Crux's crux's crux

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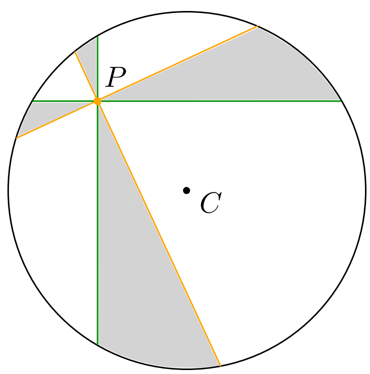
The following problem (proposed by Stanley Rabinowitz), appeared as problem 1325 in *Crux** *Mathematicorum*. We call this the *Crux*† *Problem*, since the accompanying diagram contains a shaded 'cross'=crux.

Crux Problem. Let P be any point inside a unit circle with center C. Perpendicular chords are drawn through P. Rotation of these chords counterclockwise about P through an angle θ sweep out the shaded area shown in the picture below. Show that this shaded area only depends on θ, but not on P (and hence is easily seen to be 2θ by taking $P = C$).



There were two solutions that appeared [**1**, pp. 120-122]:

(I) (Jörg Häterich) a solution using calculus and Archimedes' theorem,
(II) (Shiko Iwata) a non-calculus solution based on trigonometry.

The accompanying editor's note mentioned that Murray Klamkin generalised the problem to n chords through P with equal angles of π/n between successive chords, with the area swept out, when these chords are rotated through an angle of θ about P, then being $n\theta$. The editor's note ended with the following parenthetical remark:

(This can also be proved using the solution II. Can it be proved as in solution I?)

In this note, we present a calculus-based solution, based on a special case of a generalisation of "Archimedes' theorem", which is proved by employing vectors. We purport that this solution captures in some sense the crux‡ of the matter.

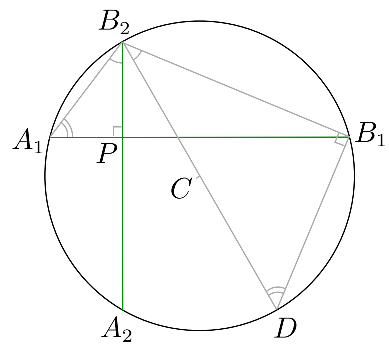
We begin with a calculus-based proof along lines similar to the first solution given in [**1**].

*First 'Crux' in the title.
†Second 'crux' in the title.
‡Third 'crux' in the title.

A calculus-based proof of the Crux problem

We will use the following result. We call it Archimedes' Theorem as it is Proposition 11 in Archimedes' work *The book of Lemmas* [**2**, p.312].

Archimedes' Theorem. If two mutually perpendicular chords A_1B_1 and A_2B_2 in a unit circle with center C meet at P, then $PA_1^2 + PB_1^2 + PA_2^2 + PB_2^2 = 4$.



Proof. ΔB_2PA_1 is similar to ΔB_2B_1D since we have $\angle B_2A_1B_1 = \angle B_2DB_1$ and $\angle B_2PA_1 = 90° = \angle B_2B_1D$. So $\angle PB_2A_1 = \angle B_1B_2D$. This implies that $\angle A_1CA_2 = \angle B_1CD$, and so $A_1A_2 = B_1D$. By Pythagoras' Theorem, we have $PB_2^2 + PB_1^2 = B_2B_1^2$ and $PA_1^2 + PA_2^2 = A_1A_2^2$. Adding these, we obtain that $PA_1^2+PB_1^2+PA_2^2+PB_2^2 = B_2B_1^2+A_1A_2^2 = B_1B_2^2+B_1D^2 = B_2D^2 = 2^2 = 4$. □

Now we give a calculus argument as follows. Rotating A_1B_1 and A_2B_2 about P through an infinitesimal angle dθ, we obtain four sectors, with areas given by

$$\frac{1}{2}PA_k^2\,\mathrm{d}\theta,\quad \frac{1}{2}PB_k^2\,\mathrm{d}\theta,\qquad k = 1, 2.$$

Upon addition, and using Archimedes' Theorem, we obtain the rate of change of area

$$\frac{\mathrm{dA}}{\mathrm{d}\theta} = \frac{1}{2}(PA_1^2 + PB_1^2 + PA_2^2 + PB_2^2) = \frac{1}{2}4 = 2,$$

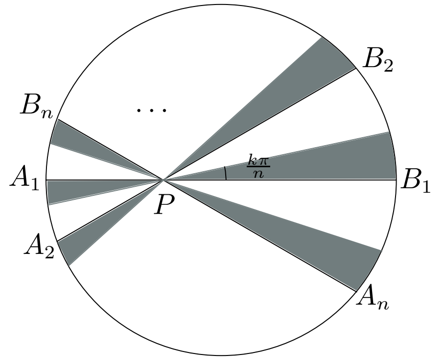
and so the total area, if the chords are rotated through an angle θ, is given by

$$\mathrm{A} = \int_0^\theta \frac{\mathrm{dA}}{\mathrm{d}\theta}\,\mathrm{d}\theta = \int_0^\theta 2\,\mathrm{d}\theta = 2\theta.$$

A vector calculus proof

We will first show the following:

Proposition 1. *Let P be any point inside a unit circle, and through P, let there be n chords $A_1B_1, \cdots, A_nB_n$, such that there are equal angles of π/n between successive chords. Suppose moreover that A_1B_1 is a diameter. If each chord is rotated counterclockwise through an angle θ, then the total area formed by the resulting sectors is $n\theta$.*



This will be shown to yield the generalisation (given in Theorem 1 below) of the Crux problem, where as opposed to the situation above, one of the chords needn't be the diameter.

In order to prove Proposition 1, we will first prove a special case of a generalisation of Archimedes' Theorem (Theorem 2 in the next section, saying that the sum of the squared distances from a point inside a unit circle to the vertices of n equally angularly spaced chords passing through that point is $2n$), when one of the chords A_1B_1 is the diameter.

Lemma 1 (Generalised Archimedes' theorem − special case)**.** *Let P be any point inside a unit circle, and let there be n chords $A_1B_1, \cdots A_nB_n$ through P such that there are equal angles of π/n between successive chords. Suppose, moreover, that A_1B_1 is a diameter. Then $PA_1^2 + PB_1^2 + \cdots + PA_n^2 + PB_n^2 = 2n$.*

Proof. Let $C_1, \cdots, C_n$ be the centers of $A_1B_1, \cdots, A_nB_n$. As A_1B_1 is the diameter, C_1 is the center of the circle. We know that for all $1 \le k \le n$,

$$\langle \overrightarrow{PA}_k - \overrightarrow{PC}_1, \overrightarrow{PA}_k - \overrightarrow{PC}_1 \rangle = \|\overrightarrow{PA}_k - \overrightarrow{PC}_1\|_2^2 = 1,$$
$$\langle \overrightarrow{PB}_k - \overrightarrow{PC}_1, \overrightarrow{PB}_k - \overrightarrow{PC}_1 \rangle = \|\overrightarrow{PB}_k - \overrightarrow{PC}_1\|_2^2 = 1.$$

Expanding these, adding, and rearranging, we obtain

$$\sum_{k=1}^{n}(PA_k^2 + PB_k^2) = 2\Big\langle \overrightarrow{PC}_1, \sum_{k=1}^{n}(\overrightarrow{PA}_k + \overrightarrow{PB}_k)\Big\rangle - 2nPC_1^2 + 2n. \qquad (1)$$

So we need to determine the inner product on the right-hand side. We have

$$\overrightarrow{PA}_k = \overrightarrow{PC}_k + \overrightarrow{C_kA}_k \text{ and } \overrightarrow{PB}_k = \overrightarrow{PC}_k + \overrightarrow{C_kB}_k.$$

But since $\overrightarrow{C_kA}_k + \overrightarrow{C_kB}_k = \mathbf{0}$, we obtain $\overrightarrow{PA}_k + \overrightarrow{PB}_k = 2\overrightarrow{PC}_k$. Hence

$$\sum_{k=1}^{n}(\overrightarrow{PA}_k + \overrightarrow{PB}_k) = 2\sum_{k=1}^{n}\overrightarrow{PC}_k = \sum_{k=1}^{n}\overrightarrow{PC}_k + \sum_{k=1}^{n}\overrightarrow{PC}_{n-k} = \sum_{k=1}^{n}(\overrightarrow{PC}_k + \overrightarrow{PC}_{n-k}).$$

By referring to Figure 1, we see that for all $1 \le k \le n$,

$$\overrightarrow{PC}_k + \overrightarrow{PC}_{n-k} = 2PC_k\Big(\cos\frac{k\pi}{n}\Big)\frac{\overrightarrow{PC}_1}{PC_1} = 2\Big(\cos\frac{k\pi}{n}\Big)PC_1\Big(\cos\frac{k\pi}{n}\Big)\frac{\overrightarrow{PC}_1}{PC_1}$$
$$= 2\Big(\cos\frac{k\pi}{n}\Big)^2\overrightarrow{PC}_1.$$

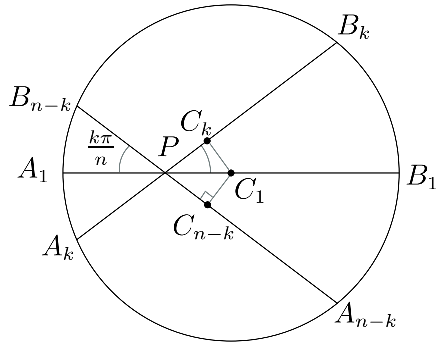


Figure 1

So

$$\sum_{k=1}^{n}(\overrightarrow{PA}_k+\overrightarrow{PB}_k)=\sum_{k=1}^{n}(\overrightarrow{PC}_k+\overrightarrow{PC}_{n-k})=\sum_{k=1}^{n}2\Big(\cos\frac{k\pi}{n}\Big)^2\overrightarrow{PC}_1. \qquad (2)$$

Now

$$\sum_{k=1}^{n}2\Big(\cos\frac{k\pi}{n}\Big)^2=\sum_{k=1}^{n}\Big(1+\Big(\cos k\frac{2\pi}{n}\Big)\Big)=n+0=n, \qquad (3)$$

where we have used

$$\sum_{k=1}^{n}\cos\Big(k\frac{2\pi}{n}\Big)=0. \qquad (4)$$

(To see (4), we first note that this sum is the horizontal component of the sum $\overrightarrow{S}$ of n vectors whose tails lie at the center of the unit circle and whose tips lie on the vertices of a regular n-gon. To see that $\overrightarrow{S}$ is zero, imagine rotating each vector counterclockwise through an angle of $\frac{2\pi}{n}$, and let the sum of the rotated vectors be $\overrightarrow{S'}$. On grounds of symmetry of the regular polygon, $\overrightarrow{S}=\overrightarrow{S'}$. On the other hand $\overrightarrow{S'}$ ought be a rotated version of $\overrightarrow{S}$ through an angle of $\frac{2\pi}{n}$. This can only happen if $\overrightarrow{S}=\mathbf{0}$. Alternative justifications of (4) can be given by first summing the geometric series

$$\sum_{k=1}^{n}e^{i\frac{2\pi}{n}k}=e^{i\frac{2\pi}{n}}\frac{1-e^{i2\pi}}{1-e^{i\frac{2\pi}{n}}}=0,$$

and taking real parts, or by noticing the sum of the nth roots of unity must add up to 0 since the coefficient of z^1 in z^n-1 is 0, and again taking real parts.)

Consequently, using (1), (2), and (3), we obtain

$$\begin{aligned}\sum_{k=1}^{n}(PA_k^2+PB_k^2)&=2\Big\langle\overrightarrow{PC}_1,\sum_{k=1}^{n}(\overrightarrow{PA}_k+\overrightarrow{PB}_k)\Big\rangle-2nPC_1^2+2n\\&=2\Big\langle\overrightarrow{PC}_1,n\overrightarrow{PC}_1\Big\rangle-2nPC_1^2+2n\\&=\cancel{2nPC_1^2}-\cancel{2nPC_1^2}+2n=2n.\end{aligned}$$

□

We are now ready to prove Proposition 1.

Proof of Proposition 1. Rotating $A_1B_1, \cdots, A_nB_n$ anticlockwise about P through an infinitesimal angle dθ, we obtain $2n$ sectors, with areas given by

$$\frac{1}{2}PA_k^2\,\mathrm{d}\theta,\ \ \frac{1}{2}PB_k^2\,\mathrm{d}\theta, \quad k = 1, \cdots, n.$$

Upon addition, and using Lemma 1, the rate of change of the total area is

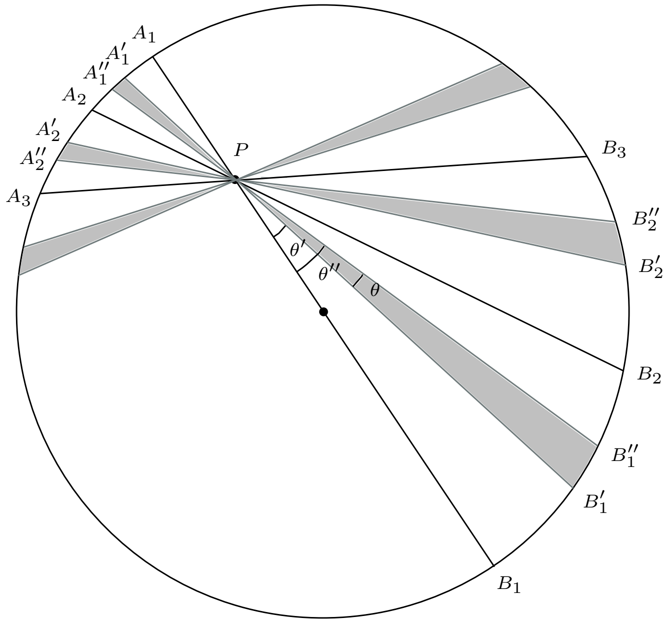
$$\frac{\mathrm{dA}}{\mathrm{d}\theta} = \frac{1}{2}\sum_{k=1}^{n}(PA_k^2 + PB_k^2)\,\mathrm{d}\theta = \frac{1}{2}2n = n,$$

and so the total area, if the chords are rotated through an angle θ, is given by

$$A = \int_0^\theta \frac{\mathrm{dA}}{\mathrm{d}\theta}\,\mathrm{d}\theta = \int_0^\theta n\,\mathrm{d}\theta = n\theta. \qquad \square$$

Theorem 1. *Let P be any point inside a unit circle, and let there be n chords through P such that there are equal angles of π/n between successive chords. If each chord is rotated counterclockwise through an angle θ, then the total area formed by the resulting sectors is $n\theta$.*

Proof. To see how this follows from Proposition 1, we first construct the diameter A_1B_1 through P, and consider successive anticlockwise rotations of this diameter through angles of π/n, resulting in the chords $A_2B_2, \cdots, A_nB_n$. Let the given chords from the theorem statement be labelled as $A_1'B_1', \cdots, A_n'B_n'$, and let their rotated versions (though an angle θ) be labelled as $A_1''B_1'', \cdots, A_n''B_n''$.



Let the angle between A_1B_1 and $A_1'B_1'$ be θ', and that between A_1B_1 and $A_1''B_1''$ be θ''. Then for all $1 \le k \le n$, we use the notation $\widehat{B_k'B_k''}$ for the sector formed by the corresponding arc with P, and denote the area of the sector by $\mathrm{A}(\widehat{B'_k B''_k})$. Then:

$$\begin{aligned}
&\sum_{k=1}^{n}[\mathrm{A}(\widehat{B'_k B''_k}) + \mathrm{A}(\widehat{A'_k A''_k})] \\
&\qquad = \sum_{k=1}^{n}[\mathrm{A}(\widehat{B_k B''_k}) - \mathrm{A}(\widehat{B_k B'_k}) + \mathrm{A}(\widehat{A_k A''_k}) - \mathrm{A}(\widehat{A_k A'_k})] \\
&\qquad = \sum_{k=1}^{n}[\mathrm{A}(\widehat{B_k B''_k}) + \mathrm{A}(\widehat{A_k A''_k})] - \sum_{k=1}^{n}[(\mathrm{A}(\widehat{B_k B'_k}) + \mathrm{A}(\widehat{A_k A'_k}))] \\
&\qquad = n\theta'' - n\theta' = n(\theta'' - \theta') = n\theta. \qquad \square
\end{aligned}$$

Archimedes' Theorem

A consequence of Theorem 1 is the following generalisation of Archimedes' Theorem from the $n = 2$ chord case considered earlier.

Theorem 2 (Generalised Archimedes' theorem)**.** *Let P be any point inside a unit circle, and let there be n chords $A_1B_1, \cdots A_nB_n$ through P such that there are equal angles of π/n between successive chords. Then*

$$PA_1^2 + PB_1^2 + \cdots + PA_n^2 + PB_n^2 = 2n.$$

Proof. By Theorem 1, we know that if the chords are rotated through an infinitesimal angle $\mathrm{d}\theta$, then the sum of the areas of the resulting sectors is $n\mathrm{d}\theta$. But this area is also equal to $\frac{1}{2}(PA_1^2 + PB_1^2 + \cdots + PA_n^2 + PB_n^2)\,\mathrm{d}\theta$. Consequently, we obtain that $PA_1^2 + PB_1^2 + \cdots + PA_n^2 + PB_n^2 = 2n$. □

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REFERENCES

1. Solution to problem 1325, *Crux Mathematicorum*, **15** (April 1989) 120–121 number 4, `https://cms.math.ca/crux/backfile/Crux_v15n04_Apr.pdf`.
2. *The works of Archimedes.* Reprint of the 1897 edition and the 1912 supplement, edited by T. L. Heath, Dover Publications, 2002.

Summary. A result of Archimedes states that for perpendicular chords passing through a point P in the interior of the unit circle, the sum of the squares of the lengths of the chord segments from P to the circle is equal to 4. A generalisation of this result to $n \geq 2$ chords. This is done in the backdrop of revisiting Problem 1325 from *Crux Mathematicorum*, for which a new solution is presented.

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