Idealness of k-wise intersecting families

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Abstract. A clutter is *k-wise intersecting* if every k members have a common element, yet no element belongs to all members. We conjecture that every 4-wise intersecting clutter is non-ideal. As evidence for our conjecture, we prove it in the binary case. Two key ingredients for our proof are Jaeger's 8-flow theorem for graphs, and Seymour's characterization of the binary matroids with the sums of circuits property. As further evidence for our conjecture, we also note that it follows from an unpublished conjecture of Seymour from 1975.

1 Introduction

Let V be a finite set of *elements*, and $\mathcal{C}$ be a family of subsets of V called *members*. The family $\mathcal{C}$ is a *clutter* over *ground set* V, if no member contains another one [11]. A *cover of* $\mathcal{C}$ is a subset $B \subseteq V$ such that $B \cap C \neq \emptyset$ for all $C \in \mathcal{C}$. A cover is *minimal* if it does not contain another cover. The family of minimal covers forms another clutter over the ground set V, called the *blocker of* $\mathcal{C}$ and denoted $b(\mathcal{C})$. It is well-known that $b(b(\mathcal{C})) = \mathcal{C}$ [15,11]. Consider for $w \in \mathbb{Z}_+^V$ the dual pair of linear programs

$$\begin{array}{llll} \min & w^\top x & \max & \mathbf{1}^\top y \\ \text{s.t.} & \sum (x_u : u \in C) \geq 1 \quad \forall C \in \mathcal{C} & \text{s.t.} & \sum (y_C : u \in C \in \mathcal{C}) \leq w_u \quad \forall u \in V \\ & x \geq \mathbf{0} & & y \geq \mathbf{0} \end{array}$$

where the left and right LPs are denoted (P) and (D), respectively. If the dual (D) has an integral optimal solution for every right-hand-side vector $w \in \mathbb{Z}_+^V$, then $\mathcal{C}$ is said to have the *max-flow min-cut (MFMC) property* [7]. By the theory of *totally dual integral* linear systems, for every MFMC clutter, the primal (P) also admits an integral optimal solution for every cost vector $w \in \mathbb{Z}_+^V$ [12]. This is why the class of MFMC clutters is a natural host to many beautiful *min-max theorems* in Combinatorial Optimization [8]. Let us elaborate.

The *packing number of* $\mathcal{C}$, denoted $\nu(\mathcal{C})$, is the maximum number of pairwise disjoint members. Note that $\nu(\mathcal{C})$ is equal to the maximum value of an integral feasible solution to (D) for $w = \mathbf{1}$. Furthermore, the covers correspond precisely to the $0-1$ feasible solutions to (P). The *covering number of* $\mathcal{C}$, denoted $\tau(\mathcal{C})$, is the minimum cardinality of a cover. Notice that $\tau(\mathcal{C})$ is equal to the minimum value of an integral feasible solution to (P) for $w = \mathbf{1}$. Also, by Weak LP Duality, $\tau(\mathcal{C}) \geq \nu(\mathcal{C})$. The clutter $\mathcal{C}$ *packs* if $\tau(\mathcal{C}) = \nu(\mathcal{C})$ [9]. Observe that if a clutter is MFMC, then it packs.

If the primal (P) has an integral optimal solution for every cost vector $w \in \mathbb{Z}_+^V$, then $\mathcal{C}$ is said to be *ideal* [10]. Ideal clutters form a rich class of clutters, one that contains the class of MFMC clutters, as discussed above. This containment is strict, and in fact, the richest examples of ideal clutters are those that are not MFMC [14,19]. Furthermore, unlike MFMC clutters, if a clutter is ideal, then so is its blocker [13,18,8].

A clutter is *intersecting* if every two members intersect yet no element belongs to every member [2]. That is, a clutter $\mathcal{C}$ is intersecting if $\tau(\mathcal{C}) \geq 2$ and $\nu(\mathcal{C}) = 1$. In particular, an intersecting clutter does not pack, and therefore is not MFMC. Intersecting clutters, however, may be ideal. For instance, the clutter

$$Q_6 := \{\{1,3,5\},\{1,4,6\},\{2,3,6\},\{2,4,5\}\},$$

whose elements are the edges and whose members are the triangles of K_4, is an intersecting clutter that is ideal [23]. In fact, Q_6 is the smallest intersecting clutter which is ideal ([1], Proposition 1.2). It is worth pointing out that if a clutter *and* its blocker are both intersecting, then the clutter must be non-ideal [3].

In this paper, we propose a sufficient condition for non-idealness that is purely combinatorial. We say that $\mathcal{C}$ is *k-wise intersecting* if every k members have a common element, yet no element belongs to all members. Note that for $k = 2$, this notion coincides with the notion of intersecting clutters. Furthermore, for $k \geq 3$, a k-wise intersecting clutter is also $(k-1)$-wise intersecting. The following is our main conjecture.

Conjecture 1. There exists an integer $k \geq 4$ such that every k-wise intersecting clutter is non-ideal.

A clutter is *binary* if the symmetric difference of any odd number of members contains a member [17]. Equivalently, a clutter is binary if every member and every minimal cover have an odd number of elements in common [17]. In particular, if a clutter is binary, then so is the blocker. Many rich classes of clutters are in fact binary [8]. For example, given a graph $G = (V,E)$ and distinct vertices s and t, the clutter of st-paths over ground set E is binary. The clutter Q_6 is also binary. As evidence for Conjecture 1, our main result is that it holds for all binary clutters.

Theorem 2. *Every 4-wise intersecting binary clutter is non-ideal.*

We also show that 4 cannot be replaced by 3 in Conjecture 1, even for binary clutters.

Proposition 3. *There exists an ideal* 3*-wise intersecting binary clutter.*

The example from Proposition 3 comes from the Petersen graph, and also coincides with the clutter T_{30} from [21], §79.3e. It has 30 elements and is the smallest such example that we are aware of.

Finally, as further evidence for Conjecture 1, we also show that it follows from an unpublished conjecture by Seymour from 1975 that was documented in [21], §79.3e.

1.1 Paper Outline

In Section 2, we show that a special class of clutters, called *cuboids* [5,1], sit at the heart of Conjecture 1. Cuboids allow us to reformulate Conjecture 1 in terms of set systems.

In Section 3, we prove Proposition 3 and Theorem 2. Two key ingredients of our proof of Theorem 2 are Jaeger's 8*-flow Theorem* [16] for graphs, and Seymour's characterization of the binary matroids with the *sums of circuits property* [25].

In Section 4, we propose a line of attack for tackling Conjecture 1, inspired by the recent work of [4]. We also discuss two applications of Theorem 2 to ideal binary clutters. Each application goes hand-in-hand with two strengthenings of Conjecture 1. One strengthening is proposed by us, which we believe is the right strategy for tackling Conjecture 1. The other conjecture is the unpublished conjecture by Seymour.

2 Cuboids

Let $n \in \mathbb{N}$ and $S \subseteq \{0,1\}^n$. The *cuboid of* S, denoted cuboid(S), is the clutter over ground set $[2n] := \{1,\dots,2n\}$ whose members have incidence vectors $(p_1, 1-p_1,\dots,p_n,1-p_n)$ over all $(p_1,\dots,p_n) \in S$. We say that a clutter is a *cuboid* if it is isomorphic to cuboid(S), for some S.

Observe that for each $C \in \text{cuboid}(S)$, $|C \cap \{2i-1, 2i\}| = 1$ for all $i \in [n]$. In particular, every member of cuboid(S) has size n (hence cuboid(S) is a clutter) and $\tau(\text{cuboid}(S)) \le 2$. Cuboids were introduced in [5] and further studied in [1].

We now describe what it means for cuboid(S) to be k-wise intersecting. We say that the points in S *agree on a coordinate* if $S \subseteq \{x : x_i = a\}$ for some coordinate $i \in [n]$ and some $a \in \{0,1\}$.

Remark 4. Let $S \subseteq \{0,1\}^n$. Then cuboid(S) is a k-wise intersecting clutter if, and only if, the points in S do not agree on a coordinate yet every k points do.

Next, we describe what it means for cuboid(S) to be ideal. Let conv(S) denote the convex hull of S. An inequality of the form $\sum_{i\in I} x_i + \sum_{j\in J}(1-x_j) \ge 1$, for some disjoint $I, J \subseteq [n]$, is called a *generalized set covering inequality* [8]. The set S is *cube-ideal* if every facet of conv(S) is defined by $x_i \ge 0$, $x_i \le 1$, or a generalized set covering inequality [1].

Theorem 5 **([1], Theorem 1.6).** *Let $S \subseteq \{0,1\}^n$. Then* cuboid(S) *is an ideal clutter if, and only if, S is a cube-ideal set.*

As a result, Conjecture 1 for cuboids reduces to the following conjecture:

Conjecture 6. There is a constant $k \geq 4$ such that for every cube-ideal set, either all the points agree on a coordinate, or there is a subset of at most k points that do not agree on a coordinate.

Surprisingly, we now show that Conjecture 6 is equivalent to Conjecture 1! Let $\mathcal{C}$ be a clutter over ground set V. To *duplicate an element u of $\mathcal{C}$* is to introduce a new element $\bar{u}$, and replace $\mathcal{C}$ by the clutter over ground set $V \cup \{\bar{u}\}$, whose members are $\{C : C \in \mathcal{C}, u \notin C\} \cup \{C \cup \{\bar{u}\} : C \in \mathcal{C}, u \in C\}$. A *duplication of* $\mathcal{C}$ is a clutter obtained from $\mathcal{C}$ by repeatedly duplicating elements. It is easily checked that a clutter is ideal if and only if some duplication of it is ideal. Moreover, a clutter is k-wise intersecting if and only if some duplication of it is k-wise intersecting.

Let I and J be disjoint subsets of V. The *minor* $\mathcal{C} \setminus I/J$ obtained after *deleting* I and *contracting* J is the clutter over ground set $V - (I \cup J)$ whose members are the minimal sets in $\{C - J : C \in \mathcal{C}, C \cap I = \emptyset\}$. It is well-known that $b(\mathcal{C} \setminus I/J) = b(\mathcal{C})/I \setminus J$ [22]. If $J = \emptyset$, then $\mathcal{C} \setminus I/J = \mathcal{C} \setminus I$ is called a *deletion minor*.

If every $k \geq 2$ members of a clutter have a common element, then so do every k members of a deletion minor. Furthermore, for every element $v \in V$, $\tau(\mathcal{C}) \geq \tau(\mathcal{C} \setminus v) \geq \tau(\mathcal{C}) - 1$, where $\tau(\mathcal{C} \setminus v) = \tau(\mathcal{C}) - 1$ if and only if v belongs to some minimum cover of $\mathcal{C}$. Motivated by these observations, we say that a clutter $\mathcal{C}$ is *tangled* if $\tau(C) = 2$ and every element belongs to a minimum cover.

We require the following facts about tangled clutters.

Proposition 7. *Let $\mathcal{C}$ be a binary tangled clutter. Then $\mathcal{C}$ is a duplication of a cuboid.*

Proof. If $\{e,f\}$ is a minimum cover, then for each $C \in \mathcal{C}$, $|C \cap \{e,f\}|$ must be odd and therefore 1, since $\mathcal{C}$ is a binary clutter. As a result, if $\{e,f\}, \{e,g\}$ are both minimum covers, then f, g must be duplicated elements. Moreover, if every element is contained in exactly one minimum cover, then $\mathcal{C}$ must be a cuboid. These two observations, along with the fact that $\mathcal{C}$ is a tangled clutter, imply that $\mathcal{C}$ is a duplication of a cuboid. □

Remark 8. Let $\mathcal{C}$ be a k-wise intersecting clutter. Let $\mathcal{C}'$ be a deletion minor of $\mathcal{C}$ that is minimal subject to $\tau(\mathcal{C}') \geq 2$. Then $\mathcal{C}'$ is a tangled k-wise intersecting clutter.

Moreover, if a clutter is ideal, then so is every minor of it [23]. Thus, it suffices to prove Conjecture 1 for tangled clutters.

Theorem 9 **([4], Theorem 5.5).** *Let $\mathcal{C}$ be an ideal tangled clutter. Then*

$$\mathcal{C}' := \{C \in \mathcal{C} : |C \cap \{u,v\}| = 1 \ \forall \ \{u,v\} \in b(\mathcal{C})\}$$

is also an ideal tangled clutter. Moreover, $\mathcal{C}'$ is a duplication of some cuboid.

We are now ready to prove that Conjectures 6 and 1 are equivalent.

Theorem 10. *Conjecture 6 is true for k if, and only if, Conjecture 1 is true for k.*

Proof. We already showed ($\Leftarrow$). It remains to prove ($\Rightarrow$). Suppose Conjecture 1 is false for some $k \geq 4$. That is, there is an ideal k-wise intersecting clutter $\mathcal{C}$. Let $\mathcal{C}'$ be a deletion minor of $\mathcal{C}$ that is minimal subject to $\tau(\mathcal{C}') \geq 2$. By Remark 8, $\mathcal{C}'$ is an ideal tangled k-wise intersecting clutter. Let

$$\mathcal{C}'' := \{C \in \mathcal{C}' : |C \cap \{u,v\}| = 1 \ \forall \ \{u,v\} \in b(\mathcal{C}')\}.$$

By Theorem 9, $\mathcal{C}''$ is an ideal tangled clutter that is a duplication of some cuboid, say cuboid(S). As every k members of $\mathcal{C}'$ have a common element, so do every k members of $\mathcal{C}''$, so the latter is k-wise intersecting. As a result, cuboid(S) is an ideal k-wise intersecting clutter, so by Remark 4 and Theorem 5, S is a cube-ideal set whose points do not agree on a coordinate yet every k points do. Therefore, S refutes Conjecture 6 for k, as required. □

3 Proof of Theorem 2

Let $S \subseteq \{0,1\}^n$. For $x, y \in \{0,1\}^n$, $x \triangle y$ denotes the coordinate-wise sum of x, y modulo 2. We say that S is a *vector space over $GF(2)$*, or simply a *binary space*, if $a \triangle b \in S$ for all $a, b \in S$. Notice that a nonempty binary space necessarily contains $\mathbf{0}$.

Remark 11 ([4], Remark 7.5). Let $\mathbf{0} \in S \subseteq \{0,1\}^n$. If cuboid($S$) is a binary clutter, then S is a binary space.

3.1 The 8-flow Theorem

Let $G = (V, E)$ be a graph where loops and parallel edges are allowed, where every loop is treated as an edge not incident to any vertex. A *cycle* is a subset $C \subseteq E$ such that every vertex is incident with an even number of edges in C. A *bridge of G* is an edge e that does not belong to any cycle. The *cycle space of G* is the set

$$\text{cycle}(G) := \{\chi_C : C \subseteq E \text{ is a cycle}\} \subseteq \{0,1\}^E$$

where χ_C denotes the incidence vector of C. As $\varnothing$ is a cycle, and the symmetric difference of any two cycles is also a cycle, it follows that cycle(G) is a binary space. We require the following two results on cycle spaces of graphs.

Remark 12. Let $G = (V, E)$ be a graph. Then the points in cycle(G) agree on a coordinate if, and only if, G has a bridge. Moreover, for all $k \in \mathbb{N}$, cycle(G) has a subset of at most $k+1$ points that do not agree on a coordinate if, and only if, G has at most k cycles the union of which is E.

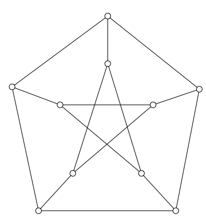
Theorem 13 ([1], Corollary 2.6 and Theorem 2.8). *The cycle space of every graph is a cube-ideal set.*

We need the following version of the celebrated 8-*Flow Theorem* of Jaeger [16].

Theorem 14 ([16]). *Every bridgeless graph* $G = (V, E)$ *contains at most* 3 *cycles the union of which is* E*. That is, given the set* $\mathrm{cycle}(G) \subseteq \{0,1\}^E$*, either all the points agree on a coordinate, or there is a subset of at most* 4 *points that do not agree on a coordinate.*

One may wonder whether the $3, 4$ in Theorem 14 may be replaced by $2, 3$? The answer is no, due to the Petersen graph (see Figure 1a):

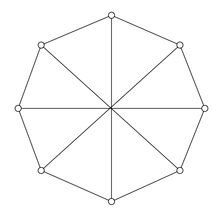
Remark 15 (see [26]). The edge set of the Petersen graph is not the union of 2 cycles.



(a) Petersen.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

(b) Fano.



(c) Wagner.

As a consequence, an ideal 3-wise intersecting clutter does exist:

Proof of Proposition 3. Let S be the cycle space of the Petersen graph, and let $\mathcal{C} := \mathrm{cuboid}(S)$. By Remark 15, the Petersen is a bridgeless graph that does not have 2 cycles the union of which is the edge set, so by Remark 12, the points in S do not agree on a coordinate, but every subset of $2+1 = 3$ points do. Moreover, S is a cube-ideal set by Theorem 13. Therefore, by Remark 4 and Theorem 5, $\mathcal{C}$ is an ideal 3-wise intersecting clutter, as required. □

The cuboid of the cycle space of the Petersen graph has already shown up in the literature, and is denoted T_{30} by Schrijver [21], §79.3e. Consider the graph obtained from Petersen by subdividing every edge once, and let T be any vertex subset of even cardinality containing all the new vertices. Then the clutter of minimal T-joins of this graft is precisely T_{30}. This construction is due to Seymour ([24], page 440).

3.2 Sums of Circuits Property

For background and notation regarding binary matroids, we refer the reader to Appendix A.

A binary matroid has the *sums of circuits property* if its cycle space is a cube-ideal set. This notion is due to Seymour [24].[5] By Theorem 13, graphic matroids have the sums of circuits property [24]. Seymour also proved a decomposition theorem [25] for binary matroids with the sums of circuits property. It turns out they can all be produced from graphic matroids and two other matroids, which we now describe. The *Fano matroid* F_7 is the binary matroid represented by the matrix in Figure 1b. The second matroid is $M(V_8)^\star$, where V_8 is the graph in Figure 1c. Seymour [25] showed that F_7 and $M(V_8)^\star$ both have the sums of circuits property.

To generate all binary matroids with the sums of circuits property, we require three composition rules. Let M_1, M_2 be binary matroids over ground sets E_1, E_2, respectively. We denote by $M_1 \triangle M_2$ the binary matroid over ground set $E_1 \triangle E_2$ whose cycles are all subsets of $E_1 \triangle E_2$ of the form $C_1 \triangle C_2$, where C_i is a cycle of M_i for $i \in [2]$. Then $M_1 \triangle M_2$ is a 1-*sum* if $E_1 \cap E_2 = \emptyset$; $M_1 \triangle M_2$ is a 2-*sum* if $E_1 \cap E_2 = \{e\}$, where e is neither a loop nor a coloop of M_1 or M_2; and $M_1 \triangle M_2$ is a Y-*sum* if $E_1 \cap E_2$ is a cocircuit of cardinality 3 in both M_1 and M_2 and contains no circuit in M_1 or M_2.

Theorem 16 ([25], (6.4), (6.7), (6.10) and (16.4)). *Let M be a binary matroid with the sums of circuits property. Then M is obtained recursively by means of 1-sums, 2-sums and Y-sums starting from copies of $F_7, M(V_8)^\star$ and graphic matroids.*

We are ready to prove Conjecture 6 for cube-ideal binary spaces.

Theorem 17. *Every binary matroid without a coloop and with the sums of circuits property has at most 3 cycles the union of which is the ground set.*

Proof. A 3-*cycle cover* of a binary matroid is three (not necessarily distinct) cycles whose union is the ground set.

Claim. Both F_7 and $M(V_8)^\star$ have 3-cycle covers.

Subproof. Given the matrix representation of F_7 in Figure 1a, label the columns $1, \dots, 7$ from left to right. Then $\emptyset, \{1,2,3,7\}, \{4,5,6\}$ a 3-cycle cover of F_7. Next, label the vertices of V_8 so that the outer 8-cycle is labelled $1, \dots, 8$. Then $M(V_8)^\star$ has a 3-cycle cover given by the following cuts of V_8: $\delta(\{1,6,7,8\})$, $\delta(\{1,7\}), \delta(\{2,4\})$, where $\delta(X)$ is the set of edges with exactly one end in X. $\diamond$

Claim. Let M, M_1, M_2 be binary matroids such that $M = M_1 \triangle M_2$ and $M_i, i \in [2]$ has a 3-cycle cover. Then the following statements hold:

(i) If M is a 1-sum of M_1, M_2, then M has a 3-cycle cover.
(ii) If M is a 2-sum of M_1, M_2, then M has a 3-cycle cover.
(iii) If M is a Y-sum of M_1, M_2, then M has a 3-cycle cover.

[5] Seymour's definition appears different from ours, but they are equivalent by [1], Corollary 2.6.

Subproof. For $i \in [2]$, let E_i be the ground set of M_i and C_1^i, C_2^i, C_3^i be a 3-cycle cover of M_i. Clearly, **(i)** holds. For **(ii)**, let $E_1 \cap E_2 = \{e\}$. We may assume $e \in C_1^i$ for all $i \in [2]$. By replacing C_2^i by $C_1^i \triangle C_2^i$ if necessary, we may assume $e \notin C_2^i$ for all $i \in [2]$. Similarly, we may assume $e \notin C_3^i$ for $i \in [2]$. But now $\{C_j^1 \triangle C_j^2 : j \in [3]\}$ is a 3-cycle cover of M. For **(iii)**, suppose $E_1 \cap E_2 = \{e, f, g\}$. Since $\{e, f, g\}$ is a cocircuit of both M_1, M_2, and since cocircuits and circuits of a binary matroid have an even number of elements in common, $|C_j^i \cap \{e, f, g\}| \in \{0, 2\}$ for all i, j. Therefore, after possibly relabeling e, f, g simultaneously in M_1 and M_2, and after possibly relabeling C_1^i, C_2^i, C_3^i for all i, we may assume that

- $C_1^i \cap \{e, f, g\} = \{e, f\}$ for all $i \in [2]$, and
- $C_2^i \cap \{e, f, g\} = \{e, g\}$ or $\{f, g\}$ for all $i = [2]$.

For $i \in [2]$, after possibly replacing C_2^i with $C_2^i \triangle C_1^i$, we may assume $C_2^i \cap \{e, f, g\} = \{e, g\}$. For $i \in [2]$, after possibly replacing C_3^i with $C_3^i \triangle C_1^i, C_3^i \triangle C_2^i$ or $C_3^i \triangle C_1^i \triangle C_2^i$, we may assume $C_3^i \cap \{e, f, g\} = \emptyset$. But now $\{C_j^1 \triangle C_j^2 : j \in [3]\}$ is a 3-cycle cover of M, as required.⋄

We leave the proof of the following claim as an easy exercise for the reader.

Claim. Let M, M_1, M_2 be binary matroids such that $M = M_1 \triangle M_2$, where $\triangle$ is either a 1-, 2- or Y-sum. If M has no coloop, then neither do M_1, M_2.⋄

The proof is complete by combining the above claims with Theorems 14 and 16. □

3.3 Proof of Theorem 2

Proof of Theorem 2. We prove the contrapositive statement. Let $\mathcal{C}$ be an ideal binary clutter such that $\tau(\mathcal{C}) \geq 2$. We need to exhibit ≤ 4 members without a common element. Let $\mathcal{C}'$ be a deletion minor of $\mathcal{C}$ that is minimal subject to $\tau(\mathcal{C}') \geq 2$. It suffices to exhibit ≤ 4 members of $\mathcal{C}'$ without a common element. Notice that $\mathcal{C}'$ is ideal, and as a minor of a binary clutter, it is also binary [22]. Moreover, by our minimality assumption, $\mathcal{C}'$ is a tangled clutter. Thus, by Proposition 7, $\mathcal{C}'$ is a duplication of a cuboid, say cuboid(S) where we may choose S so that $\mathbf{0} \in S$. It suffices to exhibit ≤ 4 members of cuboid(S) without a common element.

Note that cuboid(S) is an ideal binary cuboid with $\tau(\text{cuboid}(S)) \geq 2$. So, by Theorem 5 and Remark 11, S is a cube-ideal binary space whose points do not agree on a coordinate. By Theorem 17, S has ≤ 4 points that do not agree on a coordinate, thereby yielding ≤ 4 members of cuboid(S) without a common element, as required. □

4 Applications and two more conjectures

4.1 Embedding projective geometries

In this section, we propose a strengthening of Conjecture 1. We begin by motivating our strengthening. Conjecture 1 predicts that for some $k \geq 4$, every ideal clutter with covering number at least two has k members without a common element. By moving to a deletion minor, if necessary, we may assume that our ideal clutter is tangled. Our stronger conjecture predicts that the clutter must actually have 2^{k-1} members that "correspond to a projective geometry", and of these members, k many will not have a common element.

Let A be the $(k-1)\times(2^{k-1}-1)$ matrix whose columns are all the nonzero vectors in $\{0,1\}^{k-1}$. The binary matroid represented by A is called a *projective geometry over* $GF(2)$, and is denoted $PG(k-2,2)$. Let $r := 2^{k-1}-1$. Recall that $\mathrm{cocycle}(PG(k-2,2)) \subseteq \{0,1\}^r$ is the row space of A generated over $GF(2)$. Note that the $k-1$ points in $\mathrm{cocycle}(PG(k-2,2))$ corresponding to the rows of A agree on precisely one coordinate, which is set to 1. These $k-1$ points, together with the zero point $\mathbf{0}$, yield k points that do not agree on a coordinate. This yields the following remark.

Remark 18. There are k points of $\mathrm{cocycle}(PG(k-2,2))$ that do not agree on a coordinate. In particular, $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-2,2)))$ has k members without a common element.

Let $\mathcal{C}$ be an ideal tangled clutter. We say that $\mathcal{C}$ *embeds the projective geometry* $PG(k-2,2)$ if a subset of $\mathcal{C}$ is a *duplication* of $\mathrm{cuboid}(\mathrm{cocycle}(PG(k-2,2)))$. This notion is due to [4]. We propose the following conjecture.

Conjecture 19. There exists an integer $k \geq 4$ such that every ideal tangled clutter embeds one of $PG(0,2),\dots,PG(k-2,2)$.

In Appendix B we show that Conjecture 19 is indeed a strengthening of Conjecture 1.

Proposition 20. *If Conjecture 19 holds for k, then Conjecture 1 holds for k.*

Proposition 21 ([4], Proposition 7.4). *Let S be a binary space of $GF(2)$-rank r whose points do not agree on a coordinate. Then* $\mathrm{cuboid}(S)$ *embeds one of $PG(0,2),\dots,PG(r-1,2)$.*

As an application of Theorem 2, we now prove Conjecture 19 for $k=3$ for the class of binary clutters.

Theorem 22. *Every ideal binary tangled clutter embeds $PG(0,2)$, $PG(1,2)$, or $PG(2,2)$.*

Proof. Let $\mathcal{C}$ be a binary tangled clutter. By Proposition 7, $\mathcal{C}$ is a duplication of a cuboid, say $\mathrm{cuboid}(S)$ for some S containing $\mathbf{0}$. It suffices to show that $\mathrm{cuboid}(S)$ embeds one of the three projective geometries. Note that $\mathrm{cuboid}(S)$ is an ideal binary cuboid with $\tau(\mathrm{cuboid}(S)) \geq 2$. By Theorem 5 and Remark 11, S is a cube-ideal binary space whose points do not agree on a coordinate. By Theorem 17, S has a subset of at most 3 points that do not agree on a coordinate. Let S' be the binary space generated by these points. Note that $S' \subseteq S$, the points in S' do not agree on a coordinate, and S' has $GF(2)$-rank at most 3. By Proposition 21, $\mathrm{cuboid}(S')$, and therefore $\mathrm{cuboid}(S)$, embeds one of $PG(0,2), PG(1,2), PG(2,2)$, as desired. □

We now give an application of Theorem 22. Let G be a bridgeless graph. By applying Theorem 22 to cuboid(cycle(G)), G has 8 cycles where every edge is used in exactly 4 of the cycles. Since one of the 8 cycles may be assumed to be $\emptyset$, G has 7 cycles such that each edge is in exactly 4 of the cycles. This is Proposition 6 of [6].

4.2 Dyadic fractional packings

We finish by deriving another consequence of Theorem 2. Let $\mathcal{C}$ be a clutter over ground set V. A *fractional packing of* $\mathcal{C}$ is a vector $y \in \mathbb{R}_+^{\mathcal{C}}$ such that $\sum (y_C : C \in \mathcal{C}, v \in C) \leq 1$ for all $v \in V$. The *value of* y is $\mathbf{1}^\top y$. For $n \in \mathbb{N}$, the vector y is $\frac{1}{n}$-*integral* if every entry is $\frac{1}{n}$-integral.

Proposition 23 ([4], follows from Theorem 1.16). *For every* $k \in \mathbb{Z}_{\geq 0}$*,* cuboid(cocycle($PG(k,2)$)) *has a* $\frac{1}{2^k}$*-integral packing of value* 2*.*

This, combined with Theorem 22, implies the following:

Theorem 24. *Every ideal binary clutter* $\mathcal{C}$ *with* $\tau(\mathcal{C}) \geq 2$ *has a* $\frac{1}{4}$*-integral packing of value* 2*.*

Proof. Let $\mathcal{C}'$ be a deletion minor of $\mathcal{C}$ that is minimal subject to $\tau(\mathcal{C}') \geq 2$. Then $\mathcal{C}'$ is an ideal binary tangled clutter, so by Theorem 22, $\mathcal{C}'$ embeds one of $PG(0,2), PG(1,2), PG(2,2)$. By Proposition 23, $\mathcal{C}'$, and therefore $\mathcal{C}$, has a $\frac{1}{4}$-integral packing of value 2, as required. □

In fact, Seymour conjectures a far-reaching generalization of this theorem:

Conjecture 25 (Seymour 1975, see [21], §79.3e). Every ideal clutter $\mathcal{C}$ has a $\frac{1}{4}$-integral packing of value $\tau(\mathcal{C})$.

This conjecture is open even for binary clutters, and in particular, for the clutter of minimal T-joins of a graft ([8], Conjecture 2.15).

Proposition 26. *If Conjecture 25 is true, then so is Conjecture 1 for* $k = 5$*.*

Proof. Assume Conjecture 25 is true. Let $\mathcal{C}$ be an ideal clutter with $\tau(\mathcal{C}) \geq 2$. Let $\mathcal{C}'$ be a deletion minor of $\mathcal{C}$ with $\tau(\mathcal{C}') = 2$. Since $\mathcal{C}'$ is also ideal, it has a $\frac{1}{4}$-integral packing $y \in \mathbb{R}_+^{\mathcal{C}'}$ of value 2. Notice that $y_C \in \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$ for each $C \in \mathcal{C}'$. In particular, $|\{C : y_C > 0\}| \leq 8$. Pick a minimal subset $\mathcal{C}'' \subseteq \{C : y_C > 0\}$ such that $\sum_{C \in \mathcal{C}''} y_C > 1$. Then $|\mathcal{C}''| \leq 5$, and it is easily checked that the members of $\mathcal{C}''$ cannot have a common element. As a result, $\mathcal{C}'$, and therefore $\mathcal{C}$, has a subset of at most 5 members without a common element, as required. □

Acknowledgements

This work was supported by NSERC PDF grant 516584-2018, ONR grant 00014-18-12129, the Australian Research Council, ERC grant FOREFRONT (grant agreement no. 615640), and the Institute for Basic Science (IBS-R029-C1).

References

1. Abdi, A., Cornuéjols, G., Guričanová, N., Lee, D.: Cuboids, a class of clutters. Journal of Combinatorial Theory, Series B (2019)
2. Abdi, A., Cornuéjols, G., Lee, D.: Intersecting restrictions in clutters (2018)
3. Abdi, A., Cornuéjols, G., Lee, D.: Identically self-blocking clutters. In: Integer programming and combinatorial optimization, Lecture Notes in Comput. Sci., vol. 11480, pp. 1–12. Springer, Cham (2019)
4. Abdi, A., Cornuéjols, G., Lee, D., Superdock, M.: Projective geometries, simplices and clutters. `https://arxiv.org/abs/1908.10629` (2019)
5. Abdi, A., Pashkovich, K., Cornuéjols, G.: Ideal clutters that do not pack. Mathematics of Operations Research **43**(2), 533–553 (2017)
6. Bermond, J.C., Jackson, B., Jaeger, F.: Shortest coverings of graphs with cycles. J. Combin. Theory Ser. B **35**(3), 297–308 (1983)
7. Conforti, M., Corneujois, G.: Clutters that pack and the max flow min cut property: a conjecture. Tech. rep., The Fourth Bellairs Workshop on Combinatorial Optimization (1993)
8. Cornuéjols, G.: Combinatorial optimization: Packing and covering, vol. 74. SIAM (2001)
9. Cornuéjols, G., Guenin, B., Margot, F.: The packing property. Math. Program. **89**(1, Ser. A), 113–126 (2000)
10. Cornuéjols, G., Novick, B.: Ideal $0,1$ matrices. J. Combin. Theory Ser. B **60**(1), 145–157 (1994)
11. Edmonds, J., Fulkerson, D.R.: Bottleneck extrema. J. Combinatorial Theory **8**, 299–306 (1970)
12. Edmonds, J., Giles, R.: A min-max relation for submodular functions on graphs. In: Studies in integer programming (Proc. Workshop, Bonn, 1975). pp. 185–204. Ann. of Discrete Math., Vol. 1 (1977)
13. Fulkerson, D.R.: Blocking polyhedra. In: Graph Theory and its Applications (Proc. Advanced Sem., Math. Research Center, Univ. of Wisconsin, Madison, Wis., 1969). pp. 93–112. Academic Press, New York (1970)
14. Guenin, B.: A characterization of weakly bipartite graphs. J. Combin. Theory Ser. B **83**(1), 112–168 (2001)
15. Isbell, J.R.: A class of simple games. Duke Math. J. **25**, 423–439 (1958)
16. Jaeger, F.: Flows and generalized coloring theorems in graphs. J. Combin. Theory Ser. B **26**(2), 205–216 (1979)
17. Lehman, A.: A solution of the Shannon switching game. J. Soc. Indust. Appl. Math. **12**, 687–725 (1964)
18. Lehman, A.: On the width-length inequality. Math. Programming **16**(2), 245–259 (1979)
19. Lucchesi, C.L., Younger, D.H.: A minimax theorem for directed graphs. J. London Math. Soc. (2) **17**(3), 369–374 (1978)
20. Oxley, J.: Matroid theory, Oxford Graduate Texts in Mathematics, vol. 21. Oxford University Press, Oxford, second edn. (2011)
21. Schrijver, A.: Combinatorial optimization. Polyhedra and efficiency, Algorithms and Combinatorics, vol. 24. Springer-Verlag, Berlin (2003)
22. Seymour, P.D.: The forbidden minors of binary clutters. J. London Math. Soc. (2) **12**(3), 356–360 (1975/76)
23. Seymour, P.D.: The matroids with the max-flow min-cut property. J. Combinatorial Theory Ser. B **23**(2-3), 189–222 (1977)

24. Seymour, P.D.: Sums of circuits. In: Graph theory and related topics (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1977). pp. 341–355. Academic Press, New York-London (1979)
25. Seymour, P.D.: Matroids and multicommodity flows. European J. Combin. **2**(3), 257–290 (1981)
26. Tutte, W.T.: On the algebraic theory of graph colorings. J. Combinatorial Theory **1**, 15–50 (1966)

A Binary Matroids

For basics on matroids, we refer the reader to Oxley [20]. Let E be a finite set, $S \subseteq \{0,1\}^E$ a binary space, and $S^\perp$ the orthogonal complement of S, that is, $S^\perp = \{y \in \{0,1\}^E : y^\top x \equiv 0 \pmod 2\ \forall x \in S\}$. Notice that $S^\perp$ is another binary space, and that $(S^\perp)^\perp = S$. Therefore, there exists a 0−1 matrix A whose columns are labeled by E such that $S = \left\{x \in \{0,1\}^E : Ax \equiv \mathbf{0} \pmod 2\right\}$, and $S^\perp$ is the row space of A generated over $GF(2)$.

Let $\mathcal{S} := \{C \subseteq E : \chi_C \in S\}$. The pair $M := (E, \mathcal{S})$ is a *binary matroid*, and the matrix A is a *representation of* M. We call E the *ground set of* M. The sets in $\mathcal{S}$ are the *cycles of* M, and $\mathcal{S}$ is the *cycle space of* M, denoted by cycle(M). The minimal nonempty sets in $\mathcal{S}$ are the *circuits of* M, and the circuits of cardinality one are *loops*.

Let $\mathcal{S}^\perp := \{D \subseteq E : \chi_D \in S^\perp\}$. The binary matroid $M^\star := (E, \mathcal{S}^\perp)$ is the *dual of* M. Notice that $(M^\star)^\star = M$. The sets in $\mathcal{S}^\perp$ are the *cocycles of* M, and $\mathcal{S}^\perp$ is the *cocycle space of* M, denoted by cocycle(M). The minimal nonempty sets in $\mathcal{S}^\perp$ are the *cocircuits of* M, and the cocircuits of cardinality one are *coloops of* M.

Remark 27. Let M be a binary matroid. Then the points in cycle(M) agree on a coordinate if, and only if, M has a coloop. Moreover, for every integer $k \geq 1$, cycle(M) has a subset of at most $k+1$ points that do not agree on a coordinate if, and only if, M has at most k cycles the union of which is E.

Let $G = (V, E)$ be a graph. The binary matroid whose cycle space is cycle(G) is a *graphic matroid*, and is denoted $M(G)$. Notice the one-to-one correspondence between the cycles of $M(G)$ and the cycles of G, between the loops of $M(G)$ and the loops of G, between the cocycles of $M(G)$ and the cuts of G, and between the coloops of $M(G)$ and the bridges of G. Therefore, Remark 27 is an extension of Remark 12.

B Proof of Proposition 20

Proof of Proposition 20. Assume Conjecture 19 holds for k. Let $\mathcal{C}$ be an ideal clutter with $\tau(C) \geq 2$. Let $\mathcal{C}'$ be a deletion minor of $\mathcal{C}$ that is minimal subject to $\tau(\mathcal{C}') \geq 2$. Then $\mathcal{C}'$ is an ideal tangled clutter. Thus, $\mathcal{C}'$ embeds $PG(n-2,2)$ for some $n \in \{2, \dots, k\}$. That is, a duplication of cuboid$(PG(n-2,2))$ is a subset of $\mathcal{C}'$. By Remark 18, cuboid$(PG(n-2,2))$ has n members without a common element, so the duplication, and therefore $\mathcal{C}'$, must have n members without a common element. Thus, $\mathcal{C}$ has $n \leq k$ members without a common element, as required. □