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An analogue of Serre's conjecture for a ring of distributions

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Abstract: The set $\mathcal{A} := \mathbb{C}\delta_0 + \mathcal{D}'_+$, obtained by attaching the identity δ_0 to the set $\mathcal{D}'_+$ of all distributions on $\mathbb{R}$ with support contained in $(0, \infty)$, forms an algebra with the operations of addition, convolution, multiplication by complex scalars. It is shown that $\mathcal{A}$ is a Hermite ring, that is, every finitely generated stably free $\mathcal{A}$-module is free, or equivalently, every tall left-invertible matrix with entries from $\mathcal{A}$ can be completed to a square matrix with entries from $\mathcal{A}$, which is invertible.

Keywords: Hermite ring, Serre's conjecture, algebraic K-theory, Schwartz distribution theory

MSC: Primary 46F10; Secondary 19B10, 19K99, 46H05

1 Introduction

The aim of this note is to show that the the ring $\mathcal{A}$ is a Hermite ring. The relevant definitions are recalled below. For preliminaries on the distribution theory of L. Schwartz, we refer the reader to [1] and [2]. For the commutative algebraic terminology used below, we refer to [3] and [4].

Let $\mathcal{D}'_+$ denote the set of all distributions $T \in \mathcal{D}'(\mathbb{R})$ having their distributional support contained in the half line $(0, \infty)$. Then $\mathcal{D}'_+$ is an algebra with pointwise addition, pointwise multiplication by scalars, and with convolution taken as multiplication in the algebra. However, $\mathcal{D}'_+$ lacks the identity element with respect to multiplication. We can adjoin the identity element to the algebra $\mathcal{D}'_+$, hence obtaining the larger algebra

$$\mathcal{A} := \mathbb{C}\delta_0 + \mathcal{D}'_+,$$

whose elements are of the form $\alpha\delta_0 + T$, where $\alpha \in \mathbb{C}$ and $T \in \mathcal{D}'_+$. $\mathcal{A}$ is also an algebra with the same operations. We will denote the convolution operation henceforth by $*$.

Serre's question from 1955 was if, for the ring $R = \mathbb{F}[x_1, \cdots, x_d]$ ($\mathbb{F}$ a field), every finitely generated projective R-module is free. This was eventually settled in 1976, independently, by Quillen and by Suslin, and the considerations over this question gave rise to the subject of algebraic K-theory. In light of the Hilbert-Serre theorem, Serre's question for $R = \mathbb{F}[x_1, \cdots, x_d]$ can be reduced to the question of whether every finitely generated stably free R-module is free. A commutative unital ring R having the property that every finitely generated stably free R-module is free is called a *Hermite ring*. In terms of matrices over the ring R, one has the following characterisation of Hermite rings, see for example [3, p.VIII], [5, p.1029], [6, Lemma 8.1.20, p.290].

Let R be a commutative unital ring with multiplicative identity denoted by 1. For $m, n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, we denote by $R^{m\times n}$ the matrices with m rows and n columns having entries from R. The identity element in $R^{k\times k}$ having 1s on the diagonal and zeroes elsewhere will be denoted by I_k. A tall matrix $f \in R^{K\times k}$ is said to be *left-invertible* if there exists a $g \in R^{k\times K}$ such that $gf = I_k$. The ring R is Hermite if and only if

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> for all k and $K \in \mathbb{N}$ such that $k < K$, and
> for all $f \in R^{K\times k}$ such that there exists a $g \in R^{k\times K}$ so that $gf = I_k$,
> there exists an $f_c \in R^{K\times(K-k)}$ and there exists a $G \in R^{K\times K}$
> such that $G\begin{bmatrix} f \mid f_c \end{bmatrix} = I_K$.

In other words, the ring R is Hermite if every left invertible matrix over R can be extended to an invertible one.

Example 1.1. $R = \mathbb{C}$ is a Hermite ring. Indeed, suppose that $f \in \mathbb{C}^{K\times k}$ is left-invertible, and that $gf = I_k$ for some $g \in \mathbb{C}^{k\times K}$. Then if $v \in \mathbb{C}^k$ is such that $fv = 0$, it follows that $v = I_k v = gfv = g0 = 0$. So the columns $v_1, \cdots, v_k$ of f are linearly $\mathbb{C}^K$, and hence we can find $v_{k+1}, \cdots, v_K \in \mathbb{C}^K$ such that $\{v_1 \cdots, v_k, v_{k+1}, \cdots, v_K\}$ forms a basis for $\mathbb{C}^K$. Defining $f_c = \begin{bmatrix} v_{k+1} & \cdots & v_K \end{bmatrix} \in \mathbb{C}^{K\times(K-k)}$, we have that $\begin{bmatrix} f \mid f_c \end{bmatrix} \in \mathbb{C}^{K\times K}$ is invertible in $\mathbb{C}^{K\times K}$. ◇

The following example is well-known, see e.g. [6, Example 8.1.27, p.292].

Non-example 1.2. Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the unit sphere, and let $C(\mathbb{S}^2, \mathbb{R})$ be the ring of *real*-valued continuous functions on $\mathbb{S}^2$, with pointwise operations. Then $C(\mathbb{S}^2, \mathbb{R})$ is *not* a Hermite ring. Indeed, taking $f \in (C(\mathbb{S}^2, \mathbb{R}))^{3\times 1}$ as the map sending the point x to the normal vector at x to the manifold $\mathbb{S}^2$, that is, $x \overset{f}{\mapsto} \widehat{n}(x) = x : \mathbb{S}^2 \to \mathbb{R}^3$, we see that f is left invertible, thanks to the fact that $\langle x, x\rangle = 1$ in $\mathbb{R}^3$ for each $x \in \mathbb{S}^2$. But if $C(\mathbb{S}^2, \mathbb{R})$ were a Hermite ring, then f could be extended to an invertible matrix $\begin{bmatrix} f & g & h \end{bmatrix} \in (C(\mathbb{S}^2, \mathbb{R}))^{3\times 3}$. This results in continuous maps $v_1, v_2 : \mathbb{S}^2 \to \mathbb{R}^3$ so that $\{f(x) = n(x), v_1(x), v_2(x)\}$ forms an orthonormal basis for $\mathbb{R}^3$: indeed, we take $v_1(x), v_2(x)$ to the the projections of $g(x), h(x)$ onto the tangent space $T\mathbb{S}^2_x$ to $\mathbb{S}^2$ at x. In particular, $v_1 : \mathbb{S}^2 \to \mathbb{R}^3$ would be a continuous tangent vector field on $\mathbb{S}^2$ which is nowhere vanishing, contradicting the Hairy Ball Theorem [7]. ◇

Our result is the following:

Theorem 1.3. $(\mathbb{C}\delta_0 + \mathcal{D}'_+, +, \star)$ *is a Hermite ring.*

Proof. Let $\mathcal{A}$ be the ring $\mathbb{C}\delta_0 + \mathcal{D}'_+$. Let $f \in \mathcal{A}^{K\times k}$ be left invertible, with $gf = I_k\delta_0$ for some $g \in \mathcal{A}^{k\times K}$. Write

$$\begin{aligned} f &= \delta_0 f_0 + f_+, \\ g &= \delta_0 g_0 + g_+, \end{aligned}$$

where $f_+ \in (\mathcal{D}'_+)^{K\times k}$, $f_0 \in \mathbb{C}^{K\times k}$, and $g_+ \in (\mathcal{D}'_+)^{k\times K}$, $g_0 \in \mathbb{C}^{k\times K}$. From $gf = I_k\delta_0$, we obtain that

$$g_0 f_0 \delta_0 + (g_0 f_+ + g_+ f_0 + g_+ f_+) = \delta_0 I_k.$$

As the entries of f_+, g_+ belong to $\mathcal{D}'_+$, it follows that there exists an $\epsilon > 0$ such that each of the entries of $g_0 f_+ + g_+ f_0 + g_+ f_+$ has its support in (ϵ, ∞). So if we act both sides (entry-wise) on a test function $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\mathrm{supp}(\varphi) \subset (-\infty, \epsilon)$, then we obtain

$$g_0 f_0 \varphi(0) = I_k \varphi(0).$$

Choosing $\varphi(0) \neq 0$, this now shows that $g_0 f_0 = I_k$. But as $\mathbb{C}$ is Hermite, we can now find a $f_c \in \mathbb{C}^{K\times(K-k)}$ and a $G_0 \in \mathbb{C}^{K\times K}$ such that

$$G_0 \begin{bmatrix} f_0 \mid f_c \end{bmatrix} = I_K,$$

that is,

$$(G_0\delta_0)\begin{bmatrix} f_0\delta_0 + f_+ \mid f_c\delta_0 \end{bmatrix} = I_K\delta_0 - \underbrace{(G_0\delta_0)\begin{bmatrix} -f_+ \mid 0\delta_0 \end{bmatrix}}_{=:T}.$$

As $f_+ \in (\mathcal{D}'_+)^{K\times k}$, it follows that $T \in (\mathcal{D}'_+)^{K\times K}$. Suppose that $\epsilon' > 0$ is such that each entry of T has its support contained in (ϵ', ∞). We claim that $I_K\delta_0 - T$ is invertible in $(\mathcal{A})^{K\times K}$. Define the "geometric series"

$$S = I_K\delta_0 + T + T^2 + T^3 + \cdots.$$

We will now show that S is well-defined. We recall the theorem on supports for convolution of distributions [1, Theorem 8,p.120], namely that

$$\operatorname{supp}(T_1 \star T_2) \subset \operatorname{supp}(T_1) + \operatorname{supp}(T_2)$$

for any two distributions $T_1, T_2 \in \mathcal{D}'(\mathbb{R})$ whose supports satisfy the convolution condition. It follows that in our case, each entry T^n has its support contained in $n \cdot \operatorname{supp}(T) = n[\epsilon', \infty) = [n\epsilon', \infty)$. So it follows that the series for S converges. Indeed, given any test function $\varphi \in \mathcal{D}(\mathbb{R})$, the series (with the action $\langle T^n, \varphi\rangle$ understood to be entry-wise)

$$\sum_{n=1}^{\infty} \langle T^n, \varphi\rangle$$

contains only finitely many nonzero terms. Now if S_n denotes the nth partial sum of the series $I_K\delta_0 + T + T^2 + T^3 + \cdots$, we have

$$\begin{aligned} (I_K\delta_0 - T)S &= (I_K\delta_0 - T) \lim_{n\to\infty} S_n \\ &= \lim_{n\to\infty} (I_K\delta_0 - T)S_n \\ &= \lim_{n\to\infty} (I_K\delta_0 - T^{n+1}) \\ &= I_K\delta_0. \end{aligned}$$

The second equality above follows from the continuity of convolution in $\mathcal{D}'_+$; see [1, Theorem 7 p.120]. Now, setting

$$G = S(G_0\delta_0) = (I_K\delta_0 - T)^{-1}(G_0\delta_0) \in \mathcal{A}^{K\times K},$$

we have

$$\begin{aligned} G\left[\begin{array}{c|c} f & f_c\delta_0 \end{array}\right] &= (I_K\delta_0 - T)^{-1}(G_0\delta_0)\left[\begin{array}{c|c} f_0\delta_0 + f_+ & f_c\delta_0 \end{array}\right] \\ &= (I_K\delta_0 - T)^{-1}(I_K\delta_0 - T) \\ &= I_K\delta_0. \end{aligned}$$

This completes the proof. □

2 Remarks

2.1 A conjecture

Another natural convolution algebra is the algebra $\mathcal{E}'(\mathbb{R})$ of all compactly supported distributions, again the usual pointwise addition and convolution taken as multiplication. We have the following:

Conjecture 2.1. $(\mathcal{E}'(\mathbb{R}), +, \star)$ is a Hermite ring.

2.2 A corona-type condition for left invertibility

The famous Carleson corona theorem [8] answered Kakutani's 1942 question of whether the 'corona' $M(H^\infty)\backslash\overline{\mathbb{D}}$ is empty. Here H^∞ denotes the Banach algebra of bounded holomorphic functions in the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, with pointwise operations and the supremum norm $\|f\|_\infty := \sup\{|f(z)| : z \in \mathbb{D}\}$. Also, $M(H^\infty)$ denotes the maximal ideal space of H^∞ (the set of all multiplicative linear functionals $\varphi : H^\infty \to \mathbb{C}$, endowed with the Gelfand topology, that is the topology induced from the dual space $\mathcal{L}(H^\infty, \mathbb{C})$ equipped with the weak-$\star$ topology). From the elementary theory of Banach algebras (see e.g. [9, Lemma 9.2.6]), the answer to this question in the affirmative is equivalent to the following result (the matricial version given below is attributed to [10], and is a consequence of [8]).

Theorem 2.2. *Let $f \in (H^\infty)^{K\times k}$, where $K \geq k$. Then the following are equivalent:*
1. *There exists a $g \in (H^\infty)^{k\times K}$ such that $gf = I_k$.*
2. *There exists an $\epsilon > 0$ such that for all $z \in \mathbb{D}$, $f(z)^\star f(z) \geq \epsilon^2 I_k$.*

(Here $f(z)^\star f(z) \geq \epsilon^2 I_k$ means that $\|f(z)\mathbf{v}\|_{\mathbb{C}^k} \geq \epsilon^2 \|\mathbf{v}\|^2_{\mathbb{C}^k}$ for all $\mathbf{v} \in \mathbb{C}^k$, and $\|\cdot\|_{\mathbb{C}^k}$ denotes the usual Euclidean norm on $\mathbb{C}^k$.)

Theorem 2.3. *Let $\mathcal{A}$ be the ring $(\mathbb{C}\delta_0 + \mathcal{D}'_+, +, *)$. Then the following are equivalent for $f = f_0\delta_0 + f_+ \in \mathcal{A}^{K\times k}$, where $f_0 \in \mathbb{C}^{K\times k}$ and $f_+ \in (\mathcal{D}'_+)^{K\times k}$, $K, k \in \mathbb{N}$ with $K \geq k$:*
1. *There exists a $g \in \mathcal{A}^{k\times K}$ such that $gf = I_k\delta_0$.*
2. *There exists an $\epsilon > 0$ such that $f_0^\star f_0 \geq \epsilon^2 I_k$.*

Proof. (1)⇒(2): Write $g = g_0\delta_0 + g_+$, where $g_0 \in \mathbb{C}^{k\times K}$ and $g_+ \in (\mathcal{D}'_+)^{k\times K}$. Then

$$I_k\delta_0 = gf = g_0 f_0 \delta_0 + (g_0 f_+ + g_+ f_0 + g_+ f_+),$$

and since the bracketed expression has support in $(0, \infty)$, it follows that $I_k = g_0 f_0$. Then with $\epsilon := \|g_0^\star\|^{-2}$, where $\|g_0^\star\|$ denotes the induced operator norm of the multiplication map $v \mapsto g_0^\star v : \mathbb{C}^k \to \mathbb{C}^K$, and $\mathbb{C}^k$, $\mathbb{C}^K$ are equipped with the usual Euclidean 2-norms.

(2)⇒(1): If (2) holds, then $\ker(f_0^\star f_0) = \{0\}$, and so $f_0^\star f_0$ is invertible. Taking $g_0 := (f_0^\star f_0)^{-1} f_0^\star$, we then have $g_0 f_0 = (f_0^\star f_0)^{-1} f_0^\star f_0 = I_k$. We have

$$g_0 f = I_k\delta_0 + g_0 f_+,$$

and since the support of $T := g_0 f_+$ is contained in $(0, \infty)$, it follows that $I_k\delta_0 + T$ is invertible as an element of $\mathcal{A}^{k\times k}$, with the inverse

$$(I_k\delta_0 + g_0 f_+)^{-1} = (I_k\delta_0 + T)^{-1} = I_k\delta_0 - T + T^2 - T^3 + - \cdots .$$

So $g := (I_k\delta_0 + g_0 f_+)^{-1} g_0 \in \mathcal{A}^{k\times K}$ and

$$gf = (I_k\delta_0 + g_0 f_+)^{-1} g_0 = (I_k\delta_0 + g_0 f_+)^{-1}(I_k\delta_0 + g_0 f_+) = I_k\delta_0.$$

This completes the proof. □

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