

**The Steinhaus-Weil property:
II. The Simmons-Mospan Converse
by
N. H. Bingham and A. J. Ostaszewski**

In memory of Harry I. Miller (1939-2018)

Abstract. In this second part of a four-part series (with Parts I, III, IV referring to [BinO4,5,6]), we develop (via Propositions 1, 2 and Theorems 1, 2) a number of relatives of the Simmons-Mospan theorem, a converse to the Steinhaus-Weil theorem (for another, see [BinO1], and for yet others [BinO3, §8.5]). In Part III [BinO5, Theorems 1, 2], we link this with topologies of Weil type.

Keywords. Steinhaus-Weil property, amenability at 1, measure subcontinuity, Simmons-Mospan theorem, Weil topology, interior-points property, Haar measure, Lebesgue decomposition, left Haar null, selective measure, Cameron-Martin space.

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1 A Lebesgue Decomposition

We use the notation and terminology of Part I [BinO4] and refer also to the longer arXiv version [BinO3] combining all four parts.

We begin with definitions isolating left-handed components in Christensen's notion of Haar null sets [Chr1,2], and Solecki's left Haar null sets [Sol2]; right-handed versions have analogous properties. As far as we are aware the component notions in parts (ii)-(iv) below have not been previously studied. Below G is a Polish group, unless otherwise stated.

Definition. (i) **Left μ -null:** For $\mu \in \mathcal{M}(G)$, say that $N \subseteq G$ is *left μ -null* ($N \in \mathcal{M}_0^l(\mu)$) if it is contained in a universally measurable set $B \subseteq G$ such that

$$\mu(gB) = 0 \quad (g \in G).$$

Thus a set $S \subseteq G$ is *left Haar null* ([Sol3] after [Chr1]) if it is contained in a universally measurable set $B \subseteq G$ that is left μ -null for some $\mu \in \mathcal{M}(G)$.

(ii) **Left μ -inversion:** For $\mu \in \mathcal{M}(G)$, say that $N \in \mathcal{M}_0^L(\mu)$ is *left invertibly μ -null* ($N \in \mathcal{M}_0^{L\text{-inv}}(\mu)$) if

$$N^{-1} \in \mathcal{M}_0^L(\mu),$$

so that N^{-1} is contained in a universally measurable set B^{-1} such that

$$\mu(gB^{-1}) = 0 \quad (g \in G).$$

(iii) **Left μ -absolute continuity:** For $\mu, \nu \in \mathcal{M}(G)$, ν is *left absolutely continuous* w.r.t. μ ($\nu <^L \mu$) if $\nu(N) = 0$ for each $N \in \mathcal{M}_0^L(\mu)$, and likewise for the invertibility version: $\nu <^{L\text{-inv}} \mu$.

(iv) **Left μ -singularity:** For $\mu, \nu \in \mathcal{M}(G)$, ν is *left singular* w.r.t. μ (on B) ($\nu \perp^L \mu$ (on B)) if B is a support of ν and B is in $\mathcal{M}_0^L(\mu)$, and likewise $\nu \perp^{L\text{-inv}} \mu$.

Remark. For μ *symmetric*, since

$${}_{g^{-1}}\mu(B) = \mu_g(B^{-1})$$

if B is *left μ -null* we may conclude only that B^{-1} is *right μ -null*. The ‘inversion property’, property (ii) above, is thus quite strong (though obvious in the abelian case).

Notice that each of $\mathcal{M}_0^L(\mu)$ and $\mathcal{M}_0^{L\text{-inv}}(\mu)$ forms a σ -algebra (since $g \bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} gB_n$ and $g \left(\bigcup_{n \in \mathbb{N}} B_n \right)^{-1} = \bigcup_{n \in \mathbb{N}} gB_n^{-1}$). This implies the following left versions of the Lebesgue Decomposition Theorem (we need the second one below). The ‘pedestrian’ proof demonstrates that the Principle of Dependent Choice (DC) suffices, a further example that ‘positive’ results in measure theory follow from DC (as Solovay points out in [Solo, p. 31]).

Theorem LD. *For G a Polish group, $\mu, \nu \in \mathcal{M}(G)$, there are $\nu_a, \nu_s \in \mathcal{M}(G)$ with*

$$\nu = \nu_a + \nu_s \text{ with } \nu_a <^L \mu \text{ and } \nu_s \perp^L \mu,$$

and likewise, there are $\nu'_a, \nu'_s \in \mathcal{M}(G)$ with

$$\nu = \nu'_a + \nu'_s \text{ with } \nu_a <^{L\text{-inv}} \mu \text{ and } \nu_s \perp^{L\text{-inv}} \mu.$$

Proof. As the proof depends on σ -additivity, it will suffice to check the ‘L’ case. Write $G = \bigcup_{n \in \mathbb{N}} G_n$ with the G_n disjoint, universally measurable,

and with each $\nu(G_n)$ finite (say, with all but one term σ -compact, and their complement ν -null). Put $s_n := \sup\{\nu(E) : E \subseteq G_n, E \in \mathcal{M}_0^L(\mu)\}$. In $\mathcal{M}_0^L(\mu)$, for each n with $s_n > 0$, choose $E_{n,m} \subseteq G_n$ with $\nu(E_{n,m}) > s_n - 1/m$, and put $B_n := \bigcup_{m \in \mathbb{N}} E_{n,m} \subseteq G_n$. Then the sets B_n are disjoint and lie in $\mathcal{M}_0^L(\mu)$, as does also $B := \bigcup_{n \in \mathbb{N}} B_n$; moreover $\nu(G_n \setminus B_n) = 0$ for each n . Put $A := G \setminus B$. Then $\nu(M) = 0$ for $M \in \mathcal{M}_0^L(\mu)$ with $M \subseteq A$, since $A = \bigcup_{n \in \mathbb{N}} (G_n \setminus B_n)$. So $\nu_a := \nu|_A \ll^L \mu$, and $\nu_s := \nu|_{B \perp^L \mu}$, since $B \in \mathcal{M}_0^L(\mu)$. \square

Remark. A simpler argument rests on maximality: choose a maximal disjoint family \mathcal{B} of universally measurable sets $M \in \mathcal{M}_0^L(\mu)$ with finite positive $\nu(M)$; then, their union $B \in \mathcal{M}_0^L(\mu)$ (as \mathcal{B} would be countable, by the σ -finiteness of ν).

2 Discontinuity: the Simmons-Mospan Theorem

It is convenient to begin by repeating the gist of the Simmons-Mospan argument here, as it is short, despite its ‘near perfect disguise’, to paraphrase a phrase from Loomis [Loo, p. 85]. The result follows from their use of Theorem FN and the Lebesgue decomposition theorem of §1, but here we stress the dependence on the Fubini Null Theorem (Part I §1) and on left μ -inversion. We revert to the Weil left-sided convention and associated KK^{-1} usage.

Proposition 1 (Local almost nullity). *For G a Polish group, $\mu \in \mathcal{M}(G)$, $V \subseteq G$ open and $K \in \mathcal{K}(G) \cap \mathcal{M}_0^{L\text{-inv}}(\mu)$, so that $K, K^{-1} \in \mathcal{M}_0^L(\mu)$:
– for any $\nu \in \mathcal{M}(G)$, $\nu(tK) = 0$ for μ -almost all $t \in V$, and likewise $\nu(Kt) = 0$.*

Proof. For ν invertibly μ -absolutely continuous (as in §1 above), the conclusion is immediate; for general ν this will follow from Theorem LD (§1), once we have proved the corresponding singular version of the assertion: that is the nub of the proof.

Thus, suppose that $\nu \perp^{L\text{-inv}} \mu$ on K . For $t \in V$ let $t = uw$ be any expression for t as a group product of $u, v \in G$, and note that $\mu(uK^{-1}) = 0$, as $K^{-1} \in \mathcal{M}_0^L(\mu)$. Let H be the set

$$\bigcup_{t \in V} (\{t\} \times tK),$$

here viewed as a union of vertical t -sections. We next express it as a union of u -horizontal sections and apply the Fubini Null Theorem (Th. FN, Part I §1).

Since $u = tk = uwk$ is equivalent to $w = k^{-1}$, the u -horizontal sections of H may now be rewritten, eliminating t , as

$$\{(t, u) : uw = t \in V, u \in tK = uwK\} = \{(uw, u) : uw \in V, uw \in uK^{-1}\}.$$

So H may now be viewed as a union of u -horizontal sections as

$$\bigcup_{u \in G} (V \cap (uK^{-1})) \times \{u\},$$

all of these u -horizontal sections being μ -null. By Th. FN, μ -almost all vertical t -sections of H for $t \in V$ are ν -null. As the assumptions on K are symmetric the right-sided version follows. \square

The result here brings to mind the Dodos Dichotomy Theorem [Dod1, Th. A] for *abelian* Polish groups G : if an analytic set A is witnessed as Haar-null under one measure $\mu \in \mathcal{P}(G)$, then either A is Haar-null for quasi all $\nu \in \mathcal{P}(G)$ or else it is not Haar-null for quasi all such ν , i.e. if $A \in \mathcal{M}_0(\mu)$ (omitting the unnecessary superscript L), then either $A \in \mathcal{M}_0(\nu)$ for quasi all such ν , or $A \notin \mathcal{M}_0(\nu)$ for quasi all such ν (w.r.t. the Prokhorov-Lévy metric in $\mathcal{P}(G)$ [Dud, 11.3, cf. 9.2]). Indeed, [Dod2, Prop. 5] when A is σ -compact A is Haar-null for quasi all $\nu \in \mathcal{P}(G)$. The result is also reminiscent of [Amb, Lemma 1.1].

Before stating the Simmons-Mospan specialization to the Haar context and also to motivate one of the conditions in its subsequent generalizations, we cite (and give a direct proof of) the following known result (equivalence of Haar measure η and its inverse $\tilde{\eta}$), encapsulated in the formula

$$\eta(K^{-1}) = \int_K d\eta(t)/\Delta(t),$$

exhibiting the direct connection between η and $\tilde{\eta}$ via the (positive) modular function Δ [HewR, 15.14], or [Hal, §60.5f]; this equivalence result holds more generally between any two probability measures when one is left and the other right quasi-invariant – see [Xia, Cor. 3.1.4]; this is related to a theorem of Mackey’s [Mac], cf. the longer combined arXiv version [BinO3, §8.16]. As will be seen from the proof, in Lemma H below, there is no need to assume the

group is separable: a compact metrizable subspace (being totally bounded) is separable.

Lemma H (cf. [Hal, §50(ff); §59 Th. D]). *In a locally compact metrizable group G , for K compact, if $\eta(K) = 0$, then $\eta(K^{-1}) = 0$, and, by regularity, so also for K measurable.*

Proof. Fix a compact K . As K is compact, the modular function Δ of G is bounded away from 0 on K , say by $M > 0$; furthermore, K is separable, so pick $\{d_n : n \in \mathbb{N}\}$ dense in K . Then for any $\varepsilon > 0$ there are two (finite) sequences $m(1), \dots, m(n) \in \mathbb{N}$ and $\delta(1), \dots, \delta(n) > 0$ such that $\{B_{\delta(i)}d_{m(i)} : i \leq n\}$ covers K and

$$M \sum_{i \leq n} \eta(B_{\delta(i)}) \leq \sum_{i \leq n} \eta(B_{\delta(i)}) \Delta(d_{m(i)}) = \sum_{i \leq n} \eta(B_{\delta(i)}d_{m(i)}) < \varepsilon.$$

Then

$$\sum_{i \leq n} \eta(d_{m(i)}^{-1}B_{\delta(i)}) = \sum_{i \leq n} \eta(B_{\delta(i)}) \leq \varepsilon/M.$$

But $\{d_{m(i)}^{-1}B_{\delta(i)} : i \leq n\}$ covers K^{-1} by the symmetry of the balls B_δ (by the symmetry of the norm); so, as $\varepsilon > 0$ is arbitrary, $\eta(K^{-1}) = 0$.

As for the final assertion, if $\eta(E^{-1}) > 0$ for some measurable E , then $\eta(K^{-1}) > 0$ for some compact $K^{-1} \subseteq E^{-1}$, by regularity; then $\eta(K) > 0$, and so $\eta(E) > 0$. \square

Proposition 1 and Lemma H immediately give:

Theorem SaMo (cf. [Sak, III.11], [Mos], [BarFF, Th. 7]). *For G a locally compact group with left Haar measure η and ν a Borel measure on G , if the set S is η -null, then for η -almost all t*

$$\nu(tS) = 0.$$

In particular, this is so for S the support of a measure ν singular with respect to η .

This in turn allows us to prove the locally compact (separable) case of the Simmons-Mospan Theorem, Theorem SM ([Sim, Th. 1], [Mos, Th. 7], recently rediscovered in the abelian case [BarFF, Th. 10]). Then in Theorem 2 below we pursue a non-locally compact variant.

Theorem SM. *In a locally compact Polish group, a Borel measure has the Steinhaus-Weil property if and only if it is absolutely continuous with respect to Haar measure.*

Proof. If μ is absolutely continuous w.r.t. Haar measure η , then μ , being invariant, is subcontinuous, and Lemma 1 (in Part I §2) gives the Steinhaus-Weil property. Otherwise, decomposing μ into its singular and absolutely continuous parts w.r.t. η , choose K a compact subset of the support of the singular part of μ ; then $\mu(K) > \mu_-(K) = 0$, by Theorem SaMo above, and so Prop. 6 (Converse part – see Part I §2) applies. \square

Proposition 2 (after Simmons, cf. [Sim, Lemma] and [BarFF, Th. 8]). *For G a Polish group, $\mu, \nu \in \mathcal{M}(G)$ and $\nu \perp^{L\text{-inv}} \mu$ concentrated on a compact left invertibly μ -null set K , there is a Borel $B \subseteq K$ such that $K \setminus B$ is ν -null and both BB^{-1} and $B^{-1}B$ have empty interior.*

Proof. As we are concerned only with the subspace $KK^{-1} \cup K^{-1}K$, w.l.o.g. the group G is separable. By Prop. 1, $Z := \{x : \nu(xK) = 0\}$ is dense and so also

$$Z_1 := \{x : \nu(K \cap xK) = 0\},$$

since $\nu(K \cap xK) \leq \nu(xK) = 0$, so that $Z \subseteq Z_1$. Take a denumerable dense set $D \subseteq Z_1$ and put

$$S := \bigcup_{d \in D} K \cap dK.$$

Then $\nu(S) = 0$. Take $B := K \setminus S$. If $\emptyset \neq V \subseteq BB^{-1}$ and $d \in D \cap V$, then for some $b_1, b_2 \in B \subseteq K$

$$d = b_1 b_2^{-1} : \quad b_1 = db_2 \in K \cap dK \subseteq S,$$

a contradiction, since $B \cap S = \emptyset$. So $(K \setminus S)(K \setminus S)^{-1}$ has empty interior. A similar argument based on

$$T := \bigcup_{d \in D} Kd \cap K$$

ensures that also $(K \setminus S \setminus T)^{-1}(K \setminus S \setminus T)$ has empty interior. \square

In order to generalize the Simmons Theorem from its locally compact context we will need to cite the following result. Here $\mathbb{Q}_+ := \mathbb{Q} \cap (0, \infty)$

denotes the positive rationals, and $B_\delta^{K,\Delta}(\sigma) := \{z \in B_\delta : \sigma(Kz) > \Delta\}$ as in Part I §2.

Theorem 1 (Disaggregation Theorem, [BinO2, Th. 7.1, Prop. 7.1]). *Let G be a Polish group that is strongly amenable at 1, and let \mathbf{t} be a regular null sequence. For $\sigma = \sigma(\mathbf{t})$ there are a countable family \mathcal{H} with $\mathcal{H} \subseteq \mathcal{K}_+(\sigma)$, a countable set $D = D(\mathcal{H}) \subseteq G$ dense in G , and a dense subset $G(\sigma)$ on which the sets below are the sub-basic sets of a metrizable topology:*

$$B_\delta^{K,\Delta}(\sigma) \quad (K \in \mathcal{H}, \delta, \Delta \in \mathbb{Q}_+, \Delta < \sigma(K)).$$

In particular, the space $G(\sigma)$ is continuously and compactly embedded in G . Moreover, each such open set contains a cofinal subsequence of \mathbf{t} .

For a proof we refer the reader to [BinO2]; the result relies on I Cor. 4. The subspace $G(\sigma)$ here is a topological analogue of the *Cameron-Martin* subspace $H(\gamma)$ of a locally convex topological vector space equipped with a Radon Gaussian measure γ – see [BinO3, §8.2-3].

We are now ready for the promised generalization. This requires equivalence of the selective measure (Part I §2) and its inverse – valid at least in Polish abelian groups (see Part I, Th. 4 §2 on strong amenability at 1).

Theorem 2 (Generalized Simmons Theorem, cf. [Sim, Th. 2]). *Let G be a Polish group that is strongly amenable at 1 (e.g. if also G is abelian), let $\sigma = \sigma(\mathbf{t})$ be a selective measure corresponding to a regular null sequence \mathbf{t} , which we assume is equivalent to its inverse $\tilde{\sigma}$ (e.g. if G is abelian), and let $G(\sigma)$ be the dense subspace endowed with the refinement topology as in the preceding theorem. Then:*

$\nu \in \mathcal{M}(G)$ is left invertibly-singular w.r.t. σ iff ν has a support that is a σ -compact union of compact sets K_n with each of the compact sets $K_n K_n^{-1}$ and $K_n^{-1} K_n$ nowhere dense (equivalently: having empty interior) in the topology of $G(\sigma)$.

Proof. Suppose that $\mu \in \mathcal{M}(G)$ is subcontinuous. If $\nu \in \mathcal{M}(G)$, by Theorem LD write

$$\nu = \nu_a + \nu_s \text{ with } \nu_a \ll^{L\text{-inv}} \mu \text{ and } \nu_s \perp^{L\text{-inv}} \mu.$$

If ν is concentrated as in the statement of the theorem on a σ -compact set B with $B^{-1}B$ having empty interior in $G(\sigma)$, then $\nu_a = 0$, and so ν is left

invertibly-singular w.r.t. μ . Indeed, as ν is concentrated on B , so is ν_a . We claim that $\nu_a(B) = 0$. Otherwise, $\nu_a(K_n) > 0$ for some compact $K_n \subseteq B$. So $K := K_n \notin \mathcal{M}_0^{L\text{-inv}}(\sigma)$, as $\nu_a <^{L\text{-inv}} \sigma$. The argument now splits into two cases, according as $K \notin \mathcal{M}_0^L(\sigma)$ or $K^{-1} \notin \mathcal{M}_0^L(\sigma)$

First, suppose that $\sigma(gK) > 0$ for some $g \in G$; then, by Part I, Lemma 1, there are $\delta > 0$ and $0 < \Delta < \sigma_-^t(gK)$ with

$$B_\delta^{gK, \Delta}(\sigma) \subseteq (gK)^{-1}gK = K^{-1}K \subseteq B^{-1}B,$$

contradicting the above property of B .

Next, suppose that $\sigma(Kg) = \sigma^{-1}(g^{-1}K^{-1}) > 0$ for some $g \in G$; so $\sigma(g^{-1}K^{-1}) > 0$, as $\tilde{\sigma}$ is equivalent to σ . Then, again by Part I Lemma 1, there are $\delta > 0$ and $0 < \Delta < \sigma_-^t(g^{-1}K^{-1})$ with

$$B_\delta^{g^{-1}K^{-1}, \Delta}(\sigma) \subseteq (g^{-1}K^{-1})^{-1}g^{-1}K^{-1} = KK^{-1} \subseteq BB^{-1},$$

again contradicting the above property of B . So $\nu = \nu_a$ is invertibly singular w.r.t. μ .

The rest of the proof is as in Simmons [Sim, Th. 2], using I Prop. 6: the Baire-category argument still holds, since compactness implies closure under the $G(\sigma(\mathbf{t}))$ -topology, the latter being a finer topology; avoidance of interior points requires second countability, assured by Th. 5. \square

Corollary 5 (Simmons Theorem: [Sim, Th. 2]). *For G separable and locally compact and η left Haar measure:*

$\nu \in \mathcal{M}(G)$ is singular w.r.t. η iff ν has a support that is a σ -compact union of compact sets K_n with each of the compact sets $K_n K_n^{-1}$ nowhere dense (equivalently: having empty interior).

For the non-separable version of the above, see [BinO3, §8.1].

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Mathematics Department, Imperial College, London SW7 2AZ; n.bingham@ic.ac.uk
Mathematics Department, London School of Economics, Houghton Street,
London WC2A 2AE; A.J.Ostaszewski@lse.ac.uk