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## Supplementary Material
## Proofs of Theorems and Corollaries

### A. *Proof of Lemma III.1*

Using (11) with the definitions of $\nabla_{12}$ and $\nabla_{13}$ given in (12) and (13), a comparison of utilities from (2) and (3), (2) and (4), and (3) and (4), respectively, yields the following.

In equilibrium, a user with willingness to pay $w$ will weakly prefer:

i) Network 1 to network 2 if $w \leq \nabla_{12}$, network 2 over network 1 if $w \geq \nabla_{12}$

ii) Network 1 to option 3 if $w \leq \nabla_{13}$, option 3 over network 1 if $w \geq \nabla_{13}$

iii) Network 2 to option 3 if $\nabla_{12} \leq \nabla_{13}$, option 3 over network 2 if $\nabla_{12} \geq \nabla_{13}$, regardless of $w$ in both cases, and strict preference obtains when the corresponding inequality is strict.

Consider Case 1 of Lemma III.1. By iii, all users prefer network 2 to option 3. Thus, any users taking option 3 would migrate from network 1 to network 2. Hence $Q_{13}$ would decrease and $Q_2$ would increase, until either there is equality or until $Q_{13}$ is zero. (The preference for network 1 or network 2 is given by i.)

In Case 2 of the Lemma, by iii all users prefer option 3 to network 2; thus, any users on network 2 would migrate to network 1 to choose option 3. Hence $Q_2$ decreases and $Q_1$ increases until either there is equality, or until $Q_2$ is zero.

For the equality of Case 3 of the Lemma, it follows from iii that at equilibrium all users are indifferent between network 2 and option 3, but clearly they will not all choose one or another, since otherwise some could reduce their cost by choosing the empty network.

### B. *Proofs of Theorem III.2 and Corollary III.3*

#### 1) *Proof of Theorem III.2:*

*Proof.* We prove uniqueness and existence of $\mathbf{Q}$ by explicitly characterizing $Q_{11}, Q_{13}, Q_2$ in Theorem III.2 in the three mutually exclusive cases of Lemma III.1. In each case, we first show that $Q_{11}, Q_{13}, Q_2$ are uniquely determined—specifically, the solution of an appropriate system of linear equations; we then establish feasibility, and hence the existence of the unique solution.

**Case 1:** No users join the Content Provider, because the price is too high, hence $Q_{13}=0$. From Lemma III.1, it follows that given a vector of prices $\mathbf{p}$, the vector $\mathbf{Q}$ must satisfy

$$Q_{13} = 0 \tag{S.1}$$
$$\nabla_{12} = \frac{1}{r\widehat{C}_1}(Q_2 - rQ_{11}) + p_2 - p_1 \tag{S.2}$$
$$Q_{11} = [\nabla_{12}]_0^1 \tag{S.3}$$
$$Q_2 = 1 - Q_{11}. \tag{S.4}$$

where $[x]_l^u$ denotes the function equal to $x$ when $l \leq x \leq u$, equal to $l$ when $x < l$, and equal to $u$ when $x > u$. Solving (S.1)–(S.4) yields:

$$(Q_1, Q_2) = \begin{cases} (1,0) \\ \left(\frac{1+\widehat{C}_1(p_2-p_1)r}{1+r+\widehat{C}_1 r}, \frac{[1-\widehat{C}_1(p_2-p_1-1)]r}{1+r+\widehat{C}_1 r}\right) \\ (0,1) \end{cases} \tag{S.5}$$

if, respectively, $p_2-p_1 > 1+\frac{1}{\widehat{C}_1}$, $1+\frac{1}{\widehat{C}_1} \geq p_2-p_1 \geq -\frac{1}{r\widehat{C}_1}$, and $-\frac{1}{r\widehat{C}_1} > p_2-p_1$. The profits are:

$$\pi_1 = p_1 Q_1 \tag{S.6}$$
$$\pi_2 = p_2 Q_2 \tag{S.7}$$
$$\pi_3 = 0. \tag{S.8}$$

We have from (S.2) and (S.5):

$$\nabla_{12} = \begin{cases} -\frac{1}{\widehat{C}_1} + p_2 - p_1 & \text{if} \quad p_2-p_1 > 1+\frac{1}{\widehat{C}_1} \\ \frac{1+\widehat{C}_1(p_2-p_1)r}{1+r+\widehat{C}_1 r} & \text{if} \quad 1+\frac{1}{\widehat{C}_1} \geq p_2-p_1 \geq -\frac{1}{r\widehat{C}_1} \\ \frac{1}{r\widehat{C}_1} + p_2 - p_1 & \text{if} \quad -\frac{1}{r\widehat{C}_1} > p_2-p_1 \end{cases} \tag{S.9}$$

which correspond to the three regions of $\nabla_{12}$, respectively, being less than 1, in the closed interval $[0,1]$, and smaller than 1.

*Feasibility*: From (14) and (15) and substituting into (S.6) and (S.7) gives the necessary and sufficient conditions: $p_1 \geq 0$ and $p_2 \geq 0$. From (S.9) and (13), and using the defining expression for Case 1, which is given in terms of the user masses and the prices, we have an alternative defining expression for Case 1 in terms of the prices alone:

$$p_3 > \begin{cases} -\frac{1}{\widehat{C}_1} + p_2 - p_1 & \text{if} \quad p_2-p_1 > 1+\frac{1}{\widehat{C}_1} \\ \frac{1+\widehat{C}_1(p_2-p_1)r}{1+r+\widehat{C}_1 r} & \text{if} \quad 1+\frac{1}{\widehat{C}_1} \geq p_2-p_1 \geq -\frac{1}{r\widehat{C}_1} \\ \frac{1}{r\widehat{C}_1} + p_2 - p_1 & \text{if} \quad -\frac{1}{r\widehat{C}_1} > p_2-p_1 \end{cases}. \tag{S.10}$$

**Case 2:** No users join network 2 because the price is too high. Using (13):

$$Q_{11} = [p_3]^1$$
$$Q_{13} = 1 - [p_3]^1$$

where $[x]^u$ denotes the function equal to $x$ when $x \leq u$, and equal to $u$ when $x > u$. The requirement that firms 1 and 3 make non-negative profits, and the expressions for profits, (7), yields:

$$p_1 + t(1-[p_3]^1) \geq 0 \tag{S.11}$$
$$(p_3 - t)(1-[p_3]^1) \geq 0. \tag{S.12}$$

*Feasibility*: This region is feasible if the demand allocations are feasible and profits are non-negative, and in addition the defining condition for Case 2 holds. Using (S.12), (S.11), and the defining condition for Case 2, the conditions are:

$$p_3 \geq t \tag{S.13}$$
$$p_1 + t(1-[p_3]^1) \geq 0 \tag{S.14}$$
$$p_2 - p_1 > \frac{b + \widehat{C}_1 p_3 + (1-b)[p_3]^1}{\widehat{C}_1}. \tag{S.15}$$

**Case 3:** From (13) $\nabla_{13} = p_3$, and since $Q_{11} = [\nabla_{13}]^1$, we have:

$$Q_{11} = [p_3]^1 \tag{S.16}$$

and also have:

$$Q_2 \;=\; 1-(Q_{11}+Q_{13}). \tag{S.17}$$

*Subcase A:* $\mathbf{0 \le p_3 \le 1}$*:* Solving simultaneously the defining condition for Case 3, (S.16), (S.17) together with (6) and (10) yields:

$$Q_{11} = p_3 \tag{S.18}$$

$$Q_{13} \;=\; \frac{1+r\widehat{C}_1(p_2-p_1)-(1+r+r\widehat{C}_1)p_3}{1+br} \tag{S.19}$$

$$Q_2 \;=\; \frac{r[b-\widehat{C}_1(p_2-p_1)+(1-b+\widehat{C}_1)p_3]}{1+br} \tag{S.20}$$

$$Q_1 \;=\; \frac{1+r\widehat{C}_1(p_2-p_1)-r(1-b+\widehat{C}_1)p_3}{1+br}. \tag{S.21}$$

*Feasibility*: For the demand allocations to be feasible given the prices, we require (18) to hold. The nonnegativity conditions on the $Q$'s given by (14), (15), and (16) imply necessary and sufficient conditions for Case 3 to be feasible. These conditions are that the $p_i$ satisfy:

$$0 \;\le\; p_3 \;\le 1 \tag{S.22}$$

$$p_3 \;\le\; \frac{1+r(p_2-p_1)\widehat{C}_1}{1+r+r\widehat{C}_1} \tag{S.23}$$

$$(p_2-p_1)\widehat{C}_1 - b \;\le\; (1-b+\widehat{C}_1)p_3 \tag{S.24}$$

$$\begin{aligned} 0 \;\le\; & p_1\left[1+r(p_2-p_1)\widehat{C}_1 - r(1-b+\widehat{C}_1)p_3\right] \\ & +t\left[1+r(p_2-p_1)\widehat{C}_1-(1+r+r\widehat{C}_1)p_3\right] \end{aligned} \tag{S.25}$$

$$0 \;\le\; p_2\left[rb-r\widehat{C}_1(p_2-p_1)+r(1-b+\widehat{C}_1)p_3\right] \tag{S.26}$$

$$0 \;\le\; (p_3-t)\left[1+r\widehat{C}_1(p_2-p_1)-(1+r+r\widehat{C}_1)p_3\right]. \tag{S.27}$$

Note that the defining condition for Case 3 is satisfied by the $Q$'s by construction.

Note also that in the "fully non-boundary" case, when all the user masses are strictly positive, the conditions simplify to

$$\begin{aligned} 0 \;&<\; p_3 \;< 1 \\ p_3 \;&<\; \frac{1+r(p_2-p_1)\widehat{C}_1}{1+r+r\widehat{C}_1} \\ r[(p_2-p_1)\widehat{C}_1-b] \;&<\; r(1-b+\widehat{C}_1)p_3 \\ -p_1\frac{1+r(p_2-p_1)\widehat{C}_1-r(1-b+\widehat{C}_1)p_3}{1+r(p_2-p_1)\widehat{C}_1-(1+r+r\widehat{C}_1)p_3} \;&\le\; t \\ 0 \;&\le\; p_2 \\ t \;&\le\; p_3\,. \end{aligned}$$

*Subcase B:* $\mathbf{p_3 > 1}$*:* Solving simultaneously the defining condition for Case 3, (S.16), (S.17) together with (6) and (10) yields:

$$Q_{11} \;=\; 1 \tag{S.28}$$

$$Q_{13} \;=\; 0 \tag{S.29}$$

$$Q_2 \;=\; 0 \tag{S.30}$$

$$Q_1 \;=\; 1. \tag{S.31}$$

*Feasibility*: These conditions are that the $p_i$ satisfy:

$$p_2-p_1-p_3 \;=\; \frac{1}{\widehat{C}_1} \tag{S.32}$$

$$p_1 \;\ge\; 0 \tag{S.33}$$

$$p_3 \;>\; 1. \tag{S.34}$$

□

*2) Proof of Corollary III.3:*

*Proof.* The characterization of the constraints corresponding to (14) to (18), for each of the three cases of Lemma III.1, is given above in part 1, "Proof of Theorem III.2." When $t=0$, each of the constraints corresponds to a separating hyperplane or the space formed by intersecting hyperplanes (e.g., (S.26) is equivalent to $p_2 \ge 0$ and (S.24)). The same holds true when $t>0$, apart from the constraint for $\pi_1 \ge 0$ for Case 3, (S.25), which is quadratic in $p_1$ but which reduces to an intersection of hyperplanes. When $t<0$, it is straightforward to show the region is convex. Combining these statements proves the corollary. □

## C. *Proof of Theorem III.5*

The system {(S.18), (S.20), (S.19)} in matrix form is: $\widehat{C}_1(1+br)\mathbf{Q} = \mathbf{c}+\mathbf{q}\cdot\mathbf{p}$. Equivalently:

$$\widehat{C}_1(1+br)\begin{pmatrix} Q_{11} \\ Q_2 \\ Q_{13}\end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3\end{pmatrix} + (q_{ij}).\begin{pmatrix} p_1 \\ p_2 \\ p_3\end{pmatrix} \tag{S.35}$$

where

$$\mathbf{c} = \begin{pmatrix} 0 \\ b\widehat{C}_2 \\ \widehat{C}_1\end{pmatrix},\; \mathbf{q} = \begin{pmatrix} 0 & 0 & (1+br)\widehat{C}_1 \\ \widehat{C}_1\widehat{C}_2 & -\widehat{C}_1\widehat{C}_2 & (\widehat{C}_1-b+1)\widehat{C}_2 \\ -\widehat{C}_1\widehat{C}_2 & \widehat{C}_1\widehat{C}_2 & -(1+r+r\widehat{C}_1)\widehat{C}_1\end{pmatrix}. \tag{S.36}$$

Using (7) together with (S.35) gives:

$$\begin{aligned} \widehat{C}_1(1+br)\begin{pmatrix}\pi_1 \\ \pi_2 \\ \pi_3\end{pmatrix} \;&=\; \widehat{C}_1(1+br)\mathbf{P}\cdot\mathbf{Q} \\ &= \begin{pmatrix} p_1 & 0 & p_1+t \\ 0 & p_2 & 0 \\ 0 & 0 & p_3-t\end{pmatrix}\cdot(\mathbf{c}+\mathbf{q}\cdot\mathbf{p}). \end{aligned}$$

Taking derivatives

$$\begin{aligned} \widehat{C}_1(1+br)\begin{pmatrix}\frac{\partial\pi_1}{\partial p_1} \\ \frac{\partial\pi_2}{\partial p_2} \\ \frac{\partial\pi_3}{\partial p_3}\end{pmatrix} \;&=\; \begin{pmatrix}1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{pmatrix}\cdot(\mathbf{c}+\mathbf{q}\cdot\mathbf{p}) \\ &+\begin{pmatrix} q_{11}+q_{31} & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & q_{33}\end{pmatrix}\cdot\mathbf{p}+\mathbf{t}\begin{pmatrix} q_{31} \\ 0 \\ -q_{33}\end{pmatrix}. \end{aligned} \tag{S.37}$$

Hence at the potential N.E. where $\frac{\partial \pi_i}{\partial p_i} = 0$ for all $i$, a simultaneous turning point, the $p_i^*$ will satisfy

$$\begin{aligned} t\begin{pmatrix} -q_{31} \\ 0 \\ q_{33} \end{pmatrix} - \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \mathbf{c} &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \mathbf{q} \cdot \mathbf{p} + \begin{pmatrix} q_{11}+q_{31} & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & q_{33} \end{pmatrix} \cdot \mathbf{p} \\ &= \begin{pmatrix} 2(q_{11}+q_{31}) & q_{12}+q_{32} & q_{13}+q_{33} \\ q_{21} & 2q_{22} & q_{23} \\ q_{31} & q_{32} & 2q_{33} \end{pmatrix} \cdot \mathbf{p} \\ &= \tilde{\mathbf{q}} \cdot \mathbf{p} \end{aligned} \tag{S.38}$$

Now it follows from (S.37) that

$$\begin{pmatrix} \frac{\partial^2 \pi_1}{\partial p_1^2} \\ \frac{\partial^2 \pi_2}{\partial p_2^2} \\ \frac{\partial^2 \pi_3}{\partial p_3^2} \end{pmatrix} = 2\begin{pmatrix} q_{11}+q_{31} & 0 & 0 \\ 0 & q_{22} & 0 \\ 0 & 0 & q_{33} \end{pmatrix} \tag{S.39}$$

$$= 2\begin{pmatrix} -\widehat{C}_1\widehat{C}_2 & 0 & 0 \\ 0 & -\widehat{C}_1\widehat{C}_2 & 0 \\ 0 & 0 & -(1+r+r\widehat{C}_1)\widehat{C}_1 \end{pmatrix} \tag{S.40}$$

has strictly negative entries, (where we have used the definitions for $q_{ij}$ in (S.36)) hence the profit functions are strictly concave (in this Case 3), and hence there is a unique maximum. From (S.36), here $\tilde{\mathbf{q}}$ is given by:

$$\tilde{\mathbf{q}} = \widehat{C}_2 \begin{pmatrix} -2C_1 & \widehat{C}_1 & -\widehat{C}_1+b-1 \\ \widehat{C}_1 & -2\widehat{C}_1 & \widehat{C}_1-b+1 \\ -\widehat{C}_1 & \widehat{C}_1 & -2(\frac{1}{r}+1+\widehat{C}_1) \end{pmatrix}$$

and hence $\det \tilde{\mathbf{q}} = -2\widehat{C}_1^5 r^2[3+r(2+b+2\widehat{C}_1)] < 0$ (recall $\widehat{C}_2 = r\widehat{C}_1$). The left hand side of (S.38) is

$$\begin{pmatrix} -tq_{13}-c_1-c_3 \\ -c_2 \\ tq_{33}-c_3 \end{pmatrix} = -\widehat{C}_1 \begin{pmatrix} 1-\widehat{C}_2 t \\ br \\ 1+(1+r+\widehat{C}_2)t \end{pmatrix}.$$

Using Cramer's rule

$$p_1^* = \frac{r\widehat{C}_1^2}{\det \tilde{\mathbf{q}}} \det \begin{pmatrix} \widehat{C}_1 rt & \widehat{C}_1 & -[\widehat{C}_1-(b-1)]r \\ -(br+2)\widehat{C}_1 & -2\widehat{C}_1 & [\widehat{C}_1-(b-1)]r \\ -(1+r+r\widehat{C}_2)t\widehat{C}_1 & \widehat{C}_1 & -2(1+r+\widehat{C}_2) \end{pmatrix}.$$

Simplifying gives

$$p_1^* = \frac{1}{2\widehat{C}_1 r[3+r(2+b+2\widehat{C}_1)]}\Big((2+br)[2+br+r(1+\widehat{C}_1)] + rt\Big[(1+r)(b-1) - \widehat{C}_1\left[5(1+r)+4r\widehat{C}_1\right]\Big]\Big).$$

Hence if $t=0$, it follows that $p_1^*$ will be positive. Similarly,

$$p_2^* = \frac{1}{2\widehat{C}_1 r[3+r(2+b+2\widehat{C}_1)]}\Big(2+2(1+2b+\widehat{C}_1)r + b(3+b+3\widehat{C}_1)r^2 - tr\Big[b-1+\widehat{C}_1+(b-1-\widehat{C}_1+2b\widehat{C}_1)r\Big]\Big) \tag{S.41}$$

Hence

$$p_2^* > 0 \Leftrightarrow 0 \le t < \frac{2+2(1+2b+\widehat{C}_1)r + b(3+b+3\widehat{C}_1)r^2}{r(b-1+\widehat{C}_1+(b-1-\widehat{C}_1+2b\widehat{C}_1)r)}.$$

Similarly,

$$p_3^* = \frac{2+br+(3+3r+4\widehat{C}_1 r)t}{6+2r(2+b+2\widehat{C}_1)} \tag{S.42}$$

and hence $p_3^* > 0$ for all $t \ge 0$.

### D. *Proof of Theorem III.9 and Corollary III.10*

The following Lemma proves parts 1 and 2 of Theorem III.9. The candidate solution $\{p_i^*\}$ is a local optimum for each $i$. The requirements that the $p^*$ induce a feasible solutions result in the condition (26) together with the requirement that $p_1^* \ge 0$. If these conditions are satisfied, then either the $p_i^*$ constitute a Nash equilibrium, or, they are such that either network 1 or 2 could improve their profits by deviating, in which case $p_i^*$ is an $\epsilon$-equilibrium. In Lemma S.2 we prove part 3 of Theorem III.9, and also characterize the $\epsilon$ of part 2. Note that we first prove the theorems for general transfer price $t$, which includes the special case $t=0$ (c.f. Section F of this Supplementary Material).

#### *1) Proof of part 1 and 2 of Theorem III.9:*

**Lemma S.1.** *If a Nash equilibrium exists with positive prices $\{p_i^*\}$, given by* (20), (21), (22)*, with both networks 1 and 2 and the Content Provider having users and each making positive profit, then the transfer price satisfies*

$$-\frac{2+br}{3+r(3+4\widehat{C}_1)} < t < \frac{2+br}{3+r+2br}. \tag{S.43}$$

*Conversely, when* (26) *is satisfied and the expression for $p_1^*$ in* (20) *is positive, then the $\{p_i^*\}$ given by* (20), (21), (22) *constitute an $\epsilon$-equilibrium where $\epsilon \ge 0$. Further, all the prices $\{p_i^*\}$ are positive, and networks 1 and 2 and the Content Provider all have users.*

*Proof.* Since the prices $\{p_i^*\}$ are a local optimum for each $i$, it follows that the $p_i^*$ will be a non-degenerate $\epsilon-$equilibrium if and only if the prices are positive, the market is covered ($Q_1+Q_2=1$), the user masses are positive ($Q_{11}, Q_{13}, Q_2 > 0$), and the profits are positive. Since we solve the equations for the $Q_i$ ensuring the constraint $Q_{11}+Q_{13}+Q_2 = 1$ is met, necessary and sufficient conditions are that each $Q_{11}^*, Q_{13}^*, Q_2^*$ is in $(0,1)$, $p_i* > 0$ and $\pi_i^* > 0$.

i) Since $Q_{11}^* = p_3^*$, the condition $Q_{11}^* \in (0,1)$ is equivalent to,

$$0 < p_3^* < 1 \tag{S.44}$$

ii) Using (23), $Q_{13}^* \in (0,1)$ is equivalent to

$$0 < p_3^* - t < \frac{1+br}{1+r+r\widehat{C}_1} \tag{S.45}$$

iii) By construction $Q_2^* + Q_1^* = 1$, hence the requirement $Q_2^* \in (0,1)$ is equivalent to requiring $Q_1^* \in (0,1)$, which from (23) is equivalent to

$$0 < p_1^* + t < \frac{1+br}{r\widehat{C}_1} \tag{S.46}$$

These three conditions, together with the requirement that $p_1^* \ge 0$ also ensure that each $p_i^* \ge 0$ and each $\pi_i^* \ge 0$. Using inequality (S.44) and substituting from (S.42) gives the condition

$$-\frac{2+br}{3+r(3+4\widehat{C}_1)} < t < \frac{4+r(4+b+4\widehat{C}_1)}{3+r(3+4\widehat{C}_1)} \tag{S.47}$$

Using expression (S.45) and substituting from (S.42) gives the condition

$$-\frac{4+r(2+7b+2\widehat{C}_1+b(3+2b+3\widehat{C}_1)r)}{(3+r+2br)(1+r+\widehat{C}_1 r)} \;<\; t \;<\; \frac{2+br}{3+r+2br} \tag{S.48}$$

The conjunction of (S.47) and (S.48) gives the condition

$$-\frac{2+br}{3+r(3+4\widehat{C}_1)} \;<\; t \;<\; \frac{2+br}{3+r+2br}.$$

This condition also ensures that (S.46) is satisfied, completing the proof of the lemma. □

*2) Proof of part 3 of Theorem III.9, and characterization of $\epsilon$ of part 2.:*

**Lemma S.2.** *A Nash equilibrium exists if $t$ satisfies* (S.43) *and in addition:*

$$\Bigg(\Big[(2+br)(2+br+r(1+\widehat{C}_1))\Big] + \tag{S.49}$$

$$rt\Big[(1+r)(b-1)-\widehat{C}_1\Big(5(1+r)+4r\widehat{C}_1\Big)\Big] \geq 0 \tag{S.50}$$

*or* $\;t \leq 0\Bigg)$ *[ condition for $p_1^*$ to be positive]*

$$\textit{and}\;\Bigg(t \geq \frac{2+br}{1+r(1+\widehat{C}_1)} \quad \textit{or} \tag{S.51}$$

$$t \leq \frac{(2+br)(2+(1+\widehat{C}_1+b)r)}{6+r(11+b+15\widehat{C}_1+(b+2b\widehat{C}_1+(1+\widehat{C}_1)(5+8\widehat{C}_1))r)} \tag{S.52}$$

*or $t$ satisfies expression* (S.56)$\Bigg)$ *[ for $p_1^*$ to be optimal]*

$$\textit{and}\;\Bigg(t \leq -\frac{1+r}{br} \qquad \textit{or} \tag{S.53}$$

$$t \leq \frac{4+r\Big(-b^2r-b[2+r(1+\widehat{C}_1)]+4(1+\widehat{C}_1)[2+r(1+\widehat{C}_1)]\Big)}{6+r(11+b+9\widehat{C}_1+(b+(1+\widehat{C}_1)(5+4\widehat{C}_1))r)} \tag{S.54}$$

*or $t$ satisfies expression* (S.58) $\Bigg)$ *[ cond. for $p_2^*$ to be optimal]*

*Proof.* We prove that $p^*$ is a Nash equilibrium by fixing two of $\{p_1^*, p_2^*, p_3^*\}$ while allowing the other $p_i$ to vary, then showing conditions under which $p_i^*$ is optimal for $\pi_i$.

**$p_1^*$ is optimal for network 1**. With $p_2 = p_2^*$, $p_3 = p_3^*$, as we *increase* $p_1$ from $p_1^*$ we either stay in Region 3, or potentially move into Region 1. There are three mutually exclusive cases we need to consider:

i) For all $p_1 \geq p_1^*$ we remain in Region 3 and never move to Region 1, and hence $p_1^*$ is optimal.
The boundary between Regions 3 and 1 occurs when, from (S.23) and (S.10), $p_1 = p_1^B$ solves $p_3^* = [1+r(p_2^* - p_1)\widehat{C}_1]/[1+r(1+\widehat{C}_1)]$, (c.f. (S.73)), which substituting gives $p_1^B = [2+br-[1+r(1+2\widehat{C}_1)]t]/2r\widehat{C}_1$. Hence $p_1$ will stay in Region 3 if the boundary point is infeasible, $p_1^B \leq 0$, that is, if $t \geq \frac{2+br}{1+r(1+\widehat{C}_1)}$, i.e., (S.51).

ii) $\pi_i(p_1)$ is decreasing in Region 1 and hence $\pi_1(p_1) < \pi_1(p_1^*)$ $\forall p_1 \in$ Region 1.
Since $\pi_1(p_1)$ is convex in Region 1, a sufficient condition for this is $\frac{\partial \pi_1}{\partial p_1}|_{p_1=p_1^B} \leq 0$, which substituting and taking derivatives in (S.6), using (S.5) and substituting $p_2 = p_2^*$, $p_3 = p_3^*$, $p_1 = p_1 = p_1^B$ gives (S.52).

iii) There is a feasible local maximum for $\pi_1$ in Region 1, where the profit is given by $\breve{\pi}_1 = \breve{p}_1 \breve{Q}_1$, but

$$\pi_1^* \;=\; p_1^* Q_1^* \;+\; tQ_{13}^* \;\geq\; \breve{\pi}_1, \tag{S.55}$$

and hence again $p_1^*$ is optimal for $\pi_1$. The point $\breve{p}_1$ is where $\frac{\partial \pi_1}{\partial p_1} = 0$, that is when $0 = \frac{1+r(p_2^*-\breve{p}_1)\widehat{C}_1}{1+r(1+\widehat{C}_1)} - \breve{p}_1\frac{r\widehat{C}_1}{1+r(1+\widehat{C}_1)}$ and hence $\breve{p}_1 = \frac{1+r\widehat{C}_1 p_2^*}{2r\widehat{C}_1}$. At this point, the profit is given by $\breve{\pi}_1 = \breve{p}_1 \breve{Q}_1$, which is

$$\breve{\pi}_1 \;=\; \breve{p}_1^2 \frac{r\widehat{C}_1}{1+r(1+\widehat{C}_1)} \;=\; \frac{(1+r\widehat{C}_1 p_2^*)^2}{4r\widehat{C}_1(1+r(1+\widehat{C}_1))}.$$

Substituting for $Q_1^*$ from (23) in (S.55) gives the full condition as following the quadratic relation on $t$,

$$p_1^* \frac{r\widehat{C}_1}{1+br}(p_1^*+t) + t\frac{1+r(1+\widehat{C}_1)}{1+br}(p_3^*-t) \;\geq\; \frac{(1+r\widehat{C}_1 p_2^*)^2}{4r\widehat{C}_1(1+r(1+\widehat{C}_1))} \tag{S.56}$$

where $p_i^*$ are given in (20),(21),(22).
In the case that (S.56) does not hold (which necessarily also requires that (S.51) and (S.52) are not satisfied), define

$$\epsilon_1 \;:=\; \frac{(1+r\widehat{C}_1 p_2^*)^2}{4r\widehat{C}_1(1+r(1+\widehat{C}_1))} - p_1^* \frac{r\widehat{C}_1}{1+br}(p_1^*+t) + t\frac{1+r(1+\widehat{C}_1)}{1+br}(p_3^*-t).$$

Finally, if we *decrease*, $p_1$, we potentially move to Region 2. But we know that $\pi_1$ is decreasing in Region 2, and hence $\pi_1(p_1) \leq \pi(p_1^{B23}) < \pi_1(p_1^*)$ f or all $p_1 \in$ Region 2 where $p_1^{B23}$ is the value of $p_1$ at the boundary of Regions 2 and 3.

**$p_2^*$ is optimal for network 2**. The proof mirrors the arguments for showing $p_1^*$ is optimal for network 1. With $p_1 = p_1^*$, $p_3 = p_3^*$, as we *decrease* $p_3$ from $p_2^*$ we either stay in Region 3, or move into Region 1. There are three mutually exclusive cases we need to consider:

i) For $p_2 \leq p_2^*$ we remain in Region 3 and never move to Region 1.
At the boundary point, $p_2^B$ solves $p_3^* = \frac{1+r(p_2-p_1^*)\widehat{C}_1}{1+r(1+\widehat{C}_1)}$, that is, $p_2^B = \frac{t+r(b+t)}{2\widehat{C}_1 r}$ . The condition that is infeasible ($p_2^B < 0$) or zero gives condition (S.53)

ii) $\pi_2(p_2)$ is increasing Region 1 and hence $\pi_2(p_2) < \pi_2(p_2^*)$ $\forall p_2 \in$ Region 1, since $\pi_2(p_2)$ is concave in the interior of Region 1 .
Now using (S.7), differentiating and substituting $p_1 = p_1^*, p_2 = p_2^B, p_3 = p_3^*$ gives that $\frac{\partial \pi_2}{\partial p_2}|_{p_2^B} = 1 - p_3^* - \frac{t+r(b+t)}{2[1+r(1+\widehat{C}_1)]}$ which will be non-negative if and only if (S.54) holds.

iii) There is a feasible local maximum for $\pi_2$ in Region 1 at the point $\breve{p}_2$, with profit given by by $\breve{p}_2\breve{Q}_2$, but for which

$$\breve{\pi}_2 \;\leq\; \pi_2^* = p_2^* Q_2^* \tag{S.57}$$

and hence $p_2^*$ is optimal for $\pi_2$. $\breve{p}_2$ is the point in Region 1 at which $\frac{\partial \pi_2}{\partial p_2} = 0$, which using (S.7) and (S.5) gives the point $\breve{p}_2 = \frac{1+\widehat{C}_1+\widehat{C}_1 p_1^*}{2\widehat{C}_1}$. Using (20) gives the profit

$$\breve{\pi}_2 = \left(\frac{1+\widehat{C}_1+\widehat{C}_1 p_1^*}{2\widehat{C}_1}\right)^2 \frac{r\widehat{C}_1}{1+r(1+\widehat{C}_1)}.$$

$Q_2^*$ is given by (21), and (23) and hence substituting (S.57) is the condition

$$\left(\frac{1+\widehat{C}_1+\widehat{C}_1 p_1^*}{2\widehat{C}_1}\right)^2 \frac{r\widehat{C}_1}{1+r(1+\widehat{C}_1)} \leq p_2^* \frac{r\widehat{C}_1}{1+br} p_2^* \tag{S.58}$$

a quadratic in $t$, where the $p_i^*$ are given in (20),(21).
In the case that (S.58), (S.53), (S.54) all fail to hold, define

$$\epsilon_2 \;:=\; p_2^* \frac{r\widehat{C}_1}{1+br} p_2^* - \left(\frac{1+\widehat{C}_1+\widehat{C}_1 p_1^*}{2\widehat{C}_1}\right)^2 \frac{r\widehat{C}_1}{1+r(1+\widehat{C}_1)}.$$

Finally, if network 2 *increases* its price above $p_2^*$, it potentially moves to region 2; but network 2 receives zero profit in Region 2, hence network 2 has no incentive to increase its price above $p_2^*$.

**$p_3^*$ is optimal for the Content Provider**. We show that under the conditions of the lemma, $p_3^*$ is optimal for the Content Provider with no further restrictions.

i) If $\widehat{C}_1 \geq b-1$: As we decrease $p_3$, we remain in Region 3 and hence $p_3^*$ is optimal for all $p_3 \leq p_3^*$. Moving to Region 2 is not possible since the boundary is infeasible: the Region 2-3 boundary is the point $p_3 \geq t$ which satisfies $p_3^B = \frac{(p_2^*-p_1^*)-b}{1-b+\widehat{C}_1}$. Substituting for $p_1^*$ and $p_2^*$ gives

$$\begin{aligned} p_3^B \;=\; & \Big(-1 - br(3+(1+\widehat{C}_1+b)r) \\ & +(2\widehat{C}_1+1-b)r[1+r(1+\widehat{C}_1)]t\Big) \\ & \div \Big((\widehat{C}_1+1-b)r[3+(2+b+2\widehat{C}_1)r]\Big) \quad \text{(S.59)} \end{aligned}$$

and when $-\frac{2+br}{3+3r+4r\widehat{C}_1} \leq t \leq \frac{2+br}{3+r+2br}$, this implies $p_3^B < t$, and hence we never move to Region 2. Conversely, *increasing* $p_3$ either causes us to remain in Region 3, or move potentially move to Region 1 where the Content Provider receives zero profit, hence $p_3^*$ is optimal for all $p_3 > p_3^*$.

ii) $\widehat{C}_1 < b-1$: the only way to violate (S.24) is to *increase* $p_3$. In this scenario with $p_1^*$ and $p_2^*$ then (S.59) with the condition $0 \leq t \leq \frac{2+br}{3+r+2br}$ implies $p_3 \geq 1$ and hence $\pi_3 = 0$. If instead $-\frac{2+br}{3+3r+4r\widehat{C}_1} \leq t < 0$, then to have an interior maximum in Region 2, it is necessary for both $p_3^B < 1$ and $p_3 < (1+t)/2$, and we can show that these three conditions cannot simultaneously hold, and hence if we enter Region 2, the value of $\pi_3$ will decrease. Hence $p_3^*$ is optimal .

**Summary.** Necessary and sufficient conditions for a non-degenerate Nash equilibrium to exist at $p^*$ in are that Lemma S.2 holds, i.e., (S.43) and {(S.50) or $t \leq 0$} and {(S.51) or (S.52) or (S.56)} and { (S.53) or (S.54) or (S.58)} hold.

When only the necessary conditions hold ( (S.43) and {(S.50) or $t \leq 0$}) but not all the other conditions for Lemma S.2, (so either {(S.51) and (S.52) and (S.56)} are all false, or {(S.53) and (S.54) and (S.58)} are all false,) then $p^*$ is an $\epsilon$-equilibrium, not a Nash equilibrium, where $\epsilon = \max\{\epsilon_1, \epsilon_2\}$. ☐

### *3) Proof of Corollary III.10:*

*Proof.* From (20) and the conditions of the corollary, we obtain $p_1^* > 0$. The result follows from part 2 of Theorem III.9. ☐

## *E. Proof of Theorem III.11*

We provide here a more detailed presentation of Theorem III.11 than is given in Section III.

THEOREM III.11. There are only three possibilities for degenerate equilibria. Specifically, there exists a value $t^A$ (which can be computed) such that:

1) If $t \geq t^A$, then there exists a Nash equilibrium in which the Content Provider prices itself out of the market by setting $p_3^* = t$, $Q_{13}^* = 0$, and for networks 1 and 2, the prices, user masses, and profits are given by

$$p_1{}^* \;=\; \frac{2+r+r\widehat{C}_1}{3r\widehat{C}_1}, \qquad p_2{}^* \;=\; \frac{1+2r+2r\widehat{C}_1}{3r\widehat{C}_1} \quad \text{(S.60)}$$

where the user masses are

$$Q_1^* \;=\; \frac{2+r+r\widehat{C}_1}{3(1+r+r\widehat{C}_1)} \qquad Q_2^* \;=\; \frac{1+2r+2r\widehat{C}_1}{3(1+r+r\widehat{C}_1)} \quad \text{(S.61)}$$

and the profits are

$$\pi_1^* \;=\; \frac{(2+r+r\widehat{C}_1)^2}{9r\widehat{C}_1(1+r+r\widehat{C}_1)} \qquad \pi_2^* \;=\; \frac{[1+2(1+\widehat{C}_1)r]^2}{9r\widehat{C}_1(1+r+r\widehat{C}_1)}. \quad \text{(S.62)}$$

2) If $t < 0$, then network 1 provides a subsidy to the Content Provider for each user, i.e., let $s := -t$. Then if the subsidy $s$ is sufficiently great, viz., it is at least $\frac{2+br}{3+r(3+4\widehat{C}_1)}$, then there will be a unique Nash equilibrium where the Content Provider sets $p_3^* = 0$, and $Q_{11}^* = 0$. There are two subcases:

a) If $s \leq \frac{2+br}{r\widehat{C}_1}$, then the equilibrium is:

$$p_1^* \;=\; \frac{2+r(b+2\widehat{C}_1 s)}{3\widehat{C}_1 r}, \qquad p_2^* \;=\; \frac{1+r(2b+\widehat{C}_1 s)}{3\widehat{C}_1 r} \quad \text{(S.63)}$$

where

$$Q_{13}^* \;=\; \frac{2+r(b-\widehat{C}_1 s)}{3(1+br)}, \qquad Q_2^* \;=\; \frac{1+r(2b+\widehat{C}_1 s)}{3(1+br)} \quad \text{(S.64)}$$

with profits given by

$$\begin{aligned} \pi_1 \;&=\; \frac{[2+r(b-\widehat{C}_1 s)]^2}{9r\widehat{C}_1(1+br)}, \\ \pi_2 \;&=\; \frac{[1+r(2b+\widehat{C}_1 s)]^2}{9r\widehat{C}_1(1+br)}, \quad \text{(S.65)} \\ \pi_3 \;&=\; \frac{[2+r(b-\widehat{C}_1)]s}{3(1+br)}. \end{aligned}$$

b) If $s > \frac{2+br}{r\widehat{C}_1}$, then network 1 chooses a price of at least $s$; in consequence all users will choose network 2, $Q_2^* = 1$, and the equilibrium is: $p_1^* = s$, $p_2^* = s - \frac{1}{r\widehat{C}_1}$, with network 2 capturing all the profit, $\pi_2^* = s - \frac{1}{r\widehat{C}_1}$.

3) If $-\frac{2+br}{3+r(3+4\widehat{C}_1)} < t < t^A$, there exists a set of parametric conditions under which a Nash equilibrium exists where the optimal strategy for network 1 is to set its price to zero. Specifically, these are:

$$\widehat{C}_1 > b + 1/r \quad \text{(S.66)}$$

$$\frac{(2+br)(2+br+(1+\widehat{C}_1)r)}{r\{(1+r)(1-b)+\widehat{C}_1\,[5(1+r)+4\widehat{C}_1 r]\}} \;\leq\; t \;<\; \frac{2+br}{2+br+(1+\widehat{C}_1)r} \quad \text{(S.67)}$$

$$\frac{[(2+br)(1-t)-(1+\widehat{C}_1)rt]t}{1+br} \;\geq\; \frac{[2(2+br)+(1+\widehat{C}_1-b)rt)]^2}{4\widehat{C}_1 r(4+[3(1+\widehat{C}_1)+b]\,r)}, \quad \text{(S.68)}$$

which necessarily imply $t > 0$. The unique equilibrium is:

$$p_1^* = 0 \quad \text{(S.69)}$$

$$p_2^* = \frac{1+b+\widehat{C}_1+2br(1+\widehat{C}_1)+(1+\widehat{C}_1-b)[1+r(1+\widehat{C}_1)]t}{\widehat{C}_1\,(4+[3(1+\widehat{C}_1)+b]\,r)} \quad \text{(S.70)}$$

$$p_3^* = \frac{2+br+2[1+r(1+\widehat{C}_1)]t}{4+[3(1+\widehat{C}_1)+b]\,r}, \quad \text{(S.71)}$$

where

$$Q_{11}^* = p_3^*, \quad Q_2^* = \frac{r\widehat{C}_1}{1+br}p_2^*, \quad Q_{13}^* = \frac{1+r(1+\widehat{C}_1)}{1+br}(p_3^*-t),$$

and where the profits $\pi_1$, $\pi_2$, and $\pi_3$ can be calculated from (7).

*Proof.* First note that no Nash equilibrium is possible in Region 2, where $\nabla_{12} > \nabla_{13}$, since in that case the feasibility condition is $p_2 > p_1 + p_3 + p_3 + b(1-p_3)/\widehat{C}_1$, where the r.h.s is strictly greater than zero, and thus network 2 can decrease its price until equality holds, attracting users and moving out of Region 2 into Region 3. We first consider the three parts of the theorem, and then show that no other degenerate Nash equilibria exist.

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*1)* $\mathbf{t \geq t^A}$*:* If the transfer price $t$ is sufficiently large, the condition $p_3 \geq t$ implies $Q_{13} = 0$, and hence we are in Case 1 of Lemma III.1.

From (A.9), we have in the non-degenerate case the optimal critical value of $w$, is given by:

$$\nabla_{12} \;=\; \frac{(p_2 - p_1)r\widehat{C}_1 + 1}{r\widehat{C}_1 + r + 1}.$$

The profits are $\pi_1 = p_1 \nabla_{12} = p_1 \left[\frac{(p_2 - p_1)r\widehat{C}_1 + 1}{r\widehat{C}_1 + r + 1}\right]$, and $\pi_2 = p_2(1 - \nabla_{12}) = p_2\left[\frac{[1-(p_2-p_1)]r\widehat{C}_1 + r}{r\widehat{C}_1 + r + 1}\right]$. From the first derivatives, we obtain the unique optimal prices (S.60), (optimal since the second derivatives are negative). The optimal critical value of $w$, i.e., the optimal value of $\nabla_{12}$, is given by:

$$\nabla_{12}^* \;=\; \frac{2 + r + r\widehat{C}_1}{3(1 + r + r\widehat{C}_1)} \;=\; \frac{2 + r + \widehat{C}_2}{3(1 + r + \widehat{C}_2)}. \qquad \text{(S.72)}$$

By (S.5), the mass of users on the networks at equilibrium are given by (S.61). □

The optimal choice of $p_1$, $p_2$, namely $p_1^*$, $p_2^*$, are given by (S.60) and the $Q_i^*$ from (S.61). Since $Q_{13} = 0$ and $Q_{11} = Q_1$, we know that (S.60) and $p_3^* \geq t$ constitute a local equilibrium. To prove that these values constitute a Nash equilibrium, we need to show that for $t \geq t^A$, given $p_2^*$ and $p_3{}^*$, network 1 cannot benefit by altering its price from $p_1^*$, with corresponding statements for network 2 and the Content Provider.

$\mathbf{p_3^* = t}$ **is optimal for the Content Provider**. At this value the Content Provider has zero profit. For the Content Provider to have a nonzero profit, we require $p_3 \geq t$, hence the Content Provider cannot lower its price below this value. With the given values of $p_1^*$, $p_2^*$, rasing the price above $t$ also generates zero profit. Hence $p_3 = t$ is optimal for the Content Provider.

By Lemma III.1, we must have: $Q_2^*/r\widehat{C}_1 - gQ_1^*/\widehat{C}_1 < p_1^* + p_3 - p_2^*$. Substituting in (27) implies this will hold iff $p_3$ is greater than $[2+r(1+\widehat{C}_1)]/[3(1+r(1+\widehat{C}_1))]$. Hence we must have

$$t^A \;>\; \frac{2 + r(1+\widehat{C}_1)}{3[1 + r(1+\widehat{C}_1)]},$$

since otherwise the Content Provider could lower its price, $p_3$, below this value to attract users until equality holds.

$\mathbf{p_1^*}$ **is optimal for network 1**.

For the given values $p_1^*$ and $p_2^*$, we are in the middle subcase of the alternative defining condition for Case 1 of Lemma III.1, (S.10). If network 1 raises its price from $p_1^*$, it will remain in Case 1 and hence $p_1^*$ will remain the optimal response to $p_2^*$.

If network 1 decreases its price from $p_1^*$, it is possible that $\pi_1(p_1; p_2^*; p_3^*) > \pi_1(p_1^*; p_2^*; p_3^*)$, in which case $p_1^*$ is *not* an equilibrium. It is straightforward to shown that this can only happen if $p_1$ moves to be in Region 3, and has a greater local optimum in region 3 . We need to consider the cases (A) $p_3^* = t \leq 1$ and (B) $t > 1$ separately.

We consider subcase B first.

(B) $p_3^* > 1$. In this subcase, the first inequality in (S.10) is violated by becoming an equality as $p_1$ is decreased. By (S.31), $Q_1 = 1$, and by (S.32), at this point $p_1 = p_2^* - p_3^* - \frac{1}{\widehat{C}_1}$, with profit $\pi_1$ less than the profit at $p_1^*$, i.e.

$$\begin{aligned}\pi_1 \;=\; p_1 Q_1 \;=\; p_1 \;&<\; p_2^* - 1 - \frac{1}{\widehat{C}_1} \\ &=\; \frac{1 - r - rc}{3r\widehat{C}_1} \\ &<\; \frac{(2 + r + r\widehat{C}_1)^2}{9r\widehat{C}_1[1 + r(1+\widehat{C}_1)]} \;=\; p_1^* Q_1^*.\end{aligned}$$

If we were to reduce $p_1$ even further, then we would immediately move to Case 2 Theorem III.1, since (S.32) is an equality, where $\pi_1(p_1)$ decreasing as we decrease $p_1$. Hence no higher value of the profit is possible in case B.

(A) $p_3^* \leq 1$. For this to have a local maximum in Region 3 such that $\pi_1(p_1; p_2^*; p_3^*) > \pi_1(p_1^*; p_2^*; p_3^*)$ we require:

(i) The boundary between Region 1 and Region 3 to be feasible for $p_1$. At this boundary point the second inequality in (S.10) is violated by becoming an equality,

$$t = p_3^* \;=\; \frac{1 + r(p_2^* - p_1^{13})\widehat{C}_1}{1 + r(1+\widehat{C}_1)}. \qquad \text{(S.73)}$$

Substituting for $p_2^*$ from (27) and simplifying gives the condition that $p_1^{13} > 0$ if and only if

$$t < \frac{4 + 2r + 2r\widehat{C}_1}{3 + 3r + 3r\widehat{C}_1}. \qquad \text{(S.74)}$$

(ii) The derivative of the profit $\partial\pi_1/\partial p_1$ at the boundary $p_1^{13}$ is negative. It follows from (7), (S.21), (S.19), (S.18), and from $Q_{13} = 0$, and (S.73), that:

$$\frac{\partial \pi_1}{\partial p_1}|_{p_1^{13}} \;=\; p_1\frac{\partial Q_1}{\partial p_1} + Q_1 + t\frac{\partial Q_{13}}{\partial p_1} \;=\; t - (p_1^{13} + t)\frac{r\widehat{C}_1}{1 + br}.$$

Substituting for $p_1$ from (S.73) and for $p_2^*$ from (27) yields the condition:

$$t \;<\; \frac{2(2 + r + r\widehat{C}_1)}{3(2 + r + br)} \quad\Longleftrightarrow\quad \frac{\partial \pi_1}{\partial p_1} \;<\; 0. \qquad \text{(S.75)}$$

(iii) There is a feasible local optimum in region 3. Taking derivatives $\partial\pi_1/\partial p_1$ using (7), (S.21), (S.19), (S.18), and solving for $p_1^0$ such that $\partial_{p_1}\pi_1|p_1^0 = 0$, gives on substituting $p_3^* = t$,

$$p_1^0 = \frac{1 + p_2^* r\widehat{C}_1 + (b - 1 - 2\widehat{C}_1)rt}{2r\widehat{C}_1}$$

and hence substituting for $p_2^*$

$$t(3r[1 - b + 2\widehat{C}_1]) < 2(2 + 2r + 2r\widehat{C}_1) \quad\Longleftrightarrow\quad p_1^0 > 0. \qquad \text{(S.76)}$$

(iv) The feasible local optimum generates higher profit. That is $\pi_1(p_1^0; p_2^*; p_3^*) > \pi_1(p_1^*; p_2^*; p_3^*)$. Substituting gives the condition

$$\frac{1}{36r\widehat{C}_1(1 + br)}\Big(4(2 + r + r\widehat{C}_1)^2 + 12(b-1)r(2 + r + r\widehat{C}_1)t + 9r\left((b-1)^2 r - 4c(1 + br)\right)t^2\Big) > \frac{(2 + r + r\widehat{C}_1)^2}{9r\widehat{C}_1(1 + r + r\widehat{C}_1)}$$

which is equivalent to the condition

$$\begin{cases} t < t_l(b, r, \widehat{C}_1) \text{ OR } t > t_u(b, r, \widehat{C}_1) & \text{if } (b-1)^2 r \geq 4\widehat{C}_1(1 + br) \\ t_l(b, r, \widehat{C}_1) < t < t_u(b, r, \widehat{C}_1) & \text{if } (b-1)^2 r < 4\widehat{C}_1(1 + br) \end{cases} \qquad \text{(S.77)}$$

where $t_l, t_u$ are the upper and lower roots of the equation

$$9\left((b-1)^2 r - 4\widehat{C}_1(1+br)\right)t^2 + 12(b-1)(2+r(1+\widehat{C}_1))t - \frac{4}{1+r(1+\widehat{C}_1)}(b-1-c)(2+r(1+\widehat{C}_1))^2 = 0.$$

With a slight abuse of notation, we shall let (S.74) etc to refer to the conditions on $t$: hence $p_1^*$ is *not* optimal for network 1 only if (S.74) AND (S.75) AND (S.76) AND (S.77) hold, thus $p_1^*$ is optimal if NOT ((S.74) AND (S.75) AND (S.76) AND (S.77)).

**$p_2^*$ is optimal for network 2.**
The proof mirrors that for showing $p_1^*$ is optimal. For network 2, if we decrease $p_2$ from $p_2^*$, we remain in Case 1 of Theorem III.1, and hence cannot improve upon $\pi_2(p_2^*)$.

As we increase $p_2$ above $p_2^*$ it is possible $\pi_2(p_1^*; p_2; p_3^*) > \pi_2(p_1^*; p_2^*; p_3^*)$, in which case $p_2^*$ is *not* an equilibrium. For this to happen, $p_2$ must force a move to region 3, and network 2 must have a greater local optimum in that region. We need to consider the cases (A) $p_3^* = t \le 1$ and (B) $t > 1$ separately.

For case (B), $p_3^* = p_2 - \frac{1}{\widehat{C}_1} + p_1^*$ . At boundary point $\pi_2 = 0$ and as we increase $\pi_2$ still further we move into Case 2, hence no greater profit for network 2 is possible in this case.

(A) $\pi_2(p_1^*; p_2; p_3^*) > \pi_2(p_1^*; p_2^*; p_3^*)$ for some $p_2$ requires

(i) The boundary between Region 1 and Region 3 to be feasible for $p_2$. At the boundary $p_2 = p_2^{13}$ solves

$$t = p_3^* = \frac{1 + r(p_2^{13} - p_1^*)\widehat{C}_1}{1 + r + r\widehat{C}_1}. \tag{S.78}$$

Substituting $p_1^*$ from (27), and simplifying gives the condition that $p_2^{13}$ will be positive is

$$t > \frac{1 - r - \widehat{C}_1 r}{3 + 3r + 3\widehat{C}_1 r}. \tag{S.79}$$

(ii) The derivative $\partial\pi_2/\partial p_2$ at the boundary $p_2 = p_2^{13}$ is positive. Now

$$\frac{\partial \pi_2}{\partial p_2} = -p_2 \frac{r\widehat{C}_1}{1+br} + (1 - p_3^*).$$

Using (S.78) and (27), this will be positive if

$$t = p_3^* < \frac{4 + 3br - (1+\widehat{C}_1)r}{3[2 + br + (1+\widehat{C}_1)r]}. \tag{S.80}$$

(iii) There is a feasible local optimum in region 3. The local optimum for $\pi_2$ in region occurs at the price $p_2^0$ where $\partial_{p_2}\pi_2|p_2^0 = 0$, which substituting gives the value

$$p_2^0 = \frac{2 + r + 3br + r\widehat{C}_1 + 3(1 - b + \widehat{C}_1)rt}{6r\widehat{C}_1}$$

which will be feasible provided that

$$2 + r + 3br + r\widehat{C}_1 + 3(1 - b + \widehat{C}_1)rt > 0. \tag{S.81}$$

(iv) The feasible local optimum generates higher profit. The profit at $p_2^0$ is given by

$$\pi_2(p_1^*; p_2^0; p_3^*) = \frac{(2 + r + 3br + r\widehat{C}_1 + 3(1 - b + \widehat{C}_1)rt)^2}{36r\widehat{C}_1(1+br)}$$

and this will be greater than $\pi_2(p_1^*; p_2^*; p_3^*) = \frac{(1+2(1+\widehat{C}_1)r)^2}{9r\widehat{C}_1(1+r+r\widehat{C}_1)}$ provided that

$$t < t_l(b, r, \widehat{C}_1) \text{ OR } t > t_u(b, r, \widehat{C}_1) \tag{S.82}$$

where $t_l, t_u$ are the upper and lower roots of the quadratic in $t$

$$(2 + r + 3br + r\widehat{C}_1 + 3(1 - b + \widehat{C}_1)rt)^2 = \frac{4(1+br)(1+2(1+\widehat{C}_1)r)^2}{1+r+rc}.$$

Hence $p_2^*$ is *not* optimal only if (S.79) AND (S.80) AND (S.81) AND (S.82) hold, and hence $p_2^*$ is optimal if NOT ( (S.79) AND (S.80) AND (S.81) AND (S.82) ).

**$p_3^*$ is optimal for the Content Provider**. Trivial.

**Summarizing**, hence necessary and sufficient conditions for a Nash equilibrium to exist in this case are that NOT ( (S.79) AND (S.80) AND (S.81) AND (S.82) ) AND NOT ( (S.74) AND (S.75) AND (S.76) AND (S.77)). Hence by choosing the simpler conditions in this expression, it follows that *sufficient* conditions for a Nash equilibrium to exist with these $p_i^*$ are that $t \ge t^A$, where

$$t^A = \min\left\{1, \max\left(\frac{4+3br-(1+\widehat{C}_1)r}{3[2+br+(1+\widehat{C}_1)r]}, \frac{2+r(1+\widehat{C}_1)}{3[1+r(1+\widehat{C}_1)]}, \frac{2[2+r(1+\widehat{C}_1)]}{3[2+br+r]}\right), \max\left(\frac{4+3br-(1+\widehat{C}_1)r}{3[2+br+(1+\widehat{C}_1)r]}, \frac{2[2+r(1+\widehat{C}_1)]}{3[1+r(1+\widehat{C}_1)]}\right)\right\}. \tag{S.83}$$

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### *2)* $\mathbf{t < 0}$, $\mathbf{s = -t \ge \frac{2+br}{3+r(3+4\widehat{C}_1)}}$: e

When the subsidy is large enough, the Content Provider can set its price $p_3$ to zero. Now when $p_3 = 0$, then by (S.16), $Q_{11} = 0$, and basic service is never used in this case. First consider the case

$$\frac{2+br}{r\widehat{C}_1} \ge s \ge \frac{2+br}{3+r(3+4\widehat{C}_1)}$$

corresponding to subcase 2(a) of Theorem III.5.

*Proof.* Proof of subcase 2(a) of III.11 It is straightforward to check that the $p_i$ in the system (30) with the corresponding $Q$'s are given in (31) are consistent with being in Region 3, and moreover $p_1^*, p_2^*$ satisfy the first order conditions for $\pi_1, \pi_2$, i.e., $\frac{\partial \pi_i}{\partial p_i} = 0$, when $p_3 = 0$. The second order conditions are also satisfied for networks 1 and 2. The condition for $\pi_3$ to have a maximum at $p_3 = 0$ is that $\frac{\partial_{p_3}}{\partial \pi_3} \le 0$. For this to hold when $p = p^*$ requires:

$$\left.\frac{\partial \pi_3}{\partial p_3}\right|_{p=p^*} = s\frac{\partial Q_{13}}{\partial p_3} + Q_{13} = \frac{2 + br - (3 + 3r + 4\widehat{C}_1 r)s}{3 + 3br} \le 0 \tag{S.84}$$

and hence

$$s \ge \frac{2+br}{3+r(3+4\widehat{C}_1)}. \tag{S.85}$$

The requirement that $Q_{13} \ge 0$ necessitates that

$$s \le \frac{2+br}{r\widehat{C}_1}.$$

When both conditions, (S.84) and (S.85), on $s$ are satisfied, the remaining feasibility requirements ($Q_{13} \le 1$, $\pi \ge 0$) are also satisfied, hence we have shown that vector $(p_i^*)$ is a local maximum. It remains to prove that these are globally optimum prices.

$p_1^*$ *is optimal*. While we remain in Region 3, we know $p_1^*$ is optimal. By increasing $p_1$, we remain in Region 3 until we reach the boundary with region 1, at which point from (S.19), $Q_{13} = 0$ (recall $p_3^* = 0$), hence $Q_1 = 0$, $\pi_1 = 0$ and this profit remains zero remains so as we increase $p_1$ further (c.f. (S.5) final condition). Now consider what happens when we decrease $p_1$ below $p_1^*$. If we decrease $p_1$ sufficiently, then we potentially enter Region 2. For this to happen, since $p_2 = p_2^*$, it follows from (S.15) that

$$p_1 \; < \; p_2^* - \frac{b}{\widehat{C}_1} \; = \; \frac{1 - br + r\widehat{C}_1 s}{3r\widehat{C}_1}.$$

But then, to have a higher profit than that corresponding to the prices given in (30), (30) we require

$$\frac{1 - br - 2r\widehat{C}_1 s}{3r\widehat{C}_1} \; > \; \frac{(2 + br - r\widehat{C}_1 s)^2}{9r\widehat{C}_1(1+br)}$$

which is clearly impossible for $s > 0$ (recall that $b > 1, r > 0, c > 0$). Hence $p_1^*$ is optimal.

$p_2^*$ *is optimal*. As we increase $p_2$ and remain in Region 3 (holding $p_3 = 0$ and $p_1 = p_1^*$), the profit for network 2 remains suboptimal (less than $\pi_2^*$), until we enter Region 2. But network 2 has zero profit in Region 2. Hence network 2 cannot increase its profit by increasing its price. If network 2 were to decrease its price from $p_2^*$, then again its profit would be suboptimal while in Region 3. If it were to decrease its price further to enter Region 1; then at the boundary of region and 1 and 3, from (S.20) $Q_{13} = 0$ (recall $p_3^* = 0$) and hence $Q_2$=1. The profit at this point is less than $\pi_2^*$, and decreasing $pi_2$ further decreases the profit further.

$p_3^*$ *is optimal*. Since $p_3^* = 0$, the Content Provider can only increase its price, which potentially takes it to Region 1 where the Content Provider has no users and zero profit. Hence the Content Provider cannot increase its profit by changing its price.

Hence we have shown that $p^*$ is indeed a Nash equilibrium, thus proving the subcase 2(a). □

Now consider case 2(b) of the theorem where

$$s > \frac{2 + br}{r\widehat{C}_1}.$$

*Proof.* Proof of subcase 2(b) of Theorem III.11 With the $p_i$ given by $p_1^* = s$, $p_2^* = s - \frac{1}{r\widehat{C}_1}$, $p_3^* = 0$, we are in Region 3 with $Q_2 = 1$ .

We now show that this is indeed a Nash equilibrium. If the Content Provider were to lower its price it would become negative and potentially move the scenario to Region 2. However, $p_3 < 0$ violates Region 2 feasibility condition (S.13). If the Content Provider were to raise its price, this could potentially move the scenario to Region 1, but in that case the Content Provider still receives zero profit. Thus the Content Provider will not change its price.

Network 2 has no incentive to lower its price, since network 2 already has all the users, and this could only result in lowering its profit. network 2 has no incentive to raise its price either, since this would potentially move the scenario to Region 2, but in Region 2 we have $Q_2 = 0$, and so network 2 would have no profit. Thus network 2 will not change its price.

If network 1 were to decrease its price below $s$, this would potentially move the scenario to Region 2; but Region 2 feasibility condition (S.14) is $p_1 - s(1 - p_3) \geq 0$, which reduces here to $p_1 - s \geq 0$, a contradiction, since $p_1 - s < 0$. If network 1 were to increase its price, this would potentially move the scenario to Region 1; but in this case, (S.10) and (S.5) imply $Q_1 = 0$, and network 1 would still get no profit. Thus network 1 will not benefit by changing its price. □

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*3)* $\mathbf{t > 0}$**:**

*Proof.* When $p_1 = 0$ and we are in Region 3, then from (7), $\pi_1 = tQ_{13}$, and using (S.19) and (S.21)

$$\frac{\partial \pi_1}{\partial p_1} \; = \; Q_1 + t\frac{\partial Q_{13}}{\partial p_1} \; = \; Q_1 - t\frac{r\widehat{C}_1}{1 + br}$$
$$= \frac{1 + p_3(b - 1 - r\widehat{C}_1) + r\widehat{C}_1(p_2 - t)}{1 + br}.$$

For $p_1 = 0$ to be a Nash equilibrium for network 1, it must first be a local maximum within Region 3, which requires

$$\pi_1 \geq 0 \quad \text{and} \quad \frac{\partial \pi_1}{\partial p_1} < 0. \tag{S.86}$$

In addition $p_2$ and $p_3$ need to be local maxima, and hence solve

$$\frac{\partial \pi_2}{\partial p_2} = 0, \;\; \frac{\partial \pi_3}{\partial p_3} = 0 \quad \text{with } p_1 = 0. \tag{S.87}$$

Using equations (S.19), (S.20), then (S.87) becomes

$$\begin{aligned} b + (1 + \widehat{C}_1 - b)p_3 \; &= \; 2\widehat{C}_1 p_2 \\ 2p_3(1 + r + r\widehat{C}_1) \; &= \; 1 + t(1 + r + r\widehat{C}_1) + r\widehat{C}_1 p_2. \end{aligned}$$

Solving these equations gives (S.70) and (S.71) which is a local optimum since the $\pi_i(p_i)$ are convex. Substituting into (S.86) and using algebraic manipulation gives the conditions (S.66) and (S.67) on $b, r, \widehat{C}_1, t$ in the proposition, which also enure that $p_2^* > 0$ and $p_3^* > 0$. The profit for network 1 is then

$$\frac{1 + r(1 + \widehat{C}_1)}{1 + br} t(p_3^* - t). \tag{S.88}$$

For this to be Nash equilibrium, it must a be global optimum: in the case of network 1, network 1 can increase its price, and move the solution into Region 1. If does so, it has an optimum response to network 2 setting its price to $p_2^*$, namely the price

$$p_1 \; = \; \frac{(1 + r + r\widehat{C}_1)(4 + 2br + (1 - b + \widehat{C}_1)rt)}{2r\widehat{C}_1(4 + [b + 3(1+\widehat{C}_1)]r)}.$$

The condition that the resulting profit for network 1 at this value (calculated from (S.7)) does not exceed (S.88) gives (S.68). □

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*4) No other degenerate Nash equilibria exist:* To complete the proof of the theorem, we need only show that there cannot be a Nash equilibrium under any of the remaining boundary conditions, all of which must be in Region 3. We will have a Nash equilibrium at the point $p^* = (p_1^*, p_2^*, p_3^*)$ provided that the point $p^*$ is feasible and, for each $i$, $\pi_i(p_i)$ is maximized at $p^*$. But as we have seen, from (S.39), that the profit functions $\pi_i(p_i)$ are strictly concave, hence the only possible Nash equilibria in this case occur either at a unique interior point of the feasible region or at the boundaries of their support. The boundaries of the region are characterized in Corollary III.3 (the corresponding intervals for each $\pi_i(p_i)$ are generated by the intersection of the lines formed by fixing $p_j, j \neq i$ in the boundaries). The boundaries correspond to the hyperplanes $p_1 = 0$, $p_2 = 0$ $p_3 = 0$, and $Q_{11} = 0$, $Q_{13} = 0$, $Q_2 = 0$ (and since by construction $Q_{11} + Q_{13} + Q_2 = 1$ holds, we also consider the constraints $Q_2 = 1$). In addition, when $t \geq 0$ we have the boundary $p_3 = t$ corresponding to $\pi_3(p_3) = 0$ (note that $\pi_1 = 0, \pi_2 = 0$ are covered by other boundaries). When $t \leq 0$, there is the additional constraint boundary $\pi_1 = 0$.

**Condition** $\boldsymbol{\pi_1 = 0}$. No Nash equilibrium can exist in this case, since if $\pi_1 = p_1 Q_{11} + (p_1 + t)Q_{13} = 0$, we must have $p_1 + t < 0$ (the degenerate case $Q_{11} = 0 = Q_{13}$ is covered by subcases below), then from from (7)

$$\frac{\partial \pi_1}{\partial p_1} \; = \; Q_1 + (p_1 + t)\frac{\partial Q_{13}}{\partial p_1} > 0$$

and network 1 can increase its profit away from 0 by increasing $p_1$.

**Condition** $\mathbf{Q_{13} = 0}$. When $Q_{13} = 0$, then from (7)

$$\frac{\partial \pi_3}{\partial p_3} \; = \; p_3\frac{\partial Q_{13}}{\partial p_3} < 0$$

unless $p_3 = 0$. Hence we can decrease $p_3$, thereby increasing the Content Provider's profit away from zero, unless either $p_3 = 0$ or $t > 0$ and $p_3 = t$. We treat each of these special cases below.

**Condition $\mathbf{p_2 = 0}$**. Here, to maximize $\pi_2$ we must have $\frac{\partial \pi_2}{\partial p_2}\big|_{p_2=0} \leq 0$. From (7), $\frac{\partial \pi_2}{\partial p_2}\big|_{p_2=0} = Q_2$, hence $Q_2 = 0$. Then from (12) and (13), $p_1 + p_3 < 0$, which implies at least one price is negative, and hence there is no feasible Nash equilibrium.

**Condition $\mathbf{Q_2 = 0}$**. (7) When $Q_2 = 0$, $\pi_2 = 0$ then to be in case 3 requires that $p_2 > 0$, while from (7)

$$\frac{\partial \pi_2}{\partial p_2} = p_2 \frac{\partial Q_2}{\partial p_2} < 0$$

and hence network 2 can decrease its price and generate positive profit. Thus $Q_2 = 0$ cannot be a Nash equilibrium for network 2.

**Condition $\mathbf{p_3 = t}$** .
From (7), when $p_3 = t$, $\frac{\partial \pi_3}{\partial p_3} = Q_{13}$, and if $Q_{13} > 0$, then this cannot be a Nash equilibrium for the Content Provider, since it could increase its profit by increasing $t$. The only possibility is that both $p_3 = t$ and $Q_{13} = 0$, which is included as a particular special case of Theorem III.11 part 1.

### F. Proof of Theorem III.13 and Corollary III.14

*Proof.* Theorem III.13 is a special case of Theorem III.9 where $t = 0$, proved using Lemmas S.1 and S.2. Now (S.43), which gives conditions for prices and profits to be non-negative, is automatically satisfied when $t = 0$. This leaves Lemma S.2; when $t = 0$, the conditions of Lemma S.2 simplify: the conditions for $p_1^*$ to be positive and $p_1^*$ to be optimal are satisfied, leaving the conditions for $\pi_2$ to be optimal, ((S.54) or (S.58)), which reduce to

$$br[2 + (1+b+\widehat{C}_1)r] \;\leq\; 4(1+r+\widehat{C}_1 r)^2 \quad \text{OR} \tag{S.89}$$

$$\frac{4(2 + r(2 + 2\widehat{C}_1 + b(4 + [b + 3(1+\widehat{C}_1)]r)))^2}{1 + br} \geq \frac{\left(4 + r\left(b^2 r + 4(1+\widehat{C}_1)[2 + r(1+\widehat{C}_1)] + b(4 + 3(1+\widehat{C}_1)r)\right)\right)^2}{1 + r + \widehat{C}_1 r}. \tag{S.90}$$

We roll Theorem III.13 and Corollary III.14 into the following Lemma.

**Lemma S.3.** *When $t = 0$, sufficient conditions for a Nash equilibrium to exist are*

$$b \leq 2(1+\widehat{C}_1) + \frac{1}{r}. \tag{S.91}$$

*Sufficient conditions for a Nash equilibrium not to exist are*

$$r > 1 \textit{ AND } b > 4(1+\widehat{C}_1) + \frac{1}{r} \tag{S.92}$$

*Proof.* Proof of Lemma The first condition (S.89) is clearly satisfied if $b \leq 1 + \widehat{C}_1$. By writing $b = 2 + k\widehat{C}_1$, expanding the second inequality, taking out a factor of $(b - 1 - \widehat{C}_1)$ and equating the coefficients of $r^i$ in the remaining quartic to ensure that the resulting polynomial is always positive , then we can show that resulting inequality will always be satisfied provided $k \leq 2$. That is, if $b \leq 2(1+\widehat{C}_1)$, a Nash equilibrium will always exist. A more detailed line of reasoning will produce a broader sufficient condition:

$$b \leq 2(1+\widehat{C}_1) \quad \text{or} \quad r \leq \frac{1}{b - 2(1+\widehat{C}_1)}.$$

which can be combined into the single condition (S.91).

By substituting and simplifying, we can also show that when the condition (S.92) holds then neither (S.89) nor (S.90) are true, and hence (S.92) is a sufficient condition for a Nash equilibrium not to exist. □

### G. Proof of Theorem IV.1

We provide here a slightly more detailed presentation of Theorem IV.1 than is given in Section IV.

THEOREM IV.1 There exists a unique Nash equilibrium in the two-stage game. This occurs in one of two mutually exclusive cases, where in each case network 1 sets a positive transfer price $t^*$. In the first case all three parties make a positive profit; the second case is degenerate, where the Content Provider is shut out of the market. Specifically, either:

1) All three parties make a positive profit. The prices, user masses, and profits for this case are given by Theorem III.5 with $t = t^B$ or $t^O$, where:
   a) $t^O$ solves the first-order profit maximization conditions. This is the value of $t$ satisfying first order conditions, namely
   $$\frac{\partial}{\partial t}\pi_1(p_1^*(t), p_2^*(t), p_3^*(t)) = 0.$$
   This will yield an affine equation in $t$, with solution $t^O$. Here $t^O$ is the value of $t$ that maximizes a concave profit function and hence can in principle be found in a straightforward way by network 1.
   b) $t^B$ is the point at which network 2 is indifferent between competing with the Content Provider or lowering its price to drive it out of the market. Here $t^B < t^O$, where $t^B$ is the feasible solution to the equation in $t$ derived from (S.58), that is the positive solution to
   $$\left(\frac{1 + \widehat{C}_1 + \widehat{C}_1 p_1^*(t)}{2\widehat{C}_1}\right)^2 \frac{1}{1 + r(1+\widehat{C}_1)} = \frac{1}{1+br}(p_2^*(t))^2 \tag{S.93}$$
   a quadratic in $t$, where the $p_i^*$ are given in (20),(21). Note that from Theorem III.9, equation (26), we must have $t^B < \frac{2+br}{3+r+2br}$.

   Sufficient conditions for this case to exist are that $b^*(r, \widehat{C}_1) \leq b \leq 1 + \widehat{C}_1$ OR $(1 + \widehat{C}_1 < b \leq 2(1 + \widehat{C}_1) + 1/r$
2) The equilibrium is degenerate, with the two networks making positive profit, and the Content Provider shut-out of the market. The prices, user masses and profits are given in Theorem III.11,with $t^* \geq t^A$, where $t^A$ is defined in Theorem III.11 and $t_A$ is given explicitly in (S.83), and satisfies $t_A \geq 1$. There are two subcases:
   a) $b$ is small, satisfying $1 \leq b \leq b^*(r, \widehat{C}_1)$ where $b^*$ is the root of a cubic equation, with the bound $b^* < 1 + \widehat{C}_1$. In this case network 1 makes strictly greater profit than having a positive transfer price.
   b) $b$ is large, for which sufficient conditions are $b > 4(1 + \widehat{C}_1) + 1/r$ and $r > 1$ (c.f. Corollary III.14). In this instance network 1 cannot attain the higher profits that could be gained by setting a positive transfer price.

*Proof.* Proof We consider two subcases separately: $b \leq 1 + \widehat{C}_1$ and $b > 1 + \widehat{C}_1$ . The statements of the theorem follow by combining the results from the subcases.

**Subcase: When** $b \leq 1 + \widehat{C}_1$ Recall that in this case the profit for network 1 is smaller with $t = 0$ than without the Content Provider.

i) First note that on calculating the profits from (7) using the prices and user masses from Theorem III.5, we can see that $\pi_1(p_1^*(t), p_2^*(t), p_3^*(t))$ is quadratic in $t$. By direct calculation, it is straightforward to see that
$$\frac{\partial^2}{\partial t^2}\pi_1(p_1^*(t), p_2^*(t), p_3^*(t)) < 0$$
for $1 \leq b \leq 1 + \widehat{C}_1$ hence any local optimum satisfying the first order conditions will be a global optimum if staying within the feasible region of Case 3.

ii) The value of $t$ satisfying first order conditions is

$$t^O = (br+2)(\widehat{C}_1 r + r + 1)\left(b^2 r + 2b - \widehat{C}_1(r+1) - r - 2\right)$$
$$\div\Big[16\widehat{C}_1^3 r^2(br+1) + \widehat{C}_1^2 r(r(4b((b+8)r+10) - r + 30) + 35)$$
$$+\ 2\widehat{C}_1(r+1)\left(r([b(b+9)-1]r + 11b + 7) + 9\right)$$
$$-(b-1)^2 r(r+1)^2\Big] \quad \text{(S.94)}$$

iii) For $1 \leq b < \sqrt{\frac{\widehat{C}_1 r^2 + \widehat{C}_1 r + r^2 + 2r + 1}{r^2}} - \frac{1}{r}$, then $\frac{\partial}{\partial t}\pi_1(p_1^*(t), p_2^*(t), p_3^*(t))|_{t=0} < 0$, so network 1 can indeed increase revenue by reducing the transfer price from zero, setting a negative transfer price, thereby effectively subsidizing the Content Provider. However, network 1 can do even better by *raising* the transfer price to such a level that the Content Provider is shut out of the market—for example, by setting $t = 1$. The latter follows by first showing that $\frac{\partial}{\partial t}\pi_1(p_1^*(t), p_2^*(t), p_3^*(t))|_{t=-\frac{-2+br}{3+r(3+4\widehat{C}_1)}} > 0$, hence the value of $t_0$ is a potential Nash equilibrium (c.f. Theorem 5.3)); second, proving that $\frac{\partial}{\partial b}\pi_1(p_1^*(t^O), p_2^*(t^O), p_3^*(t^O)) > 0$ in this region, i.e., profits for network 1 increase with $b$; and third, showing that when $b = \sqrt{\frac{\widehat{C}_1 r^2 + \widehat{C}_1 r + r^2 + 2r + 1}{r^2}} - \frac{1}{r}$, network 1 is better shutting the Content Provider out of the market, hence is it better offer adopting this policy for all $b$ in this range.

iv) For $\sqrt{\frac{\widehat{C}_1 r^2 + \widehat{C}_1 r + r^2 + 2r + 1}{r^2}} - \frac{1}{r} < b \leq \widehat{C}_1 + 1$, then the optimal choice of strategy depends upon the values of $b, \widehat{C}_1, r$, or on a value $b^* = b^*(r, \widehat{C}_1)$, where $b^*$ is the root of a cubic equation involving $r$ and $\widehat{C}_1$. This is the value of $b$ at which network 1 is indifferent between choosing the optimal value of $t^* = t^O$ and choosing $t^* = 1$ to shut-out the Content Provider.

   a) If $\sqrt{\frac{cr^2 + \widehat{C}_1 r + r^2 + 2r + 1}{r^2}} - \frac{1}{r} < b < b^*$ then optimal for network 2 to shut-out the Content Provider, by raising $t$, eg $t = 1$.

   b) If $b^* \leq b \leq 1 + \widehat{C}_1$ then network 1 announces $t^* = t^O$ given by (S.94).

   Note that $b^*$ is very "close"to $1 + \widehat{C}_1$ (informally; i.e., the region is small)

v) These are the only possibilities: if a value of $t$ is chosen so that Case 3 of Theorem III.11 introduces a possible degenerate Nash, network 1 can increase profits by setting $t = 1$.

**Subcase: When** $b > 1 + \widehat{C}_1$

i) First note that *when a Nash equilibrium exists in Case 3, i.e., in Region 3*, it is always advantageous for network 1 to have the Content Provider use its network. That is, the profit for network 1 is greater with $t = 0$ than shutting the CP out.

ii) Straightforward to show that given $b > 1 + \widehat{C}_1$, $\frac{\partial}{\partial t}\pi_1(p_1^*(t), p_2^*(t), p_3^*(t))|_{t=0} > 0$ and hence if a Nash equilibrium exists with $t$ set, $t$ should be strictly positive.

iii) In this case (i.e., Nash equilibrium and in Region 3) there are two possible cases

   a) $t$ can be $t^* = t^O$, i.e., solution to first order conditions

   b) $t^* = t^B$ is on the boundary of the Nash equilibrium boundary, the critical point at which network 2 is indifferent between competing with the Content Provider, or lowering its price to drive out the Content Provider.

   Sufficient condition for one of these two cases to exist is $1+c < b \leq 2(1+c) + \frac{1}{r}$, (which follows from combining Theorem III.13 with this subcase).

iv) We know from Cor. III.10 than an $\epsilon-$equilibrium exists in this (where $b > 1 + \widehat{C}_1$) in the second stage game if $t = 0$), and hence an "optimal" $\epsilon-$equilibrium also exists -i.e.from above, it follow that $t^* = t^O$ is always an optimal $\epsilon-$equilibrium in Stackelberg game.

v) Under certain conditions, no Nash equilibrium exists in the multistage game *with the Content Provider involved* —i.e., the transfer price is raised to such a high level that it is shut out of the market, (cf Theorems III.9 and III.11 which discuss equilibria for when the Content Provider involved). Cor. III.14 gives a sufficient condition for no Nash equilibrium to exist with $t = 0$, and this implies the only Nash equilibrium is when $t$ is raised to such a level as to shut the Content Provider out of the market.

□

## *H. Proof of Theorem V.1*

THEOREM V.1 For quadratic congestion costs and equal capacities, the optimal strategy for network 1 is to set a negative transfer price $t^* = t^O(b, \widehat{C}_1)$.

*Proof.* We will show that $\frac{d}{dt}\pi_1^*(t)|_{t=0} < 0$, and that $\pi_1^*(t)$ is concave. When in Case 3 of Lemma III.1, putting $r = 1/b$, gives the defining conditions as

$$\left(\frac{bQ_2}{\widehat{C}_1}\right)^2 - \left(\frac{Q_{11} + bQ_{13}}{\widehat{C}_1}\right)^2 + p_2 - p_1 = p_3 \quad \text{(S.95)}$$

and $Q_{11} = p_3$, hence $Q_{13}$ is the solution to the quadratic

$$\left(\frac{b(1 - p_3 - Q_{13})}{\widehat{C}_1}\right)^2 - \left(\frac{p_3 + bQ_{13}}{\widehat{C}_1}\right)^2 + p_2 - p_1 = p_3 \quad \text{(S.96)}$$

which reduces to a linear equation, with solution

$$Q_{13} = \frac{b^2(1-p_3)^2 - p_3^2 - \widehat{C}_1^2(p_1 - p_2 + p_3)}{2b(b(1-p_3) + p_3)}. \quad \text{(S.97)}$$

We have seen that $\pi_1(p_1)$ and $\pi_2(p_2)$ are concave when $r = 1/b$, and that $\pi_3(p_3)$ is in general concave (or quasi-concave), hence we can find the optimum by considering the first order conditions. Equating to zero the derivatives $\frac{\partial_{\pi_i}}{\partial_{p_j}}$ for $i = 1, 2, 3$, and substituting for the partial derivatives by implicitly differentiating (S.96) or (S.97) gives, after simplifying, the equations

$$2b(p_3 + Q_{13}) = \frac{\widehat{C}_1^2(p_1 + t)}{b(1-p_3) + p_3}$$
$$2bQ_2 = \frac{\widehat{C}_1^2 p_2}{b(1-p_3) + p_3}$$
$$(b + t - bt)Q_{13} = \frac{\left(\widehat{C}_1^2 + 2p_3\right)(p_3 - t)}{2b} + b(p_3 - t)(1 - p_3), \quad \text{(S.98)}$$

whose solution give $p_i^*$, and $Q_i^*$. Eliminating $p_1$ and $p_2$, and noting that $Q_2 = 1 - p_3 - Q_{13}$, enables us to reduce the set of equations {(S.97), (S.98)} to a pair of simultaneous linear equations in $Q_{13}$, i.e.,

$$b^2(1-p_3)(3 - 5p_3 - 6Q_{13} - \widehat{C}_1^2(p_3 - t)$$
$$= p_3(b(4p_3 + 6Q_{13} - 2) + p_3)$$

$$\left(\widehat{C}_1^2 + 2p_3\right)(p_3 - t)$$
$$= 2b\left(b\left[(1-t)Q_{13} - (1-p_3)(p_3 - t)\right] + Q_{13}t\right). \quad \text{(S.99)}$$

Eliminating $Q_{13}$ reduces the equations to a cubic in $p_3$, where $p_3^*$ is the real root in $[0, 1]$. Explicitly the cubic is

$$6(b-1)^2(b+1){p_3}^3 - (b-1)\left(17b^2 + 7b + 3\widehat{C}_1^2\right){p_3}^2$$
$$+\ 2b\left(7b^2 - b + 2\widehat{C}_1^2\right)p_3 - 3b^3 = 0.$$

To show that $\frac{d}{dt}\pi_1^*(t)|_{t=0} < 0$ involves

i) Writing $\pi_1^*(t) = p_1^*(t)\,(p_3^*(t) + Q_{13}^*(t)) + tQ_{13}^*(t)$ and differentiating to obtain $\frac{d}{dt}\pi_1^*(t)|_{t=0}$

ii) Differentiating (S.98) and (S.97) implicitly w.r.t $t$, and setting $t \mapsto 0$

iii) Showing that when $t = 0$ the resulting set of equations for $\{\pi_1^*(t)|_{t=0}, (p_i^*(0), Q_i^*(0)), (\frac{d}{dt}p_i^*(t)|_{t=0}\frac{d}{dt}Q_i^*(t))|_{t=0})\}$ do not have a consistent solution unless $\frac{d}{dt}\pi_1^*(t)|_{t=0} < 0$. This is proved using Mathematica [18].

$\square$