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Information Acquisition with Heterogeneous Valuations*

by

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Abstract

We study the market for a risky asset with uncertain heterogeneous valuations. Agents seek to learn about their own valuation by acquiring private information and making inferences from the equilibrium price. As agents of one type gather more information, they pull the price closer to their valuation and further away from the valuations of other types. Thus they exert a negative learning externality on other types. This in turn implies that a lower cost of information for one type induces all agents to acquire more information. Private information production is typically not socially optimal. In the case of two types who differ in their cost of information, we can always find a Pareto improvement that entails an increase in the aggregate amount of information, with a higher proportion produced by the low-cost type.

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1 Introduction

We study the market for a risky asset in which agents have private correlated valuations for the asset. Each agent collects private information about his own valuation, and the equilibrium price reflects some of this information. Our aim is to investigate the externalities that arise in this setting, and in particular how they affect the equilibrium allocation of private information and the welfare of market participants.

Heterogeneity in valuations can be due to different uses that agents have for the asset, motivated by speculation, hedging or liquidity considerations, because of differing investment opportunities or constraints, or for purely behavioral reasons. Alternatively, we can think of the agents as producers in different industries, and interpret the asset as an input into an industry-specific stochastic production technology.[1] In these examples, each agent belongs to a group (e.g. producers who operate in the same industry) that is distinguished by its own uncertain valuation for the asset, has access to private information about this valuation, and seeks to glean the wisdom of the crowd regarding the same valuation from the equilibrium price.

We analyze competitive rational expectations equilibria in a linear-normal model. To understand the mechanics of this model, suppose there are two types of agents, with uncertain valuations θ_1 and θ_2. Agents of type i ($i = 1, 2$) choose the precision τ_i of a private signal about their valuation θ_i, at a cost that is increasing in the precision. For any given choice of precisions, τ_1 and τ_2, the price function takes the form $p = \mu[\tau_1\theta_1 + \tau_2\theta_2]$, for some constant μ.[2] The optimal choice of τ_i by agents of type i in turn depends on how much these agents learn about θ_i from the price.[3] Assuming that the correlation between θ_1 and θ_2 is nonnegative, agents of type 1 learn more, and agents of type 2 learn less, about their own valuation the greater is the ratio τ_1/τ_2.

Now consider an equilibrium (τ_1, τ_2), and suppose there is a decrease in the cost of information for type 1. This induces type 1 agents to collect more information (increasing τ_1), thus reducing price informativeness for type 2. As a result, type 2 agents collect more information as well (increasing τ_2). This reinforces the incentive of type 1 agents to accumulate more information, increasing τ_1 even further. The resulting feedback loop leads to an equilibrium in which both types gather more

[1]Rostek and Weretka (2012) provide examples of heterogeneous valuations for an asset based on group affiliations or on the geographic location of traders. Rahi and Zigrand (2018) show how diversity in valuations can be microfounded by adding hedgers to a model along the lines of Grossman and Stiglitz (1980) or Hellwig (1980). In Mendel and Shleifer (2012), one group of investors cares only about the fundamental, while a second group bases its decisions on a sentiment shock. We discuss some examples of heterogeneous valuations in a production economy with uncertain cost or demand in Section 2.

[2]We assume that agents have the same constant absolute risk aversion coefficient r. Then the aggregate trade of type i is linear in θ_i and p, with the coefficient of θ_i equal to $r^{-1}\tau_i$. In equilibrium, the sum of the trades of the two types is zero, giving us a price function that takes the stated form.

[3]While all agents of type i choose the same precision τ_i in equilibrium, an individual agent's choice has no impact on the price function.

information. The effect is more pronounced for type 1 agents: τ_1/τ_2 is higher at the new equilibrium. Consequently, type 1 agents learn more from the price and type 2 agents learn less.

More generally, this *monotone comparative statics* property, whereby a lower cost of information for one type results in more information production by all types, holds if the economy exhibits strategic complementarities in information acquisition. At the new equilibrium, the type whose cost is reduced learns more from the price while all other types learn less. There are strategic complementarities in the two-type case if and only if the correlation between the valuations of the two types exceeds a (negative) lower bound $\underline{\rho}$. With arbitrarily many types, strategic complementarities arise if pairwise correlations exceed $\underline{\rho}$ and, in addition, do not vary too much.

Next, we turn to the question of social optimality of private information acquisition. In particular, we examine the welfare effects of a change in the precision vector $\tau := (\tau_1, \ldots, \tau_N)$ in the neighborhood of an equilibrium. There are two factors at play here. All else equal, agents are better off if they are better informed. At the same time, they stand to gain more from trade the greater the distance between their own valuation and the overall market valuation, given by the equilibrium price. For each type, the overall welfare effect can be written as the sum of a *learning effect* and a *gains from trade effect*.

There is a fundamental tension between these two effects. The price p is more informative about the valuation θ_i only if it tracks θ_i more closely. To take an extreme example, if $p = \theta_i$, the price is fully revealing for type i, but type i agents gain nothing from trading at this price; indeed, their optimal trade is zero.

This tradeoff can be seen most clearly in a symmetric economy in which all types have the same cost of information and the correlation between valuations is the same for any pair of types. Such an economy has a unique equilibrium at which all types choose the same precision. At this equilibrium, the learning and gains from trade effects are collinear but opposite in sign. Moreover, the gains from trade effect dominates the learning effect, so that the types that are better off after perturbing τ are precisely those for whom price informativeness is lower. Price informativeness cannot be lower for all types, however, and hence a perturbation of τ cannot make all types better off.

The possibility of a Pareto improvement arises in the non-symmetric case. We consider an economy with two types who differ in their cost of information. For example, suppose type 2 has the lower cost. Then there is a unique equilibrium at which $\tau_1 < \tau_2$. The low-cost type produces more private information as intuition would suggest. But a Pareto improving allocation of information can always be found. It entails more information production in the aggregate, a higher proportion of which is acquired by the low-cost type, i.e. a higher $\tau_1 + \tau_2$ and a higher $\tau_2/(\tau_1 + \tau_2)$.

To summarize, our comparative statics results imply that negative learning externalities across types lead to information acquisition decisions that are clustered together — if one type gathers more or less information, other types respond by moving in the same direction. Our welfare results show that this clustering is excessive from the social point of view.

Related Literature:

A growing strand of literature starts from the premise that agents have correlated private valuations for a traded asset, and each agent has private information about his own valuation. In a seminal contribution, Vives (2011) studies strategic supply function competition among agents facing an uncertain cost.[4] Vives (2014) uses a perfectly competitive version of this model to study information revelation in the market for a risky asset. Rostek and Weretka (2012, 2015) extend the Vives (2011) setup to investigate the effect of market size on information aggregation and market power. Glebkin (2019) considers the case of two types, one of which consists of large strategic traders while the other is perfectly competitive, to analyze the interplay between liquidity and price informativeness. Bergemann et al. (2020) and Heumann (2020) introduce multidimensional signals into the Vives (2011) model; the first paper retains the strategic interaction of Vives (2011), while the second considers the perfectly competitive case. In Babus and Kondor (2018), dealers engage in bilateral trading in a network.

These papers employ a linear-Gaussian framework with exogenously specified valuations that vary across agents, just as in the present paper, but with more stringent assumptions on the correlations between these valuations. Vives (2011, 2014), Bergemann et al. (2020), Heumann (2020), and Babus and Kondor (2018) assume that the correlations are the same for any pair of agents or agent types; this is also true in Glebkin (2019) since there are only two types. Rostek and Weretka (2012, 2015) present a convincing argument for a general correlation structure, but restrict their analysis to the "equicommonal" case, wherein the average correlation between the valuation of a trader and that of the remaining traders is the same for all traders. Moreover, the symmetry assumptions imposed in all these papers ensure that price informativeness is the same for all agents, with the exception of Babus and Kondor (2018) who use an aggregate measure of constrained informational efficiency.[5] This is a key difference with respect to our setup where price informativeness can change in opposite directions for agents with different valuations, and learning spillovers play an important role in both comparative statics and welfare.

Private information is exogenous in the papers cited above, apart from Vives (2011, 2014). While these two papers differ in terms of market structure (imperfect vs perfect competition), the informational properties of the equilibrium are the same. As long as the marginal cost of information is sufficiently low, so that agents acquire at least some information, the price reveals the average signal, which for any

[4] Vives (2011) can be thought of as a generalization of Kyle (1989), incorporating a private value component in the asset payoff and thereby dispensing with the need for noise traders.

[5] Other papers in which agents have heterogeneous valuations and equilibrium prices convey information include Bernhardt and Taub (2015) on learning about common and private values in a duopoly, and Du and Zhu (2017) on the optimal frequency of trading. Kyle et al. (2018) discuss the similarities between a model with heterogeneous valuations and one with overconfident traders who agree to disagree.

individual agent is a sufficient statistic for the information of all other agents (Vives calls this a "privately revealing" equilibrium). In other words, there are no learning externalities in this setting. The perfectly competitive economy of Vives (2014) is ex post efficient regardless of how much information agents collect in equilibrium (this follows from the "private revelation" property and the first welfare theorem).

Learning externalities take center stage in Rahi and Zigrand (2018) (henceforth RZ), which serves as our point of departure, and from which we borrow some results (Lemma 2.1, Proposition 2.2 and Lemma 3.1) on rational expectations equilibrium with exogenously given signal precisions. The two papers diverge on the question of private information production. RZ study a binary information acquisition decision, wherein agents either acquire a piece of information at some cost or remain uninformed. In the present paper, we allow agents to choose the precision of their signal at a cost that increases in the precision. We assume that there is no fixed cost so that all types acquire some information. As such, our results complement those of RZ. Our welfare results, in particular, provide a different perspective on the Pareto inefficiency of the equilibrium allocation of private information. RZ show that discouraging information production can be Pareto improving if private signals are sufficiently noisy. In this paper, in contrast, the precision of private signals is endogenous, and there always exists a Pareto improvement that involves an increase in the total amount of information. We discuss this point further at the end of Section 5.

There is a large literature on the social value of public information in a pure exchange economy. More information can reduce risk-sharing opportunities (or, indeed, destroy them altogether, as in Hirshleifer (1971)). If markets are incomplete, it can also allow agents to construct better hedges. The overall impact on agents' welfare can be in any direction (Gottardi and Rahi (2014)). Much less is known about the welfare properties of asset markets in which information is endogenous and asymmetric. Most of the rational expectations literature relies on exogenous noise trade and hence does not provide a suitable framework for welfare analysis. There are some papers with fully optimizing traders but, apart from a few exceptions,[6] they do not ask if the amount of information produced by agents is socially optimal.

A number of papers feature a complementarity in information acquisition that arises because prices become less informative as more agents acquire information. The underlying mechanism differs across these papers. In Barlevy and Veronesi (2000, 2008) price informativeness falls with the incidence of informed trading because the asset payoff is negatively correlated with the noise trade, in Ganguli and Yang (2009) and Manzano and Vives (2011) because agents have two sources of information (about the asset payoff and the asset supply), in Goldstein et al. (2014) because agents with different investment opportunities trade on the same information in opposite directions, and in Breon-Drish (2012) due to non-normality of shocks. These papers use complementarity in information acquisition as a vehicle for gen-

[6]These exceptions include Vives (2014) and Rahi and Zigrand (2018) discussed above, as well as Dow et al. (2017) who study the informational feedback from asset prices to real investment.

erating multiple equilibria. In contrast, strategic complementarities in our paper actually describe the "well-behaved" case — if there are two types (which is the closest analog to the papers discussed here), the economy exhibits strategic complementarities if and only if the correlation between the valuations of the two types is not too negative; moreover, for such an economy, there is a unique equilibrium. The issue is partly a terminological one. We use strategic complementarities as a label to describe the case where higher information production by one type lowers price informativeness for the other type. But this is precisely the case in which higher information production by a given type *increases* price informativeness for that type itself. It is worth noting that the multiplicity of equilibria in Rahi and Zigrand (2018), which has the same flavor as the multiplicity found in the papers discussed above, requires a within-type complementarity which is ruled out by the across-type complementarity assumption that we make in this paper.

Another line of research investigates the interaction between private and public information in a coordination setting in which agents wish to align their actions with the actions of others. Some of this work touches on themes that run through the present paper. For example, Colombo et al. (2014)) show that increasing the informativeness of an exogenous public signal reduces the incentive of agents to collect private information. But our framework is quite different from theirs. The asset trading stage in our model cannot be reduced to a coordination game, and the public signal is the equilibrium price, which is endogenous. The welfare problem that we analyze is also more involved as agents in our setting are not ex ante identical; welfare depends not only on how much information is produced but also by whom it is produced.

We now lay out a brief road-map for the rest of the paper. In the next section, we describe the basic setup and the price function for given signal precisions for each type. We endogenize these precisions in Section 3. In Section 4 we characterize equilibrium for the two-type case. A welfare analysis follows in Section 5. In Section 6, we provide sufficient conditions on the primitives for the economy to exhibit strategic complementarities. This forms the basis of the existence and comparative statics results in Section 7. Proofs are in the Appendix.

2 The Economy

There is a single risky asset in zero net supply, and a riskfree asset with the interest rate normalized to zero. There are N types of agents, $N \geq 2$, and a continuum of agents of unit mass of each type. The valuation of an agent of type i for the risky asset is given by a random variable θ_i. Prior to trade, type i agents can acquire a private signal about θ_i. For agent n of type i (agent in for short) this signal takes the form

$$s_{in} = \theta_i + \epsilon_{in},$$

where ϵ_{in} is independent of θ_i.

For random variables x and y, let $\tau_x := [\mathrm{Var}(x)]^{-1}$, and $\tau_{x|y} := [\mathrm{Var}(x|y)]^{-1}$. We will use the shorthand notation τ_{in} for $\tau_{\epsilon_{in}}$, which is equal to $\tau_{s_{in}|\theta_i}$. Thus τ_{in} is the precision of agent in's signal, conditional on his valuation θ_i. For brevity, we will refer to it simply as the precision of his signal. The cost of this signal is $\mathscr{C}_i(\tau_{in})$. For now we will assume that all agents of type i choose the same precision τ_i, and that $\tau_i > 0$. Later, when we impose some conditions on the function $\mathscr{C}_i$ and endogenize precision choice, we will see that this assumption is indeed satisfied.

The random variables $\{\theta_i, \{\epsilon_{in}\}_{n\in[0,1]}\}_{i=1,\dots,N}$ are joint normal with mean zero. Let $\theta := (\theta_i)_{i=1}^N$. For each type i, the signal shock ϵ_{in} is independent of θ, and the signal shocks across agents, $\{\epsilon_{in}\}_{n\in[0,1]}$, are independent. Given the assumption that τ_{in} is the same for all n, these signal shocks are in fact i.i.d., and hence the average signal of agents of type i, $\int_n s_{in}dn$, is equal to θ_i. To ensure that the problem is nontrivial, we assume that the covariance matrix of θ is positive definite. We also assume that the variance of θ_i is the same for all i. We denote the correlation matrix of θ by R, with ij'th element $\rho_{ij} := \mathrm{corr}(\theta_i, \theta_j)$, and the i'th column of R by R_i. Due to the symmetry of R, the i'th row of R is $R_i^\top$.

If agent in buys q_{in} units of the risky asset at price p, his wealth is

$$W_{in} = (\theta_i - p)q_{in} - \mathscr{C}_i(\tau_{in}). \tag{1}$$

He has CARA utility with risk aversion coefficient r. He solves

$$\max_{q_{in}} E[-\exp(-rW_{in})|s_{in}, p].$$

Agents have rational expectations: they know the price function, which depends on the valuations of all agents in the economy, and condition on the price when making their portfolio decisions.

An equilibrium consists of a vector of precisions $\tau := (\tau_i)_{i=1}^N$ and a price function p such that agents optimize and markets clear. Agent optimization requires that each agent in is happy with his choice of precision τ_{in} given the price function p, and subsequently, for any realization of s_{in} and p, he chooses an optimal portfolio q_{in}. Letting $q_i := \int_n q_{in}dn$, the aggregate trade of type i, the market-clearing condition is $\sum_i q_i = 0$.

We begin by solving for a rational expectations equilibrium (REE) price function for given $\tau \in \mathbb{R}^N_{++}$. We conjecture that it takes the form

$$p = a^\top \theta,$$

for some nonzero vector a in $\mathbb{R}^N$, and that it is not fully revealing for any type, i.e. $\mathrm{Var}(\theta_i|p) > 0$, or equivalently $\tau_{\theta_i|p}$ is finite, for all i. We verify this conjecture later (in Proposition 2.2). As is standard in the CARA-normal setting, the optimal portfolio of agent in is

$$q_{in} = \frac{E(\theta_i|s_{in}, p) - p}{r\mathrm{Var}(\theta_i|s_{in}, p)}. \tag{2}$$

Using the projection theorem for normals, we have

$$\begin{aligned} E(\theta_i|s_{in},p) &= (\tau_i+\tau_{\theta_i|p})^{-1}\left[\tau_i s_{in}+\tau_{\theta_i|p}\gamma_{\theta_i,p}\,p\right], \\ \mathrm{Var}(\theta_i|s_{in},p) &= (\tau_i+\tau_{\theta_i|p})^{-1}, \end{aligned}$$

where $\gamma_{\theta_i,p} := \mathrm{Cov}(\theta_i,p)/\mathrm{Var}(p)$, the least-squares projection of θ_i on p. The details of this calculation can be found in Rahi and Zigrand (2018). Substituting into (2), we see that the optimal portfolio of an agent is linear in his signal and the price:

Lemma 2.1 (Optimal portfolios) *Agent in's optimal portfolio is given by*

$$q_{in} = r^{-1}(\tau_i s_{in} - \mu_i p),$$

where

$$\mu_i = \tau_i + \tau_{\theta_i|p}(1-\gamma_{\theta_i,p}). \tag{3}$$

Integrating over n gives us the aggregate trade of type i agents,

$$q_i = r^{-1}(\tau_i\theta_i - \mu_i p), \tag{4}$$

which is linear in θ_i and p. We can now solve for the price function using the market-clearing condition $\sum_i q_i = 0$. It has a simple characterization that involves a parameter μ defined as

$$\mu := \sum_k \beta_k \frac{R_k^\top \tau}{\tau^\top R\,\tau}, \tag{5}$$

where

$$\beta_k := \frac{\tau_k+\tau_{\theta_k|p}}{\sum_j(\tau_j+\tau_{\theta_j|p})}.$$

A key feature of this economy is that different agents extract different information from the equilibrium price. We define price informativeness for agents of type i by

$$\mathcal{V}_i := \frac{\mathrm{Var}(\theta_i)-\mathrm{Var}(\theta_i|p)}{\mathrm{Var}(\theta_i)}. \tag{6}$$

Proposition 2.2 (REE) *For a given vector of precisions $\tau \in \mathbb{R}^N_{++}$, there is a unique linear equilibrium price function,*

$$p = \mu\,\tau^\top\theta, \tag{7}$$

provided $\mu \neq 0$. Price informativeness for type i is given by

$$\mathcal{V}_i = \frac{(R_i^\top\tau)^2}{\tau^\top R\,\tau}. \tag{8}$$

The proposition describes the linear REE price function for an arbitrary exogenously given precision vector τ, with $\tau_i > 0$ for all i. There is no equilibrium if $\mu = 0$. This is clearly a knife-edge case, and we will proceed under the assumption that μ is nonzero. Later we shall see that in an economy that exhibits strategic complementarities, $R_i^\top \tau$ must be positive for all i, thus ensuring that μ is positive.

The price function does not fully reveal θ_i for any i; hence $\mathcal{V}_i \in [0, 1)$. This is a consequence of the assumption that $\tau_i > 0$ for all i and that the correlation matrix R is positive definite. Price informativeness for any type is homogeneous of degree zero in τ. Thus scaling the vector τ leaves price informativeness unchanged for all types.

Some examples of the economy described above can be found in Rahi and Zigrand (2018). In these examples, heterogeneity in valuations arises from different hedging needs. Here we provide two examples in which we interpret the asset as a good with an uncertain marginal cost of production, or as an input that can be deployed in multiple activities that generate a stochastic return. As in any linear-Gaussian model, we cannot ensure that equilibrium quantities of inputs or outputs are always positive, but it is a simple matter to relax our assumption that all random variables have zero mean and that the asset (or good) is in zero net supply in order to generate plausible outcomes. We assume for simplicity that the demand for the good in the first example, and the supply of the input in the second example, are perfectly inelastic. A straightforward extension of our setup can accommodate demand or supply functions that are affine in the price.

Example 2.1 There is a single homogeneous good supplied by perfectly competitive risk-averse producers. Demand is perfectly inelastic. There are N types of producers. Producers of type i face a constant marginal cost θ_i, which is uncertain and about which they can collect private information. The wealth of producer n of type i is given by (1), where p is the price of the good and $-q_{in}$ is the amount sold. The equilibrium price of the good is given by (7); it is a linear combination of the marginal costs $(\theta_i)_{i=1}^N$ of the different producer types. Each producer makes inferences about his own marginal cost from the price.

This example is a variant of the Vives (2011) model, but with multiple types, perfect competition, and different parametric assumptions (risk aversion instead of quadratic costs). As in Vives (2011), θ_i lends itself to several interpretations, for instance as an ex post pollution levy that depends on the production technology. $\|$

Example 2.2 There are N goods (outputs) all of which are produced from the same input or resource using a constant returns to scale technology. For good i, one unit of the input yields b_i units of the good. The demand for the good is perfectly elastic at price v_i. Thus the revenue per unit of input is $\theta_i := b_i v_i$. There is a continuum of risk-averse producers of good i each of whom collects private information about θ_i before deciding how much of the input to buy. We can think of this as information about either b_i (information about a productivity shock) or v_i (information about a demand shock), with the other parameter taken to be fixed. The supply of the input is perfectly inelastic. The wealth of producer n of type i is given by (1), where p is the price of the input and q_{in} is the amount bought. The equilibrium price of the input is given by (7). It aggregates (imperfectly) the information of different producers on the stochastic return from using the input. ‖

3 Information Acquisition

We now endogenize the choice of precision. Agent in pays the cost $\mathscr{C}_i(\tau_{in})$ for a signal of precision τ_{in}. We assume that the function $\mathscr{C}_i$ takes the form $\mathscr{C}_i(\tau_{in}) := \alpha_i C_i(\tau_{in})$, for some $\alpha_i \in [\underline{\alpha}, \bar{\alpha}]$, $\bar{\alpha} > \underline{\alpha} > 0$. Let $\alpha := (\alpha_i)_{i=1}^N$ and let $\mathcal{A}$ denote the N-fold Cartesian product of $[\underline{\alpha}, \bar{\alpha}]$. We specify cost functions in this way as we will be interested in comparative statics with respect to $\alpha \in \mathcal{A}$. We impose the following conditions on C_i:

The function $C_i : [0, \infty) \to [0, \infty)$ is twice-differentiable and satisfies

i. $C_i(0) = 0$;

ii. $C_i'(0) = 0$, and $C_i'(x) > 0$ for $x > 0$;

iii. $C_i'' > 0$.

In particular, we assume that there are no fixed costs and that obtaining a small amount of information is cheap. This ensures that each agent acquires some information.

It is convenient to use the following monotonic transformation of ex ante expected utility:

$$\mathcal{U}_{in} := \big(E[\exp(-rW_{in})]\big)^{-2}.$$

From Lemma 6.1 in Rahi and Zigrand (2018), we have:

Lemma 3.1 (Utilities) *For a given vector of precisions $\tau \in \mathbb{R}^N_{++}$, agent in's utility is*

$$\mathcal{U}_{in} = \exp\big[-2r\alpha_i C_i(\tau_{in})\big]\,(\tau_{in} + \tau_{\theta_i|p})\operatorname{Var}(\theta_i - p).$$

This is the indirect utility of agent in for an exogenously specified precision vector τ, and the REE price function associated with this τ. It depends on the agent's cost of acquiring information, through the term $\exp[-2r\alpha_i C_i(\tau_{in})]$, on how much he learns about his valuation, given by $[\operatorname{Var}(\theta_i|s_{in}, p)]^{-1} = \tau_{in} + \tau_{\theta_i|p}$, and on $\operatorname{Var}(\theta_i - p)$, which captures his "gains from trade". We defer a discussion of gains from trade to Section 5. They play no role in the agent's choice of precision, which is governed solely by the tradeoff between learning and the cost of information.

Maximizing $\mathcal{U}_{in}$ with respect to τ_{in}, we obtain:

Lemma 3.2 (Optimal precisions) *Agent in's optimal precision is the unique solution τ_{in} to*

$$2r\alpha_i C_i'(\tau_{in})(\tau_{in} + \tau_{\theta_i|p}) = 1. \tag{9}$$

This solution is positive and the same for all n.

Equation (9) follows directly from the first-order condition for the agent's maximization problem. We write the solution as $\tau_{in}(\tau)$ to indicate the dependence on τ. In equilibrium $\tau_i = \tau_{in}(\tau)$ for all i, so that all agents of type i have the same utility, which we denote by $\mathcal{U}_i$:

$$\mathcal{U}_i = \exp\left[-2r\alpha_i C_i(\tau_i)\right] (\tau_i + \tau_{\theta_i|p}) \operatorname{Var}(\theta_i - p). \tag{10}$$

Moreover, from (9),

$$2r\alpha_i C_i'(\tau_i)(\tau_i + \tau_{\theta_i|p}) = 1. \tag{11}$$

From (6), we see that

$$\tau_{\theta_i|p} = \tau_\theta[1 - \mathcal{V}_i(\tau)]^{-1}, \tag{12}$$

where $\tau_\theta = \tau_{\theta_i}$, assumed to be the same for all i, and $\mathcal{V}_i(\tau)$ is given by (8). Substituting this expression into (11) gives us:

Lemma 3.3 (Equilibrium precisions) *A vector $\tau \in \mathbb{R}^N_{++}$ is an equilibrium vector of precisions if and only if it is a solution to the following system of equations:*

$$2r\alpha_i C_i'(\tau_i)\left[\tau_i + \tau_\theta[1 - \mathcal{V}_i(\tau)]^{-1}\right] = 1, \qquad i = 1, \ldots, N. \tag{13}$$

We are now ready to characterize the equilibrium allocation of private information, in particular the relationship between the cost of information production, precision choice, and price informativeness. We study the two-type case in the next section, deferring the general case to later sections.

4 Equilibrium: The Two-Type Case

Suppose $N = 2$ and let ρ be the correlation between the valuations of the two types. From Proposition 2.2, for any given precisions τ_1 and τ_2 there is a unique linear equilibrium price function,

$$p = \mu[\tau_1\theta_1 + \tau_2\theta_2],$$

and price informativeness for the two types is given by

$$\mathcal{V}_1 = \frac{(\tau_1 + \rho\tau_2)^2}{\tau_1^2 + \tau_2^2 + 2\rho\tau_1\tau_2}, \qquad \text{and} \qquad \mathcal{V}_2 = \frac{(\rho\tau_1 + \tau_2)^2}{\tau_1^2 + \tau_2^2 + 2\rho\tau_1\tau_2}. \tag{14}$$

Price informativeness depends on (τ_1, τ_2) only through the ratio τ_1/τ_2. Differentiating with respect to τ_1/τ_2, we get the following result:

Proposition 4.1 (Learning externalities: two types) *Suppose $N = 2$ and $\rho \geq 0$. Then*

$$\frac{\partial \mathcal{V}_1}{\partial(\tau_1/\tau_2)} > 0 \qquad \text{and} \qquad \frac{\partial \mathcal{V}_2}{\partial(\tau_1/\tau_2)} < 0. \tag{15}$$

If ρ is nonnegative, agents of type 1 learn more from the price the greater is τ_1/τ_2, while the opposite is true for type 2 agents. A change in (τ_1, τ_2) either leaves price informativeness unchanged for both types (if τ_1/τ_2 is unchanged), or price informativeness moves in opposite directions for the two types. There are negative learning externalities across types: if agents of one type gather more information, price informativeness goes down for the other type. This property underlies the results in this section, and also the inefficiency result in the next section (Proposition 5.4). The assumption that $\rho \geq 0$ is sufficient for negative learning externalities but it is not necessary. We will provide a weaker condition, which is necessary and sufficient, in Section 6.

The discussion so far applies for any exogenously given precisions τ_1 and τ_2. We now characterize an equilibrium (τ_1, τ_2).

Proposition 4.2 (Equilibrium precisions: two types) *Suppose $N = 2$ and $\rho \geq 0$. Then there is a unique equilibrium $(\tau_1, \tau_2) \in \mathbb{R}^2_{++}$. If the functions C_1 and C_2 are the same, the equilibrium (τ_1, τ_2) satisfies the following properties:*

i. $\alpha_1 = \alpha_2 \;\Leftrightarrow\; \tau_1 = \tau_2 \;\Leftrightarrow\; \mathcal{V}_1 = \mathcal{V}_2$;

ii. $\alpha_1 > \alpha_2 \;\Leftrightarrow\; \tau_1 < \tau_2 \;\Leftrightarrow\; \mathcal{V}_1 < \mathcal{V}_2$.

The equivalence between the statements about precisions and price informativeness in (i) and (ii) follows from (14). The equivalence of these with the statement about the cost parameters α_1 and α_2 follows from (13), using the assumption that the functions C_1 and C_2 are the same. Thus if the two types have the same cost function, they gather the same amount of information and learn the same amount from the equilibrium price. If, on the other hand, the cost function of one type is uniformly higher than that of the other, the higher cost type gathers less information and, as a consequence, the price is less informative about the valuation of this type.

Next, we study the comparative statics of precision choice and price informativeness with respect to the cost parameters. We consider a change in α_1; comparative statics with respect to α_2 are analogous. For this result we do not need to assume that C_1 and C_2 are the same.

Proposition 4.3 (Comparative statics: two types) *Suppose $N = 2$ and $\rho \geq 0$. Let $\tau(\alpha_1, \alpha_2)$ be the equilibrium vector of precisions for cost parameters (α_1, α_2). We have the following comparative statics with respect to α_1, $\alpha_1 \in (\underline{\alpha}, \bar{\alpha})$:*

i.

$$\frac{\partial \tau_1}{\partial \alpha_1} < 0, \qquad \frac{\partial \tau_2}{\partial \alpha_1} < 0, \qquad \text{and} \qquad \frac{\partial(\tau_1/\tau_2)}{\partial \alpha_1} < 0;$$

ii.
$$\frac{\partial \mathcal{V}_1}{\partial \alpha_1} < 0 \qquad \textit{and} \qquad \frac{\partial \mathcal{V}_2}{\partial \alpha_1} > 0.$$

Consider a decrease in α_1. This represents a downward shift in the cost of information for type 1. In response, type 1 agents increase τ_1. The resulting price function is less informative for type 2 agents, who react by increasing τ_2. But this reduces price informativeness for type 1 agents, inducing them to increase τ_1 even further. While both types accumulate more information, the impact on type 1 is greater, due to the lower cost borne by this type. This means that τ_1/τ_2 is higher, and consequently price informativeness is higher for type 1 and lower for type 2 (from (15)).

Proposition 4.3 generalizes to the case of arbitrarily many types. This requires a careful, and somewhat technical, analysis of strategic complementarities, however. So we present our welfare analysis first.

5 Welfare

The plan for this section is to begin with a general analysis, for arbitrary N, of how welfare depends on the vector of precisions in the neighborhood of an equilibrium, followed by an inefficiency result for the two-type case. We will need some more notation. Consider a function $f : X \to \mathbb{R}$, where X is an open subset of $\mathbb{R}^n$. We denote by $\partial_z f(x)$ the directional derivative of f at x in the direction $z \in \mathbb{R}^n$, i.e.

$$\partial_z f(x) := \sum_k z_k \frac{\partial f}{\partial x_k}(x).$$

We say that $A \propto B$ if A and B have the same sign ($A = \lambda B$, for some $\lambda > 0$).

Given an equilibrium vector of precisions τ, we investigate the welfare effects that arise when we perturb this vector, and thereby perturb the REE associated with it (for any given vector of precisions there is a unique linear REE, by Proposition 2.2). In other words, a hypothetical planner chooses the signal precision for each type, after which the market mechanism takes over. The cost functions $\{\mathscr{C}_i\}_{i=1}^N$ remain unaltered in this exercise. Each agent pays the cost associated with the precision assigned to him by the planner.

We define the gains from trade for type i as

$$G_i := \frac{\mathrm{Var}(\theta_i - p)}{\mathrm{Var}(\theta_i)}.$$

Agents of type i have more profitable trading opportunities the greater the distance between their own valuation θ_i and the overall market valuation, given by the equilibrium price.[7] We can decompose the effect of a local change in τ on the welfare of

[7] In our model, the distance between two random variables is measured by the variance of the difference between these random variables.

type i into two components, one arising from a change in price informativeness $\mathcal{V}_i$ (the *learning effect*) and the other from a change in G_i (the *gains from trade effect*):

Lemma 5.1 (Welfare effects) *At an equilibrium τ, we have*

$$\partial_z \log \mathcal{U}_i(\tau) = \underbrace{\frac{\tau_{\theta_i|p}}{(1-\mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})}\partial_z \mathcal{V}_i(\tau)}_{\text{learning effect}} \quad + \quad \underbrace{\frac{1}{G_i}\partial_z G_i(\tau)}_{\text{gains from trade effect}}.$$

All else equal, agents are better off if they are better informed. They are also better off if they can reap higher gains from trade. The sign of the learning effect cannot be the same for all types:

Lemma 5.2 (Learning effects) *Consider a vector of precisions $\tau \in \mathbb{R}^N_{++}$ such that $R_i^\top \tau > 0$ for all i. Then $\partial_z \mathcal{V}_i(\tau) \propto \partial_z \mathcal{V}_j(\tau)$, for all i, j, if and only if $\partial_z \mathcal{V}_k(\tau) = 0$, for all k.*

In other words, a local change in τ cannot increase price informativeness for all types, nor can it reduce price informativeness for all. The vector τ need not be an equilibrium; the result applies for any $\tau \in \mathbb{R}^N_{++}$ satisfying $R_i^\top \tau > 0$ for all i. We will have more to say about this condition in the next section. For now we just note that it holds if $\rho_{ij} \geq 0$ for all i, j.

Returning to Lemma 5.1, the key question is how the learning and gains from trade effects interact. Indeed, there is a fundamental tension between them. If the equilibrium price tracks the valuation of an agent closely, it will reveal more information about that valuation. But the agent has more to gain from trade the further his valuation is from the price. To take a stark example, suppose $p = \theta_i$. Then price informativeness is maximal for type i agents ($\mathcal{V}_i = 1$), but there are no gains from trade for these agents ($G_i = 0$); their optimal trade is zero. Such a price function cannot arise in our model, of course (the price does not fully reveal θ_i for any i, and gains from trade are positive for all i), but it serves to illustrate the tradeoff between learning from prices and gains from trade.

A useful benchmark for investigating this tradeoff is a symmetric economy, by which we mean an economy in which cost functions and pairwise correlations are the same across types ($\mathscr{C}_i$ is the same for all i, and $\rho_{ij} = \rho$ for all $i \neq j$).

Proposition 5.3 (Symmetric economy) *Consider a symmetric economy with correlation parameter $\rho \geq 0$. Then there is a unique equilibrium τ with the following properties:*

i. $\tau_i = \tau_j$*, for all i, j;*

ii. $\partial_z G_i(\tau) = -\partial_z \mathcal{V}_i(\tau)$ *and* $\partial_z \mathcal{U}_i(\tau) \propto -\partial_z \mathcal{V}_i(\tau)$*, for all i.*

At the equilibrium of a symmetric economy, the learning and gains from trade effects are collinear but of the opposite sign,[8] with the latter dominating the first. A local

[8] An alternative way to express this relationship is $\nabla G_i(\tau) = -\nabla \mathcal{V}_i(\tau)$, where ∇ denotes the gradient vector with respect to τ.

change in τ makes type i agents better off if and only if these agents learn *less* from the price. This also means that at least one type must be worse off since price informativeness cannot go down for all types (by Lemma 5.2). In other words, different types exert externalities on each other in an exactly offsetting way, so that the equilibrium allocation of private information is locally Pareto efficient.[9]

The welfare changes described in Proposition 5.3 are reminiscent of the so-called Hirshleifer effect, insofar as better informed agents are worse off. However, in the symmetric economy considered here, this effect does not go in the same direction for all types. Indeed, a local change in τ either has no welfare effect, or the welfare effects are of opposite sign for at least two types.

One way to interpret the local efficiency result for a symmetric economy is that the planner has only $N-1$ tools to control the N-dimensional objective $\{\mathcal{U}_i\}_{i=1}^{N}$. Locally, utility depends only on price informativeness (the gains from trade effect being collinear with the learning effect), which is homogeneous of degree zero in τ. While the planner is free to choose any τ in a neighborhood of the equilibrium, scaling this vector has no effect on the welfare of any agent. The aggregate amount of information $\sum_i \tau_i$ is welfare-neutral; only the relative values of the τ_i's matter.

Once we depart from symmetry, however, the possibility of a local (and hence also global) Pareto improvement arises. In a non-symmetric economy, the planner has an additional tool, the amount of information $\sum_i \tau_i$, which affects the gains from trade for each type. Our next result shows that a Pareto improvement can always be found in the non-symmetric two-type case (with $\tau_1 \neq \tau_2$; the assumption in the proposition that $\tau_1 < \tau_2$ is without loss of generality):

Proposition 5.4 (Welfare) *Suppose $N = 2$ and $\rho \geq 0$. Consider an equilibrium at which $\tau_1 < \tau_2$. Then there exists a strict local Pareto improvement which entails an increase in both $\tau_1 + \tau_2$ and $\tau_2/(\tau_1 + \tau_2)$.*

By Proposition 4.2, there is a unique equilibrium. For ease of interpretation, let us assume that $C_1 = C_2$. Then we have $\tau_1 < \tau_2$ at the equilibrium if and only if $\alpha_1 > \alpha_2$. Thus type 2 can unambiguously be identified as the low-cost type. Proposition 5.4 says that we can always find a local Pareto improvement that entails more information production in the aggregate, with a higher proportion of that information produced by the low-cost type.

Under the assumption of nonnegative ρ, there are negative learning externalities across types. If one type acquires more information, the other type has an incentive to acquire more information as well. Conversely, if one type cuts back on information production, the other type will also want to do that. This suggests that, in equilibrium, information acquisition decisions are more closely aligned than is socially optimal. Proposition 5.4 tells us that this is indeed true: a Pareto improvement entails a greater differentiation between the two types in terms of the proportion of

[9] However, it is not Pareto efficient in general. It is easy to check numerically that, for some parameter values, a large (non-local) change in τ can make all types better off.

information they collect, with the low-cost type acquiring a greater proportion of the total than in equilibrium.[10]

It is instructive to compare this inefficiency result with the one in Rahi and Zigrand (2018)(RZ). In RZ, all private signals have the same exogenously specified precision. Agents of type i can either pay a fixed cost c_i for their signal, or remain uninformed; the highest cost is borne by type N. In equilibrium, the proportion of informed agents of type i is λ_i. RZ consider a corner equilibrium in which $\lambda_i = 1$ for all $i \neq N$, and perturb λ_N while keeping the remaining λ_i's fixed at 1. They show that, if private signals are sufficiently noisy, a reduction in λ_N is Pareto improving. This suggests that there is overproduction of information in equilibrium, while Proposition 5.4 points to the opposite conclusion. The two results should be considered complementary rather than contradictory, however, since they apply in different settings (binary vs continuous choice of information). The results do share one common feature — the proposed Pareto improvement involves pushing the information choices of different types further apart (increasing $\lambda_i - \lambda_N$, $i \neq N$, in RZ's result, and $(\tau_2 - \tau_1)/(\tau_1 + \tau_2)$ in Proposition 5.4).

6 Strategic Complementarities

In this section we lay the groundwork for a general analysis of equilibrium with arbitrarily many types. Recall that $\tau_{in}(\tau)$ is the optimal precision for agent in, given the precision choices of all other agents.

Definition 6.1 *The economy exhibits strategic complementarities in information acquisition if $\partial\tau_{in}(\tau)/\partial\tau_j > 0$, for all $i \neq j$.*

Henceforth, we will drop the modifier "in information acquisition" for the sake of brevity. There are strategic complementarities if the optimal precision choice for any agent is increasing in the precision choices of agents of other types. This is a key property that underlies (most of) our results. The following lemma is immediate from (9) and (12):

Lemma 6.1 (Strategic complementarities) *The economy exhibits strategic complementarities if and only if $\partial\mathcal{V}_i/\partial\tau_j < 0$ for all $i \neq j$.*

Thus strategic complementarities are equivalent to negative learning externalities across types. The assumption of nonnegative ρ is sufficient for strategic complementarities in the two-type case (Proposition 4.1). We will generalize this condition to

[10] Since there are negative learning externalities, one might surmise that agents collect too much information in equilibrium, and welfare gains can be realized by curbing this activity. This reasoning is flawed on several counts. First, scaling down (τ_1, τ_2) has no effect on price informativeness. Second, a change in the ratio τ_1/τ_2 leads to price informativeness going up for one type and down for the other. Third, higher price informativeness does not necessarily make agents better off; indeed, in the symmetric case, agents who learn more from prices are worse off (Proposition 5.3).

allow for negative correlation, and provide conditions for three or more types. These conditions involve restrictions on the correlation matrix R that get more stringent as the number of types increases. Roughly speaking, strategic complementarities arise if the correlations between types exceed a lower bound $\underline{\rho}$ (which is negative), and are close to each other.

A change in the precision of type i's signal affects price informativeness for all types. We refer to $\partial \mathcal{V}_i/\partial \tau_i$ as an "own-effect" and to $\partial \mathcal{V}_i/\partial \tau_j$ for $i \neq j$ as a "cross-effect". Lemma 6.1 says that the economy exhibits strategic complementarities if and only if all cross-effects are negative. Since $\mathcal{V}_i$ is homogeneous of degree zero in τ, we have

$$\sum_j \tau_j \frac{\partial \mathcal{V}_i}{\partial \tau_j} = 0,$$

by Euler's theorem. It follows that, in an economy that exhibits strategic complementarities, $\partial \mathcal{V}_i/\partial \tau_i > 0$ for all i. In other words, if all cross-effects are negative, then all own-effects must be positive. For $N = 2$, the converse is true as well.

We will analyze these effects shortly. But first we need to take a closer look at the relationship between correlations and precisions.

Lemma 6.2 *Suppose the economy exhibits strategic complementarities for any vector of precisions $\tau \in \mathbb{R}^N_{++}$. Then, if $N = 2$, we must have $\rho \geq 0$. If $N \geq 3$, we must have $\rho_{ij} = 0$ for all $i \neq j$.*

In other words, if we insist on strategic complementarities for any (positive) vector of precisions, we cannot improve upon the nonnegative correlation condition that we used in our earlier results for the two-type case. Moreover, if there are three or more types, we have to reconcile ourselves to the rather stringent requirement of uncorrelated valuations. Lemma 6.2 is essentially a consequence of having to allow precisions that are arbitrarily close to zero. Fortunately, by imposing a mild condition on R that bounds it away from singularity, we can ensure that an equilibrium vector of precisions is bounded away from zero. We can then provide additional conditions on R such that the economy exhibits strategic complementarities. These conditions are much weaker than those in Lemma 6.2.

Let $\mathcal{R}$ be the set of N-dimensional positive definite correlation matrices, an open convex set. The closure of $\mathcal{R}$, denoted by $\mathrm{cl}(\mathcal{R})$, is the set of positive semidefinite correlation matrices. The boundary of $\mathcal{R}$ is the set of correlation matrices in $\mathrm{cl}(\mathcal{R})$ with zero determinant. Let $\mathcal{R}_\eta$ be the subset of $\mathcal{R}$ consisting of correlation matrices R for which $\det(R) \geq \eta$ for some $\eta \in (0, 1)$. As η goes to zero, $\mathcal{R}_\eta$ approaches $\mathcal{R}$.

Lemma 6.3 (Precision bounds) *Suppose $R \in \mathcal{R}_\eta$ for some $\eta \in (0, 1)$. Then there are positive scalars $\underline{\tau}$ and $\bar{\tau}$, that are independent of $R \in \mathcal{R}_\eta$ and $\alpha \in \mathcal{A}$, such that $\tau_i \in [\underline{\tau}, \bar{\tau}]$ for all i. We have $\lim_{\eta \to 0} \underline{\tau} = 0$. Furthermore, if C_i is the same for all types, then $\lim_{\tau_\theta \to 0}(\underline{\tau}/\bar{\tau}) > \underline{\alpha}/\bar{\alpha}$.*

Let $\mathcal{T} := \times_{i=1}^N [\underline{\tau}, \bar{\tau}]$, the N-fold Cartesian product of the interval $[\underline{\tau}, \bar{\tau}]$. Lemma 6.3 tells us that if $R \in \mathcal{R}_\eta$, then $\tau \in \mathcal{T}$. If η is close to zero, so is the lower bound $\underline{\tau}$.

This is essentially a Grossman-Stiglitz paradox in the limit: an agent has very little incentive to gather information if his valuation is almost perfectly correlated with that of some other agent who does acquire information.

Henceforth, when we invoke the assumption that $R \in \mathcal{R}_\eta$, we presuppose some fixed level of $\eta \in (0,1)$. If $\rho_{ij} = \rho$ for all $i \neq j$, R is positive definite if and only if

$$-\frac{1}{N-1} < \rho < 1 \tag{16}$$

(see Rahi and Zigrand (2018), Lemma 6.5); in this case $R \in \mathcal{R}_\eta$ amounts to the assumption that $\rho \in [\underline{\kappa}, \bar{\kappa}]$, for some $\underline{\kappa}, \bar{\kappa}$ satisfying $-(N-1)^{-1} < \underline{\kappa} < \bar{\kappa} < 1$. Specializing further to the case of $N = 2$, the condition $R \in \mathcal{R}_\eta$ is equivalent to $|\rho| \leq \sqrt{1-\eta}$.

With τ restricted to $\mathcal{T}$, we look for conditions on R under which the economy exhibits strategic complementarities, or equivalently negative cross-effects.[11] A necessary condition is positive own-effects. Own-effects are positive on $\mathcal{T}$ if correlations exceed a threshold level $\underline{\rho}$ given by

$$\underline{\rho} := -\frac{\underline{\tau}}{(N-1)\bar{\tau}}.$$

Lemma 6.4 (Positive own-effects) *$\partial \mathcal{V}_i / \partial \tau_i > 0$ if and only if $R_i^\top \tau > 0$. Furthermore, if $R \in \mathcal{R}_\eta$, a sufficient condition for $R_i^\top \tau > 0$ for all i and all $\tau \in \mathcal{T}$ is $\rho_{ij} > \underline{\rho}$ for all i, j.*

The lower bound $\underline{\rho}$ satisfies

$$-\frac{1}{N-1} < \underline{\rho} < 0.$$

Note that if all pairwise correlations are the same, this common value must be greater than $-(N-1)^{-1}$ for R to be positive definite (condition (16)). Also, $\underline{\rho}$ converges to zero as η goes to zero (due to the corresponding limiting property of $\underline{\tau}$ noted in Lemma 6.3).

Positive own-effects imply that the coefficient μ of the price function $p = \mu \tau^\top \theta$ is positive:

Lemma 6.5 (Price function) *Consider a vector of precisions $\tau \in \mathbb{R}^N_{++}$ such that $R_i^\top \tau > 0$ for all i. Then the coefficient μ of the price function is positive.*

This is immediate from the expression for μ, given by (5). Moreover, since $\mathrm{Cov}(\theta_i, p) = \mu R_i^\top \tau$, positive own-effects imply that the valuation of any type is positively correlated with the REE price function. Another consequence of positive own-effects, from Lemma 5.2, is that price informativeness cannot change in the same direction for all types when we perturb τ.

[11] Even though we restrict τ to lie in the compact set $\mathcal{T}$, the derivative $\partial \mathcal{V}_i / \partial \tau_j$ is well-defined on an open set containing $\mathcal{T}$, and hence on the boundary of $\mathcal{T}$.

To summarize the discussion so far, strategic complementarities are equivalent to negative cross-effects (if one type gathers more information, price informativeness goes down for all other types). A necessary condition for this is positive own-effects (if a given type gathers more information, price informativeness goes up for that type). Positive own-effects imply that the coefficient μ of the price function is positive. They also imply that a local change in precisions leads to price informativeness going up for some types and down for others (or remaining unchanged for all types, as in the case of the precision vector τ being scaled up or down).

As we noted earlier, in the two-type case positive own-effects are not only necessary for strategic complementarities, but also sufficient (due to Euler's theorem). By Lemma 6.4, the condition that all correlations exceed $\underline{\rho}$ suffices for positive own-effects on $\mathcal{T}$. In the two-type case, this lower bound condition is in fact necessary:

Proposition 6.6 (Complementarities: two types) *Suppose $N = 2$ and $|\rho| \leq \sqrt{1-\eta}$, for some $\eta \in (0,1)$. Then the economy exhibits strategic complementarities on $\mathcal{T}$ if and only if $\rho > \underline{\rho}$.*

Propositions 4.1, 4.2, 4.3 and 5.4 rely on strategic complementarities, for which $\rho \geq 0$ is a sufficient condition. In light of Proposition 6.6, we can generalize these results by replacing the nonnegative ρ assumption with the weaker assumption that $|\rho| \leq \sqrt{1-\eta}$ for some $\eta \in (0,1)$, and $\rho > \underline{\rho}$. The latter condition reduces to $\rho \geq 0$ if we take η to be arbitrarily small, since $\lim_{\eta \to 0} \underline{\rho} = 0$.[12]

When there are three or more types, we need additional restrictions on R to ensure that cross-effects are negative. Recall that, under the assumption that $R \in \mathcal{R}_\eta$, we have $\tau_i \in [\underline{\tau}, \bar{\tau}]$ for all i. Let

$$\delta := \frac{\underline{\tau}}{\bar{\tau}}.$$

The parameter δ lies in $(0,1)$.

Proposition 6.7 (Complementarities: three or more types) *Suppose $N \geq 3$ and $R \in \mathcal{R}_\eta$. Then the economy exhibits strategic complementarities on $\mathcal{T}$ if any of the following conditions is satisfied:*

i. For all $i \neq j$,

$$\underline{\rho} < \rho_{ij} \leq \frac{\underline{\rho}^2}{\underline{\rho}^2 + (1-\underline{\rho}^2)(N-2)^2}; \tag{17}$$

ii. $\delta \geq 1/2$ and, for all $i \neq j$,

$$\check{\rho} \leq \rho_{ij} \leq 1 - \frac{(1-\check{\rho}^2)(N-2)^2}{(1+\check{\rho})\delta^2 + 2\check{\rho}(N-2)\delta + (N-2)^2}, \tag{18}$$

for some $\check{\rho} \in [0,1)$;

[12] We can also relax the assumption of nonnegative ρ in Proposition 5.3, assuming instead that $R \in \mathcal{R}_\eta$ and $\rho > \underline{\rho}$, which implies that own-effects are positive on $\mathcal{T}$. We do not need strategic complementarities for this result.

iii. For all $i \neq j$, $\rho_{ij} = \rho > \underline{\rho}$. Furthermore, if $\delta < 1/2$, then

$$\rho \leq \frac{\delta^2}{(1-2\delta)(N-1)}. \tag{19}$$

Given $R \in \mathcal{R}_\eta$, the condition that $\rho_{ij} > \underline{\rho}$ for all i, j ensures that all own-effects are positive on $\mathcal{T}$, by Lemma 6.4. For cross-effects to be negative on $\mathcal{T}$, low or negative correlations suffice (condition (i)). An alternative condition is that $\delta \geq 1/2$, and all correlations are nonnegative and close to each other (condition (ii)); we provide examples below. Note that the upper bound in (18) is decreasing in N. If $\rho_{ij} = \rho$ for all $i \neq j$, two cases arise. If $\delta \geq 1/2$, there is no further restriction. If $\delta < 1/2$, we require an upper bound on ρ, given by (19).

The conditions in Proposition 6.7 involve the precision bounds $\underline{\tau}$ and $\bar{\tau}$, since $\delta = \underline{\tau}/\bar{\tau}$ and $\underline{\rho} = -(N-1)^{-1}\underline{\tau}/\bar{\tau}$. This is to be expected since these are conditions for negative cross-effects on $\mathcal{T} := \times_{i=1}^{N}[\underline{\tau}, \bar{\tau}]$. The precision bounds depend on the exogenous primitives $r, \underline{\alpha}, \bar{\alpha}, \{C_i\}_{i=1}^N, \tau_\theta$, and η (see the proof of Lemma 6.3). By Lemma 6.3, if C_i is the same for all types, then $\lim_{\tau_\theta \to 0} \delta > \underline{\alpha}/\bar{\alpha}$. Therefore, $\delta \geq 1/2$ if $\underline{\alpha}/\bar{\alpha} \geq 1/2$, the functions $\{C_i\}$ are not very dissimilar across types, and the uncertainty regarding valuations is large (τ_θ is small).

While there is no explicit solution for δ or $\underline{\rho}$ in terms of the primitives, we can check if the conditions in the proposition are satisfied for a particular economy, as we illustrate in the following example:

Example 6.1 For this example, we need to refer to the proof of Lemma 6.3, in particular to equations (35)–(37).

Suppose $r = 1$, $C_i(\tau_i) = (1/12)\tau_i^2$ for all i, and $\underline{\alpha} = 3$. Suppose further that $\tau_\theta = 1 - \bar{\mathcal{V}}$, where $\bar{\mathcal{V}}$ is the maximal price informativeness for any type, as defined by (36).[13] From (35) and (37), the bounds for τ_i are given by

$$\bar{\tau} = 1, \qquad \underline{\tau} = \frac{1}{2}\left[-1 + \sqrt{1 + \frac{12}{\bar{\alpha}}}\right].$$

Therefore, $\delta = \underline{\tau}/\bar{\tau} \geq 1/2$ if and only if $\bar{\alpha} \in (\underline{\alpha}, 4] = (3, 4]$.

Now suppose $N = 3$. If we take $\bar{\alpha} = 4$, we get $\delta = 1/2$ and $\underline{\rho} = -\delta/2 = -1/4$. Condition (17) reduces to

$$\underline{\rho} < \rho_{ij} \leq \underline{\rho}^2,$$

while condition (18) becomes

$$0 \leq \check{\rho} \leq \rho_{ij} \leq 0.2 + 0.8\check{\rho}.$$

The economy exhibits strategic complementarities if $R \in \mathcal{R}_\eta$ and any of the following conditions holds for all $i \neq j$:

[13] This choice of τ_θ is convenient since it implies that $\underline{\tau}$, and hence $\underline{\rho}$, does not depend on η. It does mean, however, that τ_θ depends on η, with $\lim_{\eta \to 0} \tau_\theta = 0$.

i. $\rho_{ij} \in (-0.25, 0.0625]$;

ii. $\rho_{ij} \in [0.25, 0.4]$, or $\rho_{ij} \in [0.75, 0.8]$;

iii. $\rho_{ij} = \rho \in (-0.25, 1)$.

In (ii), we have chosen two values of $\check{\rho}$, 0.25 and 0.75, for illustration. The restriction $R \in \mathcal{R}_\eta$ can be written as $\det(R) \geq \eta$, or equivalently

$$\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2\rho_{12}\rho_{13}\rho_{23} \leq 1 - \eta. \tag{20}$$

Suppose we choose $\eta = 0.1$. Then it is easy to check that any R satisfying condition (i) or (ii) also satisfies (20). If $\rho_{ij} = \rho$ for $i \neq j$, (20) is satisfied if $\rho \in [-0.47, 0.8]$; for comparison, the positive definiteness condition is $\rho \in (-0.5, 1)$, from (16). Thus, while any R satisfying (iii) is positive definite, imposing (20) restricts ρ to the interval $(-0.25, 0.8]$. $\|$

7 Equilibrium: The General Case

In this section we characterize the equilibrium allocation of private information for arbitrarily many types, providing conditions for existence of equilibrium, and studying comparative statics with respect to the cost parameters $\alpha \in \mathcal{A}$. Our comparative statics results generalize those that we obtained earlier for the two-type case (Proposition 4.3).

We say that an equilibrium τ (which must be in $\mathbb{R}^N_{++}$) satisfies the monotone comparative statics (MCS) property if τ is decreasing in α, $\alpha \in \mathcal{A}$. It satisfies the strong MCS property if $\partial\tau_i/\partial\alpha_j < 0$, for all $\alpha_j \in (\underline{\alpha}, \bar{\alpha})$, and for all i, j. An equilibrium $\hat{\tau}$ is the highest equilibrium if $\hat{\tau} \geq \tau$ for any equilibrium τ. Similarly, an equilibrium $\check{\tau}$ is the lowest equilibrium if $\check{\tau} \leq \tau$ for any equilibrium τ.

Proposition 7.1 (Existence, comparative statics) *Suppose $R \in \mathcal{R}_\eta$ and the economy exhibits strategic complementarities on $\mathcal{T}$. Then there exists a highest equilibrium $\hat{\tau}$ and a lowest equilibrium $\check{\tau}$. Both $\hat{\tau}$ and $\check{\tau}$ satisfy the MCS property.*

The highest and lowest equilibrium may coincide, in which case there is a unique equilibrium (as in the two-type case; see Proposition 4.2). The MCS property is valid for any change of α in $\mathcal{A}$, not just for a local change. It relies on a theorem in Milgrom and Shannon (1994). In our model the MCS property can be strengthened to the strong MCS property:

Lemma 7.2 (Strong MCS) *Suppose $R \in \mathcal{R}_\eta$ and the economy exhibits strategic complementarities on $\mathcal{T}$. Then an equilibrium τ satisfies the MCS property if and only if it satisfies the strong MCS property.*

Thus $\partial\tau_i/\partial\alpha_j < 0$, for all $\alpha_j \in (\underline{\alpha}, \bar{\alpha})$, and for all i, j. A decrease in α_j induces all agents to collect more private information. Type j agents increase τ_j in response to the lower α_j. This makes prices less informative for all other types, who in turn gather more information. This feeds back into less informative prices for type j agents, causing them to increase τ_j further.

It is apparent from (13) that for types other than j a higher precision choice must be accompanied by a decrease in price informativeness — since there is no change in the cost function of these agents, they can only be induced to collect more private information if they learn less from the price. But price informativeness cannot decrease for all types (Lemma 5.2), so it must go up for type j. This is formalized in our last result (remember that the thought experiment here involves a decrease in α_j):

Proposition 7.3 (Comparative statics II) *Suppose $R \in \mathcal{R}_\eta$ and the economy exhibits strategic complementarities on $\mathcal{T}$. Then, for any equilibrium that satisfies the MCS property, we have $\partial\mathcal{V}_j/\partial\alpha_j < 0$ and $\partial\mathcal{V}_i/\partial\alpha_j > 0$, for all $\alpha_j \in (\underline{\alpha}, \bar{\alpha})$ and for all $i \neq j$.*

8 Concluding Remarks

In an economy with heterogeneous valuations, agents make inferences about their own valuation from the equilibrium price. Under natural conditions, more information acquisition by one type leads to lower price informativeness for all other types. One consequence of this externality is that a lower cost of information for one type induces all agents to produce more information.

Lower price informativeness tends to raise welfare. This is because gains from trade for an agent are higher the further away his valuation is from the equilibrium price. In general, an equilibrium allocation of private information is Pareto inefficient. In the case of two types who differ in their cost of information, we can always find a Pareto improvement that entails an increase in the aggregate amount of information, with a higher proportion produced by the low-cost type.

Our existence and comparative statics results are fairly general (for the linear-normal setting). But the welfare analysis is harder and leaves a number of questions unanswered. For the two-type case we describe a Pareto improvement in the neighborhood of an equilibrium. Characterizing the Pareto frontier remains an open question, however, as does the case of more than two types.

Appendix

In the proofs it will often be useful to work with relative precisions, given by

$$\mathring{\tau}_i := \frac{\tau_i}{\sum_k \tau_k},$$

for type i. We denote the vector of relative precisions by $\mathring{\tau} := (\mathring{\tau}_i)_{i=1}^N$.

Proof of Proposition 2.2 Conjecturing that the price function takes the form $p = a^\top \theta$ for some nonzero vector a, and that $\mathrm{Var}(\theta_i|p) > 0$ for all i, we showed in the main text that q_i is given by (4). Using the market-clearing condition $\sum_i q_i = 0$, we obtain the price function $p = \hat{\mu}\,\tau^\top\theta$, where $\hat{\mu} := (\sum_i \mu_i)^{-1}$. This verifies the conjecture, provided $\hat{\mu}$ is well-defined (i.e. finite) and nonzero. Assuming that these conditions are satisfied, we proceed to calculate $\hat{\mu}$. From (3),

$$\mu_i = \tau_i + \tau_{\theta_i|p}\left[1 - \hat{\mu}^{-1}\frac{R_i^\top\tau}{\tau^\top R\,\tau}\right].$$

Therefore,

$$\hat{\mu}^{-1} = \sum_i \mu_i = \sum_i (\tau_i + \tau_{\theta_i|p}) - \hat{\mu}^{-1}\sum_i \tau_{\theta_i|p}\frac{R_i^\top\tau}{\tau^\top R\,\tau}.$$

It follows that

$$\begin{aligned}
\hat{\mu} &= \frac{1 + \sum_k \tau_{\theta_k|p}\left[\frac{R_k^\top\tau}{\tau^\top R\,\tau}\right]}{\sum_j(\tau_j + \tau_{\theta_j|p})} \\
&= \frac{\sum_k(\tau_k + \tau_{\theta_k|p})\frac{R_k^\top\tau}{\tau^\top R\,\tau}}{\sum_j(\tau_j + \tau_{\theta_j|p})} \\
&= \sum_k \beta_k \frac{R_k^\top\tau}{\tau^\top R\,\tau},
\end{aligned}$$

which is equal to μ (defined in (5)). This expression is clearly finite. We do, however, need to assume that it is nonzero. If $\mu = 0$, there is no equilibrium; calculating q_i when $p \equiv 0$, we can easily verify that this price function does not clear markets.

The price function can also be obtained directly from Proposition 3.2 of Rahi and Zigrand (2018) (RZ), applying this result for the case in which all types are differentially informed and all agents of all types acquire information (in RZ's notation, we set $\sigma_{\eta_i}^2 = 0$, $\lambda_i = 1$, and $r_i = r$, for all i).[14] RZ show that this is the unique linear REE. In particular, there is no linear equilibrium in which the price is fully revealing for some type. Price informativeness $\mathcal{V}_i$ is the same as in RZ, with τ playing the role

[14] RZ's result requires the assumption that the coefficient μ of the price function (which is k in their notation) is nonzero, a point that they overlook.

of RZ's $\boldsymbol{\lambda}$. $\square$

Proof of Lemma 3.2 Equation (9) follows from the first-order condition:

$$\frac{\partial \mathcal{U}_{in}}{\partial \tau_{in}} = \exp\big[-2r\alpha_i C_i(\tau_{in})\big]\big[1 - 2r\alpha_i C_i'(\tau_{in})(\tau_{in} + \tau_{\theta_i|p})\big] \operatorname{Var}(\theta_i - p) = 0.$$

Since C' is increasing and $C'(0) = 0$, there is a unique solution τ_{in}^* to (9), and it is positive. Furthermore,

$$\frac{\partial^2 \mathcal{U}_{in}}{\partial \tau_{in}^2} \propto -C'(\tau_{in})\big[1 - 2r\alpha_i C_i'(\tau_{in})(\tau_{in} + \tau_{\theta_i|p})\big] - C_i''(\tau_{in})(\tau_{in} + \tau_{\theta_i|p}) - C_i'(\tau_{in}),$$

which is negative at τ_{in}^* since $C' \geq 0$ and $C'' > 0$ (the symbol $\propto$ means "has the same sign as"). Hence, τ_{in}^* is a local maximum. Indeed, it must be the global maximum because $\partial \mathcal{U}_{in}/\partial \tau_{in}$ is positive for $\tau_{in} \in [0, \tau_{in}^*)$ and negative for $\tau_{in} \in (\tau_{in}^*, \infty)$. $\square$

Proof of Proposition 4.2 Existence follows from Proposition 7.1 which we prove later. The argument for properties (i) and (ii) is in the main text. To show uniqueness, consider equilibria (τ_1, τ_2) and (τ_1', τ_2'), with corresponding price informativeness $(\mathcal{V}_1, \mathcal{V}_2)$ and $(\mathcal{V}_1', \mathcal{V}_2')$. Suppose $\tau_1/\tau_2 \neq \tau_1'/\tau_2'$; without loss of generality, suppose that $\tau_1/\tau_2 > \tau_1'/\tau_2'$. This gives us the following chain of implications (the subscripted number indicates the equations/inequalities from which the implication follows):

$$\frac{\tau_1}{\tau_2} > \frac{\tau_1'}{\tau_2'} \quad \Rightarrow_{(15)} \quad \mathcal{V}_1 > \mathcal{V}_1',\, \mathcal{V}_2 < \mathcal{V}_2' \quad \Rightarrow_{(13)} \quad \tau_1 < \tau_1',\, \tau_2 > \tau_2' \quad \Rightarrow \quad \frac{\tau_1}{\tau_2} < \frac{\tau_1'}{\tau_2'},$$

a contradiction. Hence,

$$\frac{\tau_1}{\tau_2} = \frac{\tau_1'}{\tau_2'} \quad \Rightarrow_{(14)} \quad \mathcal{V}_1 = \mathcal{V}_1',\, \mathcal{V}_2 = \mathcal{V}_2' \quad \Rightarrow_{(13)} \quad \tau_1 = \tau_1',\, \tau_2 = \tau_2',$$

thus proving uniqueness. $\square$

Proof of Proposition 4.3 Suppose $\partial\tau_2/\partial\alpha_1 \geq 0$. Then we have the following chain of implications (the subscripted number indicates the equations/inequalities from which the implication follows):

$$\frac{\partial \tau_2}{\partial \alpha_1} \geq 0 \quad \Rightarrow_{(13)} \quad \frac{\partial \mathcal{V}_2}{\partial \alpha_1} \leq 0 \quad \Rightarrow_{(15)} \quad \frac{\partial \mathcal{V}_1}{\partial \alpha_1} \geq 0 \quad \Rightarrow_{(13)} \quad \frac{\partial \tau_1}{\partial \alpha_1} < 0.$$

The first and last inequalities together imply that

$$\frac{\partial(\tau_1/\tau_2)}{\partial \alpha_1} < 0 \quad \Rightarrow_{(15)} \quad \frac{\partial \mathcal{V}_1}{\partial \alpha_1} < 0,$$

which contradicts the sign of $\partial\mathcal{V}_1/\partial\alpha_1$ in the previous chain. Hence $\partial\tau_2/\partial\alpha_1 < 0$, giving us the following chain:

$$\frac{\partial \tau_2}{\partial \alpha_1} < 0 \quad \Rightarrow_{(13)} \quad \frac{\partial \mathcal{V}_2}{\partial \alpha_1} > 0 \quad \Rightarrow_{(15)} \quad \frac{\partial \mathcal{V}_1}{\partial \alpha_1} < 0 \quad \Rightarrow_{(15)} \quad \frac{\partial(\tau_1/\tau_2)}{\partial \alpha_1} < 0.$$

The first and last inequalities in this chain imply that $\partial\tau_1/\partial\alpha_1 < 0$. $\square$

Proof of Lemma 5.1 The indirect utility of type i, for any $\tau \in \mathbb{R}^N_{++}$, is given by (10). Using the definition of G_i and taking logs, we have

$$\log \mathcal{U}_i = \log(\tau_i + \tau_{\theta_i|p}) + \log G_i - 2r\alpha_i C_i(\tau_i) - \log \tau_\theta.$$

Differentiating this expression with respect to τ_k, and using (11), we obtain (the indicator function $\mathbb{1}_{i=k}$ takes value 1 when $i = k$, and is 0 otherwise):

$$\begin{aligned}
\frac{\partial \log \mathcal{U}_i}{\partial \tau_k} &= (\tau_i + \tau_{\theta_i|p})^{-1}\left[\mathbb{1}_{i=k} + \frac{\partial \tau_{\theta_i|p}}{\partial \tau_k}\right] + G_i^{-1}\frac{\partial G_i}{\partial \tau_k} - 2r\alpha_i C_i'(\tau_i)\mathbb{1}_{i=k} \\
&= (\tau_i + \tau_{\theta_i|p})^{-1}\frac{\partial \tau_{\theta_i|p}}{\partial \tau_k} + G_i^{-1}\frac{\partial G_i}{\partial \tau_k}.
\end{aligned}$$

Recalling that $\tau_{\theta_i|p} = \tau_\theta(1-\mathcal{V}_i)^{-1}$, we have

$$\begin{aligned}
\frac{\partial \tau_{\theta_i|p}}{\partial \tau_k} &= \tau_\theta(1-\mathcal{V}_i)^{-2}\,\frac{\partial \mathcal{V}_i}{\partial \tau_k} \\
&= \tau_{\theta_i|p}(1-\mathcal{V}_i)^{-1}\,\frac{\partial \mathcal{V}_i}{\partial \tau_k}.
\end{aligned}$$

Hence,

$$\frac{\partial \log \mathcal{U}_i}{\partial \tau_k} = \left[\frac{\tau_{\theta_i|p}}{(1-\mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})}\right]\frac{\partial \mathcal{V}_i}{\partial \tau_k} + G_i^{-1}\frac{\partial G_i}{\partial \tau_k}.$$

The result follows. $\square$

Proof of Lemma 5.2 Differentiating (8), we obtain:

$$\begin{aligned}
\frac{\partial \mathcal{V}_i}{\partial \tau_j} &= \frac{2\rho_{ij}(\tau^\top R\,\tau)(R_i^\top \tau) - 2(R_i^\top \tau)^2(R_j^\top \tau)}{(\tau^\top R\,\tau)^2} \\
&= \frac{2}{\tau^\top R\,\tau}\left[\rho_{ij}R_i^\top \tau - \mathcal{V}_i(R_j^\top \tau)\right]. \qquad (21)
\end{aligned}$$

Hence we can write $\partial_z\mathcal{V}_i$ as follows:

$$\begin{aligned}
\partial_z\mathcal{V}_i &:= \sum_j z_j \frac{\partial \mathcal{V}_i}{\partial \tau_j} \\
&= \frac{2}{\tau^\top R\,\tau}\left[(R_i^\top \tau)(R_i^\top z) - \mathcal{V}_i(\tau^\top R\,z)\right] \\
&= \frac{2(R_i^\top \tau)(\tau^\top R\,z)}{\tau^\top R\,\tau}\left[\frac{R_i^\top z}{\tau^\top R\,z} - \frac{R_i^\top \tau}{\tau^\top R\,\tau}\right].
\end{aligned}$$

Invoking the assumption that $R_i^\top \tau > 0$ for all i, we have

$$\sum_i \frac{\tau_i}{R_i^\top \tau}(\partial_z\mathcal{V}_i) = 0.$$

The result follows. $\square$

Proof of Proposition 5.3 Consider a symmetric economy with cost function $\mathscr{C}$ for all types. From Lemma 3.3, τ is an equilibrium vector of precisions if and only if

$$2r\mathscr{C}'(\tau_i)\big[\tau_i + \tau_\theta(1-\mathcal{V}_i)^{-1}\big] = 1, \tag{22}$$

for all i. Also, since $\rho_{ij} = \rho$ for all $i \neq j$, price informativeness for type i (from (8)) is given by

$$\mathcal{V}_i = \frac{\big[(1-\rho)\tau_i + \rho\sum_k \tau_k\big]^2}{(1-\rho)\sum_k \tau_k^2 + \rho\big(\sum_k \tau_k\big)^2}. \tag{23}$$

We claim that at an equilibrium τ, $\tau_i = \tau_j$ for all i, j. Suppose not. Then $\tau_i > \tau_j$ for some i, j, and hence, from (23), $\mathcal{V}_i > \mathcal{V}_j$ (using the assumption that $\rho \geq 0$). But then (22) implies that $\tau_i < \tau_j$, a contradiction. This proves that τ_i is the same for all i, which we denote by τ^*. From (23), $\mathcal{V}_i$ is also the same for all i, and is given by

$$\mathcal{V}^* := \frac{1+\rho(N-1)}{N},$$

which does not depend on τ^*. From (22), τ^* is the unique solution to

$$2r\mathscr{C}'(\tau^*)\big[\tau^* + \tau_\theta(1-\mathcal{V}^*)^{-1}\big] = 1.$$

In order to prove statement (ii) of the proposition, we first compute G_i. We do this for an arbitrary $\tau \in \mathbb{R}^N_{++}$, as we will need the general expression for our later results. It is convenient to write the price function as $p = \xi\,\mathring{\tau}^\top\theta$, where $\xi := \mu\sum_k \tau_k$ (recall the $\mathring{\tau}$ is the vector of relative precisions, defined at the beginning of the Appendix). We have

$$\begin{aligned}\mathrm{Var}(\theta_i - p) &= \mathrm{Var}(\theta_i) + \mathrm{Var}(p) - 2\mathrm{Cov}(\theta_i, p)\\ &= \mathrm{Var}(\theta_i)\big[1 + \xi^2\mathring{\tau}^\top R\,\mathring{\tau} - 2\xi R_i^\top\mathring{\tau}\big],\end{aligned}$$

so that

$$\begin{aligned}G_i := \frac{\mathrm{Var}(\theta_i - p)}{\mathrm{Var}(\theta_i)} &= 1 + \mathring{\tau}^\top R\,\mathring{\tau}\left[\xi^2 - 2\xi\frac{R_i^\top\mathring{\tau}}{\mathring{\tau}^\top R\,\mathring{\tau}}\right]\\ &= 1 - \frac{(R_i^\top\mathring{\tau})^2}{\mathring{\tau}^\top R\,\mathring{\tau}} + \mathring{\tau}^\top R\,\mathring{\tau}\left[\xi^2 - 2\xi\frac{R_i^\top\mathring{\tau}}{\mathring{\tau}^\top R\,\mathring{\tau}} + \left(\frac{R_i^\top\mathring{\tau}}{\mathring{\tau}^\top R\,\mathring{\tau}}\right)^2\right]\\ &= 1 - \mathcal{V}_i + (\mathring{\tau}^\top R\,\mathring{\tau})\phi_i^2, \end{aligned}\tag{24}$$

where

$$\phi_i := \xi - \frac{R_i^\top\mathring{\tau}}{\mathring{\tau}^\top R\,\mathring{\tau}}.$$

From (5),

$$\xi := \mu \sum_k \tau_k = \sum_k \beta_k \frac{R_k^\top \mathring{\tau}}{\mathring{\tau}^\top R \mathring{\tau}}.$$

Hence we can write ϕ_i as follows:

$$\phi_i = \frac{\sum_k \beta_k (R_k - R_i)^\top \mathring{\tau}}{\mathring{\tau}^\top R \mathring{\tau}}. \tag{25}$$

We now apply these results to a symmetric economy with equilibrium τ. Since all pairwise correlations are the same, and all types choose the same precision, $R_k^\top \mathring{\tau}$ is the same for all k. From (25), $\phi_i = 0$, and hence, from (24), $\partial G_i/\partial \tau_j = -\partial \mathcal{V}_i/\partial \tau_j$ for all j, or equivalently $\partial_z G_i = -\partial_z \mathcal{V}_i$ for all directions $z \in \mathbb{R}^N$. Using Lemma 5.1, we have

$$\partial_z \log \mathcal{U}_i(\tau) = \left[\frac{\tau_{\theta_i|p}}{(1-\mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})} - \frac{1}{G_i}\right] \partial_z \mathcal{V}_i(\tau).$$

Since $G_i = 1 - \mathcal{V}_i$, from (24), $\partial_z \log \mathcal{U}_i(\tau) \propto -\partial_z \mathcal{V}_i(\tau)$. $\square$

Proof of Proposition 5.4 For this proof it is convenient to parametrize agents' welfare by $\mathring{\tau}_1 := \tau_1/(\tau_1+\tau_2)$ and $\psi := \tau_1 + \tau_2$ (instead of τ_1 and τ_2), and define $\partial_z f := z_1 \frac{\partial f}{\partial \mathring{\tau}_1} + z_2 \frac{\partial f}{\partial \psi}$. We have

$$R_i^\top \mathring{\tau} = (1-\rho)\mathring{\tau}_i + \rho,$$
$$\mathring{\tau}^\top R \mathring{\tau} = (1-\rho)(\mathring{\tau}_1^2 + \mathring{\tau}_2^2) + \rho,$$

so that

$$1 - \mathcal{V}_i = 1 - \frac{(R_i^\top \mathring{\tau})^2}{\mathring{\tau}^\top R \mathring{\tau}} = \frac{(1-\rho^2)\mathring{\tau}_j^2}{\mathring{\tau}^\top R \mathring{\tau}}, \qquad j \neq i, \tag{26}$$

which does not depend on ψ. Differentiating with respect to $\mathring{\tau}_1$, we obtain

$$\partial_z \mathcal{V}_1 = \frac{\partial \mathcal{V}_1}{\partial \mathring{\tau}_1} z_1 = \frac{2(1-\rho^2)(R_1^\top \mathring{\tau})\mathring{\tau}_2}{(\mathring{\tau}^\top R \mathring{\tau})^2} z_1 = \frac{2(1-\mathcal{V}_1)(R_1^\top \mathring{\tau})}{\mathring{\tau}_2(\mathring{\tau}^\top R \mathring{\tau})} z_1, \tag{27}$$

and, similarly,

$$\partial_z \mathcal{V}_2 = \frac{\partial \mathcal{V}_2}{\partial \mathring{\tau}_1} z_1 = -\frac{2(1-\mathcal{V}_2)(R_2^\top \mathring{\tau})}{\mathring{\tau}_1(\mathring{\tau}^\top R \mathring{\tau})} z_1. \tag{28}$$

From (25),

$$\phi_1 = -\frac{(1-\rho)(\mathring{\tau}_1 - \mathring{\tau}_2)\beta_2}{\mathring{\tau}^\top R \mathring{\tau}},$$
$$\phi_2 = \frac{(1-\rho)(\mathring{\tau}_1 - \mathring{\tau}_2)\beta_1}{\mathring{\tau}^\top R \mathring{\tau}}.$$

Hence, from (24),

$$G_i = 1 - \mathcal{V}_i + H\beta_j^2, \qquad j \neq i,$$

where

$$H := \frac{(1-\rho)^2(\mathring{\tau}_1 - \mathring{\tau}_2)^2}{\mathring{\tau}^\top R\,\mathring{\tau}}.$$

Since H does not depend on ψ, we have

$$\partial_z H = \frac{\partial H}{\partial \mathring{\tau}_1} z_1 = \frac{2(1-\rho)^2(1+\rho)(\mathring{\tau}_1 - \mathring{\tau}_2)}{(\mathring{\tau}^\top R\,\mathring{\tau})^2} z_1,$$

and hence (for $j \neq i$),

$$\begin{aligned}
\partial_z G_i &= -\partial_z \mathcal{V}_i + \beta_j^2 \partial_z H + 2H\beta_j \partial_z \beta_j \\
&= -\partial_z \mathcal{V}_i + \frac{2(1-\rho)^2(\mathring{\tau}_1 - \mathring{\tau}_2)\beta_j\big[(1+\rho)\beta_j z_1 + (\mathring{\tau}_1 - \mathring{\tau}_2)(\mathring{\tau}^\top R\,\mathring{\tau})\partial_z \beta_j\big]}{(\mathring{\tau}^\top R\,\mathring{\tau})^2}.
\end{aligned}$$

Using Lemma 5.1 (for $j \neq i$),

$$\begin{aligned}
\partial_z \mathcal{U}_i &\propto \tau_{\theta_i|p} G_i\, \partial_z \mathcal{V}_i + (1-\mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})\, \partial_z G_i \\
&= \big[\tau_{\theta_i|p}(1 - \mathcal{V}_i + H\beta_j^2) - (1-\mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})\big]\partial_z \mathcal{V}_i + (1-\mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})\big[\partial_z \mathcal{V}_i + \partial_z G_i\big] \\
&= \big[H\beta_j^2 \tau_{\theta_i|p} - (1-\mathcal{V}_i)\tau_i\big]\partial_z \mathcal{V}_i + (1-\mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})\big[\partial_z \mathcal{V}_i + \partial_z G_i\big] \\
&\propto \big[(1-\rho)(\mathring{\tau}_1 - \mathring{\tau}_2)^2 \beta_j^2 \tau_{\theta_i|p} - (1+\rho)\tau_i \mathring{\tau}_j^2\big](\mathring{\tau}^\top R\,\mathring{\tau})\partial_z \mathcal{V}_i \\
&\quad + 2(1-\rho)(1-\mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})(\mathring{\tau}_1 - \mathring{\tau}_2)\beta_j\big[(1+\rho)\beta_j z_1 + (\mathring{\tau}_1 - \mathring{\tau}_2)(\mathring{\tau}^\top R\,\mathring{\tau})\partial_z \beta_j\big] \\
&\propto \big[(1-\rho)(\tau_1 - \tau_2)^2 \beta_j^2 \tau_{\theta_i|p} - (1+\rho)\tau_i \tau_j^2\big](\mathring{\tau}^\top R\,\mathring{\tau})\partial_z \mathcal{V}_i \\
&\quad + 2(1-\rho)(1-\mathcal{V}_i)(\tau_i + \tau_{\theta_i|p})(\tau_1^2 - \tau_2^2)\beta_j\big[(1+\rho)\beta_j z_1 + (\mathring{\tau}_1 - \mathring{\tau}_2)(\mathring{\tau}^\top R\,\mathring{\tau})\partial_z \beta_j\big].
\end{aligned}$$

Substituting from (27) and (28), we obtain $\partial_z \mathcal{U}_i \propto L_i$, where

$$\begin{aligned}
L_1 &:= \big[(1-\rho)(\tau_1 - \tau_2)^2 \beta_2^2 \tau_{\theta_1|p} - (1+\rho)\tau_1 \tau_2^2\big](R_1^\top \tau)\tau_1 z_1 \\
&\quad + (1-\rho)(\tau_1 + \tau_{\theta_1|p})(\tau_1^2 - \tau_2^2)\tau_1\tau_2\beta_2\big[(1+\rho)\beta_2 z_1 + (\mathring{\tau}_1 - \mathring{\tau}_2)(\mathring{\tau}^\top R\,\mathring{\tau})\partial_z \beta_2\big] \\
&= \Big((1-\rho)(\tau_1 - \tau_2)\tau_1\beta_2^2\big[(\tau^\top R\,\tau)\tau_{\theta_1|p} + (1+\rho)(\tau_1 + \tau_2)\tau_1\tau_2\big] - (1+\rho)(R_1^\top \tau)\tau_1^2\tau_2^2\Big)z_1 \\
&\quad + (1-\rho)(\tau_1 + \tau_{\theta_1|p})(\tau_1 - \tau_2)^2(\mathring{\tau}^\top R\,\mathring{\tau})\tau_1\tau_2\beta_2(\partial_z \beta_2), \\
L_2 &:= -\big[(1-\rho)(\tau_1 - \tau_2)^2 \beta_1^2 \tau_{\theta_2|p} - (1+\rho)\tau_1^2 \tau_2\big](R_2^\top \tau)\tau_2 z_1 \\
&\quad + (1-\rho)(\tau_2 + \tau_{\theta_2|p})(\tau_1^2 - \tau_2^2)\tau_1\tau_2\beta_1\big[(1+\rho)\beta_1 z_1 + (\mathring{\tau}_1 - \mathring{\tau}_2)(\mathring{\tau}^\top R\,\mathring{\tau})\partial_z \beta_1\big] \\
&= \Big((1-\rho)(\tau_1 - \tau_2)\tau_2\beta_1^2\big[(\tau^\top R\,\tau)\tau_{\theta_2|p} + (1+\rho)(\tau_1 + \tau_2)\tau_1\tau_2\big] + (1+\rho)(R_2^\top \tau)\tau_1^2\tau_2^2\Big)z_1 \\
&\quad + (1-\rho)(\tau_2 + \tau_{\theta_2|p})(\tau_1 - \tau_2)^2(\mathring{\tau}^\top R\,\mathring{\tau})\tau_1\tau_2\beta_1(\partial_z \beta_1) \\
&= \Big((1-\rho)(\tau_1 - \tau_2)\tau_2\beta_1^2\big[(\tau^\top R\,\tau)\tau_{\theta_2|p} + (1+\rho)(\tau_1 + \tau_2)\tau_1\tau_2\big] + (1+\rho)(R_2^\top \tau)\tau_1^2\tau_2^2\Big)z_1 \\
&\quad - (1-\rho)(\tau_1 + \tau_{\theta_1|p})(\tau_1 - \tau_2)^2(\mathring{\tau}^\top R\,\mathring{\tau})\tau_1\tau_2\beta_2(\partial_z \beta_2).
\end{aligned}$$

The last equality follows from the observation that $(\tau_1 + \tau_{\theta_1|p})\beta_2 = (\tau_2 + \tau_{\theta_2|p})\beta_1$, and

$\partial_z\beta_1 + \partial_z\beta_2 = 0$. We can write the equations for L_1 and L_2 compactly as follows:

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} a_1 & b \\ a_2 & -b \end{bmatrix} \begin{bmatrix} z_1 \\ \partial_z\beta_2 \end{bmatrix} \tag{29}$$

$$= \begin{bmatrix} a_1 & b \\ a_2 & -b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\partial\beta_2}{\partial\mathring{\tau}_1} & \frac{\partial\beta_2}{\partial\psi} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \tag{30}$$

where

$$\begin{aligned} a_1 &= (1-\rho)(\tau_1-\tau_2)\tau_1\beta_2^2\big[(\tau^\top R\,\tau)\tau_{\theta_1|p} + (1+\rho)(\tau_1+\tau_2)\tau_1\tau_2\big] - (1+\rho)(R_1^\top\tau)\tau_1^2\tau_2^2, \\ a_2 &= (1-\rho)(\tau_1-\tau_2)\tau_2\beta_1^2\big[(\tau^\top R\,\tau)\tau_{\theta_2|p} + (1+\rho)(\tau_1+\tau_2)\tau_1\tau_2\big] + (1+\rho)(R_2^\top\tau)\tau_1^2\tau_2^2, \\ b &= (1-\rho)(\tau_1+\tau_{\theta_1|p})(\tau_1-\tau_2)^2(\mathring{\tau}^\top R\,\mathring{\tau})\tau_1\tau_2\beta_2. \end{aligned}$$

A direction z is strictly Pareto improving if and only if both L_1 and L_2 are positive.

We claim that both the 2×2 matrices in (30) are nonsingular. As to the first matrix, we have $b \neq 0$, given our assumption that $\tau_1 < \tau_2$, so it suffices to show that $a_1 + a_2 \neq 0$. Indeed,

$$\begin{aligned} a_1 + a_2 &= (1-\rho)(\tau_1-\tau_2) \\ &\quad\cdot\Big((\tau^\top R\,\tau)\big[\tau_1\beta_2^2\tau_{\theta_1|p} + \tau_2\beta_1^2\tau_{\theta_2|p}\big] + (1+\rho)\tau_1\tau_2\big[(\tau_1+\tau_2)(\tau_1\beta_2^2+\tau_2\beta_1^2) - \tau_1\tau_2\big]\Big). \end{aligned}$$

Moreover,

$$\begin{aligned} (\tau_1+\tau_2)(\tau_1\beta_2^2+\tau_2\beta_1^2) - \tau_1\tau_2 &\propto \mathring{\tau}_1\beta_2^2 + \mathring{\tau}_2\beta_1^2 - \mathring{\tau}_1\mathring{\tau}_2 \\ &= \mathring{\tau}_1\beta_2^2 + (1-\mathring{\tau}_1)\beta_1^2 - \mathring{\tau}_1(1-\mathring{\tau}_1) \\ &= \mathring{\tau}_1(\beta_2^2 - \beta_1^2 - 1 + \mathring{\tau}_1) + \beta_1^2 \\ &= \mathring{\tau}_1(\mathring{\tau}_1 - 2\beta_1) + \beta_1^2 \\ &= (\mathring{\tau}_1 - \beta_1)^2, \end{aligned}$$

which is nonnegative. Therefore, $a_1 + a_2 \propto \tau_1 - \tau_2 < 0$. Turning now to the second matrix, we have

$$\begin{aligned} \beta_2 &= \frac{\tau_2 + \tau_{\theta_2|p}}{\tau_1+\tau_2+\tau_{\theta_1|p}+\tau_{\theta_2|p}} \\ &= \frac{\psi(1-\mathring{\tau}_1) + \tau_{\theta_2|p}}{\psi + \tau_{\theta_1|p} + \tau_{\theta_2|p}}. \end{aligned} \tag{31}$$

Using (12) and (26),

$$\begin{aligned} \frac{\partial\beta_2}{\partial\psi} &= \frac{\beta_1 - \mathring{\tau}_1}{\psi + \tau_{\theta_1|p} + \tau_{\theta_2|p}} \\ &\propto \frac{\tau_1 + \tau_{\theta_1|p}}{\psi + \tau_{\theta_1|p} + \tau_{\theta_2|p}} - \frac{\tau_1}{\psi} \\ &\propto \tau_2\tau_{\theta_1|p} - \tau_1\tau_{\theta_2|p} \\ &\propto \tau_2(1-\mathcal{V}_1)^{-1} - \tau_1(1-\mathcal{V}_2)^{-1} \\ &\propto \tau_1 - \tau_2, \end{aligned} \tag{32}$$

which is nonzero. This completes the verification of the claim that both the 2×2 matrices in (30) are nonsingular. Hence, there exists a vector z such that L_1 and L_2 are both positive. This is a strictly Pareto improving direction. Moreover, for any such direction z, $L_1 + L_2$ must be positive. Since

$$L_1 + L_2 = (a_1 + a_2)z_1 \propto (\tau_1 - \tau_2)z_1,$$

it follows that $z_1 < 0$.

In order to determine the sign of z_2, we invoke the assumption that $\rho \geq 0$, which implies that both $R_1^\top \mathring{\tau}$ and $R_2^\top \mathring{\tau}$ are positive. Then $a_1 < 0$. We also have $b > 0$ (this is true even without the assumption of nonnegative ρ), while the sign of a_2 is not pinned down. From (31), we have

$$\frac{\partial \beta_2}{\partial \mathring{\tau}_1} = \frac{-\psi + \beta_1 \frac{\partial \tau_{\theta_2|p}}{\partial \mathring{\tau}_1} - \beta_2 \frac{\partial \tau_{\theta_1|p}}{\partial \mathring{\tau}_1}}{\psi + \tau_{\theta_1|p} + \tau_{\theta_2|p}}.$$

Using (12), (27) and (28),

$$\frac{\partial \tau_{\theta_1|p}}{\partial \mathring{\tau}_1} \propto \frac{\partial \mathcal{V}_1}{\partial \mathring{\tau}_1} > 0,$$
$$\frac{\partial \tau_{\theta_2|p}}{\partial \mathring{\tau}_1} \propto \frac{\partial \mathcal{V}_2}{\partial \mathring{\tau}_1} < 0.$$

It follows that $\partial \beta_2 / \partial \mathring{\tau}_1 < 0$. From (32), $\partial \beta_2 / \partial \psi < 0$ as well.

Now consider a strictly Pareto improving direction z. We have already established that such a direction exists, and it has the property that $z_1 < 0$. Recall that $a_1 < 0$ and $b > 0$. Two cases arise, depending on the sign of a_2. If $a_2 < 0$, we can choose z_2 such that $\partial_z \beta_2 = 0$. From (29), $L_i = a_i z_1$, which is positive for both types. If, on the other hand, $a_2 \geq 0$, we have $a_2 z_1 \leq 0$. Hence, in this case, a necessary condition for z to be strictly Pareto improving is $\partial_z \beta_2 < 0$.

Thus we have shown that there is a Pareto improving direction z with the property that $z_1 < 0$ and $\partial_z \beta_2 \leq 0$. Since

$$\partial_z \beta_2 = z_1 \frac{\partial \beta_2}{\partial \mathring{\tau}_1} + z_2 \frac{\partial \beta_2}{\partial \psi},$$

and both the partial derivatives are negative, $z_1 < 0$ and $\partial_z \beta_2 \leq 0$ together imply that $z_2 > 0$. $\square$

Proof of Lemma 6.2 Suppose the economy exhibits strategic complementarities. Then own-effects must be positive. From (21), the own-effect for type i is given by

$$\frac{\partial \mathcal{V}_i}{\partial \tau_i} = \frac{2R_i^\top \tau(1 - \mathcal{V}_i)}{\tau^\top R \tau}, \tag{33}$$

which is positive if and only if $R_i^\top \tau > 0$. Hence, a necessary condition for strategic complementarities is $R_i^\top \tau > 0$ for all i. For this to be true for any $\tau \in \mathbb{R}^N_{++}$, we must have $\rho_{ij} \geq 0$ for all i, j.

Now suppose $N \geq 3$, and consider cross-effects. From (21),

$$\begin{aligned}
\frac{\partial \mathcal{V}_i}{\partial \tau_j} &= \frac{2R_i^\top \tau}{\tau^\top R \tau}\left[\rho_{ij} - \mathcal{V}_i \frac{R_j^\top \tau}{R_i^\top \tau}\right] \\
&= \frac{2R_i^\top \tau}{\tau^\top R \tau}\left[\rho_{ij} - \mathcal{V}_i \sqrt{\frac{\mathcal{V}_j}{\mathcal{V}_i}}\right] \\
&= \frac{2R_i^\top \tau}{\tau^\top R \tau}\left[\rho_{ij} - \sqrt{\mathcal{V}_i \mathcal{V}_j}\right]. \qquad (34)
\end{aligned}$$

Since $R_i^\top \tau > 0$, $\partial \mathcal{V}_i / \partial \tau_j < 0$ if and only if $\rho_{ij} < \sqrt{\mathcal{V}_i \mathcal{V}_j}$. Let i and j be a pair of types such that ρ_{ij} is the highest pairwise correlation, and let k be any other type. We consider limits as all precisions go to zero except for τ_k. From (8),

$$\lim_{\tau_\ell \to 0, \forall \ell \neq k} \mathcal{V}_k = 1, \qquad \text{and} \qquad \lim_{\tau_\ell \to 0, \forall \ell \neq k} \mathcal{V}_m = \rho_{mk}^2, \quad m \neq k.$$

If the economy exhibits strategic complementarities for all $\tau \in \mathbb{R}_{++}^N$, all pairwise correlations must be nonnegative, and $\rho_{ij} < \sqrt{\mathcal{V}_i \mathcal{V}_j}$ for all $\tau \in \mathbb{R}_{++}^N$. It follows that

$$\begin{aligned}
\rho_{ij} &\leq \lim_{\tau_\ell \to 0, \forall \ell \neq k} \sqrt{\mathcal{V}_i \mathcal{V}_j} \\
&= \rho_{ik}\rho_{jk} \\
&\leq \rho_{ij}^2,
\end{aligned}$$

which implies that $\rho_{ij} = 0$. But ρ_{ij} is the highest pairwise correlation. Therefore, all pairwise correlations are zero. $\square$

Proof of Lemma 6.3 An equilibrium value of τ_i satisfies (13). In particular, for any given $\mathcal{V}_i$, τ_i is decreasing in α_i and τ_θ. Thus $\tau_i \leq \bar{\tau}_i$, where $\bar{\tau}_i$ is the solution to

$$2r\underline{\alpha} C_i'(\tau_i)\tau_i = 1. \qquad (35)$$

Let $\bar{\tau} := \max_i \bar{\tau}_i$.

We now show that there exists $\zeta_i > 0$ such that $\tau_i \geq \zeta_i$ for all $(R, \alpha) \in \mathcal{R}_\eta \times \mathcal{A}$. Suppose not. Then we can find a sequence of economies $\{(R(k), \alpha(k))\}$ in $\mathcal{R}_\eta \times \mathcal{A}$, and a corresponding sequence of precisions for type i, $\{\tau_{i,k}\}$, such that $\lim_{k\to\infty} \tau_{i,k} = 0$. Let $\mathcal{V}_{i,k}$ be the price informativeness for type i in the economy $(R(k), \alpha(k))$. Using (13), and the assumption that $R(k) \in \mathcal{R}_\eta$ for all k,

$$\begin{aligned}
\lim_{k\to\infty} \tau_{i,k} = 0 \quad &\Rightarrow \quad \lim_{k\to\infty} \mathcal{V}_{i,k} = 1 \\
&\Rightarrow \quad \lim_{k\to\infty} \tau_{j,k} = 0, \qquad \forall j \neq i \\
&\Rightarrow \quad \lim_{k\to\infty} \mathcal{V}_{j,k} = 1, \qquad \forall j \neq i.
\end{aligned}$$

(The first and last implications follow from (13), and the second implication follows from the observation that the price function cannot fully reveal θ_i if it puts any weight on θ_j for $j \neq i$ (given that $\tau_{i,k}$ is bounded)). Thus in the limit the price function is fully revealing for all types, a contradiction.

While the above argument establishes a lower bound for τ_i, we will go one step further and choose a lower bound that can be explicitly characterized, in order to prove the limit results in the lemma. Let $\hat{\mathcal{T}} := \times_{i=1}^{N}[\zeta_i, \bar{\tau}]$ and let

$$\bar{\mathcal{V}} := \max_{\tau \in \hat{\mathcal{T}}, R \in \mathcal{R}_\eta} \mathcal{V}_i. \tag{36}$$

This maximum exists since both $\hat{\mathcal{T}}$ and $\mathcal{R}_\eta$ are compact, and is strictly less than one. Since τ_i solves (13) and is decreasing in α_i and $\mathcal{V}_i$, it follows that $\tau_i \geq \underline{\tau}_i$, where $\underline{\tau}_i$ solves

$$2r\bar{\alpha}C_i'(\tau_i)\big[\tau_i + \tau_\theta(1 - \bar{\mathcal{V}})^{-1}\big] = 1. \tag{37}$$

Let $\underline{\tau} := \min_i \underline{\tau}_i$. As $\eta \to 0, \bar{\mathcal{V}} \to 1$, and hence $\underline{\tau} \to 0$.

Now suppose that $C_i = C$ for all i. Let $\lim_{\tau_\theta \to 0} \underline{\tau} := t$ (note that $\bar{\tau}$ does not depend on τ_θ). Then, from (35) and (37), we have,

$$2r\underline{\alpha}C'(\bar{\tau})\bar{\tau} = 1,$$
$$2r\bar{\alpha}C'(t)t = 1.$$

Clearly, $t < \bar{\tau}$. Therefore,

$$\lim_{\tau_\theta \to 0}\left(\frac{\underline{\tau}}{\bar{\tau}}\right) = \frac{t}{\bar{\tau}} = \frac{\underline{\alpha}}{\bar{\alpha}}\frac{C'(\bar{\tau})}{C'(t)} > \frac{\underline{\alpha}}{\bar{\alpha}}.$$

This proves the result. Note that the assumption that $R \in \mathcal{R}_\eta$ is only needed to obtain the lower bound $\underline{\tau}$, not the upper bound $\bar{\tau}$. $\square$

Proof of Lemma 6.4 From equation (33) we already know that $\partial\mathcal{V}_i/\partial\tau_i > 0$ if and only if $R_i^\top\tau > 0$. Now suppose $R \in \mathcal{R}_\eta$. Using the bounds for τ_i given by Lemma 6.3, we have

$$\mathring{\tau}_i \geq \mathring{\underline{\tau}} := \frac{\underline{\tau}}{\underline{\tau} + (N-1)\bar{\tau}}.$$

We can write $\underline{\rho}$ in terms of $\mathring{\underline{\tau}}$ as follows:

$$\underline{\rho} := -\frac{\underline{\tau}}{(N-1)\bar{\tau}} = -\frac{\mathring{\underline{\tau}}}{1 - \mathring{\underline{\tau}}}.$$

Let $\check{\rho} := \min_{i,j} \rho_{ij}$. Then,

$$\begin{aligned}
R_i^\top \mathring{\tau} &= \mathring{\tau}_i + \sum_{k \neq i} \rho_{ik} \mathring{\tau}_k \\
&\geq \mathring{\tau}_i + \check{\rho}(1 - \mathring{\tau}_i) \\
&= (1 - \mathring{\tau}_i) \left[\check{\rho} + \frac{\mathring{\tau}_i}{1 - \mathring{\tau}_i} \right] \\
&\geq (1 - \mathring{\tau}_i) \left[\check{\rho} + \frac{\underline{\mathring{\tau}}}{1 - \underline{\mathring{\tau}}} \right] \\
&= (1 - \mathring{\tau}_i)(\check{\rho} - \underline{\rho}).
\end{aligned}$$

Therefore, $R_i^\top \tau > 0$ for all i and all $\tau \in \mathcal{T}$ if $\check{\rho} > \underline{\rho}$, or equivalently if $\rho_{ij} > \underline{\rho}$ for all i, j. $\square$

Proof of Proposition 6.6 Suppose $N = 2$. Then, by Euler's theorem, the economy exhibits strategic complementarities if and only if both own-effects are positive on $\mathcal{T}$, which is equivalent to $R_i^\top \tau > 0$ for both values of i and for all $\tau \in \mathcal{T}$ (Lemma 6.4). We have, for $j \neq i$,

$$\begin{aligned}
R_i^\top \tau &= \tau_i + \rho \tau_j \\
&\propto \rho + \frac{\tau_j}{\tau_i} \\
&\geq \rho + \frac{\underline{\tau}}{\overline{\tau}} \\
&= \rho - \underline{\rho}.
\end{aligned}$$

Hence, $R_i^\top \tau$ is positive for both values of i and all $\tau \in \mathcal{T}$ if and only if $\rho > \underline{\rho}$. $\square$

Proof of Proposition 6.7 In all three conditions in the statement of the proposition, $\rho_{ij} > \underline{\rho}$ for all i, j. By Lemma 6.4, $R_i^\top \tau > 0$ for all i and for all $\tau \in \mathcal{T}$. From (34), $\partial \mathcal{V}_i / \partial \tau_j < 0$ if and only if $\rho_{ij} < \sqrt{\mathcal{V}_i \mathcal{V}_j}$, or equivalently

$$D := \left[\sqrt{\mathcal{V}_i \mathcal{V}_j} - \rho_{ij} \right] \tau^\top R \tau > 0.$$

This inequality is clearly satisfied if $\rho_{ij} \leq 0$; hence we only need to consider the case where $\rho_{ij} > 0$ $(i \neq j)$. We have

$$\begin{aligned}
D &= (R_i^\top \tau)(R_j^\top \tau) - \rho_{ij}(\tau^\top R\, \tau) \\
&= \Big[\tau_i + \rho_{ij}\tau_j + \sum_{k \neq i,j} \rho_{ik}\tau_k\Big]\Big[\tau_j + \rho_{ij}\tau_i + \sum_{k \neq i,j} \rho_{jk}\tau_k\Big] \\
&\qquad - \rho_{ij}\tau_i\Big[\tau_i + \rho_{ij}\tau_j + \sum_{k \neq i,j} \rho_{ik}\tau_k\Big] - \rho_{ij}\tau_j\Big[\tau_j + \rho_{ij}\tau_i + \sum_{k \neq i,j} \rho_{jk}\tau_k\Big] \\
&\qquad - \rho_{ij}\sum_{\ell \neq i,j} \tau_\ell\Big[\rho_{\ell i}\tau_i + \rho_{\ell j}\tau_j + \sum_{k \neq i,j} \rho_{\ell k}\tau_k\Big] \\
&= (\tau_i + \rho_{ij}\tau_j)(\tau_j + \rho_{ij}\tau_i) + (\tau_i + \rho_{ij}\tau_j)\sum_{k \neq i,j} \rho_{jk}\tau_k + (\tau_j + \rho_{ij}\tau_i)\sum_{k \neq i,j} \rho_{ik}\tau_k \\
&\qquad + \Big[\sum_{k \neq i,j} \rho_{ik}\tau_k\Big]\Big[\sum_{k \neq i,j} \rho_{jk}\tau_k\Big] \\
&\qquad - \rho_{ij}\tau_i(\tau_i + \rho_{ij}\tau_j) - \rho_{ij}\tau_i\sum_{k \neq i,j} \rho_{ik}\tau_k - \rho_{ij}\tau_j(\tau_j + \rho_{ij}\tau_i) - \rho_{ij}\tau_j\sum_{k \neq i,j} \rho_{jk}\tau_k \\
&\qquad - \rho_{ij}\tau_i\sum_{\ell \neq i,j} \rho_{i\ell}\tau_\ell - \rho_{ij}\tau_j\sum_{\ell \neq i,j} \rho_{j\ell}\tau_\ell - \rho_{ij}\sum_{\ell \neq i,j} \tau_\ell\Big[\sum_{k \neq i,j} \rho_{\ell k}\tau_k\Big] \\
&= (1 - \rho_{ij}^2)\tau_i\tau_j + (\tau_i - \rho_{ij}\tau_j)\sum_{k \neq i,j} \rho_{jk}\tau_k + (\tau_j - \rho_{ij}\tau_i)\sum_{k \neq i,j} \rho_{ik}\tau_k \\
&\qquad + \Big[\sum_{k \neq i,j} \rho_{ik}\tau_k\Big]\Big[\sum_{k \neq i,j} \rho_{jk}\tau_k\Big] - \rho_{ij}\sum_{\ell \neq i,j} \tau_\ell\Big[\sum_{k \neq i,j} \rho_{\ell k}\tau_k\Big] \\
&= (1 - \rho_{ij}^2)\tau_i\tau_j + (\tau_i - \rho_{ij}\tau_j)S_j + (\tau_j - \rho_{ij}\tau_i)S_i + S_iS_j - \rho_{ij}\sum_{\ell \neq i,j} \tau_\ell S_\ell, \qquad (38)
\end{aligned}$$

where

$$S_m := \sum_{k \neq i,j} \rho_{mk}\tau_k.$$

Note that $S_m < \sum_{k \neq i,j} \tau_k$, for all m. We consider condition (ii) of the proposition first, followed by (i) and (iii).

Proof of (ii):
Let $\check{\rho} := \min_{k,\ell} \rho_{k\ell}$, and suppose that $\check{\rho} \geq 0$. Then, for $m = i, j$,

$$S_m \geq \check{\rho} \sum_{k \neq i,j} \tau_k \geq \check{\rho}(N-2)\underline{\tau}.$$

From (38),

$$\begin{aligned}
\frac{\partial D}{\partial S_i} &= \tau_j - \rho_{ij}\tau_i + S_j \\
&\geq \underline{\tau} - \rho_{ij}\bar{\tau} + \check{\rho}(N-2)\underline{\tau} \\
&\propto \big[1 + \check{\rho}(N-2)\big]\delta - \rho_{ij},
\end{aligned}$$

which is nonnegative if

$$\rho_{ij} \le \big[1 + \check{\rho}(N-2)\big]\delta. \tag{39}$$

Assuming that (39) holds, D is increasing in S_i, and by symmetry in S_j as well. Also, D is decreasing in S_ℓ, $\ell \ne i, j$. Hence, from (38),

$$\begin{aligned}
D > D_1 &:= (1-\rho_{ij}^2)\tau_i\tau_j + \check{\rho}(\tau_i - \rho_{ij}\tau_j)\sum_{k\ne i,j}\tau_k + \check{\rho}(\tau_j - \rho_{ij}\tau_i)\sum_{k\ne i,j}\tau_k \\
&\qquad + \check{\rho}^2\Big[\sum_{k\ne i,j}\tau_k\Big]^2 - \rho_{ij}\Big[\sum_{k\ne i,j}\tau_k\Big]^2 \\
&= (1-\rho_{ij}^2)\tau_i\tau_j + \check{\rho}(1-\rho_{ij})(\tau_i+\tau_j)\sum_{k\ne i,j}\tau_k + (\check{\rho}^2 - \rho_{ij})\Big[\sum_{k\ne i,j}\tau_k\Big]^2.
\end{aligned} \tag{40}$$

Moreover, since D_1 is increasing in τ_i and τ_j,

$$D_1 \ge D_2 := (1-\rho_{ij}^2)\underline{\tau}^2 + 2\check{\rho}(1-\rho_{ij})\underline{\tau}\sum_{k\ne i,j}\tau_k + (\check{\rho}^2-\rho_{ij})\Big[\sum_{k\ne i,j}\tau_k\Big]^2. \tag{41}$$

For $m \ne i, j$,

$$\begin{aligned}
\frac{\partial D_2}{\partial \tau_m} &= 2\check{\rho}(1-\rho_{ij})\underline{\tau} + 2(\check{\rho}^2-\rho_{ij})\sum_{k\ne i,j}\tau_k \\
&\le 2\check{\rho}(1-\rho_{ij})\underline{\tau} + 2(\check{\rho}^2-\rho_{ij})(N-2)\underline{\tau} \\
&\propto \check{\rho}(1-\rho_{ij}) + (\check{\rho}^2-\rho_{ij})(N-2) \\
&\le \check{\rho}(1-\check{\rho}) + (\check{\rho}^2-\check{\rho})(N-2) \\
&= -\check{\rho}(1-\check{\rho})(N-3) \\
&\le 0.
\end{aligned}$$

Therefore, from (41),

$$D_2 \ge D_3 := (1-\rho_{ij}^2)\underline{\tau}^2 + 2\check{\rho}(1-\rho_{ij})\underline{\tau}(N-2)\bar{\tau} + (\check{\rho}^2-\rho_{ij})(N-2)^2\bar{\tau}^2. \tag{42}$$

Combining (40), (41), and (42), we have

$$\begin{aligned}
D > D_3 &\propto (1-\rho_{ij}^2)\delta^2 + 2\check{\rho}(1-\rho_{ij})(N-2)\delta + (\check{\rho}^2-\rho_{ij})(N-2)^2 \\
&\ge (1-\rho_{ij})(1+\check{\rho})\delta^2 + 2\check{\rho}(1-\rho_{ij})(N-2)\delta + (\check{\rho}^2-\rho_{ij})(N-2)^2,
\end{aligned}$$

which is greater than equal to zero if

$$\rho_{ij} \le 1 - \frac{(1-\check{\rho}^2)(N-2)^2}{(1+\check{\rho})\delta^2 + 2\check{\rho}(N-2)\delta + (N-2)^2}. \tag{43}$$

Thus $D > 0$, for all $i \ne j$ and all $\tau \in \mathcal{T}$, if (39) and (43) hold.

Finally, we show that if $\delta \geq 1/2$, we can drop condition (39), since in this case the upper bound in (39) exceeds the upper bound in (43), i.e.

$$\big[1+\check{\rho}(N-2)\big]\delta \geq 1 - \frac{(1-\check{\rho}^2)(N-2)^2}{(1+\check{\rho})\delta^2 + 2\check{\rho}(N-2)\delta + (N-2)^2}.$$

Letting $M := N-2$, we can write this inequality as follows:

$$K := (1+\check{\rho})(1+\check{\rho}M)\delta^3 + \big[2\check{\rho}M(1+\check{\rho}M) - (1+\check{\rho})\big]\delta^2 + M\big[M(1+\check{\rho}M) - 2\check{\rho}\big]\delta - \check{\rho}^2M^2 \geq 0.$$

We have

$$\begin{aligned}
\frac{\partial K}{\partial \delta} &= 3(1+\check{\rho})(1+\check{\rho}M)\delta^2 + 2\big[2\check{\rho}M(1+\check{\rho}M) - (1+\check{\rho})\big]\delta + M\big[M(1+\check{\rho}M) - 2\check{\rho}\big],\\
\frac{\partial^2 K}{\partial \delta^2} &= 6(1+\check{\rho})(1+\check{\rho}M)\delta + 2\big[2\check{\rho}M(1+\check{\rho}M) - (1+\check{\rho})\big].
\end{aligned}$$

Henceforth, we assume that $\delta \geq 1/2$. Then, $\partial K/\partial \delta$ is increasing in δ, and therefore

$$\begin{aligned}
\frac{\partial K}{\partial \delta} &\geq \frac{\partial K}{\partial \delta}\Big|_{\delta=\frac{1}{2}}\\
&= \frac{3}{4}(1+\check{\rho})(1+\check{\rho}M) + \big[2\check{\rho}M(1+\check{\rho}M) - (1+\check{\rho})\big] + M\big[M(1+\check{\rho}M) - 2\check{\rho}\big]\\
&= \frac{3}{4}(1+\check{\rho})(1+\check{\rho}M) + 2\check{\rho}^2M^2 + (M^2-1) + \check{\rho}(M^3-1)\\
&> 0.
\end{aligned}$$

Thus K is increasing in δ, and therefore

$$\begin{aligned}
K &\geq K\big|_{\delta=\frac{1}{2}}\\
&\propto (1+\check{\rho})(1+\check{\rho}M) + 2\big[2\check{\rho}M(1+\check{\rho}M) - (1+\check{\rho})\big] + 4M\big[M(1+\check{\rho}M) - 2\check{\rho}\big] - 8\check{\rho}^2M^2\\
&= (1+\check{\rho})(\check{\rho}M - 1) + 4M(1+\check{\rho}M)(M-\check{\rho})\\
&\geq (1+\check{\rho})\big[(\check{\rho}M-1) + 4M(M-\check{\rho})\big]\\
&\propto 4M^2 - 3\check{\rho}M - 1,
\end{aligned}$$

which is nonnegative as desired.

Proof of (i):
Now we assume only that $\rho_{k\ell} > \underline{\rho}$. Recall that $\underline{\rho} = -(N-1)^{-1}\delta$. The proof proceeds along the same lines as for condition (ii), with $\tilde{D}_1$, $\tilde{D}_2$ and $\tilde{D}_3$ taking the place of D_1, D_2 and D_3, respectively; $\tilde{D}_i$ is the same expression as D_i except that $\check{\rho}$ is replaced by $\underline{\rho}$.

For $m = i, j$, we have

$$S_m \geq \underline{\rho} \sum_{k \neq i,j} \tau_k \geq \underline{\rho}(N-2)\bar{\tau}.$$

From (38),

$$
\begin{aligned}
\frac{\partial D}{\partial S_i} &= \tau_j - \rho_{ij}\tau_i + S_j \\
&\geq \underline{\tau} - \rho_{ij}\bar{\tau} + \underline{\rho}(N-2)\bar{\tau} \\
&\propto \delta + \underline{\rho}(N-2) - \rho_{ij} \\
&= -\underline{\rho} - \rho_{ij}.
\end{aligned}
$$

Assuming that $\rho_{ij} \leq -\underline{\rho}$, D is increasing in S_i and, by symmetry, in S_j as well. Moreover, D is decreasing in S_ℓ, $\ell \neq i,j$. Hence, from (38),

$$
D > \tilde{D}_1 := (1-\rho_{ij}^2)\tau_i\tau_j + \underline{\rho}(1-\rho_{ij})(\tau_i+\tau_j)\sum_{k\neq i,j}\tau_k + (\underline{\rho}^2 - \rho_{ij})\Big[\sum_{k\neq i,j}\tau_k\Big]^2. \tag{44}
$$

Differentiating, we see that

$$
\begin{aligned}
\frac{\partial \tilde{D}_1}{\partial \tau_i} &= (1-\rho_{ij}^2)\tau_j + \underline{\rho}(1-\rho_{ij})\sum_{k\neq i,j}\tau_k \\
&\geq (1-\rho_{ij}^2)\underline{\tau} + \underline{\rho}(1-\rho_{ij})(N-2)\bar{\tau} \\
&\propto (1+\rho_{ij})\delta + \underline{\rho}(N-2) \\
&= \delta\rho_{ij} - \underline{\rho} \\
&> 0.
\end{aligned}
$$

Thus $\tilde{D}_1$ is increasing in τ_i and, by symmetry, in τ_j as well. Hence, from (44),

$$
\tilde{D}_1 \geq \tilde{D}_2 := (1-\rho_{ij}^2)\underline{\tau}^2 + 2\underline{\rho}(1-\rho_{ij})\underline{\tau}\sum_{k\neq i,j}\tau_k + (\underline{\rho}^2 - \rho_{ij})\Big[\sum_{k\neq i,j}\tau_k\Big]^2. \tag{45}
$$

For $m \neq i,j$,

$$
\frac{\partial \tilde{D}_2}{\partial \tau_m} = 2\underline{\rho}(1-\rho_{ij})\underline{\tau} + 2(\underline{\rho}^2 - \rho_{ij})\sum_{k\neq i,j}\tau_k,
$$

which is negative if $\rho_{ij} \geq \underline{\rho}^2$. If $\rho_{ij} < \underline{\rho}^2$, we have

$$
\begin{aligned}
\frac{\partial \tilde{D}_2}{\partial \tau_m} &\leq 2\underline{\rho}(1-\rho_{ij})\underline{\tau} + 2(\underline{\rho}^2 - \rho_{ij})(N-2)\bar{\tau} \\
&\propto \underline{\rho}(1-\rho_{ij})\delta + (\underline{\rho}^2 - \rho_{ij})(N-2) \\
&= \underline{\rho}\big[\delta + \underline{\rho}(N-2)\big] - \rho_{ij}\big[(1+\underline{\rho}\delta) + (N-3)\big] \\
&= -\underline{\rho}^2 - \rho_{ij}\big[(1+\underline{\rho}\delta) + (N-3)\big],
\end{aligned}
$$

which again is negative. Thus $\tilde{D}_2$ is decreasing in τ_m, for $m \neq i,j$. Therefore, from (45),

$$
\tilde{D}_2 \geq \tilde{D}_3 := (1-\rho_{ij}^2)\underline{\tau}^2 + 2\underline{\rho}(1-\rho_{ij})\underline{\tau}(N-2)\bar{\tau} + (\underline{\rho}^2 - \rho_{ij})(N-2)^2\bar{\tau}^2. \tag{46}
$$

Combining (44), (45), and (46), we have

$$\begin{aligned}D > \tilde{D}_3 &\propto (1-\rho_{ij}^2)\delta^2 + 2\underline{\rho}(1-\rho_{ij})(N-2)\delta + (\underline{\rho}^2 - \rho_{ij})(N-2)^2 \\ &> (1-\rho_{ij})\delta^2 + 2\underline{\rho}(1-\rho_{ij})(N-2)\delta + (\underline{\rho}^2 - \rho_{ij})(N-2)^2,\end{aligned}$$

which is greater than equal to zero if

$$\begin{aligned}\rho_{ij} &\le \frac{\delta^2 + 2\underline{\rho}(N-2)\delta + \underline{\rho}^2(N-2)^2}{\delta^2 + 2\underline{\rho}(N-2)\delta + (N-2)^2} \\ &= \frac{\big[\delta + \underline{\rho}(N-2)\big]^2}{\big[\delta + \underline{\rho}(N-2)\big]^2 + (1-\underline{\rho}^2)(N-2)^2} \\ &= \frac{\underline{\rho}^2}{\underline{\rho}^2 + (1-\underline{\rho}^2)(N-2)^2}. \qquad (47)\end{aligned}$$

Notice that (47) strengthens our earlier assumption that $\rho_{ij} \le -\underline{\rho}$. Thus $D > 0$, for all $i \neq j$ and all $\tau \in \mathcal{T}$, if (47) holds. This gives us condition (17) in the proposition.

Proof of (iii):
Suppose $\rho_{ij} = \rho$ for all $i \neq j$. Then,

$$S_i = S_j = \rho \sum_{k \neq i,j} \tau_k,$$

and for $\ell \neq i, j$,

$$S_\ell = (1-\rho)\tau_\ell + \rho \sum_{k \neq i,j} \tau_k.$$

There is nothing to prove if $\rho \le 0$. If $\rho > 0$, we have, from (38),

$$\begin{aligned}D &= (1-\rho^2)\tau_i\tau_j + \rho(1-\rho)(\tau_i+\tau_j)\sum_{k\neq i,j}\tau_k - \rho(1-\rho)\sum_{k\neq i,j}\tau_k^2 \\ &\propto (1+\rho)\tau_i\tau_j + \rho(\tau_i+\tau_j)\sum_{k\neq i,j}\tau_k - \rho\sum_{k\neq i,j}\tau_k^2 \qquad (48)\\ &\ge (1+\rho)\underline{\tau}^2 + 2\rho\underline{\tau}\sum_{k\neq i,j}\tau_k - \rho\sum_{k\neq i,j}\tau_k^2 \qquad (49)\\ &\ge (1+\rho)\underline{\tau}^2 + 2\rho\underline{\tau}(N-2)\bar{\tau} - \rho(N-2)\bar{\tau}^2 \\ &\propto \delta^2 + \big[\delta^2 + (2\delta-1)(N-2)\big]\rho \qquad (50)\\ &> \delta^2 + (2\delta-1)(N-1)\rho,\end{aligned}$$

where we have used the fact that (48) is increasing in τ_i and τ_j, while (49) is decreasing in $\tau_k, k \neq i, j$. Hence, $D > 0$ if $\delta \ge \frac{1}{2}$. If $\delta < \frac{1}{2}$, $D > 0$ if (19) holds. Note that a less stringent condition on ρ can be derived from (50), but (19) is easier to interpret.
□

Proof of Proposition 7.1 Substituting (12) into (9), we obtain:

$$2r\alpha_i C_i'(\tau_{in})\left[\tau_{in} + \tau_\theta[1-\mathcal{V}_i(\tau)]^{-1}\right] = 1.$$

Since the economy exhibits strategic complementarities on $\mathcal{T}$, τ_{in} is decreasing in τ_i and increasing in $\tau_{-i} := (\tau_j)_{j\neq i}$, for $\tau \in \mathcal{T}$. Moreover, τ_{in} is decreasing in α_i.

The following argument is similar to the one used by Milgrom and Shannon (1994) for general equilibrium with gross substitutes. For given α, consider the fictitious game $\Gamma(\alpha)$ with N players in which player i chooses $\tau_i \in [\underline{\tau}, \bar{\tau}]$ and has payoff

$$\pi_i(\tau_i, \tau_{-i}, \alpha) = -|\tau_i - \tau_{in}(\tau_i, \tau_{-i}, \alpha)|.$$

Let $t_i := (\tau_{-i}, -\alpha)$ and $f_i(\tau_i, t_i) := \tau_i - \tau_{in}(\tau_i, \tau_{-i}, \alpha)$. Player i's payoff can then be written as

$$\pi_i(\tau_i, t_i) = -|f_i(\tau_i, t_i)|.$$

For any given t_i, f_i is continuous in τ_i, with $f(\underline{\tau}, t_i) \leq 0$ and $f(\bar{\tau}, t_i) \geq 0$. Hence, there exists a τ_i such that $\pi_i(\tau_i, t_i) = 0$. It follows that a profile of precisions τ is an equilibrium of our economy if and only if it is a pure strategy Nash equilibrium of $\Gamma(\alpha)$. Note that the function f_i is strictly increasing in τ_i, and decreasing in t_i.

We claim that π_i satisfies the single-crossing property in (τ_i, t_i), i.e. for all $\breve{\tau}_i > \tau_i, \breve{t}_i > t_i$:

$$\pi_i(\breve{\tau}_i, t_i) - \pi_i(\tau_i, t_i) \ \geq (>) \ 0 \qquad \Rightarrow \qquad \pi_i(\breve{\tau}_i, \breve{t}_i) - \pi_i(\tau_i, \breve{t}_i) \ \geq (>) \ 0,$$

or, equivalently,

$$-|f_i(\breve{\tau}_i, t_i)| + |f_i(\tau_i, t_i)| \ \geq (>) \ 0 \qquad \Rightarrow \qquad -|f_i(\breve{\tau}_i, \breve{t}_i)| + |f_i(\tau_i, \breve{t}_i)| \ \geq (>) \ 0. \quad (51)$$

Since f_i is strictly increasing in τ_i, $f_i(\breve{\tau}_i, t_i) > f_i(\tau_i, t_i)$. Hence, in order for the supposition in (51) to be true, we must have $f_i(\tau_i, t_i) < 0$. In fact, since f_i is decreasing in t_i, $f_i(\tau_i, \breve{t}_i) \leq f_i(\tau_i, t_i) < 0$. Therefore, we can write (51) as follows:

$$|f_i(\breve{\tau}_i, t_i)| + f_i(\tau_i, t_i) \ \leq (<) \ 0 \qquad \Rightarrow \qquad |f_i(\breve{\tau}_i, \breve{t}_i)| + f_i(\tau_i, \breve{t}_i) \ \leq (<) \ 0. \quad (52)$$

Now note that $f_i(\breve{\tau}_i, \breve{t}_i) \leq f_i(\breve{\tau}_i, t_i)$. Thus if $f_i(\breve{\tau}_i, \breve{t}_i) \geq 0$, (52) is satisfied (both terms in the implication are lower than the corresponding terms in the supposition). If $f_i(\breve{\tau}_i, \breve{t}_i) < 0$, we must have $f_i(\tau_i, \breve{t}_i) < f_i(\breve{\tau}_i, \breve{t}_i) < 0$, so the implication in (52) holds. This verifies the single-crossing property.

Thus $\{\Gamma(\alpha)\}_\alpha$ is a family of games with strategic complementarities satisfying the single-crossing property, as defined by Milgrom and Shannon (1994). Hence there is a highest equilibrium $\hat{\tau}(\alpha)$ and a lowest equilibrium $\check{\tau}(\alpha)$, and both satisfy the MCS property. $\square$

Proof of Lemma 7.2 Consider an economy that exhibits strategic complementarities, and an equilibrium τ that satisfies the MCS property. Thus we have, for all i, j,

$$\frac{\partial \mathcal{V}_i}{\partial \tau_i} > 0; \qquad \frac{\partial \mathcal{V}_i}{\partial \tau_k} < 0, \quad k \neq i; \qquad \frac{\partial \tau_i}{\partial \alpha_j} \leq 0, \quad \alpha_j \in (\underline{\alpha}, \bar{\alpha}). \quad (53)$$

We show that the last inequality is strict. Differentiating (13) with respect to α_j, we obtain:

$$C''(\tau_i)\frac{\partial\tau_i}{\partial\alpha_j}\left[\tau_i+\tau_\theta(1-\mathcal{V}_i)^{-1}\right]+C'(\tau_i)\left[\frac{\partial\tau_i}{\partial\alpha_j}+\tau_\theta(1-\mathcal{V}_i)^{-2}\frac{\partial\mathcal{V}_i}{\partial\alpha_j}\right]=0, \qquad i\neq j, \tag{54}$$

and

$$\begin{aligned}\left[\alpha_j C_j''(\tau_j)\frac{\partial\tau_j}{\partial\alpha_j}+C_j'(\tau_j)\right]&\left[\tau_j+\tau_\theta(1-\mathcal{V}_j)^{-1}\right]\\&+\alpha_j C_j'(\tau_j)\left[\frac{\partial\tau_j}{\partial\alpha_j}+\tau_\theta(1-\mathcal{V}_j)^{-2}\frac{\partial\mathcal{V}_j}{\partial\alpha_j}\right]=0.\end{aligned} \tag{55}$$

Suppose $\partial\tau_j/\partial\alpha_j=0$. Then,

$$\frac{\partial\mathcal{V}_j}{\partial\alpha_j}=\sum_{k\neq j}\frac{\partial\mathcal{V}_j}{\partial\tau_k}\frac{\partial\tau_k}{\partial\alpha_j},$$

which is nonnegative due to (53). But then (55) is not satisfied, a contradiction. It follows that $\partial\tau_j/\partial\alpha_j<0$. Now suppose there is an $i\neq j$ such that $\partial\tau_i/\partial\alpha_j=0$. Then,

$$\frac{\partial\mathcal{V}_i}{\partial\alpha_j}=\sum_{k\neq i}\frac{\partial\mathcal{V}_i}{\partial\tau_k}\frac{\partial\tau_k}{\partial\alpha_j},$$

which is positive due to (53) and the fact that $\partial\tau_j/\partial\alpha_j<0$. This implies that (54) is not satisfied, a contradiction. Therefore, $\partial\tau_i/\partial\alpha_j<0$ for all i. $\square$

Proof of Proposition 7.3 Consider an equilibrium τ that satisfies the MCS property, and hence the strong MCS property by Lemma 7.2. From (54), it follows that $\partial\mathcal{V}_i/\partial\alpha_j>0$, for all $\alpha_j\in(\underline{\alpha},\bar{\alpha})$, and for all $i\neq j$. Hence, $\partial\mathcal{V}_j/\partial\alpha_j<0$ for all $\alpha_j\in(\underline{\alpha},\bar{\alpha})$, due to Lemma 5.2 (note that, since $\mathcal{V}_i$ depends on α_j only through τ, $\partial\mathcal{V}_i/\partial\alpha_j=\partial_z\mathcal{V}_i$ for some direction z, for all i). $\square$

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