s', \pi) \quad \text{is continuous at} \quad \pi' &\text{(4.2)}
\end{aligned}
$$

for every $(s', \pi') \in (0, K/\alpha] \times (0, 1)$ given and fixed with some $\varepsilon > 0$ small enough.

In order to derive the property of (4.1), let us fix some $s_1 \le s_2$ in $[s' - \varepsilon, s' + \varepsilon]$ and $\pi \in [\pi' - \varepsilon, \pi' + \varepsilon]$ such that the associated square belongs to $(0, K/\alpha] \times [0, 1]$. We consider $\tau_* = \tau_*(s_2, \pi)$ the optimal stopping time in (2.7) for the starting point (s_2, π) of the process (S, Π). Then, taking into account the explicit form of the process S in (2.1)-(2.3), we get

$$
\begin{aligned}
0 &\le V_*(s_2, \pi) - V_*(s_1, \pi) &\text{(4.3)}\\
&\le \mathbb{E}_{s_2,\pi}\bigg[\int_0^{\tau_*} e^{-ru}\,(c + \nu S_u)\,du + e^{-r\tau_*}\,\alpha S_{\tau_*}\bigg] - \mathbb{E}_{s_1,\pi}\bigg[\int_0^{\tau_*} e^{-ru}\,(c + \nu S_u)\,du + e^{-r\tau_*}\,\alpha S_{\tau_*}\bigg]\\
&= \mathbb{E}\bigg[\int_0^{\tau_*} e^{-ru}\,\nu\big(S_u^{(s_2,\pi)} - S_u^{(s_1,\pi)}\big)\,du + e^{-r\tau_*}\,\alpha\big(S_{\tau_*}^{(s_2,\pi)} - S_{\tau_*}^{(s_1,\pi)}\big)\bigg]\\
&= (s_2 - s_1)\,\mathbb{E}\bigg[\int_0^{\tau_*} e^{-ru}\,\nu S_u^{(1,\pi)}\,du + e^{-r\tau_*}\,\alpha S_{\tau_*}^{(1,\pi)}\bigg]
\end{aligned}
$$

where the last expectation is finite, for all s_1 and s_2 from $[s' - \varepsilon, s' + \varepsilon]$. Here we recall that $(S^{(s,\pi)}, \Pi^{(\pi)})$ indicates the dependence of the processes S and Π on the starting point $(s, \pi) \in (0, K/\alpha] \times [0, 1]$. Observe that the right-hand side in (4.3) converges monotonically to zero as s_1 approaches s_2, independently of $\pi \in [\pi' - \varepsilon, \pi' + \varepsilon]$ for any $\varepsilon > 0$ fixed, so that the property in (4.1) holds.

In order to derive the property of (4.2), let us fix $\pi_1 \le \pi_2$ in $[\pi' - \varepsilon, \pi' + \varepsilon]$ such that the associated interval belongs to $[0, 1]$. We now denote by $\tau_* = \tau_*(s', \pi_2)$ the optimal stopping time in (2.7) for the starting point (s', π_2) of the process (S, Π). Then, taking into account the explicit form of the process S in (2.1)-(2.3), we get

$$
\begin{aligned}
0 &\le V_*(s',\pi_2) - V_*(s',\pi_1) && (4.4)\\
&\le \mathbb{E}_{s',\pi_2}\bigg[\int_0^{\tau_*} e^{-ru}\,(c+\nu S_u)\,du + e^{-r\tau_*}\,\alpha S_{\tau_*}\bigg] - \mathbb{E}_{s',\pi_1}\bigg[\int_0^{\tau_*} e^{-ru}\,(c+\nu S_u)\,du + e^{-r\tau_*}\,\alpha S_{\tau_*}\bigg]\\
&= \mathbb{E}\bigg[\int_0^{\tau_*} e^{-ru}\,\nu\big(S_u^{(s',\pi_2)} - S_u^{(s',\pi_1)}\big)\,du + e^{-r\tau_*}\,\alpha\big(S_{\tau_*}^{(s',\pi_2)} - S_{\tau_*}^{(s',\pi_1)}\big)\bigg]
\end{aligned}
$$

where the last expectation is finite, for all π_1 and π_2 from $[\pi'-\varepsilon, \pi'+\varepsilon]$, and we have under the assumption $\delta_0 > \delta_1$, that

$$
\begin{aligned}
S_t^{(s',\pi_2)} - S_t^{(s',\pi_1)} &= S_t^{(s',\pi_1)}\left(\frac{S_t^{(s',\pi_2)}}{S_t^{(s',\pi_1)}} - 1\right) && (4.5)\\
&= S_t^{(s',\pi_1)}\left(\exp\left(\int_0^t (\delta_0-\delta_1)\,\big(\Pi_u^{(\pi_2)} - \Pi_u^{(\pi_1)}\big)\,du\right) - 1\right)
\end{aligned}
$$

for all $t \ge 0$. Here and in the rest of this subsection, we also indicate by $\Pi^{(\pi)}$ the dependence of the process Π on the starting point $\pi \in [0,1]$. Hence, by using the comparison results for strong solutions of stochastic differential equations and applying the Lebesgue dominated convergence theorem, we may conclude that the right-hand side in (4.4) converges to zero as π_1 approaches π_2, for any $\varepsilon > 0$ fixed, so that the property in (4.2) holds. Note that the Lebesgue dominated convergence theorem can be applied here, because of the fact that the inequality $S_t \le K/\alpha$ holds, for any $0 \le t \le \tau_*$.

(ii) *The value function $V_*(s,\pi)$ is continuously differentiable at the boundary ∂C_*.* We prove below that the partial derivatives $(V_*)_s(s,\pi)$ and $(V_*)_\pi(s,\pi)$ are continuous functions at ∂C_* under the assumption that the points of the optimal exercise boundary ∂C_* in (2.23) are regular for the stopping region D_* in (2.24) relative to the process (S,Π) defined in (2.3)-(2.5), in the sense that a sequence $(\tau_*(s_n,\pi_n))_{n\in\mathbb{N}}$ tends to zero ($\mathbb{P}$-a.s.), whenever the sequence $(s_n,\pi_n)_{n\in\mathbb{N}}$ from C_* tends to (s,π) from ∂C_* such that either $s = a_*(\pi) < K/\alpha$ or $s = b_*(\pi) < K/\alpha$ holds, for $\pi \in (0,1)$, respectively. Note that this assumption does not follow from the analysis of Subsection 2.3 above. The global C^1-regularity of the value functions of optimal stopping problems for multi-dimensional diffusions which are equivalent to either parabolic- or elliptic-type free-boundary problems was extensively studied in [15]. Thus, taking into account the fact that $V_*(s,\pi)$ is explicitly known in D_* from (2.10), let us first establish the properties

$$
(V_*)_s(s,\pi)\big|_{s=a_*(\pi)-} = 0 \ \text{ if } a_*(\pi) < K/\alpha, \quad \text{or} \quad (V_*)_s(s,\pi)\big|_{s=b_*(\pi)-} = \alpha \ \text{ if } b_*(\pi) < K/\alpha \tag{4.6}
$$

$$
(V_*)_\pi(s,\pi)\big|_{s=a_*(\pi)-} = 0 \ \text{ if } a_*(\pi) < K/\alpha, \quad \text{or} \quad (V_*)_\pi(s,\pi)\big|_{s=b_*(\pi)-} = 0 \ \text{ if } b_*(\pi) < K/\alpha \tag{4.7}
$$

for $\pi \in (0,1)$. For this purpose, we further consider the case of $(s,\pi) \in (0,K/\alpha)\times(0,1)$ fixed such that $s = b_*(\pi) < K/\alpha$ and $V_*(s,\pi) = \alpha s$ holds, while the case of $s = a_*(\pi) < K/\alpha$ and $V_*(s,\pi) = K$ can be dealt with using similar arguments. We will show the existence of other directional derivatives of $V_*(s,\pi)$ along the boundary ∂C_* in part (iii) of this subsection, by means of the arguments presented below combined with the ones applied in [37; Section 11].

Smooth fit in the variable s. In order to derive the property in the right-hand part of (4.6), we first observe directly from the structure of the continuation region in (2.9) and (2.23) that the inequality

$$
\limsup_{\varepsilon\downarrow 0} \frac{V_*(s-\varepsilon,\pi) - V_*(s,\pi)}{-\varepsilon} \le \alpha \tag{4.8}
$$

is satisfied, due to the fact that $V_*(s-\varepsilon,\pi) \ge \alpha(s-\varepsilon)$ holds. Let us now denote by $\tau_\varepsilon^1 = \tau_*(s-\varepsilon,\pi)$ the optimal stopping time in (2.7) for the starting point $(s-\varepsilon,\pi)$ of the process (S,Π), with some

$\varepsilon > 0$ small enough. Then, taking into account the explicit form of the process S in (2.1)-(2.3), we get

$$
\begin{aligned}
&V_*(s-\varepsilon,\pi) - V_*(s,\pi) &(4.9)\\
&\le \mathbb{E}_{s-\varepsilon,\pi}\bigg[\int_0^{\tau^1_\varepsilon} e^{-ru}\,(c+\nu S_u)\,du + e^{-r\tau^1_\varepsilon}\,\alpha S_{\tau^1_\varepsilon}\bigg] - \mathbb{E}_{s,\pi}\bigg[\int_0^{\tau^1_\varepsilon} e^{-ru}\,(c+\nu S_u)\,du + e^{-r\tau^1_\varepsilon}\,\alpha S_{\tau^1_\varepsilon}\bigg]\\
&= \mathbb{E}\bigg[\int_0^{\tau^1_\varepsilon} e^{-ru}\,\nu\big(S^{(s-\varepsilon,\pi)}_u - S^{(s,\pi)}_u\big)\,du + e^{-r\tau^1_\varepsilon}\,\alpha\big(S^{(s-\varepsilon,\pi)}_{\tau^1_\varepsilon} - S^{(s,\pi)}_{\tau^1_\varepsilon}\big)\bigg]\\
&= -\varepsilon\,\mathbb{E}\bigg[\int_0^{\tau^1_\varepsilon} e^{-ru}\,\nu S^{(1,\pi)}_u\,du + e^{-r\tau^1_\varepsilon}\,\alpha S^{(1,\pi)}_{\tau^1_\varepsilon}\bigg]
\end{aligned}
$$

where the last expectation is positive and finite, for any $\varepsilon > 0$ small enough. Hence, by using the fact that $\tau^1_\varepsilon \to 0$ ($\mathbb{P}$-a.s.) as $\varepsilon \downarrow 0$ due to the assumption of regularity of the boundary ∂C_* for the region D_* relative to (S, Π), and applying the Lebesgue dominated convergence theorem, we obtain that the inequality

$$
\liminf_{\varepsilon\downarrow 0} \frac{V_*(s-\varepsilon,\pi) - V_*(s,\pi)}{-\varepsilon} \ge \alpha \tag{4.10}
$$

holds. Thus, getting the inequalities in (4.8) and (4.10) together, we conclude that the smooth-fit conditions in the right-hand part of (4.6) are satisfied.

Smooth fit in the variable π. In order to derive the property in the right-hand part of (4.7), we first observe directly from the structure of the continuation region in (2.9) and (2.23) and the obvious fact that the value function $V_*(s,\pi)$ from (2.7) is increasing in π on $(0,1)$ under $\delta_0 > \delta_1$ that the inequality

$$
\liminf_{\varepsilon\downarrow 0} \frac{V_*(s,\pi+\varepsilon) - V_*(s,\pi)}{\varepsilon} \ge 0 \tag{4.11}
$$

holds. Let us finally denote by $\tau^2_\varepsilon = \tau_*(s,\pi+\varepsilon)$ the optimal stopping time in (2.7) for the starting point $(s,\pi+\varepsilon)$ of the process (S,Π), with some $\varepsilon > 0$ small enough. Then, taking into account the explicit form of the process S in (2.1)-(2.3) and applying Itô's formula to the process $\Pi/(1-\Pi)$, we get

$$
\begin{aligned}
&V_*(s,\pi+\varepsilon) - V_*(s,\pi) &(4.12)\\
&\le \mathbb{E}_{s,\pi+\varepsilon}\bigg[\int_0^{\tau^2_\varepsilon} e^{-ru}\,(c+\nu S_u)\,du + e^{-r\tau^2_\varepsilon}\,\alpha S_{\tau^2_\varepsilon}\bigg] - \mathbb{E}_{s,\pi}\bigg[\int_0^{\tau^2_\varepsilon} e^{-ru}\,(c+\nu S_u)\,du + e^{-r\tau^2_\varepsilon}\,\alpha S_{\tau^2_\varepsilon}\bigg]\\
&= \mathbb{E}\bigg[\int_0^{\tau^2_\varepsilon} e^{-ru}\,\nu\big(S^{(s,\pi+\varepsilon)}_u - S^{(s,\pi)}_u\big)\,du + e^{-r\tau^2_\varepsilon}\,\alpha\big(S^{(s,\pi+\varepsilon)}_{\tau^2_\varepsilon} - S^{(s,\pi)}_{\tau^2_\varepsilon}\big)\bigg]
\end{aligned}
$$

where the last expectation is positive and finite, for any $\pi \in (0,1)$ and $\varepsilon > 0$ small enough, under the assumption that $\delta_0 > \delta_1$, and we have

$$
\begin{aligned}
&S^{(s,\pi+\varepsilon)}_t - S^{(s,\pi)}_t = S^{(s,\pi)}_t \Bigg(\frac{S^{(s,\pi+\varepsilon)}_t}{S^{(s,\pi)}_t} - 1\Bigg) &(4.13)\\
&= S^{(s,\pi)}_t \bigg(\exp\bigg(\int_0^t (\delta_0-\delta_1)\,\big(\Pi^{(\pi+\varepsilon)}_u - \Pi^{(\pi)}_u\big)\,du\bigg) - 1\bigg)\\
&= S^{(s,\pi)}_t \Bigg(\exp\Bigg(\int_0^t (\delta_0-\delta_1)\,\Pi^{(\pi)}_u(1-\Pi^{(\pi+\varepsilon)}_u)\,\Bigg(\frac{\Pi^{(\pi+\varepsilon)}_u(1-\Pi^{(\pi)}_u)}{(1-\Pi^{(\pi+\varepsilon)}_u)\Pi^{(\pi)}_u} - 1\Bigg)\,du\Bigg) - 1\Bigg)
\end{aligned}
$$

and

$$\frac{\Pi_t^{(\pi+\varepsilon)}(1-\Pi_t^{(\pi)})}{(1-\Pi_t^{(\pi+\varepsilon)})\Pi_t^{(\pi)}} = \frac{(\pi+\varepsilon)(1-\pi)}{(1-\pi-\varepsilon)\pi} \tag{4.14}$$
$$\times \exp\bigg(\int_0^t \bigg(\frac{\lambda(1-2\Pi_u^{(\pi+\varepsilon)})}{\Pi_u^{(\pi+\varepsilon)}(1-\Pi_u^{(\pi+\varepsilon)})} - \frac{\lambda(1-2\Pi_u^{(\pi)})}{\Pi_u^{(\pi)}(1-\Pi_u^{(\pi)})} + \frac{(\delta_0-\delta_1)^2}{\sigma^2} \big(\Pi_u^{(\pi+\varepsilon)} - \Pi_u^{(\pi)}\big) \bigg)\, du \bigg)$$

for all $t \geq 0$. Hence, taking into account the sample-path properties of the process Π which solves the stochastic differential equation in (2.4)-(2.5), providing Taylor's expansions for the appropriate exponential functions, by using the fact that $\tau_\varepsilon^2 \to 0$ ($\mathbb{P}$-a.s.) as $\varepsilon \downarrow 0$ due to the assumption of regularity of the boundary ∂C_* for the region D_* relative to (S, Π), and applying the Lebesgue dominated convergence theorem, we obtain that the inequality

$$\limsup_{\varepsilon \downarrow 0} \frac{V_*(s, \pi+\varepsilon) - V_*(s,\pi)}{\varepsilon} \leq 0 \tag{4.15}$$

holds. Thus, getting the inequalities in (4.11) and (4.15) together, we conclude that the smooth-fit conditions in the right-hand part of (4.6) are satisfied.

(iii) *The derivative $(V_*)_s(s,\pi)$ is continuous at the boundary ∂C_*.* For this purpose, we need to show that the property

$$\lim_{n\to\infty} (V_*)_s(s_n, \pi_n) = \alpha \tag{4.16}$$

holds, for any sequence $(s_n, \pi_n)_{n\in\mathbb{N}}$ tending to (s,π) as $n \to \infty$ such that $s = b_*(\pi) < K/\alpha$. Since we have $V_*(s_n, \pi_n) = \alpha s_n$, for $(s_n, \pi_n) \in D_*$ under $c < (\delta_0 - \nu/\alpha)K$, and the conditions of (4.6) hold at $s = b_*(\pi) < K/\alpha$, there is no restriction to assume that $(s_n,\pi_n) \in C_*$, for every $n \in \mathbb{N}$. Let us first show that the inequality

$$\limsup_{n\to\infty} (V_*)_s(s_n,\pi_n) = \limsup_{n\to\infty} \lim_{\varepsilon\downarrow 0} \frac{V_*(s_n - \varepsilon, \pi_n) - V_*(s_n,\pi_n)}{-\varepsilon} \leq \alpha \tag{4.17}$$

holds. In this case, we observe from the first identity in (4.17) that one can choose subsequences $(s_{n_k}, \pi_{n_k})_{k\in\mathbb{N}}$ and $(\varepsilon_k)_{k\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} (V_*)_s(s_n,\pi_n) = \lim_{k\to\infty} \frac{V_*(s_{n_k} - \varepsilon_k, \pi_{n_k}) - V_*(s_{n_k}, \pi_{n_k})}{-\varepsilon_k} \tag{4.18}$$

with $(s_{n_k} - \varepsilon_k, \pi_{n_k})_{k\in\mathbb{N}}$ tending to (s,π) as $k \to \infty$. Let us consider $\tau_k^1 = \tau_*(s_{n_k}, \pi_{n_k})$ the optimal stopping time for the value function $V_*(s_{n_k}, \pi_{n_k})$, for every $k \in \mathbb{N}$. Then, taking into account the structure of the continuation region in (2.9) and (2.23) as well as explicit form of the process S in (2.1)-(2.3), we find that

$$V_*(s_{n_k} - \varepsilon_k, \pi_{n_k}) - V_*(s_{n_k}, \pi_{n_k}) \tag{4.19}$$
$$\geq \mathbb{E}_{s_{n_k}-\varepsilon_k, \pi_{n_k}} \bigg[\int_0^{\tau_k^1} e^{-ru}\, (c + \nu S_u)\, du + e^{-r\tau_k^1}\, \alpha S_{\tau_k^1} \bigg] - \mathbb{E}_{s_{n_k}, \pi_{n_k}} \bigg[\int_0^{\tau_k^1} e^{-ru}\, (c + \nu S_u)\, du + e^{-r\tau_k^1}\, \alpha S_{\tau_k^1} \bigg]$$
$$= \mathbb{E}\bigg[\int_0^{\tau_k^1} e^{-ru}\, \nu \big(S_u^{(s_{n_k}-\varepsilon_k, \pi_{n_k})} - S_u^{(s_{n_k}, \pi_{n_k})} \big)\, du + e^{-r\tau_k^1}\, \alpha \big(S_{\tau_k^1}^{(s_{n_k}-\varepsilon_k, \pi_{n_k})} - S_{\tau_k^1}^{(s_{n_k}, \pi_{n_k})} \big) \bigg]$$
$$= -\varepsilon_k\, \mathbb{E}\bigg[\int_0^{\tau_k^1} e^{-ru}\, \nu S_u^{(1,\pi_{n_k})}\, du + e^{-r\tau_k^1}\, \alpha S_{\tau_k^1}^{(1,\pi_{n_k})} \bigg]$$

where the last expectation is positive and finite, for every $k \in \mathbb{N}$. Hence, letting $k \to \infty$ and recalling the fact that $\tau_k^1 \to 0$ ($\mathbb{P}$-a.s.) as $k \to \infty$ due to the assumption of the regularity of the boundary ∂C_* for the region D_* relative to (S, Π), we see by the Lebesgue dominated convergence theorem that the expression in (4.19) combined with the one in (4.18) implies the desired one in (4.17). Thus, it remains to show that the inequality

$$\liminf_{n\to\infty} (V_*)_s(s_n, \pi_n) = \liminf_{n\to\infty} \lim_{\varepsilon \downarrow 0} \frac{V_*(s_n - \varepsilon, \pi_n) - V_*(s_n, \pi_n)}{-\varepsilon} \geq \alpha \tag{4.20}$$

holds too. In this case, we observe from the first identity in (4.20) that one can choose subsequences $(s_{n_k}, \pi_{n_k})_{k\in\mathbb{N}}$ and $(\varepsilon_k)_{k\in\mathbb{N}}$ such that

$$\liminf_{n\to\infty} (V_*)_s(s_n, \pi_n) = \lim_{k\to\infty} \frac{V_*(s_{n_k} - \varepsilon_k, \pi_{n_k}) - V_*(s_{n_k}, \pi_{n_k})}{-\varepsilon_k} \tag{4.21}$$

with $(s_{n_k} - \varepsilon_k, \pi_{n_k})_{k\in\mathbb{N}}$ tending to (s, π) as $k \to \infty$. Let us consider $\tau_k^2 = \tau_*(s_{n_k} - \varepsilon_k, \pi_{n_k})$ the optimal stopping time for the value function $V_*(s_{n_k} - \varepsilon_k, \pi_{n_k})$, for every $k \in \mathbb{N}$. Then, taking into account the explicit form of the process S in (2.1)-(2.3), we find in the same way as in (4.9) above that

$$V_*(s_{n_k} - \varepsilon_k, \pi_{n_k}) - V_*(s_{n_k}, \pi_{n_k}) \leq -\varepsilon_k \, \mathbb{E}\bigg[\int_0^{\tau_k^2} e^{-ru} \, \nu S_u^{(1,\pi_{n_k})} \, du + e^{-r\tau_k^2} \, \alpha S_{\tau_k^2}^{(1,\pi_{n_k})} \bigg] \tag{4.22}$$

where the last expectation is positive and finite, for every $k \in \mathbb{N}$. Hence, letting $k \to \infty$ and recalling the fact that $\tau_k^2 \to 0$ ($\mathbb{P}$-a.s.) as $k \to \infty$ due to the assumption of regularity of the boundary ∂C_* for the region D_* relative to (S, Π), we see by the Lebesgue dominated convergence theorem that the expression in (4.22) combined with the one in (4.21) implies the desired one in (4.20). Therefore, getting the inequalities in (4.17) and (4.20) together, we obtain the property of (4.16).

The derivative $(V_*)_\pi(s, \pi)$ is continuous at the boundary ∂C_*. For this purpose, we need to show that the property

$$\lim_{n\to\infty} (V_*)_\pi(s_n, \pi_n) = 0 \tag{4.23}$$

holds, for any sequence $(s_n, \pi_n)_{n\in\mathbb{N}}$ tending to (s, π) as $n \to \infty$ such that $s = b_*(\pi) < K/\alpha$. Since we have $V_*(s_n, \pi_n) = \alpha s_n$, for $(s_n, \pi_n) \in D_*$ under $c < (\delta_0 - \nu/\alpha)K$, and the conditions of (4.7) hold at $s = b_*(\pi) < K/\alpha$, we assume again that $(s_n, \pi_n) \in C_*$, for every $n \in \mathbb{N}$. Then, we conclude from the structure of the continuation region in (2.9) and (2.23) and the fact that the value function $V_*(s, \pi)$ from (2.7) is increasing in π on $(0, 1)$ under $\delta_0 > \delta_1$ that the inequality

$$\liminf_{n\to\infty} (V_*)_\pi(s_n, \pi_n) = \liminf_{n\to\infty} \lim_{\varepsilon \downarrow 0} \frac{V_*(s_n, \pi_n + \varepsilon) - V_*(s_n, \pi_n)}{\varepsilon} \geq 0 \tag{4.24}$$

holds. Thus, it remains to show that the inequality

$$\limsup_{n\to\infty} (V_*)_\pi(s_n, \pi_n) = \limsup_{n\to\infty} \lim_{\varepsilon \downarrow 0} \frac{V_*(s_n, \pi_n + \varepsilon) - V_*(s_n, \pi_n)}{\varepsilon} \leq 0 \tag{4.25}$$

holds too. In this case, we observe from the first identity in (4.25) that one can choose subsequences $(s_{n_k}, \pi_{n_k})_{k\in\mathbb{N}}$ and $(\varepsilon_k)_{k\in\mathbb{N}}$ such that

$$\limsup_{n\to\infty} (V_*)_\pi(s_n, \pi_n) = \lim_{k\to\infty} \frac{V_*(s_{n_k}, \pi_{n_k} + \varepsilon_k) - V_*(s_{n_k}, \pi_{n_k})}{\varepsilon_k} \tag{4.26}$$

with $(s_{n_k}, \pi_{n_k} + \varepsilon_k)_{k\in\mathbb{N}}$ tending to (s, π) as $k \to \infty$. Let us consider $\tau_k^3 = \tau_*(s_{n_k}, \pi_{n_k} + \varepsilon_k)$ the optimal stopping time for the value function $V_*(s_{n_k}, \pi_{n_k} + \varepsilon_k)$, for every $k \in \mathbb{N}$. Then, taking into account the explicit form of the process S in (2.1)-(2.3) and applying Itô's formula to the process $\Pi/(1-\Pi)$, we find in the same way as in (4.12) with (4.13)-(4.14) above that

$$
\begin{aligned}
&V_*(s_{n_k}, \pi_{n_k} + \varepsilon_k) - V_*(s_{n_k}, \pi_{n_k}) \qquad (4.27)\\
&\le \mathbb{E}\bigg[\int_0^{\tau_k^3} e^{-ru}\, \nu\big(S_u^{(s_{n_k}, \pi_{n_k}+\varepsilon_k)} - S_u^{(s_{n_k}, \pi_{n_k})}\big)\, du + e^{-r\tau_k^3}\, \alpha\big(S_{\tau_k^3}^{(s_{n_k}, \pi_{n_k}+\varepsilon_k)} - S_{\tau_k^3}^{(s_{n_k}, \pi_{n_k})}\big)\bigg]
\end{aligned}
$$

where the last expectation is positive and finite under $\delta_0 > \delta_1$, for every $k \in \mathbb{N}$. Hence, letting $k \to \infty$ and recalling the fact that $\tau_k^3 \to 0$ ($\mathbb{P}$-a.s.) as $k \to \infty$ due to the assumption of regularity of the boundary ∂C_* for the region D_* relative to (S, Π), by means of arguments similar to the ones used in the end of smooth fit analysis above, we see by the Lebesgue dominated convergence theorem that the expression in (4.27) combined with the one in (4.26) implies the desired one in (4.25). Therefore, getting the inequalities in (4.24) and (4.25) together, we obtain the property of (4.23).

4.2 The change of variables. In order to be able to apply the change-of-variable formula from [50] in the verification assertion below, we introduce an appropriate change of variables to reduce the infinitesimal operator of the process (S, Π) to the normal form. For this purpose, let us equivalently define the process $Q = (Q_t)_{t \ge 0}$ by

$$
Q_t = \frac{S_t^{-\eta} \Pi_t}{1 - \Pi_t} \quad \text{with} \quad \eta = \frac{\delta_0 - \delta_1}{\sigma^2} \quad \text{so that} \quad \Pi_t = \frac{S_t^{\eta} Q_t}{1 + S_t^{\eta} Q_t} \tag{4.28}
$$

for all $t \ge 0$. Then, by applying Itô's formula to the expressions in (4.28), we get from the representations in (2.3)-(2.4) that the process (S, Q) solves the system of stochastic differential equations

$$
dS_t = \bigg(r - \delta_0 - (\delta_1 - \delta_0)\, \frac{S_t^{\eta} Q_t}{1 + S_t^{\eta} Q_t} \bigg)\, S_t\, dt + \sigma\, S_t\, d\overline{B}_t \quad (S_0 = s) \tag{4.29}
$$

and

$$
dQ_t = \bigg(\frac{\lambda(1 - S_t^{2\eta} Q_t^2)}{S_t^{\eta} Q_t} - \xi \bigg)\, Q_t\, dt \quad \text{with} \quad \xi = \frac{\eta}{2}\, (2r - \delta_0 - \delta_1 - \sigma^2) \quad \bigg(Q_0 = q \equiv \frac{s^{-\eta}\pi}{1 - \pi} \bigg) \tag{4.30}
$$

for any $(s, q) \in (0, \infty)^2$ (see, e.g. [27], [30], or [37] for similar transformations of variables). It is seen from the form of the stochastic differential equation in (4.30) that the process Q started at $q > 0$ is of bounded variation. More precisely, if the inequality $S_t^{\eta} Q_t < (\sqrt{\xi^2 + 4\lambda^2} - \xi)/(2\lambda)$ holds, for $t \ge 0$, then the process Q is increasing, while if the inequality $S_t^{\eta} Q_t > (\sqrt{\xi^2 + 4\lambda^2} - \xi)/(2\lambda)$ holds, for $t \ge 0$, then the process Q is decreasing.

Note that, for any $(s, \pi) \in (0, K/\alpha] \times (0, 1)$ fixed, the value function of the optimal stopping problem in (2.7) takes the form $V_*(s, \pi) = \widetilde{V}_*(s, s^{-\eta}\pi/(1 - \pi))$ with

$$
\widetilde{V}_*(s, q) = \inf_{\zeta} \sup_{\tau} \mathbb{E}_{s,q}\big[Y_\tau\, I(\tau < \zeta) + Z_\zeta\, I(\zeta \le \tau) \big] = \sup_{\tau} \inf_{\zeta} \mathbb{E}_{s,q}\big[Y_\tau\, I(\tau < \zeta) + Z_\zeta\, I(\zeta \le \tau) \big] \tag{4.31}
$$

where $\mathbb{E}_{s,q}$ denotes the expectation taken under the assumption that the two-dimensional Markov process (S, Q) satisfying the stochastic differential equations in (4.29)-(4.30) starts at some $(s, q) \in (0, K/\alpha] \times (0, \infty)$. We observe from the relations in (4.28) that there exists a one-to-one correspondence between the processes (S, Π) and (S, Q), so that the infimum and supremum in (4.31) is equivalently taken over all stopping times ζ and τ with respect to the natural filtration of (S, Q) which coincides with $(\mathcal{F}_t)_{t \ge 0}$.

Taking into account the arguments around [37; Formula 11.20], we further assume that, for each $q > 0$ fixed, there exists a unique $\pi \in (0, 1)$ such that $s = a_*(\pi) = (\pi/(q(1 - \pi)))^{1/\eta} = \widetilde{a}(q)$ or

$s = b_*(\pi) = (\pi/(q(1-\pi)))^{1/\eta} = \widetilde{b}(q)$ holds. In this respect, by virtue of the fact proved in Lemma 2.1 that the boundaries $a_*(\pi)$ and $b_*(\pi)$ in the expressions of (2.22) and (2.23) are monotone, we actually assume that the arithmetic equations

$$a(q) = a_*\big(a^\eta(q)q/(1+a^\eta(q)q)\big) \quad \text{and} \quad b(q) = b_*\big(b^\eta(q)q/(1+b^\eta(q)q)\big) \tag{4.32}$$

admit unique solutions $\widetilde{a}(q)$ and $\widetilde{b}(q)$, for each $q>0$, respectively. This assumption inherently yields the property that the boundaries $\widetilde{a}(q)$ and $\widetilde{b}(q)$ in the expressions of (4.33) and (4.34) are monotone. It thus follows from the expressions in (2.22) that the stopping times ζ_* and τ_* providing the Nash equilibrium for the optimal stopping game in (4.31) are of the form

$$\zeta_* = \inf\big\{t \ge 0 \,\big|\, S_t \ge \widetilde{a}(Q_t)\big\} \quad \text{and} \quad \tau_* = \inf\big\{t \ge 0 \,\big|\, S_t \ge \widetilde{b}(Q_t)\big\} \tag{4.33}$$

so that the continuation region in (2.9) and (2.23) takes either of the forms

$$\widetilde{C} = \big\{(s,q) \in (0,K/\alpha)\times(0,\infty) \,\big|\, s < \widetilde{a}(q)\big\} \quad \text{or} \quad \widetilde{C} = \big\{(s,q) \in (0,K/\alpha)\times(0,\infty) \,\big|\, s < \widetilde{b}(q)\big\} \tag{4.34}$$

for some functions $0 < \widetilde{a}(q), \widetilde{b}(q) \le K/\alpha$ to be determined, respectively.

4.3 The free-boundary problem. By means of standard arguments based on the application of Itô's formula (see, e.g. [47; Chapter IV, Theorem 4.4]), it is shown that the infinitesimal operator $\mathbb{L}_{(S,Q)}$ of the process (S,Q) solving the stochastic differential equations in (4.29)-(4.30) acts on an arbitrary locally bounded function $\widetilde{F}(s,q)$ from the class $C^{2,1}$ according to the rule

$$\begin{aligned}(\mathbb{L}_{(S,Q)}\widetilde{F})(s,q) = \Bigg(\bigg(r - \delta_0 - (\delta_1-\delta_0)\,\frac{s^\eta q}{1+s^\eta q}\bigg)\, s\,\widetilde{F}_s + \frac{\sigma^2 s^2}{2}\,\widetilde{F}_{ss} \\ + \bigg(\frac{\lambda(1-s^{2\eta}q^2)}{s^\eta q} - \frac{\eta}{2}\,(2r-\delta_0-\delta_1-\sigma^2)\bigg)\, q\,\widetilde{F}_q\Bigg)(s,q)\end{aligned} \tag{4.35}$$

for all $(s,q) \in (0,K/\alpha)\times(0,\infty)$. In order to characterise the unknown value function $\widetilde{V}_*(s,q)$ from (4.31) and the unknown boundaries $\widetilde{a}(q)$ and $\widetilde{b}(q)$ from (4.33) we may use the results of general theory of optimal stopping problems for continuous time Markov processes (see, e.g. [54; Chapter IV, Section 8]) and formulate the following associated free-boundary problem

$$(\mathbb{L}_{(S,Q)}\widetilde{V} - r\widetilde{V})(s,q) = -(c+\nu s) \ \text{ for } \ 0 < s < \widetilde{a}(q) \le K/\alpha \quad \text{or} \quad 0 < s < \widetilde{b}(q) \le K/\alpha \tag{4.36}$$

$$\widetilde{V}(s,q)\big|_{s=\widetilde{a}(q)-} = K \ \text{ if } \ \widetilde{a}(q) \le K/\alpha, \quad \text{or} \quad \widetilde{V}(s,q)\big|_{s=\widetilde{b}(q)-} = \alpha\,\widetilde{b}(q) \ \text{ if } \ \widetilde{b}(q) \le K/\alpha \tag{4.37}$$

$$\widetilde{V}_s(s,q)\big|_{s=\widetilde{a}(q)-} = 0 \ \text{ if } \ \widetilde{a}(q) < K/\alpha, \quad \text{or} \quad \widetilde{V}_s(s,q)\big|_{s=\widetilde{b}(q)-} = \alpha \ \text{ if } \ \widetilde{b}(q) < K/\alpha \tag{4.38}$$

$$\widetilde{V}_q(s,q)\big|_{s=\widetilde{a}(q)-} = 0 \ \text{ if } \ \widetilde{a}(q) < K/\alpha, \quad \text{or} \quad \widetilde{V}_q(s,q)\big|_{s=\widetilde{b}(q)-} = 0 \ \text{ if } \ \widetilde{b}(q) < K/\alpha \tag{4.39}$$

$$\widetilde{V}(s,q)\big|_{s=0+} \ \text{ is finite} \tag{4.40}$$

$$\widetilde{V}(s,q) = K \vee (\alpha s) \ \text{ for } s > \widetilde{a}(q) \ \text{ if } \widetilde{a}(q) \le K/\alpha, \ \text{ or } \ \widetilde{V}(s,q) = \alpha s \ \text{ for } s > \widetilde{b}(q) \ \text{ if } \widetilde{b}(q) \le K/\alpha \tag{4.41}$$

$$\alpha s < \widetilde{V}(s,q) < K \ \text{ for } \ 0 < s < \widetilde{a}(q) \le K/\alpha \quad \text{or} \quad 0 < s < \widetilde{b}(q) \le K/\alpha \tag{4.42}$$

$$(\mathbb{L}_{(S,Q)}\widetilde{V} - r\widetilde{V})(s,q) > -(c+\nu s) \ \text{ for } \ \widetilde{a}(q) < s < K/\alpha \ \text{ if } \ \widetilde{a}(q) \le K/\alpha, \quad \text{or} \tag{4.43}$$

$$(\mathbb{L}_{(S,Q)}\widetilde{V} - r\widetilde{V})(s,q) < -(c+\nu s) \ \text{ for } \ \widetilde{b}(q) < s < K/\alpha \ \text{ if } \ \widetilde{b}(q) \le K/\alpha \tag{4.44}$$

for all $q>0$. Note that the inequalities in (4.43) and (4.44) follow directly from the assertion of Lemma 2.1 proved in Subsection 2.3 above.

We recall that the existence of the value of the optimal stopping game in (4.31) follows from the results of [11; Theorem 4.1] and the one-to-one correspondence between the processes (S,Π)

and (S, Q). Then, by virtue of the strong Markov property of the process (S, Q), it is shown that the value function $\widetilde{V}_*(s, q)$ from (4.31) solves the parabolic-type partial differential equation in (4.35)+(4.36). Hence, taking into account the assumption of regularity of the points of the optimal exercise boundary ∂C_* for the stopping region D_* relative to the process (S, Π), we may conclude from the results on parabolic-type partial differential equations (see, e.g. [46; Chapter V]) combined with standard applications of Itô's formula and Doob's optional sampling theorem (see, e.g. [54; Chapter III, Section 7]) that the value function $\widetilde{V}_*(s, q)$ belongs to the class $C^{2,1}$ in the regions $\widetilde{C} \cap (((0, K/\alpha) \times (0, \infty)) \setminus E)$ and to the class $C^{2,0}$ in $\widetilde{C} \cap E$, where we set $E = \{(s, q) \in (0, K/\alpha) \times (0, \infty) \,|\, s^\eta q = (\sqrt{\xi^2 + 4\lambda^2} - \xi)/(2\lambda)\}$, and the regions $\widetilde{C}$ take the form of (4.34). Moreover, by virtue of the regularity of the value function proved in Subsection 2.4 and the bijective and smooth change of variables introduced in Subsection 2.5, it follows that the instantaneous-stopping and smooth-fit conditions of (4.37) and (4.38)-(4.39) hold for the value function $\widetilde{V}_*(s, q)$.

It particularly follows from the results of Lemma 2.1 and the structure of the free-boundary problem that we can find an explicit solution to the system in (4.35)+(4.36)-(4.44) under certain relations between the parameters of the model. More precisely, if the inequalities $(\delta_0 - \nu/\alpha)K \le c \le (r - \nu/\alpha)K$ hold, then the results of Lemma 2.1 yield that $\widetilde{a}(q) \equiv \widetilde{b}(q) \equiv K/\alpha$, so that the system in (4.35)+(4.36)-(4.44) becomes a fixed-boundary problem. It is seen that the solution of the partial differential equation in (4.35)+(4.36) satisfying the conditions of (4.37) admits the representation

$$\widetilde{V}(s, q; K/\alpha) = U(s, 0; K/\alpha) \, \frac{1}{1 + s^\eta q} + U(s, 1; K/\alpha) \, \frac{s^\eta q}{1 + s^\eta q} \tag{4.45}$$

where the functions $U(s, i; K/\alpha)$, for $i = 0, 1$, given by the expressions in (3.48)-(3.49) are solutions of the ordinary differential equations in (3.3)+(3.4) satisfying the conditions of (3.5) and (3.8) from the *full information* case considered in the next section.

4.4 The main result. In order to formulate and prove the main results of this section, taking into account the structure of the partial differential equation in (4.35)-(4.36) as well as the instantaneous-stopping and smooth-fit conditions in (4.37) and (4.38)-(4.39), we observe that the second-order derivative $(\widetilde{V}_*)_{ss}(s, q)$ admits a continuous extension to the closure of the appropriate continuation region $\widetilde{C}$ from (4.34). This fact means that, by virtue of the assumption of regularity of the boundary ∂C_* for D_* relative to (S, Π), the function $\widetilde{V}_*(s, q)$ admits a natural extension in the class $C^{2,1}$ in the closure of the region $\widetilde{C} \cap (((0, K/\alpha) \times (0, \infty)) \setminus E)$ and in the class $C^{2,0}$ in the closure of the appropriate region $\widetilde{C} \cap E$ with $E = \{(s, q) \in (0, K/\alpha) \times (0, \infty) \,|\, s^\eta q = (\sqrt{\xi^2 + 4\lambda^2} - \xi)/(2\lambda)\}$. Moreover, by virtue of the results of [53; Theorem 3], it follows from the regularity of the value function $\widetilde{V}_*(s, q)$, along with the assumption that the arithmetic equations in (4.32) admit unique solutions, that the boundaries $\widetilde{a}(q)$ and $\widetilde{b}(q)$ in (4.34) are continuous (monotone) functions. Recall the property that the process Q is monotone off the curve $E = \{(s, q) \in (0, \infty)^2 \,|\, s^\eta q = (\sqrt{\xi^2 + 4\lambda^2} - \xi)/(2\lambda)\}$. In this case, it can be shown by means of arguments similar to the ones applied in part 1 of the proof of [37; Theorem 19] that there exists a sequence of piecewise-monotone processes $Q^n = (Q^n_t)_{t \ge 0}$, $n \in \mathbb{N}$, which converges to Q in $\mathbb{P}$-probability on compact time intervals from $[0, \infty)$, and the sequence of total variations of Q^n, $n \in \mathbb{N}$, also converges in $\mathbb{P}$-probability to the one of Q, on each interval $[0, T]$, for any $T > 0$ fixed, as $n \to \infty$. Note that, without loss of generality, the processes Q^n, $n \in \mathbb{N}$, can be assumed to be continuous, by virtue of possible applications of standard straight-line approximations. Since each of the resulting continuous processes $\widetilde{a}(Q^n)$ and $\widetilde{b}(Q^n)$ are of bounded variation, thus continuous semimartingales, the change-of-variable formula from [50; Theorem 3.1] can be applied to the process $e^{-rt}\widetilde{V}(S, Q^n)$, for every $n \in \mathbb{N}$, and thus, to the process $e^{-rt}\widetilde{V}(S, Q)$, by virtue of the appropriate convergence relations mentioned above and the regularity of the candidate value function $\widetilde{V}(s, q)$, which is defined by means of the right-hand side of the expression in (4.46) below (see part 1 of the proof of [37; Theorem 19] for further arguments).

We continue with the following *verification* assertion related to the free-boundary problem in (4.35)+(4.36)-(4.44).

Lemma 4.1 *Let the processes S and Q solve the stochastic differential equations in (4.29)-(4.30). Suppose that the inequalities $0 < \delta_1 < \delta_0 < r$ and $0 < \nu < \alpha\delta_1 < \alpha\delta_0$ as well as $c < rK$ hold, and the boundary ∂C_* is regular for the region D_* relative to the process (S, Π) defined in (2.3)-(2.5). Assume that the equations in (4.32) admit unique solutions $\widetilde{a}(q)$ and $\widetilde{b}(q)$, for each $q > 0$ fixed. Then, the value function $\widetilde{V}_*(s, q)$ of the optimal stopping game in (4.31) admits the representation*

$$\widetilde{V}_*(s,q) = \begin{cases} \widetilde{V}(s,q;\widetilde{a}(q)), & \text{if } 0 < s < \widetilde{a}(q) \le K/\alpha \\ \widetilde{V}(s,q;\widetilde{b}(q)), & \text{if } 0 < s < \widetilde{b}(q) \le K/\alpha \\ K \vee (\alpha s), & \text{if } s \ge \widetilde{a}(q) \text{ and } \widetilde{a}(q) \le K/\alpha \\ \alpha s, & \text{if } s \ge \widetilde{b}(q) \text{ and } \widetilde{b}(q) \le K/\alpha \end{cases} \tag{4.46}$$

and the stopping times ζ_* and τ_* from (4.33) form a Nash equilibrium, where the functions $\widetilde{V}(s,q;\widetilde{a}(q))$ and $\widetilde{V}(s,q;\widetilde{b}(q))$ as well as the continuous boundaries $\widetilde{a}(q)$ and $\widetilde{b}(q)$ of bounded variation are specified as follows:

(i) If $(r - \nu/\alpha)K < c < rK$ holds, then $\widetilde{V}(s,q;\widetilde{a}(q))$ and $\widetilde{a}(q) \le K/\alpha$ are determined as the unique solution to the left-hand system of (4.35)+(4.36)-(4.42)+(4.43), whenever it exists, and otherwise, by the expression in (4.45) with $\widetilde{a}(q) \equiv K/\alpha$.

(ii) If $c < (\delta_0 - \nu/\alpha)K$ holds, then $\widetilde{V}(s,q;\widetilde{b}(q))$ and $\widetilde{b}(q) \le K/\alpha$ are determined as the unique solution to the right-hand system of (4.35)+(4.36)-(4.42)+(4.44), whenever it exists, and otherwise, by the expression in (4.45) with $\widetilde{b}(q) \equiv K/\alpha$.

(iii) If $(\delta_0 - \nu/\alpha)K \le c \le (r - \nu/\alpha)K$ holds, then we have $\widetilde{a}(q) \equiv \widetilde{b}(q) \equiv K/\alpha$, and $\widetilde{V}(s,q;K/\alpha)$ is explicitly given by the expression in (4.45).

Proof. (Part I). Recall from the line of arguments before the formulation of the theorem that the functions $\widetilde{V}(s,q;\widetilde{a}(q))$ and $\widetilde{V}(s,q;\widetilde{b}(q))$ belong to the class $C^{1,1}$ at the appropriate boundaries $\partial\widetilde{C} \setminus E$ and to $C^{1,0}$ at $\partial\widetilde{C} \cap E$, and thus, to the class $C^{2,1}$ in the closures of the appropriate regions $\widetilde{C} \cap (((0, K/\alpha) \times (0,\infty)) \setminus E)$ and to $C^{2,0}$ in the closures of $\widetilde{C} \cap E$, where we have $E = \{(s,q) \in (0,K/\alpha) \times (0,\infty) \,|\, s^\eta q = (\sqrt{\xi^2 + 4\lambda^2} - \xi)/(2\lambda)\}$ and the regions $\widetilde{C}$ are given by (4.34). It is also assumed that these functions solve the partial differential equation in (4.35)+(4.36) as well as satisfy the conditions of (4.37)-(4.40) at the continuous boundaries $\widetilde{a}(q)$ and $\widetilde{b}(q)$. Hence, we can apply the change-of-variable formula with local time on surfaces from [50; Theorem 3.1] (see also [54; Chapter II, Section 3.5] for a summary of the related results and further references) to obtain

$$\begin{aligned} &\int_0^t e^{-ru}\,(c + \nu S_u)\,du + e^{-rt}\,\widetilde{V}(S_t, Q_t) = \widetilde{V}(s,q) + \widetilde{N}_t + \widetilde{L}_t \\ &+ \int_0^t e^{-ru}\,(\mathbb{L}_{(S,Q)}\widetilde{V} - r\widetilde{V} + c + \nu S_u)(S_u, Q_u)\, I\big(S_u \neq \widetilde{a}(Q_u) \vee (K/\alpha), S_u \neq \widetilde{b}(Q_u) \vee (K/\alpha)\big)\,du \end{aligned} \tag{4.47}$$

where the process $\widetilde{N} = (\widetilde{N}_t)_{t \ge 0}$ defined by:

$$\widetilde{N}_t = \int_0^t e^{-ru}\,\widetilde{V}_s(S_u, Q_u)\, I\big(S_u \neq \widetilde{a}(Q_u) \vee (K/\alpha), S_u \neq \widetilde{b}(Q_u) \vee (K/\alpha)\big)\,\sigma S_u\, d\overline{B}_u \tag{4.48}$$

is a local martingale with respect to the probability measure $\mathbb{P}_{s,q}$. Here, the process $\widetilde{L} = (\widetilde{L}_t)_{t \ge 0}$ is defined by

$$\widetilde{L}_t = \frac{1}{2} \int_0^t e^{-ru} \, \Delta_s \widetilde{V}(\widetilde{a}(Q_u) \vee (K/\alpha), Q_u) \, I\big(S_u = \widetilde{a}(Q_u) \vee (K/\alpha)\big) \, d\ell_u^{\widetilde{a} \vee (K/\alpha)}(S) \tag{4.49}$$
$$+ \frac{1}{2} \int_0^t e^{-ru} \, \Delta_s \widetilde{V}(\widetilde{b}(Q_u) \vee (K/\alpha), Q_u) \, I\big(S_u = \widetilde{b}(Q_u) \vee (K/\alpha)\big) \, d\ell_u^{\widetilde{b} \vee (K/\alpha)}(S)$$

where $\Delta_s \widetilde{V}(\widetilde{a}(q) \vee (K/\alpha), q) = \widetilde{V}_s(\widetilde{a}(q) \vee (K/\alpha)+, q) - \widetilde{V}_s(\widetilde{a}(q) \vee (K/\alpha)-, q)$, $\Delta_s \widetilde{V}(\widetilde{b}(q) \vee (K/\alpha), q) = \widetilde{V}_s(\widetilde{b}(q) \vee (K/\alpha)+, q) - \widetilde{V}_s(\widetilde{b}(q) \vee (K/\alpha)-, q)$, and the processes $\ell^{\widetilde{a} \vee (K/\alpha)}(S) = (\ell_t^{\widetilde{a} \vee (K/\alpha)}(S))_{t \geq 0}$, $\ell^{\widetilde{b} \vee (K/\alpha)}(S) = (\ell_t^{\widetilde{b} \vee (K/\alpha)}(S))_{t \geq 0}$ defined by

$$\ell_t^{\widetilde{a} \vee (K/\alpha)}(S) = \mathbb{P}_{s,q} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I\big(-\varepsilon < S_u - \widetilde{a}(Q_u) \vee (K/\alpha) < \varepsilon\big) \, \sigma^2 S_u^2 \, du \tag{4.50}$$

and

$$\ell_t^{\widetilde{b} \vee (K/\alpha)}(S) = \mathbb{P}_{s,q} - \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t I\big(-\varepsilon < S_u - \widetilde{b}(Q_u) \vee (K/\alpha) < \varepsilon\big) \, \sigma^2 S_u^2 \, du \tag{4.51}$$

are the local times of S at the surfaces $\widetilde{a}(q) \vee (K/\alpha)$ and $\widetilde{b}(q) \vee (K/\alpha)$, at which $\widetilde{V}_s(s,q)$ may not exist. It follows from the structure of the lower and upper processes Y and Z in (2.6) and the stopping times ζ_* and τ_* in (4.33), that either the inequality $\Delta_s \widetilde{V}(\widetilde{a}(q), q) \leq 0$ or $\Delta_s \widetilde{V}(\widetilde{b}(q), q) \geq 0$ should hold, for all $q > 0$, when either the case $\widetilde{a}(q) \leq K/\alpha$ or $\widetilde{b}(q) \leq K/\alpha$ is realised, respectively. This feature particularly implies that the continuous process $\widetilde{L}$ defined in (4.49) is monotone. We may therefore conclude that $\widetilde{L}_{(\zeta_* \vee \tau_*) \wedge t} = 0$ can hold, for all $t \geq 0$, if and only if the smooth-fit conditions of (4.38) are satisfied.

It follows from the properties in (2.25) and (2.26) that the inequalities in (4.43)-(4.44) hold for the gain functions K and αs, respectively. Thus, the inequality $(\mathbb{L}_{(S,Q)} \widetilde{V} - r\widetilde{V})(s,q) \geq -(c + \nu s)$ holds, for any $\widetilde{a}(q) < s < K/\alpha$, if $\widetilde{a}(q) \leq K/\alpha$, or the inequality $(\mathbb{L}_{(S,Q)} \widetilde{V} - r\widetilde{V})(s,q) \leq -(c + \nu s)$ holds, for any $\widetilde{b}(q) < s < K/\alpha$, if $\widetilde{b}(q) \leq K/\alpha$, for all $q > 0$. Moreover, by virtue of the assumptions of (4.41) and (4.42), we see that either the inequality $\widetilde{V}(s,q) \leq K$ or $\widetilde{V}(s,q) \geq \alpha s$ holds, for all $(s,q) \in (0, K/\alpha] \times (0, \infty)$, if either the case $\widetilde{a}(q) \leq K/\alpha$ or $\widetilde{b}(q) \leq K/\alpha$ is realised, respectively. Since the time spent by S at the surfaces $\widetilde{a}(q)$ and $\widetilde{b}(q)$ is of Lebesgue measure zero, the indicators which appear in the integrals in the second line of (4.47) and in (4.48) can be ignored. Let $(\tau_n)_{n \in \mathbb{N}}$ be a localising sequence for the process $\widetilde{N}$ such that $\tau_n = \inf\{t \geq 0 \,|\, |\widetilde{N}_t| \geq n\}$, for each $n \in \mathbb{N}$. Then, the expression in (4.47) yields that the inequalities

$$Z_{\zeta \wedge \tau_* \wedge \tau_n} - \widetilde{L}_{\zeta \wedge \tau_* \wedge \tau_n} \tag{4.52}$$
$$\geq \int_0^{\zeta \wedge \tau_* \wedge \tau_n} e^{-ru} \, (c + \nu S_u) \, du + e^{-r(\zeta \wedge \tau_* \wedge \tau_n)} \, \widetilde{V}(S_{\zeta \wedge \tau_* \wedge \tau_n}, Q_{\zeta \wedge \tau_* \wedge \tau_n}) - \widetilde{L}_{\zeta \wedge \tau_* \wedge \tau_n} \geq \widetilde{V}(s,q) + \widetilde{N}_{\zeta \wedge \tau_* \wedge \tau_n}$$

and

$$Y_{\zeta_* \wedge \tau \wedge \tau_n} - \widetilde{L}_{\zeta_* \wedge \tau \wedge \tau_n} \tag{4.53}$$
$$\leq \int_0^{\zeta_* \wedge \tau \wedge \tau_n} e^{-ru} \, (c + \nu S_u) \, du + e^{-r(\zeta_* \wedge \tau \wedge \tau_n)} \, \widetilde{V}(S_{\zeta_* \wedge \tau \wedge \tau_n}, Q_{\zeta_* \wedge \tau \wedge \tau_n}) - \widetilde{L}_{\zeta_* \wedge \tau \wedge \tau_n} \leq \widetilde{V}(s,q) + \widetilde{N}_{\zeta_* \wedge \tau \wedge \tau_n}$$

hold, for any stopping times ζ and τ of the process (S,Q) started at $(s,q) \in (0, K/\alpha] \times (0, \infty)$. Taking the expectations with respect to the probability measure $\mathbb{P}_{s,q}$ in (4.52) and (4.53), by means of Doob's optional sampling theorem, we get that the inequalities

$$\begin{aligned}
&\mathbb{E}_{s,q}\big[Y_{\tau_*}\,I(\tau_*<\zeta\wedge\tau_n)+Z_{\zeta\wedge\tau_n}\,I(\zeta\wedge\tau_n\le\tau_*)-\widetilde{L}_{\zeta\wedge\tau_*\wedge\tau_n}\big] \\
&\ge \mathbb{E}_{s,q}\bigg[\int_0^{\zeta\wedge\tau_*\wedge\tau_n} e^{-ru}\,(c+\nu S_u)\,du+e^{-r(\zeta\wedge\tau_*\wedge\tau_n)}\,\widetilde{V}(S_{\zeta\wedge\tau_*\wedge\tau_n},Q_{\zeta\wedge\tau_*\wedge\tau_n})-\widetilde{L}_{\zeta\wedge\tau_*\wedge\tau_n}\bigg] \\
&\ge \widetilde{V}(s,q)+\mathbb{E}_{s,q}\big[\widetilde{N}_{\zeta\wedge\tau_*\wedge\tau_n}\big]=\widetilde{V}(s,q)
\end{aligned} \tag{4.54}$$

and

$$\begin{aligned}
&\mathbb{E}_{s,q}\big[Y_{\tau\wedge\tau_n}\,I(\tau\wedge\tau_n<\zeta_*)+Z_{\zeta_*}\,I(\zeta_*\le\tau\wedge\tau_n)-\widetilde{L}_{\zeta_*\wedge\tau\wedge\tau_n}\big] \\
&\le \mathbb{E}_{s,q}\bigg[\int_0^{\zeta_*\wedge\tau\wedge\tau_n} e^{-ru}\,(c+\nu S_u)\,du+e^{-r(\zeta_*\wedge\tau\wedge\tau_n)}\,V(S_{\zeta_*\wedge\tau\wedge\tau_n},Q_{\zeta_*\wedge\tau\wedge\tau_n})-\widetilde{L}_{\zeta_*\wedge\tau\wedge t}\bigg] \\
&\le \widetilde{V}(s,q)+\mathbb{E}_{s,q}\big[\widetilde{N}_{\zeta_*\wedge\tau\wedge\tau_n}\big]=\widetilde{V}(s,q)
\end{aligned} \tag{4.55}$$

hold, for all $(s,q)\in(0,K/\alpha]\times(0,\infty)$ and each $n\in\mathbb{N}$. According to the structure of the lower and upper processes Y and Z in (2.6) and the stopping times in (2.8), it is obvious that the property

$$\mathbb{E}_{s,q}\Big[\sup_{t\ge0} Y_{(\zeta_*\vee\tau_*)\wedge t}\Big]\le\mathbb{E}_{s,q}\Big[\sup_{t\ge0} Z_{(\zeta_*\vee\tau_*)\wedge t}\Big]<\infty \tag{4.56}$$

holds, for all $(s,q)\in(0,K/\alpha]\times(0,\infty)$, and the variables $Y_{\zeta_*\vee\tau_*}$ and $Z_{\zeta_*\vee\tau_*}$ are bounded on the set $\{\zeta_*\vee\tau_*=\infty\}$. Hence, letting n go to infinity and using Fatou's lemma, we obtain that the inequalities

$$\begin{aligned}
&\mathbb{E}_{s,q}\big[Y_{\tau}\,I(\tau<\zeta_*)+Z_{\zeta_*}\,I(\zeta_*\le\tau)-\widetilde{L}_{\zeta_*\wedge\tau}\big] \\
&\le\widetilde{V}(s,q)\le\mathbb{E}_{s,q}\big[Y_{\tau_*}\,I(\tau_*<\zeta)+Z_{\zeta}\,I(\zeta\le\tau_*)-\widetilde{L}_{\zeta\wedge\tau_*}\big]
\end{aligned} \tag{4.57}$$

are satisfied, for any stopping times ζ and τ and all $(s,q)\in(0,K/\alpha]\times(0,\infty)$. Thus, taking into account the fact that the function $\widetilde{V}(s,q)$ and the continuous boundaries $\widetilde{a}(q)$ and $\widetilde{b}(q)$ solve the partial differential equation in (4.36) and satisfy the conditions of (4.37)-(4.41), inserting ζ_* in place of ζ and τ_* in place of τ into the expression of (4.57), we obtain that the equality

$$\mathbb{E}_{s,q}\big[Y_{\tau_*}\,I(\tau_*<\zeta_*)+Z_{\zeta_*}\,I(\zeta_*\le\tau_*)\big]=\widetilde{V}(s,q) \tag{4.58}$$

holds, for all $(s,q)\in(0,K/\alpha]\times(0,\infty)$, since $\widetilde{L}_{\zeta_*\wedge\tau_*}=0$ holds for the stopping times $\zeta_*\wedge\tau_*$ in (4.33). We may therefore conclude that the candidate function $\widetilde{V}(s,q)$ coincides with the value function $\widetilde{V}_*(s,q)$ of the optimal stopping game in (4.31), and the optimal stopping times ζ_* and τ_* form a Nash equilibrium of the game.

(Part II). In order to prove the uniqueness of the value function $\widetilde{V}(s,q)$ and boundary $\widetilde{a}(q)$ or $\widetilde{b}(q)$ as solutions to the free-boundary problem in (4.36)-(4.42) with the smooth-fit conditions of (4.38), let us assume that there exist other continuous boundaries of bounded variation $\widehat{a}(q)$ and $\widehat{b}(q)$ such that $0<\widehat{a}(q),\widehat{b}(q)\le K/\alpha$ holds, for all $q>0$, and the inequalities in (4.43)-(4.44) are satisfied, respectively. Then, define the function $\widehat{V}(s,q)$ as in (4.46) with $\widehat{V}(s,q;\widehat{a}(q))$ and $\widehat{V}(s,q;\widehat{b}(q))$ and the stopping times $\widehat{\zeta}$ and $\widehat{\tau}$ as in (4.33) at $\widehat{a}(q)$ or $\widehat{b}(q)$ instead of $\widetilde{a}(q)$ or $\widetilde{b}(q)$, respectively. Following the arguments from the previous part of the proof and using the facts that the function $\widehat{V}(s,q)$ belongs to the class $C^{2,1}$ in the closures of the appropriate regions $\widehat{C}\cap(((0,K/\alpha)\times(0,\infty))\setminus E)$ and to $C^{2,0}$ in the closures of $\widehat{C}\cap E$, where we have $E=\{(s,q)\in(0,K/\alpha)\times(0,\infty)\,|\,s^\eta q=(\sqrt{\xi^2+4\lambda^2}-\xi)/(2\lambda)\}$ and the regions $\widehat{C}$ are defined as in (4.34) with $\widehat{a}(q)$ or $\widehat{b}(q)$ instead of $\widetilde{a}(q)$ or $\widetilde{b}(q)$, and solves the partial differential equation in (4.35)+(4.36) as well as satisfies the conditions of (4.37)-(4.41) at $\widehat{a}(q)$

or $\widehat{b}(q)$ instead of $\widetilde{a}(q)$ or $\widetilde{b}(q)$ by construction, we apply the change-of-variable formula from [50] to get

$$\int_0^t e^{-ru}\,(c+\nu S_u)\,du + e^{-rt}\,\widehat{V}(S_t,Q_t) = \widehat{V}(s,q) + \widehat{N}_t + \widehat{L}_t \tag{4.59}$$
$$+ \int_0^t e^{-ru}\,(\mathbb{L}_{(S,Q)}\widehat{V} - r\widehat{V} + c + \nu S_u)(S_u,Q_u)\, I\big(S_u > \widehat{a}(Q_u)\vee(K/\alpha) \text{ or } S_u > \widehat{b}(Q_u)\vee(K/\alpha)\big)\,du$$

where the process $\widehat{N}=(\widehat{N}_t)_{t\geq 0}$ defined as in (4.48) with $\widehat{V}(s,q)$ instead of $\widetilde{V}(s,q)$ is a local martingale with respect to the probability measure $\mathbb{P}_{s,q}$, and $\widehat{L}=(\widehat{L}_t)_{t\geq 0}$ is defined as in (4.49), with $\widehat{V}(s,q)$ instead of $\widetilde{V}(s,q)$ and $\widehat{a}(q)$ or $\widehat{b}(q)$ instead of $\widetilde{a}(q)$ or $\widetilde{b}(q)$, respectively. Let $(\zeta_n)_{n\in\mathbb{N}}$ be a localising sequence for the process $\widehat{N}$ such that $\zeta_n = \inf\{t\geq 0 \,|\, |\widehat{N}_t| \geq n\}$, for each $n\in\mathbb{N}$. Thus, inserting $\zeta\wedge\widehat{\tau}\wedge\zeta_n$ and $\widehat{\zeta}\wedge\tau\wedge\zeta_n$ instead of t into (4.59) and applying arguments similar to the ones used for the derivations of the formulas in (4.52)-(4.58), we conclude that the equality

$$\mathbb{E}_{s,q}\big[Y_{\widehat{\tau}}\, I(\widehat{\tau}<\widehat{\zeta}) + Z_{\widehat{\zeta}}\, I(\widehat{\zeta}\leq\widehat{\tau})\big] = \widehat{V}(s,q) \tag{4.60}$$

is satisfied. Therefore, recalling the fact that ζ_* and τ_* are the optimal stopping times in (2.7), and comparing the expressions in (4.58) and (4.60), we see that either the inequality $\widehat{V}(s,q)\geq\widetilde{V}(s,q)$ or $\widehat{V}(s,q)\leq\widetilde{V}(s,q)$ should hold, for all $(s,q)\in(0,K/\alpha]\times(0,\infty)$, if either the case $\widehat{a}(q)\leq K/\alpha$ or $\widehat{b}(q)\leq K/\alpha$ is realised, respectively.

In order to prove the fact that $\widehat{a}(q)\leq\widetilde{a}(q)$ and $\widehat{b}(q)\leq\widetilde{b}(q)$ holds, let us take a point $\widetilde{a}(q)< s < K/\alpha$ or $\widetilde{b}(q) < s < K/\alpha$, for which we have $\widehat{V}(s,q)=\widetilde{V}(s,q)=K$ when $\widetilde{a}(q)\leq K/\alpha$, or $\widehat{V}(s,q)=\widetilde{V}(s,q)=\alpha s$ when $\widetilde{b}(q)\leq K/\alpha$, respectively. For this purpose, we consider the stopping time

$$\upsilon_* = \inf\{t\geq 0 \mid S_t\leq\widetilde{a}(Q_t) \text{ or } S_t\leq\widetilde{b}(Q_t)\}. \tag{4.61}$$

Then, inserting $(\zeta_*\vee\tau_*)\wedge\upsilon_*\wedge\zeta_n$ and $(\zeta_*\vee\tau_*)\wedge\upsilon_*\wedge\zeta_n$ into (4.47) and (4.59) in place of t, respectively, and using the fact that the variable $Y_{\upsilon_*} I(\upsilon_*<\zeta_*) + Z_{\upsilon_*} I(\upsilon_*\leq\tau_*)$ is bounded on the event $\{\upsilon_*=\infty\}$ ($\mathbb{P}_{s,q}$-a.s.), by means of arguments similar to the ones applied above, we obtain

$$\mathbb{E}_{s,q}\bigg[\int_0^{(\zeta_*\vee\tau_*)\wedge\upsilon_*} e^{-ru}(c+\nu S_u)du + e^{-r((\zeta_*\vee\tau_*)\wedge\upsilon_*)}\widetilde{V}(S_{(\zeta_*\vee\tau_*)\wedge\upsilon_*}, Q_{(\zeta_*\vee\tau_*)\wedge\upsilon_*})\bigg] \tag{4.62}$$
$$= \widetilde{V}(s,q) + \mathbb{E}_{s,q}\big[\widehat{L}_{(\zeta_*\vee\tau_*)\wedge\upsilon_*}\big]$$
$$+ \mathbb{E}_{s,q}\bigg[\int_0^{(\zeta_*\vee\tau_*)\wedge\upsilon_*} e^{-ru}\,(\mathbb{L}_{(S,Q)}\widetilde{V} - r\widetilde{V} + c + \nu S_u)(S_u,Q_u)\, I\big(S_u>\widetilde{a}(Q_u) \text{ or } S_u>\widetilde{b}(Q_u)\big)\,du\bigg]$$

and

$$\mathbb{E}_{s,q}\bigg[\int_0^{(\zeta_*\vee\tau_*)\wedge\upsilon_*} e^{-ru}(c+\nu S_u)du + e^{-r((\zeta_*\vee\tau_*)\wedge\upsilon_*)}\widehat{V}(S_{(\zeta_*\vee\tau_*)\wedge\upsilon_*}, Q_{(\zeta_*\vee\tau_*)\wedge\upsilon_*})\bigg] \tag{4.63}$$
$$= \widehat{V}(s,q) + \mathbb{E}_{s,q}\big[\widehat{L}_{(\zeta_*\vee\tau_*)\wedge\upsilon_*}\big]$$
$$+ \mathbb{E}_{s,q}\bigg[\int_0^{(\zeta_*\vee\tau_*)\wedge\upsilon_*} e^{-ru}\,(\mathbb{L}_{(S,Q)}\widehat{V} - r\widehat{V} + c + \nu S_u)(S_u,Q_u)\, I\big(S_u>\widehat{a}(Q_u) \text{ or } S_u>\widehat{b}(Q_u)\big)\,du\bigg]$$

for all $(s,q)\in(0,K/\alpha]\times(0,\infty)$. Hence, taking into account the fact that the inequality $\widehat{V}(\widetilde{a}(q),q)\geq\widetilde{V}(\widetilde{a}(q),q)$ or $\widehat{V}(\widetilde{b}(q),q)\leq\widetilde{V}(\widetilde{b}(q),q)$ holds, for all $q>0$, we get from (4.62) and (4.63) that the inequality

$$\mathbb{E}_{s,q}\bigg[\int_0^{(\zeta_*\vee\tau_*)\wedge\upsilon_*} e^{-ru}\,(\mathbb{L}_{(S,Q)}\widetilde{V} - r\widetilde{V} + c + \nu S_u)(S_u,Q_u)\, I\big(S_u>\widetilde{a}(Q_u) \text{ or } S_u>\widetilde{b}(Q_u)\big)\,du\bigg] \tag{4.64}$$
$$\lesseqgtr \mathbb{E}_{s,q}\bigg[\int_0^{(\zeta_*\vee\tau_*)\wedge\upsilon_*} e^{-ru}\,(\mathbb{L}_{(S,Q)}\widehat{V} - r\widehat{V} + c + \nu S_u)(S_u,Q_u)\, I\big(S_u>\widehat{a}(Q_u) \text{ or } S_u>\widehat{b}(Q_u)\big)\,du\bigg]$$

is satisfied, when either the case $\widetilde{a}(q) \le K/\alpha$ or $\widetilde{b}(q) \le K/\alpha$ is realised, respectively. Thus, by virtue of the continuity of $\widehat{a}(q)$ and $\widehat{b}(q)$, we see from (4.64) and (4.43)-(4.44) that $\widehat{a}(q) \le \widetilde{a}(q)$ or $\widehat{b}(q) \le \widetilde{b}(q)$ holds, for all $q > 0$, respectively.

We finally show that $\widehat{a}(q)$ and $\widehat{b}(q)$ should coincide with $\widetilde{a}(q)$ and $\widetilde{b}(q)$. For this purpose, we take $s \in (\widehat{a}(q), \widetilde{a}(q))$ or $s \in (\widehat{b}(q), \widetilde{b}(q))$, for some $q > 0$. Hence, inserting $\zeta_* \wedge \tau_* \wedge \zeta_n$ into (4.59) in place of t and using the fact that the variables $Y_{\zeta_* \vee \tau_*}$ and $Z_{\zeta_* \vee \tau_*}$ are bounded on the event $\{\zeta_* \vee \tau_* = \infty\}$ ($\mathbb{P}_{s,q}$-a.s.), by means of arguments similar to the ones applied above, we obtain

$$\mathbb{E}_{s,q}\bigg[\int_0^{\zeta_* \wedge \tau_*} e^{-ru}\,(c + \nu S_u)\,du + e^{-r(\zeta_* \wedge \tau_*)}\,\widehat{V}(S_{\zeta_* \wedge \tau_*}, Q_{\zeta_* \wedge \tau_*})\bigg] = \widehat{V}(s,q) \tag{4.65}$$

$$+\,\mathbb{E}_{s,q}\bigg[\int_0^{\zeta_* \wedge \tau_*} e^{-ru}\,(\mathbb{L}_{(S,Q)}\widehat{V} - r\widehat{V} + c + \nu S_u)(S_u, Q_u)\, I\big(S_u > \widehat{a}(Q_u) \text{ or } S_u > \widehat{b}(Q_u)\big)\,du + \widehat{L}_{\zeta_* \wedge \tau_*}\bigg]$$

for all $(s,q) \in (0, K/\alpha] \times (0,\infty)$. Thus, since we have $\widehat{V}(s,q) = \widetilde{V}(s,q)$ for $s = \widetilde{a}(q)$ and $s = \widetilde{b}(q)$, and either the inequality $\widehat{V}(s,q) \ge \widetilde{V}(s,q)$ or $\widehat{V}(s,q) \le \widetilde{V}(s,q)$ holds, we see from the expressions in (4.58) and (4.65) that the inequalities

$$\mathbb{E}_{s,q}\bigg[\int_0^{\zeta_* \wedge \tau_*} e^{-ru}\,(\mathbb{L}_{(S,Q)}\widehat{V} - r\widehat{V} + c + \nu S_u)(S_u, Q_u)\, I\big(S_u > \widehat{a}(Q_u) \text{ or } S_u > \widehat{b}(Q_u)\big)\,du + \widehat{L}_{\zeta_* \wedge \tau_*}\bigg] \lesseqgtr 0 \tag{4.66}$$

should hold, if either the case $\widehat{a}(q) \le K/\alpha$ or $\widehat{b}(q) \le K/\alpha$ is realised, respectively. However, if the function $\widehat{V}(s,q)$ satisfies the smooth-fit conditions of (4.6) at the boundaries $\widehat{a}(q)$ and $\widehat{b}(q)$, then $\widehat{L}_{\zeta_* \wedge \tau_*} = 0$ holds, and thus, the strict inequalities in (4.66) cannot be satisfied due to the continuity of $\widetilde{a}(q)$ and $\widetilde{b}(q)$. We may therefore conclude that $\widetilde{a}(q) = \widehat{a}(q)$ and $\widetilde{b}(q) = \widehat{b}(q)$, so that $\widehat{V}(s,q)$ coincides with $\widetilde{V}(s,q)$, for all $(s,q) \in (0, K/\alpha] \times (0,\infty)$. $\square$

Getting together the assertions of Lemmata 2.1 and 4.1 as well as the arguments above, we can formulate the main result of this section concerning the solution to the convertible bond pricing problem under *partial information*.

Theorem 4.2 *Let the processes S and Π be defined by (2.1)-(2.3) and (2.4)-(2.5). Suppose that the inequalities $0 < \delta_1 < \delta_0 < r$ and $0 < \nu < \alpha\delta_1 < \alpha\delta_0$ as well as $c < rK$ hold, and the points of the boundary ∂C_* are regular for the region D_* relative to the process (S, Π). Assume that the equations in (4.32) admit unique solutions $\widetilde{a}(q)$ and $\widetilde{b}(q)$, for each $q > 0$ fixed. Then, the value function of the optimal stopping game in (2.7) takes the form $V_*(s,\pi) = \widetilde{V}_*(s, s^{-\eta}\pi/(1-\pi))$ and the optimal stopping times ζ_* and τ_* from (2.22) form a Nash equilibrium, where the function $\widetilde{V}_*(s,q)$ admits the representation in (4.46), while the monotone optimal stopping boundaries $a_*(\pi)$ and $b_*(\pi)$ are uniquely specified by the equations $a(\pi) = \widetilde{a}(a^{-\eta}(\pi)\pi/(1-\pi))$ and $b(\pi) = \widetilde{b}(b^{-\eta}(\pi)\pi/(1-\pi))$, where the continuous functions $\widetilde{a}(q)$ and $\widetilde{b}(q)$ are determined in parts (i)-(iii) of Lemma 4.1 above.*

4.5 Solution in a particular case. In order to underline the complexity in the structure of solutions to the optimal stopping game in the model with partial information, we present a specific choice of parameters under which the game admits a closed-form solution. More precisely, let us assume until the end of this section that $\lambda = 0$ and $\delta_0 + \delta_1 = 2r - \sigma^2$ holds. The first equality means that $\Theta_t = \Theta_0$, for all $t \ge 0$, where $\mathbb{P}(\Theta_0 = 1) = \pi$ and $\mathbb{P}(\Theta_0 = 0) = 1 - \pi$, for $\pi \in [0,1]$ (see, e.g. [54; Chapter VI, Section 21] and [27; Section 4]). Such a situation occurs when the firm issuing the asset does not change the dividend policy which is unknown to small investors during the whole infinite time interval. In this case, the parabolic-type partial differential equation in (4.35)+(4.36) degenerates into an ordinary one, and the general solution to that equation takes the form

$$\widetilde{V}(s,q) = \sum_{j=1}^{2} \widetilde{C}_j(q)\, s^{\gamma_{0,j}}\, F(\psi_{j,1}, \psi_{j,2}; \varphi_j; -s^\eta q) + \widetilde{P}(s,q) \tag{4.67}$$

where $\widetilde{C}_j(q)$, $j = 1, 2$, are some arbitrary twice continuously differentiable functions. Here, $\widetilde{P}(s,q)$ is a particular bounded solution of the second-order ordinary differential equation resulting from (4.35)-(4.36) under the assumptions $\lambda = 0$ and $\delta_0 + \delta_1 = 2r - \sigma^2$, and we additionally set

$$\psi_{k,l} = \frac{\gamma_{0,k} - \gamma_{1,l}}{\eta} \quad \text{and} \quad \varphi_k = 1 + \frac{2}{\eta}\left(\gamma_{0,k} - \frac{1}{2} + \frac{r-\delta_0}{\sigma^2}\right) \tag{4.68}$$

for every $k, l = 1, 2$, where $\gamma_{0,j}$ is given by the equation in (3.15) below with $\lambda = 0$. Then $F(\alpha, \beta; \gamma; x)$ denotes Gauss' hypergeometric function, which is defined by means of the expansion

$$F(\alpha, \beta; \gamma; x) = 1 + \sum_{m=1}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m} \frac{x^m}{m!} \tag{4.69}$$

for $\gamma \neq 0, -1, -2, \ldots$, and $(\gamma)_m = \gamma(\gamma+1)\cdots(\gamma+m-1)$, $m \in \mathbb{N}$ (see, e.g. [1; Chapter XV] and [2; Chapter II]). Taking into account the fact that $\gamma_{0,2} < 0 < 1 < \gamma_{0,1}$, we observe that $\widetilde{C}_2(q) = 0$ should hold in (4.67) under $s \downarrow 0$ and the assumption of $\delta_0 > \delta_1$, since otherwise $\widetilde{V}(s,q) \to \pm\infty$, that must be excluded by virtue of the obvious fact that the value function in (2.7) is bounded under $s \downarrow 0$, for any $\pi \in [0,1]$ fixed. Then, we can apply the conditions of (4.37) and (4.38) to the function in (4.67) at the boundaries $\widetilde{a}(q)$ and $\widetilde{b}(q)$, for each $q > 0$. It therefore remains to determine the value function $\widetilde{V}(s,q)$ and the boundaries $\widetilde{a}(q)$ and $\widetilde{b}(q)$ in every particular combination of the parameters of the model indicated above.

In the case of part (i) of Lemma 4.1, by solving the resulting second-order ordinary differential equation in (4.35)+(4.36) with the conditions in the left-hand sides of (4.37)-(4.40), we get that when $(r - \nu/\alpha)K < c < rK$ holds, the solution to the free-boundary problem is given by

$$\widetilde{V}(s,q;\widetilde{a}(q)) = \big(K - \widetilde{P}(\widetilde{a}(q),q)\big)\left(\frac{s}{\widetilde{a}(q)}\right)^{\gamma_{0,1}} \frac{F(\psi_{1,1}, \psi_{1,2}; \varphi_1; -s^\eta q)}{F(\psi_{1,1}, \psi_{1,2}; \varphi_1; -\widetilde{a}^\eta(q) q)} + \widetilde{P}(s,q) \tag{4.70}$$

for all $0 < s < \widetilde{a}(q)$, which represents an expression for the candidate value function, whenever the inequalities in (4.42) and (4.43) hold. Here, the candidate boundary $\widetilde{a}(q) \leq K/\alpha$ is determined as a unique solution to the equation

$$\frac{F(1+\psi_{1,1}, 1+\psi_{1,2}; 1+\varphi_1; -a^\eta(q)q)}{\varphi_1 F(\psi_{1,1}, \psi_{1,2}; \varphi_1; -a^\eta(q)q)} = \frac{a(q)\widetilde{P}_s(a(q),q) + \gamma_{0,1}(K - \widetilde{P}(a(q),q))}{\psi_{1,1}\psi_{1,2}\eta q (K - \widetilde{P}(a(q),q)) a^\eta(q)} \tag{4.71}$$

whenever it exists, for each $q > 0$ fixed. Moreover, in the case of part (ii) of Lemma 4.1, by applying the conditions in the right-hand sides of (4.37)-(4.40), we get that, when $c < (\delta_0 - \nu/\alpha)K$ holds, the solution to the free-boundary problem is given by

$$\widetilde{V}(s,q;\widetilde{b}(q)) = \big(\alpha\, \widetilde{b}(q) - \widetilde{P}(\widetilde{b}(q),q)\big)\left(\frac{s}{\widetilde{b}(q)}\right)^{\gamma_{0,1}} \frac{F(\psi_{1,1}, \psi_{1,2}; \varphi_1; -s^\eta q)}{F(\psi_{1,1}, \psi_{1,2}; \varphi_1; -\widetilde{b}^\eta(q) q)} + \widetilde{P}(s,q) \tag{4.72}$$

for all $0 < s < \widetilde{b}(q)$, which represents an expression for the candidate value function, whenever the inequalities in (4.42) and (4.44) hold. Here, the candidate boundary $\widetilde{b}(q) \leq K/\alpha$ is determined as a unique solution to the equation

$$\frac{F(1+\psi_{1,1}, 1+\psi_{1,2}; 1+\varphi_1; -b^\eta(q)q)}{\varphi_1 F(\psi_{1,1}, \psi_{1,2}; \varphi_1; -b^\eta(q)q)} = \frac{b(q)(\widetilde{P}_s(b(q),q) - 1) + \gamma_{0,1}(\alpha\, b(q) - \widetilde{P}(b(q),q))}{\psi_{1,1}\psi_{1,2}\eta q(\alpha b(q) - \widetilde{P}(b(q),q)) b^\eta(q)} \tag{4.73}$$

whenever it exists, for each $q > 0$ fixed. The uniqueness of solutions to the equations in (4.71) and (4.73), whenever they exist, as well as the validity of the inequalities (4.42)-(4.44) follow from the uniqueness of the solution to the system in (4.35)+(4.36)-(4.40), which is proved in Lemma 4.1 above.

We summarise the above results in the following assertion.

Corollary 4.3 *Suppose that the assumptions of Lemma 4.1 are satisfied with $\lambda = 0$ and $\delta_0 + \delta_1 = 2r - \sigma^2$. Then, the value function of the optimal stopping game in (4.31) admits the representation of (4.46), where the functions $\widetilde{V}(s, q; \widetilde{a}(q))$ and $\widetilde{V}(s, q; \widetilde{b}(q))$ with the boundaries $\widetilde{a}(q)$ and $\widetilde{b}(q)$ are determined as follows:*

(i) If $(r-\nu/\alpha)K < c < rK$ holds, then $\widetilde{V}(s, q; \widetilde{a}(q))$ is given by the expression in (4.70), whenever the equation in (4.71) admits a unique solution $\widetilde{a}(q) \le K/\alpha$ and the inequalities in (4.42) and (4.43) hold, and otherwise, by the expression in (4.45) with $\widetilde{a}(q) \equiv K/\alpha$.

(ii) If $c < (\delta_0 - \nu/\alpha)K$ holds, then $\widetilde{V}(s, q; \widetilde{b}(q))$ is given by the expression in (4.72), whenever the equation in (4.73) admits a unique solution $\widetilde{b}(q) \le K/\alpha$ and the inequalities in (4.42) and (4.44) hold, and otherwise, by the expression in (4.45) with $\widetilde{b}(q) \equiv K/\alpha$.

(iii) If $(\delta_0 - \nu/\alpha)K \le c \le (r - \nu/\alpha)K$ holds, then $\widetilde{V}(s, q; K/\alpha)$ is explicitly given by the expression in (4.45) with $\widetilde{a}(q) \equiv \widetilde{b}(q) \equiv K/\alpha$.

5 The case of asymmetric information (Conclusions)

In this section, we finally consider the optimal stopping game in the associated model with *asymmetric information*. In this model, the dividend policy of the firm issuing the asset is accessible to the writer, but inaccessible to the holder of the convertible bond. Then, a rational value the bond from the point of view of the holder, as an uninformed player, is given by the value of the optimal stopping game

$$W_*(s, \pi) = \inf_{\zeta'} \sup_{\tau} \mathbb{E}_{s,\pi}\big[Y_\tau\, I(\tau < \zeta') + Z_{\zeta'}\, I(\zeta' \le \tau)\big] = \sup_{\tau} \inf_{\zeta'} \mathbb{E}_{s,\pi}\big[Y_\tau\, I(\tau < \zeta') + Z_{\zeta'}\, I(\zeta' \le \tau)\big] \tag{5.1}$$

where the infimum and supremum are taken over all stopping times $\zeta' = \zeta'(S, \Theta)$ and $\tau = \tau(S, \Pi)$ with respect to the natural filtrations of (S, Θ) and S, respectively. The latter fact literally means that the continuous-time Markov chain Θ is observable by the writer but not by the holder of the bond in this formulation. In this case, the writer of the bond can be thought of as someone with the complete knowledge of the dividend policy of the underlying asset. Note that, since the infimum and supremum are taken over stopping times with respect to different filtrations, the problem of (5.1) falls outside the scope of the classical theory of optimal stopping games for Markov processes. However, the additive structure of the reward functionals in (5.1) as well as in (2.7) and (3.1) and the results presented in Theorems 3.1 and 4.2 allow us to conclude that the associated optimal stopping game has a value and the optimal stopping times ζ'_* and τ_* have the forms of (2.22) and (3.2) above.

More precisely, we observe that when $(r - \nu/\alpha)K < c < rK$ holds, it follows from the results of part (i) in Theorem 3.1 and Theorem 4.2 (see also part (i) in Lemma 4.1) that the writer should recall the bond first, so that the value function in (5.1) admits the representation

$$\begin{aligned}
W_*(s, \pi) &= \inf_{\zeta'} \mathbb{E}_{s,\pi}\big[Y_{\tau'_*}\, I(\tau'_* < \zeta') + Z_{\zeta'}\, I(\zeta' \le \tau'_*)\big] \\
&= \inf_{\zeta'} \mathbb{E}_{s,0}\big[Z_{\zeta'}\, I(\zeta' \le \tau'_*)\big]\,(1 - \pi) + \inf_{\zeta'} \mathbb{E}_{s,1}\big[Z_{\zeta'}\, I(\zeta' \le \tau'_*)\big]\,\pi \\
&= U_*(s, 0)\,(1 - \pi) + U_*(s, 1)\,\pi
\end{aligned}$$

where $\tau_* = \tau_*(S, \Pi)$ coincides with $\tau'_* = \tau'_*(S, \Theta)$, and we thus have $\tau_* = \tau'_* = \zeta'_* \vee \tau'_*$. Then, we may conclude directly from the structure of the value functions in (3.1) and (5.1) that the inequality

$W_*(s,\pi) = U_*(s,0)(1-\pi) + U_*(s,1)\pi \le V_*(s,\pi)$ is satisfied, so that the standard comparison arguments imply that $g_*(0) \ge a_*(\pi)$ holds, for all $(s,\pi) \in (0,K/\alpha] \times [0,1]$. On the other hand, we observe that when $c < (\delta_0 - \nu/\alpha)K$ holds, it follows from the results of part (ii) in Theorem 3.1 and Theorem 4.2 (see also part (ii) in Lemma 4.1) that the holder should convert the bond first, so that the value function in (5.1) admits the representation

$$W_*(s,\pi) = \sup_{\tau} \mathbb{E}_{s,\pi}\big[Y_\tau\, I(\tau < \zeta_*) + Z_{\zeta_*}\, I(\zeta_* \le \tau)\big] = V_*(s,\pi)$$

where $\zeta'_* = \zeta'_*(S,\Theta)$ coincides with $\zeta_* = \zeta_*(S,\Pi)$, and we thus have $\zeta'_* = \zeta_* = \zeta_* \vee \tau_*$. Then, we may conclude directly from the structure of the value functions in (3.1) and (5.1) that the inequality $W_*(s,\pi) = V_*(s,\pi) \le U_*(s,0)(1-\pi) + U_*(s,1)\pi$ is satisfied, so that the standard comparison arguments imply that $b_*(\pi) \le h_*(1)$ holds, for all $(s,\pi) \in (0,K/\alpha] \times [0,1]$. Finally, we note that when $(\delta_0 - \nu/\alpha)K \le c \le (r - \nu/\alpha)K$ holds, it follows from the results of part (iii) in Theorem 3.1 and Theorem 4.2 (see also part (iii) in Lemma 4.1) that the writer and the holder should exercise the contract simultaneously, so that the value function in (5.1) admits the representation $W_*(s,\pi) = U_*(s,0)(1-\pi) + U_*(s,1)\pi = V_*(s,\pi)$, as well as $a_*(\pi) = g_*(i) = K/\alpha$ and $b_*(\pi) \equiv h_*(i) = K/\alpha$, for all $(s,\pi) \in (0,K/\alpha] \times [0,1]$ and every $i = 0,1$.

We now summarise these facts in the following assertion.

Corollary 5.1 *Suppose that the assumptions of Theorems 3.1 and 4.2 hold with the inequalities $0 < \delta_1 < \delta_0 < r$ and $0 < \nu < \alpha\delta_1 < \alpha\delta_0$ as well as $c < rK$. Then, the stopping times ζ'_* and τ_* from (3.2) and (2.22) form a Nash equilibrium in the optimal stopping game in (5.1), where the value function $W_*(s,\pi)$ and the boundaries $a_*(\pi)$ and $b_*(\pi)$, for $\pi \in [0,1]$, as well as $g_*(i)$ and $h_*(i)$, for $i = 0,1$, satisfy the following properties:*

(i) If $(r-\nu/\alpha)K < c < rK$ holds, then we have $W_*(s,\pi) = U_*(s,0)(1-\pi) + U_*(s,1)\pi \le V_*(s,\pi)$ and $a_*(\pi) \le g_*(0) \le K/\alpha$.

(ii) If $c < (\delta_0 - \nu/\alpha)K$ holds, then we have $W_*(s,\pi) = V_*(s,\pi) \le U_*(s,0)(1-\pi) + U_*(s,1)\pi$ and $b_*(\pi) \le h_*(1) \le K/\alpha$.

(iii) If $(\delta_0 - \nu/\alpha)K \le c \le (r - \nu/\alpha)K$ holds, then we have $W_*(s,\pi) = V_*(s,\pi) = U_*(s,0)(1-\pi) + U_*(s,1)\pi$ as well as $a_*(\pi) \equiv g_*(i) = K/\alpha$ and $b_*(\pi) \equiv h_*(i) = K/\alpha$.

It is seen from the results stated above that the rational value $W_*(s,\pi)$ from (5.1) of the convertible bond computed in the model with *asymmetric information* does not generally exceed the value $V_*(s,\pi)$ from (2.7) computed in the model with *partial information.* More precisely, if the property $(r-\nu/\alpha)K < c < rK$ holds, then the inequality $W_*(s,\pi) \le V_*(s,\pi)$ is satisfied, while if the property $c \le (r-\nu/\alpha)K$ holds, then the equality $W_*(s,\pi) = V_*(s,\pi)$ is satisfied, for all $(s,\pi) \in (0,K/\alpha] \times [0,1]$. Thus, the difference $V_*(s,\pi) - W_*(s,\pi)$ may be interpreted as the value of the (additional) information of the writer compared to the one of the holder in the given perpetual convertible bond pricing problem.

6 Appendix

Let us now prove the existence and uniqueness of solutions to the systems of arithmetic equations in (3.27), (3.31), (3.42), and (3.46).

(i) It is shown by means of standard arguments that the system in (3.27) is equivalent to

$$I_{1,1}(h(0)) = J_{1,1}(h(1)) \quad \text{and} \quad I_{1,2}(h(0)) = J_{1,2}(h(1)) \tag{6.1}$$

with

$$I_{1,k}(s) = \sum_{j=1}^{2} (-1)^j \bigg(\frac{c}{r} \, \gamma_{1,3-k} \, \beta_{3-j} \bigg(R_0(\beta_j) - \frac{\lambda^2}{\lambda + r} \bigg) - \frac{c}{r} \, \beta_1 \, \beta_2 \, R_0(\beta_j) \, s^{-\gamma_{1,k}} \tag{6.2}$$
$$+ \frac{(\alpha\delta_0 + \alpha\delta_1 - 2\nu) \, \lambda + \delta_1(\alpha\delta_0 - \nu)}{(\delta_0 + \lambda)(\delta_1 + \lambda) - \lambda^2} \, (\beta_{3-j} - 1) \, (\beta_j - \gamma_{1,3-k}) \bigg(R_0(\beta_j) - \frac{\lambda^2}{\lambda + \delta_1} \bigg) \, s^{1-\gamma_{1,k}} \bigg)$$

and

$$J_{1,k}(s) = \frac{\lambda(\beta_1 - \beta_2)(\gamma_{1,1} - \gamma_{1,2})}{s^{\gamma_{1,k}}} \bigg((1 - \gamma_{1,3-k}) \, \frac{\nu - \alpha\delta_1}{\delta_1 + \lambda} \, s - \gamma_{1,3-k} \, \frac{c}{r + \lambda} \bigg) \tag{6.3}$$

for $k = 1, 2$, where the functions $R_i(\beta)$ as well as the constants β_j and $\gamma_{1,k}$, for $i = 0, 1$ and $j, k = 1, 2$, are defined in (3.14)-(3.15). Given the expressions for the derivatives of the functions in (6.2)-(6.3), and the facts that $1 < \beta_2 < \gamma_{1,1} < \beta_1$, $R_0(\beta_1) < 0 < R_0(\beta_2)$, and $\lambda^2/(\delta_1 + \lambda) < R_0(\beta_2)$, we observe that the function $I_{1,1}(s)$ is increasing on $(0, M_{1,1})$, with $I_{1,1}(0+) = -\infty$ and $I_{1,1}(M_{1,1}) > 0$, and decreasing on $(M_{1,1}, \infty)$, with $I_{1,1}(\infty) = +0$. Here, $M_{1,1}$ is the unique point at which the function $I_{1,1}(s)$ attains its maximum. Moreover, it is shown that the function $J_{1,k}(s)$ is decreasing on $(0, c/(\alpha\delta_1 - \nu))$, with $J_{1,1}(0+) = \infty$, $J_{1,2}(0) = 0$, and $J_{1,k}(c/(\alpha\delta_1 - \nu)) < 0$, $k = 1, 2$, and increasing on $(c/(\alpha\delta_1 - \nu), \infty)$, with $J_{1,1}(\infty) = -0$ and $J_{1,2}(\infty) = \infty$. We further distinguish three cases generated by the shape of the function $I_{1,2}(s)$ and specified by the location of the point $R_0(\beta_2)$ with respect to the points $((\gamma_{1,1} - 1)L_1(\delta_1) + (\beta_2 - 1)L_2)/(\beta_1 - 1)$ and $(\gamma_{1,1}L_1(r) + \beta_2 L_2)/\beta_1$, where the function $L_1(\delta)$ and the constant L_2 are defined by

$$L_1(\delta) = \frac{\lambda^2}{\delta + \lambda} \, \frac{\beta_1 - \beta_2}{\gamma_{1,1} - \beta_2} > 0 \quad \text{and} \quad L_2 = R_0(\beta_1) \, \frac{\gamma_{1,1} - \beta_1}{\gamma_{1,1} - \beta_2} > 0 \tag{6.4}$$

for all $\delta > 0$. For instance, we assume that the property $(\gamma_{1,1}L_1(\delta_1) + \beta_2 L_2)/\beta_1 < R_0(\beta_2)$ holds, and the two other cases are analysed using arguments similar to the ones that follow. It is shown that the function $I_{1,2}(s)$ is increasing on $(0, M_{1,2})$, with $I_{1,2}(0) = 0$ and $I_{1,2}(M_{1,2}) > 0$, and decreasing on $(M_{1,2}, \infty)$, with $I_{1,2}(\infty) = -\infty$, where $M_{1,2}$ is the unique point at which the function $I_{1,2}(s)$ attains its maximum. Hence, taking into account the shape of the functions in (6.1), as well as the fact that $h(0) \le h(1) \le K/\alpha$ holds in this case, it can be shown that every equation in (6.1) implies that, for each appropriate $h(1)$, there exists a unique $h(0)$. Therefore, the equations in (6.1) uniquely define an increasing function $h_+(1; h(0))$ and a decreasing function $h_-(1; h(0))$ with the appropriate ranges, so that the curves associated with these functions can have at most one intersection point, which has the coordinates $h_*(0)$ and $h_*(1)$ such that $0 < h_+(1; h_*(0)) = h_*(1) = h_-(1; h_*(0)) \le K/\alpha$ holds.

(ii) It is shown by means of standard arguments that the system in (3.31) is equivalent to

$$I_{1,1}(h(0)) = J_{2,1}(f(1)) \quad \text{and} \quad I_{1,2}(h(0)) = J_{2,2}(f(1)) \tag{6.5}$$

with $I_{1,k}(h(0))$, $k = 1, 2$, given by the equation in (6.2), as well as

$$J_{2,1}(f(1)) = \lambda \, (\beta_1 - \beta_2) \, (\gamma_{1,1} - \gamma_{1,2}) \, f(1) \tag{6.6}$$

and

$$J_{2,2}(f(1)) = \lambda \, (\beta_1 - \beta_2) \, (\gamma_{1,1} - \gamma_{1,2}) \, (K/\alpha)^{-\gamma_{1,2}} \, (f(1) \, (K/\alpha)^{\gamma_{1,1}} + B_1(K/\alpha) - K). \tag{6.7}$$

We recall that the properties of the function $I_{1,1}(s)$ in (6.2) are analysed in part (i) of this section, while the functions $J_{2,k}(s)$, $k = 1, 2$, in (6.6)-(6.7) are linear and increasing. We further consider a structurally different case generated by the shape of the function $I_{1,2}(s)$, than the related one studied in part (i) above. Namely, assume that $R_0(\beta_2) < ((\gamma_{1,1}-1)L_1(\lambda)+(\beta_2-1)L_2)/(\beta_1-1)$ holds, where $L_1(\delta)$ and L_2 are given in (6.4), and the two other cases are analysed using arguments similar to the ones that follow. It is shown that $I_{1,2}(s)$ is decreasing on $(0, N_{1,2})$, with $I_{1,2}(0)=0$ and $I_{1,2}(N_{1,2})<0$, and increasing on $(N_{1,2}, \infty)$, with $I_{1,2}(\infty)=\infty$, where $N_{1,2}$ is the unique point at which the function $I_{1,2}(s)$ attains its minimum. Taking into account the shape of the functions in (6.5), as well as the fact that $h(0) \le K/\alpha$ holds in this case, it can be shown that every equation in (6.5) implies that, for each appropriate $f(1)$, there exists a unique $h(0)$. We may therefore conclude that if $c/(\alpha\delta_0-\nu) < K/\alpha \le M_{1,1} \wedge N_{1,2}$ holds, the equations in (6.5) uniquely define an appropriate increasing function $h_{1,+}(0; f(1))$ and an appropriate decreasing function $h_{1,-}(0; f(1))$, with the same range $(0, K/\alpha]$. The curves associated with these functions can have at most one intersection point which has the coordinates $f_*(1)$ and $h_*(0)$ such that $0 < h_{1,+}(0; f_*(1)) = h_*(0) = h_{1,-}(0; f_*(1)) \le K/\alpha$ holds. The other subcases, in which $K/\alpha > M_{1,1} \vee N_{1,2} \vee (c/(\alpha\delta_0-\nu))$, and either $(c/(\alpha\delta_0-\nu)) \vee M_{1,1} < K/\alpha \le N_{1,2}$ or $(c/(\alpha\delta_0-\nu)) \vee N_{1,2} < K/\alpha \le M_{1,1}$ holds, are considered similarly.

(iii) It is shown by means of standard arguments that the system in (3.42) is equivalent to

$$I_{3,1}(g(1)) = J_{3,1}(g(0)) \quad \text{and} \quad I_{3,2}(g(1)) = J_{3,2}(g(0)) \tag{6.8}$$

with

$$I_{3,k}(s) = \sum_{j=1}^{2}(-1)^j \left(\frac{\lambda(rK-c)}{r}\, \beta_j\, (\beta_{3-j}-\gamma_{0,3-k})\, R_0(\beta_j) \left(\frac{R_0(\beta_{3-j})}{\lambda+r} - 1 \right) s^{-\gamma_{0,k}} \right. \tag{6.9}$$
$$\left. + \frac{(\beta_j-1)(\delta_0+2\lambda)\lambda\nu}{(\delta_0+\lambda)(\delta_1+\lambda)-\lambda^2}\, R_0(\beta_j) \left(\beta_{3-j}-\gamma_{0,3-k}-(1-\gamma_{0,3-k})\, \frac{R_0(\beta_{3-j})}{\lambda+\delta_0} \right) s^{1-\gamma_{0,k}} \right)$$

and

$$J_{3,k}(s) = \frac{R_0(\beta_1)R_0(\beta_2)(\beta_1-\beta_2)}{s^{\gamma_{0,k}}} \left(\frac{(\gamma_{0,3-k}-1)\nu}{\delta_0+\lambda}\, s - \frac{\gamma_{0,3-k}\,(rK-c)}{r+\lambda} \right) \tag{6.10}$$

for $k=1,2$. We observe from the expressions for the derivatives of the functions in (6.9)-(6.10), together with the relations between the parameters indicated in the previous parts of this section, that the function $I_{3,1}(s)$ is increasing on $(0, M_{3,1})$, with $I_{3,1}(0+)=-\infty$ and $I_{3,1}(M_{3,1})>0$, and decreasing on $(M_{3,1}, \infty)$, with $I_{3,1}(\infty)=+0$, where $M_{3,1}$ is the unique point at which the function $I_{3,1}(s)$ attains its maximum. Moreover, it is shown that the functions $J_{3,k}(s)$, $k=1,2$, are increasing on $(0, (rK-c)/\nu)$, with $J_{3,1}(0+)=-\infty$, $J_{3,2}(0)=0$, and $J_{3,k}((rK-c)/\nu)>0$, $k=1,2$, and decreasing on $((rK-c)/\nu, \infty)$, with $J_{3,1}(\infty)=+0$ and $J_{3,2}(\infty)=-\infty$. We further distinguish three cases generated by the shape of the function $I_{3,2}(s)$ and specified by the location of the point $(\beta_2-\beta_1)R_0(\beta_1)R_0(\beta_2)>0$ with respect to the points $((\beta_1-1)L_3(\delta_0)+(\beta_2-1)L_4(\delta_0))/(\gamma_{0,1}-1)>0$ and $(\beta_1 L_3(r)+\beta_2 L_4(r))/\gamma_{0,1}>0$, for the function $L_{i+2}(\delta)$ defined by

$$L_{i+2}(\delta) = (-1)^i\, (\delta+\lambda)\, (\gamma_{0,1}-\beta_{3-i})\, R_0(\beta_i) > 0 \tag{6.11}$$

for all $\delta>0$ and $i=1,2$. For instance, we assume that the property $(\beta_2-\beta_1)R_0(\beta_1)R_0(\beta_2) > ((\beta_1-1)L_3(r)+(\beta_2-1)L_4(r))/(\gamma_{0,1}-1)$ holds, and the two other cases are analysed using arguments similar to the ones that follow. It is shown that $I_{3,2}(s)$ is decreasing on $(0, N_{3,2})$, with $I_{3,2}(0)=0$ and $I_{3,2}(N_{3,2})<0$, and increasing on $(N_{3,2}, \infty)$, with $I_{3,2}(\infty)=\infty$, where $N_{3,2}$ is the unique point at which the function $I_{3,2}(s)$ attains its minimum. Taking into account the shape of the functions in (6.8), as well as the fact that $g(1) \le g(0) \le K/\alpha$ holds in this case, it can be shown that every equation in (6.8) implies that, for each appropriate $g(0)$, there exists a unique $g(1)$. We may therefore conclude that the left-hand equation in (6.8) uniquely defines an appropriate increasing function $g_{1,+}(1; g(0))$ or an appropriate decreasing function $g_{1,-}(1; g(0))$, and the right-hand equation in (6.8) uniquely defines an appropriate decreasing function $g_{2,-}(1; g(0))$. These facts directly imply that, when the function $g_{1,+}(1; g(0))$ is defined, the curves associated with the functions $g_{1,+}(1; g(0))$ and $g_{2,-}(1; g(0))$ can have at most one intersection point which has the coordinates $g_*(0)$ and $g_*(1)$, such that $0 < g_{1,+}(1; g_*(0)) = g_*(1) = g_{2,-}(1; g_*(0)) \le K/\alpha$. On the other hand, when the function $g_{1,-}(1; g(0))$ is defined, the curves associated with the functions $g_{1,-}(0; g(1))$ and $g_{2,-}(0; g(1))$ can have several intersection points. In the latter case, we choose the couple $g_*(0)$ and $g_*(1)$, that satisfies the inequalities $g_*(1) \le g_*(0) \le K/\alpha$ as well as (3.9)–(3.10).

(iv) It is shown by means of standard arguments that the system in (3.46) is equivalent to

$$I_{3,1}(g(1)) = J_{4,1}(f(0)) \quad \text{and} \quad I_{3,2}(g(1)) = J_{4,2}(f(0)) \tag{6.12}$$

with $I_{3,k}(g(1))$, $k = 1, 2$, given by the equation in (6.9), as well as

$$J_{4,1}(f(0)) = R_0(\beta_1)\, R_0(\beta_2)\, (\beta_1 - \beta_2)\, f(0) \tag{6.13}$$

and

$$J_{4,2}(f(0)) = R_0(\beta_1)\, R_0(\beta_2)\, (\beta_1 - \beta_2)\, (K/\alpha)^{-\gamma_{0,2}}\, \big(f(0)\, (K/\alpha)^{\gamma_{0,1}} + B_0(K/\alpha) - K\big). \tag{6.14}$$

We recall that the properties of the function $I_{3,1}(s)$ in (6.9) are analysed in part (iii) of this section, while the functions $J_{4,k}(f(0))$, $k = 1, 2$, in (6.13)-(6.14) are linear and increasing. We further consider the same structural case for the function $I_{3,2}(s)$ as in part (iii) above, and the two other cases are analysed using arguments similar to the ones that follow. Taking into account the shape of the functions in (6.12), as well as the fact that $g(1) \le K/\alpha$ holds in this case, it can be shown that every equation in (6.12) implies that, for each appropriate $f(0)$, there exists a unique $g(1)$. We may therefore conclude that if $c/(\alpha r) < K/\alpha \le M_{3,1} \wedge N_{3,2} \wedge (c/(\alpha r - \nu))$ holds, the equations in (6.12) uniquely define an appropriate decreasing function $g_{1,-}(1; f(1))$ and an appropriate increasing function $g_{2,+}(1; f(1))$, with the same range $(0, K/\alpha]$. The curves associated with these functions can have at most one intersection point which has the coordinates $f_*(0)$ and $g_*(1)$ such that $0 < g_{1,-}(1; f_*(0)) = g_*(1) = g_{2,+}(1; f_*(0)) \le K/\alpha$ holds. The other subcases, in which $M_{3,1} \vee N_{3,2} \vee (c/(\alpha r)) < K/\alpha < c/(\alpha r - \nu)$, and either $(c/(\alpha r)) \vee N_{3,2} < K/\alpha \le M_{3,1}$ or $(c/(\alpha r)) \vee M_{3,1} < K/\alpha \le N_{3,2}$ holds, are considered similarly.

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