1. SUPPLEMENTARY MATERIAL. For the sake of easy reference we write the regularity conditions and some expressions used in the proof of the results. Appendix A gives the proof of the main results of the paper which employs a series of lemmas given in Appendix B. Finally Appendix C presents tables from the Monte Carlo experiment.

We denote

$$(1-L)^d = \sum_{k=0}^{\infty} \pi_k(d) L^k; \quad \pi_k(-d) = \frac{\Gamma(k+d)}{\Gamma(d)\Gamma(k+1)}, \quad k \in \mathbb{N}$$

$$\left(1 - 2\cos\omega L + L^2\right)^d = \sum_{k=0}^{\infty} \tau_k(\cos\omega; d) L^k,$$

where $\Gamma(\cdot)$ denotes the gamma function such that $\Gamma(c) = \infty$ for $c = 0$ with $\Gamma(0)/\Gamma(0) = 1$ and the coefficients $\tau_k(\cos\omega; d)$ follow the second order homogeneous difference equation

$$\tau_k(z; d) = 2z\left(\frac{k-d-1}{k}\right)\tau_{k-1}(z; d) - \left(\frac{k-2d-2}{k}\right)\tau_{k-2}(z; d),$$

see Section 8.93 in Gradshteyn and Ryzhik (2000).

CONDITION 1. *$\{x_t\}_{t\in\mathbb{Z}}$ and $\{u_t\}_{t\in\mathbb{Z}}$ are two mutually independent sequences of random variables such that*

$$x_t = \sum_{j=0}^{\infty} \varphi_j \varrho_{t-j}; \quad \sum_{j=0}^{\infty} \varphi_j^2 < \infty, \quad \varphi_0 = 1,$$

$$u_t = \sum_{j=0}^{\infty} \vartheta_j \varepsilon_{t-j}; \quad \sum_{j=0}^{\infty} \vartheta_j^2 < \infty, \quad \vartheta_0 = 1,$$

where $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ and $\{\varrho_t\}_{t\in\mathbb{Z}}$ are zero mean sequences with finite variance. Denote $E\left(\varepsilon_t^2\right) = \sigma_\varepsilon^2$. Also

$$\vartheta_j = \sum_{k=0}^{j} \xi_k(-d_1, -d_2) b_{k-j} = \sum_{k=0}^{j} \xi_{k-j}(-d_1, -d_2) b_k$$

where $\xi_k(d_1; d_2) = \sum_{\ell=0}^{k} \tau_\ell(\cos\omega; d_2)\pi_{k-\ell}(d_1) =: \sum_{\ell=0}^{k} \tau_{k-\ell}(\cos\omega; d_2)\pi_\ell(d_1)$ and $\sum_{k=0}^{\infty} k^2 |b_k| < \infty$. Finally, $\left|\varphi_j\right| = O\left(j^{d_x-1}\right)$ with $d_x \in \left[0, \frac{1}{2}\right)$.

Next we denote

$$g(\lambda; d_1, d_2) =: \left(1 - e^{-i\lambda}\right)^{d_1}\left(1 - (2\cos\omega)e^{-i\lambda} + e^{-2i\lambda}\right)^{d_2} = \sum_{j=0}^{\infty} \xi_j(d_1, d_2) e^{-ij\lambda}$$

$$B(\lambda) = \sum_{j=0}^{\infty} b_j e^{-ij\lambda},$$

so that $f_u(\lambda) = \frac{\sigma_\varepsilon^2}{2\pi}|g(\lambda; d_1, d_2)|^{-2}|B(\lambda)|^2$. Finally $\{u_t\}_{t\in\mathbb{Z}}$ admits the AR representation

$$u_t = \sum_{j=1}^{\infty} \phi_j u_{t-j} + \varepsilon_t; \qquad \phi_j =: \sum_{k=0}^{j} \xi_k(d_1, d_2) a_{k-j} =: \sum_{k=0}^{j} \xi_{k-j}(d_1, d_2) a_k,$$

where $\sum_{k=0}^{\infty} k^2 |a_k| < \infty$ and $B^{-1}(\lambda) =: A(\lambda) = \sum_{j=0}^{\infty} a_j e^{-ij\lambda}$. So,

$$f_u(\lambda) = \frac{\sigma_\varepsilon^2}{2\pi} |g(\lambda; d_1, d_2)|^{-2} |A(\lambda)|^{-2}.$$

Finally denote $g(\lambda, d_1, d_2; L) = \sum_{j=0}^{L} \xi_j (d_1, d_2) e^{-ij\lambda}$.

CONDITION 2. *$\{\varrho_t\}_{t\in\mathbb{Z}}$ and $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ are independent identically distributed sequences of random variables with finite 8th moments. In addition, denoting $\phi(x)$ as the probability density function of x_t, we have that*

$$\int_{\mathbb{R}} \sum_{p=1}^{4} \left| \frac{\partial^p \phi}{\partial x^p}(x) \right| dx < \infty.$$

CONDITION 3. *As $n \to \infty$,*

$$\frac{m^4}{n^3} + \frac{n^2}{m^3} \log n \to 0.$$

CONDITION 4. *As Condition 1 but with $d_2 = 0$.*

We denote the least squares residuals

$$\widehat{u}_t = (u_t - \overline{u}_n) - \left(\widehat{\beta} - \beta\right)(x_t - \overline{x}_n), \tag{1}$$

where $\overline{x}_n = n^{-1} \sum_{t=1}^{n} x_t$, $\overline{u}_n = n^{-1} \sum_{t=1}^{n} u_t$ and

$$\widehat{\beta} - \beta = \left(\sum_{t=1}^{n} (x_t - \overline{x}_n)^2 \right)^{-1} \sum_{t=1}^{n} (x_t - \overline{x}_n) u_t. \tag{2}$$

Also we have that

$$\sum_{t=1}^{n} \widehat{u}_t = \sum_{t=1}^{n} x_t \widehat{u}_t = 0. \tag{3}$$

$$\mathcal{T}_n(x) = \frac{1}{n} \sum_{t=1}^{n} \mathcal{I}(x_t < x) \widehat{u}_t, \tag{4}$$

$$\mathcal{T}_n(x) = \frac{1}{n} \sum_{t=1}^{n} \{\mathcal{I}(x_t < x) - F(x) + \phi(x) x_t\} \widehat{u}_t, \tag{5}$$

We shall write

$$\mathbf{1}_t(x) = \mathcal{I}(x_t < x) - F(x); \quad \mathring{\mathbf{1}}_t(x) = \mathbf{1}_t(x) + \phi(x) x_t, \tag{6}$$

where $F(x) = \int_{-\infty}^{x} \phi(z)\, dz$.

$$\zeta(d_1) = 2\Gamma(1 - 2d_1) \cos\left(\pi \left(\frac{1}{2} - d_1\right)\right). \tag{7}$$

$\mathcal{G}(x)$ denotes a Gaussian process in the real line with covariance structure given by

$$Cov(\mathcal{G}(x),\mathcal{G}(y)) = \gamma_u(0)\,E\left(\mathring{\mathbf{i}}_0(x)\,\mathring{\mathbf{i}}_0(y)\right) + \sum_{\ell=1}^{\infty}\gamma_u(\ell)\,E\left(\mathring{\mathbf{i}}_0(x)\,\mathring{\mathbf{i}}_\ell(y)\right)$$

$$+\sum_{\ell=1}^{\infty}\gamma_u(\ell)\,E\left(\mathring{\mathbf{i}}_0(y)\,\mathring{\mathbf{i}}_\ell(x)\right)\quad x,y\in\mathbb{R}. \tag{8}$$

Denoting $\widehat{\Psi}(\lambda) = g\left(\lambda,-\widehat{d}_1,-\widehat{d}_2;n\right)\widehat{B}(\lambda)$,

$$\ddot{u}_t^* = \frac{1}{(3n)^{1/2}}\sum_{j=1}^{3n-1} e^{it\widetilde{\lambda}_j}\,\widehat{\Psi}\left(\widetilde{\lambda}_j\right) w_{\varepsilon^*}\left(\widetilde{\lambda}_j\right),\quad t=1,...,3n. \tag{9}$$

Next for any integer $k\geq 0$,

$$\begin{aligned}
\pi_k &=: \pi_k(d_1);\ \widehat{\pi}_k =: \pi_k\left(\widehat{d}_1\right);\\
\overline{\pi}_k &=: \pi_k(-d_1);\ \check{\pi}_k =: \pi_k\left(-\widehat{d}_1\right);\\
\tau_k &=: \tau_k(d_2);\ \widehat{\tau}_k =: \tau_k\left(\widehat{d}_2\right);\\
\overline{\tau}_k &=: \tau_k(-d_2);\ \check{\tau}_k =: \tau_k\left(-\widehat{d}_2\right);\\
\xi_k &=: \xi_k(d_1,d_2);\ \widehat{\xi}_k =: \xi_k\left(\widehat{d}_1,\widehat{d}_2\right);\\
\overline{\xi}_k &=: \xi_k(-d_1,-d_2);\ \check{\xi}_k =: \xi_k\left(-\widehat{d}_1,-\widehat{d}_2\right).
\end{aligned} \tag{10}$$

Also $\widehat{\phi}_0 = \mathring{\phi}_0 = \widehat{\vartheta}_0 = \mathring{\vartheta}_0 = 1$ and

$$\widehat{\phi}_\ell = \begin{cases} \sum_{k=0}^{\ell\wedge M}\widehat{\xi}_{\ell-k}\widehat{a}_k, & 1\leq\ell\leq n-M\\ \sum_{k=\ell-(n-M)}^{M}\widehat{\xi}_{\ell-k}\widehat{a}_k, & n-M<\ell\leq n,\end{cases} \tag{11}$$

$$\mathring{\phi}_\ell = \begin{cases} \sum_{k=0}^{\ell\wedge M}\xi_{\ell-k}a_k, & 1\leq\ell\leq n-M\\ \sum_{k=\ell-(n-M)}^{M}\xi_{\ell-k}a_k, & n-M<\ell\leq n,\end{cases} \tag{12}$$

$$\widehat{\vartheta}_\ell = \begin{cases} \sum_{p=0}^{\ell\wedge M}\widehat{b}_p\check{\xi}_{\ell-p}, & 1\leq\ell\leq n\\ \sum_{p=\ell-n}^{M}\widehat{b}_p\check{\xi}_{\ell-p}, & n<\ell\leq n+M,\end{cases} \tag{13}$$

$$\mathring{\vartheta}_\ell = \begin{cases} \sum_{p=0}^{\ell\wedge M}b_p\overline{\xi}_{\ell-p}, & 1\leq\ell\leq n\\ \sum_{p=\ell-n}^{M}b_p\overline{\xi}_{\ell-p}, & n<\ell\leq n+M.\end{cases} \tag{14}$$

Next $\ddot{u}_t^*$ in (9) is

$$\begin{aligned}
\ddot{u}_t^* &= \sum_{\ell=0}^{t-1}\widehat{\vartheta}_\ell\varepsilon_{t-\ell}^* + \sum_{\ell=t}^{n+M}\widehat{\vartheta}_\ell\varepsilon_{n+t-\ell}^*\quad\text{if}\quad t<n+M\\
&= \sum_{\ell=0}^{n+M}\widehat{\vartheta}_\ell\varepsilon_{t-\ell}^*\quad\text{if}\quad n+M\leq t
\end{aligned} \tag{15}$$

and $u_s^* = \ddot{u}_{2n+s}^*$, $s = 1, ..., n$. Because for any sequence $\{\zeta_j\}_{j\in\mathbb{Z}}$

$$\frac{1}{n^{1/2}} \sum_{j=1}^{n} e^{it\lambda_j} \left(\sum_{\ell=0}^{n} \zeta_\ell e^{-i\ell\lambda_j} \right) w_z(\lambda_j) =: \sum_{\ell=0}^{t-1} \zeta_\ell z_{t-\ell} + \sum_{\ell=t}^{n} \zeta_\ell z_{n-(\ell-t)}, \tag{16}$$

we obtain that

$$\widehat{\varepsilon}_t = \sum_{\ell=0}^{t-1} \widehat{\phi}_\ell u_{t-\ell} + \sum_{\ell=t}^{n} \widehat{\phi}_\ell u_{n+t-\ell}. \tag{17}$$

Finally it is worth recalling that for any sequence $\{\zeta_j\}_{j\geq 1}$ and a martingale difference sequence $\{\eta_j\}_{j\in\mathbb{Z}}$ with finite p moments, we have that

$$E\left|\sum_{j=a}^{b} \zeta_j \eta_j\right|^p \leq KE\left|\sum_{j=a}^{b} \zeta_j^2\right|^{p/2-1} \sum_{j=a}^{b} \zeta_j^2 E|\eta_j|^p \tag{18}$$

by Burkholder and then Hölder's inequalities.

1.1. *APPENDIX A*.

For simplicity and without loss of generality we shall assume that $\sigma_x^2 = 1$. Also herewith K denotes a generic positive and finite constant.

PROPOSITION 1. *Under Conditions* 1 *and* 3*, for any* $p \geq 1$ *such that* $E|\varepsilon_t|^p < \infty$*, we have that*

$$\frac{1}{n} \sum_{t=1}^{n} |\breve{\varepsilon}_t - \varepsilon_t|^p = o_p(1). \tag{19}$$

PROOF. First observe that because

$$\frac{1}{n} \sum_{t=1}^{n} |\breve{\varepsilon}_t - \varepsilon_t|^p \leq 2^{p-1} \left(\frac{1}{n} \sum_{t=1}^{n} |\widehat{\varepsilon}_t - \varepsilon_t|^p + \left| \frac{1}{n} \sum_{t=1}^{n} \widehat{\varepsilon}_t \right|^p \right),$$

it suffices to show that the first term on the right is $o_p(1)$.

We shall first examine the case when $B(L) = 1$. Denoting

$$\widetilde{\varepsilon}_t = \frac{1}{n^{1/2}} \sum_{j=1}^{n} e^{it\lambda_j} \left(\sum_{\ell=0}^{n} \xi_\ell e^{-i\ell\lambda_j} \right) w_u(\lambda_j),$$

and observing that when $B(L) = A(L) = 1$, $\widehat{\varepsilon}_t$ in *STEP 1* becomes

$$\widehat{\varepsilon}_t = \frac{1}{n^{1/2}} \sum_{j=1}^{n-1} e^{it\lambda_j} \left(\sum_{\ell=0}^{n} \widehat{\xi}_\ell e^{-i\ell\lambda_j} \right) w_u(\lambda_j), \quad t = 1, ..., n,$$

we then have that (16) and Lemma 3 imply that

$$\begin{aligned}
\widehat{\varepsilon}_t - \widetilde{\varepsilon}_t &= \sum_{\ell=0}^{t-1} \left(\widehat{\xi}_\ell - \xi_\ell \right) u_{t-\ell} + \sum_{\ell=t}^{n} \left(\widehat{\xi}_\ell - \xi_\ell \right) u_{n-(\ell-t)} \\
&= \sum_{h=1}^{H-1} \left\| \widehat{d} - d \right\|^h \sum_{\ell=0}^{t-1} g_\ell^h(d) \xi_\ell u_{t-\ell} + \log^H n \left\| \widehat{d} - d \right\|^H \sum_{\ell=0}^{t-1} \left| \frac{u_{t-\ell}}{\ell^{1+(d_1 \wedge d_2)}} \right| \\
&\quad + \sum_{h=1}^{H-1} \left\| \widehat{d} - d \right\|^h \sum_{\ell=t}^{n} g_\ell^h(d) \xi_\ell u_{n-(\ell-t)} + \log^H n \left\| \widehat{d} - d \right\|^H \sum_{\ell=t}^{n} \left| \frac{u_{n-(\ell-t)}}{\ell^{1+(d_1 \wedge d_2)}} \right|.
\end{aligned}$$

By standard inequalities, it suffices to show that

$$\frac{1}{n}\sum_{t=1}^{n}\left|\widehat{\varepsilon}_t-\widetilde{\varepsilon}_t\right|^p=o_p\left(1\right) \tag{20}$$

$$\frac{1}{n}\sum_{t=1}^{n}\left|\varepsilon_t-\widetilde{\varepsilon}_t\right|^p=o_p\left(1\right). \tag{21}$$

That (21) holds true follows because $\varepsilon_t-\widetilde{\varepsilon}_t=:\sum_{\ell=t}^{\infty}\xi_\ell u_{t-\ell}-\sum_{\ell=t}^{n}\xi_\ell u_{n-(\ell-t)}$ so that the expectation of the left side of (21) is bounded by

$$\frac{K}{n}\sum_{t=1}^{n}\left(\sum_{\ell=t}^{\infty}\left|\xi_k\right|\right)^p\leq\frac{K}{n}\sum_{t=1}^{n}\frac{\mathcal{I}\left(d_1\wedge d_2>0\right)}{t^{p(d_1\wedge d_2)}}=o\left(1\right),$$

because $\left|\xi_k\right|=O\left(k^{-1-(d_1\wedge d_2)}\mathcal{I}\left(d_1\wedge d_2>0\right)\right)$, that $E\left|u_t\right|^p<K$ and Hölder's inequality yields that

$$E\left|\sum_{k=a}^{b}\zeta_k\eta_k\right|^p\leq\left(\sum_{k=a}^{b}\left|\zeta_k\right|\right)^{p-1}\sum_{k=a}^{b}\left|\zeta_k\right|E\left|\eta_k\right|^p. \tag{22}$$

Next (20) also holds true after we observe that (22) implies that

$$E\left|\sum_{\ell=0}^{t-1}g_\ell^h\left(d\right)\xi_\ell u_{t-\ell}\right|^p+E\left|\sum_{\ell=t}^{n}g_\ell^h\left(d\right)\xi_\ell u_{n-(t-\ell)}\right|^p\leq K\sum_{\ell=0}^{n}\left|g_\ell^h\left(d\right)\xi_\ell\right|\leq K,$$

$\left\|\widehat{d}-d\right\|=O_p\left(m^{-1/2}\right)$ and choosing H large enough in Lemma 3.

Now, we examine the general case when $B\left(L\right)\neq 1$. To that end, denote

$$\mathring{\varepsilon}_t=\sum_{\ell=0}^{t-1}\mathring{\phi}_\ell u_{t-\ell}+\sum_{\ell=t}^{n}\mathring{\phi}_\ell u_{n-(\ell-t)},$$

where

$$\mathring{\phi}_\ell=\begin{cases}\sum_{k=0}^{\ell\wedge M}\xi_{\ell-k}a_k, & 1\leq\ell\leq n-M\\ \sum_{k=\ell-(n-M)}^{M}\xi_{\ell-k}a_k, & n-M<\ell\leq n.\end{cases} \tag{23}$$

As we proceed when we assumed that $B\left(L\right)=1$, by standard inequalities, it suffices to show that

$$\frac{1}{n}\sum_{t=1}^{n}\left|\widehat{\varepsilon}_t-\mathring{\varepsilon}_t\right|^p=o_p\left(1\right) \tag{24}$$

$$\frac{1}{n}\sum_{t=1}^{n}\left|\mathring{\varepsilon}_t-\widetilde{\varepsilon}_t\right|^p+\frac{1}{n}\sum_{t=1}^{n}\left|\varepsilon_t-\widetilde{\varepsilon}_t\right|^p=o_p\left(1\right), \tag{25}$$

where $\widetilde{\varepsilon}_t=\sum_{\ell=0}^{t-1}\phi_\ell u_{t-\ell}+\sum_{\ell=t}^{n}\phi_\ell u_{n-(\ell-t)}$ and $\widehat{\varepsilon}_t$ given in (17).

Now, that the second term on the left of (25) is $o_p\left(1\right)$ follows proceeding as with the proof of (21). Next, the first term on the left of (25). Because $\mathring{\phi}_\ell-\phi_\ell=0$ if $\ell\leq M$, we have that that its expectation is bounded by

$$\frac{1}{n}\sum_{t=M+1}^{n}\left\{E\left|\sum_{\ell=M+1}^{t-1}\left(\mathring{\phi}_\ell-\phi_\ell\right)u_{t-\ell}\right|^p+E\left|\sum_{\ell=t}^{n}\left(\mathring{\phi}_\ell-\phi_\ell\right)u_{n-(\ell-t)}\right|^p\right\}$$

$$
\begin{aligned}
&+\frac{1}{n}\sum_{t=1}^{M}E\left|\sum_{\ell=M+1}^{n}\left(\mathring{\phi}_{\ell}-\phi_{\ell}\right)u_{n-(\ell-t)}\right|^{p}\\
&\leq\frac{1}{n}\sum_{t=M+1}^{n}\left\{\left(\sum_{\ell=M+1}^{t-1}\left|\mathring{\phi}_{\ell}-\phi_{\ell}\right|\right)^{p}+\left(\sum_{\ell=t}^{n}\left|\mathring{\phi}_{\ell}-\phi_{\ell}\right|\right)^{p}\right\}\\
&+\frac{1}{n}\sum_{t=1}^{M}\left(\sum_{\ell=M+1}^{n}\left|\mathring{\phi}_{\ell}-\phi_{\ell}\right|\right)^{p}\\
&=o(1)
\end{aligned}
$$

by (22) and that $E|u_t|^p<K$ and then because $\left|\mathring{\phi}_{\ell}-\phi_{\ell}\right|$ is a summable sequence by Lemma 7.

Finally (24), whose left term is bounded by

$$
\begin{aligned}
&\frac{K}{n}\sum_{t=1}^{M}\left\{\left|\sum_{\ell=0}^{t-1}\ddot{\phi}_{\ell}u_{t-\ell}\right|^{p}+\left|\sum_{\ell=t}^{M}\ddot{\phi}_{\ell}u_{n-(\ell-t)}\right|^{p}+\left|\sum_{\ell=M+1}^{n}\ddot{\phi}_{\ell}u_{n-(\ell-t)}\right|^{p}\right\}\\
&+\frac{K}{n}\sum_{t=M+1}^{n}\left\{\left|\sum_{\ell=0}^{M}\ddot{\phi}_{\ell}u_{t-\ell}\right|^{p}+\left|\sum_{\ell=M+1}^{t-1}\ddot{\phi}_{\ell}u_{t-\ell}\right|^{p}+\left|\sum_{\ell=t}^{n}\ddot{\phi}_{\ell}u_{n-(\ell-t)}\right|^{p}\right\},
\end{aligned}
$$

where we have abbreviated $\widehat{\phi}_{\ell}-\mathring{\phi}_{\ell}=:\ddot{\phi}_{\ell}$. Now, that the fourth term is $o_p(1)$ follows because (65) yields that this term is

$$
O_p\left(m^{-p/2}\right)\frac{K}{n}\sum_{t=M+1}^{n}\left|\sum_{\ell=0}^{M}|u_{t-\ell}|\right|^{p}=O_p\left(\left(\frac{M^2}{m}\right)^{p/2}\right)=o_p(1)
$$

by Condition 3. Next by Lemma 6 part (ii), the sixth term is

$$
O_p\left(m^{-p/2}\right)\frac{K}{n}\sum_{t=M+1}^{n}\left|\sum_{\ell=t}^{n}|\xi_{\ell-M}|\,|u_{t-\ell}|\right|^{p}=O_p\left(m^{-p/2}\right)
$$

because (22) and $E|u_t|^p<K$ imply that

$$
E\sum_{t=M+1}^{n}\left|\sum_{\ell=t}^{n}|\xi_{\ell-M}|\,|u_{t-\ell}|\right|^{p}\leq K\sum_{t=M+1}^{n}\left\{\left(\sum_{\ell=t}^{n}(\ell-M)^{-1-(d_1\wedge d_2)}\right)^{p}\right\}=O(n)
$$

because $(d_1\wedge d_2)>0$. The fifth term follows by the same argument, whereas the first three terms are $o_p(1)$ proceeding as with the last three terms and that $M/n=o(1)$. ∎

COROLLARY 1. *Assume Conditions* 1 *and* 3. *Then, for any* $p\geq 1$ *such that* $E|\varepsilon_t|^p<\infty$, *we have that*

$$
E^{*}\varepsilon_t^{*p}=:\frac{1}{n}\sum_{t=1}^{n}\breve{\varepsilon}_t^{p}\overset{P}{\to}E\varepsilon_t^{p}.
$$

PROOF. By standard equalities, we have that

$$
\frac{1}{n}\sum_{t=1}^{n}\breve{\varepsilon}_t^{p}=\frac{1}{n}\sum_{t=1}^{n}\left(\breve{\varepsilon}_t-\varepsilon_t\right)^{p}+\sum_{k=1}^{p-1}\frac{1}{n}\sum_{t=1}^{n}\binom{p}{k}\left(\breve{\varepsilon}_t-\varepsilon_t\right)^{k}\varepsilon_t^{p-k}+\frac{1}{n}\sum_{t=1}^{n}\varepsilon_t^{p}.
$$

The first term is $o_p(1)$ by Proposition 1. The third term converges in probability to $E\varepsilon_t^p$ by Condition 1. From here the conclusion follows by Hölder's inequality. ∎

PROPOSITION 2. *Under Conditions* 1 *and* 3*, for any* $p \geq 1$ *such that* $E|\varepsilon_t|^p < \infty$*, we have that*

$$E^* |u_t^* - \widetilde{u}_t^*|^p = o_p(1), \tag{26}$$

where $\widetilde{u}_t^* = \sum_{\ell=0}^{n+M} \vartheta_\ell \varepsilon_{t-\ell}^*$.

PROOF. Defining $\mathring{u}_t^* = \sum_{\ell=0}^{n+M} \mathring{\vartheta}_\ell \varepsilon_{t-\ell}^*$, where

$$\mathring{\vartheta}_\ell = \begin{cases} \sum_{k=0}^{\ell \wedge M} b_p \overline{\xi}_{\ell-p}, & 1 \leq \ell \leq n \\ \sum_{k=\ell-n}^{M} b_p \overline{\xi}_{\ell-p}, & n < \ell \leq n+M, \end{cases} \tag{27}$$

it suffices to show that

$$E^* |u_t^* - \mathring{u}_t^*|^p = o_p(1) \tag{28}$$

$$E^* |\mathring{u}_t^* - \widetilde{u}_t^*|^p = o_p(1). \tag{29}$$

We first examine (29). Because $\vartheta_\ell - \mathring{\vartheta}_\ell = 0$ if $\ell \leq M$, we have that $\mathring{u}_t^* - \widetilde{u}_t^* = \sum_{\ell=M+1}^{n+M} \left(\mathring{\vartheta}_\ell - \vartheta_\ell\right) \varepsilon_{t-\ell}^*$, so that the left side of (29) becomes

$$E^* \left| \sum_{\ell=M+1}^{n+M} \left(\mathring{\vartheta}_\ell - \vartheta_\ell\right) \varepsilon_{t-\ell}^* \right|^p \leq K \left(\sum_{\ell=M+1}^{n+M} \left|\mathring{\vartheta}_\ell - \vartheta_\ell\right|^2 \right)^{p/2-1} \sum_{\ell=M+1}^{n+M} \left|\mathring{\vartheta}_\ell - \vartheta_\ell\right|^2 E^* \left|\varepsilon_{t-\ell}^*\right|^p$$

by (18) because $\{\varepsilon_t^*\}_{t=1}^{n+M}$ is a random sequence with zero mean. Now, by definition we have that $\mathring{\vartheta}_\ell - \vartheta_\ell =: \overline{\xi}_\ell O\left(M^{-2}\right)$ by Lemma 9, so that we conclude that (29) holds true because $\left\{\overline{\xi}_\ell^2\right\}_{\ell \geq 1}$ is a summable sequence and Corollary 1 implies that $E^* \left|\varepsilon_{t-\ell}^*\right|^p - E\left|\varepsilon_{t-\ell}\right|^p = o_p(1)$. So, to complete the proof it remains to show that (28) holds true. But this is the case because $u_t^* - \mathring{u}_t^* = \sum_{\ell=0}^{n+M} \left(\widehat{\vartheta}_\ell - \mathring{\vartheta}_\ell\right) \varepsilon_{t-\ell}^*$ and hence

$$E^* |u_t^* - \mathring{u}_t^*|^p \leq K \left(\sum_{\ell=0}^{n+M} \left|\widehat{\vartheta}_\ell - \mathring{\vartheta}_\ell\right|^2 \right)^{p/2-1} \sum_{\ell=0}^{n+M} \left|\widehat{\vartheta}_\ell - \mathring{\vartheta}_\ell\right|^2 E^* \left|\varepsilon_{t-\ell}^*\right|^p$$

by (18). From here the proof follows by Lemma 8 since $\overline{\xi}_k^2$ is a summable sequence and Corollary 1. ∎

PROPOSITION 3. *Under Conditions 1 and 3 , as* $n \to \infty$*, in probability,*

$$\varepsilon_t^* \xrightarrow{d^*} \varepsilon_t$$

PROOF. Denote by $d_2(\cdot, \cdot)$ the Mallows metric as defined for example by Bickel and Freedman (1981). Let $\widehat{F}_n(x) = \frac{1}{n} \sum_{t=1}^{n} \mathcal{I}(\widehat{\varepsilon}_t \leq x)$, $F_n(x) = \frac{1}{n} \sum_{t=1}^{n} \mathcal{I}(\varepsilon_t \leq x)$ and $F(x) = P(\varepsilon_t \leq x)$. Then

$$d_2\left(\widehat{F}_n, F\right) \leq d_2\left(\widehat{F}_n, F_n\right) + d_2(F_n, F). \tag{30}$$

Let W be a random variable distributed uniformly on $\{1,2,\ldots,n\}$. Then

$$d_2\left(\widehat{F}_n, F_n\right) \leq E_W\left(\varepsilon_W - \widehat{\varepsilon}_W\right)^2 = \frac{1}{n}\sum_{t=1}^{n}\left(\widehat{\varepsilon}_t - \varepsilon_t\right)^2.$$

By Proposition 1, the last expression converges to zero in probability. The second term of (30) converges to zero almost surely by Lemma 8.4 of Bickel and Freedman (1981). Therefore $d_2\left(\widehat{F}_n, F\right) = o_p(1)$ and the proposition holds. ∎

PROPOSITION 4. *Assume Conditions 1 and 3. Then, as $n \to \infty$, in probability,*

$$u_t^* \overset{d}{\to}{}^* u_t.$$

By the definition of u_t^* in (9), we obtain that

(31)

$$u_t^* = \sum_{\ell=0}^{n+M} \vartheta_\ell \varepsilon_{t-\ell}^* + \sum_{\ell=0}^{M}\left(\widehat{\vartheta}_\ell - \mathring{\vartheta}_\ell\right)\varepsilon_{t-\ell}^* + \sum_{\ell=M+1}^{n+M}\left(\widehat{\vartheta}_\ell - \mathring{\vartheta}_\ell\right)\varepsilon_{t-\ell}^* + \sum_{\ell=M+1}^{n+M}\left(\mathring{\vartheta}_\ell - \vartheta_\ell\right)\varepsilon_{t-\ell}^*,$$

where $\widehat{\vartheta}_\ell$ was given in (13) and because $\mathring{\vartheta}_\ell - \vartheta_\ell = 0$ if $\ell \leq M$. The last term on the right of (31) is $o_{p^*}(1)$, since its second moment is

$$\sum_{\ell=M+1}^{n+M}\left(\mathring{\vartheta}_\ell - \vartheta_\ell\right)^2 E^* \varepsilon_{t-\ell}^{*2} = o_p(1)$$

by summability of $\left\{\left|\mathring{\vartheta}_\ell - \vartheta_\ell\right|^2\right\}_{\ell\geq 1}$ and Corollary 1 implies that $E^*\varepsilon_{t-\ell}^{*2} - E\varepsilon_{t-\ell}^2 = o_p(1)$. Next, the second and third terms on the right of (31) are also $o_{p^*}(1)$ as we now show. Indeed, the second moment is

$$\sum_{\ell=0}^{M}\left(\widehat{\vartheta}_\ell - \mathring{\vartheta}_\ell\right)^2 E^*\varepsilon_{t-\ell}^{*2} + \sum_{\ell=M+1}^{n+M}\left(\widehat{\vartheta}_\ell - \mathring{\vartheta}_\ell\right)^2 E^*\varepsilon_{t-\ell}^{*2}.$$

From here we conclude by Lemma 8 parts (i) and (ii) respectively and again by Corollary 1. Thus, Markov inequality implies that the last three terms on the right of (31) are $o_{p^*}(1)$, so that

$$u_t^* = \sum_{\ell=0}^{n+M} \vartheta_\ell \varepsilon_{t-\ell}^* + o_{p^*}(1).$$

Now, Proposition 3 and Cramér-Wold's device imply that for any $u \in \mathbb{R}$,

$$P^*\left(\sum_{\ell=0}^{n+M} \vartheta_\ell \varepsilon_{t-\ell}^* \leq u\right) = P\left(\sum_{\ell=0}^{n+M} \vartheta_\ell \varepsilon_{t-\ell} \leq u\right) + o_p(1).$$

Since $\left\{\vartheta_\ell^2\right\}_{\ell\geq 1}$ is summable, $E\left(\sum_{\ell=n+M+1}^{\infty} \vartheta_\ell \varepsilon_{t-\ell}\right)^2 = o(1)$ and Markov's inequality yields that

$$u_t = \sum_{\ell=0}^{\infty} \vartheta_\ell \varepsilon_{t-\ell} = \sum_{\ell=0}^{n+M} \vartheta_\ell \varepsilon_{t-\ell} + o_p(1).$$

Gathering the last three displayed expressions we conclude the proof of the proposition.

THEOREM 1. *Assume Conditions* 2 *and* 4. *Then, if* $d_1 + 2d_x < 1$,

$$\mathcal{G}_n(x) := \frac{1}{n^{1/2}} \sum_{t=1}^{n} \mathring{\mathbf{1}}_t(x) u_t \overset{weakly}{\Rightarrow} \mathcal{G}(x) \quad x \in \mathbb{R}.$$

PROOF. By Lemma 12, we have that $\mathcal{G}_n(x)$ is tight. So, it suffices to show that the finite dimensional distributions converge to a normal random variable with covariance structure given in (8). First we observe that

$$\begin{aligned} E(\mathcal{G}_n(x)\mathcal{G}_n(y)) &= \gamma_u(0) E\left(\mathring{\mathbf{1}}_0(x)\mathring{\mathbf{1}}_0(y)\right) \\ &\quad + \sum_{\ell=1}^{n} \gamma_u(\ell) E\left(\mathring{\mathbf{1}}_0(x)\mathring{\mathbf{1}}_\ell(y)\right) + \sum_{\ell=1}^{n} \gamma_u(\ell) E\left(\mathring{\mathbf{1}}_0(y)\mathring{\mathbf{1}}_\ell(x)\right) \\ &\underset{n\nearrow\infty}{\to} E(\mathcal{G}(x)\mathcal{G}(y)). \end{aligned}$$

To complete the proof, we need to show that for any $x \in \mathbb{R}$,

$$\mathcal{G}_n(x) \overset{d}{\Rightarrow} \mathcal{N}\left(0, E\left(\mathcal{G}^2(x)\right)\right).$$

The proof uses a Central Limit Theorem of Scott (1973) very much in the way employed in Robinson and Hidalgo (1997), see also Giraitis et al.'s (2012) Proposition 11.5.4 for a different approach. Indeed, we first write

$$\frac{1}{n^{1/2}} \sum_{t=1}^{n} \mathring{\mathbf{1}}_t(x) u_t = \frac{1}{n^{1/2}} \sum_{s=-N}^{n} \boldsymbol{\upsilon}_s(x)\varepsilon_s + \frac{1}{n^{1/2}} \sum_{s=-\infty}^{-N+1} \boldsymbol{\upsilon}_s(x)\varepsilon_s, \tag{32}$$

where $\boldsymbol{\upsilon}_s(x) = \sum_{t=1}^{n} \vartheta_{t-s}\mathring{\mathbf{1}}_t(x)$ with $\vartheta_{t-s} = 0$ if $t < s$. Because $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ is an iid sequence, we have that the second moment of the second term on the right of (32), by Condition 4, is

$$\begin{aligned} \frac{\sigma_\varepsilon^2}{n} \sum_{s=-\infty}^{-N+1} E\boldsymbol{\upsilon}_s^2(x) &= \frac{\sigma_\varepsilon^2}{n} \sum_{s=-\infty}^{-N+1} \sum_{t,r=1}^{n} E\left(\mathring{\mathbf{1}}_t(x)\mathring{\mathbf{1}}_r(x)\right)\vartheta_{t-s}\vartheta_{r-s} \\ &= \frac{K}{n} \sum_{s=-\infty}^{-N+1} \sum_{t,r=1}^{n} \frac{1}{|t-r|_+^{2-4d_x}} \frac{1}{|t-s|^{1-d_1}} \frac{1}{|r-s|^{1-d_1}} \\ &= \frac{K}{n} \sum_{t=1\le r}^{n} \frac{1}{(r-t)_+^{2-4d_x}} \sum_{s=N-1}^{\infty} \frac{1}{(t+s)^{1-d_1}} \frac{1}{(r+s)^{1-d_1}} \\ &= \frac{K}{n} \sum_{t=1\le r}^{n} \frac{1}{(r-t)_+^{2-4d_x}} \frac{1}{(t+N)^{1-2d_1}} \\ &= o(1) \end{aligned}$$

because $d_1 + 2d_x < 1$ and choosing N large enough. Recall that $\mathring{\mathbf{1}}_t(x) = \mathbf{1}_t(x) + \phi(x)x_t$ has a covariance structure $K\gamma_x^2(\ell)$, as $-\phi(x)x_t$ is the first term of the "Hermite/Appell" expansion of $\mathbf{1}_t(x)$. So, the asymptotic behaviour of $\mathcal{G}_n(x)$ is governed by that of the first term on the right of (32) which converges in distribution to $\mathcal{N}\left(0, E\left(\mathcal{G}^2(x)\right)\right)$ proceeding as in the proof of Robinson and Hidalgo's Theorem 1 (1997). Notice that Robinson and Hidalgo (1997) did not assume the regressors to be a linear process, only that they are mean zero with covariance structure $\gamma_x(j)$ such that $\{\gamma_x(j)\gamma_u(j)\}_{j\ge1}$ is summable. This concludes the proof of the theorem. ∎

PROPOSITION 5. *Assume Conditions 2 and 4 with $\{x_t\}_{t\in\mathbb{Z}}$ being a sequence of Gaussian random variables. Then, under the null hypothesis H_0, if $d_1+2d_x<1$, we have that uniformly in $x\in\mathbb{R}$*

$$n^{1/2}\mathcal{T}_n(x)=\mathcal{G}_n(x)+o_p(1).$$

PROOF. First of all, using (3) and (1), standard algebra yields that

$$\begin{aligned}\mathcal{T}_n(x)&=\frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)u_t-\overline{u}_n\frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\\&-\left(\widetilde{\beta}-\beta\right)\left\{\frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)x_t-\overline{x}_n\frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\right\}\\&+\frac{n^{-1}\sum_{t=1}^{n}\left(x_t^2-1\right)}{n^{-1}\sum_{t=1}^{n}x_t^2}\left(\widetilde{\beta}-\beta\right)\left\{\frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)x_t-\overline{x}_n\frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\right\},\end{aligned}$$

where by orthogonality of the Hermite polynomials $E\left(\mathring{\mathbf{1}}_t(x)x_t\right)=0$ and

$$\widetilde{\beta}-\beta=\frac{1}{n}\sum_{t=1}^{n}x_tu_t-\overline{x}_n\overline{u}_n. \tag{33}$$

Condition 4 and Markov's inequality imply that

$$\overline{u}_n=O_p\left(n^{d_1-1/2}\right);\ \overline{x}_n=O_p\left(n^{d_x-1/2}\right) \tag{34}$$

$$\frac{1}{n}\sum_{t=1}^{n}\left(x_t^2-1\right)=O_p\left(n^{-1/2}+n^{-1+2d_x}\mathcal{I}(d_x>1/4)\right).$$

On the other hand, using results in Robinson and Hidalgo (1997),

$$\widetilde{\beta}-\beta=O_p\left(n^{-1/2}+n^{d_x+d_1-1}\mathcal{I}(d_x+d_1>1/2)\right),$$

whereas Dehling and Taqqu (1989) and Taqqu (1975) yield respectively that

$$\sup_{x\in\mathbb{R}}\left|\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\right|=O_p\left(n^{1/2}\right)\ \text{ if } d_x<1/4 \tag{35}$$

$$\frac{1}{n^{2d_x}}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\to\mathcal{R}_2(x),\ \text{ if } d_x>1/4,\ \text{(weakly)}$$

a Rosenblatt process of order 2. Notice that, as shown in Wu's (2003) Theorem 3, (35) holds true if we drop the assumption of Gaussianity from the sequence $\{x_t\}_{t\in\mathbb{Z}}$. So because $d_1+2d_x<1$, (34) and (35) imply that

$$\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\right||\overline{u}_n|=o_p\left(n^{-1/2}\right). \tag{36}$$

Finally, Lemmas 17 and 18 imply that

$$\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)x_t\right|=O_p\left(n^{-1/2}+n^{3d_x-3/2}\mathcal{I}(d_x>1/3)\right). \tag{37}$$

So, gathering (34) to (37) we conclude. ∎

COROLLARY 2. *Under Conditions of Proposition 5, we have that for any continuous functional* $\varphi(\cdot) : \mathbb{R} \to \mathbb{R}^{+}$,

$$\varphi\left(n^{1/2}\mathcal{T}_n(x)\right) \overset{d}{\to} \varphi(\mathcal{G}_n(x)).$$

PROOF. It follows by the continuous mapping theorem, Theorem 1 and Proposition 5, so it is omitted. ∎

PROPOSITION 6. *Under Conditions* 2 *to* 4*, if* $d_1 + d_x < 1/2$, $\widehat{d}_1 - d_1 = O_p\left(m^{-1/2}\right)$.

PROOF. Similar to Robinson (1997), the properties of the estimator are not affected by using the residuals $\widehat{u}_t$ instead of the errors u_t. Indeed

$$I_{\widehat{u}\widehat{u}}(\lambda_j) - I_{uu}(\lambda_j) = \left(\widehat{\beta} - \beta\right) I_{ux}(\lambda_j) + \left(\widehat{\beta} - \beta\right)^2 I_{xx}(\lambda_j).$$

Since $\widehat{\beta} - \beta = O_p\left(n^{-1/2}\right)$ and proceeding as in Robinson (1995),

$$\begin{aligned} E\left|I_{ux}(\lambda_j) I_{ux}(\lambda_k)\right| &= E\left(w_u(\lambda_j)\overline{w}_u(\lambda_k)\right) E\left(w_x(\lambda_j)\overline{w}_x(\lambda_k)\right) \\ &\simeq \lambda_j^{-d_1-d_x}\lambda_k^{-d_1-d_x}\max\left(k^{-1}, j^{-1}\right). \end{aligned}$$

So, we conclude that

$$\frac{1}{m}\sum_{j=1}^{m}\lambda_j^{2d_1}\left(I_{\widehat{u}\widehat{u}}(\lambda_j) - I_{uu}(\lambda_j)\right) = o_p\left(m^{-1/2}\right).$$

From here the proof proceeds as in Robinson (1997). ∎

PROPOSITION 7. *Assuming Conditions* 2 *to* 4*, for any* $0 \le \ell \le n$,

$$E^{*}\left(u_t^{*}u_{t+\ell}^{*}\right) - \gamma_u(\ell) = o_p(1)\,\ell^{2d_1-1} + O_p\left(\frac{n^{2d_1}}{m^{3/2+d_1}}\right) + O\left(n^{2d_1-1}\right).$$

PROOF. It is immediate. Indeed, Lemma 5 implies that the left side of the last displayed expression is

$$\begin{aligned} &\widehat{\sigma}_{\varepsilon}^2\sum_{k=0}^{n+M-\ell}\left(\widehat{\vartheta}_k\widehat{\vartheta}_{k+\ell} - \mathring{\vartheta}_k\mathring{\vartheta}_{k+\ell}\right) - \left(\widehat{\sigma}_{\varepsilon}^2 - \sigma_{\varepsilon}^2\right)\sum_{k=0}^{n+M-\ell}\mathring{\vartheta}_k\mathring{\vartheta}_{k+\ell} \\ &\sigma_{\varepsilon}^2\sum_{k=0}^{n+M-\ell}\left(\mathring{\vartheta}_k\mathring{\vartheta}_{k+\ell} - \vartheta_k\vartheta_{k+\ell}\right) - \sigma_{\varepsilon}^2\sum_{k=n+M-\ell}^{\infty}\vartheta_k\vartheta_{k+\ell}. \end{aligned}$$

From here we conclude because $\sum_{k=n+M-\ell}^{\infty}|\vartheta_k\vartheta_{k+\ell}| = O\left(n^{2d_1-1}\right)$, Hidalgo and Yajima's (2002) Corollary 1 and Theorem 1, and that respectively Lemmas 10 and 11 yield that the first and third terms of the last displayed expression are $o(1)\,\ell^{2d_1-1}$. Observe that $\sum_{k=0}^{n+M-\ell}\vartheta_k\vartheta_{k+\ell} = K\ell^{2d_1-1}(1+o(1))$. ∎

We introduce some notation. By K_n we denote a sequence of nonnegative random variables.

PROPOSITION 8. *Assuming Conditions* 2 *to* 4*, if* $d_1 + d_x < 1/2$*, (in probability)*

$$n^{1/2}\left(\widehat{\beta}^{*} - \widehat{\beta}\right) \overset{d^{*}}{\to} \mathcal{N}(0, \mathcal{V}),$$

where $\mathcal{V} = \left(\sigma_x^2\right)^{-2}\int_{-\pi}^{\pi} f_u(\lambda) f_x(\lambda)\, d\lambda$ *is the asymptotic variance of the LSE.*

PROOF. By definition,

$$\widehat{\beta}^{*}-\widehat{\beta}=\frac{1}{n}\sum_{t=1}^{n}\left(x_t-\overline{x}_n\right)u_t^{*}-\frac{\frac{1}{n}\sum_{t=1}^{n}\left(x_t^2-1\right)}{\frac{1}{n}\sum_{t=1}^{n}x_t^2}\left(\frac{1}{n}\sum_{t=1}^{n}\left(x_t-\overline{x}_n\right)u_t^{*}\right).$$

Because $\sum_{t=1}^{n}\left(x_t^2-1\right)=o\left(n\right)K_n$, it suffices to consider the first term on the right of last displayed equation. Now $\widehat{\sigma}_{\varepsilon}^2-\sigma_{\varepsilon}^2=O_p\left(n^{-1/2}+m/n\right)$ by Hidalgo and Yajima's (2002) Corollary and Theorem 1, so

$$E^{*}\left\{\frac{1}{n^{1+2\widehat{d}_1}}\sum_{t,r=1}^{n}u_t^{*}u_r^{*}\right\}\overset{P}{\to}\sigma_{\varepsilon}^2\left(\lim_{n\to\infty}\frac{1}{n^{2d_1}}\sum_{\ell=-n}^{n}\gamma_u\left(\ell\right)+O\left(1\right)\right),$$

by Propositions 6 and 7. So we conclude that $\sum_{t=1}^{n}u_t^{*}=O_{p^{*}}\left(n^{1/2+\widehat{d}_1}\right)$, which together with $\overline{x}_n=O_p\left(n^{d_x-1/2}\right)$ implies that $n^{1/2}\left(\widehat{\beta}^{*}-\widehat{\beta}\right)$ is governed by $n^{-1/2}\sum_{t=1}^{n}x_tu_t^{*}$. Now, standard algebra implies that

$$E^{*}\left(\frac{1}{n^{1/2}}\sum_{t=1}^{n}x_tu_t^{*}\right)^2=\frac{1}{n}\sum_{t,r=1}^{n}x_tx_rE^{*}\left(u_t^{*}u_r^{*}\right)=\widehat{\sigma}_{\varepsilon}^2\frac{1}{n}\sum_{t,r=1}^{n}x_tx_r\sum_{k=0}^{n+M-|t-r|}\widehat{\vartheta}_k\widehat{\vartheta}_{k+|t-r|}, \tag{38}$$

using (63) in Lemma 3. As we mentioned above, the expression on the right of (38) is $\widehat{\sigma}_{\varepsilon}^2=\sigma_{\varepsilon}^2\left(1+o_p\left(1\right)\right)$ times

$$\frac{1}{n}\sum_{t,r=1}^{n}x_tx_r\sum_{k=0}^{n+M-|t-r|}\left(\widehat{\vartheta}_k\widehat{\vartheta}_{k+|t-r|}-\mathring{\vartheta}_k\mathring{\vartheta}_{k+|t-r|}\right)+\frac{1}{n}\sum_{t,r=1}^{n}x_tx_r\sum_{k=0}^{n+M-|t-r|}\vartheta_k\vartheta_{k+|t-r|} \tag{39}$$

$$+\frac{1}{n}\sum_{t,r=1}^{n}x_tx_r\sum_{k=0}^{n+M-|t-r|}\left(\mathring{\vartheta}_k\mathring{\vartheta}_{k+|t-r|}-\vartheta_k\vartheta_{k+|t-r|}\right). \tag{40}$$

That (40) is $o_p\left(1\right)$ follows by Lemmas 13 and 11. The second term of (39) is

$$\sum_{\ell=1-n}^{n-1}\left(1-\frac{|\ell|}{n}\right)\gamma_x\left(|\ell|\right)\sum_{k=0}^{n+M-|\ell|}\vartheta_k\vartheta_{k+|\ell|}$$
$$+\frac{1}{n}\sum_{t,r=1}^{n}\left(x_tx_r-\gamma_x\left(|t-r|\right)\right)\sum_{k=0}^{n+M-|t-r|}\vartheta_k\vartheta_{k+|t-r|}$$
$$\overset{P}{\to}\sum_{\ell=-\infty}^{\infty}\gamma_x\left(|\ell|\right)\gamma_u\left(|\ell|\right)/\sigma_{\varepsilon}^2$$

because proceeding as with the proof of Lemma 13, we have that the second term of last displayed expression is $o_p\left(1\right)$ after observing that Condition 4 implies $\sum_{k=0}^{n+M-|t-r|}\vartheta_k\vartheta_{k+|t-r|}=O\left(|t-r|^{2d_1-1}\right)$. It is worth mentioning that alternatively we could have used arguments similar to those in (46).

So, to complete the proof that the second moments of $\frac{1}{n^{1/2}}\sum_{t=1}^{n} x_t u_t^*$ converge in probability to $\sum_{\ell=-\infty}^{\infty}\gamma_x(|\ell|)\gamma_u(|\ell|)$, we need to show that the first term of (39) is $o_p(1)$. But this follows easily by Lemma 10. Indeed, this term is

$$
\begin{aligned}
&\varsigma_n \frac{1}{n}\sum_{t,r=1}^{n} x_t x_r |t-r|^{2d_1-1} + \frac{1}{n}\sum_{t,r=1}^{n} |x_t x_r| \, |t-r|^{2d_1-1} O_p\left(n^{-1}\right)\\
&= \varsigma_n W_n + O_p\left(n^{2d_1-1}\right)\\
&= o_p(1)
\end{aligned}
$$

because EW_n^2 is, except multiplicative constants, equal to

$$
\begin{aligned}
&\frac{1}{n^2}\sum_{t_1<r_1=2;t_2<r_2=2}^{n} \gamma_x(|t_1-r_1|)\gamma_x(|t_2-r_2|)|t_1-r_1|^{2d_1-1}|t_2-r_2|^{2d_1-1}\\
&+\frac{2}{n^2}\sum_{t_1<r_1=2;t_2<r_2=2}^{n} \gamma_x(|t_1-r_2|)\gamma_x(|t_2-r_1|)|t_1-r_1|^{2d_1-1}|t_2-r_2|^{2d_1-1}\\
&+\frac{1}{n^2}\sum_{t_1<r_1=2;t_2<r_2=2}^{n} cum\left(x_{t_1};x_{t_2};x_{r_1};x_{r_2}\right)|t_1-r_1|^{2d_1-1}|t_2-r_2|^{2d_1-1}
\end{aligned}
$$

which is $O(1)$ since $2d_1+2d_x-1<0$, $\gamma_x(|t|)=O\left(|t|^{2d_x-1}\right)$ and

$$
cum\left(x_{t_1};...,x_{t_4}\right) = \kappa_{4,\varrho}\sum_{j=0}^{\infty}\prod_{\ell=1}^{4}\varphi_{j+|t_\ell-t_1|};\quad \varphi_j = O\left(j^{d_x-1}\right), \tag{41}
$$

where $\kappa_{4,\varrho}$ is the fourth cumulant of $\{\varrho_t\}_{t\in\mathbb{Z}}$. Note that we could have invoked Lemma 13 to reach the same conclusion.

We are left to show the Lindeberg's condition. To that end, we notice that in view of (15), we have that

$$
\frac{1}{n^{1/2}}\sum_{t=1}^{n} x_t u_t^* = \frac{1}{n^{1/2}}\sum_{s=1}^{n+M}\widehat{\boldsymbol{v}}_{s,1}\varepsilon^*_{n+M-s} + \frac{1}{n^{1/2}}\sum_{s=1}^{n}\widehat{\boldsymbol{v}}_{s,2}\varepsilon^*_{n+M+s},
$$

where

$$
\begin{aligned}
\widehat{\boldsymbol{v}}_{s,1} &= \sum_{t=1}^{n}\widehat{\vartheta}_{t+s}x_t\mathcal{I}(s\le M) + \sum_{t=1}^{n+M-s+1}\widehat{\vartheta}_{t+s}x_t\mathcal{I}(M<s)\\
\widehat{\boldsymbol{v}}_{s,2} &= \sum_{t=s+1}^{n}\widehat{\vartheta}_{t-s}x_t.
\end{aligned} \tag{42}
$$

So, a sufficient condition is that

$$
\frac{1}{n^2}\sum_{s=1}^{n+M}E^*\left(\widehat{\boldsymbol{v}}_{s,1}\varepsilon^*_{n+M-s}\right)^4 + \frac{1}{n^2}\sum_{s=1}^{n}E^*\left(\widehat{\boldsymbol{v}}_{s,2}\varepsilon^*_{n+M+s}\right)^4 = o_p(1)K_n. \tag{43}
$$

Now, the first term on the left of (43) is bounded by

$$
\frac{1}{n^2}\left\{\sum_{s=1}^{M}\left(\sum_{t=1}^{n}\widehat{\vartheta}_{t+s}x_t\right)^4 + \sum_{s=M+1}^{n+M}\left(\sum_{t=1}^{n+M-s+1}\widehat{\vartheta}_{t+s}x_t\right)^4\right\}K_n
$$

because $E^*\left(\varepsilon_s^{*4}\right)=K_n$. We examine the first term being the second similarly handled. Now, that term is bounded by

$$\frac{8}{n^2}\sum_{s=1}^{M}\left(\sum_{t=1}^{n}\left(\widehat{\vartheta}_{t+s}-\mathring{\vartheta}_{t+s}\right)x_t\right)^4+\frac{8}{n^2}\sum_{s=1}^{M}\left(\sum_{t=1}^{n}\mathring{\vartheta}_{t+s}x_t\right)^4. \tag{44}$$

The second term of (44) has first moment

$$\begin{aligned}&\frac{24}{n^2}\sum_{s=1}^{M}\left(\sum_{t_1,t_2=1}^{n}\mathring{\vartheta}_{t_1+s}\mathring{\vartheta}_{t_2+s}\gamma_x\left(|t_1-t_2|\right)\right)^2\\&+\frac{24}{n^2}\sum_{s=1}^{M}\sum_{t_1,\dots,t_4=1}^{n}\left(\prod_{\ell=1}^{4}\mathring{\vartheta}_{t_\ell+s}\right)cum\left(x_{t_1};\dots,x_{t_4}\right)\\=\;&o(1)\end{aligned}$$

because $\mathring{\vartheta}_s=O\left(s^{d_1-1}\right)$ by Lemma 9 and Condition 4, $\gamma_x(s)=O\left(s^{2d_x-1}\right)$ and (41). Recall that $1-2d_x-2d_1>0$.

Now, assuming for notational simplicity that $\vartheta_t=\pi_t$, Lemma 1 implies that the first term of (44) is bounded by

$$\begin{aligned}&8\left(\sum_{h=1}^{H}\left(\widehat{d}_1-d_1\right)^h g_k^h\left(d_1\right)\right)^4\frac{1}{n^2}\sum_{s=1}^{M}\left(\sum_{t=1}^{n}\vartheta_{t+s}x_t\right)^4+\\&O_p\left(\left|\widehat{d}_1-d_1\right|^{4H+4}\log^{4H+4}k\right)\frac{1}{n^2}\sum_{s=1}^{M}\left(\sum_{t=1}^{n}\left|\vartheta_{t+s}x_t\right|\right)^4\end{aligned}$$

which is $o_p(1)$ taken $H=3$ and Condition 3. Notice that $\left(\sum_{t=1}^{n}|\vartheta_{t+s}x_t|\right)^4\le n^3\sum_{t=1}^{n}\left|\vartheta_{t+s}^4x_t^4\right|=O_p\left(n^3\right)$. This concludes the proof that the first term on the left of (43) is $o_p(1)K_n$. Similarly the second term on the left of (43) is $o_p(1)K_n$. So using (34), we conclude that (in probability)

$$n^{1/2}\left(\widehat{\beta}^*-\widehat{\beta}\right)\overset{d^*}{\Rightarrow}\mathcal{N}\left(0,\sum_{\ell=-\infty}^{\infty}\gamma_x(\ell)\gamma_u(\ell)/E^2\left(x_t^2\right)\right)$$

since $\sum_{\ell=-\infty}^{\infty}\gamma_x(\ell)\gamma_u(\ell)=(2\pi)^{-1}\int_{-\pi}^{\pi}f_u(x)\,dF_x(\lambda)$. This concludes the proof of the proposition. ∎

THEOREM 2. *Assuming Conditions 2 to 4, if $d_1+2d_x<1$, (in probability)*

$$\mathcal{G}_n^*(x):=\frac{1}{n^{1/2}}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\,u_t^*\overset{d^*}{\Rightarrow}\mathcal{G}(x)\quad x\in\mathbb{R}\,.$$

PROOF. Since by Lemma 19, $\mathcal{G}_n^*(x)$ is tight, it suffices to show that the finite dimensional distributions converge to a normal random variable with covariance structure given in (8). The proof proceeds similarly to that in Proposition 8, after we notice that $E\left(\mathring{\mathbf{1}}_t(x)\,\mathring{\mathbf{1}}_s(x)\right)=O\left(|t-s|^{4d_x-2}\right)$, given in Theorem 1. Indeed, we first notice that

$$\frac{1}{n^{1/2}}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\,u_t^*=\frac{1}{n^{1/2}}\sum_{s=1}^{n+M}\widehat{\boldsymbol{v}}_{s,1}(x)\;\varepsilon_{n+M-s}^*+\frac{1}{n^{1/2}}\sum_{s=1}^{n}\widehat{\boldsymbol{v}}_{s,2}(x)\;\varepsilon_{n+M+s}^*, \tag{45}$$

where $\widehat{\boldsymbol{v}}_{s,1}(x)$ and $\widehat{\boldsymbol{v}}_{s,2}(x)$ are defined as in (42) but with x_t being replaced by $\mathring{\mathbf{i}}_t(x)$. Because independence of the sequence $\{\varepsilon_t^*\}_{t=1}^n$, the second moment of the second term on the right of (45) is

$$
\begin{aligned}
\frac{\widehat{\sigma}_\varepsilon^2}{n}\sum_{s=1}^n \widehat{\boldsymbol{v}}_{s,2}^2(x) &= \frac{\widehat{\sigma}_\varepsilon^2}{n}\sum_{s=1}^n\sum_{t,r=s+1}^n \mathring{\mathbf{i}}_t(x)\,\mathring{\mathbf{i}}_r(x)\,\widehat{\vartheta}_{t-s}\widehat{\vartheta}_{r-s} \\
&= \frac{\widehat{\sigma}_\varepsilon^2}{n}\sum_{s=1}^n\sum_{t,r=s+1}^n \mathring{\mathbf{i}}_t(x)\,\mathring{\mathbf{i}}_r(x)\left(\widehat{\vartheta}_{t-s}-\mathring{\vartheta}_{t-s}\right)\left(\widehat{\vartheta}_{r-s}-\mathring{\vartheta}_{r-s}\right) \\
&\quad +\frac{2\widehat{\sigma}_\varepsilon^2}{n}\sum_{s=1}^n\sum_{t,r=s+1}^n \mathring{\mathbf{i}}_t(x)\,\mathring{\mathbf{i}}_r(x)\,\mathring{\vartheta}_{t-s}\left(\widehat{\vartheta}_{r-s}-\mathring{\vartheta}_{r-s}\right) \\
&\quad +\frac{\widehat{\sigma}_\varepsilon^2}{n}\sum_{s=1}^n\sum_{t,r=s+1}^n \mathring{\mathbf{i}}_t(x)\,\mathring{\mathbf{i}}_r(x)\,\mathring{\vartheta}_{t-s}\mathring{\vartheta}_{r-s}.
\end{aligned}
\tag{46}
$$

The first two terms on the right of (46) are $o_p(1)$. Indeed, Lemma 8 implies that the second term is

$$
\begin{aligned}
&\frac{\widehat{\sigma}_\varepsilon^2}{n}\sum_{s=1}^n\sum_{t,r=s+1}^n \mathring{\mathbf{i}}_t(x)\,\mathring{\mathbf{i}}_r(x)\,\mathring{\vartheta}_{t-s}\mathring{\vartheta}_{r-s}(\varsigma_n) \\
&+O_p\left(n^{-1}\right)\frac{\widehat{\sigma}_\varepsilon^2}{n}\sum_{s=1}^n\sum_{t,r=s+1}^n \left|\mathring{\mathbf{i}}_t(x)\,\mathring{\mathbf{i}}_r(x)\right|\mathring{\vartheta}_{t-s}\mathring{\vartheta}_{r-s} \\
=\;& O\left(n^{2d_1-1}+\varsigma_n\right)K_n,
\end{aligned}
$$

since $E\left(\mathring{\mathbf{i}}_t(x)\,\mathring{\mathbf{i}}_r(x)\right) = O\left(|t-r|^{4d_x-2}\right)$ and Lemma 9 implies that $\mathring{\vartheta}_t = O\left(t^{d_1-1}\right)$. Likewise, the first term on the right of (46) is $O\left(n^{2d_1-1}+\varsigma_n\right)K_n$. So, the second moment of second term on the right of (45) is

$$
\frac{\widehat{\sigma}_\varepsilon^2}{n}\sum_{s=1}^n\sum_{t,r=s+1}^n \mathring{\mathbf{i}}_t(x)\,\mathring{\mathbf{i}}_r(x)\,\mathring{\vartheta}_{r-s}\mathring{\vartheta}_{t-s} + O\left(n^{2d_1}m^{-2}+m^{-1/2}\right)K_n.
$$

Similarly the contribution of the second moment due to the first term on the right of (45) is, then given by that of

$$
\frac{\widehat{\sigma}_\varepsilon^2}{n}\sum_{s=1}^n\sum_{t,r=0}^s \mathring{\mathbf{i}}_t(x)\,\mathring{\mathbf{i}}_r(x)\,\mathring{\vartheta}_{r-s}\mathring{\vartheta}_{t-s} + O\left(n^{2d_1-1}+\varsigma_n\right)K_n.
$$

So, the last two displayed expressions imply that the second moment of the left side of (45) is

$$
\frac{\sigma_\varepsilon^2}{n}\sum_{s=1}^n\sum_{t,r=0}^n \mathring{\mathbf{i}}_t(x)\,\mathring{\mathbf{i}}_r(y)\,\mathring{\vartheta}_{r-s}\mathring{\vartheta}_{t-s}\left(1+o_p(1)\right)K_n,
$$

as $\widehat{\sigma}_\varepsilon^2 - \sigma_\varepsilon^2 = o_p(1)$ by Hidalgo and Yajima's (2002) Corollary 1. But the last displayed expression is asymptotically

$$
\frac{\sigma_\varepsilon^2}{n}\sum_{t,r=0}^n \left\{\mathring{\mathbf{i}}_t(x)\,\mathring{\mathbf{i}}_r(y) - E\left(\mathring{\mathbf{i}}_t(x)\,\mathring{\mathbf{i}}_r(y)\right)\right\}\left(\sum_{s=1}^n \mathring{\vartheta}_{r-s}\mathring{\vartheta}_{t-s}\right)
$$

$$+\frac{\sigma_\varepsilon^2}{n}\sum_{t,r=0}^{n} E\left(\mathring{\mathbf{1}}_t(x)\,\mathring{\mathbf{1}}_r(y)\right)\left\{\sum_{s=1}^{\infty}\mathring{\vartheta}_{r-s}\mathring{\vartheta}_{t-s}-\sum_{s=n+1}^{\infty}\mathring{\vartheta}_{r-s}\mathring{\vartheta}_{t-s}\right\}.$$

The second term converges to $E(\mathcal{G}(x)\mathcal{G}(y))$, whereas the first term is $o_p(1)$ by Lemma 13 because $\sum_{s=1}^{n}\vartheta_{r-s}\vartheta_{t-s}=O\left(|t-r|^{2d_1-1}\right)$ and as indicated above $\mathring{\vartheta}_t=\vartheta_t(1+o(1))$ by Lemma 9. So, we have shown that the (bootstrap) second moment structure of the left hand of (45) converges in probability to $E(\mathcal{G}(x)\mathcal{G}(y))$.

We are left to show the Lindeberg's condition. Proceeding as with Proposition 8, it suffices to show that

$$\frac{1}{n^2}\sum_{s=1}^{M}\left(\sum_{t=1}^{n}\widehat{\vartheta}_{t+s}\mathring{\mathbf{1}}_t(x)\right)^4=o_p(1). \tag{47}$$

Now the left side of the last displayed expression is

$$\frac{8}{n^2}\sum_{s=1}^{M}\left(\sum_{t=1}^{n}\left(\widehat{\vartheta}_{t+s}-\mathring{\vartheta}_{t+s}\right)\mathring{\mathbf{1}}_t(x)\right)^4+\frac{8}{n^2}\sum_{s=1}^{M}\left(\sum_{t=1}^{n}\mathring{\vartheta}_{t+s}\mathring{\mathbf{1}}_t(x)\right)^4. \tag{48}$$

The second term of (48) is, except constants,

$$\frac{1}{n^2}\sum_{s=1}^{M}\left\{\left(\sum_{t=1}^{n}\mathring{\vartheta}_{t+s}^2\mathring{\mathbf{1}}_t^2(x)+\sum_{t_1<t_2}^{n}\mathring{\vartheta}_{t_1+s}\mathring{\vartheta}_{t_2+s}E\left(\mathring{\mathbf{1}}_{t_1}(x)\,\mathring{\mathbf{1}}_{t_2}(x)\right)\right)^2\right\}$$
$$+\frac{1}{n^2}\sum_{s=1}^{M}\left\{\left(\sum_{t_1<t_2}^{n}\mathring{\vartheta}_{t_1+s}\mathring{\vartheta}_{t_2+s}\left(\mathring{\mathbf{1}}_{t_1}(x)\,\mathring{\mathbf{1}}_{t_2}(x)-E\left(\mathring{\mathbf{1}}_{t_1}(x)\,\mathring{\mathbf{1}}_{t_2}(x)\right)\right)\right)^2\right\}.$$

The first term of the last displayed expression is $o_p(1)$ as $E\left(\mathring{\mathbf{1}}_{t_1}(x)\,\mathring{\mathbf{1}}_{t_2}(x)\right)=O\left(|t_1-t_2|^{4d_x-2}\right)$ and $E\mathring{\mathbf{1}}_t^4(x)<C$. The second term is also $o(1)$ by Lemma 13 because Lemma 9 and Condition 4 imply that

$$\left|\mathring{\vartheta}_{t_1+s}\mathring{\vartheta}_{t_2+s}\right|\le C\,|t_2-t_1|^{2d_1-1}\frac{|t_2-t_1|^{1-2d_1}}{|t_1+s|^{1-d_1}\,|t_2+s|^{1-d_1}}$$
$$\le C\frac{|t_2-t_1|^{2d_1-1}}{s^{1-d_1}t_2^{d_1}}$$

and $\sum_{s=1}^{M}s^{2d_1-2}<C$. The first term of (48) is also $o_p(1)$ arguing as in the proof of Proposition 8, i.e. (44), so (47) holds true and hence (in probability)

$$\mathcal{G}_n^*(x)\overset{d^*}{\Rightarrow}\mathcal{N}\left(0,E\left(\mathcal{G}^2(x)\right)\right)$$

This concludes the proof of the theorem. ∎

PROPOSITION 9. *Assuming Conditions 2 to 4 with $\{x_t\}_{t\in\mathbb{Z}}$ being Gaussian, we have that uniformly in $x\in\mathbb{R}$, if $d_1+2d_x<1$, (in probability)*

$$\text{(a)}\quad n^{1/2}\mathcal{T}_n^*(x)=\mathcal{G}_n^*(x)+o_{p^*}(1),$$

$$\text{(b)}\quad \varphi\left(n^{1/2}\mathcal{T}_n^*(x)\right)\overset{d^*}{\to}\varphi(\mathcal{G}(x)),$$

for any continuous functional $\varphi(\cdot):\mathbb{R}\to\mathbb{R}^+$.

PROOF. Part $(\mathbf{a})$. Proceeding as in Proposition 5,

$$\mathcal{T}_n^*(x) = \frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\,u_t^* - \frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\,\frac{1}{n}\sum_{t=1}^{n}u_t^*$$
$$-\left(\widetilde{\beta}^* - \widehat{\beta}\right)\left\{\frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\,x_t - \frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\,\frac{1}{n}\sum_{t=1}^{n}x_t\right\}$$
$$+\frac{\left(\widetilde{\beta}^* - \widehat{\beta}\right)\frac{1}{n}\sum_{t=1}^{n}\left(x_t^2-1\right)}{\frac{1}{n}\sum_{t=1}^{n}x_t^2}\left\{\frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\,x_t - \frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\,\frac{1}{n}\sum_{t=1}^{n}x_t\right\}.$$

By Propositions 7 and/or 8, $\widetilde{\beta}^* - \widehat{\beta} = O_{p^*}\left(n^{-1/2} + n^{d_x+d_1-1}\mathcal{I}\left(d_x+d_1>1/2\right)\right)$. Also, because $d_1 + 2d_x < 1$, we have

$$\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\right|\left|\frac{1}{n}\sum_{t=1}^{n}u_t^*\right| = o_{p^*}\left(n^{-1/2}\right)$$

since (35) implies that $\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x) = \left(n^{1/2}\mathcal{I}\left(d_x<1/4\right)+n^{2d_x}\mathcal{I}\left(d_x>1/4\right)\right)K_n$ and $\sum_{t=1}^{n}u_t^* = O_{p^*}\left(n^{1/2+d_1}\right)$ as a consequence of Proposition 7. So using (37), we conclude that, uniformly in $x\in\mathbb{R}$,

$$\mathcal{T}_n^*(x) = \frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{1}}_t(x)\,u_t^* + o_{p^*}\left(n^{-1/2}\right)$$

since $d_1 + 2d_x < 1$, and the conclusion follows by Theorem 2.

Part $(\mathbf{b})$ follows by part $(\mathbf{a})$ and the continuous mapping theorem. ∎

1.2. *APPENDIX B*.

LEMMA 1. *For any $H \geq 3$ and under Condition 1, we have that if $d_1 > 0$*

$$(\mathbf{a})\quad \frac{\widehat{\pi}_k - \pi_k}{\pi_k} = \left(\sum_{h=1}^{H-1}\left(\widehat{d}_1 - d_1\right)^h g_k^h(d_1)\right) + O_p\left(\left|\widehat{d}_1 - d_1\right|^H\right)\log^H k$$

$$(\mathbf{b})\quad \frac{\check{\pi}_k - \overline{\pi}_k}{\overline{\pi}_k} = \left(\sum_{h=1}^{H-1}\left(\widehat{d}_1 - d_1\right)^h g_k^h(d_1)\right) + O_p\left(\left|\widehat{d}_1 - d_1\right|^H\right)\log^H k,$$

where $g_k(d_1) = d_1^{-1} + \sum_{\ell=2}^{k}\ell^{-1}$, whereas if $d_1 = 0$

$$(\mathbf{c})\quad \widehat{\pi}_k = \frac{\log k}{k}\widehat{d}_1 + \frac{1}{k}O_p\left(\left|\widehat{d}_1\right|^2\right) + O_p\left(\left|\widehat{d}_1\right|^H\right)\log^H k$$

and the $O_p(\circ)$ is uniformly in k.

PROOF. We first notice that by definition, for any $0 \leq |d| < 1/2$

$$\pi_k(d) = c_k(d)\,\pi_{k-1}(d), \qquad \pi_1(d) = d, \tag{49}$$

where

$$c_k(d) = \frac{k-1-d}{k};\ c_0(d) = 1.$$

We shall explicitly examine part $(\mathbf{a})$, being part $(\mathbf{b})$ identically handled. After standard algebra, (49) implies that

$$\widehat{\pi}_k - \pi_k = c_k(d_1)(\widehat{\pi}_{k-1} - \pi_{k-1}) + \left(\widehat{d}_1 - d_1\right)\frac{\widehat{\pi}_{k-1}}{k}.$$

Now recalling the solution of an equation in difference $\xi_k = \kappa_k \xi_{k-1} + h_k$ is

$$\xi_k = \left(\prod_{p=2}^{k} \kappa_p\right)\xi_1 + \sum_{\ell=1}^{k-1}\left({\prod_{p=0}^{\ell-1}}' \kappa_{k-p}\right) h_{k-\ell}, \tag{50}$$

where ${\prod_{p=0}^{\ell-1}}' = 1$ if $\ell = 0$, we obtain that

$$\begin{aligned}\widehat{\pi}_k - \pi_k &= \left(\widehat{d}_1 - d_1\right)\left\{\left(\prod_{p=2}^{k} c_p(d_1)\right) + \sum_{\ell=1}^{k-1}\left({\prod_{p=0}^{\ell-1}}' c_{k-p}(d_1)\right)\frac{\widehat{\pi}_{k-\ell}}{k-\ell}\right\}\\ &= \left(\widehat{d}_1 - d_1\right)\pi_k\left(\frac{1}{d_1} + \sum_{\ell=1}^{k-1}\frac{\widehat{\pi}_{k-\ell}}{\pi_{k-\ell}}\frac{1}{k-\ell}\right) \qquad (51)\end{aligned}$$

because $\widehat{\pi}_1 - \pi_1 = \widehat{d}_1 - d_1$ and by definition

$$\prod_{p=2}^{k} c_p(d_1) = \frac{\pi_k}{d_1}; \quad {\prod_{p=0}^{\ell-1}}' c_{k-p}(d_1) = \pi_k/\pi_{k-\ell}.$$

Now, because $\left|\frac{\widehat{\pi}_{k-\ell}}{\pi_{k-\ell}}\right| \leq K$ and $\sum_{\ell=1}^{k-1}\left|\frac{1}{k-\ell}\right| \leq K\log k$, we conclude that

$$|\widehat{\pi}_k - \pi_k| = O\left(\left|\widehat{d}_1 - d_1\right|\pi_k \log(k+1)\right).$$

But the last displayed expression suggests that we can give better rates of convergence of $\widehat{\pi}_k - \pi_k$. Indeed, using (51) and an obvious change of subindex, we have that

$$\begin{aligned}\widehat{\pi}_k - \pi_k &= \left(\widehat{d}_1 - d_1\right)\pi_k\left(\frac{1}{d_1} + \sum_{\ell=0}^{k-2}\frac{\widehat{\pi}_{k-\ell}}{\pi_{k-\ell}}\frac{1}{k-\ell}\right)\\ &= \left(\widehat{d}_1 - d_1\right)\pi_k\left(g_k(d_1) + \sum_{\ell=2}^{k}\frac{1}{\ell}\left(\frac{\widehat{\pi}_\ell}{\pi_\ell} - 1\right)\right).\end{aligned}$$

Now, using recursively the right side of the last displayed expression for $\frac{\widehat{\pi}_\ell}{\pi_\ell} - 1$, we have that

$$\frac{\widehat{\pi}_k - \pi_k}{\pi_k} = \sum_{h=1}^{H-1}\left(\widehat{d}_1 - d_1\right)^h g_k^h(d_1) + O\left(\left|\widehat{d}_1 - d_1\right|^H \log^H k\right),$$

which concludes the proof of part $(\mathbf{a})$.

Now we examine part $(\mathbf{c})$. After standard algebra, (49) implies that

$$\widehat{\pi}_k = \widehat{\pi}_{k-1}\frac{k-1}{k} + \widehat{d}_1\frac{\widehat{\pi}_{k-1}}{k}$$

So proceeding as with did in part $(\mathbf{a})$, we have that

$$\widehat{\pi}_k = \widehat{d}_1 \left(\prod_{p=2}^{k} \frac{p-1}{p} \right) + \widehat{d}_1 \sum_{\ell=0}^{k-2} \left({\prod_{p=0}^{\ell-1}}' \frac{k-p-1}{k-p} \right) \frac{\widehat{\pi}_{k-\ell}}{k-\ell}$$

$$= \frac{1}{k}\widehat{d}_1 + \frac{1}{k}\widehat{d}_1 \sum_{\ell=2}^{k} \widehat{\pi}_\ell$$

$$= \frac{\widehat{d}_1}{k} \log k + O\left(\frac{\left|\widehat{d}_1\right|^2}{k} + \left|\widehat{d}_1\right|^H \log^H k \right),$$

for any $H \geq 3$. This completes the proof of the lemma. ∎

REMARK 1. The proof of part $(\mathbf{c})$ can also be done after observing that

$$k\widehat{\pi}_k = (k-1)\,\widehat{\pi}_{k-1} + \widehat{d}_1$$

is a first order nonhomogeneous difference equation with constant coefficients by replacing $k\widehat{\pi}_k$ by, say, κ_k.

For the next lemma "$a \propto b$" denotes $K^{-1}a \leq b \leq Ka$ and $\widetilde{\tau}_k = \tau_k\left(\widehat{d}_2\right)$ and $\tau_k = \tau_k\,(d_2)$.

LEMMA 2. *For any $H \geq 2$ and Condition 1, we have that if $0 < |d_2| < 1/2$*

$$(\mathbf{a}) \quad \widetilde{\tau}_k - \tau_k = \left(\sum_{h=1}^{H-1} \left(\widehat{d}_2 - d_2\right)^h g_k^h\,(d_2) \right) \tau_k + O_p\left(\left|\widehat{d}_2 - d_2\right|^H \frac{\log^H k}{k^{1-d_2}} \right)$$

where $g_k\,(d_2) \propto \left(d_2^{-1} + \log k\right)$ for k large enough. When $d_2 = 0$, we have that

$$(\mathbf{b}) \quad \widetilde{\tau}_k = \sum_{h=1}^{H-1} \frac{\log^h k}{k} \left|\widehat{d}_2\right|^h + O_p\left(\left|\widehat{d}_2 - d_2\right|^H \frac{\log^H k}{k} \right).$$

PROOF. We begin with part $(\mathbf{a})$. First using expression 8.93 in Gradshteyn and Ryzhik's (2000), for any $0 < |d| < 1/2$,

$$\left(1 - 2\cos(\omega)\,t + t^2\right)^{-d} = \sum_{k=0}^{\infty} C_k\,(\cos(\omega)\,; d)\,t^k,$$

where the coefficients $C_k\,(x; d)$ satisfies the second order difference equation

$$C_k\,(x; d) = 2x \left(\frac{k+d-1}{k} \right) C_{k-1}\,(x; d) - \left(\frac{k+2d-2}{k} \right) C_{k-2}\,(x; d)\,. \tag{52}$$

Now, using Elaydi's (2005) expression (2.2.18), (52) has 2 linearly independent solutions $h_{k,1}$ and $h_{k,2}$, so that the homogeneous solution of (52) becomes

$$C_k\,(x; d) = g_1 h_{k,1} + g_2 h_{k,2}, \tag{53}$$

where the coefficients g_1 and g_2 depend on the initial conditions $C_0\,(x; d) = 1$ and $C_1\,(x; d) = 2dx$.

On the other hand, it is known that

$$C_k(x;d) = \frac{\Gamma(k+2d)\,\Gamma(d+1/2)}{\Gamma(2d)\,\Gamma(k+d+1/2)}\mathcal{P}_k^{(d-1/2,d-1/2)}(x),$$

where $\mathcal{P}_k^{(d-1/2,d-1/2)}(x)$ denotes the Jacobi's polynomials, see Gradshteyn and Ryzhik's (2000) $8.962(4)$. In addition, formulae 8.966 in Gradshteyn and Ryzhik's (2000) yields that, with $x = \cos\omega$, for k large $C_k(\cos\omega; d) \propto k^{d-1}\cos(k\omega)$, which implies that $h_{k,1}, h_{k,2} \propto k^{d-1}\cos(k\omega)$.

Now by definition of $\widetilde{\tau}_k$ and τ_k and using (52), standard algebra yields that

$$\widetilde{\tau}_k - \tau_k = \left\{2\cos(\omega)\left(\frac{k+d_2-1}{k}\right)(\widetilde{\tau}_{k-1}-\tau_{k-1}) - \left(\frac{k+2d_2-2}{k}\right)(\widetilde{\tau}_{k-2}-\tau_{k-2})\right\}$$

$$+\left(\widehat{d}_2 - d_2\right)\frac{2}{k}\left\{\cos(\omega)\,\widetilde{\tau}_{k-1} - \widetilde{\tau}_{k-2}\right\} \tag{54}$$

which is a nonhomogeneous second order difference equation with nonconstant coefficients. Consider (54) but with the second term on the right replaced by $\left(\widehat{d}_2 - d_2\right)\frac{2}{k}\left\{\cos(\omega)\,\tau_{k-1} - \tau_{k-2}\right\}$, that is

$$a_k = \left\{2\cos(\omega)\left(\frac{k+d_2-1}{k}\right)a_{k-1} - \left(\frac{k+2d_2-2}{k}\right)a_{k-2}\right\}$$

$$+\left(\widehat{d}_2 - d_2\right)\frac{2}{k}\left\{\cos(\omega)\,\tau_{k-1} - \tau_{k-2}\right\}, \tag{55}$$

where we have abbreviated $\widetilde{\tau}_k - \tau_k$ by a_k. It is known that the solution to (60) is

$$a_k =: a_{k,h} + a_{k,p},$$

where $a_{k,h}$ and $a_{k,p}$ are respectively its homogeneous and particular solutions. From (52) and because $h_{k,1}, h_{k,2} \propto k^{d_2-1}\cos(k\omega)$, we obtain that

$$a_{k,h} \propto \left(\frac{\widehat{d}_2 - d_2}{d_2}\right)k^{d_2-1}\cos(k\omega) \propto: \left(\frac{\widehat{d}_2 - d_2}{d_2}\right)\tau_k \tag{56}$$

since the initial conditions for the difference equation (60) are given by

$$\widetilde{\tau}_1 - \tau_1 =: a_1 = 2\left(\widehat{d}_2 - d_2\right)\cos(\omega) =: \left(\frac{\widehat{d}_2 - d_2}{d_2}\right)\tau_1$$

$$\widetilde{\tau}_2 - \tau_2 =: a_2 = \left(\frac{\widehat{d}_2 - d_2}{d_2}\right)(\tau_2 + \tau_1).$$

We now examine the behaviour of $a_{k,p}$. Using Elaydi's (2005) Section $2.4.1$, we have that

$$a_{k,p} = \zeta_{k,1}h_{k,1} + \zeta_{k,2}h_{k,2}, \tag{57}$$

where

$$\zeta_{k,1} = -2\left(\widehat{d}_2 - d_2\right)\sum_{j=1}^{k-1}\frac{h_{j,2}\left(\cos(\omega)\,\tau_{j-1} - \tau_{j-2}\right)}{jW_{j+1}} \tag{58}$$

$$\zeta_{k,2} = 2\left(\widehat{d}_2 - d_2\right)\sum_{j=1}^{k-1}\frac{h_{j,1}\left(\cos(\omega)\,\tau_{j-1} - \tau_{j-2}\right)}{jW_{j+1}}$$

and where W_j, $j = 0, 1, 2, ...$, denotes the Casoratian of $h_{k,1}$ and $h_{k,2}$, which satisfies the recursive expression

$$W_{j+1} = \frac{j + 2d_2 - 2}{j} W_j; \;\; W_1 = 1$$
$$\propto j^{2d_2 - 2}.$$

So, because $h_{k,1}, h_{k,2} \propto \cos(k\omega) k^{d_2 - 1}$, we obtain that

$$\sum_{j=1}^{k-1} \frac{h_{j,\ell}(\cos(\omega)\tau_{j-1} - \tau_{j-2})}{jW_{j+1}} \propto \sum_{j=1}^{k-1} j^{-1} \propto \log k, \quad \ell = 1, 2, \tag{59}$$

which yields that $a_{k,p}$ given in (57) satisfies

$$a_{k,p} \propto \left(\widehat{d}_2 - d_2\right) \tau_k \log k.$$

Hence, the latter displayed expression and (56) imply that the solution to (60) satisfies

$$a_k \propto: \left(\widehat{d}_2 - d_2\right) \{1 + \log k\} \tau_k.$$

The lemma now proceeds by iteration as we did in the proof of Lemma 1 after we observe that

$$a_k =: \widetilde{\tau}_k - \tau_k = a_{k,h} + a_{k,p} + \left(\widehat{\zeta}_{k,1} - \zeta_{k,1}\right) h_{k,1} + \left(\widehat{\zeta}_{k,2} - \zeta_{k,2}\right) h_{k,2},$$

where

$$\widehat{\zeta}_{k,1} - \zeta_{k,1} = -2\left(\widehat{d}_2 - d_2\right) \sum_{j=1}^{k-1} \frac{h_{j,2}(\cos(\omega) a_{j-1} - a_{j-2})}{jW_{j+1}}$$

$$\widehat{\zeta}_{k,2} - \zeta_{k,2} = -2\left(\widehat{d}_2 - d_2\right) \sum_{j=1}^{k-1} \frac{h_{j,1}(\cos(\omega) a_{j-1} - a_{j-2})}{jW_{j+1}}.$$

We now examine part $(\mathbf{b})$. The proof is similar to part $(\mathbf{a})$. First when $d_2 = 0$, we have that (54) becomes

$$\widetilde{\tau}_k = \left\{2\cos(\omega)\left(\frac{k-1}{k}\right)\widetilde{\tau}_{k-1} - \left(\frac{k-2}{k}\right)\widetilde{\tau}_{k-2}\right\} + \widehat{d}_2 \frac{2}{k}\{\cos(\omega)\widetilde{\tau}_{k-1} - \widetilde{\tau}_{k-2}\}$$

or equivalently,

$$g_k = 2\cos(\omega) g_{k-1} - g_{k-2} + \widehat{d}_2 \frac{2}{k}\{\cos(\omega)\widetilde{\tau}_{k-1} - \widetilde{\tau}_{k-2}\}. \tag{60}$$

The proof now proceeds as that of part $(\mathbf{a})$, if not easier, after observing that $g_k = 2\cos(\omega) g_{k-1} - g_{k-2}$ is a second order difference equation with constant coefficients, whose characteristic roots are complex but with unit moduli. ∎

LEMMA 3. *Under Condition* 1*, we have that for any integer* $H \geq 3$

$$(a)\;\; \widehat{\xi}_k - \xi_k = \xi_k \sum_{h=1}^{H-1} \left\|\widehat{d} - d\right\|^h g_k^h(d) + O_p\left(\left\|\widehat{d} - d\right\|^H \frac{\log^H k}{k^{1+(d_1 \wedge d_2)}}\right); \; d_1, d_2 > 0$$

$$(b)\;\; \check{\xi}_k - \bar{\xi}_k = \bar{\xi}_k \sum_{h=1}^{H-1} \left\|\widehat{d} - d\right\|^h g_k^h(d) + O_p\left(\left\|\widehat{d} - d\right\|^H \frac{\log^H k}{k^{1+(d_1 + d_2)}}\right); \; d_1, d_2 < 0,$$

where $d=(d_1,d_2)$, $g_k(\|d\|)\propto\left(\|d\|^{-1}+\log k\right)$, $\|a\|$ *denotes the norm of the vector* a *and* $O_p(\circ)$ *is uniformly in* k.

PROOF. We begin with part (a). By definition, we have that

$$\begin{aligned}\widehat{\xi}_k-\xi_k&=\sum_{\ell=0}^{k}\left(\widehat{\pi}_\ell\widehat{\tau}_{k-\ell}-\pi_\ell\tau_{k-\ell}\right)\\&=\sum_{\ell=0}^{k}\tau_{k-\ell}\left(\widehat{\pi}_\ell-\pi_\ell\right)+\sum_{\ell=0}^{k}\pi_\ell\left(\widehat{\tau}_{k-\ell}-\tau_{k-\ell}\right)\\&\quad+\sum_{\ell=0}^{k}\left(\widehat{\pi}_\ell-\pi_\ell\right)\left(\widehat{\tau}_{k-\ell}-\tau_{k-\ell}\right).\end{aligned}$$

Now, using Lemmas 1 and 2, standard algebra yields that the right side of the last displayed expression behaves as

$$\sum_{h=1}^{H-1}\left\|\widehat{d}-d\right\|^h\sum_{\ell=0}^{k}\tau_{k-\ell}\pi_\ell g_\ell^h(d)+O_p\left(\left\|\widehat{d}-d\right\|^H\log^H k\right)\sum_{\ell=0}^{k}\left|\tau_{k-\ell}\pi_\ell\right|. \tag{61}$$

But because $\sum_{\ell=0}^{k}\tau_{k-\ell}\pi_\ell g_\ell^h(d)\propto\log^h k\sum_{\ell=0}^{k}\tau_{k-\ell}\pi_\ell=\xi_k\log^h k$ and

$$\begin{aligned}\sum_{\ell=0}^{k}\left|\tau_{k-\ell}\pi_\ell\right|&\propto\sum_{\ell=0}^{k}\ell^{-d_1-1}(k-\ell)^{-d_2-1}\\&\propto\ell^{-(d_1\wedge d_2)-1},\end{aligned}$$

we easily conclude part (a) by standard arguments.

Next, part (b) proceeds as with the proof of part (a), except that now

$$\sum_{\ell=0}^{k}\left|\tau_{k-\ell}\pi_\ell\right|\propto\sum_{\ell=0}^{k}\ell^{-d_1-1}(k-\ell)^{-d_2-1}\propto\ell^{-(d_1+d_2)-1}.$$

This concludes the proof of the lemma. ∎

For the next lemma we shall denote by $\{\delta_\ell\}_{\ell=1}^M$ and $\left\{\widehat{\delta}_\ell\right\}_{\ell=1}^M$ either $\{b_\ell\}_{\ell=1}^M$ and $\left\{\widehat{b}_\ell\right\}_{\ell=1}^M$ or $\{a_\ell\}_{\ell=1}^M$ and $\{\widehat{a}_\ell\}_{\ell=1}^M$. The next lemma is an immediate consequence of Hidalgo and Yajima's (2002) Theorem 3 which we give for easy reference.

LEMMA 4. *Under Conditions* 1 *to* 3, *we have that, uniformly in* $\ell=1,...,M$,

$$\widehat{\delta}_\ell-\delta_\ell=\sum_{p=1}^{M}\kappa_{n,p}\upsilon_{n,\ell,p}+\frac{1}{M}\sum_{p=1}^{M}\left|\kappa_{n,p}\right|+O_p\left(m^{-3/2}\right),$$

where $|\upsilon_{n,\ell,p}|<K$, $\{\kappa_{n,p}\}_{p=1}^M$ *is a triangular array sequence of random variables such that* $E\left|\sum_{p=1}^{M}\kappa_{n,p}\upsilon_{n,\ell,p}\right|^r=O\left(n^{-r/2}\right)$ *for any* $r\geq 2$ *such that* $E|\varepsilon_t|^{2r}<\infty$, $E\sum_{p=1}^{M}|\kappa_{n,p}|=O\left(m^{-1/2}\right)$ *and* $E\left(\sum_{p=1}^{M}\kappa_{n,p}\upsilon_{n,\ell_1,p}\sum_{p=1}^{M}\kappa_{n,p}\upsilon_{n,\ell_2,p}\right)=O\left(n^{-1}\delta_{|\ell_2-\ell_1|}\right)$.

REMARK 1. *One consequence of Lemma 4 is that*

$$\sum_{\ell=1}^{M}\left(\widehat{b}_{\ell}-b_{\ell}\right)=O_p\left(m^{-1/2}\right). \tag{62}$$

LEMMA 5. *For any* $0\le r\le n$,

$$E^*\left(u_t^*u_{t+r}^*\right)=\widehat{\sigma}_{\varepsilon}^2\sum_{k=0}^{n+M-r}\widehat{\vartheta}_k\widehat{\vartheta}_{k+r}. \tag{63}$$

PROOF. The proof follows immediately from (9). Indeed, by construction

$$\begin{aligned}E^*\left(u_t^*u_{t+r}^*\right)&=\frac{\widehat{\sigma}_{\varepsilon}^2}{3n}\sum_{j=1}^{3n}e^{ir\widetilde{\lambda}_j}\left|\sum_{\ell=0}^{n}\widehat{\xi}_{\ell}e^{-i\ell\widetilde{\lambda}_j}\sum_{\ell=0}^{M}\widehat{b}_{\ell}e^{-i\ell\widetilde{\lambda}_j}\right|^2\\&=\frac{\widehat{\sigma}_{\varepsilon}^2}{3n}\sum_{j=1}^{3n}e^{ir\widetilde{\lambda}_j}\left|\sum_{k=0}^{n+M}\widehat{\vartheta}_ke^{-ik\widetilde{\lambda}_j}\right|^2.\end{aligned}$$

From here the proof of (63) is standard. ∎

Before we present the next lemma, we first observe that by definitions of $\widehat{\phi}_k$ and $\mathring{\phi}_k$ in (11) and (23) respectively, we have that

$$\widehat{\phi}_k-\mathring{\phi}_k=\sum_{\ell=1\vee k-(n-M)}^{k\wedge M}\widehat{\xi}_{k-\ell}\left(\widehat{a}_{\ell}-a_{\ell}\right)+\sum_{\ell=1\vee k-(n-M)}^{k\wedge M}\left(\widehat{\xi}_{k-\ell}-\xi_{k-\ell}\right)a_{\ell},\qquad k\le n. \tag{64}$$

LEMMA 6. *Under Conditions* 1 *and* 3*, we have that*

$$\begin{aligned}&\text{(i)}\quad \widehat{\phi}_k-\mathring{\phi}_k=\widetilde{\varsigma}_{n,M}+\varsigma_{n,M}\xi_k, && k\le M\\&\text{(ii)}\quad \widehat{\phi}_k-\mathring{\phi}_k=\varsigma_{n,M}\xi_{k-M}, && M<k<n,\end{aligned}$$

where $\varsigma_{n,M}=O_p\left(m^{-1/2}\right)$ *and* $\varsigma_{n,M}=O_p\left(n^{-1/2}\right)$ *independent of* k.

PROOF. First we have that Lemma 3 and (64) yields that $\widehat{\phi}_k-\mathring{\phi}_k=:\ddot{\phi}_k$ is

$$\begin{aligned}\ddot{\phi}_k=&\sum_{\ell=1\vee k-(n-M)}^{k\wedge M}\left(\widehat{a}_{\ell}-a_{\ell}\right)\left\{\xi_{k-\ell}\left\{\sum_{h=0}^{3}\left\|\widehat{d}-d\right\|^h g_{k-\ell}^h(d)+O_p\left(\frac{\log^4 n}{m^2}\right)\right\}\right\}\\&+\sum_{\ell=1\vee k-(n-M)}^{k\wedge M}a_{\ell}\xi_{k-\ell}\left\{\sum_{h=1}^{3}\left\|\widehat{d}-d\right\|^h g_{k-\ell}^h(d)+O_p\left(\frac{\log^4 n}{m^2}\right)\right\}.\end{aligned}$$

We begin with part (i), so that $\sum_{\ell=1\vee k-(n-M)}^{k\wedge M}=:\sum_{\ell=1}^{k}$. When $k\le M$, the contribution of the second term on the right of the last displayed expression is $O_p\left(m^{-1/2}\right)\xi_k$ because $\left\|\widehat{d}-d\right\|^h=O_p\left(m^{-h/2}\right)$, by Robinson (1995$b$) and Arteche (2000), and that

$\sum_{\ell=1}^{k} a_\ell \xi_{k-\ell} = O(\xi_k)$ since $a_\ell = O(\ell^{-3})$. Next the first term on the right of the last displayed expression, whose contribution is dominated by that of $\sum_{\ell=1}^{k} (\widehat{a}_\ell - a_\ell)\xi_{k-\ell}$ because $\left\|\widehat{d} - d\right\| = O_p(m^{-1/2})$ independent of k. Now

$$\sum_{\ell=1}^{k} (\widehat{a}_\ell - a_\ell)\xi_{k-\ell} = \sum_{\ell=1}^{k} \xi_{k-\ell} \left\{ \sum_{p=1}^{M} \kappa_{n,p} \upsilon_{n,\ell,p} + \frac{1}{M} \sum_{p=1}^{M} |\kappa_{n,p}| \right\} + O_p\left(m^{-3/2}\right) \sum_{\ell=1}^{k} \xi_{k-\ell}$$
$$= O_p\left(n^{-1/2}\right) + O_p\left(m^{-3/2}\right)$$

because

$$E\left(\sum_{\ell=1}^{k}\sum_{p=1}^{M} \xi_{k-\ell}\kappa_{n,p}\upsilon_{n,\ell,p}\right)^2 = Kn^{-1} \sum_{\ell_1,\ell_2=1}^{k} a_{|\ell_1-\ell_2|}\xi_{k-\ell_1}\xi_{k-\ell_2} = Kn^{-1},$$

since $\sum_{\ell=1}^{k} \xi_{k-\ell} < K$ and $M^{-1}\sum_{\ell=1}^{k} \xi_{k-\ell} \sum_{p=1}^{M} |\kappa_{n,p}\upsilon_{n,\ell,p}| = O_p(M^{-1}m^{-1/2})$. Now we conclude by Condition 3.

Next part (ii). When $M < k \le n - M$, it proceeds as with the proof of part (i), although now we use that

$$\frac{K}{n} \sum_{\ell_1,\ell_2=1}^{M} a_{|\ell_1-\ell_2|}\xi_{k-\ell_1}\xi_{k-\ell_2} = O\left(\frac{M}{n}\xi_{k-M}^2\right)$$

$$\frac{1}{M}\sum_{\ell=1}^{M} \xi_{k-\ell} \sum_{p=1}^{M} |\kappa_{n,p}| = O_p\left(m^{-1/2}\right) \frac{1}{M} \sum_{\ell=1}^{M} \xi_{k-\ell} = O_p\left(m^{-1/2}\right)\xi_{k-M}.$$

Finally, when $n - M \le k \le n$ the proof follows similarly as before with standard modifications after we notice that we have that $\sum_{\ell=1\vee k-(n-M)}^{k\wedge M} =: \sum_{\ell=1\vee k-(n-M)}^{M}$. ∎

REMARK 2. *A consequence of Lemma 6 together with (62) is that*

$$\sup_{k=1,\ldots,n} \left|\widehat{\phi}_k - \mathring{\phi}_k\right| = O_p\left(m^{-1/2}\right). \tag{65}$$

The next lemma examines the behaviour of

$$\begin{aligned} \mathring{\phi}_k - \phi_k &= \sum_{\ell=M+1}^{k} \xi_{k-\ell} a_\ell, & M < k \le n - M \\ &= -\sum_{\ell=0}^{k-(n-M)} \xi_{k-\ell} a_\ell - \sum_{\ell=M+1}^{k} \xi_{k-\ell} a_\ell, & n - M < k \le n. \end{aligned} \tag{66}$$

LEMMA 7. *Under Condition 1, we have that*

(i) $\mathring{\phi}_k - \phi_k = O\left(M^{-3/2}\right)\xi_k, \quad M < k \le n - M$

(ii) $\mathring{\phi}_k - \phi_k = O(1)\xi_k, \quad n - M < k < n.$

PROOF. We begin with part (i). When $M<k\leq 2M$, we have that

$$\sum_{\ell=M+1}^{k}\xi_{k-\ell}a_{\ell}=O\left(M^{-3}\right)=O\left(M^{-3/2}\right)\xi_{k}$$

by Condition 1 and that $M^{-3/2}\leq M^{-1-d_1}=O\left(\xi_k\right)$, whereas when $2M<k\leq n-M$, we have that

$$\begin{aligned}\sum_{\ell=M+1}^{k}\xi_{k-\ell}a_{\ell}&=\left\{\sum_{\ell=M+1}^{k/2}+\sum_{\ell=k/2+1}^{k}\right\}\xi_{k-\ell}a_{\ell}\\&=\xi_{k/2}O\left(M^{-2}\right)+O\left(k^{-3}\right)=o\left(M^{-3/2}\right)\xi_{k}.\end{aligned}$$

This concludes the proof of part (i). Next, part (ii). The right side of the equality is

$$\begin{aligned}&-\sum_{\ell=0}^{k-(n-M)}\pi_{k-\ell}a_{\ell}-\sum_{\ell=M+1}^{k/2}\xi_{\ell-k}a_{\ell}-\sum_{\ell=k/2+1}^{k}\xi_{k-\ell}a_{\ell}\\&=\xi_{n-M}O\left(\sum_{\ell=0}^{k-(n-M)}|a_{\ell}|\right)+\xi_{k/2}O\left(\sum_{\ell=M+1}^{k/2}|a_{k}|\right)+O\left(k^{-3}\right).\end{aligned}$$

From here the conclusion is standard because $\left|\xi_{n-M}/\xi_{k}\right|+\left|\xi_{k/2}/\xi_{k}\right|<K$. ∎

Before we present the next lemma, we first observe that by definition of $\widehat{\vartheta}_k$ and $\mathring{\vartheta}_k$ in (13) and (14) respectively, we have that

$$\widehat{\vartheta}_{k}-\mathring{\vartheta}_{k}=\sum_{\ell=1}^{k\wedge M}\check{\xi}_{k-\ell}\left(\widehat{b}_{\ell}-b_{\ell}\right)+\sum_{\ell=1}^{k\wedge M}\left(\check{\xi}_{k-\ell}-\overline{\xi}_{k-\ell}\right)b_{\ell},\qquad 1\leq k\leq n+M \tag{67}$$

LEMMA 8. *Under Conditions* 1 *and* 3*, we have that*

(i) $$\widehat{\vartheta}_{k}-\mathring{\vartheta}_{k}=\sum_{\ell=0}^{k\wedge M}\overline{\xi}_{k-\ell}\left(\widehat{b}_{\ell}-b_{\ell}\right)+\left(\varsigma_{n}+o_{p}\left(n^{-1}\right)\right)\overline{\xi}_{k}\qquad 0\leq k\leq M$$

(ii) $$\widehat{\vartheta}_{k}-\mathring{\vartheta}_{k}=\sum_{\ell=0}^{k\wedge M}\overline{\xi}_{k-\ell}\left(\widehat{b}_{\ell}-b_{\ell}\right)+\left(\varsigma_{n}+o_{p}\left(n^{-1}\right)\right)\overline{\xi}_{k-M}\quad M<k\leq n+M,$$

where $\varsigma_n=O_p\left(m^{-1/2}\right)$ *independent of* k.

PROOF. First we have that Lemma 1 yields that

$$\begin{aligned}\widehat{\vartheta}_{k}-\mathring{\vartheta}_{k}=&\sum_{\ell=1}^{k\wedge M}\left(\widehat{b}_{\ell}-b_{\ell}\right)\left\{\overline{\xi}_{k-\ell}\left\{\sum_{h=0}^{3}\left\|\widehat{d}-d\right\|^{h}g_{k-\ell}^{h}\left(d\right)+o_{p}\left(n^{-1}\right)\right\}\right\}\\&+\sum_{\ell=1}^{k\wedge M}b_{\ell}\overline{\xi}_{k-\ell}\left\{\sum_{h=1}^{3}\left\|\widehat{d}-d\right\|^{h}g_{k-\ell}^{h}\left(d\right)+o_{p}\left(n^{-1}\right)\right\}.\end{aligned}\tag{68}$$

We begin with part (i). The contribution due to the second term on the right of (68) is

$$\left(\varsigma_n + o_p\left(n^{-1}\right)\right)\left|\sum_{\ell=1}^{k} b_\ell \overline{\xi}_{k-\ell}\right| = \left(\varsigma_n + o_p\left(n^{-1}\right)\right)\overline{\xi}_k$$

because $b_\ell = O\left(\ell^{-3}\right)$ and $\overline{\xi}_k = O\left(k^{(d_1\wedge d_2)-1}\right)$. From here we conclude because the first term is dominated by $\sum_{\ell=0}^{k\wedge M}\overline{\xi}_{k-\ell}\left(\widehat{b}_\ell - b_\ell\right)$.

We next examined part (ii). But it follows by the same arguments as those for part (i), except that now $\left|\sum_{\ell=1}^{k} b_\ell\overline{\xi}_{k-\ell}\right| = K\overline{\xi}_{k-M}$. This concludes the proof of the lemma. ∎

REMARK 3. *It is obvious to see that one consequence of the previous lemma together with* (62) *is that*

$$\sup_{k=1,\ldots,n}\left|\widehat{\vartheta}_k - \mathring{\vartheta}_k\right| = O_p\left(m^{-1/2}\right). \tag{69}$$

Before we present the next lemma, we first observe that by definition of ϑ_k in Condition 1 and $\mathring{\vartheta}_k$ in (14) respectively, we have that

$$\begin{aligned}\mathring{\vartheta}_k - \vartheta_k &= \sum_{\ell=M+1}^{k}\overline{\xi}_{k-\ell}b_\ell, && M < k \le n \\ &= -\sum_{\ell=0}^{k-n}\overline{\xi}_{k-\ell}b_\ell - \sum_{\ell=M+1}^{k}\overline{\xi}_{k-\ell}b_\ell, && n < k \le n+M.\end{aligned} \tag{70}$$

LEMMA 9. *Under Condition* 1, *we have that*

(i) $\mathring{\vartheta}_k - \vartheta_k = O\left(M^{-2}\right)\overline{\xi}_k, \qquad M < k \le n$

(ii) $\mathring{\vartheta}_k - \vartheta_k = O\left(1\right)\overline{\xi}_k, \qquad n < k < n+M.$

PROOF. We begin with part (i). When $M < k \le 2M$, we have that

$$\sum_{\ell=M+1}^{k}\overline{\xi}_{k-\ell}b_\ell = O\left(M^{(d_1\wedge d_2)-3}\right) = O\left(M^{-2}\right)M^{(d_1\wedge d_2)-1} = O\left(M^{-2}\right)\xi_k$$

by Condition 1, whereas when $2M < k \le n-M$, we have that

$$\begin{aligned}\sum_{\ell=M+1}^{k}\overline{\xi}_{k-\ell}b_\ell &= \left\{\sum_{\ell=M+1}^{k/2} + \sum_{\ell=k/2+1}^{k}\right\}\overline{\pi}_{k-\ell}b_\ell \\ &= \overline{\xi}_{k/2}O\left(M^{-2}\right) + O\left(k^{(d_1\wedge d_2)-3}\right) = o\left(M^{-2}\right)\overline{\xi}_k.\end{aligned}$$

This concludes the proof of part (i). Next, part (ii). The right side of the equality is

$$\begin{aligned}&-\sum_{\ell=0}^{k-(n-M)}\overline{\xi}_{k-\ell}b_\ell - \sum_{\ell=M+1}^{k/2}\overline{\xi}_{k-\ell}b_\ell - \sum_{\ell=k/2+1}^{k}\overline{\xi}_{k-\ell}b_\ell \\ &= \overline{\xi}_{n-M}O\left(\sum_{\ell=0}^{k-(n-M)}|b_\ell|\right) + \overline{\xi}_{k/2}O\left(\sum_{\ell=M+1}^{k/2}|b_k|\right) + O\left(k^{-3}\right).\end{aligned}$$

From here the conclusion proceeds as that of Lemma 7. ∎

LEMMA 10. *Under Conditions* 1 *to* 3*, we have that for* $0 \le p \le n$*,*

$$\text{(i)}\quad \sum_{k=0}^{M-1}\left(\widehat{\vartheta}_k\widehat{\vartheta}_{k+p}-\vartheta_k\vartheta_{k+p}\right)=\left(\widetilde{\varsigma}_n+\varsigma_n\right)\vartheta_p+o_p\left(n^{-1}\right)h_p$$

$$\text{(ii)}\quad \sum_{k=M}^{n+M-p}\left(\widehat{\vartheta}_k\widehat{\vartheta}_{k+p}-\vartheta_k\vartheta_{k+p}\right)=\left\{\sum_{\ell=1}^{M}\left(\widehat{b}_\ell-b_\ell\right)+\varsigma_n\right\}h_p+o_p\left(n^{-1}\right)h_p,$$

where $\{h_p\}_{p\ge1}$ *is a sequence such that* $h_p=O\left(p^{2(d_1\wedge d_2)-1}\right)$ *and where* $\varsigma_n=O_p\left(m^{-1/2}\right)$ *and* $\widetilde{\varsigma}_n=O_p\left(m^{-1/2}\log n\right)$ *independent of* p*.*

PROOF. We begin with part (ii), which standard algebra yields that the left side is

$$\sum_{k=M}^{n+M-p}\left(\widehat{\vartheta}_k-\mathring{\vartheta}_k\right)\left(\widehat{\vartheta}_{k+p}-\mathring{\vartheta}_{k+p}\right)+\sum_{k=M}^{n+M-p}\mathring{\vartheta}_{k+p}\left(\widehat{\vartheta}_k-\mathring{\vartheta}_k\right)+\sum_{k=M}^{n+M-p}\mathring{\vartheta}_k\left(\widehat{\vartheta}_{k+p}-\mathring{\vartheta}_{k+p}\right). \tag{71}$$

We shall examine the third term of (71), the first two terms are similarly handled. By Lemma 8, this term is

$$\sum_{\ell=1}^{M}\left(\widehat{b}_\ell-b_\ell\right)\sum_{k=M}^{n+M-p}\mathring{\vartheta}_k\overline{\xi}_{k+p-\ell}+\varsigma_n\sum_{k=M}^{n+M-p}\mathring{\vartheta}_k\overline{\xi}_{k-M+p}+o_p\left(n^{-1}\right)\left|\sum_{k=M}^{n+M-p}\mathring{\vartheta}_k\overline{\xi}_{k-M+p}\right|$$
$$=\left\{\sum_{\ell=1}^{M}\left(\widehat{b}_\ell-b_\ell\right)+\varsigma_n\right\}h_p+o_p\left(n^{-1}\right)h_p.$$

Next we examine part (i). The left side is

$$\sum_{k=0}^{M-1}\left(\widehat{\vartheta}_k-\mathring{\vartheta}_k\right)\left(\widehat{\vartheta}_{k+p}-\mathring{\vartheta}_{k+p}\right)+\sum_{k=0}^{M-1}\mathring{\vartheta}_k\left(\widehat{\vartheta}_{k+p}-\mathring{\vartheta}_{k+p}\right)+\sum_{k=0}^{M-1}\mathring{\vartheta}_{k+p}\left(\widehat{\vartheta}_k-\mathring{\vartheta}_k\right).$$

We only examine the third term, the first two terms are similarly handled. Again, using Lemma 8, this term is

$$\sum_{k=0}^{M-1}\mathring{\vartheta}_{k+p}\sum_{\ell=1}^{k}\overline{\xi}_{k-\ell}\left(\widehat{b}_\ell-b_\ell\right)+\varsigma_n\sum_{k=0}^{M-1}\mathring{\vartheta}_{k+p}\overline{\xi}_k+o_p\left(n^{-1}\right)\left|\sum_{k=0}^{M-1}\mathring{\vartheta}_{k+p}\overline{\xi}_k\right|. \tag{72}$$

Now, by an obvious change of subindexes, the first term of the last displayed expression is

$$\sum_{\ell=1}^{M}\left(\widehat{b}_\ell-b_\ell\right)\overline{\xi}_\ell\sum_{k=\ell}^{M-1}\mathring{\vartheta}_{\ell+p}=Kh_pM\sum_{\ell=1}^{M}\left(\widehat{b}_\ell-b_\ell\right)\ell^{-1}$$
$$=\widetilde{\varsigma}_nh_p,$$

which by Lemma 4, $\widetilde{\varsigma}_n=O_p\left(m^{-1/2}\log M\right)$. This concludes the proof of the lemma. ∎

LEMMA 11. *Under Conditions* 2 *to* 4*, we have that for any* $0 \le p \le n$*,*

$$\sum_{k=0}^{n+M-p}\left(\vartheta_k\vartheta_{k+p}-\mathring{\vartheta}_k\mathring{\vartheta}_{k+p}\right)=O\left(M^{-2}\right)p^{2(d_1\wedge d_2)-1}.$$

PROOF. The left side of the last displayed expression is

$$\sum_{k=0}^{M}\left(\vartheta_k\vartheta_{k+p}-\mathring{\vartheta}_k\mathring{\vartheta}_{k+p}\right)+\sum_{k=M+1}^{n+M-p}\left(\vartheta_k\vartheta_{k+p}-\mathring{\vartheta}_k\mathring{\vartheta}_{k+p}\right).$$

The first term of the last displayed expression is $\sum_{k=M-p}^{M}\left(\vartheta_{k+p}-\mathring{\vartheta}_{k+p}\right)\vartheta_k=O\left(M^{-2}\right)p^{2(d_1\wedge d_2)-1}$ because $\vartheta_k=\mathring{\vartheta}_k$ if $k\le M$ and Lemma 9.

So, it remains to examine the behaviour of the second term of the last displayed expression, which standard algebra yields

$$\sum_{k=M+1}^{n+M-p}\left(\vartheta_{k+p}-\mathring{\vartheta}_{k+p}\right)\vartheta_k+\sum_{k=M+1}^{n+M-p}\left(\vartheta_k-\mathring{\vartheta}_k\right)\vartheta_{k+p}$$
$$+\sum_{k=M+1}^{n+M-p}\left(\vartheta_k-\mathring{\vartheta}_k\right)\left(\vartheta_{k+p}-\mathring{\vartheta}_{k+p}\right).$$

Again Lemma 9 will imply that, by standard algebra, the last displayed expression is also $O\left(M^{-2}\right)p^{2(d_1\wedge d_2)-1}$. ∎

Let's introduce some notation.

$\wp_{t-k}\left[\mathring{\mathbf{1}}_t\right]=E\left[\mathring{\mathbf{1}}_t\mid\mathcal{F}_{t-k}\right]-E\left[\mathring{\mathbf{1}}_t\mid\mathcal{F}_{t-k-1}\right]$, where we abbreviate $\mathring{\mathbf{1}}_t(x)$ by $\mathring{\mathbf{1}}_t$. $E(\circ)$ denotes the expectation of the random variable that precedes, for instance, in $\eta\varphi-E(\circ)$, $E(\circ)$ stands for $E(\eta\varphi)$. In addition for a generic function $g(x)$, we shall denote $g(y;x)=g(y)-g(x)$. And finally $\mathcal{F}_t$ and $\mathcal{J}_t$ denote respectively the σ-algebras of events generated by $\{\varrho_s, s\le t\}$ and $\{\varepsilon_s, s\le t\}$.

LEMMA 12. *Under Conditions* 2 *and* 4 *and* $d_1+2d_x<1$, $\mathcal{G}_n(x)$ *is tight.*

PROOF. First, denoting $\varsigma_t(x)=\left(\mathring{\mathbf{1}}_t(x)-E\left(\mathring{\mathbf{1}}_t(x)\mid\mathcal{F}_{t-1}\right)\right)u_t$,

$$\mathcal{G}_n(x)=\frac{1}{n^{1/2}}\sum_{t=1}^{n}\varsigma_t(x)+\frac{1}{n^{1/2}}\sum_{t=1}^{n}E\left(\mathring{\mathbf{1}}_t(x)\mid\mathcal{F}_{t-1}\right)u_t$$
$$=:A_n(x)+B_n(x).$$

Because $\varsigma_t(x)$ is a martingale difference with respect to $\mathcal{F}_{t-1}\cup\mathcal{J}_{t-1}$, we have that Lemma 14 of Wu (2003) implies that for each $\epsilon,\eta>0$, there exists a $\delta>0$, such that

$$\Pr\left\{\sup_{|x-y|<\delta}|A_n(x;y)|>\epsilon\right\}<\eta, \tag{73}$$

which implies that $A_n(x)$ is tight by Billingsley's (1968) Theorem 8.3.

We next examine the tightness of $B_n(x)$. To that end, write

$$B_n(x)=\frac{1}{n^{1/2}}\sum_{s=-N}^{n}\boldsymbol{\upsilon}_s(x)\varepsilon_s+\frac{1}{n^{1/2}}\sum_{s=-\infty}^{-N+1}\boldsymbol{\upsilon}_s(x)\varepsilon_s$$
$$=B_{n1}(x)+B_{n2}(x),$$

where $\boldsymbol{\upsilon}_s(x)=\sum_{t=1}^{n}E\left(\mathring{\mathbf{1}}_t(x)\mid\mathcal{F}_{t-1}\right)\vartheta_{t-s}$ with the convention that $\vartheta_{t-s}=0$ if $t<s$ and N is to be chosen later. We now proceed similarly as in the proof of Wu's (2003) Theorem

3. That is, using Wu's (2003) Lemma 4,

$$E\sup_x\left|\frac{\partial B_{n2}(x)}{\partial x}\right|^2\leq 2\int_{\mathbb{R}}E\left|\frac{\partial B_{n2}(x)}{\partial x}\right|^2dx+2\int_{\mathbb{R}}E\left|\frac{\partial^2 B_{n2}(x)}{\partial^2 x}\right|^2dx.$$

Because $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ are iid$\left(0,\sigma_\varepsilon^2\right)$, we have that

$$\int_{\mathbb{R}}E\left|\frac{\partial B_{n2}(x)}{\partial x}\right|^2dx=\frac{\sigma_\varepsilon^2}{n}\sum_{s=-\infty}^{-N+1}\int_{\mathbb{R}}\left|\sum_{t=1}^{n}\frac{\partial E\left(\mathring{\mathbf{1}}_t(x)\mid\mathcal{F}_{t-1}\right)}{\partial x}\vartheta_{t-s}\right|^2dx.$$

Now using Lemma 11 and expression (42) in Lemma 12 of Wu (2003) but with $L(\cdot;s)$, or $J(\cdot;s)$, there being replaced by $E\left(\mathring{\mathbf{1}}_t(y)\mid\mathcal{F}_{t-1}\right)\vartheta_{t-s}$ and noticing that his quantity $\lambda_n=:\theta_n$ is $O\left(\varphi_n^{3/2}\right)$ in our case, it implies that the right side of the last displayed expression is bounded by

$$\frac{K}{n}\sum_{s=-\infty}^{-N+1}\sum_{j=-\infty}^{n}\left(\sum_{t=\max(1,j)}^{n}\varphi_{t-j+1}^{3/2}\vartheta_{t-j-s+1}\right)^2=o(1),\tag{74}$$

when $N\nearrow\infty$ as we now show. Indeed when $2d_x\geq d_1$, Condition 1 implies that the left side of (74) is

$$K\sum_{s=N}^{\infty}\frac{1}{n}\sum_{j=-\infty}^{n}\left(\sum_{t=\max(1,j)}^{n}\frac{1}{(t-j+s)^{1-d_1}}\frac{1}{(t-j+1)^{3/2(1-d_x)}}\right)^2$$
$$\leq K\sum_{s=N}^{\infty}\frac{1}{s^{1+\zeta}}\frac{1}{n}\sum_{j=-\infty}^{n}\left(\sum_{t=\max(1,j)}^{n}\frac{1}{(t-j)^{1+\zeta/2}}\right)^2,$$

where $\zeta=1-2d_x-d_1>0$ by assumption. When $2d_x<d_1$, it implies that $d_x<1/4$ in which case the left side of the last displayed expression is

$$K\sum_{s=N}^{\infty}\frac{1}{s^{2-2d_1}}\frac{1}{n}\sum_{j=-\infty}^{n}\left(\sum_{t=\max(1,j)}^{n}\frac{1}{(t-j)^{1+\varsigma/2}}\right)^2$$

with $\varsigma=1-4d_x$. Thus (74) holds true. Similarly

$$\int_{\mathbb{R}}E\left|\frac{\partial^2 B_{n2}(x)}{\partial^2 x}\right|^2dx=o(1).$$

Hence, because $|B_{n2}(x)-B_{n2}(y)|\leq|x-y|\sup_{z\in\mathbb{R}}\left|\frac{\partial B_{n2}(z)}{\partial z}\right|$, we conclude that $B_{n2}(x)$ is tight. Finally, we need to show the tightness of $B_{n1}(x)$ which follows similarly as that of $B_{n2}(x)$, with the only difference that for instance we have now that

$$\int_{\mathbb{R}}E\left|\frac{\partial B_{n1}(x)}{\partial x}\right|^2dx=\frac{K}{n}\sum_{s=-N}^{n}\sum_{j=-\infty}^{n}\left(\sum_{t=\max(1,j)}^{n}\varphi_{t-j+1}^{3/2}\vartheta_{t-j+1-s}\right)^2\leq K.$$

So $B_n(x)$ is tight and then so it is $\mathcal{G}_n(x)$. ∎

We now examine a series of lemmas which are needed to show the validity of $\mathcal{G}_n^*(x)$. Lemmas 13 and 16 will show that

$$\frac{1}{n}\sum_{t\le s} E^*(u_t^* u_s^*)\left\{\mathring{\mathbf{i}}_t \mathring{\mathbf{i}}_s - E(\circ)\right\} = o_p(1). \tag{75}$$

A consequence of (75) is that

$$E^*\left(\frac{1}{n^{1/2}}\sum_{t=1}^n \mathring{\mathbf{i}}_t u_t^*\right)^2 \overset{P}{\to} Cov(\mathcal{G}(x), \mathcal{G}(x))$$

given in (8). Finally Lemma 19 shows the tightness of $\mathcal{G}_n^*(x)$. It is worth noticing that $\gamma_u(r) \simeq Cr^{2d_1-1}$ as $r \nearrow \infty$ by Condition 4.

LEMMA 13. *Under Conditions* 2 *to* 4*, we have that for any sequence* $h(t)$ *such that* $h(t) = O\left(t^{2d_1-1}\right)$, $t \ge 1$, $h(0) = 1$,

$$E\left(\frac{1}{n}\sum_{t_1\le t_2} h(t_2-t_1)\left\{\mathring{\mathbf{i}}_{t_1}\mathring{\mathbf{i}}_{t_2} - E(\circ)\right\}\right)^2 = o(1). \tag{76}$$

PROOF. First, because $E\left(n^{-1}\sum_{t=1}^n \mathring{\mathbf{i}}_t^2 - E\left(\mathring{\mathbf{i}}_t^2\right)\right)^2 \to 0$ by ergodicity, it suffices to examine (76) for $t_1 < t_2$ only, that is

$$\begin{aligned}
&\frac{1}{n}\sum_{t_1<t_2} h(t_2-t_1)\sum_{k_1,k_2=0}^{\infty}\left(\wp_{t_1-k_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\wp_{t_2-k_2}\left[\mathring{\mathbf{i}}_{t_2}\right] - E(\circ)\right)\\
&= \frac{1}{n}\sum_{t_1<t_2} h(t_2-t_1)\,\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\wp_{t_2}\left[\mathring{\mathbf{i}}_{t_2}\right]\\
&\quad + \frac{1}{n}\sum_{t_1<t_2} h(t_2-t_1)\sum_{k_1=1}^{\infty}\left(\wp_{t_2}\left[\mathring{\mathbf{i}}_{t_2}\right]\wp_{t_1-k_1}\left[\mathring{\mathbf{i}}_{t_1}\right] - E(\circ)\right)\\
&\quad + \frac{1}{n}\sum_{t_1<t_2} h(t_2-t_1)\sum_{k_2=1}^{\infty}\left(\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\wp_{t_2-k_2}\left[\mathring{\mathbf{i}}_{t_2}\right] - E(\circ)\right)\\
&\quad + \frac{1}{n}\sum_{t_1<t_2} h(t_2-t_1)\sum_{k_1,k_2=1}^{\infty}\left(\wp_{t_1-k_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\wp_{t_2-k_2}\left[\mathring{\mathbf{i}}_{t_2}\right] - E(\circ)\right)\\
&=: A_{n1} + A_{n2} + A_{n3} + A_{n4},
\end{aligned}$$

after observing that by martingale difference, $E\left\{\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\wp_{t_2}\left[\mathring{\mathbf{i}}_{t_2}\right]\right\} = 0$ if $t_1 < t_2$. The conclusion follows by Lemmas 14 and 15. Notice that the contributions due to A_{n2} and A_{n3} into the left side of (76) follow similarly to that of A_{n4} if not easier. ∎

LEMMA 14. $EA_{n1}^2 = o(1)$.

PROOF. Because $\wp_t\left[\mathring{\mathbf{i}}_t\right]$ is a martingale difference, EA_{n1}^2 is

$$\frac{1}{n^2}\sum_{t_1,t_3<t_2} E\left(\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\wp_{t_3}\left[\mathring{\mathbf{i}}_{t_3}\right]\wp_{t_2}^2\left[\mathring{\mathbf{i}}_{t_2}\right]\right)\prod_{i=1}^{2} h(t_2-t_{2i-1}), \tag{77}$$

where we shall take $x = 0$ without loss of generality to simplify the notation. Now from the definition of $\mathring{\mathbf{1}}_t$ in (6), we have that

$$\wp_t\left[\mathring{\mathbf{1}}_t\right] =: \left(\mathbf{1}_t - E\left(\mathbf{1}_t \mid \mathcal{F}_{t-1}\right)\right) + f\left(0\right)\varrho_t, \tag{78}$$

so that (77) becomes

$$\begin{aligned}
&\frac{1}{n^2}\sum_{t_1,t_3<t_2} E\left(\wp_{t_1}\left[\mathring{\mathbf{1}}_{t_1}\right]\wp_{t_3}\left[\mathring{\mathbf{1}}_{t_3}\right]E\left(\mathbf{1}_{t_2}\mid\mathcal{F}_{t_2-1}\right)\right)\prod_{i=1}^{2}h\left(t_2-t_{2i-1}\right)\\
&-\frac{1}{n^2}\sum_{t_1,t_3<t_2} E\left(\wp_{t_1}\left[\mathring{\mathbf{1}}_{t_1}\right]\wp_{t_3}\left[\mathring{\mathbf{1}}_{t_3}\right]E^2\left(\mathbf{1}_{t_2}\mid\mathcal{F}_{t_2-1}\right)\right)\prod_{i=1}^{2}h\left(t_2-t_{2i-1}\right)\\
&+\frac{2f\left(0\right)}{n^2}\sum_{t_1,t_3<t_2} E\left(\wp_{t_1}\left[\mathring{\mathbf{1}}_{t_1}\right]\wp_{t_3}\left[\mathring{\mathbf{1}}_{t_3}\right]\mathbf{1}_{t_2}\varrho_{t_2}\right)\prod_{i=1}^{2}h\left(t_2-t_{2i-1}\right)\\
&+\frac{f^2\left(0\right)}{n^2}\sum_{t_1,t_3<t_2} E\left(\wp_{t_1}\left[\mathring{\mathbf{1}}_{t_1}\right]\wp_{t_3}\left[\mathring{\mathbf{1}}_{t_3}\right]\right)E\varrho_{t_2}^2\prod_{i=1}^{2}h\left(t_2-t_{2i-1}\right).
\end{aligned} \tag{79}$$

It suffices to consider only the sums for $t_1 \le t_3$. The fourth term of (79) is easily seen to be $o\left(1\right)$ because $E\left(\wp_{t_1}\left[\mathring{\mathbf{1}}_{t_1}\right]\wp_{t_3}\left[\mathring{\mathbf{1}}_{t_3}\right]\right) = O\left(\left|t_3-t_1\right|^{4d_x-2}\right)$ and $1 > 2d_x + d_1$. Next, denoting

$$E\left(\mathbf{1}_{t_2}\xi_{t_2}\mid\mathcal{F}_{t_2-1}\right) = \widetilde{F}\left(-\sum_{\ell\ge1}\varphi_\ell\varrho_{t_2-\ell}\right), \tag{80}$$

the third term of (79) is, after standard algebra, proportional to

$$\frac{1}{n^2}\sum_{t_1\le t_3<t_2} E\left(\wp_{t_1}\left[\mathring{\mathbf{1}}_{t_1}\right]\wp_{t_3}\left[\mathring{\mathbf{1}}_{t_3}\right]\widetilde{F}\left(-\sum_{\ell\ge1}\varphi_\ell\varrho_{t_2-\ell}\right)\right)\left(\prod_{i=1}^{2}h\left(t_2-t_{2i-1}\right)\right). \tag{81}$$

Now because $\wp_{t_3}\left[\mathring{\mathbf{1}}_{t_3}\right]$ is a martingale difference,

$$E\left(\wp_{t_1}\left[\mathring{\mathbf{1}}_{t_1}\right]\wp_{t_3}\left[\mathring{\mathbf{1}}_{t_3}\right]\widetilde{F}\left(-\sum_{\ell\ge1;\ne t_2-t_3}\varphi_\ell\varrho_{t_2-\ell}\right)\right) = 0,$$

and hence Taylor's expansion of $\widetilde{F}\left(\cdot\right)$ implies that (81) is

$$\begin{aligned}
&\frac{1}{n^2}\sum_{t_1\le t_3<t_2}\left\{\varphi_{t_2-t_3}v\left(t_1,t_2,t_3\right)\left(\prod_{i=1}^{2}h\left(t_2-t_{2i-1}\right)\right)\right\}\\
&+\frac{K}{n^2}\sum_{t_1\le t_3<t_2}\left(t_2-t_3\right)^{2d_x-2}\prod_{i=1}^{2}h\left(t_2-t_{2i-1}\right)\\
&=\frac{1}{n^2}\sum_{t_1\le t_3<t_2}\left\{\varphi_{t_2-t_3}v\left(t_1,t_2,t_3\right)\left(\prod_{i=1}^{2}h\left(t_2-t_{2i-1}\right)\right)\right\}+O\left(n^{2d_1-1}\right)
\end{aligned} \tag{82}$$

using that $\varphi_t = O\left(t^{d_x-1}\right)$ and where

$$v\left(t_1,t_2,t_3\right) =: E\left(\wp_{t_1}\left[\mathring{\mathbf{1}}_{t_1}\right]\wp_{t_3}\left[\mathring{\mathbf{1}}_{t_3}\right]\varrho_{t_3}\widetilde{F}^{(1)}\left(-\sum_{\ell\ge1;\ne t_2-t_3}\varphi_\ell\varrho_{t_2-\ell}\right)\right)$$

with $\widetilde{F}^{(\ell)}(s) = \partial^{(\ell)}\widetilde{F}(s)/\partial s^{(\ell)}$ for $\ell = 1, 2, ...$ Because $2d_x + d_1 - 1 < 0$, it suffices to examine the first term on the right of (82) for $t_1 < t_3$ which, except multiplicative constants, is

$$\frac{1}{n^2}\sum_{t_1<t_3<t_2}\varphi_{t_2-t_3}\prod_{i=1}^{2}h(t_2-t_{2i-1})\left\{E\left(\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\widetilde{F}^{(1)}\left(-\sum_{\ell\geq 1;\neq t_2-t_3}\varphi_\ell\varrho_{t_2-\ell}\right)\right)\right.$$

$$\left.+E\left(\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\widetilde{F}\left(-\sum_{\ell\geq 1}\varphi_\ell\varrho_{t_3-\ell}\right)\widetilde{F}^{(1)}\left(-\sum_{\ell\geq 1;\neq t_2-t_3}\varphi_\ell\xi_{t_2-\ell}\right)\right)\right\}, \tag{83}$$

using (78) and (80). Again as $\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]$ is a martingale difference,

$$E\left(\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\widetilde{F}^{(1)}\left(-\sum_{\ell\geq 1;\neq t_2-t_3}^{t_2-t_1-1}\varphi_\ell\varrho_{t_2-\ell}\right)\right) = 0,$$

so that Taylor's expansion implies that

$$\begin{aligned}
&E\left(\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\widetilde{F}^{(1)}\left(-\sum_{\ell\geq 1;\neq t_2-t_3}\varphi_\ell\varrho_{t_2-\ell}\right)\right)\\
&= E\left(\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\widetilde{F}^{(2)}\left(-\sum_{\ell\geq 1;\neq t_2-t_3}^{t_2-t_1-1}\varphi_\ell\varrho_{t_2-\ell}\right)\sum_{\ell\geq t_2-t_1}\varphi_\ell\varrho_{t_2-\ell}\right) + E\left(\sum_{\ell\geq t_2-t_1}\varphi_\ell\varrho_{t_2-\ell}\right)^2\\
&= \varphi_{t_2-t_1}E\left(\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\widetilde{F}^{(2)}\left(-\sum_{\ell\geq 1;\neq t_2-t_3}^{t_2-t_1-1}\varphi_\ell\varrho_{t_2-\ell}\right)\varrho_{t_1}\right) + E\left(\sum_{\ell\geq t_2-t_1}\varphi_\ell\varrho_{t_2-\ell}\right)^2\\
&= O\left((t_2-t_1)^{2d_x-1}\right).
\end{aligned}$$

So, because $1 - 2d_x - d_1 > 0$, we conclude that the first term of (83) is

$$\frac{1}{n^2}\sum_{t_1<t_3<t_2}\frac{1}{(t_2-t_3)^{2-d_x-2d_1}(t_2-t_1)^{2-2d_x-2d_1}} = o(1).$$

Next the behaviour of the second term of (83). By Taylor's expansion of $\widetilde{F}^{(1)}\left(-\sum_{\ell\geq 1;\neq t_2-t_3}\varphi_\ell\varrho_{t_2-\ell}\right) - \widetilde{F}^{(1)}\left(-\sum_{\ell\geq 1}^{t_2-t_3-1}\varphi_\ell\varrho_{t_2-\ell}\right)$, the expectation factor is, except multiplicative constants,

$$\begin{aligned}
&E\left(\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\widetilde{F}\left(-\sum_{\ell\geq 1}\varphi_\ell\varrho_{t_3-\ell}\right)\right)\\
&+E\left(\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\widetilde{F}\left(-\sum_{\ell\geq 1}\varphi_\ell\varrho_{t_3-\ell}\right)\sum_{\ell\geq t_2-t_3+1}\varphi_\ell\varrho_{t_2-\ell}\right)\\
&+E\left(\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\widetilde{F}\left(-\sum_{\ell\geq 1}\varphi_\ell\varrho_{t_3-\ell}\right)\left(\sum_{\ell\geq t_2-t_3+1}\varphi_\ell\varrho_{t_2-\ell}\right)^2\right).
\end{aligned} \tag{84}$$

Now the third term of (84) is $O\left((t_2-t_3+1)^{2d_x-1}\right)$ so that its contribution into the the second term of (83) is

$$\frac{1}{n^2}\sum_{t_1<t_3<t_2}\frac{1}{(t_2-t_3)^{3-3d_x-2d_1}(t_2-t_1)^{1-2d_1}}=o(1)$$

because $1-2d_x-d_1>0$. Now because

$$\left|E\left(\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\left(\widetilde{F}\left(-\sum_{\ell\geq 1}\varphi_\ell\varrho_{t_3-\ell}\right)-\widetilde{F}\left(-\sum_{\ell\geq 1;\neq t_3-t_1}\varphi_\ell\varrho_{t_3-\ell}\right)\right)\right)\right|\leq K\varphi_{t_3-t_1}$$

the contribution of the first term of (84) into the second term of (83) is also $o(1)$. Notice that $E\left(\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\widetilde{F}\left(-\sum_{\ell\geq 1;\neq t_3-t_1}\varphi_\ell\varrho_{t_3-\ell}\right)\right)=0$. Finally the contribution of the second term of (84) into the second term of (83) is also $o(1)$ proceeding similarly.

So, to complete the proof of the lemma, it suffices to show that the second term of (79) is $o(1)$, since the first term of (79) is similarly handled. Now, by Taylor's expansion, we obtain that the second term of (79) is

$$\begin{aligned}
&\frac{1}{n^2}\sum_{t_1<t_3<t_2}\left(\prod_{i=1}^{2}h(t_2-t_{2i-1})\right)E\left\{\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\wp_{t_3}\left[\mathring{\mathbf{i}}_{t_3}\right]\right.\\
&\left.\times\left(\widetilde{F}\left(\sum_{\ell\geq 1;\neq t_2-t_3}\varphi_\ell\varrho_{t_2-\ell}\right)+\varphi_{t_2-t_3}\varrho_{t_3}\widetilde{F}^{(1)}\left(\sum_{\ell\geq 1;\neq t_2-t_3}\varphi_\ell\varrho_{t_2-\ell}\right)+K\varphi^2_{t_2-t_3}\varrho^2_{t_3}\right)^2\right\}\\
=&\frac{2}{n^2}\sum_{t_1<t_3<t_2}\varphi_{t_2-t_3}\prod_{i=1}^{2}h_u(t_2-t_{2i-1})E\left\{\wp_{t_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\wp_{t_3}\left[\mathring{\mathbf{i}}_{t_3}\right]\varrho_{t_3}\right.\\
&\left.\times\widetilde{F}\left(\sum_{\ell\geq 1;\neq t_2-t_3}\varphi_\ell\varrho_{t_2-\ell}\right)\widetilde{F}^{(1)}\left(\sum_{\ell\geq 1;\neq t_2-t_3}\varphi_\ell\varrho_{t_2-\ell}\right)\right\}\\
&+o(1)
\end{aligned}$$

because $\sum_{t_1<t_3<t_2}\varphi^2_{t_2-t_3}\prod_{i=1}^{2}h(t_2-t_{2i-1})=o\left(n^{1+2d_1}\right)$. Now proceed similarly again as we did with the first term of (82) but with $\widetilde{F}^{(1)}(s)$ replaced by $\widetilde{F}(s)\widetilde{F}^{(1)}(s)$ there, to conclude that the right side of the last displayed expression is $o(1)$. This finishes the proof of the lemma. ∎

LEMMA 15. $EA^2_{n4}=o(1)$.

PROOF. Now by standard inequalities EA^2_{n4} is bounded by

$$E\left(\frac{1}{n}\sum_{t_1<t_2}h(t_2-t_1)\sum_{k=1}^{\infty}\left(\wp_{t_1-k}\left[\mathring{\mathbf{i}}_{t_1}\right]\wp_{t_1-k}\left[\mathring{\mathbf{i}}_{t_2}\right]-E(\circ)\right)\right)^2 \tag{85}$$

$$+E\left(\frac{1}{n}\sum_{t_1<t_2}h(t_2-t_1)\left\{\sum_{\substack{k_1,k_2=1;\\ t_1-k_1<t_2-k_2}}^{\infty}+\sum_{\substack{k_1,k_2=1;\\ t_2-k_2<t_1-k_1}}^{\infty}\right\}\wp_{t_1-k_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\wp_{t_2-k_2}\left[\mathring{\mathbf{i}}_{t_2}\right]\right)^2.$$

Before we examine (85), we introduce the following notation in the spirit of that given in Wu's (2003) Lemma 9. Define

$$\mathcal{R}_{t-k}\left[\mathbf{1}_t\right]=\wp_{t-k}\left[\mathbf{1}_t\right]+\varphi_k\varrho_{t-k}F_k^{(1)}\left(-\sum_{\ell\geq 1}\varphi_{\ell+k}\varrho_{t-k-\ell}\right) \tag{86}$$

$$\mathcal{M}_{t-k}^{(1)}\left[\mathring{\mathbf{1}}_t\right]=F_k^{(1)}\left(-\sum_{\ell>k}\varphi_\ell\varrho_{t-\ell}\right)-E\left(F_k^{(1)}\left(-\sum_{\ell>k}\varphi_\ell\varrho_{t-\ell}\right)\right)$$

$$=:\breve{F}_k^{(1)}\left(-\sum_{\ell>k}\varphi_\ell\varrho_{t-\ell}\right). \tag{87}$$

where $F_k(z)=E\left(\mathcal{I}\left(x_t<z\right)\mid\mathcal{F}_{t-k}\right)$ and $F^{(i)}(z)=\partial^i F(z)/\partial z^i$. Observe that $E\left(F_k^{(1)}\left(-\sum_{\ell>k}\varphi_\ell\varrho_{t-\ell}\right)\right)=F^{(1)}(0)$.

By definition, we have that

$$\wp_{t-k}\left[\mathring{\mathbf{1}}_t\right]=\mathcal{R}_{t-k}\left[\mathbf{1}_t\right]-\varphi_k\varrho_{t-k}\mathcal{M}_{t-k}^{(1)}\left[\mathring{\mathbf{1}}_t\right], \tag{88}$$

where according to Wu 's (2003) Lemma 9 and expression (37) there, we have that

$$E\left|\mathcal{R}_{t-k}\left[\mathbf{1}_t\right]\right|^q\leq Kk^{2q(d_x-1)} \tag{89}$$

$$E\left|\breve{F}_k^{(\ell)}\left(-\sum_{\ell>k}\varphi_\ell\varrho_{t-\ell}\right)\right|^q\leq Kk^{q(d_x-1/2)},\quad \ell=1,2,3. \tag{90}$$

We first show the first term of (85) is $o(1)$, for which it suffices to show that this is the case, using (88), for the next two displayed expressions

$$E\left(\frac{1}{n}\sum_{t_1<t_2}h\left(t_2-t_1\right)\sum_{k=1}^{\infty}\left(\mathcal{R}_{t_1-k}\left[\mathbf{1}_{t_1}\right]\mathcal{R}_{t_1-k}\left[\mathbf{1}_{t_2}\right]-E\left(\circ\right)\right)\right)^2 \tag{91}$$

$$E\left(\frac{1}{n}\sum_{t_1<t_2}h\left(t_2-t_1\right)\sum_{k=1}^{\infty}\varphi_k\varphi_{t_2-t_1+k}\left(\varrho_{t_1-k}^2\mathcal{M}_{t_1-k}^{(1)}\left[\mathring{\mathbf{1}}_{t_1}\right]\mathcal{M}_{t_1-k}^{(1)}\left[\mathring{\mathbf{1}}_{t_2}\right]-E\left(\circ\right)\right)\right)^2.$$

The second term in (91), using notation in (87), is bounded by

$$2E\left\{\frac{1}{n}\sum_{t_1<t_2}h\left(t_2-t_1\right)\sum_{k=1}^{\infty}\varphi_k\varphi_{t_2-t_1+k}\right.$$
$$\left.\times\left(\left(\varrho_{t_1-k}^2-\sigma_\varrho^2\right)\breve{F}_k^{(1)}\left(\sum_{\ell>k}\varphi_\ell\varrho_{t_1-\ell}\right)\breve{F}_k^{(1)}\left(\sum_{\ell>k}\varphi_{\ell+t_2-t_1}\varrho_{t_1-\ell}\right)\right)\Bigg]\right\}^2$$
$$+2E\left\{\frac{\sigma_\varrho^2}{n}\sum_{t_1<t_2}h\left(t_2-t_1\right)\sum_{k=1}^{\infty}\varphi_k\varphi_{t_2-t_1+k}\right.$$
$$\left.\times\left(\breve{F}_k^{(1)}\left(\sum_{\ell>k}\varphi_\ell\varrho_{t_1-\ell}\right)\breve{F}_k^{(1)}\left(\sum_{\ell>k}\varphi_{\ell+t_2-t_1}\varrho_{t_1-\ell}\right)-E\left(\circ\right)\right)\Bigg]^2\right\}.$$

Because $\varrho_{t_1-k}^2-\sigma_\varrho^2$ is iid, it is easy to show that the first term of the last displayed expression is $o(1)$, after one notices that $\sum_{k=1}^{\infty}\left|\varphi_k\varphi_{t_2-t_1+k}\right| = O\left(\left|t_2-t_1\right|^{2d_x-1}\right)$, whereas the second term, except constants, equals

$$
\begin{aligned}
&\frac{1}{n^2}\sum_{\substack{t_1<t_2;\\ t_3<t_4}}\prod_{i=1}^{2}h\left(t_{2i}-t_{2i-1}\right)\sum_{k_1,k_3=1}^{\infty}\varphi_{k_1}\varphi_{t_2-t_1+k_1}\varphi_{k_3}\varphi_{t_4-t_3+k_3}\\
&\left\{E\left\{\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_\ell\varrho_{t_1-\ell}\right)\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>k_3}\varphi_\ell\varrho_{t_3-\ell}\right)\right\}\right.\\
&\times E\left\{\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_{\ell+t_2-t_1}\varrho_{t_1-\ell}\right)\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>k_3}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right)\right\}\\
&+E\left\{\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_\ell\varrho_{t_1-\ell}\right)\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>k_3}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right)\right\}\\
&\times E\left\{\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_{\ell+t_2-t_1}\varrho_{t_1-\ell}\right)\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>k_3}\varphi_\ell\varrho_{t_3-\ell}\right)\right\}\\
&\left.+Cum\begin{pmatrix}\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_\ell\varrho_{t_1-\ell}\right);\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_{\ell+t_2-t_1}\varrho_{t_1-\ell}\right);\\ \breve{F}_{k_3}^{(1)}\left(\sum_{\ell>k_3}\varphi_\ell\varrho_{t_3-\ell}\right);\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>k_3}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right)\end{pmatrix}\right\}.
\end{aligned}
\tag{92}
$$

We examine the scenario $t_1-k_1<t_3-k_3$, being similarly handled when $t_3-k_3<t_1-k_1$. Using that

$$
E\left(\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>k_3}\varphi_\ell\varrho_{t_3-\ell}\right)|\mathcal{F}_{t_1-k_1}|\right)=\breve{F}_{t_3-t_1+k_1}^{(1)}\left(\sum_{\ell\geq t_3-t_1+k_1}\varphi_\ell\varrho_{t_3-\ell}\right)
\tag{93}
$$

we obtain that

$$
\begin{aligned}
&E\left|F_{k_3}^{(1)}\left(\sum_{\ell>k_3}\varphi_\ell\varrho_{t_3-\ell}\right)-F_{t_3-t_1+k_1}^{(1)}\left(\sum_{\ell\geq t_3-t_1+k_1}\varphi_\ell\varrho_{t_3-\ell}\right)\right|^2\\
&=O\left(\left(t_3-t_1+k_1\right)^{3(d_x-1)}\right)
\end{aligned}
\tag{94}
$$

see Wu (2003) and then Cauchy-Schwarz inequality together with (90), we have that

$$
\begin{aligned}
&\left|E\left\{\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_\ell\varrho_{t_1-\ell}\right)\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>k_3}\varphi_\ell\varrho_{t_3-\ell}\right)\right\}\right|\\
&=\left|E\left\{\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_\ell\varrho_{t_1-\ell}\right)\breve{F}_{t_3-t_1+k_1}^{(1)}\left(\sum_{\ell\geq t_3-t_1+k_1}\varphi_\ell\varrho_{t_3-\ell}\right)\right\}\right|\\
&=O\left(k_1^{d_x-1/2}\left(t_3-t_1+k_1\right)^{d_x-1/2}\right).
\end{aligned}
\tag{95}
$$

So, the contribution of the first and second terms of (92) is bounded by

$$
\frac{1}{n^2}\sum_{\substack{t_1<t_2;\\ t_3<t_4}}\prod_{i=1}^{2}h\left(t_{2i}-t_{2i-1}\right)\sum_{\substack{k_1,k_3=1;\\ t_1-k_1<t_3-k_3}}^{\infty}\left\{\varphi_{k_1}\varphi_{t_2-t_1+k_1}\varphi_{k_3}\varphi_{t_4-t_3+k_3}\right.
$$

$$\times k_1^{d_x-1/2}\left(t_3-t_1+k_1\right)^{d_x-1/2}\left(t_4-t_1+k_1\right)^{d_x-1/2}\left(t_2-t_1+k_1\right)^{d_x-1/2}\Big\}$$

$$=\frac{1}{n^2}\sum_{\substack{t_1<t_2;\\ t_3<t_4}}\frac{1}{\left(t_2-t_1\right)^{1-2d_1}\left(t_4-t_3\right)^{1-2d_1}}\sum_{\substack{k_1,k_3=1;\\ t_1-k_1<t_3-k_3}}^{\infty}\frac{1}{k_3^{1-d_x}\left(t_4-t_3+k_3\right)^{1-d_x}}$$

$$\times\frac{1}{k_1^{3/2-2d_x}\left(t_3-t_1+k_1\right)^{1/2-d_x}\left(t_4-t_1+k_1\right)^{1/2-d_x}\left(t_2-t_1+k_1\right)^{3/2-2d_x}}$$

which is $o(1)$ by standard algebra using the fact that $1=2d_x+d_1+\upsilon$ for some $\upsilon>0$.

Next we examine the contribution due to the cumulant. For that purpose, by standard algebra and the definition of the fourth cumulant, we have that it is

$$Cum\left(\begin{array}{c}\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_{\ell+t_2-t_1}\varrho_{t_1-\ell}\right);\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>t_3-t_1+k_1}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right);\\ \breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_{\ell}\varrho_{t_1-\ell}\right);\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>t_3-t_1+k_1}\varphi_{\ell}\varrho_{t_3-\ell}\right)\end{array}\right)$$

$$+Cum\left(\begin{array}{c}\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_{\ell+t_2-t_1}\varrho_{t_1-\ell}\right);\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_{\ell}\varrho_{t_1-\ell}\right);\\ \breve{F}_{k_3}^{(1)}\left(\sum_{\ell>t_3-t_1+k_1}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right);\\ \breve{F}_{k_3}^{(1)}\left(\sum_{\ell>k_3}\varphi_{\ell}\varrho_{t_3-\ell}\right)-\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>t_3-t_1+k_1}\varphi_{\ell}\varrho_{t_3-\ell}\right)\end{array}\right) \tag{96}$$

$$+Cum\left(\begin{array}{c}\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_{\ell+t_2-t_1}\varrho_{t_1-\ell}\right);\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_{\ell}\varrho_{t_1-\ell}\right);\\ \breve{F}_{k_3}^{(1)}\left(\sum_{\ell>t_3-t_1+k_1}\varphi_{\ell}\varrho_{t_3-\ell}\right);\\ \breve{F}_{k_3}^{(1)}\left(\sum_{\ell>k_3}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right)-\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>t_3-t_1+k_1}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right)\end{array}\right)$$

$$+Cum\left(\begin{array}{c}\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_{\ell+t_2-t_1}\varrho_{t_1-\ell}\right);\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_{\ell}\varrho_{t_1-\ell}\right);\\ \breve{F}_{k_3}^{(1)}\left(\sum_{\ell>k_3}\varphi_{\ell}\varrho_{t_3-\ell}\right)-\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>t_3-t_1+k_1}\varphi_{\ell}\varrho_{t_3-\ell}\right);\\ \breve{F}_{k_3}^{(1)}\left(\sum_{\ell>k_3}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right)-\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>t_3-t_1+k_1}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right)\end{array}\right).$$

Because

$$cum\left(z_{t_1},z_{t_2},z_{t_3},z_{t_4}\right)=E\prod_{\ell=1}^{4}z_{t_\ell}-3E\left(z_{t_1}z_{t_2}\right)E\left(z_{t_3}z_{t_3}\right), \tag{97}$$

and (93) – (94), the contribution of the first term of (96) into the third term of (92) is $o(1)$ using (90), whereas the contribution due to the second term of (96) is also $o(1)$ after we notice that mean value theorem that $cum\left(z_{t_1},z_{t_2},z_{t_3},z_{t_4}\upsilon\right)=0$ if υ is independent of z_{t_ℓ}, implies that the second term of (96) is

$$Cum\left(\begin{array}{c}\breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_{\ell+t_2-t_1}\varrho_{t_1-\ell}\right);\breve{F}_{k_3}^{(1)}\left(\sum_{\ell=t_3-t_1+k_1}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right);\\ \breve{F}_{k_1}^{(1)}\left(\sum_{\ell>k_1}\varphi_{\ell}\varrho_{t_1-\ell}\right);\\ \left(\sum_{\ell=k_3}^{t_3-t_1+k_1}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right)^2\breve{F}_{k_3}^{(3)}\left(\sum_{\ell>k_3}\varphi_{\ell}\varrho_{t_3-\ell}\right)\end{array}\right)$$

and then that for instance

$$E\left|\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>k_3}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right)-\breve{F}_{k_3}^{(1)}\left(\sum_{\ell>t_3-t_1+k_1}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right)\right|^q$$

$$\leq CE\left|\sum_{\ell>k_3}^{t_3-t_1+k_1}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right|^q$$

$$=O\left(\left(t_4-t_3+k_3\right)^{q(d_x-1/2)}\right).$$

Likewise the contribution due to the third and fourth are also $o(1)$. So, this finishes the proof that the second term in (91) is $o(1)$.

Next we examine the first term in (91), which is

$$\frac{1}{n^2}\sum_{\substack{t_1<t_2;\\ t_3<t_4}}\left\{\left(\prod_{i=1}^{2}h(t_{2i}-t_{2i-1})\right)\right.$$

$$\times\sum_{k_1,k_3=1}^{\infty}\left\{E\left(\mathcal{R}_{t_1-k_1}[\mathbf{1}_{t_1}]\mathcal{R}_{t_3-k_3}[\mathbf{1}_{t_3}]\right)E\left(\mathcal{R}_{t_1-k_1}[\mathbf{1}_{t_2}]\mathcal{R}_{t_3-k_3}[\mathbf{1}_{t_4}]\right)\right.$$

$$(98)\qquad +E\left(\mathcal{R}_{t_1-k_1}[\mathbf{1}_{t_1}]\mathcal{R}_{t_3-k_3}[\mathbf{1}_{t_4}]\right)E\left(\mathcal{R}_{t_1-k_1}[\mathbf{1}_{t_2}]\mathcal{R}_{t_3-k_3}[\mathbf{1}_{t_3}]\right)$$

$$\left.\left.+Cum\left(\mathcal{R}_{t_1-k_1}[\mathbf{1}_{t_1}];\mathcal{R}_{t_1-k_1}[\mathbf{1}_{t_2}];\mathcal{R}_{t_3-k_3}[\mathbf{1}_{t_3}];\mathcal{R}_{t_3-k_3}[\mathbf{1}_{t_4}]\right)\right\}\right\}.$$

Because $\mathcal{R}_s[\mathbf{1}_t]$ is a martingale difference, the first and second terms are different than zero only if $t_1-k_1=t_3-k_3$, in which case is

$$\frac{1}{n^2}\sum_{\substack{t_1<t_2;\\ t_3<t_4}}\prod_{i=1}^{2}h(t_{2i}-t_{2i-1})\sum_{k=1}^{\infty}E\left(\mathcal{R}_{t_1-k}[\mathbf{1}_{t_1}]\mathcal{R}_{t_1-k}[\mathbf{1}_{t_3}]\right)E\left(\mathcal{R}_{t_1-k}[\mathbf{1}_{t_2}]\mathcal{R}_{t_1-k}[\mathbf{1}_{t_4}]\right)$$

which is $o(1)$. Indeed, (89) and Cauchy-Schwarz inequality imply that the last displayed expression is bounded by

$$\frac{1}{n^2}\sum_{t_1<t_2;t_3<t_4}\left\{\frac{1}{(t_2-t_1)^{1-2d_1}(t_4-t_3)^{1-2d_1}}\right.$$

$$\times\sum_{k=1}^{\infty}\frac{1}{k^{1-d_x}(k+t_3-t_1)^{1-d_x}(k+t_4-t_1)^{1-d_x}(k+t_2-t_1)^{1-d_x}}$$

$$\leq\frac{K}{n^2}\sum_{t_1<t_2;t_3<t_4}\frac{1}{(t_2-t_1)^{2-2d_x-2d_1}(t_4-t_3)^{1-2d_1}(t_3-t_1)^{1-d_x}(t_4-t_1)^{1-d_x}}$$

$$\leq\frac{K}{n}\left(\sum_{t=1}^{n}\frac{1}{t^{2-2d_x-2d_1}}\right)^2=o(1)$$

because $d_1+2d_x<1$. Assuming that $t_1-k_1<t_3-k_3$ without loss of generality, the third term of (98) is also $o(1)$. Indeed, observing that if we replace $\mathcal{R}_{t_3-k_3}[\mathbf{1}_{t_4}]$ by

$$g_{t_3-k_3}[\mathbf{1}_{t_4}]=:F_{k_3}\left(\sum_{\ell=k_3}^{t_3-t_1+k_1-1}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right)-F_{k_3+1}\left(\sum_{\ell=k_3+1}^{t_3-t_1+k_1-1}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right)$$

$$-\varphi_{k_3+t_4-t_3}\varrho_{t_3-k_3}F^{(1)}_{k_3}\left(\sum_{\ell=k_3+1}^{t_3-t_1+k_1-1}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right),$$

there, the cumulant is zero as the last displayed expression is independent of $\mathcal{R}_{t_1-k_1}[\mathbf{1}_{t_1}]$, $\mathcal{R}_{t_1-k_1}[\mathbf{1}_{t_2}]$ and $\mathcal{R}_{t_3-k_3}[\mathbf{1}_{t_3}]$. So we have then that the cumulant is

$$Cum\left(\mathcal{R}_{t_1-k_1}[\mathbf{1}_{t_1}];\mathcal{R}_{t_1-k_1}[\mathbf{1}_{t_2}];\mathcal{R}_{t_3-k_3}[\mathbf{1}_{t_3}];\mathcal{R}_{t_3-k_3}[\mathbf{1}_{t_4}]-g_{t_3-k_3}[\mathbf{1}_{t_4}]\right).$$

But $\mathcal{R}_{t_3-k_3}[\mathbf{1}_{t_4}] - g_{t_3-k_3}[\mathbf{1}_{t_4}]$ is $\sum_{\ell=t_3-t_1+k_1}^{\infty}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}$ times

$$F_{k_3}^{(1)}\left(\sum_{\ell=k_3}^{t_3-t_1+k_1-1}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right) - F_{k_3+1}^{(1)}\left(\sum_{\ell=k_3+1}^{t_3-t_1+k_1-1}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right)$$
$$-\varphi_{k_3+t_4-t_3}\varrho_{t_3-k_3}F_{k_3}^{(2)}\left(\sum_{\ell=k_3+1}^{t_3-t_1+k_1-1}\varphi_{\ell+t_4-t_3}\varrho_{t_3-\ell}\right)$$

by Taylor's expansion, and whose qth moment is $O\left((t_4-t_3+k_3)^{2q(d_x-1)}\right)$ by routine extension of (33) to (35) of Wu (2003). So, proceeding as we did with the contribution due to the cumulant in expression (92) yields that the third term of (98) is also $o(1)$. So, this finishes that the first term of (91), and hence the first term of (85), is $o_p(1)$.

To complete the proof of the lemma, we have to show that the second term of (85) is $o(1)$. We shall only examine explicitly the contribution due to the first sum, being the second term identically handled. Now because $\wp_t[\mathring{\mathbf{1}}_t]$ is a martingale difference, the term is

$$\frac{1}{n^2}\sum_{\substack{t_1<t_2;\\ t_3<t_4}}\left\{\left(\prod_{i=1}^{2}h(t_{2i}-t_{2i-1})\right)\right.$$
$$\sum_{\substack{k_1,k_3,k_4=1;\\ t_1-k_1<t_4-k_4\\ t_3-k_3<t_4-k_4}}^{\infty}\left\{E\left(\wp_{t_1-k_1}[\mathring{\mathbf{1}}_{t_1}]\wp_{t_3-k_3}[\mathring{\mathbf{1}}_{t_3}]\right)E\left(\wp_{t_4-k_4}[\mathring{\mathbf{1}}_{t_2}]\wp_{t_4-k_4}[\mathring{\mathbf{1}}_{t_4}]\right)\right. \tag{99}$$
$$\left.\left.+Cum\left(\wp_{t_1-k_1}[\mathring{\mathbf{1}}_{t_1}];\wp_{t_3-k_3}[\mathring{\mathbf{1}}_{t_3}];\wp_{t_4-k_4}[\mathring{\mathbf{1}}_{t_2}];\wp_{t_4-k_4}[\mathring{\mathbf{1}}_{t_4}]\right)\right\}\right\}.$$

The first term is clearly $o(1)$, proceeding as with the first term in (85) after we notice that $E\left(\wp_{t_1-k_1}[\mathring{\mathbf{1}}_{t_1}]\wp_{t_3-k_3}[\mathring{\mathbf{1}}_{t_3}]\right)=0$ unless $t_1-k_1=t_3-k_3$, say. It goes without saying that we have taken without loss of generality that $t_4-k_4<t_2$. If otherwise, we would have considered the same expression but reversing the roles of the subindeces 4 and 2.

So, to complete the proof of the lemma we are left to examine the contribution due to the cumulant is $o(1)$, which in view of (90) it suffices to do so for

$$Cum\left(\mathcal{R}_{t_1-k_1}[\mathbf{1}_{t_1}];\mathcal{R}_{t_3-k_3}[\mathbf{1}_{t_3}];\mathcal{R}_{t_4-k_4}[\mathbf{1}_{t_2}];\mathcal{R}_{t_4-k_4}[\mathbf{1}_{t_4}]\right) \tag{100}$$
$$\varphi_{k_1}\varphi_{k_3}\varphi_{k_4}\varphi_{t_2-t_4+k_4}Cum\left(\varrho_{t_1-k_1}\mathcal{M}_{t_1-k_1}^{(1)}[\mathring{\mathbf{1}}_{t_1}];\varrho_{t_3-k_3}\mathcal{M}_{t_3-k_3}^{(1)}[\mathring{\mathbf{1}}_{t_3}];\right.$$
$$\left.\varrho_{t_4-k_4}\mathcal{M}_{t_4-k_4}^{(1)}[\mathring{\mathbf{1}}_{t_2}];\varrho_{t_4-k_4}\mathcal{M}_{t_4-k_4}^{(1)}[\mathring{\mathbf{1}}_{t_4}]\right). \tag{101}$$

Because (89), (97), Holder's inequality and $t_4-t_2<k_4$, the contribution due to (100) is

$$\frac{K}{n^2}\sum_{t_1<t_2<t_3<t_4}\left(\prod_{i=1}^{2}h(t_{2i}-t_{2i-1})\right)$$
$$\times\sum_{k_1,k_3}\frac{1}{k_1^{2(1-d_x)}k_3^{2(1-d_x)}}\sum_{t_4-t_2<k_4}\frac{1}{k_4^{2(1-d_x)}(t_2-t_4+k_4)^{2(1-d_x)}}$$
$$\leq\frac{K}{n^2}\sum_{t_1<t_2<t_3<t_4}\frac{1}{(t_2-t_1)^{1-2d_1}(t_4-t_3)^{1-2d_1}(t_4-t_2)^{3-4d_x}}$$

which is $o(1)$ because $d_1+2d_x-1<0$ and $h(t)=O\left(t^{2d_1-1}\right)$.

Finally the contribution due to (101). We shall examine the case when $t_1 - k_1 \neq t_3 - k_3$ and assume that $t_1 - k_1 < t_3 - k_3$, being the case $t_3 - k_3 < t_1 - k_1$ symmetrically handled and when $t_1 - k_1 = t_3 - k_3$ it follows by identical if not easier arguments. For that purpose we shall use very often that for any twice continuous differentiable function $g(\cdot)$ we have that

$$E\left|g\left(\sum_{\ell\geq 0}\varphi_\ell\varrho_{t-\ell}\right) - g\left(\sum_{\ell\geq 0;\ell\neq\ell_0}\varphi_\ell\varrho_{t-\ell}\right) - \varphi_{\ell_0}\varrho_{t-\ell_0}g^{(1)}\left(\sum_{\ell\geq 0;\ell\neq\ell_0}\varphi_\ell\varrho_{t-\ell}\right)\right|^q$$

$$= O\left(\varphi_{\ell_0}^{2q}\right)E\left|g^{(2)}\left(\sum_{\ell\geq 0;\ell\neq\ell_0}\varphi_\ell\varrho_{t-\ell}\right)\right|^q \tag{102}$$

and the observation that the cumulant factor in (101) is σ_ϱ^2 times

$$E\left(\varrho_{t_1-k_1}\mathcal{M}^{(1)}_{t_1-k_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\varrho_{t_3-k_3}\mathcal{M}^{(1)}_{t_3-k_3}\left[\mathring{\mathbf{i}}_{t_3}\right]\mathcal{M}^{(1)}_{t_4-k_4}\left[\mathring{\mathbf{i}}_{t_2}\right]\mathcal{M}^{(1)}_{t_4-k_4}\left[\mathring{\mathbf{i}}_{t_4}\right]\right). \tag{103}$$

When $t_1 - k_1 = t_3 - k_3$ the only difference is that we have the extra term

$$E\left(\mathcal{M}^{(1)}_{t_1-k_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\mathcal{M}^{(1)}_{t_1-k_1}\left[\mathring{\mathbf{i}}_{t_3}\right]\right)E\left(\mathcal{M}^{(1)}_{t_4-k_4}\left[\mathring{\mathbf{i}}_{t_2}\right]\mathcal{M}^{(1)}_{t_4-k_4}\left[\mathring{\mathbf{i}}_{t_4}\right]\right).$$

However the contribution of this term becomes negligible proceeding similarly. Now, for the remaining of the lemma we employ the convention that say

$$\mathcal{M}^{(1)}_{t_3-k_3}\left[\mathring{\mathbf{i}}_{t_3};\neq t_1-k_1\right] =: \breve{F}^{(1)}_{k_3}\left(-\sum_{\ell>k_3;\neq t_3-t_1+k_1}\varphi_\ell\varrho_{t_3-\ell}\right).$$

That is it is just (87) where we have removed the dependence on $\varrho_{t_1-k_1}$. Next

$$E\left(\varrho_{t_1-k_1}\mathcal{M}^{(1)}_{t_1-k_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\varrho_{t_3-k_3}\prod_{\ell=2}^{4}\mathcal{M}^{(1)}_{t_\ell-k_\ell}\left[\mathring{\mathbf{i}}_{t_\ell};\neq t_1-k_1\right]\right) = 0, \tag{104}$$

because ϱ_t is independent sequence by Condition 4 and also recall the equality

$$\prod_{\ell=2}^{4}\upsilon_\ell - \prod_{\ell=2}^{4}\zeta_\ell = \prod_{\ell=2}^{4}(\upsilon_\ell-\zeta_\ell) + \sum_{j=2}^{4}\zeta_j\prod_{\ell=1;\neq j}^{4}(\upsilon_\ell-\zeta_\ell) + \sum_{j=2}^{4}(\upsilon_j-\zeta_j)\prod_{\ell=2;\neq j}^{4}\zeta_\ell.$$

Thus, once we identify say υ_ℓ and ζ_ℓ respectively as $\mathcal{M}^{(1)}_{t_\ell-k_\ell}\left[\mathring{\mathbf{i}}_{t_\ell}\right]$ and $\mathcal{M}^{(1)}_{t_\ell-k_\ell}\left[\mathring{\mathbf{i}}_{t_\ell};\neq t_1-k_1\right]$, we have that a typical term in (103) minus the left side of (104) is

$$E\left(\varrho_{t_1-k_1}\mathcal{M}^{(1)}_{t_1-k_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\varrho_{t_3-k_3}\left(\mathcal{M}^{(1)}_{t_3-k_3}\left[\mathring{\mathbf{i}}_{t_3}\right]-\mathcal{M}^{(1)}_{t_3-k_3}\left[\mathring{\mathbf{i}}_{t_3};\neq t_1-k_1\right]\right)\right.$$
$$\left.\mathcal{M}^{(1)}_{t_4-k_4}\left[\mathring{\mathbf{i}}_{t_2};\neq t_1-k_1\right]\mathcal{M}^{(1)}_{t_4-k_4}\left[\mathring{\mathbf{i}}_{t_4};\neq t_1-k_1\right]\right).$$

The other terms in the latter difference proceeds similarly if not easily handled.

Because (102) and (90), Hölder's inequality implies that the last displayed expression is

$$\begin{aligned}&\varphi_{t_3-t_1+k_1}E\varrho^2_{t_1-k_1}E\left(\mathcal{M}^{(1)}_{t_1-k_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\varrho_{t_3-k_3}\left(\mathcal{M}^{(2)}_{t_3-k_3}\left[\mathring{\mathbf{i}}_{t_3};\neq t_1-k_1\right]\right)\right.\\&\left.\mathcal{M}^{(1)}_{t_4-k_4}\left[\mathring{\mathbf{i}}_{t_2};\neq t_1-k_1\right]\mathcal{M}^{(1)}_{t_4-k_4}\left[\mathring{\mathbf{i}}_{t_4};\neq t_1-k_1\right]\right)\\&+O\left(\varphi^2_{t_3-t_1+k_1}\left(k_1k_3k_4\left(t_2-t_4+k_4\right)\right)^{d_x-1/2}\right).\end{aligned} \tag{105}$$

The contribution into (99) of the second term of (105) is

$$\frac{1}{n^2}\sum_{\substack{t_1<t_2;\\ t_3<t_4}}\left\{\prod_{i=1}^{2} h\left(t_{2i}-t_{2i-1}\right)\right.$$

$$\left.\sum_{\substack{k_1,k_3,k_4=1\\ t_1-k_1\leq t_3-k_3<t_4-k_4<t_2}}^{\infty}\left\{\left[k_1k_3k_4\left(t_2-t_4+k_4\right)\right]^{2d_x-3/2}\left(t_3-t_1+k_1\right)^{2d_x-2}\right\}\right.$$
$$=o\left(1\right)$$

because $1-2d_x-d_1>0$. So we are left examining the contribution due to the first term of (105). But that contribution is also $o\left(1\right)$ repeating the same steps again but now when we look at its difference with

$$\varphi_{t_3-t_1+k_1}E\varrho^2_{t_1-k_1}E\left(\mathcal{M}^{(1)}_{t_1-k_1}\left[\mathring{\mathbf{i}}_{t_1}\right]\varrho_{t_3-k_3}\left(\mathcal{M}^{(2)}_{t_3-k_3}\left[\mathring{\mathbf{i}}_{t_3};\neq t_1-k_1\right]\right)\right.$$
$$\left.\mathcal{M}^{(1)}_{t_4-k_4}\left[\mathring{\mathbf{i}}_{t_2};\neq t_1-k_1,t_3-k_3\right]\mathcal{M}^{(1)}_{t_4-k_4}\left[\mathring{\mathbf{i}}_{t_4};\neq t_1-k_1,t_3-k_3\right]\right)$$

which is 0 by Condition 4. This concludes the proof of the lemma. ∎

LEMMA 16. *Under Conditions* 2 *to* 4 *and* $d_1+2d_x<1$,

$$\frac{1}{n}\sum_{t\leq s}E^*\left(u^*_tu^*_s\right)\left(\mathring{\mathbf{i}}_t\mathring{\mathbf{i}}_s-E\left(\mathring{\mathbf{i}}_t\mathring{\mathbf{i}}_s\right)\right)=o_{p^*}\left(1\right).$$

PROOF. By Lemma 13, it suffices to show

$$\frac{1}{n}\sum_{t\leq s}\left(E^*\left(u^*_tu^*_s\right)-\gamma_u\left(\left|t-s\right|\right)\right)\left(\mathring{\mathbf{i}}_t\mathring{\mathbf{i}}_s-E\left(\mathring{\mathbf{i}}_t\mathring{\mathbf{i}}_s\right)\right)=o_{p^*}\left(1\right). \tag{106}$$

We will examine explicitly the scenario where in Condition 4 we assume $b_j=0$ for all j in which case $\pi_j=\vartheta_j$. First, by definition $\gamma_u\left(\left|t-s\right|\right)=\sum_{j=0}^{\infty}\pi_j\pi_{j+|t-s|}$ and since $\left\{\varepsilon^*_t\right\}_{t=1}^n$ is independent and identically distributed with variance $\widehat{\sigma}^2_{\varepsilon}$, we have that, for any $L\left(n\right)\geq Cn$,

$$E^*\left(u^*_tu^*_s\right)=E^*\left(\sum_{j=0}^{L(n)}\breve{\pi}_j\varepsilon^*_{t-j}\sum_{k=0}^{L(n)}\breve{\pi}_k\varepsilon^*_{s-k}\right)=\widehat{\sigma}^2_{\varepsilon}\sum_{j=0}^{L(n)}\breve{\pi}_j\breve{\pi}_{j+|t-s|}=:\widehat{\gamma}_u\left(\left|t-s\right|\right).$$

Now the contribution due to $\sum_{j=L(n)+1-|t-s|}^{\infty}\overline{\pi}_j\overline{\pi}_{j+|t-s|}$ into the left of (106) is $o_p\left(1\right)$ by Lemma 13 because

$$\sum_{j=L(n)+1-|t-s|}^{\infty}\overline{\pi}_j\overline{\pi}_{j+|t-s|}\simeq K\sum_{j=L(n)+1-|t-s|}^{\infty}\frac{1}{j^{1-d_1}}\frac{1}{\left(j+\left|t-s\right|\right)^{1-d_{11}}}$$
$$\simeq\frac{K}{\left(L\left(n\right)-\left|t-s\right|\right)^{1-2d_1}}$$

and then because $L\left(n\right)\geq Kn$ and that $\widehat{\sigma}^2_{\varepsilon}=\sigma^2_{\varepsilon}\left(1+o_p\left(1\right)\right)$.

Next, the contribution due to

$$\frac{1}{n}\sum_{t\leq s}\left(\sum_{j=0}^{L(n)-|t-s|}\widehat{\pi}_j\widehat{\pi}_{j+|t-s|}-\pi_j\pi_{j+|t-s|}\right)\left(\mathring{\mathbf{i}}_t\mathring{\mathbf{i}}_s-E\left(\mathring{\mathbf{i}}_t\mathring{\mathbf{i}}_s\right)\right)$$

into the left of (106). By standard algebra, it is

$$\frac{1}{n}\sum_{t\leq s}\left\{\left(\sum_{j=0}^{L(n)-|t-s|}\left(\breve{\pi}_j-\overline{\pi}_j\right)\pi_{j+|t-s|}\right)+\left(\sum_{j=0}^{L(n)-|t-s|}\left(\breve{\pi}_{j+|t-s|}-\overline{\pi}_{j+|t-s|}\right)\pi_j\right)\right.$$

$$\left.+\left(\sum_{j=0}^{L(n)-|t-s|}\left(\breve{\pi}_j-\overline{\pi}_j\right)\left(\breve{\pi}_{j+|t-s|}-\overline{\pi}_{j+|t-s|}\right)\right)\right\}\left(\mathring{\mathbf{i}}_t\mathring{\mathbf{i}}_s-E\left(\mathring{\mathbf{i}}_t\mathring{\mathbf{i}}_s\right)\right). \tag{107}$$

By Lemma 1, the third term of (107) is bounded by

$$\left|\widehat{d}_1-d_1\right|^2\frac{1}{n}\sum_{t\leq s}\left\{\left(\sum_{j=0}^{L(n)-|t-s|}\overline{\pi}_j\overline{\pi}_{j+|t-s|}\right)\left(\mathring{\mathbf{i}}_t\mathring{\mathbf{i}}_s-E\left(\mathring{\mathbf{i}}_t\mathring{\mathbf{i}}_s\right)\right)\right\}$$

$$+\left|\widehat{d}_1-d_1\right|^4\frac{1}{n}\sum_{t\leq s}\left(\sum_{j=0}^{L(n)-|t-s|}\frac{1}{j\left(j+|t-s|\right)}\right)E\left|\mathring{\mathbf{i}}_t\mathring{\mathbf{i}}_s\right|$$

which is $o_p(1)$ because $\widehat{d}_1-d_1=O_p\left(m^{-1/2}\right)$ by Proposition 6 and then Condition 4 and Lemma 13. Now, as the first and second terms of (107) are similar, we shall only handle the first one explicitly, which is bounded by

$$\left(\widehat{d}_1-d_1\right)\frac{1}{n}\sum_{t\leq s}\left\{\left(\sum_{j=0}^{L(n)-|t-s|}\overline{\pi}_j\overline{\pi}_{j+|t-s|}\right)\left(\mathring{\mathbf{i}}_t\mathring{\mathbf{i}}_s-E\left(\mathring{\mathbf{i}}_t\mathring{\mathbf{i}}_s\right)\right)\right\}$$

$$+\left|\widehat{d}_1-d_1\right|^2\frac{1}{n}\sum_{t\leq s}\left(\sum_{j=0}^{L(n)-|t-s|}\frac{\overline{\pi}_{j+|t-s|}}{j}\right)E\left|\mathring{\mathbf{i}}_t\mathring{\mathbf{i}}_s\right|.$$

Clearly the first term of the last displayed expression is $o_p(1)$, whereas the second is bounded by

$$\sum_{t=1}^{n}\left(\sum_{j=0}^{L(n)-|t-s|}\frac{\overline{\pi}_{j+t}}{j}\right)O_p\left(m^{-1}\right)=O_p\left(m^{-1}\log n\right)\sum_{t=1}^{n}\overline{\pi}_t$$

$$=O_p\left(n^{d_1}m^{-1}\log n\right)=o_p(1)$$

by Condition 4. This completes the proof of the lemma. ∎

LEMMA 17. *Under Condition 4 and that $\{x_t\}_{t\in\mathbb{Z}}$ is a Gaussian sequence, if $d_x\leq 1/3$, then*

$$\mathcal{J}_n(x)=\frac{1}{d_n}\sum_{t=1}^{n}x_t\mathring{\mathbf{i}}_t(x)$$

is tight, where $d_n=n^{1/2}\mathcal{I}(d_x<1/3)+n^{1/2}/\log^{1/2}n\mathcal{I}(d_x=1/3)$.

PROOF. To that end, we shall show the stronger statement that for all $\epsilon,\eta>0$, there is a $\delta>0$ such that

$$\sum_{\ell\in\mathbb{Z}}\Pr\left\{\sup_{\ell\delta\leq x\leq(\ell+1)\delta}\left|\mathcal{J}_n(x)-\mathcal{J}_n(\ell\delta)\right|>\epsilon\right\}<\eta \tag{108}$$

for large n. We shall consider explicitly only $d_x < 1/3$. To that end, we first notice that due to the properties of Hermite polynomials, we have that

$$E\left|\mathcal{J}_n(y)-\mathcal{J}_n(x)\right|^2 \le K\Xi(y;x)\frac{1}{n}\sum_{t,s=1}^{n}\left|\gamma_x(|t-s|)\right|^3 \quad (109)$$
$$= K\Xi(y;x)$$

because $E\left(\mathring{\mathbf{i}}_t(x)\,x_s\right)=0$ for all $t,s=1,...,n$ and $\left|E\left(\mathring{\mathbf{i}}_t(x)\,\mathring{\mathbf{i}}_2(x)\right)\right| = O\left(\gamma_x(|t-s|)^2\right)$ by Condition 1, where $\Xi(x)=K\{\Phi(x)+\phi(x)\}$. Now using Wu's (2003) Lemma 4, we have that

$$\sup_{\ell\delta\le z\le(\ell+1)\delta}\Xi^2(z) \le \frac{2}{\delta}\int_{\ell\delta}^{(\ell+1)\delta}\Xi^2(z)\,dz + 2\delta\int_{\ell\delta}^{(\ell+1)\delta}\left(\frac{\partial\Xi(z)}{\partial z}\right)^2 dz$$
$$=: \upsilon_\ell(\delta)$$

which implies that $\Xi^2((\ell+1)\delta;\ell\delta) \le \delta^2\upsilon_\ell(\delta)$, where

$$\delta\sum_{\ell\in\mathbb{Z}}\upsilon_\ell(\delta) \le 2\int_{\mathbb{R}}\Xi^2(z)\,dz + 2\delta^2\int_{\mathbb{R}}\left(\frac{\partial\Xi(z)}{\partial z}\right)^2 dz.$$

On the other hand, proceeding as in Ho and Sun (1987), we obtain that

$$E\left(\mathcal{J}_n(y)-\mathcal{J}_n(x)\right)^4 = \frac{K}{n}E\left(\mathring{\mathbf{i}}_t^4(y;x)\,x_t^4\right) + C\left(\frac{1}{n}\sum_{t,s=1}^{n}E\left(\mathring{\mathbf{i}}_t(y;x)\,\mathring{\mathbf{i}}_s(y;x)\,x_t x_s\right)\right)^2$$
$$= \frac{K}{n}\Xi(y;x) + K\Xi^2(y;x).$$

So after we identify x with $(k-1)p+\ell\delta$ and y with $kp+\ell\delta$, and denoting $\mathcal{K}_k = \mathcal{J}_n(kp+\ell\delta)-\mathcal{J}_n((k-1)p+\ell\delta)$, we conclude that

$$E\left(\sum_{k=\iota}^{j}\mathcal{K}_k\right)^4 \le K\left(\sum_{k=\iota}^{j}\nu_k\right)^2,$$

where $\nu_k = p\upsilon_\ell^{1/2}(\delta) + \left(F((k-1)p+\ell\delta;kp+\ell\delta)/n\right)^{1/2}$, and then Billingsley's (1969) Theorem 12.2 implies that

$$\Pr\left(\max_{k=1,\dots,m}|\mathcal{K}_k| \ge \epsilon/8\right) \le \frac{K}{\epsilon^4}\left(\sum_{k=1}^{m}\nu_k\right)^2$$
$$\le K\eta\left(\frac{m}{n}\Xi(\ell\delta;(\ell+1)\delta) + m^2p^2\upsilon_\ell(\delta)\right).$$

So proceeding as usual, see Lemma 14 of Wu (2003) for instance, we conclude that the left side of (108) is bounded by

$$\frac{K}{\epsilon^4}\left(\frac{m}{n}+\delta\sum_{\ell\in\mathbb{Z}}\upsilon_\ell(\delta)\right)$$

by choosing p and m such that $p=\delta/m<\epsilon/\left(8c_0n^{1/2}\right)$, where $m=\left[8\left(\sup_x f_1(x)=:c_0\right)\eta n^{1/2}\right]+1$. Now choose $\delta=\epsilon^4\eta$ to finish the proof. ∎

The next lemma is an extension of Dehling and Taqqu's (1989) uniform reduction principle, which we state for simplicity.

LEMMA 18. *Under Condition* 4 *and that* $\{x_t\}_{t\in\mathbb{Z}}$ *is a standard Gaussian sequence, if* $1/3 < d_x$, *then*

$$\Pr\left\{\sup_x \left|\frac{1}{d_n}\sum_{t=1}^{n} \breve{\mathbf{i}}_t(x)\, x_t\right| > \epsilon\right\} \le Cn^{-\zeta},$$

for some $\zeta > 0$, *where* $d_n = n^{3d_x - 1/2}$ *and* $\breve{\mathbf{i}}_t(x) = \mathring{\mathbf{i}}_t(x) - \frac{1}{2}f'(x)\left(x_t^2 - 1\right)$.

PROOF. We first notice that $f'(x)\left(x_t^2-1\right)$ is nothing but the second term in the Hermite expansion of the indicator function $\mathcal{I}(x_t < x)$. Now denoting $\Xi(x) = K\{\Phi(x) + \phi(x)\}$, orthogonality of the Hermite polynomials imply that

$$\begin{aligned} E\left|\frac{1}{d_n}\sum_{t=1}^{n}\breve{\mathbf{i}}_t(y;x)\, x_t\right|^2 &\le \Xi(y;x)\,\frac{1}{d_n^2}\sum_{t,s=1}^{n}\left|\gamma_x(|t-s|)\right|^4 \\ &= Kn^{-\zeta} \end{aligned}$$

as in of Dehling and Taqqu's (1989) Lemma 3.1. Now the proof proceeds step by step to that of Dehling and Taqqu's (1989) Lemma 3.2 after one partitions $\mathbb{R}$ as

$$-\infty = x_0(k) \le x_1(k) \le \ldots \le x_{2^k}(k) = \infty,\ k = 0, 1, \ldots, K,$$

$x_i(k) = \inf\left\{x : \Lambda(x) \ge \Lambda(\infty)\, i2^{-k}\right\}$, $i = 0, 1, \ldots, 2^k - 1$ and where their $\Lambda(x)$ is replaced by

$$\Lambda(x) = F_x(x) + \int_{-\infty}^{x}\left(w + \frac{w^2-1}{2}\right) f(w)\, dw.$$

∎

REMARK 4. *Lemma 18 indicates that when* $1/3 < d_x$,

$$\sup_{x\in\mathbb{R}}\left|\frac{1}{n}\sum_{t=1}^{n}\mathring{\mathbf{i}}_t(x)\, x_t - \frac{1}{2}f'(x)\,\frac{1}{n}\sum_{t=1}^{n}\left(x_t^2-1\right)x_t\right| = o_p\left(n^{3(d_x - 1/2)}\right).$$

LEMMA 19. *Under Conditions* 2 *to* 4 *and* $d_1 + 2d_x < 1$, *(in probability)* $\mathcal{G}_n^*(x)$ *is tight.*

PROOF. First, denoting $\varsigma_t^*(x) = \left(\mathring{\mathbf{i}}_t(x) - E\left(\mathring{\mathbf{i}}_t(x) \mid \mathcal{F}_{t-1}\right)\right)u_t^*$, we have that

$$\begin{aligned} \mathcal{G}_n^*(x) &= \frac{1}{n^{1/2}}\sum_{t=1}^{n}\varsigma_t^*(x) + \frac{1}{n^{1/2}}\sum_{t=1}^{n}E\left(\mathring{\mathbf{i}}_t \mid \mathcal{F}_{t-1}\right)u_t^* \\ &=: A_n^*(x) + B_n^*(x). \end{aligned}$$

Thus, we need to show that $A_n^*(x)$ and $B_n^*(x)$ are tight. We examine the tightness of $B_n^*(x)$ first. To that end, we notice that

$$\begin{aligned} B_n^*(x) &= \frac{1}{n^{1/2}}\sum_{s=1}^{n+M}\breve{\boldsymbol{\upsilon}}_{s,1}(x)\,\varepsilon_{n+M-s}^* + \frac{1}{n^{1/2}}\sum_{s=1}^{n}\breve{\boldsymbol{\upsilon}}_{s,2}(x)\,\varepsilon_{n+M+s}^* \\ &= B_{n1}^*(x) + B_{n2}^*(x), \end{aligned}$$

where $\breve{\boldsymbol{v}}_{s,1}(x)$ and $\breve{\boldsymbol{v}}_{s,2}(x)$ are as defined in (42) but with x_t there being replaced by $E\left(\mathring{\mathbf{1}}_t(x) \mid \mathcal{F}_{t-1}\right)$. We now adapt the proof of Wu's (2003) Theorem 3. First using Lemma 4 of Wu (2003), we have that

$$E^* \sup_{x\in\mathbb{R}} \left|\frac{\partial B^*_{n2}(x)}{\partial x}\right|^2 \leq 2\int_{\mathbb{R}} E^* \left|\frac{\partial B^*_{n2}(x)}{\partial x}\right|^2 dx + 2\int_{\mathbb{R}} E^* \left|\frac{\partial^2 B^*_{n2}(x)}{\partial^2 x}\right|^2 dx.$$

Next because ε^*_t are iid$\left(0, \widehat{\sigma}^2_\varepsilon\right)$, we have that

$$\int_{\mathbb{R}} E^* \left|\frac{\partial B^*_{n2}(x)}{\partial x}\right|^2 dx = \frac{1}{n}\sum_{s=1}^{n}\int_{\mathbb{R}} \left|\sum_{t=1+s}^{n} \frac{\partial E\left(\mathring{\mathbf{1}}_t(x) \mid \mathcal{F}_{t-1}\right)}{\partial x}\widehat{\vartheta}_{t-s}\right|^2 dx.$$

Now, arguing as we did with $\int_{\mathbb{R}} E\left|\frac{\partial B_{n2}(x)}{\partial x}\right|^2 dx$ in Lemma 12, we obtain that the right side of the last displayed expression is bounded by

$$\frac{K}{n}\sum_{s=1}^{n}\sum_{j=-\infty}^{n}\left(\sum_{t=s+\max(1,j)}^{n}\varphi^{3/2}_{t-j+1}\widehat{\vartheta}_{t-s-j+1}\right)^2,$$

where, taking for simplicity $b_\ell = 0$, Lemma 1 implies that it is

$$\frac{K}{n}\sum_{s=1}^{n}\sum_{j=-\infty}^{n}\left(\sum_{t=s+\max(1,j)}^{n}\varphi^{3/2}_{t-j+1}\vartheta_{t-s-j+1}\right)^2 (1+o_p(1)) = o_p(1)$$

as we argued in Lemma 1 because $1-2d_x-d_1>0$. Similarly $\int_{\mathbb{R}} E^*\left|\frac{\partial^2 B^*_{n2}(x)}{\partial^2 x}\right|^2 dx = o_p(1)$. So, because

$$|B^*_{n2}(x) - B^*_{n2}(y)| \leq |x-y| \sup_{z\in\mathbb{R}}\left|\frac{\partial B^*_{n2}(z)}{\partial z}\right| = |x-y|\, K_n,$$

we conclude that $B^*_{n2}(x)$ is tight.

Finally, the tightness of

$$B^*_{n1}(x) =: B^*_{n1,1}(x) + B^*_{n1,2}(x),$$

where

$$B^*_{n1,1}(x) = \frac{1}{n^{1/2}}\sum_{s=1}^{M}\left(\sum_{t=1}^{n}\widehat{\vartheta}_{t+s}x_t\right)\varepsilon^*_{n+M-s}$$
$$B^*_{n1,2}(x) = \frac{1}{n^{1/2}}\sum_{s=M+1}^{n+M}\left(\sum_{t=1}^{n+M-s+1}\widehat{\vartheta}_{t+s}x_t\right)\varepsilon^*_{n+M-s}$$

proceeds similarly to that of $B^*_{n2}(x)$, with the only difference that we now have, say, that

$$\int_{\mathbb{R}} E^*\left|\frac{\partial B^*_{n1,2}(x)}{\partial x}\right|^2 dx = \frac{K}{n}\sum_{s=M+1}^{n+M}\sum_{j=-\infty}^{n}\left(\sum_{t=\max(1,j)}^{n}\varphi^{3/3}_{t-j+1}\widehat{\vartheta}_{t+s}\right)^2 = O_p(1),$$

using Lemma 1.

Next the tightness of $A_n^*(x)$. To that end, we first use the same decomposition as with $B_n^*(x)$, that is

$$A_n^*(x) = \frac{1}{n^{1/2}}\sum_{s=1}^{n+M}\breve{\boldsymbol{v}}_{s,1}(x)\,\varepsilon_{n+M-s}^* + \frac{1}{n^{1/2}}\sum_{s=1}^{n}\breve{\boldsymbol{v}}_{s,2}(x)\,\varepsilon_{n+M+s}^*$$
$$= A_{n1}^*(x) + A_{n2}^*(x),$$

where $\breve{\boldsymbol{v}}_{s,1}(x)$ and $\breve{\boldsymbol{v}}_{s,2}(x)$ are as defined in (42) but with x_t there being replaced by $\breve{\mathbf{1}}_t(x) =: \mathring{\mathbf{1}}_t(x) - E\left(\mathring{\mathbf{1}}_t(x) \mid \mathcal{F}_{t-1}\right)$. Because $\breve{\boldsymbol{v}}_{s,1}(x)\,\varepsilon_{n+M-s}^*$ and $\breve{\boldsymbol{v}}_{s,2}(x)\,\varepsilon_{n+M+s}^*$ are independent zero mean sequences, we have that

$$E^*A_{n1}^*(y;x)^4 = H_n\left(\frac{1}{n}\breve{\boldsymbol{v}}_{s,1}^4(y;x) + \left(\frac{1}{n}\sum_{s=1}^{n+M}\breve{\boldsymbol{v}}_{s,1}^2(y;x)\right)^2\right)$$
$$E^*A_{n2}^*(y;x)^4 = H_n\left(\frac{1}{n}\breve{\boldsymbol{v}}_{s,2}^4(y;x) + \left(\frac{1}{n}\sum_{s=1}^{n}\breve{\boldsymbol{v}}_{s,2}^2(y;x)\right)^2\right)$$

since $E^*\varepsilon_t^{*4} = H_n = O_p(1)$. Now the proof proceeds after we notice that Lemma 1 yields

$$\frac{1}{n}\sum_{s=1}^{n}\breve{\boldsymbol{v}}_{s,2}^2(y;x) = \frac{1}{n}\sum_{s=1}^{n}\left(\sum_{t=s+1}^{n}\vartheta_{t-s}\breve{\mathbf{1}}_t(y;x)\right)^2(1+o_p(1))$$

and that $\breve{\mathbf{1}}_t(x)$ is a martingale difference with respect the $\sigma-$algebra generated by $\mathcal{F}_t$, it implies that $E\left(\sum_{t=s+1}^{n}\vartheta_{t-s}\breve{\mathbf{1}}_t(y;x)\right)^2 = O(|y-x|)$ and hence

$$E^*A_{n2}^*(y;x)^4 = \left(\frac{1}{n}\breve{\mathbf{1}}_t(x,y) + (y-x)^2\right)K_n.$$

Then since $E\breve{\mathbf{1}}_t(x,y)^2 < K(F_x(y) - F_x(x))$, proceeding as in Lemma 17, i.e. Theorems 12.2 and 15.5 of Billingsley (1968), we conclude that

$$\textit{Pr}^*\left\{\sup_{|x-y|<\delta}|A_{n2}^*(x;y)| > \epsilon\right\} < \eta K_n. \tag{110}$$

Similarly, we can conclude that $\textit{Pr}^*\left\{\sup_{|x-y|<\delta}|A_{n1}^*(x;y)| > \epsilon\right\} < \eta K_n$. Hence $A_n^*(x)$ is also tight so it is $\mathcal{G}_n^*(x)$. ∎

1.3. *APPENDIX C*.

We now present a Monte Carlo experiment to shed some light on the behaviour of our test, and in particular how well the bootstrap algorithm performs in finite samples even in cases where we do not have formal theoretical results, i.e. when $2d_x + d_1 > 1$ or $d_x + d_1 > 1/2$ and x_t is a non-Gaussian sequence.

To address the performance under the null hypothesis, we have generated the linear regression model

$$y_t = \alpha + \beta x_t + u_t,\ t = 1, ..., n, \tag{111}$$

where $\alpha = \beta = 1$ for three different sample sizes $n = 128, 256, 512$ and for different combinations of d_x and d_1. All throughout the errors $\{u_t\}_{t=1}^n$ were generated as a sequence of Gaussian random variables with mean 0, and we have chosen the 16 different combinations

$d_1, d_x = 0.1, .., 0.4$. For each sample size and each combination of d_1 and d_x, we have simulated (111) when the regressor $\{x_t\}_{t=1}^{n}$ were generated as a linear sequence for two sets of innovations, i.e. when $\{\varrho_t\}_{t=1}^{n}$ is standard Gaussian or a χ_1^2 centered around its mean. The reason to consider these two scenarios is due to the different theoretical results that we have obtained in Propositions 1 and 2, so we can address the question of how sensitive the distributional conditions of the regressor is on the performance of the test. The statistic $\mathcal{T}_n(x)$ were computed in the range $x \in [-1.0, 1.0]$ with a mesh width of 0.1 and we have chosen the *Kolmogorov's* type of functional for $\varphi(\cdot)$. That is,

$$\mathcal{KS}_n = \max_{\ell=1,\dots,21} \left| n^{1/2} \mathcal{T}_n(x_\ell) \right|,$$

where $\{x_\ell\}_{\ell=1}^{21}$, $x_\ell = -1.0 + (\ell - 1)\, 0.1$.

In order to save computational time, for each sample we compute only one bootstrap counterpart according to Section 3 and equations (3.1) and (3.2). The stacked bootstrapped statistics are then used to construct critical values and confidence regions at appropriate levels. For each combination of models and/or samples sizes n, 1000 iterations were performed. This is the idea behind the WARP algorithm of Giacomini et al. (2013). Finally, to implement the bootstrap algorithm we need to choose the smoothing parameter m. Although an algorithm as that described at then end of the previous section can be implemented, in this Monte-Carlo experiment we have considered two different choices of m, namely $m = n/4$ and $m = n/8$. Likewise in the expression $\widehat{C}(\lambda) = \exp\left\{\sum_{r=1}^{[n/4m]} \widehat{c}_r e^{-ir\lambda}\right\}$, we have chosen $\widehat{c}_r = 0$ for $r \geq 1$ and the case $\widehat{c}_r = 0$ for $r > 1$ with $\widehat{c}_1 \neq 0$. The first scenario uses the fact that we know that there is no SM component whereas in the second we have taken $[n/4m] = 1$, after we notice that in almost all cases $[n/4m] \leq 1$. Finally, in all the tables, the first row in each cell presents the results of the test for the 10% size whereas the second row are those for the 5% size.

TABLES 1 TO 4

A general conclusion that we can draw from *Tables 1* to *4* is the good performance of the test even for samples sizes as small as $n = 128$. This performance is regardless the distribution of the regressor x_t and the choice of $m = n/4$ appears to perform slightly better for moderate sample sizes, i.e. when $n = 128$. Also the tables suggests that even when we choose $\widehat{c}_1 \neq 0$, there is no visible deterioration of the finite sample performance when compared to the case of $\widehat{c}_1 = 0$. Another conclusion that we can draw from the above tables is that the bootstrap appears to approximate the finite sample distribution of the test even in scenarios for which we do not have formal theoretical results, say when $d_x = 0.4$ and $d_1 = 0.3$ or 0.4 with Gaussian regressor or when $d_x + d_1 > 1/2$ when we have a centered χ_1^2 distribution.

To address the power of the test we have simulated the following two alternative regression models

(112) $$y_t = \alpha + \beta x_t + \gamma x_t^2 + u_t,\ t = 1, ..., n$$

(113) $$y_t = \alpha + \beta x_t + \gamma \sin(x_t) + u_t,\ t = 1, ..., n$$

with $\gamma = 0.5$ and 1.5. The second model being more difficult to detect as the function $\sin(x_t)$ is periodic and bounded by 1 and so its variability is smaller than that compared to a regressor of the type x_t^2. That is the signal/noise ratio for the second model is far smaller than for the first. We present the results of the Monte Carlo experiment in *Tables 5-16* below.

TABLES 5 TO 16

We should caution that when we consider the alternative model in (112), we only present the results for $\gamma = 0.5$, as the power of the test for all different combinations had always a 100% rejection rate regardless whether x_t is or it is not Gaussian and for all combinations of d_1/d_x. The results for (113) illustrate a very good power performance although smaller than that obtained for (112). However this is somehow expected as the power depends among other issues on the "distance" between the null and alternative, being bigger for model (112) than for model (113).

TABLE 1
Size when x_t is Gaussian and $\widehat{C}(\lambda) = 1$

				d_1=.1						d_1=.2		
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	8.5	5.0	10.4	6.2	9.2	9.9	7.8	6.2	10.0	8.9	10.5	7.8
	4.5	1.4	5.7	3.0	3.8	6.0	4.0	3.5	5.3	3.6	5.2	4.0
.2	11.1	7.8	8.4	9.0	12.2	8.8	10.1	9.8	8.5	7.6	11.9	10.6
	5.8	3.9	4.7	3.9	5.6	4.2	4.8	4.9	5.2	3.8	5.8	5.9
.3	9.9	8.8	8.8	11.3	9.0	10.8	10.0	10.3	10.1	9.8	8.7	7.8
	5.0	4.6	3.7	4.5	5.4	4.6	5.3	6.0	4.7	3.8	3.9	4.2
.4	12.1	10.0	9.9	9.1	7.6	12.3	9.3	7.0	8.2	9.5	9.7	8.0
	6.2	5.1	4.9	4.6	3.9	5.2	4.5	3.1	4.3	4.7	4.7	5.1
				d_u=.3						d_u=.4		
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	7.8	7.2	8.8	8.8	8.6	7.9	10.1	6.7	7.7	6.9	11.8	7.9
	3.6	4.2	4.5	3.9	4.0	4.8	5.5	3.1	3.5	3.4	4.6	3.3
.2	8.5	7.1	8.6	7.6	14.2	7.2	8.1	7.9	9.5	8.0	10.2	8.6
	5.2	3.4	4.3	3.0	6.6	4.3	4.4	3.6	5.6	3.7	5.3	4.4
.3	9.1	8.5	9.8	6.9	9.3	8.9	9.4	8.3	10.5	7.8	9.8	8.0
	5.1	3.4	5.3	1.8	5.6	3.0	5.0	3.6	5.9	4.0	5.5	3.2
.4	9.8	6.3	7.9	6.8	9.6	10.7	10.9	9.3	9.0	8.0	10.4	9.2
	3.7	3.0	3.4	2.8	4.7	5.8	5.5	3.3	4.0	4.1	5.1	4.5

TABLE 2

Size when x_t is χ^2_2 and $\widehat{C}(\lambda) = 1$

	$d_1=.1$						$d_1=.2$					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	13.1	11.0	11.3	10.7	8.0	12.6	9.3	7.4	14.1	10.2	11.4	8.0
	6.2	4.0	5.6	4.8	3.7	6.5	5.0	3.1	5.9	5.2	3.7	2.9
.2	9.3	9.4	9.9	9.2	11.6	8.2	9.0	9.9	9.6	9.7	9.9	8.9
	3.8	4.8	4.5	4.9	5.7	3.8	3.7	4.7	5.6	3.4	6.2	3.7
.3	12.9	9.3	9.0	10.5	10.1	10.0	8.3	8.5	10.2	9.5	8.2	9.5
	5.8	4.1	5.8	5.4	4.1	7.4	3.7	3.6	4.9	4.7	4.9	5.6
.4	7.7	8.0	8.8	8.6	9.9	10.8	11.2	9.1	9.4	9.9	10.0	10.5
	3.5	4.1	4.6	4.2	3.3	3.8	4.7	4.2	4.8	5.9	5.8	5.0

	$d_u=.3$						$d_u=.4$					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	8.9	8.3	10.3	9.3	10.6	10.8	11.1	8.9	7.2	8.7	9.7	10.4
	5.0	3.7	5.2	4.1	6.5	4.6	5.2	3.1	4.6	4.6	5.4	4.6
.2	9.8	8.5	10.7	9.2	9.8	11.2	6.9	8.4	9.8	8.5	8.9	8.1
	5.8	3.9	5.4	3.0	5.5	3.7	3.4	3.0	4.6	4.2	4.5	3.4
.3	8.8	7.0	7.9	8.9	11.0	8.4	8.9	7.2	9.1	10.0	10.1	6.9
	4.1	3.8	4.6	4.4	6.6	3.3	4.4	2.7	3.7	4.9	4.8	2.4
.4	10.5	8.6	11.1	6.5	10.8	8.3	10.1	11.0	10.1	10.0	9.8	8.3
	5.1	2.9	5.4	3.6	5.5	4.6	4.2	2.9	5.9	5.1	5.0	3.7

TABLE 3

Size when x_t is Gaussian and $\widehat{C}(\lambda) = \exp\left\{\widehat{c}_1 e^{-ir\lambda}\right\}$

	d_1=.1						d_1=.2					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	10.4	5.9	7.4	7.7	9.9	9.5	12.5	9.8	9.8	12.7	11.1	9.1
	6.2	2.2	3.8	3.9	5.0	3.7	6.1	6.1	4.9	7.1	6.9	3.5
.2	9.0	7.2	11.5	6.3	9.5	5.9	11.1	8.0	9.8	8.1	10.2	8.5
	4.5	3.2	7.4	3.5	5.5	2.0	4.7	4.2	6.5	4.1	4.9	4.5
.3	10.8	7.6	9.0	9.0	9.4	5.6	11.8	8.1	10.1	10.8	14.2	8.6
	4.9	3.7	4.8	4.1	4.0	2.4	6.6	4.0	6.3	5.4	7.5	4.8
.4	8.8	9.8	11.5	9.3	12.4	7.2	12.4	11.7	8.8	8.8	10.7	12.8
	4.7	4.4	6.0	4.0	5.4	3.1	6.8	5.5	4.5	5.0	4.5	7.3

	d_u=.3						d_u=.4					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	10.3	8.3	11.5	12.0	13.4	13.9	14.3	12.0	11.1	16.1	13.4	11.8
	5.4	4.3	5.7	4.6	6.9	7.5	6.7	6.7	5.9	8.0	5.0	4.7
.2	10.8	9.4	10.5	11.7	12.7	10.5	9.9	14.1	12.1	12.5	12.5	13.3
	5.6	5.8	5.4	5.8	5.6	5.9	5.1	6.1	6.0	6.8	6.8	6.4
.3	10.1	10.1	12.5	10.5	14.7	12.7	12.7	11.9	11.7	13.8	14.2	13.7
	3.3	4.2	5.9	5.6	6.1	9.0	7.0	5.1	6.4	7.7	6.9	6.0
.4	12.2	10.2	11.2	14.9	14.1	13.9	12.9	14.3	13.2	14.1	12.5	15.6
	5.5	5.1	6.4	7.9	7.4	7.6	6.5	6.7	7.0	8.4	6.1	8.4

TABLE 4

Size when x_t is χ_2^2 and $\widehat{C}(\lambda) = \exp\left\{\widehat{c}_1 e^{-ir\lambda}\right\}$

	d_1=.1						d_1=.2					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	8.1	8.7	8.7	9.0	9.1	8.1	10.6	11.4	11.8	10.0	9.7	8.8
	4.1	4.0	3.6	3.9	4.1	3.3	6.2	5.7	6.5	5.5	4.2	6.5
.2	9.8	8.5	9.4	7.8	8.9	9.0	9.9	8.6	10.1	9.1	9.3	11.7
	6.3	2.9	3.9	4.4	5.0	4.8	5.1	4.3	6.3	3.6	5.2	5.2
.3	8.4	9.4	10.1	10.5	11.0	9.5	9.8	10.4	12.0	8.2	10.3	11.8
	4.8	4.7	4.7	7.0	5.8	3.7	4.7	6.4	7.0	4.6	5.0	6.7
.4	10.7	8.0	11.4	8.8	11.0	9.2	10.9	8.5	13.3	10.4	12.7	11.7
	5.7	3.6	5.8	4.4	7.0	3.5	5.1	3.4	6.0	5.7	7.5	5.8

	d_u=.3						d_u=.4					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	10.9	8.2	11.4	12.6	12.6	10.0	13.8	10.9	11.3	16.0	11.0	11.6
	6.5	4.1	5.4	7.0	7.0	4.3	6.1	5.2	6.1	9.9	6.6	5.9
.2	13.7	11.5	12.2	10.2	12.8	10.9	12.3	11.9	12.4	14.5	15.2	13.3
	6.0	5.7	6.3	4.9	7.1	6.2	5.4	5.9	6.5	7.3	8.6	7.9
.3	8.4	10.8	12.3	10.6	12.2	12.7	13.4	12.3	12.1	12.1	15.7	12.0
	3.6	6.2	7.0	6.0	6.5	6.6	5.3	6.2	7.1	6.5	8.3	6.4
.4	13.4	11.2	11.3	11.0	11.9	15.2	10.1	13.1	15.2	16.5	15.0	15.6
	7.0	5.9	5.2	6.3	5.8	7.7	5.2	6.1	8.5	8.0	7.0	7.8

TABLE 5

Power when x_t is Gaussian for model (112)
and $\widehat{C}(\lambda) = 1$

			d$_1$=.1						d$_1$=.2			
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	98.9	97.6	100	100	100	100	99.1	97.7	100	100	100	100
	96.7	95.0	100	100	100	100	95.6	93.3	100	100	100	100
.2	98.9	97.6	100	100	100	100	98.8	98.8	100	100	100	100
	97.0	96.1	100	99.9	100	100	96.6	95.5	100	100	100	100
.3	99.0	99.1	100	100	100	100	99.1	97.4	100	100	100	100
	97.0	97.6	100	100	100	100	97.4	95.9	100	100	100	100
.4	99.6	98.4	100	100	100	100	98.7	99.1	100	100	100	100
	97.2	96.9	100	100	100	100	97.7	95.3	100	100	100	100

			d$_1$=.3						d$_1$=.4			
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	98.4	97.1	100	100	100	100	95.1	93.4	99.8	100	100	100
	95.2	92.7	99.9	99.8	100	100	90.4	83.3	99.5	99.1	100	100
.2	98.0	96.4	100	100	100	100	95.8	94.4	100	100	100	100
	95.1	93.0	100	100	100	100	91.4	86.8	99.7	99.8	100	100
.3	98.0	98.5	100	100	100	100	95.9	95.7	100	100	100	100
	95.1	94.7	100	100	100	100	91.1	87.6	99.9	99.8	100	100
.4	97.8	98.4	100	99.9	100	100	96.7	96.7	100	100	100	100
	96.0	95.8	100	99.9	100	100	94.6	92.5	100	99.8	100	100

TABLE 6
Power when x_t is Gaussian for model (113) with $\gamma = 0.5$
and $\widehat{C}(\lambda) = 1$

	d₁=.1						d₁=.2					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	20.9	20.3	36.5	31.6	55.5	53.3	19.9	17.4	35.9	29.3	53.8	52.9
	12.1	12.0	24.8	20.3	46.0	39.2	8.5	8.7	24.5	18.9	43.4	40.0
.2	26.2	18.8	40.5	38.7	61.3	60.4	23.5	17.3	40.3	32.8	59.3	61.2
	12.5	11.2	29.4	28.1	48.5	43.7	13.5	8.2	26.9	18.8	42.1	45.3
.3	29.0	24.9	44.2	50.2	71.3	70.9	25.9	21.3	44.8	44.7	70.0	70.4
	18.7	15.7	31.6	38.1	59.3	59.8	16.0	12.0	29.6	32.9	57.3	60.0
.4	35.6	31.5	60.0	64.1	87.9	89.9	37.3	34.0	60.2	52.9	85.9	85.5
	22.5	18.9	49.7	50.4	78.8	85.3	26.9	21.6	43.4	40.2	73.7	77.7

	d₁=.3						d₁=.4					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	20.0	11.1	29.0	25.5	46.1	41.8	18.7	15.9	21.9	18.3	41.5	34.2
	11.8	6.6	18.8	15.7	29.8	30.5	10.8	8.2	12.3	10.7	27.7	22.3
.2	21.2	19.5	30.2	26.3	51.5	48.0	16.1	15.9	28.1	23.8	40.9	42.6
	12.4	10.3	19.2	19.2	41.4	37.1	9.0	9.2	15.0	13.2	27.2	26.4
.3	26.9	20.8	40.2	34.1	65.7	62.5	19.0	17.8	32.9	24.8	52.8	48.3
	15.5	11.4	28.8	21.4	53.4	47.9	12.3	10.3	20.8	16.7	37.4	33.2
.4	28.1	23.1	51.5	52.2	83.0	78.2	26.6	24.0	41.8	41.4	71.2	68.8
	16.3	16.4	35.9	36.3	72.5	64.3	17.4	14.2	29.0	25.2	54.3	51.9

TABLE 7
Power when x_t is Gaussian for model (113) with $\gamma = 1.5$
and $\widehat{C}(\lambda) = 1$

	d$_1$=.1						d$_1$=.2					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	77.1	73.2	96.2	95.0	100	100	74.8	69.9	94.6	95.7	99.7	100
	67.0	57.0	93.7	90.5	100	99.9	61.8	57.2	91.8	92.1	99.6	99.8
.2	80.0	76.6	97.2	97.5	99.9	100	79.0	75.9	96.2	97.4	100	100
	71.9	58.3	94.5	95.7	99.8	99.9	69.4	60.1	92.0	93.6	100	99.9
.3	86.0	86.8	99.5	99.2	100	100	83.5	82.0	99.5	98.9	100	100
	78.3	77.2	99.0	98.2	100	99.9	75.8	72.1	97.1	96.9	100	100
.4	91.9	91.7	99.9	99.6	100	100	89.9	88.1	99.7	99.4	100	100
	87.7	86.5	99.6	99.2	100	100	82.6	80.7	99.6	98.5	100	100

	d$_1$=.3						d$_1$=.4					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	71.3	63.5	92.6	92.4	99.8	99.8	58.5	59.6	87.7	85.4	99.7	99.2
	58.2	50.2	84.4	86.5	99.7	99.4	43.0	43.6	81.0	75.3	98.6	98.4
.2	72.1	68.2	97.4	93.0	99.9	100	69.9	59.9	91.4	89.2	100	99.7
	57.2	56.2	92.4	85.3	99.7	99.8	55.2	42.5	82.5	78.2	99.0	98.9
.3	83.9	78.5	98.9	97.9	100	100	76.0	71.3	97.4	93.7	100	99.9
	71.1	64.6	96.9	95.3	99.9	99.9	60.9	54.4	93.2	89.9	100	99.8
.4	89.4	86.2	99.7	99.3	100	100	86.3	83.9	97.8	98.5	100	99.9
	82.5	74.9	98.9	98.4	100	100	81.2	72.8	96.1	96.2	100	99.8

TABLE 8

Power when x_t is Gaussian for model (112) and $\widehat{C}\left(\lambda\right)=\exp\left\{\widehat{c}_1 e^{-ir\lambda}\right\}$

	$d_1=.1$						$d_1=.2$					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	98.7	98.9	100	100	100	100	98.8	98.2	100	100	100	100
	97.0	96.1	100	100	100	100	95.1	95.1	99.9	100	100	100
.2	98.6	98.5	100	100	100	100	98.9	99.0	100	100	100	100
	96.0	95.6	100	100	100	100	97.5	96.2	100	100	100	100
.3	99.2	98.8	100	100	100	100	98.8	99.0	100	100	100	100
	97.3	97.3	100	100	100	100	96.6	96.5	100	100	100	100
.4	99.3	99.1	100	100	100	100	99.1	99.0	100	100	100	100
	97.5	98.1	100	99.9	100	100	98.0	97.3	100	100	100	100

	$d_1=.3$						$d_1=.4$					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	98.4	97.0	100	100	100	100	97.7	96.7	100	100	100	100
	96.1	94.1	100	99.9	100	100	94.5	92.4	99.7	100	100	100
.2	98.8	98.3	100	100	100	100	97.3	96.2	100	100	100	100
	96.9	95.4	100	100	100	100	94.7	91.5	99.9	100	100	100
.3	98.0	98.5	100	100	100	100	98.6	96.9	100	100	100	100
	95.1	94.7	100	100	100	100	96.3	93.4	100	100	100	100
.4	97.9	97.9	100	100	100	100	97.3	97.8	100	100	100	100
	95.4	95.2	100	99.8	100	100	95.3	95.0	100	100	100	100

TABLE 9
Power when x_t is Gaussian for model (113) with $\gamma = 0.5$
and $\widehat{C}(\lambda) = \exp\left\{\widehat{c}_1 e^{-ir\lambda}\right\}$

				d_1=.1						d_1=.2		
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	20.9	20.3	36.5	31.6	55.5	53.3	19.9	17.4	35.9	29.3	53.8	52.9
	12.1	12.0	24.8	20.3	46.0	39.2	8.5	8.7	24.5	18.9	43.4	40.0
.2	26.2	18.8	40.5	38.7	61.3	60.4	23.5	17.3	40.3	32.8	59.3	61.2
	12.5	11.2	29.4	28.1	48.5	43.7	13.5	8.2	26.9	18.8	42.1	45.3
.3	29.0	24.9	44.2	50.2	71.3	70.9	25.9	21.3	44.8	44.7	70.0	70.4
	18.7	15.7	31.6	38.1	59.3	59.8	16.0	12.0	29.6	32.9	57.3	60.0
.4	35.6	31.5	60.0	64.1	87.9	89.9	37.3	34.0	60.2	52.9	85.9	85.5
	22.5	18.9	49.7	50.4	78.8	85.3	26.9	21.6	43.4	40.2	73.7	77.7

				d_1=.3						d_1=.4		
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	20.0	11.1	29.0	25.5	46.1	41.8	18.7	15.9	21.9	18.3	41.5	34.2
	11.8	6.6	18.8	15.7	29.8	30.5	10.8	8.2	12.3	10.7	27.7	22.3
.2	21.2	19.5	30.2	26.3	51.5	48.0	16.1	15.9	28.1	23.8	40.9	42.6
	12.4	10.3	19.2	19.2	41.4	37.1	9.0	9.2	15.0	13.2	27.2	26.4
.3	26.9	20.8	40.2	34.1	65.7	62.5	19.0	17.8	32.9	24.8	52.8	48.3
	15.5	11.4	28.8	21.4	53.4	47.9	12.3	10.3	20.8	16.7	37.4	33.2
.4	28.1	23.1	51.5	52.2	83.0	78.2	26.6	24.0	41.8	41.4	71.2	68.8
	16.3	16.4	35.9	36.3	72.5	64.3	17.4	14.2	29.0	25.2	54.3	51.9

TABLE 10
Power when x_t is Gaussian for model (113) with $\gamma = 1.5$
and $\widehat{C}(\lambda) = \exp\left\{\widehat{c}_1 e^{-ir\lambda}\right\}$

		d_1=.1			d_1=.2	
	n=128	n=256	n=512	n=128	n=256	n=512
d_x	m= $\frac{n}{4}$ $\frac{n}{8}$	$\frac{n}{4}$ $\frac{n}{8}$	$\frac{n}{4}$ $\frac{n}{8}$	$\frac{n}{4}$ $\frac{n}{8}$	$\frac{n}{4}$ $\frac{n}{8}$	$\frac{n}{4}$ $\frac{n}{8}$
.1	77.1 73.2 67.0 57.0	96.2 95.0 93.7 90.5	100 100 100 99.9	74.8 69.9 61.8 57.2	94.6 95.7 91.8 92.1	99.7 100 99.6 99.8
.2	80.0 76.6 71.9 58.3	97.2 97.5 94.5 95.7	99.9 100 99.8 99.9	79.0 75.9 69.4 60.1	96.2 97.4 92.0 93.6	100 100 100 99.9
.3	86.0 86.8 78.3 77.2	99.5 99.2 99.0 98.2	100 100 100 99.9	83.5 82.0 75.8 72.1	99.5 98.9 97.1 96.9	100 100 100 100
.4	91.9 91.7 87.7 86.5	99.9 99.6 99.6 99.2	100 100 100 100	89.9 88.1 82.6 80.7	99.7 99.4 99.6 98.5	100 100 100 100

		d_1=.3			d_1=.4	
	n=128	n=256	n=512	n=128	n=256	n=512
d_x	m= $\frac{n}{4}$ $\frac{n}{8}$	$\frac{n}{4}$ $\frac{n}{8}$	$\frac{n}{4}$ $\frac{n}{8}$	$\frac{n}{4}$ $\frac{n}{8}$	$\frac{n}{4}$ $\frac{n}{8}$	$\frac{n}{4}$ $\frac{n}{8}$
.1	71.3 63.5 58.2 50.2	92.6 92.4 84.4 86.5	99.8 99.8 99.7 99.4	58.5 59.6 43.0 43.6	87.7 85.4 81.0 75.3	99.7 99.2 98.6 98.4
.2	72.1 68.2 57.2 56.2	97.4 93.0 92.4 85.3	99.9 100 99.7 99.8	69.9 59.9 55.2 42.5	91.4 89.2 82.5 78.2	100 99.7 99.0 98.9
.3	83.9 78.5 71.1 64.6	98.9 97.9 96.9 95.3	100 100 99.9 99.9	76.0 71.3 60.9 54.4	97.4 93.7 93.2 89.9	100 99.9 100 99.8
.4	89.4 86.2 82.5 74.9	99.7 99.3 98.9 98.4	100 100 100 100	86.3 83.9 81.2 72.8	97.8 98.5 96.1 96.2	100 99.9 100 99.8

TABLE 11
Power when x_t is χ^2_2 for model (112)
and $\widehat{C}(\lambda) = 1$

	d_1=.1						d_1=.2					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	99.9	99.9	100	100	100	100	99.6	99.8	100	100	100	100
	98.0	98.6	100	100	100	100	99.1	98.1	100	100	100	100
.2	99.9	99.6	100	100	100	100	100	99.8	100	100	100	100
	98.3	98.0	100	100	100	100	98.5	98.8	100	100	100	100
.3	99.7	99.7	100	100	100	100	99.6	99.3	100	100	100	100
	98.6	98.5	100	100	100	100	98.8	98.1	100	100	100	100
.4	99.2	99.3	100	100	100	100	99.3	99.1	100	100	100	100
	96.3	96.0	100	100	100	100	96.6	96.0	100	100	100	100

	d_1=.3						d_1=.4					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	100	99.9	100	100	100	100	100	99.7	100	100	100	100
	99.4	98.9	100	100	100	100	98.0	98.3	100	100	100	100
.2	99.7	99.7	100	100	100	100	99.8	99.6	100	100	100	100
	98.1	98.6	100	100	100	100	98.5	98.4	100	100	100	100
.3	99.8	99.6	100	100	100	100	99.6	99.6	100	100	100	100
	97.8	97.9	100	100	100	100	98.5	97.1	100	100	100	100
.4	99.4	98.8	100	100	100	100	98.9	98.9	100	100	100	100
	98.1	95.0	100	100	100	100	95.8	94.4	100	100	100	100

TABLE 12
Power when x_t is χ_2^2 for model (113) with $\gamma = 0.5$
and $\widehat{C}(\lambda) = 1$

				d$_1$=.1						d$_1$=.2		
		n=128		n=256		n=512		n=128		n=256		n=512
d$_x$	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	70.6	70.4	95.4	95.3	100	100	77.6	67.2	96.7	94.7	99.9	100
	58.3	57.3	88.5	91.4	99.9	99.9	64.4	52.4	91.7	88.4	99.8	99.8
.2	72.6	72.0	93.9	96.5	100	100	73.1	65.6	94.2	93.4	100	99.9
	58.2	53.5	88.5	91.9	99.9	99.9	58.7	53.4	88.0	88.8	99.4	99.6
.3	77.6	72.0	96.5	95.5	100	100	67.9	70.4	94.1	93.4	99.9	100
	64.2	60.6	91.6	92.8	99.9	99.8	53.0	57.1	91.6	87.0	99.9	99.5
.4	77.6	73.7	96.9	96.7	99.9	100	72.0	71.6	96.1	95.0	100	100
	65.1	56.2	93.4	94.0	99.9	99.9	58.8	56.0	90.5	87.8	100	100

				d$_1$=.3						d$_1$=.4		
		n=128		n=256		n=512		n=128		n=256		n=512
d$_x$	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	66.5	64.3	92.3	92.3	99.9	99.2	60.9	59.2	89.8	85.1	98.9	98.5
	55.6	51.8	83.8	85.6	99.6	97.8	45.9	43.8	77.0	72.7	96.8	94.9
.2	69.8	63.1	92.3	89.6	99.7	99.6	63.7	54.4	81.8	84.0	98.8	98.1
	56.3	46.3	87.0	79.9	99.4	99.1	43.8	37.8	72.0	70.9	96.7	93.6
.3	59.2	63.8	92.6	86.6	99.8	99.7	58.5	53.7	83.8	81.9	98.7	96.7
	43.9	45.8	85.7	74.3	99.7	98.9	40.4	38.8	69.4	66.1	97.1	89.5
.4	70.8	63.1	90.2	91.4	99.7	99.8	60.9	60.7	83.6	78.9	97.2	96.9
	56.5	42.8	81.9	84.5	99.3	99.1	42.4	42.1	72.6	68.2	93.1	93.5

TABLE 13
Power when x_t is χ_2^2 for model (113) with $\gamma = 1.5$
and $\widehat{C}\,(\lambda) = 1$

			d_1=.1							d_1=.2		
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	100	100	100	100	100	100	100	100	100	100	100	100
	100	100	100	100	100	100	100	100	100	100	100	100
.2	100	100	100	100	100	100	100	100	100	100	100	100
	100	100	100	100	100	100	100	100	100	100	100	100
.3	100	100	100	100	100	100	100	100	100	100	100	100
	100	100	100	100	100	100	100	100	100	100	100	100
.4	100	100	100	100	100	100	100	100	100	100	100	100
	100	100	100	100	100	100	100	100	100	100	100	100

			d_1=.3							d_1=.4		
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	100	100	100	100	100	100	100	99.9	100	100	100	100
	100	100	100	100	100	100	100	99.7	100	100	100	100
.2	99.9	99.9	100	100	100	100	100	99.9	100	100	100	100
	99.9	99.8	100	100	100	100	99.8	99.9	100	100	100	100
.3	100	100	100	100	100	100	99.9	99.9	100	100	100	100
	100	100	100	100	100	100	99.9	99.6	100	100	100	100
.4	100	100	100	100	100	100	100	99.9	100	100	100	100
	100	100	100	100	100	100	99.8	99.8	100	100	100	100

TABLE 14
Power when x_t is χ^2_2 for model (112)
and $\widehat{C}(\lambda) = \exp\left\{\widehat{c}_1 e^{-ir\lambda}\right\}$

				d$_1$=.1						d$_1$=.2		
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	100	99.9	100	100	100	100	100	99.6	100	100	100	100
	99.7	98.5	100	100	100	100	99.3	98.0	100	100	100	100
.2	99.5	99.9	100	100	100	100	99.6	99.8	100	100	100	100
	98.3	98.5	100	100	100	100	98.8	98.2	100	100	100	100
.3	99.9	99.6	100	100	100	100	99.9	99.6	100	100	100	100
	99.4	98.4	100	100	100	100	99.0	97.5	100	100	100	100
.4	99.3	99.1	100	100	100	100	99.4	99.0	100	100	100	100
	97.7	97.6	100	100	100	100	96.8	96.6	100	100	100	100

				d$_1$=.3						d$_1$=.4		
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	99.6	99.8	100	100	100	100	99.9	99.6	100	100	100	100
	98.0	98.5	100	100	100	100	99.3	98.9	100	100	100	100
.2	99.9	99.6	100	100	100	100	99.6	99.9	100	100	100	100
	98.4	96.8	100	100	100	100	98.5	98.0	100	100	100	100
.3	99.8	99.5	100	100	100	100	99.9	99.6	100	100	100	100
	98.8	98.0	100	100	100	100	98.0	98.6	100	100	100	100
.4	99.9	99.6	100	100	100	100	99.7	99.5	100	100	100	100
	98.2	98.0	100	100	100	100	97.3	98.0	100	100	100	100

TABLE 15
Power when x_t is χ^2_2 for model (113) with $\gamma = 0.5$
and $\widehat{C}(\lambda) = \exp\left\{\widehat{c}_1 e^{-ir\lambda}\right\}$

	d_1=.1						d_1=.2					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	70.6	70.4	95.4	95.3	100	100	77.6	67.2	96.7	94.7	99.9	100
	58.3	57.3	88.5	91.4	99.9	99.9	64.4	52.4	91.7	88.4	99.8	99.8
.2	72.6	72.0	93.9	96.5	100	100	73.1	65.6	94.2	93.4	100	99.9
	58.2	53.5	88.5	91.9	99.9	99.9	58.7	53.4	88.0	88.8	99.4	99.6
.3	77.6	72.0	96.5	95.5	100	100	67.9	70.4	94.1	93.4	99.9	100
	64.2	60.6	91.6	92.8	99.9	99.8	53.0	57.1	91.6	87.0	99.9	99.5
.4	77.6	73.7	96.9	96.7	99.9	100	72.0	71.6	96.1	95.0	100	100
	65.1	56.2	93.4	94.0	99.9	99.9	58.8	56.0	90.5	87.8	100	100

	d_1=.3						d_1=.4					
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	66.5	64.3	92.3	92.3	99.9	99.2	60.9	59.2	89.8	85.1	98.9	98.5
	55.6	51.8	83.8	85.6	99.6	97.8	45.9	43.8	77.0	72.7	96.8	94.9
.2	69.8	63.1	92.3	89.6	99.7	99.6	63.7	54.4	81.8	84.0	98.8	98.1
	56.3	46.3	87.0	79.9	99.4	99.1	43.8	37.8	72.0	70.9	96.7	93.6
.3	59.2	63.8	92.6	86.6	99.8	99.7	58.5	53.7	83.8	81.9	98.7	96.7
	43.9	45.8	85.7	74.3	99.7	98.9	40.4	38.8	69.4	66.1	97.1	89.5
.4	70.8	63.1	90.2	91.4	99.7	99.8	60.9	60.7	83.6	78.9	97.2	96.9
	56.5	42.8	81.9	84.5	99.3	99.1	42.4	42.1	72.6	68.2	93.1	93.5

TABLE 16

Power when x_t is χ_2^2 for model (113) with $\gamma = 1.5$ and $\widehat{C}(\lambda) = \exp\left\{\widehat{c}_1 e^{-ir\lambda}\right\}$

		d₁=.1						d₁=.2				
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	100	100	100	100	100	100	100	100	100	100	100	100
	100	100	100	100	100	100	100	100	100	100	100	100
.2	100	100	100	100	100	100	100	100	100	100	100	100
	100	100	100	100	100	100	100	100	100	100	100	100
.3	100	100	100	100	100	100	100	100	100	100	100	100
	100	100	100	100	100	100	100	100	100	100	100	100
.4	100	100	100	100	100	100	100	100	100	100	100	100
	100	100	100	100	100	100	100	100	100	100	100	100

		d₁=.3						d₁=.4				
	n=128		n=256		n=512		n=128		n=256		n=512	
d_x	m= $\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$	$\frac{n}{4}$	$\frac{n}{8}$
.1	100	100	100	100	100	100	100	99.9	100	100	100	100
	100	100	100	100	100	100	100	99.7	100	100	100	100
.2	99.9	99.9	100	100	100	100	100	99.9	100	100	100	100
	99.9	99.8	100	100	100	100	99.8	99.9	100	100	100	100
.3	100	100	100	100	100	100	99.9	99.9	100	100	100	100
	100	100	100	100	100	100	99.9	99.6	100	100	100	100
.4	100	100	100	100	100	100	100	99.9	100	100	100	100
	100	100	100	100	100	100	99.8	99.8	100	100	100	100

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