ON THE EXISTENCE OF SPATIALLY TEMPERED NULL SOLUTIONS TO LINEAR CONSTANT COEFFICIENT PDES

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Abstract. Given a linear, constant coefficient partial differential equation in $\mathbb{R}^{d+1}$, where one independent variable plays the role of 'time', a distributional solution is called a null solution if its past is zero. Motivated by physical considerations, distributional solutions that are tempered in the spatial directions alone (with no restriction in the time direction) are considered. An algebraic-geometric characterization is given, in terms of the polynomial describing the PDE, for the null solution space to be trivial (that is, consisting only of the zero distribution).

1. Introduction

Given a polynomial $p \in \mathbb{C}[X_1, \cdots, X_d, T] =: \mathbb{C}[\mathbf{X}, T]$, we associate with it the linear constant coefficient differential operator D_p by making the replacements $X_k \rightsquigarrow \frac{\partial}{\partial x_k}$, $k = 1, \cdots, d$, $T \rightsquigarrow \frac{\partial}{\partial t}$. A *solution space* is a subspace S of the space of distributions $\mathcal{D}'(\mathbb{R}^{d+1})$. Unless otherwise indicated, we will use the standard distribution theory notation from Schwartz [11] or Tréves [12]. Fixing a solution space S, $p \in \mathbb{C}[\mathbf{X}, T]$ gives rise to the differential operator $D_p : S \to \mathcal{D}'(\mathbb{R}^{d+1})$, defined by $D_p u := p(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t})u$, $u \in S$. Let $p \in \mathbb{C}[\mathbf{X}, T]$, and S be a solution space. A *null solution in S associated with p* is a distribution $u \in S$ such that $D_p u = \mathbf{0}$ and $u|_{t<0} = \mathbf{0}$. We denote by $N_S(p)$ the subspace of S consisting of all null solutions in S associated with p: $N_S(p) := \{u \in S : D_p u = \mathbf{0} \text{ and } u|_{t<0} = \mathbf{0}\}$. The notion of a null solution was considered in [4] and [5].

We are interested in giving an algebraic-geometric characterisation of the polynomials p for which $N_S(p)$ is the subspace $\{\mathbf{0}\}$, consisting of only the zero distribution $\mathbf{0}$. Such a characterization is expected to depend on the solution space S, as illustrated by Propositions 1.1 and 1.2 below. In the following, $\mathcal{E}'(\mathbb{R}^{d+1})$ denotes the space of compactly supported distributions.

Proposition 1.1. *Let $S = \mathcal{E}'(\mathbb{R}^{d+1})$ or $S = \mathcal{D}(\mathbb{R}^{d+1})$. Let $p \in \mathbb{C}[\mathbf{X}, T]$. Then $N_S(p) = \{\mathbf{0}\}$ if and only if $p \neq \mathbf{0}$.*

Proof. ('If' part): Let $D_p u = \mathbf{0}$ and $p \neq \mathbf{0}$. By the Payley-Wiener-Schwartz theorem [12, Prop. 29.1,p. 307], the Fourier transform $\mathcal{F}u$ of u (with respect to *all* the variables) can be extended to an entire function on $\mathbb{C}^{d+1}$. So $D_p u = \mathbf{0}$ yields $p(i\mathbf{z}) \cdot (\mathcal{F}u)(\mathbf{z}) = 0$, $\mathbf{z} \in \mathbb{C}^{d+1}$. But the ring $\mathrm{A}(\mathbb{C}^{d+1})$ of entire functions in $d+1$ complex variables is an integral domain. As $p(i\cdot) \neq \mathbf{0}$ in $\mathrm{A}(\mathbb{C}^{d+1})$, $\mathcal{F}u = \mathbf{0}$, and so $u = \mathbf{0}$. Thus $N_S(p) = \{\mathbf{0}\}$. ('Only if'): Suppose that $p = \mathbf{0}$. Then clearly $N_S(p) = S \neq \{\mathbf{0}\}$. ☐

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When $S = \mathrm{C}^\infty(\mathbb{R}^{d+1})$ or $\mathcal{D}'(\mathbb{R}^{d+1})$, using [7, Thm. 8.6.7, 8.6.8], one can show the result below. Here, $\deg(\cdot)$ is used to denote the total degree.

Proposition 1.2. *Let $S = \mathcal{D}'(\mathbb{R}^{d+1})$ or $S = \mathrm{C}^\infty(\mathbb{R}^{d+1})$. Let $p \in \mathbb{C}[\mathbf{X}, T]$. Then $N_S(p) = \{\mathbf{0}\}$ if and only if $\deg(p(\mathbf{X}, T)) = \deg(p(\mathbf{0}, T))$.*

Proof. [7, Thm. 8.6.7] says that for a characteristic plane with normal $\mathbf{n}$, there exists a solution in C^∞ whose support is $\{\mathbf{y} : \langle \mathbf{y}, \mathbf{n}\rangle_{\mathbb{R}^{d+1}} \leqslant 0\}$. The hyperplane with the normal $\mathbf{n} := (\mathbf{0}, 1) \in \mathbb{R}^{d+1}$ is characteristic for D_p if and only if $\deg(p) \neq \deg(p(\mathbf{0}, T))$. This gives the 'only if' part.

For the 'if' part, we use [7, Theorem 8.6.8], which says that if X_1, X_2 are open convex sets such that $X_1 \subset X_2$, then the following are equivalent:
- If $u \in \mathcal{D}'(X_2)$ satisfies $D_p u = \mathbf{0}$ in X_2 and $u|_{X_1} = \mathbf{0}$, then $u = \mathbf{0}$ in X_2.
- Every characteristic hyperplane which intersects X_2 also intersects X_1.

Taking $X_1 = \{(\mathbf{x}, t) : \langle (\mathbf{x}, t), \mathbf{n}\rangle_{\mathbb{R}^{d+1}} = t < 0\}$, where $\mathbf{n} := (\mathbf{0}, 1) \in \mathbb{R}^{d+1}$, and with $X_2 := \mathbb{R}^{d+1}$, the above yields the 'if' part of the proposition. □

Example 1.3. The diffusion equation is $(\frac{\partial}{\partial t} - \Delta)u = \mathbf{0}$, that is, $D_p u = \mathbf{0}$, where $p(\mathbf{X}, T) = T - (X_1^2 + \cdots + X_d^2)$. As $\deg(p(\mathbf{X}, T)) = 2$, but $\deg(p(\mathbf{0}, T)) = \deg(T) = 1$, Proposition 1.2 implies that $N_{\mathcal{D}'(\mathbb{R}^{d+1})}(p) \neq \{\mathbf{0}\}$ and $N_{\mathrm{C}^\infty(\mathbb{R}^{d+1})}(p) \neq \{\mathbf{0}\}$. ◇

In this example, the outcome is physically unexpected, for example while considering matter diffusion and u is the density of matter: then zero density up to time $t = 0$ should mean that the density stays zero in the future as well. However, the above example shows that there are 'pathological' null solutions in C^∞ or in $\mathcal{D}'$ that are nonzero in the future. Choosing a different, physically motivated solution space, namely where at each time instant the spatial profile belongs to $\mathrm{L}^1(\mathbb{R}^d)$, the associated null solution space is then trivial, as expected. It is well-known that the reason that the null solution space is nontrivial in the above example when $S = \mathrm{C}^\infty(\mathbb{R}^{d+1})$ or $\mathcal{D}'(\mathbb{R}^{d+1})$ is that there is no growth restriction on the spatial profiles of the solutions at each time instant, and 'rapid' growth (roughly, faster than $e^{\|\mathbf{x}\|^2}$ [3, Theorem, p.44]) is allowed. Indeed, in most physical situations, we expect that at each time instant, the spatial profile is typically in some L^p space or at most polynomially growing, etc. This motivates the following solution space we consider. Below, $\mathcal{S}'(\mathbb{R}^d)$ denotes the space of tempered distributions.

The *space of distributions on $\mathbb{R}^{d+1}$ tempered in the spatial directions*, is the space $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$ of all continuous linear maps from $\mathcal{D}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R}^d)$, where $\mathcal{D}(\mathbb{R})$ is endowed with its inductive limit topology and $\mathcal{S}'(\mathbb{R}^d)$ is equipped with the weak dual topology $\sigma(\mathcal{S}', \mathcal{S})$. $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$ has the topology $\mathcal{L}_\sigma(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$ of pointwise convergence. For $u \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$, $\frac{\partial u}{\partial x_k}, \frac{\partial u}{\partial t} \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$ are defined by $\langle \frac{\partial u}{\partial x_k}(\varphi), \psi\rangle = -\langle u(\varphi), \frac{\partial \psi}{\partial x_k}\rangle$ and $\langle \frac{\partial u}{\partial t}(\varphi), \psi\rangle = -\langle u(\varphi'), \psi\rangle$ for $\varphi \in \mathcal{D}(\mathbb{R})$, $\psi \in \mathcal{S}(\mathbb{R}^d)$, $k = 1, \cdots, d$. For $u \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$, its 'spatial' Fourier transform $\widehat{u} \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$ is given by $\langle \widehat{u}(\varphi), \psi\rangle = \langle u(\varphi), \widehat{\psi}\rangle$. $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$ is a subspace of $\mathcal{D}'(\mathbb{R}^{d+1})$ as follows: For $u \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$, define U by $\langle U, \varphi \otimes \psi\rangle = \langle u(\varphi), \psi\rangle$ for $\varphi \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{D}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$. By the Schwartz kernel theorem [7, Thm. 5.2.1, p.128], there is a unique such distribution $U \in \mathcal{D}'(\mathbb{R}^{d+1})$. $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$ is also isomorphic to the completed projective- (or epsilon-)tensor product $\mathcal{D}'(\mathbb{R})\widehat{\otimes}_\pi \mathcal{S}'(\mathbb{R}^d)$ of $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R}^d)$.

We will study the set of null solutions with respect to the space of distributions tempered in the spatial directions, and give an algebraic-geometric characterization of those polynomials p for which the corresponding null solution space consists of just the zero solution.

Given a set I of polynomials from $\mathbb{C}[X_1, \cdots, X_d]$, we denote its variety in $\mathbb{C}^d$ by $V(I)$. We make the following two observations, used later.

- If $p \in \mathbb{C}[\mathbf{X}]$, $u \in \mathcal{S}'(\mathbb{R}^d)$ satisfy $D_p u = \mathbf{0}$, then $p(i\boldsymbol{\xi})\widehat{u} = \mathbf{0}$, and so we have that $\text{supp}(\widehat{u}) \subset \{\boldsymbol{\xi} \in \mathbb{R}^d : p(i\boldsymbol{\xi}) = 0\}$. Thus, $V(p) \cap i\mathbb{R}^d = \varnothing$ implies $u = 0$.
- If $p \in \mathbb{C}[T]$, $u \in \mathcal{D}'(\mathbb{R})$ satisfy $D_p u = \mathbf{0}$, then $u \in \text{span}\{t^k e^{\lambda t} : \lambda \in \mathbb{C}, k \in \mathbb{Z}_+\}$. Thus $u|_{t<0} = \mathbf{0}$ implies $u = \mathbf{0}$. Here $\mathbb{Z}_+ = \{0, 1, 2, 3, \cdots\}$.

As our solution space $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d)) \simeq \mathcal{D}'(\mathbb{R})\widehat{\otimes}_\pi \mathcal{S}'(\mathbb{R}^d)$, we expect our algebraic-geometric characterisation to reduce to above extreme cases when the polynomial belongs either to $\mathbb{C}[\mathbf{X}]$ or to $\mathbb{C}[T]$. For formulating this algebraic-geometric condition, we give the following definition. For $p = a_0 + a_1 T + \cdots + a_n T^n \in \mathbb{C}[\mathbf{X}, T] = \mathbb{C}[\mathbf{X}][T]$, where $a_0, \cdots, a_n \in \mathbb{C}[\mathbf{X}]$, the $\mathbf{X}$-*content* $C_{\mathbf{X}}(p)$ *of* p is the ideal in $\mathbb{C}[\mathbf{X}]$ generated by $a_0, \cdots, a_n$. We show that if $V(C_{\mathbf{X}}(p))$ meets $i\mathbb{R}^d$, then the null solution space is nontrivial.

Theorem 1.4. *Let* $p \in \mathbb{C}[\mathbf{X}, T]$. *If* $N_{\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))}(p) = \{\mathbf{0}\}$, *then* $V(C_{\mathbf{X}}(p)) \cap i\mathbb{R}^d = \varnothing$.

Proof. Let $V(C_{\mathbf{X}}(p)) \cap i\mathbb{R}^d \neq \varnothing$, and $\boldsymbol{\xi}_0 \in \mathbb{R}^d$ be such that $i\boldsymbol{\xi}_0 \in V(C_{\mathbf{X}}(p))$. Let $u := e^{i\langle \mathbf{x}, \boldsymbol{\xi}_0 \rangle_{\mathbb{R}^d}} \otimes \Theta$, where $\Theta \in C^\infty(\mathbb{R})$ is nonzero and has a zero past, e.g. $e^{-1/t}$ if $t > 0$ and 0 otherwise. Then $u \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$ and $u|_{t<0} = \mathbf{0}$. If $p = a_0 + \cdots + a_n T^n$, with the $a_k \in \mathbb{C}[\mathbf{X}]$, then $a_k \in C_{\mathbf{X}}(p)$, and $a_k(i\boldsymbol{\xi}_0) = 0$ for all k. Consequently, we have that $D_p u = a_0(i\boldsymbol{\xi}_0) e^{i\langle \mathbf{x}, \boldsymbol{\xi}_0 \rangle_{\mathbb{R}^d}} \otimes \Theta + \cdots + a_n(i\boldsymbol{\xi}_0) e^{i\langle \mathbf{x}, \boldsymbol{\xi}_0 \rangle_{\mathbb{R}^d}} \otimes \Theta^{(n)} = \mathbf{0}$. Hence $u \in N_{\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))}(p)$. But $u \neq \mathbf{0}$, and so $N_{\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}))}(p) \neq \{\mathbf{0}\}$. $\square$

Our main result (Thm. 4.1) is to show the sufficiency of $V(C_{\mathbf{X}}(p)) \cap i\mathbb{R}^d = \varnothing$ for $N_{\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))}(p) = \{\mathbf{0}\}$. Thus, Theorems 1.4 and 4.1 together give:

Theorem 1.5. *Let* $p \in \mathbb{C}[\mathbf{X}, T]$. *Then we have* $N_{\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))}(p) = \{\mathbf{0}\}$ *if and only if* $V(C_{\mathbf{X}}(p)) \cap i\mathbb{R}^d = \varnothing$.

We will also consider distributions which have spatial profiles at each time instant lying in certain Besov spaces. Summarising we have the following results:

	Solution space S	Test on p for $N_S(p) = \{\mathbf{0}\}$	Result reference
1.	$C^\infty(\mathbb{R}^{d+1})$	$\deg(p) = \deg(p(\mathbf{0}, T))$	Prop. 1.2
2.	$\mathcal{D}'(\mathbb{R}^{d+1})$	$\deg(p) = \deg(p(\mathbf{0}, T))$	Prop. 1.2
3.	$\mathcal{D}(\mathbb{R}^{d+1})$	$p \neq \mathbf{0}$	Prop. 1.1
4.	$\mathcal{E}'(\mathbb{R}^{d+1})$	$p \neq \mathbf{0}$	Prop. 1.1
5.	$\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$	$V(C_{\mathbf{X}}(p)) \cap i\mathbb{R}^d = \varnothing$	Thm. 1.4, 4.1
6.	$\mathcal{L}(\mathcal{D}(\mathbb{R}), B_{p,q}(\mathbb{R}^d))$	$p \neq \mathbf{0}$	Thm. 5.1
7.	$\mathcal{L}(\mathcal{D}(\mathbb{R}), H_s(\mathbb{R}^d))$	$p \neq \mathbf{0}$	Cor. 5.2
8.	$\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}(\mathbb{R}^d))$	$p \neq \mathbf{0}$	Cor. 5.2
9.	$\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{E}'(\mathbb{R}^d))$	$p \neq \mathbf{0}$	Thm. 5.3
10.	$\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$	$\forall \mathbf{v} \in A^{-1} 2\pi \mathbb{Z}^d$, $\exists t \in \mathbb{C} : p(i\mathbf{v}, t) \neq 0$.	Thm. 6.1

The key idea used in proving the sufficiency part of (5) above is as follows. By taking Fourier transform, the partial derivatives ∂_{x_k} with respect to the spatial variables x_k are converted into $i\xi_k$, and so $p(i\boldsymbol{\xi}, \partial_t)\widehat{u} = \mathbf{0}$, a family (parameterised by $\boldsymbol{\xi} \in \mathbb{R}^d$) of equations involving ∂_t^k with the polynomial coefficients $a_k(i\boldsymbol{\xi})$. One would like to 'freeze' a $\boldsymbol{\xi} \in \mathbb{R}^d$, to get an ODE for $(\widehat{u}(\cdot))(\boldsymbol{\xi}) \in \mathcal{D}'(\mathbb{R})$, where for such a solution to an ODE we can indeed say that zero past implies zero future, and so the proof can be completed easily by varying the arbitrarily fixed $\boldsymbol{\xi}$. This is possible if the spatial Fourier transform is a function, so that the evaluation at $\boldsymbol{\xi}$ is allowed, and this is essentially how one shows the results in the second half of the table above.

For showing our main result for $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$, where spatial Fourier transform will not result in a function of $\boldsymbol{\xi}$, but rather a distribution, the idea is as follows. Using Holmgren's uniqueness principle, the support of $\widehat{u}$ is contained in $V \times [0, \infty)$, where V is the real zero set of the leading coefficient a_n. If $d = 1$, so that a_n were a polynomial of just one variable, then the real zeroes are isolated points, and we can complete the proof using a structure theorem of Schwartz, which says that distributions supported on a line must be essentially the Dirac delta and its derivatives, tensored with distributions T_k of one variable (time). We can then boil down $p(i\boldsymbol{\xi}, \partial_t)\widehat{u} = \mathbf{0}$ to give an ODE for these distributions T_k of time, and since each T_k can be shown to have zero past, we can conclude that the T_ks must be zero. So this is how the proof works when $d = 1$ and when a_n was a polynomial of just one variable. In the general case, to handle the case when a_n may be a polynomial of d variables, we proceed inductively on the number of spatial dimensions d. It is too much to hope that at each inductive step we end up with polynomials as coefficients of ∂_t^k, since polynomial parametrisations of the zero sets of the polynomial a_n may not be possible (e.g. $\{(X, Y) : X^2 + Y^2 - 1 = 0\}$ does not possess a polynomial parametrisation). But the $d = 1$ case just relied on the discreteness of the zero set of a_n, which is also guaranteed if a_n were real analytic instead of being a polynomial. So to carry out the induction, we use the set up where we make sure that the coefficients of ∂_t^k obtained at each inductive step are real analytic functions. To begin with, polynomials are real analytic, real analytic varieties do possess locally real analytic parametrisations (Łojaciewicz structure theorem for real analytic varieties), and composition of real analytic functions is real analytic. This allows us to complete the induction step, by again appealing to a structure theorem of Schwartz, now for distributions with support in a smooth manifold. The technical details are carried out in Lemma 3.1. The organisation of the article is as follows.

- In Section 2, we recall some preliminaries needed for the proofs.
- In Section 3, we will prove the central technical result in Lemma 3.1, which will lead to the proof of Theorem 4.1 on the sufficiency.
- In Section 4, we will prove Theorem 4.1.
- In Section 5, we consider distributions which have spatial profiles at each time instant lying in certain Besov spaces.
- In Section 6 we consider spatially periodic distributions.
- Finally, a class of open problems on the null solution theme is mentioned.

2. Preliminaries

Here we recall three results used in proving Lemma 3.1: Holmgren's uniqueness theorem, Schwartz structure theorem for distributions supported on a manifold, and Łojaciewicz structure theorem for real analytic varieties.

2.1. **Holmgren's uniqueness theorem.** Let $\Omega \subset \mathbb{R}^d$ be open, and P be a differential operator of order N with coefficients $a_{\mathbf{n}}$ that are real analytic functions in Ω: $P(x, \boldsymbol{\partial}) = \sum_{|\mathbf{n}| \leqslant N} a_{\mathbf{n}}(\mathbf{x}) \frac{\partial^{n_1}}{\partial x_1^{n_1}} \cdots \frac{\partial^{n_d}}{\partial x_d^{n_d}}$. For $\mathbf{n} = (n_1, \cdots, n_d)$, $|\mathbf{n}| = n_1 + \cdots + n_d$. Recall the uniqueness theorem of Holmgren [6, Lemma 5.3.2, p.125]:

Proposition 2.1. *In an open $\Omega \subset \mathbb{R}^d$, let $P(\mathbf{x}, \boldsymbol{\partial}) = \sum_{|\mathbf{n}| \leqslant N} a_n(\mathbf{x}) \frac{\partial^{n_1}}{\partial x_1^{n_1}} \cdots \frac{\partial^{n_d}}{\partial x_d^{n_d}}$ be a differential operator having coefficients real analytic on Ω. Assume that the coefficient of $\frac{\partial^N}{\partial x_d^N}$ never vanishes in Ω. If $u \in \mathcal{D}'(\Omega)$ and $P(\mathbf{x}, \boldsymbol{\partial})u = 0$ in $\Omega_c := \{\mathbf{x} \in \Omega : x_d < c\}$ for some c, then $u = \mathbf{0}$ in Ω_c provided that $\Omega_c \cap (\mathrm{supp}(u))$ is relatively compact in Ω.*

We will use the following consequence of this.

Lemma 2.2. *Let $U \subset \mathbb{R}^d$ be an open set, $c_0, \cdots, c_N$ be real analytic functions in U, $u \in \mathcal{D}'(U \times \mathbb{R})$, $u|_{t<0} = \mathbf{0}$, and $c_0(\boldsymbol{\xi})u + c_1(\boldsymbol{\xi})\frac{\partial u}{\partial t} + \cdots + c_N(\boldsymbol{\xi})\frac{\partial^N u}{\partial t^N} = \mathbf{0}$. Then we have that $\mathrm{supp}(u) \subset \{(\boldsymbol{\xi}, t) \in U \times \mathbb{R} : c_N(\boldsymbol{\xi}) = 0\}$.*

Proof. Let $\boldsymbol{\xi}_0 \in U$ be such that $c_N(\boldsymbol{\xi}_0) \neq 0$. Let $r > 0$ be such that the open ball $B(\boldsymbol{\xi}_0, 2r) \subset U$ and $c_N(\boldsymbol{\xi}) \neq 0$ in $B(\boldsymbol{\xi}_0, 2r)$. We will use Holmgren's uniqueness theorem with $\Omega_{\boldsymbol{\xi}_0} := B(\boldsymbol{\xi}_0, 2r) \times \mathbb{R}$ and $P((\boldsymbol{\xi}, t), \boldsymbol{\partial}) := c_0(\boldsymbol{\xi}) + c_1(\boldsymbol{\xi})\frac{\partial}{\partial t} + \cdots + c_N(\boldsymbol{\xi})\frac{\partial^N}{\partial t^N}$. The coefficient $c_N(\boldsymbol{\xi})$ of $\frac{\partial^N}{\partial t^N}$ never vanishes in $\Omega_{\boldsymbol{\xi}_0}$. Let $\widetilde{u}$ be the restriction of u to the set $\Omega_{\boldsymbol{\xi}_0} = B(\boldsymbol{\xi}_0, 2r) \times \mathbb{R} \subset U \times \mathbb{R}$. We know $\mathrm{supp}(\widetilde{u}) \subset B(\boldsymbol{\xi}_0, 2r) \times [0, \infty)$ since $\widetilde{u}|_{t<0} = \mathbf{0}$ (as $u|_{t<0} = \mathbf{0}$). For $c > 0$, with $\Omega_c := \{(\boldsymbol{\xi}, t) : \boldsymbol{\xi} \in B(\boldsymbol{\xi}_0, r),\ t < c\}$, we have $\Omega_c \cap \mathrm{supp}(\widetilde{u})$ is relatively compact in $\Omega_{\boldsymbol{\xi}_0}$. Hence $\widetilde{u} = \mathbf{0}$ in Ω_c. As $c > 0$ was arbitrary, $\widetilde{u} = \mathbf{0}$ in $B(\boldsymbol{\xi}_0, r) \times \mathbb{R}$. By varying the $\boldsymbol{\xi}_0$ having the property that $c_N(\boldsymbol{\xi}_0) \neq 0$, we obtain that the restriction of u to the set $V := \{\boldsymbol{\xi} \in U : c_N(\boldsymbol{\xi}) \neq 0\} \times \mathbb{R}$ is the zero distribution $\mathbf{0} \in \mathcal{D}'(V)$. So $\mathrm{supp}(u) \subset \{(\boldsymbol{\xi}, t) \in U \times \mathbb{R} : c_N(\boldsymbol{\xi}) = 0\}$. □

2.2. **Schwartz structure theorem for distributions with support in a submanifold of $\mathbb{R}^d$.** We will need a local structure result [11, Thm. XXXVII, p. 102], for distributions with support contained in a smooth submanifold of $\mathbb{R}^d$. Here, for a multi-index $\mathbf{k} = (k_{d'+1}, \cdots, k_d)$ of nonnegative integers, we define $|\mathbf{k}| = k_{d'+1} + \cdots + k_d$, and $\partial_{\mathbf{y}}^{\mathbf{k}} = (\frac{\partial}{\partial y_{d'+1}})^{k_{d'+1}} \cdots (\frac{\partial}{\partial y_d})^{k_d}$.

Proposition 2.3. *Suppose that M is a submanifold of $\mathbb{R}^d$ of dimension d', $\boldsymbol{\xi}_0 \in M$, and $\mathbf{y}$ are coordinates in $B(\boldsymbol{\xi}_0, R) = \{\boldsymbol{\xi} \in \mathbb{R}^d : \|\boldsymbol{\xi} - \boldsymbol{\xi}_0\|_2 < R\}$ in $\mathbb{R}^d$, such that $B(\boldsymbol{\xi}_0, R) \cap M = \{\boldsymbol{\xi} \in B(\boldsymbol{\xi}_0, R) : y_{d'+1}(\boldsymbol{\xi}) = \cdots = y_d(\boldsymbol{\xi}) = 0\}$. Then a distribution T on $\mathbb{R}^d$ with support in M can be locally decomposed as $T = \sum_{|\mathbf{k}| \leqslant K} \partial_{\mathbf{y}}^{\mathbf{k}} T_{\mathbf{k}}$, for some distributions $T_{\mathbf{k}}$ on M.*

2.3. **Łojasiewicz structure theorem for real analytic varieties.** We now give Lemma 2.4, which is a consequence of a more elaborate structure theorem for real analytic varieties due to S. Łojaciewicz [10] (see also [1]). We will not recall this theorem here, but refer the reader to the exposition given in [9, Theorem 6.3.3, p.168], and we use the same terminology and notation here. This result given in [9, Theorem 6.3.3, p.168] is stronger than what we need. We only require the following decomposition into lower dimensional real analytic varieties and the local analytic parametrisation.

Lemma 2.4. *Let $V = \{\mathbf{x} \in \mathbb{R}^d : f(\mathbf{x}) = 0\}$ be the variety of a real analytic function $f : \mathbb{R}^d \to \mathbb{R}$. Then for each point $\mathbf{x}_0$ of V, there exists a neighbourhood Ω such that $V \cap \Omega = M_{d-1} \cup \cdots \cup M_0$, where some of the $M_{d'}$s may be empty, and where the $M_{d'}$ are real analytic varieties which are analytic submanifolds of $\mathbb{R}^d$ of dimension d', admitting real analytic parametrisations as follows: For every $\mathbf{x}_0 \in M_{d'}$, there exists a neighbourhood U of $\mathbf{x}_0 \in \mathbb{R}^d$ and a neighbourhood W of $\mathbf{0} \in \mathbb{R}^d$, with a homeomorphism $\varphi : W \to U$ which is real analytic, $\mathbb{R}^{d'} \times \mathbb{R}^{d-d'} \supset W \ni (\boldsymbol{\tau}, \boldsymbol{\sigma}) \mapsto \varphi(\boldsymbol{\tau}, \boldsymbol{\sigma}) \in U$, such that $M_{d'} \cap U = \{\varphi(\boldsymbol{\tau}, \mathbf{0}) : \boldsymbol{\tau} \in \mathbb{R}^{d'}$ such that $(\boldsymbol{\tau}, \mathbf{0}) \in W\}$.*

Proof. This follows immediately from [9, Theorem 6.3.3, p.168], since its part (2) guarantees the local decomposition, and the real analytic parametrisation, namely $(x_1, \cdots, x_k) \mapsto (x_1, \cdots, x_k, \chi_{\eta^k_{k+1}}(x_1, \cdots, x_k), \cdots, \chi_{\eta^k_d}(x_1, \cdots, x_k))$ corresponds to the one we need in our lemma if we take $\boldsymbol{\tau} = (x_1, \cdots, x_k)$, $\boldsymbol{\sigma} = (x_{k+1}, \cdots, x_d)$, and $\boldsymbol{\xi}(\boldsymbol{\tau}, \boldsymbol{\sigma}) = (\boldsymbol{\tau}, \chi_{\eta^k_{k+1}}(x_1, \cdots, x_k) - x_{k+1}, \cdots, \chi_{\eta^k_d}(x_1, \cdots, x_k) - x_d)$. Then we note that the differential of $\boldsymbol{\xi}$ has the form

$$d\boldsymbol{\xi} = \begin{bmatrix} I_{d'} & \mathbf{0} \\ * & -I_{d-d'} \end{bmatrix},$$

which is clearly invertible. Here I_k denotes the $k \times k$ identity matrix. ☐

3. The main technical lemma

In this section, we will show the main technical result in Lemma 3.1, which will enable us to show our result on sufficiency, namely Theorem 4.1.

Lemma 3.1. *Let $U \subset \mathbb{R}^d$ be open, $\mathbf{0} \neq p = c_0(\boldsymbol{\xi}) + \cdots + c_n(\boldsymbol{\xi})T^n \in \mathrm{C}^\omega(U)[T]$, $c_n \neq \mathbf{0}$, $V(c_0, c_1, \cdots, c_n) \cap U = \varnothing$, $w \in \mathcal{D}'(U \times \mathbb{R})$, $w|_{t<0} = \mathbf{0}$, and*

$$c_0(\boldsymbol{\xi}) + c_1(\boldsymbol{\xi})\frac{\partial}{\partial t}w + \cdots + c_n(\boldsymbol{\xi})\frac{\partial^n}{\partial t^n}w = \mathbf{0}. \tag{1}$$

Then $w = \mathbf{0}$.

Proof. We prove this inductively on the number of spatial dimensions d.

Step 1. Let $d = 1$. Holmgren's uniqueness theorem (Lemma 2.2) implies that $\mathrm{supp}(w) \subset \{(\xi, t) \in U \times \mathbb{R} : c_n(\xi) = 0,\ t \geqslant 0\}$. If c_n is constant (which must necessarily be $\neq 0$, since c_n was nonzero), then $w = \mathbf{0}$, and we are done.

Let c_n be not a constant. Let $w \neq \mathbf{0}$. Let $(\xi_k)_{k \in \mathbb{N}}$ be the real zeros of c_n in U. Then each ξ_k is isolated in U. We have $\mathrm{supp}(w) \subset \bigcup_{k\in\mathbb{N}} \{\xi_k\} \times [0, +\infty)$. Each of the half lines above carries a solution of the equation $c_0(\boldsymbol{\xi}) + c_1(\boldsymbol{\xi})\frac{\partial}{\partial t}w + \cdots + c_n(\boldsymbol{\xi})\frac{\partial^n}{\partial t^n}w = \mathbf{0}$, and w is a sum of these.

Let $T \in (0, \infty)$. Take a $\xi_* \in \{\xi_1, \xi_2, \cdots\}$, U a neighbourhood of ξ_* not containing the other ξ_ks, and an $\alpha \in \mathcal{D}(\mathbb{R})$ which is identically 1 in a neighbourhood of $[-T, T]$ such that the distribution $\alpha w \in \mathcal{D}'(U \times \mathbb{R})$ is nonzero. Then αw has compact support, and by the structure theorem for distributions (e.g. [7, Theorem 2.3.5, p.47] or the result from Subsection 2.2), it follows that there exist distributions $T_0, \cdots, T_K \in \mathcal{D}'(\mathbb{R})$ ('of the time variable'), with $T_K \neq \mathbf{0}$, such that $\alpha w = \sum_{k=0}^{K}((\frac{\partial}{\partial \xi})^k \delta_{\xi_*}) \otimes T_k$. Here δ_{ξ_*} is the Dirac delta of the spatial variable ξ, supported at ξ_*. From the above, it can be shown that also $w = \sum_{k=0}^{K}((\frac{\partial}{\partial\xi})^k\delta_{\xi_*})\otimes T_k$ in the strip $U \times (-T, T)$.

We claim that $T_K|_{(-T,0)} = \mathbf{0}$. For if not, then there is a $\varphi \in \mathcal{D}(\mathbb{R})$ with support in $(-T, 0)$ such that $\langle T_K, \varphi\rangle \neq 0$. Hence the sum $\sum_{k=0}^{K}\langle T_k, \varphi\rangle(\frac{\partial}{\partial\xi})^k\delta_{\xi_*}$ is a nonzero distribution in $\mathcal{D}'(U)$. Otherwise, we get the contradiction that $\delta_{\xi_*}, \cdots, \delta_{\xi_*}^{(K)}$ are linearly dependent in $\mathcal{D}'(U)$. So there exists a $\psi \in \mathcal{D}(U)$ such that $\langle\sum_{k=0}^{K}\langle T_k, \varphi\rangle(\frac{\partial}{\partial\xi})^k\delta_{\xi_*}, \psi\rangle \neq 0$, that is, $\langle w, \psi\otimes\varphi\rangle \neq 0$. But the support of $\psi\otimes\varphi$ is in $U \times (-T, 0)$, and so we have arrived at a contradiction to $w|_{t<0} = \mathbf{0}$. This proves $T_K|_{(-T,0)} = \mathbf{0}$.

Using $\mathbf{0} = \sum_{\ell=0}^{n} c_\ell(\xi)(\frac{\partial}{\partial t})^\ell w$, we have, for $(\xi-\xi_*)^K \in \mathrm{C}^\infty(U)$ and $\varphi \in \mathcal{D}(\mathbb{R})$, that

$$\begin{aligned} 0 &= \left\langle \sum_{\ell=0}^{n} c_\ell(\xi)\Big(\frac{\partial}{\partial t}\Big)^\ell \sum_{k=0}^{K}\Big(\frac{\partial}{\partial \xi}\Big)^k \delta_{\xi_*}\otimes T_k,\ (\xi-\xi_*)^K\otimes\varphi \right\rangle \\ &= \sum_{\ell=0}^{n}\sum_{k=0}^{K}\left\langle \Big(\frac{\partial}{\partial t}\Big)^\ell T_k, \varphi\right\rangle(-1)^k\left\langle \delta_{\xi_*}, \Big(\frac{\partial}{\partial \xi}\Big)^k\big(c_\ell(\xi)(\xi-\xi_*)^K\big)\right\rangle. \end{aligned} \tag{2}$$

But $(\frac{\partial}{\partial\xi})^k(c_\ell(\xi)(\xi-\xi_*)^K) = \sum_{m=0}^{k}\binom{k}{m}((\xi-\xi_*)^K)^{(m)}(\frac{\partial}{\partial\xi})^{k-m}c_\ell(\xi)$, and if $k < K$, then for all $m = 0, \cdots, k$, the mth derivative of $(\xi-\xi_*)^K$ will be zero at $\xi = \xi_*$ as $K - m \geqslant K - k \geqslant 1$. So $(\frac{\partial}{\partial\xi})^k(c_\ell(\xi)(\xi-\xi_*)^K)|_{\xi=\xi_*} = 0$ for $k < K$. Hence the sum over $k = 0, \cdots, K$ in (2) collapses to one over $k = K$, giving

$$\begin{aligned} 0 &= \sum_{\ell=0}^{n}\left\langle \Big(\frac{\partial}{\partial t}\Big)^\ell T_K, \varphi\right\rangle(-1)^K\left\langle \delta_{\xi_*}, \Big(\frac{\partial}{\partial \xi}\Big)^K\big(c_\ell(\xi)(\xi-\xi_*)^K\big)\right\rangle \\ &= \sum_{\ell=0}^{n}\left\langle \Big(\frac{\partial}{\partial t}\Big)^\ell T_K, \varphi\right\rangle(-1)^K c_\ell(\xi_*)K! = (-1)^K K!\left\langle \sum_{\ell=0}^{n} c_\ell(\xi_*)\Big(\frac{\partial}{\partial t}\Big)^\ell T_K, \varphi\right\rangle. \end{aligned}$$

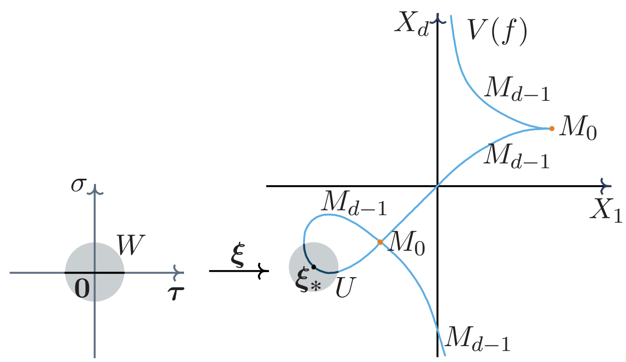
As $\varphi \in \mathcal{D}(\mathbb{R})$ was arbitrary, $(c_0(\xi_*)+c_1(\xi_*)\frac{d}{dt}+\cdots+c_n(\xi_*)(\frac{d}{dt})^n)T_K = \mathbf{0}$. Owing to our condition that $V(c_0, c_1, \cdots, c_n) \cap U = \varnothing$, we know that at least one of the coefficients $c_0(\xi_*), \cdots, c_n(\xi_*)$ is nonzero (we know that $c_n(\xi_*) = 0$ since ξ_* was one of the roots of c_n). Thus we now have a solution T_K to an ODE with constant coefficients. But then T_K is a classical smooth solution expressible as a linear combination of analytic functions of the type $t^k e^{\lambda t}$ for some nonnegative integers k and some complex numbers λ. The zero past condition $T_K|_{(-T,0)} = \mathbf{0}$, furthermore implies that this analytic function must in fact be identically zero, that is $T_K = \mathbf{0}$ in $(-T, T)$, a contradiction. Hence our assumption that w is nonzero can't be true. Consequently, $w = \mathbf{0}$. This completes the proof of the lemma when $d = 1$.

Step 2. Suppose now that $d > 1$, and that the statement of the lemma holds for all spatial dimensions strictly less than d. We wish to prove the induction step that then the result holds for d-many spatial dimensions too. Let w be a solution to

$$c_0(\boldsymbol{\xi}) + c_1(\boldsymbol{\xi})\frac{\partial}{\partial t}w + \cdots + c_n(\boldsymbol{\xi})\frac{\partial^n}{\partial t^n}w = \mathbf{0}, \tag{3}$$

with zero past. Suppose that w is nonzero. Holmgren's uniqueness theorem (Lemma 2.2) implies that $\mathrm{supp}(w) \subset \{(\boldsymbol{\xi}, t) \in U \times \mathbb{R} : c_n(\boldsymbol{\xi}) = 0,\ t \geqslant 0\}$. If c_n is constant (which must necessarily be nonzero, since c_n is nonzero), then $w = \mathbf{0}$, a contradiction, and so we are done.

Suppose c_n is not a constant. Then $V(c_n) := M_{d-1} \cup \cdots \cup M_0$, where M_k is the union of k-dimensional real analytic varieties, each possessing an analytic parametrisation.



Suppose that $\boldsymbol{\xi}_* \in M_{d'}$, where $0 < d' \leqslant d-1$. Then there exists an open neighbourhood Ω of $\boldsymbol{\xi}_*$, an open neighbourhood W of $\mathbf{0} \in \mathbb{R}^d$, and also a homeomorphism, namely $\mathbb{R}^{d'} \times \mathbb{R}^{d-d'} \supset W \ni (\boldsymbol{\tau}, \boldsymbol{\sigma}) \mapsto \boldsymbol{\xi}(\boldsymbol{\tau}, \boldsymbol{\sigma}) : W \to \Omega$, with $\boldsymbol{\tau} \mapsto \boldsymbol{\xi}(\boldsymbol{\tau}, \mathbf{0}) \in \mathrm{C}^\omega(\widetilde{\Omega})$, where $\widetilde{\Omega} := \{\boldsymbol{\tau} \in \mathbb{R}^{d'} : (\boldsymbol{\tau}, \mathbf{0}) \in W\}$, and $M_{d-1} \cap \Omega = \{\boldsymbol{\xi}(\boldsymbol{\tau}, \mathbf{0}) : (\boldsymbol{\tau}, \mathbf{0}) \in W\}$. Suppose that w is nonzero in $\Omega \times \mathbb{R}$. Then there is a large enough $T > 0$ such that w is nonzero on $\Omega \times (-T, T)$. By the Schwartz structure theorem (§2.2), we can decompose w locally in a neighbourhood $\Omega \times (-T, T)$ of $(\boldsymbol{\xi}_*, 0) \in \mathbb{R}^{d+1}$ as $w = \sum_{|\mathbf{k}| \leqslant K} \partial_{\boldsymbol{\sigma}}^{\mathbf{k}} T_{\mathbf{k}}$, for some distributions $T_{\mathbf{k}}$ on $(M_{d'} \cap \Omega) \times (-T, T)$, such that not all $T_{\mathbf{k}} = \mathbf{0}$ when $|\mathbf{k}| = K$. Also, as $w|_{t<0} = \mathbf{0}$, $T_{\mathbf{k}}|_{t<0} = \mathbf{0}$ for all $\mathbf{k}$. For a $d-d'$ tuple $\mathbf{k} = (k_{d'+1}, \cdots, k_d)$ of nonnegative integers, with $|\mathbf{k}| = K$, let $\psi_{\mathbf{k}}$ be the smooth function given by $\psi_{\mathbf{k}} := \sigma_1^{k_{d'+1}} \cdots \sigma_{d-d'}^{k_d}$. Then we have that $\partial_{\boldsymbol{\sigma}}^{\mathbf{k}'} \psi_{\mathbf{k}}\big|_{\boldsymbol{\sigma}=\mathbf{0}} = k_{d'+1}! \cdots k_d! \cdot \delta_{k_{d'+1}, k'_{d'+1}} \cdots \delta_{k_d, k'_d}$, where $\delta_{\ell, \ell'}$ is equal to 1 if $\ell = \ell'$, and 0 otherwise. Using $c_0(\boldsymbol{\xi}) + c_1(\boldsymbol{\xi}) \frac{\partial}{\partial t} w + \cdots + c_n(\boldsymbol{\xi}) \frac{\partial^n}{\partial t^n} w = \mathbf{0}$, it follows that for all $\varphi \in \mathcal{D}(\widetilde{\Omega} \times (-T, T))$, where $\widetilde{\Omega}$ is as defined above, we have with $\kappa := (-1)^K k_{d'+1}! \cdots k_d!$ that

$$0 = \left\langle \sum_{\ell=0}^{n} \sum_{|\mathbf{k}'| \leqslant K} c_\ell(\boldsymbol{\xi}(\boldsymbol{\tau}, \boldsymbol{\sigma})) \left(\frac{\partial}{\partial t}\right)^{\ell} \partial_{\boldsymbol{\sigma}}^{\mathbf{k}'} T_{\mathbf{k}'}, \psi_{\mathbf{k}} \otimes \varphi \right\rangle = \sum_{\ell=0}^{n} \left\langle \kappa c_\ell(\boldsymbol{\xi}(\boldsymbol{\tau}, \mathbf{0})) \Big(\frac{\partial}{\partial t}\Big)^{\ell} T_{\mathbf{k}}, \varphi \right\rangle.$$

As $\boldsymbol{\tau} \mapsto \boldsymbol{\xi}(\boldsymbol{\tau}, \mathbf{0}) \in \mathrm{C}^\omega(\widetilde{\Omega})$, and each $c_\ell \in \mathrm{C}^\omega(U)$, their (well-defined) composition, namely $\boldsymbol{\tau} \mapsto c_\ell(\boldsymbol{\xi}(\boldsymbol{\tau}, \mathbf{0}))$ is real analytic. S As $V(c_0, c_1, \cdots, c_n) \cap U = \varnothing$, we obtain that with $\widetilde{c}_\ell(\boldsymbol{\tau}) := c_\ell(\boldsymbol{\xi}(\boldsymbol{\tau}, \mathbf{0}))$, $\ell = 0, \cdots, n$, and with $p_0 := \sum_{\ell=0}^{n} \widetilde{c}_\ell(\boldsymbol{\xi}(\boldsymbol{\tau}, 0)) T^\ell \in \mathrm{C}^\omega(\widetilde{\Omega})[T]$, we have $V(\widetilde{c}_0, \widetilde{c}_1, \cdots, \widetilde{c}_n) \cap \mathbb{R}^{d'} = \varnothing$, and $D_{p_0} T_{\mathbf{k}} = \mathbf{0}$. We also recall from the above that $T_{\mathbf{k}}|_{t<0} = \mathbf{0}$. By the induction hypothesis, we conclude that $T_{\mathbf{k}} = \mathbf{0}$. Repeating this argument for each $\mathbf{k}$ satisfying $|\mathbf{k}| = K$, gives $T_{\mathbf{k}} = \mathbf{0}$ whenever $|\mathbf{k}| = K$, a contradiction. So $M_{d'} \cap \operatorname{supp}(w) = \varnothing$. As d' such that $0 < d' \leqslant d-1$ was arbitrary, we conclude that $\operatorname{supp}(w) \subset M_0$. But now we repeat the same argument above from Step 1, when w was supported on isolated lines, to conclude that $\operatorname{supp}(w) = \varnothing$, that is, $w = \mathbf{0}$. This completes the induction step. □

4. Proof of sufficiency

Theorem 1.4 says $V(C_{\mathbf{X}}(p)) \cap i\mathbb{R}^d = \varnothing$ is necessary for $N_{\mathcal{L}(\mathcal{D}(\mathbb{R}),\mathcal{S}'(\mathbb{R}^d))}(p) = \{\mathbf{0}\}$. We now show sufficiency.

Theorem 4.1. *Suppose that* $p = a_0(\mathbf{X}) + a_1(\mathbf{X})T + \cdots + a_n(\mathbf{X})T^n \in \mathbb{C}[\mathbf{X}][T]$ *and that* $a_n \neq \mathbf{0} \in \mathbb{C}[\mathbf{X}]$. *If* $V(C_{\mathbf{X}}(p)) \cap i\mathbb{R}^d = \varnothing$, *then* $N_{\mathcal{L}(\mathcal{D}(\mathbb{R}),\mathcal{S}'(\mathbb{R}^d))}(p) = \{\mathbf{0}\}$.

Proof. Suppose that $V(C_{\mathbf{X}}(p)) \cap i\mathbb{R}^d = \varnothing$. Let $u \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$ be such that $u|_{t<0} = \mathbf{0}$, $D_p u = \mathbf{0}$ and such that $u \neq \mathbf{0}$. Fourier transformation with respect to the spatial variables in $D_p u = \mathbf{0}$ yields $a_0(i\boldsymbol{\xi})\widehat{u} + a_1(i\boldsymbol{\xi})\frac{\partial}{\partial t}\widehat{u} + \cdots + a_n(i\boldsymbol{\xi})(\frac{\partial}{\partial t})^n\widehat{u} = \mathbf{0}$. By Lemma 3.1, $\widehat{u} = \mathbf{0}$. Taking the inverse Fourier transform gives $u = \mathbf{0}$. □

Example 4.2. Consider again the diffusion equation $(\frac{\partial}{\partial t} - \Delta)u = \mathbf{0}$, that is, $D_p u = \mathbf{0}$, where $p(\mathbf{X}, T) = T - (X_1^2 + \cdots + X_d^2)$. The constant polynomial $a_1 = \mathbf{1}$ is nonzero, and so $V(C_{\mathbf{X}}(p)) \cap i\mathbb{R}^d = \varnothing$. Theorem 4.1 implies that $N_{\mathcal{L}(\mathcal{D}(\mathbb{R}),\mathcal{S}'(\mathbb{R}^d))}(p) = \{\mathbf{0}\}$, in conformity with our physical intuition. ◇

Modern physics rejects the diffusion equation as an accurate model of physical reality since it is not 'Lorentz invariant', admitting infinite propagation speeds. This can already be seen in the case of classical solutions to the initial value problem to the diffusion equation, where the solution is given by a (spatial) convolution of the initial data f with the Gaussian kernel, and so for arbitrarily small time instants $t > 0$ and at $\mathbf{x} = \mathbf{0}$, even arbitrarily far away initial data has an influence, which violates the special relativistic tenet that nothing travels faster than the speed of light. With this in mind, we illustrate our theorem with the Lorentz-invariant Klein-Gordon equation.

Example 4.3. For $m \in \mathbb{R}$, consider $(\frac{\partial^2}{\partial t^2} - \Delta + m^2)u = \mathbf{0}$, that is, $D_p u = \mathbf{0}$, where $p(\mathbf{X}, T) = T^2 - (X_1^2 + \cdots + X_d^2) + m^2$. The constant polynomial $a_2 = \mathbf{1}$ is nonzero, and so $V(C_{\mathbf{X}}(p)) \cap i\mathbb{R}^d = \varnothing$. Theorem 4.1 implies that $N_{\mathcal{L}(\mathcal{D}(\mathbb{R}),\mathcal{S}'(\mathbb{R}^d))}(p) = \{\mathbf{0}\}$.

We remark that Proposition 1.2 also gives a sensible result here, since we have $\deg(p) = \deg(T^2 - (X_1^2 + \cdots + X_d^2) + m^2) = 2 = \deg(T^2 + m^2) = \deg(p(\mathbf{0}, T))$, and so $N_{\mathcal{D}'(\mathbb{R}^{d+1})}(p) = \{\mathbf{0}\}$ and $N_{C^\infty(\mathbb{R}^{d+1})}(p) = \{\mathbf{0}\}$.

If $\eta_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$ are the Minkowski metric tensor components in the Cartesian/inertial coordinates, then the only Lorentz-invariant scalar linear constant coefficient differential operator one can build has the form $\sum_{n=0}^{N} c_n(\eta^{\mu\nu}\partial_\mu\partial_\nu)^n$, where $[\eta^{\mu\nu}]$ denotes the inverse of the metric matrix $[\eta_{\mu\nu}]$, and $c_k \in \mathbb{C}$. Then we have that $p = \sum_{n=0}^{N} c_n(T^2 - (X_1^2 + \cdots + X_d^2))^n$, and so $\deg(p) = \deg(p(\mathbf{0}, T))$ is *always* satisfied for such Lorentz invariant partial differential operators. Thus Hörmander's Proposition 1.2 is physically sound from the spacetime perspective of special relativity. ◇

5. Spatial profile in Besov spaces

Besides the space $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}'(\mathbb{R}^d))$, one may consider also other natural solution spaces with some growth restriction in the spatial directions. As an example, we consider $\mathcal{L}(\mathcal{D}(\mathbb{R}), B_{p,k}(\mathbb{R}^d))$, where $B_{p,k}(\mathbb{R}^d)$, defined below, is a subspace of $\mathcal{S}'(\mathbb{R}^d)$. We follow [8, §10.1]. A function $k : \mathbb{R}^d \to (0, \infty)$ is a *temperate weight function* if there exist $C, N > 0$ such that for all $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathbb{R}^d$, $k(\boldsymbol{\xi} + \boldsymbol{\eta}) \leqslant (1 + C\|\boldsymbol{\xi}\|_2)^N k(\boldsymbol{\eta})$, where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^d$. The set of all such functions is denoted by $\mathcal{K}$. If $k \in \mathcal{K}$ and

$1 \leqslant p \leqslant \infty$, then the *Besov space* $B_{p,k}$ is the set of all distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ such that the Fourier transform $\widehat{u}$ is a function and $\|u\|_{p,k} = (\int_{\mathbb{R}^d} |k(\boldsymbol{\xi})\widehat{u}(\boldsymbol{\xi})|^p \mathrm{d}^d\boldsymbol{\xi})^{1/p} < \infty$. When $p = \infty$, $\|u\|_{p,k} := \text{ess.sup } |k(\cdot)\widehat{u}(\cdot)|$. Then $B_{p,k}$ is a Banach space with the above norm. The scale of Sobolev spaces $H_s(\mathbb{R}^d)$, parameterised by $s \in \mathbb{R}$, corresponds to $\mathcal{K}_{\text{Sob}} := \{k_s : s \in \mathbb{R}\} \subset \mathcal{K}$, where $k_s(\boldsymbol{\xi}) := (1 + \|\boldsymbol{\xi}\|^2)^{s/2}$. The space $\mathcal{L}(\mathcal{D}(\mathbb{R}), B_{p,k}(\mathbb{R}^d))$ is a subspace of $\mathcal{D}'(\mathbb{R}^{d+1})$: if $u \in \mathcal{L}(\mathcal{D}(\mathbb{R}), B_{p,k}(\mathbb{R}^d))$, then we define $U \in \mathcal{D}'(\mathbb{R}^{d+1})$ by $\langle U, \psi \otimes \varphi\rangle = \int_{\mathbb{R}^d} (\langle u, \varphi\rangle)(\boldsymbol{\xi})\psi(\boldsymbol{\xi})\mathrm{d}^d\boldsymbol{\xi}$ for $\varphi \in \mathcal{D}(\mathbb{R})$, $\psi \in \mathcal{D}(\mathbb{R}^d)$. We prove the following result. Despite again using the Fourier transform as the main tool, akin to the proof of Lemma 3.1, the proof is markedly simpler, thanks to the possibility of 'evaluation' at $\boldsymbol{\xi}$ (as for each 'time' test function $\varphi \in \mathcal{D}(\mathbb{R})$, $\widehat{u}(\varphi)$ is a function of the variable $\boldsymbol{\xi} \in \mathbb{R}^d$).

Theorem 5.1. *Let* $p \in \mathbb{C}[\mathbf{X}, T]$. *Then* $N_{\mathcal{L}(\mathcal{D}(\mathbb{R}), B_{p,k}(\mathbb{R}^d))}(p) = \{\mathbf{0}\}$ *if and only if* $p \neq \mathbf{0}$.

Proof. ('Only if' part:) Let $p = \mathbf{0}$. Take any nonzero $\psi \in B_{p,k}$, e.g. any nonzero $\psi \in \mathcal{D}(\mathbb{R}^d)$. Let $\Theta \in \mathrm{C}^\infty(\mathbb{R})$ be the nonzero function with zero past as in the proof of Theorem. 1.4. Define u by $u(x,t) := \psi(x)\Theta(t)$ ($x \in \mathbb{R}^d$, $t \in \mathbb{R}$). Then $u \in \mathcal{L}(\mathcal{D}(\mathbb{R}), B_{p,k}(\mathbb{R}^d))$, $u|_{t<0} = \mathbf{0}$ and $D_p u = \mathbf{0}$. But $u \neq \mathbf{0}$, and so $N_{\mathcal{L}(\mathcal{D}(\mathbb{R}), B_{p,k}(\mathbb{R}^d))}(p) \neq \{\mathbf{0}\}$.

('If':) Let $p \neq \mathbf{0}$, and $u \in \mathcal{L}(\mathcal{D}(\mathbb{R}), B_{p,k}(\mathbb{R}^d))$ satisfy $u|_{t<0} = \mathbf{0}$, $D_p u = \mathbf{0}$. Let $p = a_0 + a_1 T + \cdots + a_n T^n \in \mathbb{C}[\mathbf{X}][T]$, where $a_0, a_1, \cdots, a_n \in \mathbb{C}[\mathbf{X}]$ and $a_n \neq \mathbf{0}$ in $\mathbb{C}[\mathbf{X}]$. Fourier transformation with respect to the spatial variables in $D_p u = \mathbf{0}$ gives $a_0(i\boldsymbol{\xi})\widehat{u} + a_1(i\boldsymbol{\xi})\frac{\partial}{\partial t}\widehat{u} + \cdots + a_n(i\boldsymbol{\xi})(\frac{\partial}{\partial t})^n\widehat{u} = \mathbf{0}$. Let $\boldsymbol{\xi} \in \mathbb{R}^d$ be such that $a_n(i\boldsymbol{\xi}) \neq 0$. Then $(\widehat{u}(\varphi))(\boldsymbol{\xi}) = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R})$. Since the Lebesgue measure of the set of zeros of the polynomial $a_n(i\boldsymbol{\xi})$ is zero, it follows that for each $\varphi \in \mathcal{D}(\mathbb{R})$, the function $\mathbb{R}^d \ni \boldsymbol{\xi} \mapsto (\widehat{u}(\varphi))(\boldsymbol{\xi})$ is 0 almost everywhere, and so $\widehat{u}(\varphi) = \mathbf{0}$. But then $\widehat{u} = \mathbf{0}$ too, and so $u = \mathbf{0}$. ☐

As the $H_s(\mathbb{R}^d)$ are special instances of the spaces $B_{p,q}(\mathbb{R}^d)$ [8, Example 10.1.2, p.5], and also $\mathcal{S}(\mathbb{R}^d) \subset B_{p,k}(\mathbb{R}^d)$ [8, Thm. 10.1.7, p.7], we have:

Corollary 5.2. *Let* $p \in \mathbb{C}[\mathbf{X}, T]$ *and* $S = \mathcal{L}(\mathcal{D}(\mathbb{R}), H_s(\mathbb{R}^d))$ *or* $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{S}(\mathbb{R}^d))$. *Then* $N_S(p) = \{\mathbf{0}\}$ *if and only if* $p \neq \mathbf{0}$.

By the Payley-Wiener-Schwartz theorem [12, Prop. 29.1, p. 307], the Fourier transform of elements of $\mathcal{E}'(\mathbb{R}^d)$ can be extended to entire functions on $\mathbb{C}^d$. Thus the same proof, mutatis mutandis, as of Thm. 5.1 gives:

Theorem 5.3. *Let* $p \in \mathbb{C}[\mathbf{X}, T]$. *Then* $N_{\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{E}'(\mathbb{R}^d))}(p) = \{\mathbf{0}\}$ *if and only if* $p \neq \mathbf{0}$.

6. Spatially periodic distributions

In this section, we consider the space $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$, which is, roughly speaking, the set of all distributions on $\mathbb{R}^{d+1}$ that are periodic in the spatial directions with a discrete set $\mathbb{A}$ of periods. We now give the definition of $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$. For $\mathbf{a} \in \mathbb{R}^d$, the *translation operation* $\mathbf{S_a}$ on distributions in $\mathcal{D}'(\mathbb{R}^d)$ is defined by $\langle \mathbf{S_a}(T), \varphi\rangle = \langle T, \varphi(\cdot + \mathbf{a})\rangle$ for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$. $T \in \mathcal{D}'(\mathbb{R}^d)$ is said to be *periodic with a period* $\mathbf{a} \in \mathbb{R}^d$ if $T = \mathbf{S_a}(T)$. Let $\mathbb{A} := \{\mathbf{a_1}, \cdots, \mathbf{a_d}\}$ be an independent set vectors in $\mathbb{R}^d$. We define $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$ to be the set of all distributions T that satisfy $\mathbf{S_{a_k}}(T) = T$, $k = 1, \cdots, d$. From [2, §34], T is a tempered distribution, and taking Fourier transforms, $(1 - e^{i\mathbf{a_k}\cdot\boldsymbol{\xi}})\widehat{T} = 0$ for $k = 1, \cdots, d$. It can be seen that $\widehat{T} = \sum_{\mathbf{v} \in A^{-1}2\pi\mathbb{Z}^d} \alpha_{\mathbf{v}}(T)\delta_{\mathbf{v}}$, for some scalars $\alpha_{\mathbf{v}} \in \mathbb{C}$, and where A is the matrix with its rows equal to the transposes of the column vectors $\mathbf{a_1}, \cdots, \mathbf{a_d}$: $A^\top := \begin{bmatrix} \mathbf{a_1} & \cdots & \mathbf{a_d} \end{bmatrix}$. By the Schwartz Kernel Theorem [7, p. 128, Thm. 5.2.1], $\mathcal{D}'(\mathbb{R}^{d+1})$ is isomorphic as a topological space to $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{D}'(\mathbb{R}^d))$, the space of all continuous linear maps from $\mathcal{D}(\mathbb{R})$ to $\mathcal{D}'(\mathbb{R}^d)$, thought of as vector-valued distributions. In this section, we indicate this isomorphism by putting an arrow on top of elements of $\mathcal{D}'(\mathbb{R}^{d+1})$. Thus for $u \in \mathcal{D}'(\mathbb{R}^{d+1})$, we set $\vec{u} \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{D}'(\mathbb{R}^d))$ to be the vector valued distribution defined by $\langle \vec{u}(\varphi), \psi \rangle = \langle u, \psi \otimes \varphi \rangle$ for $\varphi \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{D}(\mathbb{R}^d)$. We define $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1}) = \{u \in \mathcal{D}'(\mathbb{R}^{d+1}) : \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}),\ \vec{u}(\varphi) \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)\}$. Then for $u \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$, $\frac{\partial}{\partial x_k} u \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$ for $k = 1, \cdots, d$, and $\frac{\partial}{\partial t} u \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$. Also, for $u \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$, we define $\widehat{u} \in \mathcal{D}'(\mathbb{R}^{d+1})$ by $\langle \widehat{u}, \psi \otimes \varphi \rangle = \langle \vec{u}(\varphi), \widehat{\psi} \rangle$, for $\varphi \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{D}(\mathbb{R}^d)$. We have the following characterisation for the space of null solutions to be trivial.

Theorem 6.1. *Suppose that* $\mathbb{A} = \{\mathbf{a_1}, \cdots, \mathbf{a_d}\}$ *is a linearly independent set of vectors in* $\mathbb{R}^d$*. Let* $S = \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$ *and* $p \in \mathbb{C}[\mathbf{X}, T]$*. Then* $N_S(p) = \{\mathbf{0}\}$ *if and only if for all* $\mathbf{v} \in A^{-1} 2\pi \mathbb{Z}^d$*, there exists a* $t \in \mathbb{C}$ *such that* $p(i\mathbf{v}, t) \neq 0$*.*

Proof. ('Only if' part:) Let $\mathbf{v_0} \in A^{-1} 2\pi \mathbb{Z}^d$ be such that for all $t \in \mathbb{C}$, $p(i\mathbf{v_0}, t) = \mathbf{0}$. Then $p(i\mathbf{v_0}, T) = 0 \in \mathbb{C}[T]$. Let $\Theta \in C^\infty(\mathbb{R})$ be any nonzero smooth function such that $\Theta|_{t<0} = \mathbf{0}$. Define $u := e^{i\mathbf{v_0} \cdot \mathbf{x}} \otimes \Theta$. Here $\mathbf{v_0} \cdot \mathbf{x}$ is the usual real Euclidean inner product of $\mathbf{v_0}, \mathbf{x} \in \mathbb{R}^d$. Then $u \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$, as $\mathbf{S_{a_k}} u = e^{i\mathbf{v_0} \cdot (\mathbf{x} + \mathbf{a_k})} \otimes \Theta = e^{i\mathbf{v_0} \cdot \mathbf{a_k}} e^{i\mathbf{v_0} \cdot \mathbf{x}} \otimes \Theta = u$. We have $u|_{t<0} = \mathbf{0}$, because $\Theta|_{t<0} = \mathbf{0}$. Also, $u \in N_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p) \backslash \{\mathbf{0}\}$ since $\Theta \neq \mathbf{0}$ and $p(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t}) u = e^{i\mathbf{v_0} \cdot \mathbf{x}} p(i\mathbf{v_0}, \frac{d}{dt}) \Theta = e^{i\mathbf{v_0} \cdot \mathbf{x}} \cdot \mathbf{0} = \mathbf{0}$.
('If':) Suppose that for each $\mathbf{v} \in A^{-1} 2\pi \mathbb{Z}^d$, there exists a $t \in \mathbb{C}$ such that $p(i\mathbf{v}, t) \neq 0$. Then $p(i\mathbf{v}, T) \neq 0 \in \mathbb{C}[T]$. So $N_{\mathcal{D}'(\mathbb{R})}(p(i\mathbf{v}, T)) = \{\mathbf{0}\}$. Thus for each $\mathbf{v} \in A^{-1} 2\pi \mathbb{Z}^d$, whenever $T \in \mathcal{D}'(\mathbb{R})$ is such that $T|_{t<0} = \mathbf{0}$ and satisfies $p(i\mathbf{v}, \frac{d}{dt}) T = \mathbf{0}$, there holds that $T = \mathbf{0}$. Suppose that $u \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$ satisfies $u|_{t<0} = \mathbf{0}$ and $p(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t}) u = \mathbf{0}$. Taking Fourier transformation with respect to the spatial variables, $p(i\boldsymbol{\xi}, \frac{\partial}{\partial t}) \widehat{u} = \mathbf{0}$. For each fixed $\varphi \in \mathcal{D}(\mathbb{R})$, $\vec{u}(\varphi) \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$, and so $\widehat{\vec{u}(\varphi)} = \sum_{\mathbf{v} \in A^{-1} 2\pi \mathbb{Z}^d} \delta_{\mathbf{v}}\, \alpha_{\mathbf{v}}(\widehat{u}, \varphi)$, for appropriate coefficients $\alpha_{\mathbf{v}}(\widehat{u}, \varphi) \in \mathbb{C}$. So the support of $\widehat{u}$ is contained in $A^{-1} 2\pi \mathbb{Z}^d \times [0, +\infty)$. Thus each of the half lines in $A^{-1} 2\pi \mathbb{Z}^d \times [0, +\infty)$ carries a solution of $p(i\boldsymbol{\xi}, \frac{\partial}{\partial t}) \widehat{u} = \mathbf{0}$, and $\widehat{u}$ is a sum of these. We show that each of these summands is zero. The map $\varphi \mapsto \alpha_{\mathbf{v}}(\widehat{u}, \varphi) : \mathcal{D}(\mathbb{R}) \to \mathbb{C}$ defines a distribution $T^{(\mathbf{v})}$ in $\mathcal{D}'(\mathbb{R})$. Moreover, the support of $T^{(\mathbf{v})}$ is contained in $[0, +\infty)$. For a small enough neighbourhood N of $\mathbf{v} \in A^{-1} 2\pi \mathbb{Z}^d$ in $\mathbb{R}^d$, we have $\delta_{\mathbf{v}} \otimes p(i\mathbf{v}, \frac{d}{dt}) T^{(\mathbf{v})} = \mathbf{0}$ in $N \times \mathbb{R}$. But our algebraic hypothesis implies that the set $N_{\mathcal{D}'(\mathbb{R})}(p(i\mathbf{v}, T))$ of null solutions is trivial, i.e. $N_{\mathcal{D}'(\mathbb{R})}(p(i\mathbf{v}, T)) = \{\mathbf{0}\}$, and so $T^{(\mathbf{v})} = \mathbf{0}$. As this happens with each $\mathbf{v} \in A^{-1} 2\pi \mathbb{Z}^d$, we conclude that $\widehat{u} = \mathbf{0}$ and hence also $u = \mathbf{0}$. Consequently, $N_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p) = \{\mathbf{0}\}$. □

7. Open question: For which p is the set of futures of null solutions dense in the set of futures of all solutions?

It follows from [5], that the set $\{u|_{t>0} : u \in N_{\mathrm{C}^\infty(\mathbb{R}^{d+1})}(p)\}$ of futures of smooth null solutions, is dense in the set $\{u|_{t>0} : u \in \mathrm{C}^\infty(\mathbb{R}^{d+1}) \text{ and } D_p u = \mathbf{0}\}$ of futures of all smooth solutions, if each irreducible factor p' of p satisfies $\deg(p') \neq \deg(p'(\mathbf{0}, T))$. In our alternative solution spaces, one could ask a similar question, that is, if it is possible to give a characterisation in terms of the polynomial p so that the set of futures of null solutions is dense in the set of futures of all solutions. We leave this class of an open questions for future investigation.

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References

[1] E. Bierstone and P. Milman. Semianalytic and subanalytic sets. *Institut des Hautes Études Scientifiques, Publications Mathématiques*, 67:5-42, 1988.
[2] W.F. Donoghue, Jr. *Distributions and Fourier Transforms.* Pure and Applied Mathematics 32, Academic Press, New York and London, 1969.
[3] G. Hellwig. *Partial differential equations. An introduction. Second edition.* B.G. Teubner, 1977.
[4] L. Hörmander. On the theory of general partial differential operators. *Acta Mathematica*, 94:161-248, 1955.
[5] L. Hörmander. Null solutions of partial differential equations. *Archive for Rational Mechanics and Analysis*, 255-261 no. 4, 1960.
[6] L. Hörmander. *Linear Partial Differential Operators.* Fourth printing. Springer, 1976.
[7] L. Hörmander. *The Analysis of Linear Partial Differential Operators. I.* Second edition. Springer, 1990.
[8] L. Hörmander. *The analysis of linear partial differential operators. II.* Reprint of the 1983 original. Springer, 2005.
[9] S. Krantz and H. Parks. *A Primer of Real Analytic Functions.* Second edition. Birkhäuser, 2002.
[10] S. Łojasiewicz. *Introduction to Complex Analytic Geometry.* Translated from the Polish by Maciej Klimek. Birkhäuser, 1991.
[11] L. Schwartz. *Théorie des Distributions.* Hermann, Paris, 1966.
[12] F. Trèves. *Topological Vector Spaces, Distributions and Kernels.* Dover, New York, 2006.
[13] H. Whitney. Elementary structure of real algebraic varieties. *Annals of Mathematics, Second Series*, 66:545-556, 1957.

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